

Processing and interpreting potential-field data via Equivalent-Layer Technique



Vanderlei C. Oliveira Jr.

Feb/2022

Summary

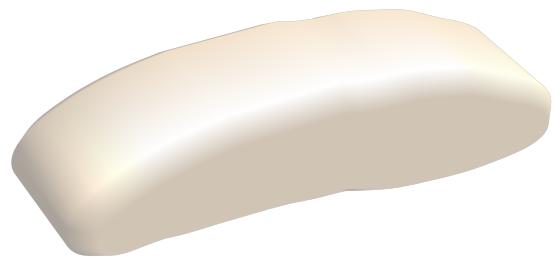
- Motivation
- Potential-field data
- The Equivalent-Layer (EqL) Technique
- Theoretical aspects
- Some open questions

Summary

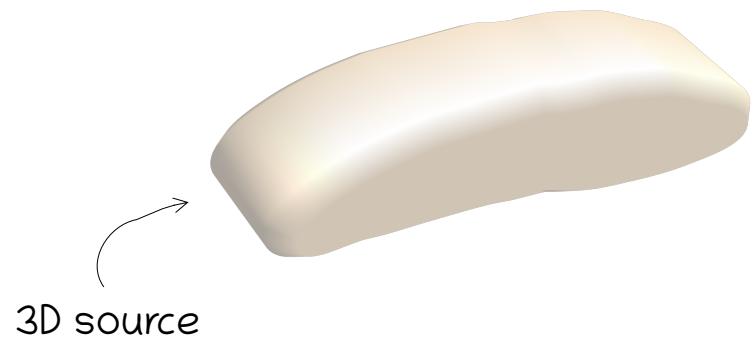
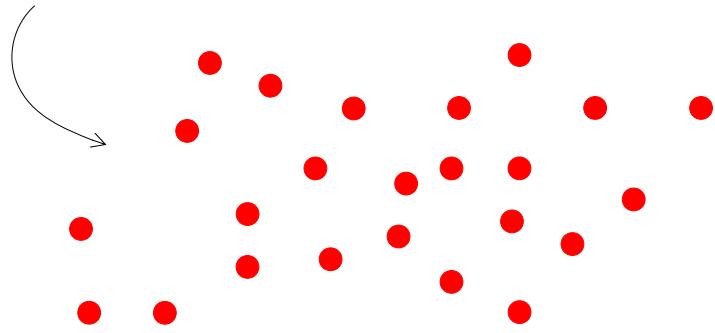
- **Motivation**
- Potential-field data
- The Equivalent-Layer (EqL) Technique
- Theoretical aspects
- Some open questions



