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Grupo de problemas inversos em geofísica

Camada equivalente aplicada ao processamento e interpretação de dados de campos potenciais

Vanderlei C. Oliveira Jr.

2016

Observatório Nacional

SBGf

Integral de continuação para cima

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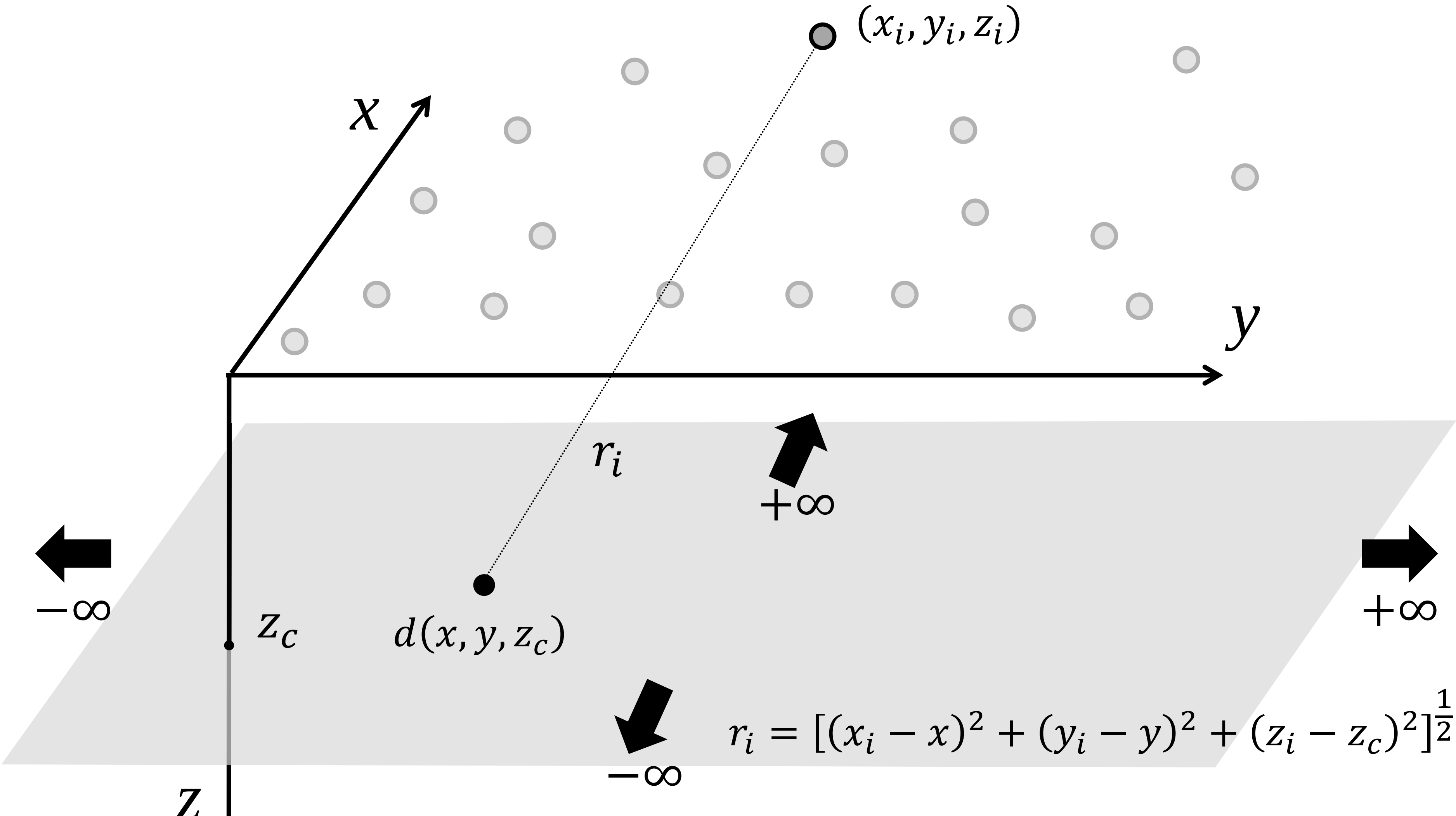
2016

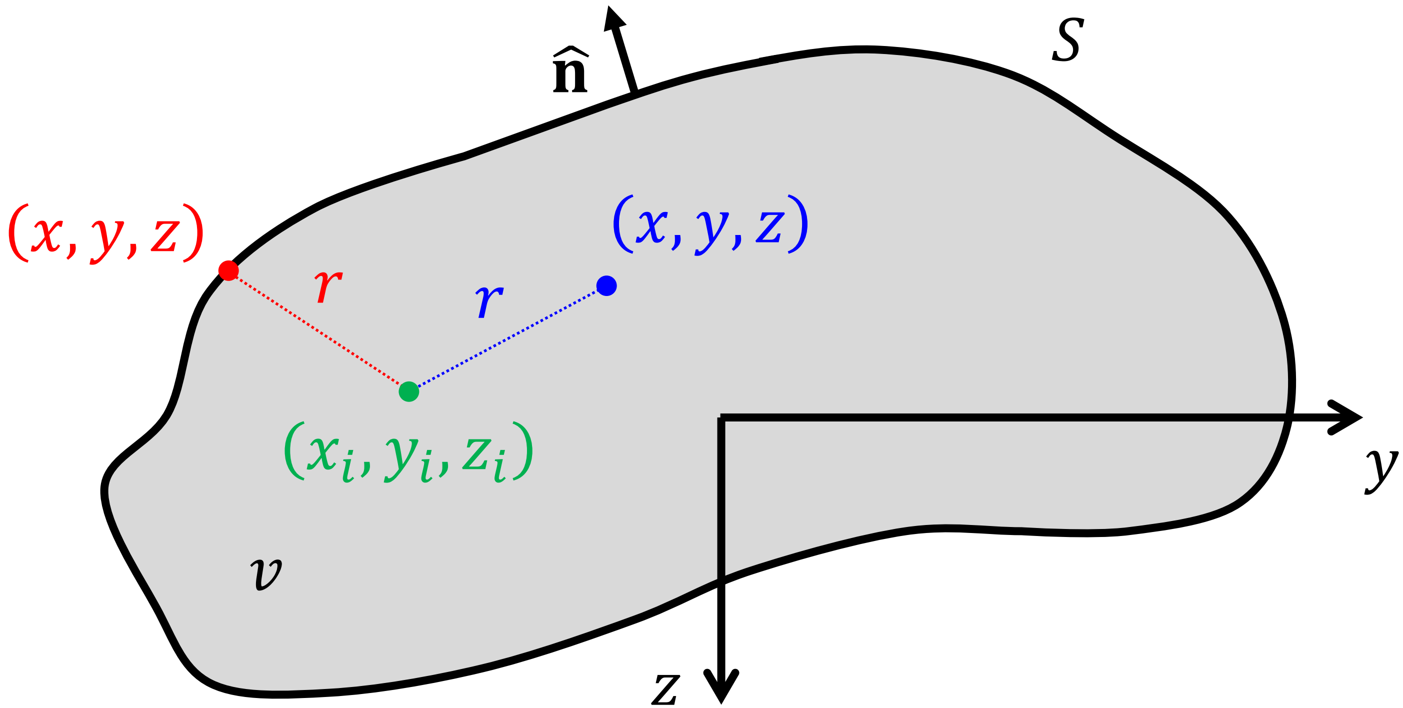
Tal como mencionado anteriormente, a técnica da camada equivalente é baseada em uma equação integral chamada **integral de continuação para cima** (Skeels, 1947; Henderson and Zietz, 1949; Henderson, 1960; Roy, 1962; Bhattacharyya, 1967; Henderson, 1970; Twomey, 1977; Blakely, 1996)



Integral de continuação para cima

$$d(x_i, y_i, z_i) = \frac{z_c - z_i}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d(x, y, z_c)}{r_i^3} dxdy \ , \qquad z_c > z_i$$