3D source

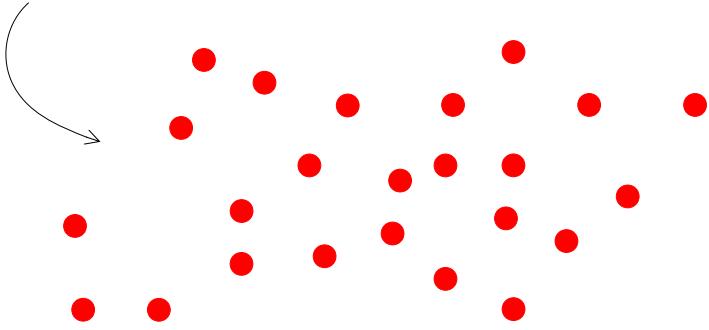


Potential-field data

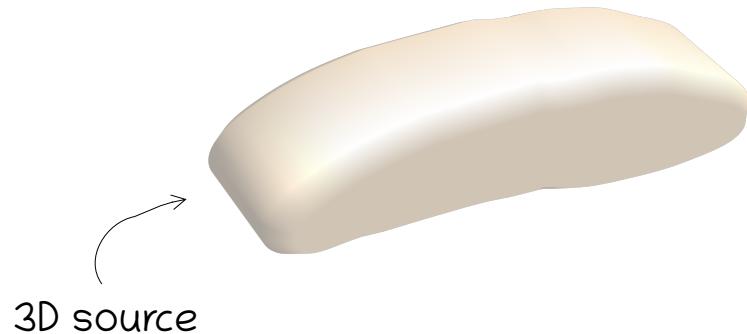


3D source

Potential-field data

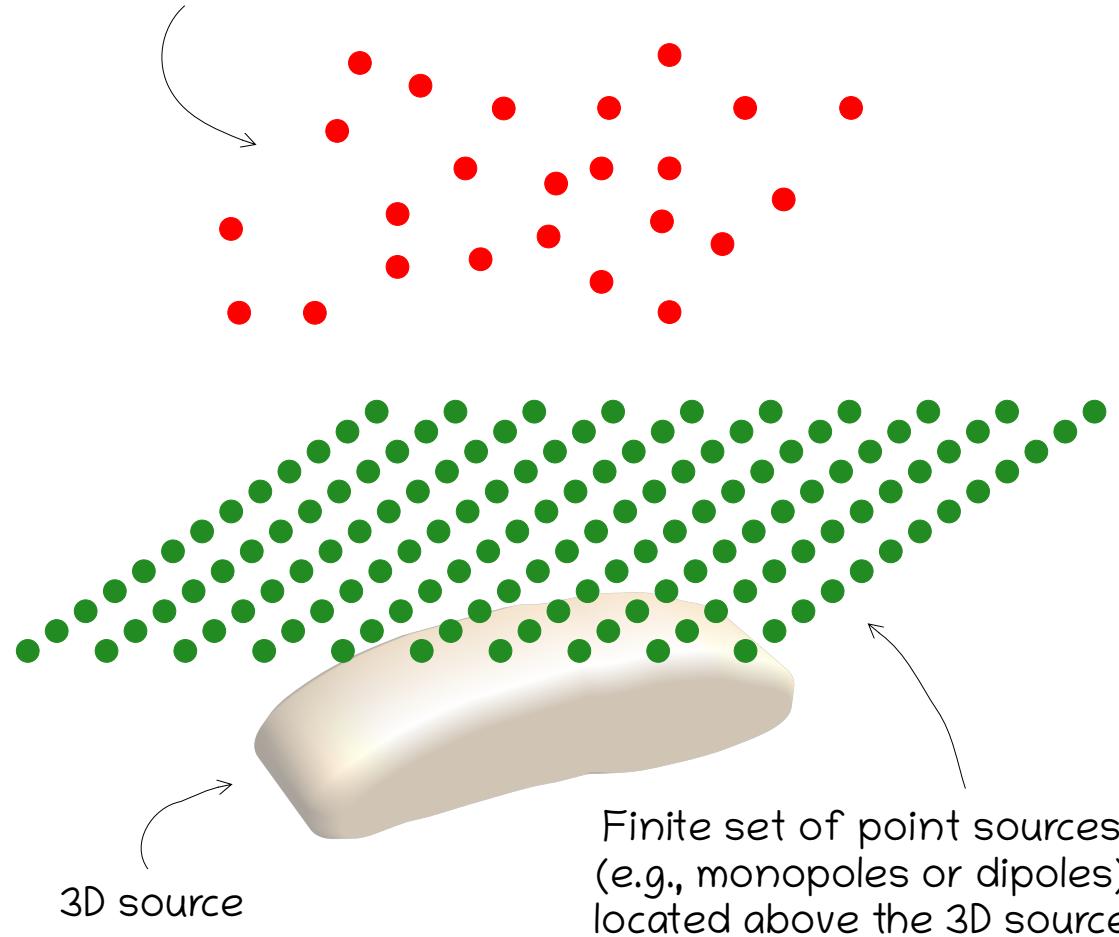


Throughout this presentation, I consider that potential-field data can be represented by harmonic functions

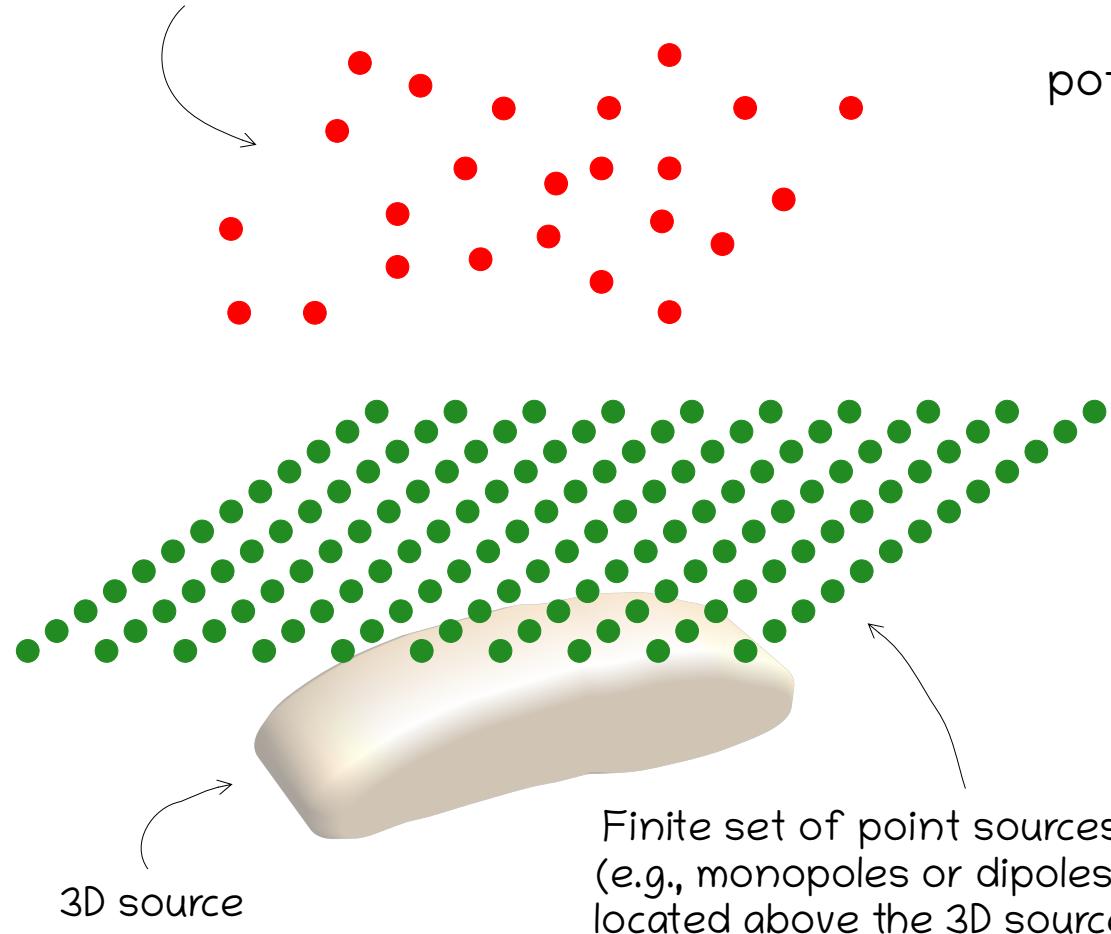


3D source

Potential-field data

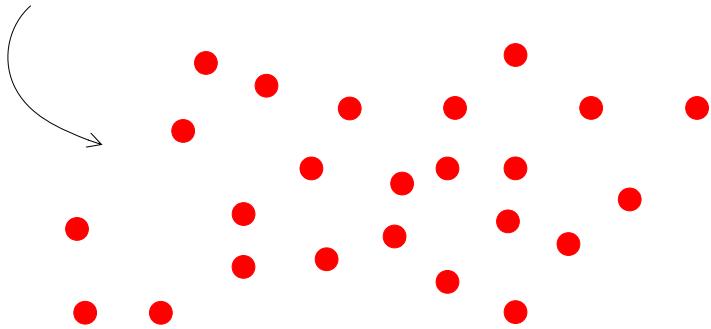


Potential-field data



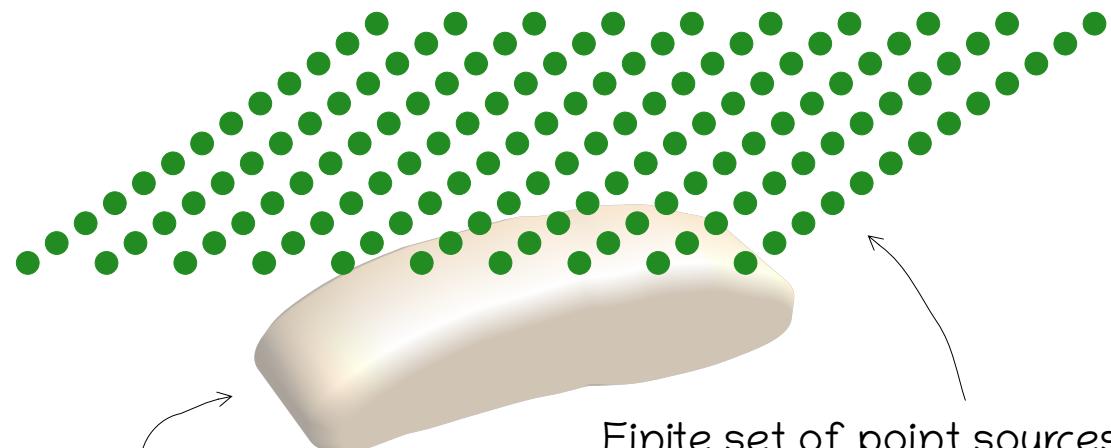
The Equivalent-Layer (EqL) Technique consists in first estimating the physical-property distribution of these **point sources** so that they satisfactorily reproduce the potential-field data caused by the 3D source at the **observation points**

Potential-field data



The Equivalent-Layer (EqL) Technique consists in first estimating the physical-property distribution of these **point sources** so that they satisfactorily reproduce the potential-field data caused by the 3D source at the **observation points**