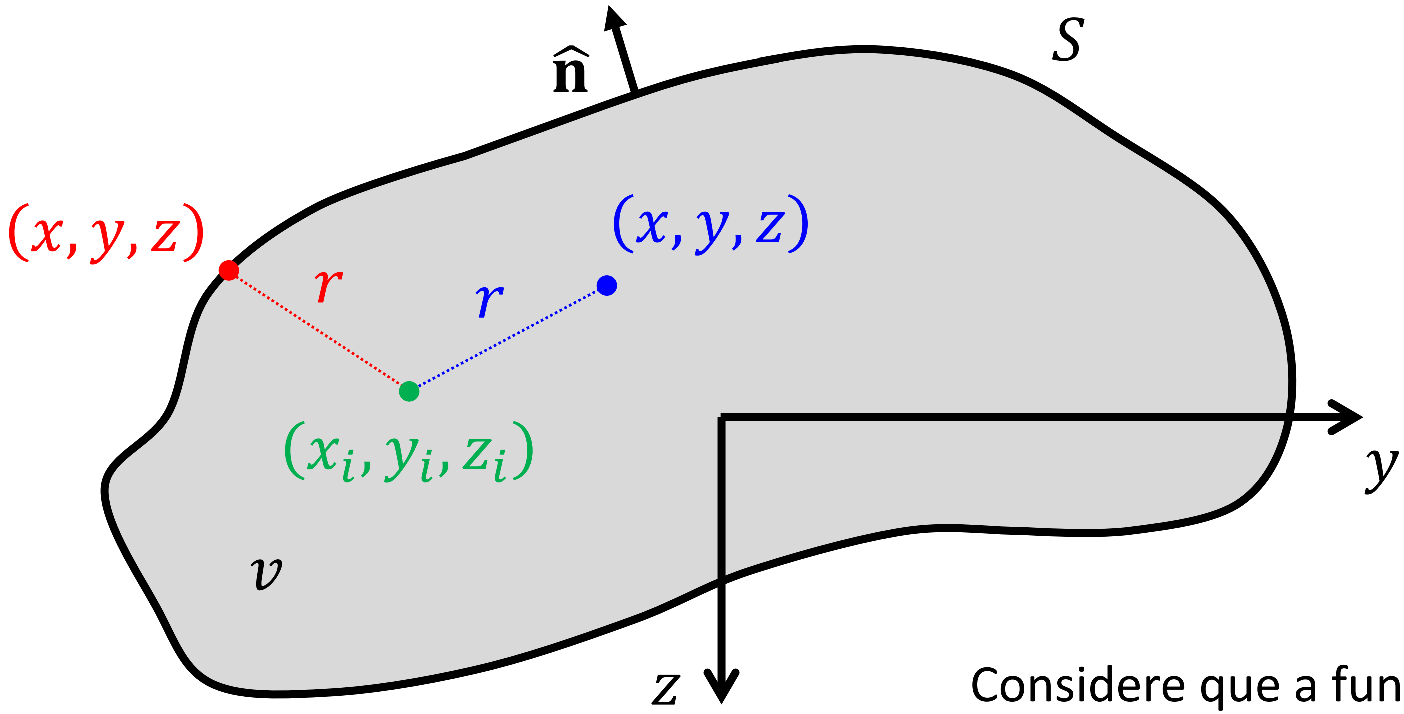
A integral de continuação para cima é deduzida a partir das **identidades de Green** (Green, 1871; Kellogg, 1929)



$$U_i \equiv U(x_i, y_i, z_i) \qquad r = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}$$

$$U_i = -\frac{1}{4\pi} \iiint\limits_{v} \frac{1}{r} \nabla^2 U \, dv - \frac{1}{4\pi} \iint\limits_{S} U \left(\nabla \frac{1}{r} \right)^{\mathrm{T}} \hat{\mathbf{n}} - \frac{1}{r} \nabla U^{\mathrm{T}} \hat{\mathbf{n}} \; dS$$

Terceira identidade de Green (Kellogg, 1929)

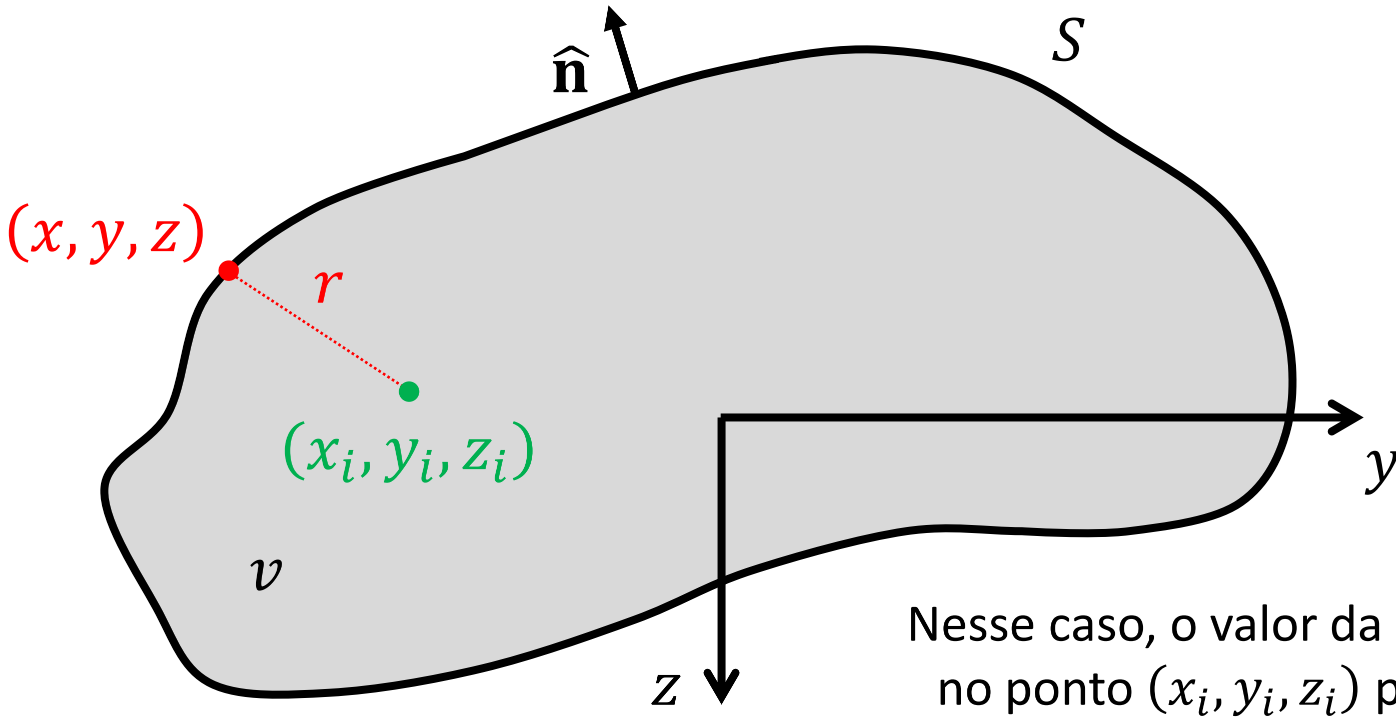


Considere que a função U seja harmônica no volume v

$$U_i \equiv U(x_i, y_i, z_i)$$

$$U_i = -\frac{1}{4\pi} \iiint\limits_v \frac{1}{r} \cancelto{0}{\nabla^2 U} \, dv - \frac{1}{4\pi} \iint\limits_S U \left(\nabla \frac{1}{r} \right)^{\mathrm{T}} \hat{\mathbf{n}} - \frac{1}{r} \nabla U^{\mathrm{T}} \hat{\mathbf{n}} \; dS$$

Terceira identidade de Green (Kellogg, 1929)

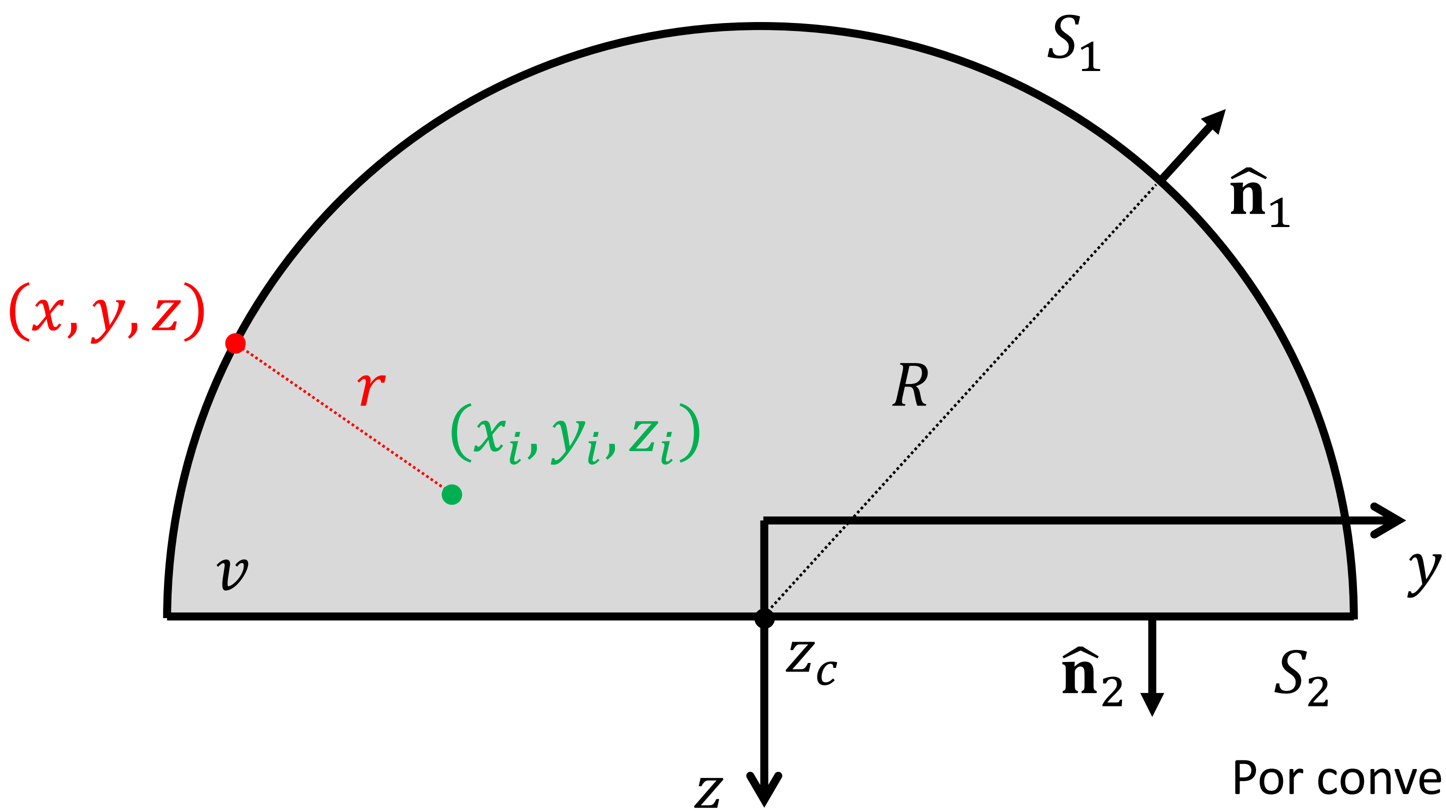


Nesse caso, o valor da função U no ponto (x_i, y_i, z_i) pode ser determinado a partir de uma combinação entre os valores de U e de suas primeiras derivadas sobre a superfície S

$$U_i \equiv U(x_i, y_i, z_i)$$

$$U_i = -\frac{1}{4\pi} \iint_S U \left(\nabla \frac{1}{r} \right)^{\mathrm{T}} \hat{\mathbf{n}} - \frac{1}{r} \nabla U^{\mathrm{T}} \hat{\mathbf{n}} \; dS$$

Terceira identidade de Green (Kellogg, 1929)

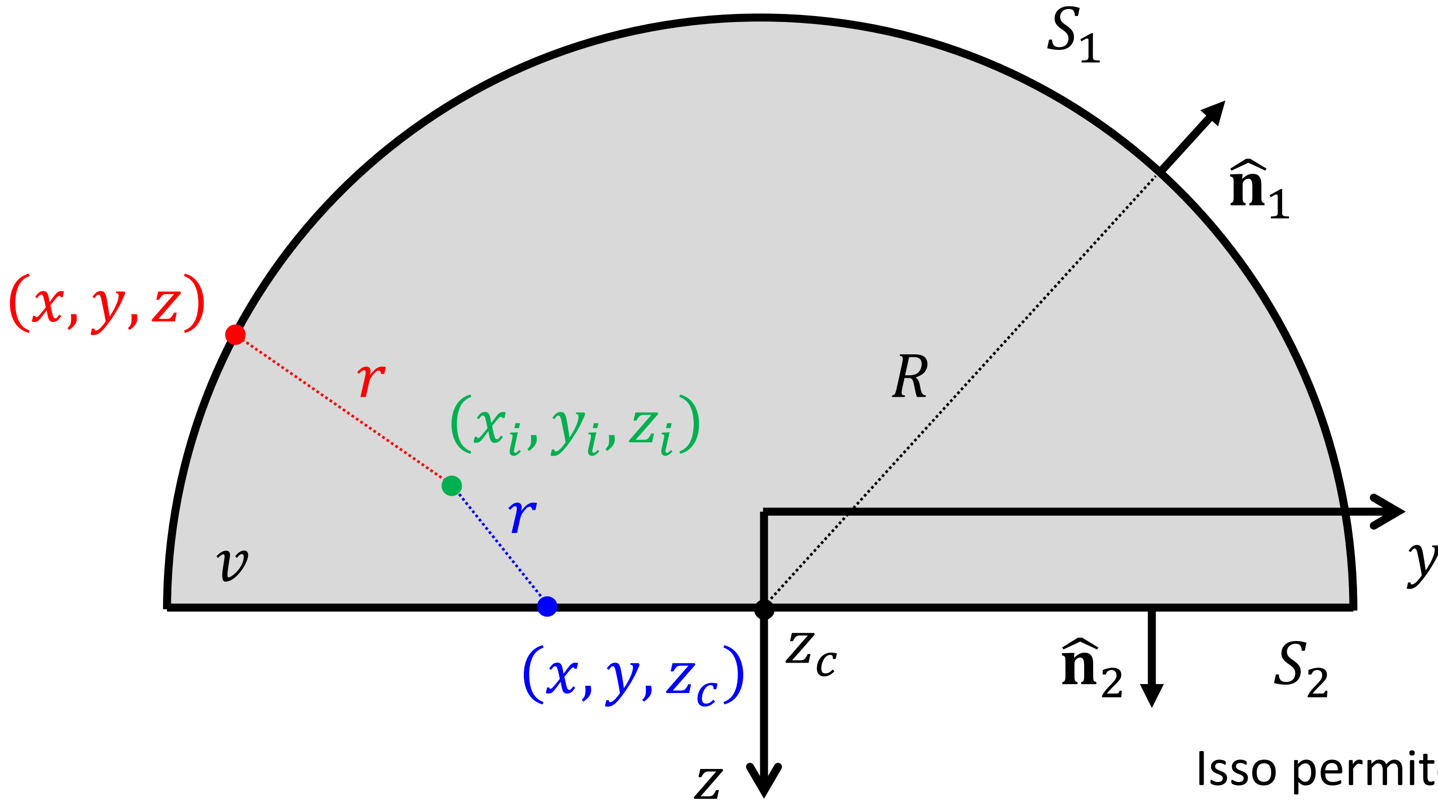


Por conveniência, considere uma região v limitada por uma semiesfera S_1 de raio R e um plano horizontal S_2 localizado em z_c , acima das fontes

$$U_i \equiv U(x_i, y_i, z_i)$$

$$U_i = -\frac{1}{4\pi} \iint_S U \left(\nabla \frac{1}{r} \right)^{\mathrm{T}} \hat{\mathbf{n}} - \frac{1}{r} \nabla U^{\mathrm{T}} \hat{\mathbf{n}} \; dS$$

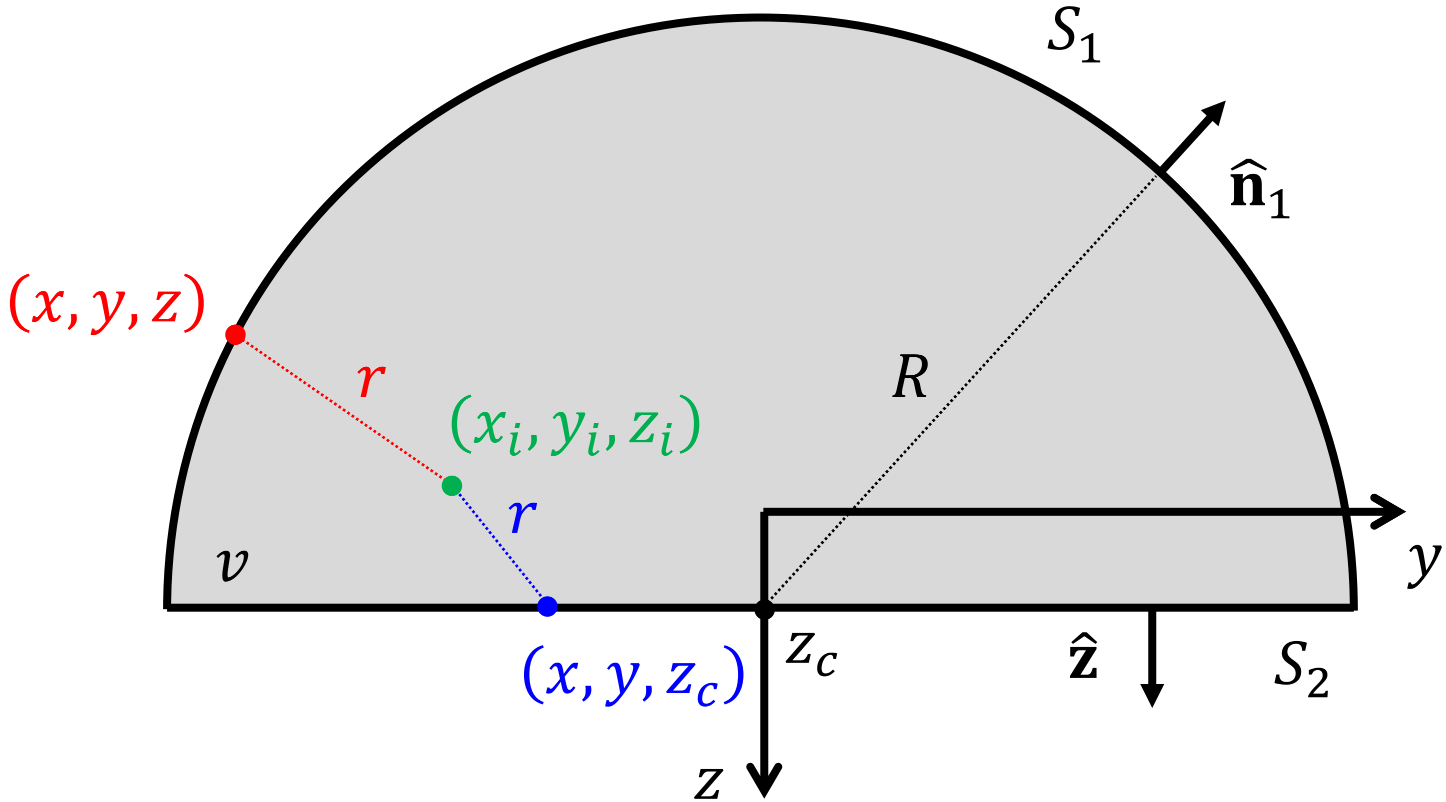
Terceira identidade de Green (Kellogg, 1929)