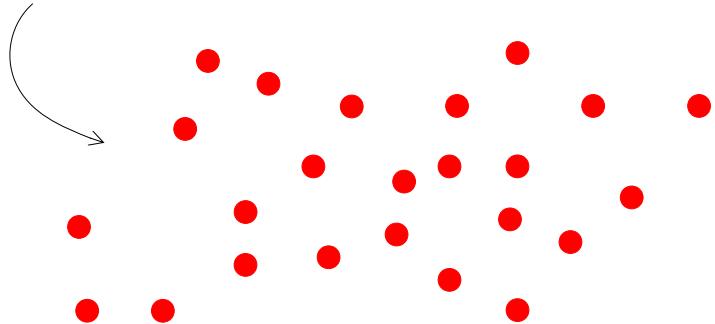
Then, the estimated physical property distribution can be used, for example, to:



3D source

Finite set of point sources
(e.g., monopoles or dipoles)
located above the 3D source

Potential-field data

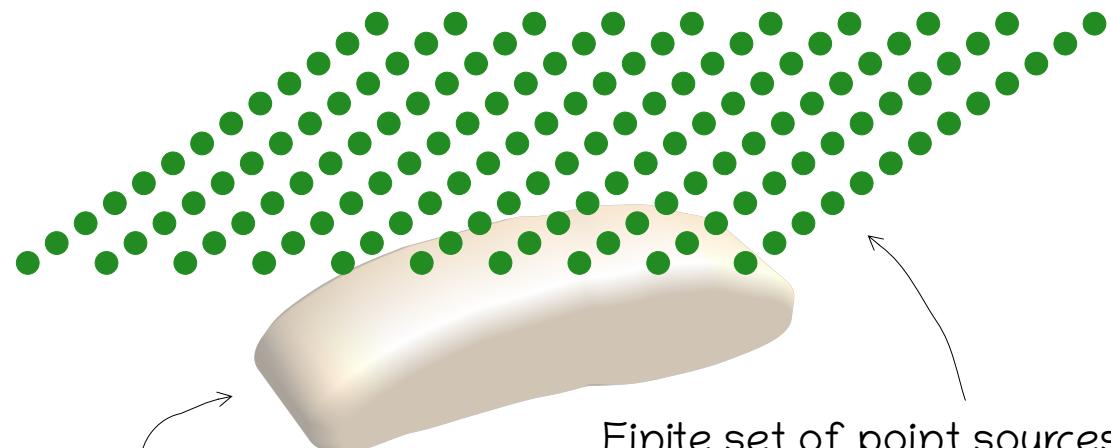


The Equivalent-Layer (EqL) Technique consists in first estimating the physical-property distribution of these **point sources** so that they satisfactorily reproduce the potential-field data caused by the 3D source at the **observation points**