Isso permite dividir a integral de superfície em duas partes, uma avaliada sobre S_1 e a outra avaliada sobre S_2

$$U_i = -\frac{1}{4\pi} \iint_{S_1} U \left(\nabla \frac{1}{r} \right)^{\mathrm{T}} \hat{\mathbf{n}}_1 - \frac{1}{r} \nabla U^{\mathrm{T}} \hat{\mathbf{n}}_1 \; dS_1 +$$

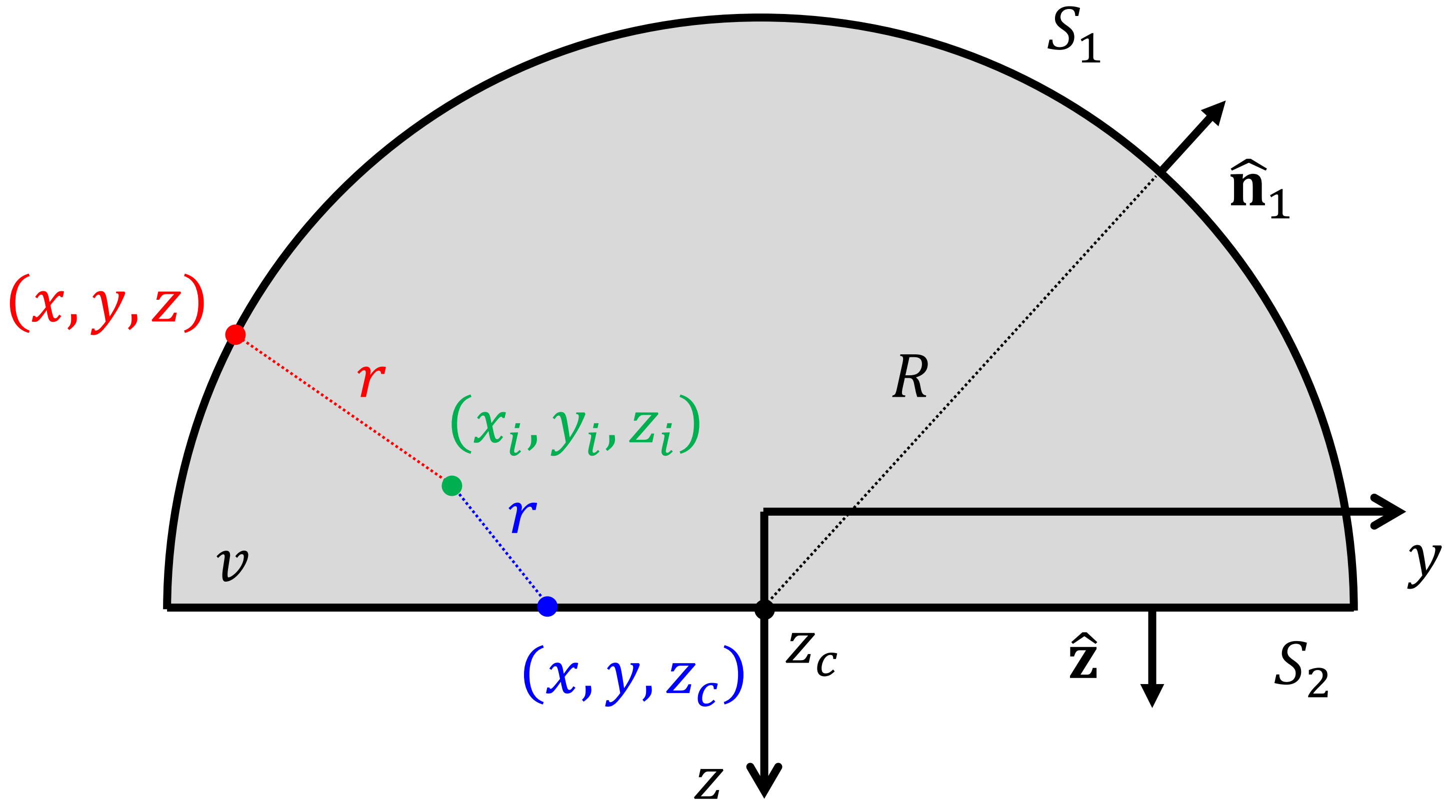
$$-\frac{1}{4\pi} \iint_{S_2} U \left(\nabla \frac{1}{r} \right)^{\mathrm{T}} \hat{\mathbf{n}}_2 - \frac{1}{r} \nabla U^{\mathrm{T}} \hat{\mathbf{n}}_2 \; dS_2$$



$$U_i = -\frac{1}{4\pi} \iint_{S_1} U \left(\nabla \frac{1}{r}\right)^{\mathrm{T}} \hat{\mathbf{n}}_1 - \frac{1}{r} \nabla U^{\mathrm{T}} \hat{\mathbf{n}}_1 \; dS_1 +$$

$$-\frac{1}{4\pi} \iint_{S_2} U \left(\nabla \frac{1}{r}\right)^{\mathrm{T}} \hat{\mathbf{z}} - \frac{1}{r} \nabla U^{\mathrm{T}} \hat{\mathbf{z}} \; dxdy$$

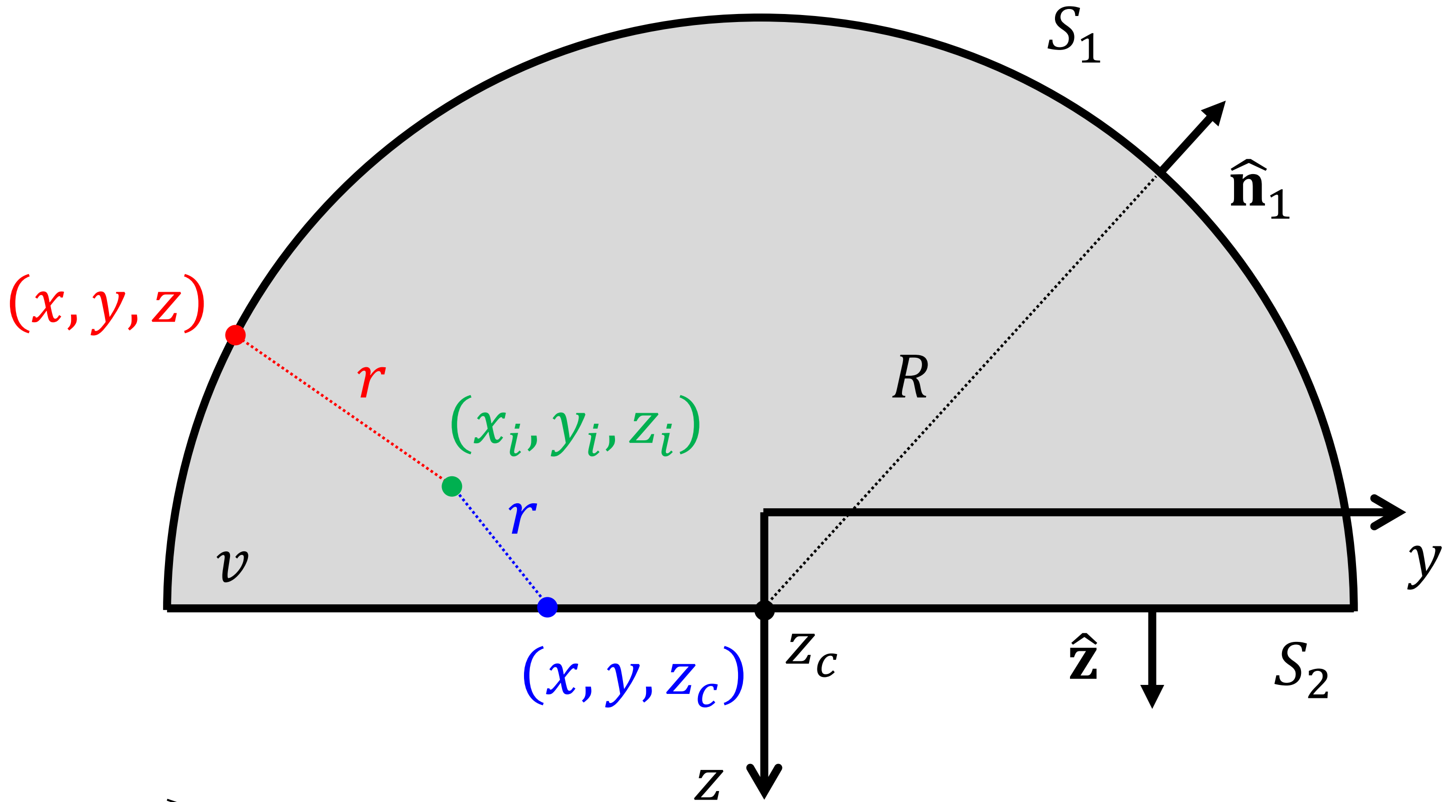
Note que a normal $\hat{\mathbf{n}}_2$ coincide com o vetor unitário $\hat{\mathbf{z}}$ na direção do eixo z e que o elemento de área $dS_2 = dxdy$



$$U_i = -\frac{1}{4\pi} \iint_{S_1} U \left(\nabla \frac{1}{r} \right)^{\mathrm{T}} \hat{\mathbf{n}}_1 - \frac{1}{r} \nabla U^{\mathrm{T}} \hat{\mathbf{n}}_1 \; dS_1 +$$

$$- \frac{1}{4\pi} \iint_{S_2} U \frac{\partial}{\partial z} \frac{1}{r} - \frac{1}{r} \frac{\partial U}{\partial z} dxdy$$

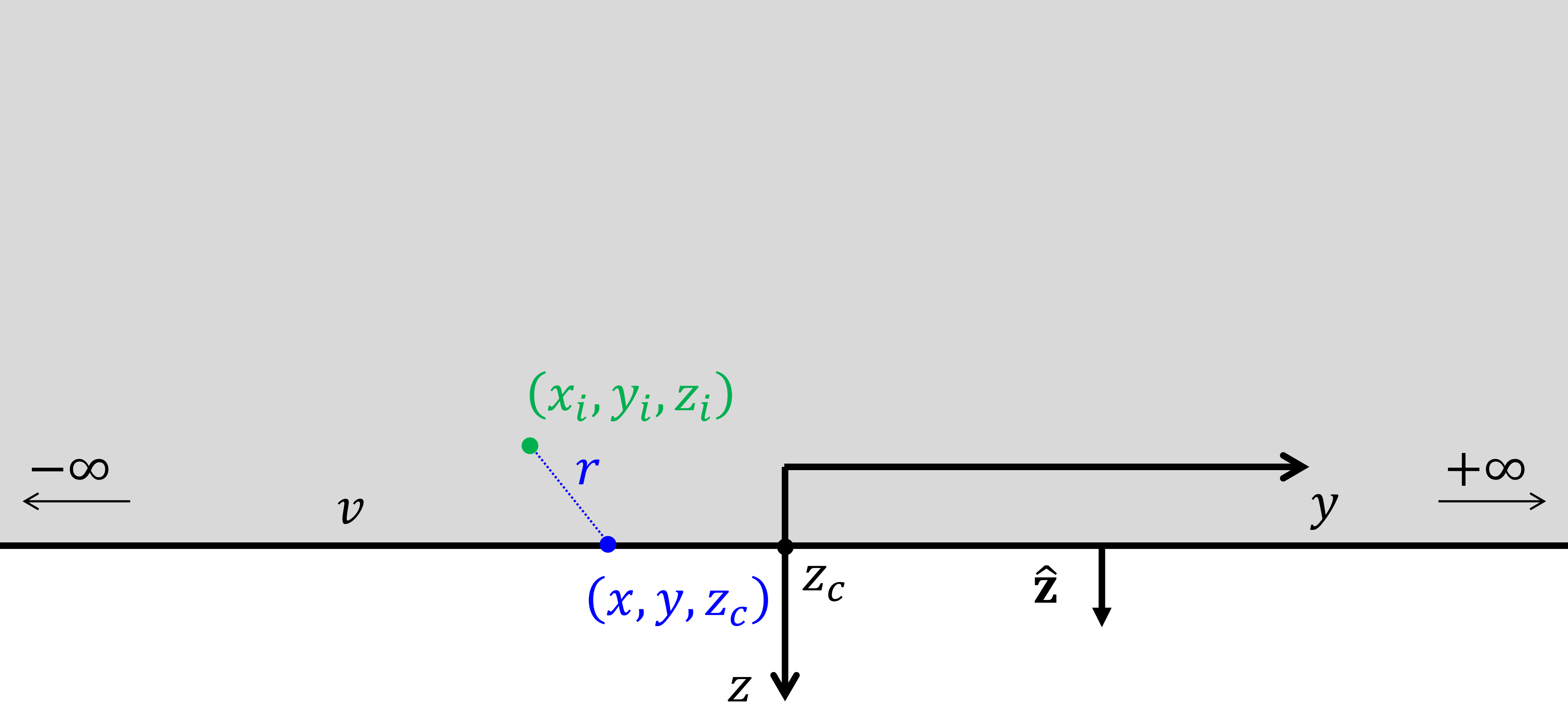
Consequentemente, as derivadas na direção de $\hat{\mathbf{n}}_2$ equivalem à derivada parcial em relação à coordenada z



$$U_i = -\frac{1}{4\pi} \iint_{S_1} \underbrace{U \left(\nabla \frac{1}{r} \right)^{\mathrm{T}} \hat{\mathbf{n}}_1 - \frac{1}{r} \nabla U^{\mathrm{T}} \hat{\mathbf{n}}_1}_{= 0} \, dS_1 +$$

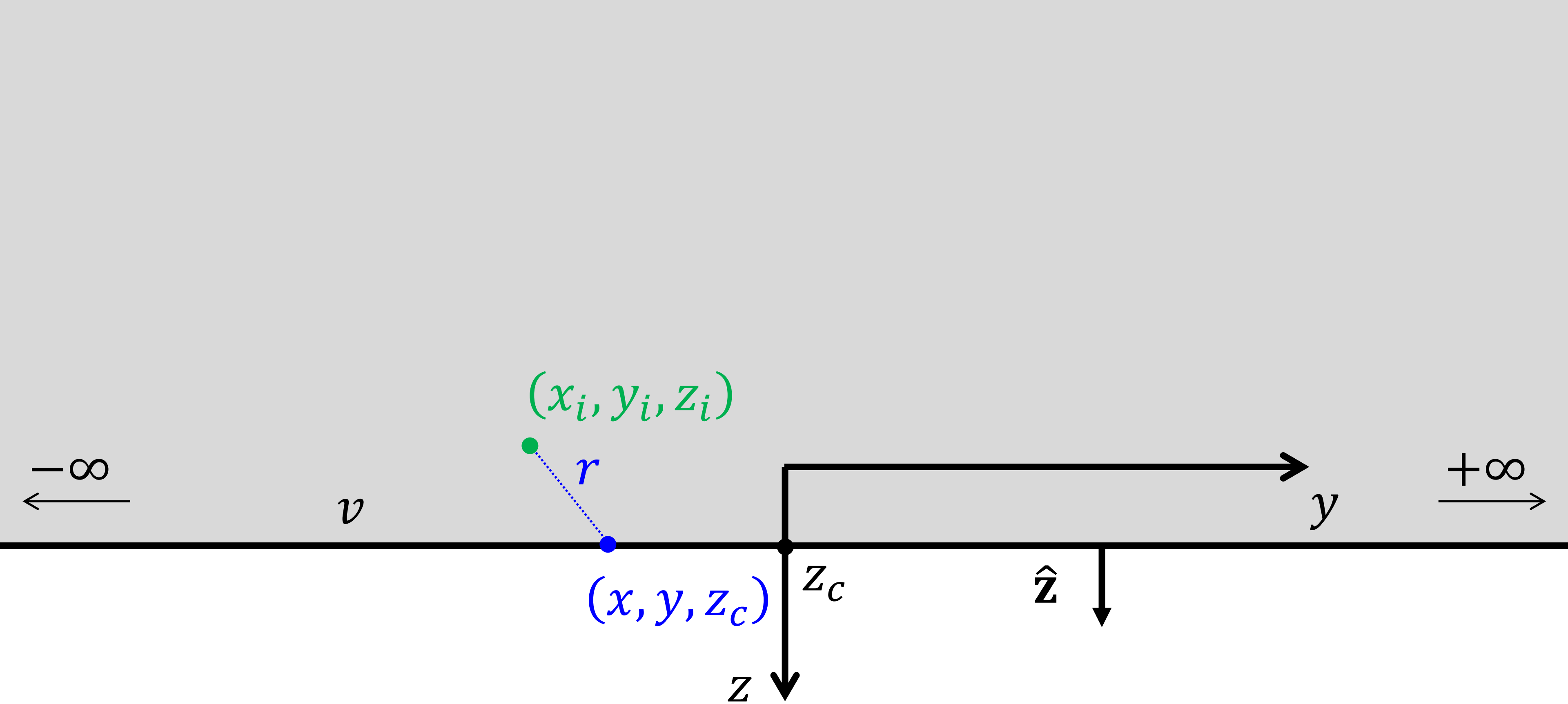
$$- \frac{1}{4\pi} \iint_{S_2} U \frac{\partial}{\partial z} \frac{1}{r} - \frac{1}{r} \frac{\partial U}{\partial z} \, dx dy$$

No limite $R \to \infty$, a integral sobre a superfície S_1 tende a zero



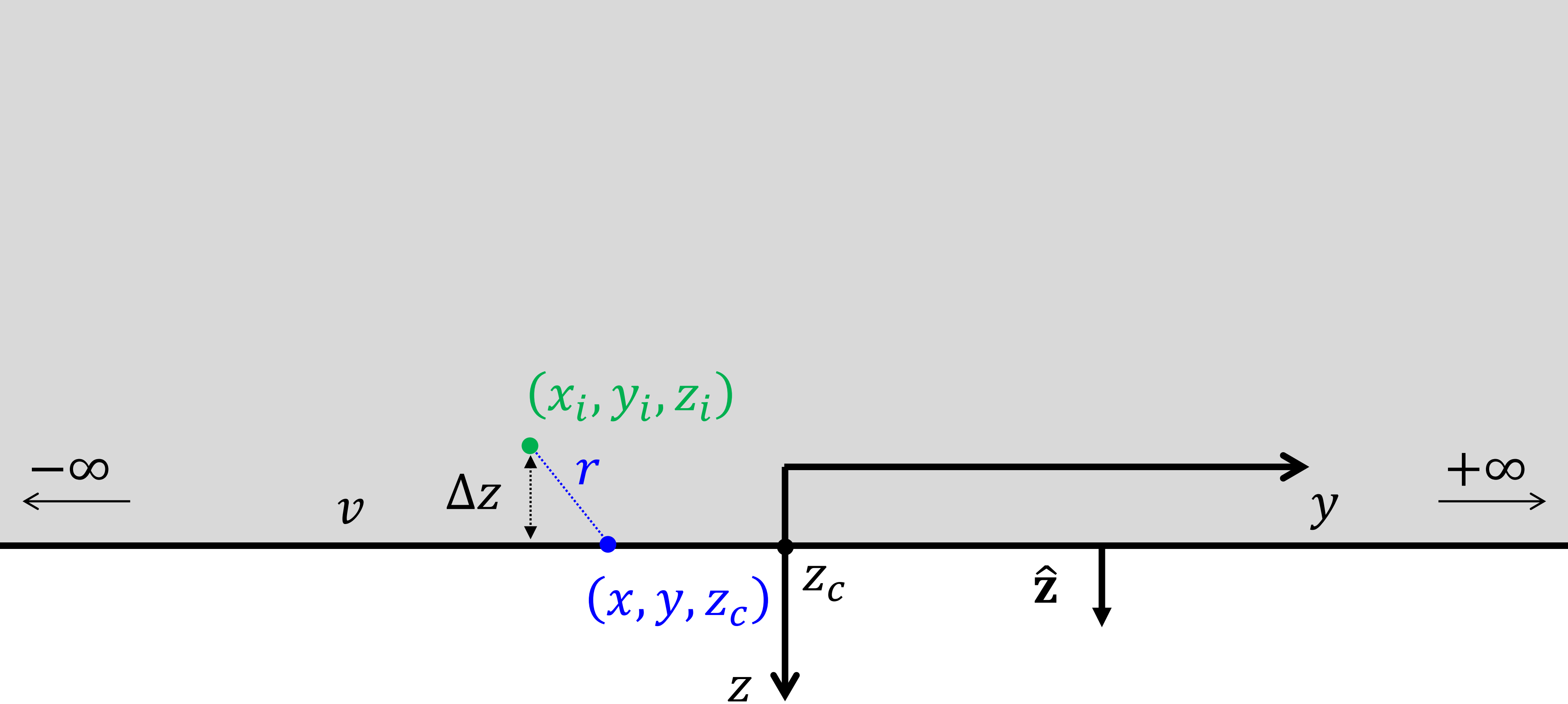
$$U_i = -\frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} U \frac{\partial}{\partial z} \frac{1}{r} - \frac{1}{r} \frac{\partial U}{\partial z} \, dxdy$$

Note que, neste caso, o valor da função U no ponto (x_i, y_i, z_i) é determinado por uma combinação entre os seus valores e os de sua derivada parcial em relação à coordenada z sobre o plano horizontal z_c



$$U_i = -\frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} U\frac{\partial}{\partial z}\frac{1}{r} - \frac{1}{r}\frac{\partial U}{\partial z}\, dxdy$$

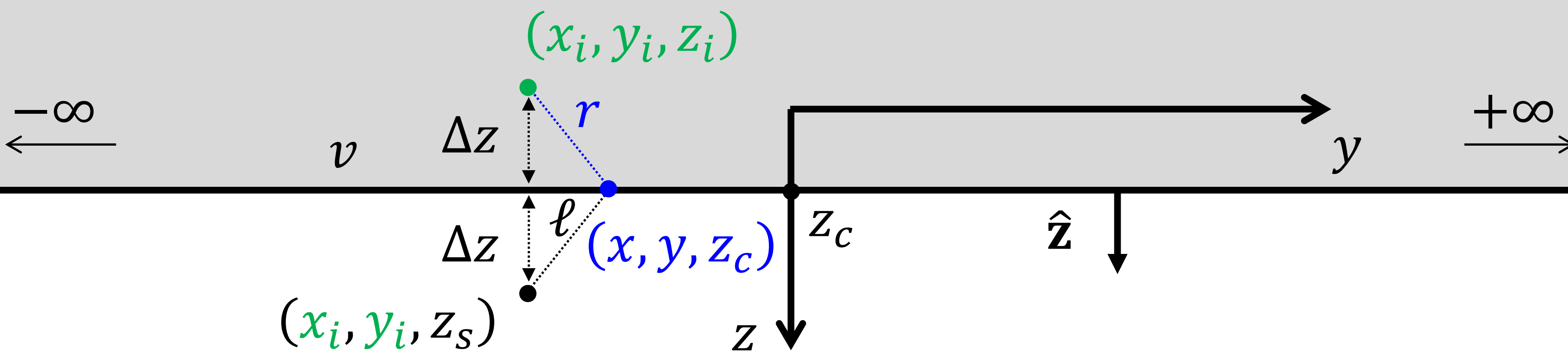
Para deduzir a integral de continuação para cima, é necessário remover o termo que envolve a derivada parcial de U



$$U_i = -\frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} U \frac{\partial}{\partial z} \frac{1}{r} - \frac{1}{r} \frac{\partial U}{\partial z} \, dxdy$$

Para tanto, considere
$z_i = z_c - \Delta z$, em que $\Delta z > 0$

$$\frac{1}{\ell} = \frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z_c - z_s)^2}}$$

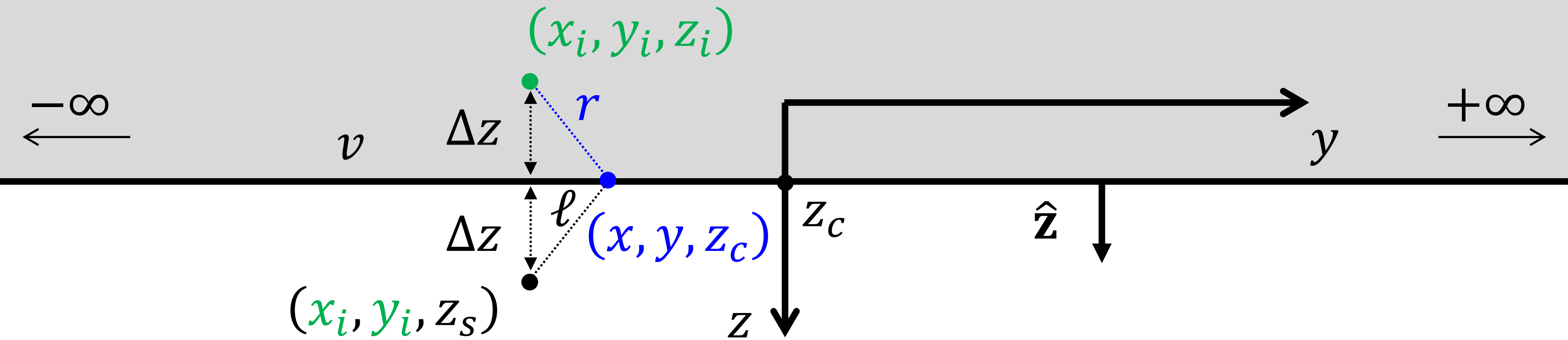


$$U_i = -\frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} U\frac{\partial}{\partial z}\frac{1}{r} - \frac{1}{r}\frac{\partial U}{\partial z}\,dxdy$$

O próximo passo consiste em definir a função inverso da distância entre um ponto (x, y, z_c) e o ponto (x_i, y_i, z_s)

$$\frac{1}{\ell} = \frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z_c - z_s)^2}}$$

$$\frac{1}{r} = \frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z_c - z_i)^2}}$$



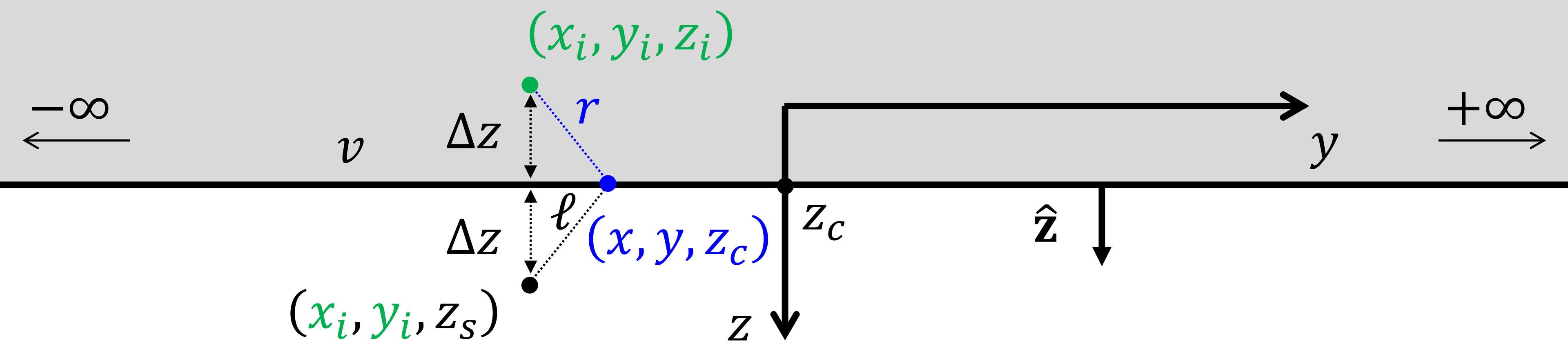
$$U_i = -\frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} U \frac{\partial}{\partial z} \frac{1}{r} - \frac{1}{r} \frac{\partial U}{\partial z} dxdy$$

O próximo passo consiste em definir a função inverso da distância entre um ponto (x, y, z_c) e o ponto (x_i, y_i, z_s)

Note a diferença entre esta nova função $1/\ell$ e a função $1/r$

$$\frac{1}{\ell} = \frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z_c - z_s)^2}}$$

$$\frac{1}{r} = \frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z_c - z_i)^2}}$$



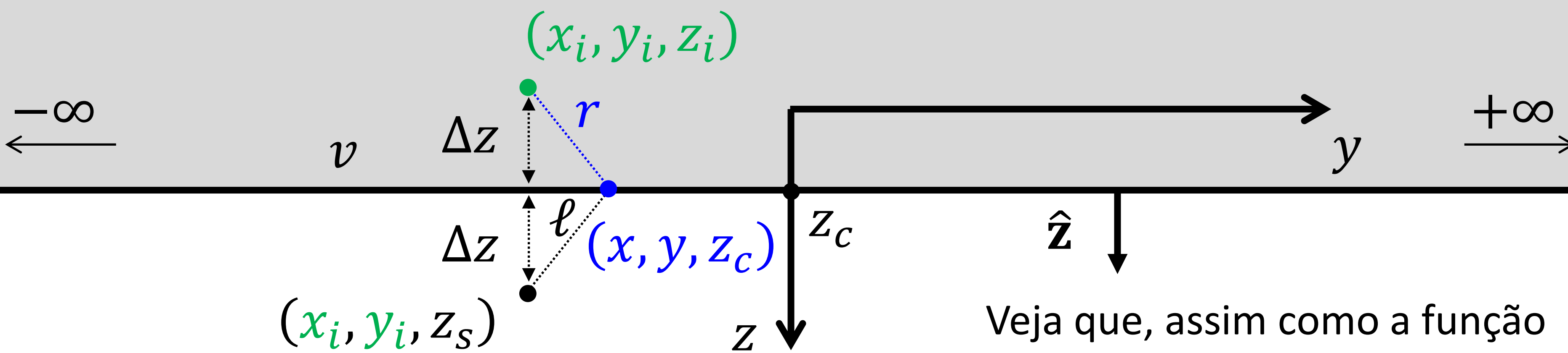
$$U_i = -\frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} U \frac{\partial}{\partial z} \frac{1}{r} - \frac{1}{r} \frac{\partial U}{\partial z} \, dxdy$$

Sendo assim, de acordo com a segunda identidade de Green

$$\iiint_v U \, \nabla^2 \frac{1}{\ell} - \frac{1}{\ell} \, \nabla^2 U \, dv = \iint_S U \left(\nabla \frac{1}{\ell} \right)^{\mathrm{T}} \hat{\mathbf{n}} - \frac{1}{\ell} \nabla U^{\mathrm{T}} \hat{\mathbf{n}} \; dS$$

$$\frac{1}{\ell} = \frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z_c - z_s)^2}}$$

$$\frac{1}{r} = \frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z_c - z_i)^2}}$$



Veja que, assim como a função U, a função $1/\ell$ é harmônica em todo o volume v

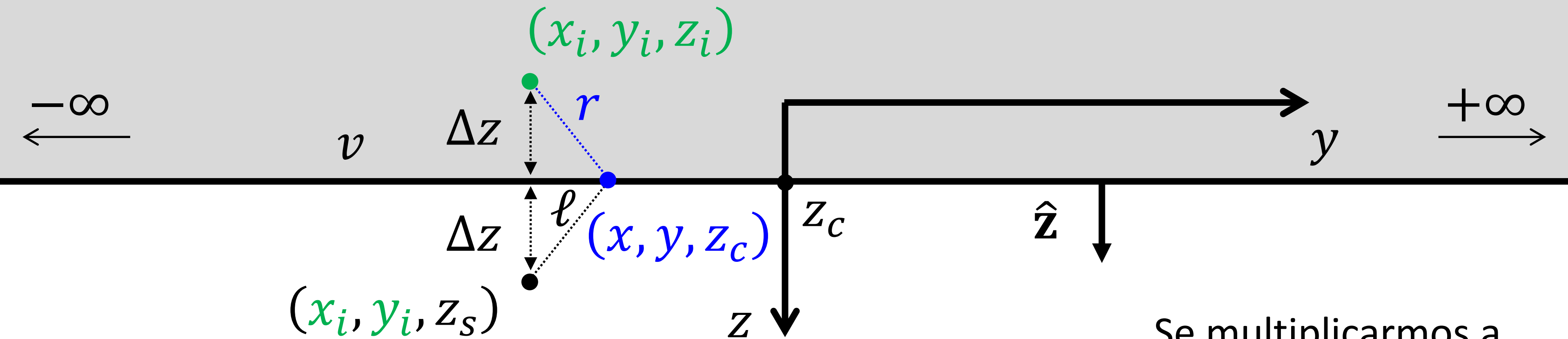
$$U_i = -\frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} U \frac{\partial}{\partial z} \frac{1}{r} - \frac{1}{r} \frac{\partial U}{\partial z} \, dxdy$$

Além disso, assim como fizemos anteriormente, considere $R \to \infty$

$$\iiint_v U \cancelto{0}{\nabla^2 \frac{1}{\ell}} - \frac{1}{\ell} \cancelto{0}{\nabla^2 U} \, dv = \iint_S U \left(\nabla \frac{1}{\ell} \right)^{\mathrm{T}} \hat{\mathbf{n}} - \frac{1}{\ell} \nabla U^{\mathrm{T}} \hat{\mathbf{n}} \; dS$$

$$\frac{1}{\ell} = \frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z_c - z_s)^2}}$$

$$\frac{1}{r} = \frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z_c - z_i)^2}}$$



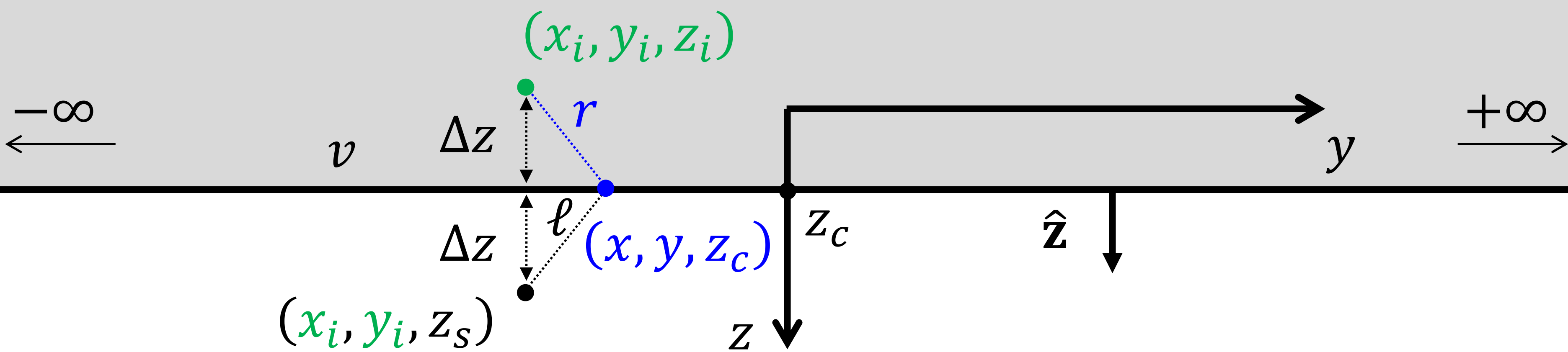
Se multiplicarmos a equação abaixo por $1/4\pi$ e somarmos o resultado na equação acima...

$$U_i = -\frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} U \frac{\partial}{\partial z} \frac{1}{r} - \frac{1}{r} \frac{\partial U}{\partial z} \, dx dy$$

$$0 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} U \frac{\partial}{\partial z} \frac{1}{\ell} - \frac{1}{\ell} \frac{\partial U}{\partial z} \, dx dy$$

$$\frac{1}{\ell} = \frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z_c - z_s)^2}}$$

$$\frac{1}{r} = \frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z_c - z_i)^2}}$$

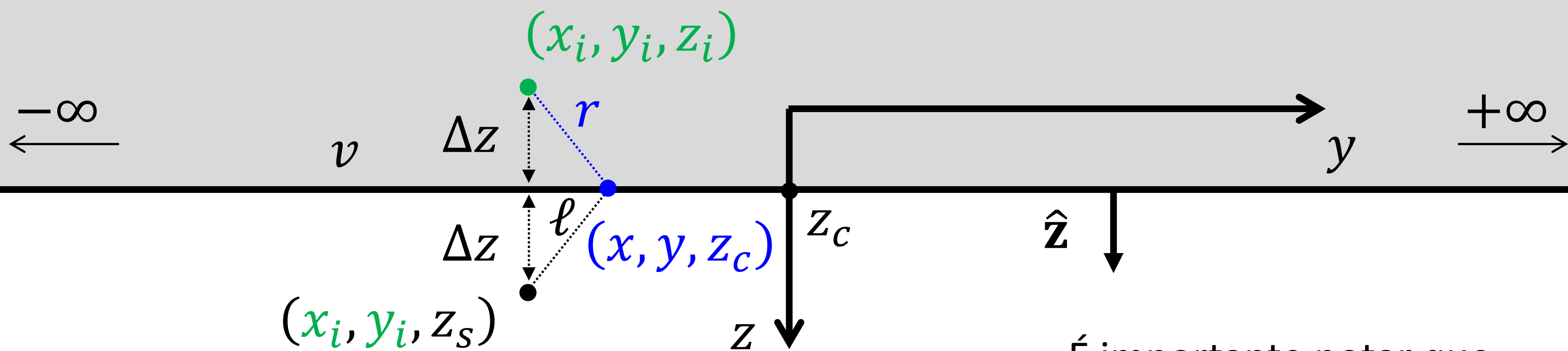


$$U_i = -\frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} U \frac{\partial}{\partial z} \frac{1}{r} - \frac{1}{r} \frac{\partial U}{\partial z} \, dx dy$$

$$U_i = -\frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} U \frac{\partial}{\partial z} \left(\frac{1}{r} - \frac{1}{\ell} \right) - \left(\frac{1}{r} - \frac{1}{\ell} \right) \frac{\partial U}{\partial z} \, dx dy$$

$$\frac{1}{\ell} = \frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z_c - z_s)^2}}$$

$$\frac{1}{r} = \frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z_c - z_i)^2}}$$



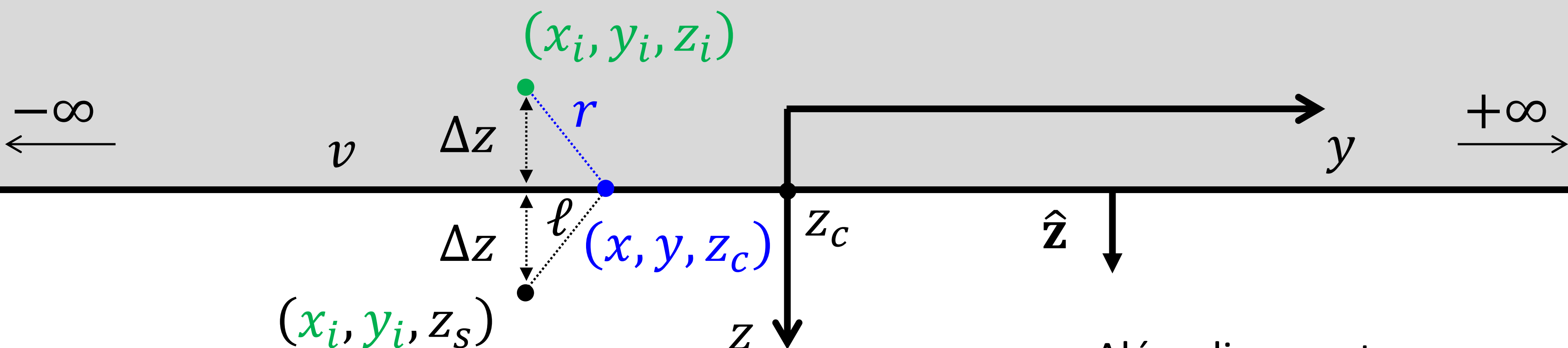
É importante notar que, como $z_s = z_c + \Delta z$ e $z_i = z_c - \Delta z$, então $1/\ell = 1/r$

$$U_i = -\frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} U\frac{\partial}{\partial z}\frac{1}{r} - \frac{1}{r}\frac{\partial U}{\partial z}\,dxdy$$

$$U_i = -\frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} U\frac{\partial}{\partial z}\left(\frac{1}{r} - \frac{1}{\ell}\right) - \cancelto{0}{\left(\frac{1}{r} - \frac{1}{\ell}\right)}\frac{\partial U}{\partial z}\,dxdy$$

$$\frac{1}{\ell} = \frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z_c - z_s)^2}}$$

$$\frac{1}{r} = \frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z_c - z_i)^2}}$$



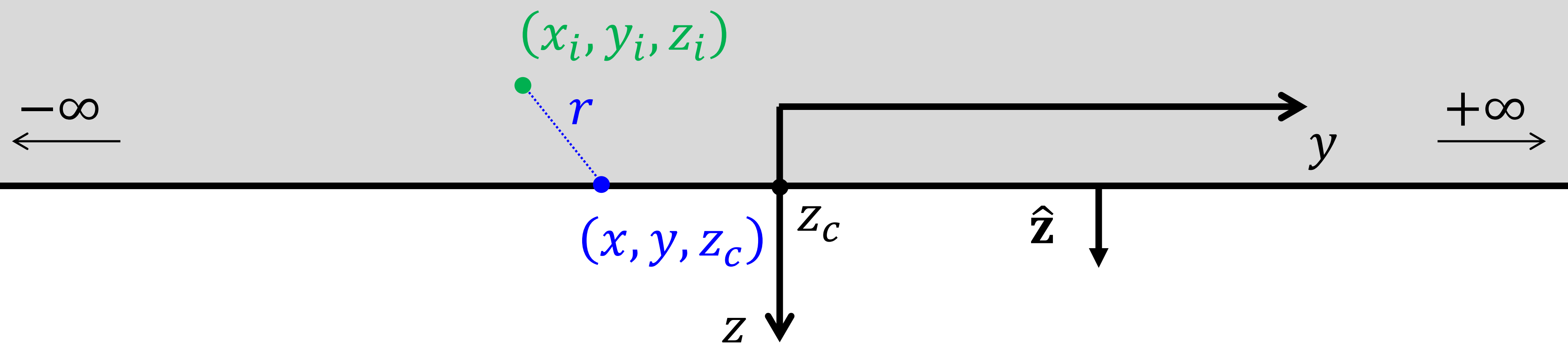
Além disso, note que

$$\frac{\partial}{\partial z}\frac{1}{\ell} = -\frac{\partial}{\partial z}\frac{1}{r}$$

$$U_i = -\frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} U\frac{\partial}{\partial z}\frac{1}{r} - \frac{1}{r}\frac{\partial U}{\partial z}\,dxdy$$

$$U_i = -\frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} U\cancelto{= 2\frac{\partial}{\partial z}\frac{1}{r}}{\frac{\partial}{\partial z}\left(\frac{1}{r} - \frac{1}{\ell}\right)} - \cancelto{= 0}{\left(\frac{1}{r} - \frac{1}{\ell}\right)}\frac{\partial U}{\partial z}\,dxdy$$

$$\frac{1}{r} = \frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z_c - z_i)^2}}$$



$$U_i = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} U \frac{\partial}{\partial z} \frac{1}{r} \, dx dy$$

Integral de continuação para cima

(Henderson, 1960, 1970)

$$= \frac{z_c - z_i}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{U(x, y, z_c)}{\left[(x_i - x)^2 + (y_i - y)^2 + (z_i - z_c)^2\right]^{\frac{3}{2}}} \, dx dy$$

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