Then, the estimated physical property distribution can be used, for example, to:

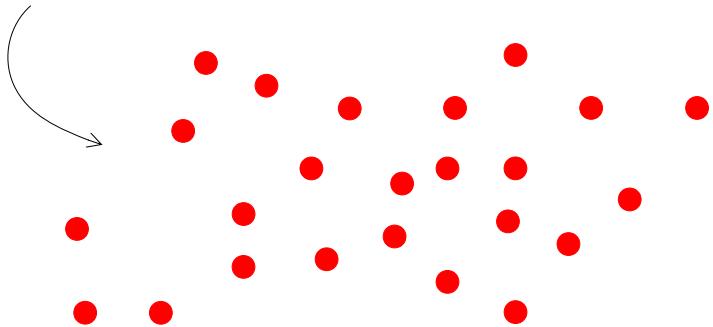
- * predict the potential field at points without measurements (interpolation, upward/downward continuation)

3D source



Finite set of point sources
(e.g., monopoles or dipoles)
located above the 3D source

Potential-field data



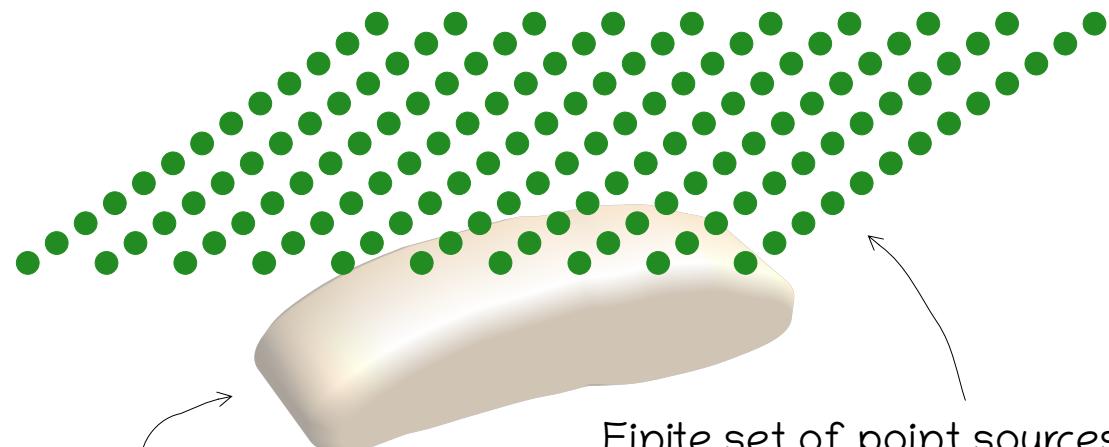
The Equivalent-Layer (EqL) Technique consists in first estimating the physical-property distribution of these **point sources** so that they satisfactorily reproduce the potential-field data caused by the 3D source at the **observation points**

Then, the estimated physical property distribution can be used, for example, to:

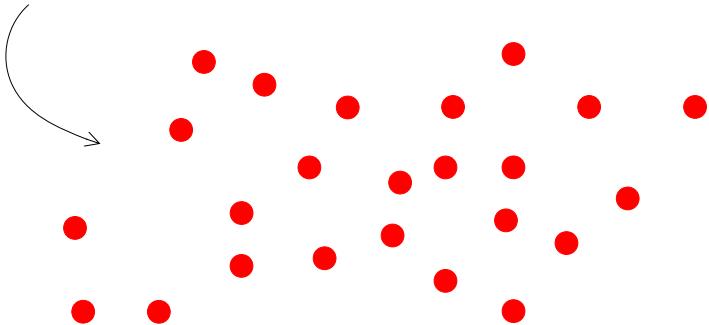
- * predict the potential field at points without measurements (interpolation, upward/downward continuation)
- * reduce total-field anomalies to the pole

3D source

Finite set of point sources
(e.g., monopoles or dipoles)
located above the 3D source

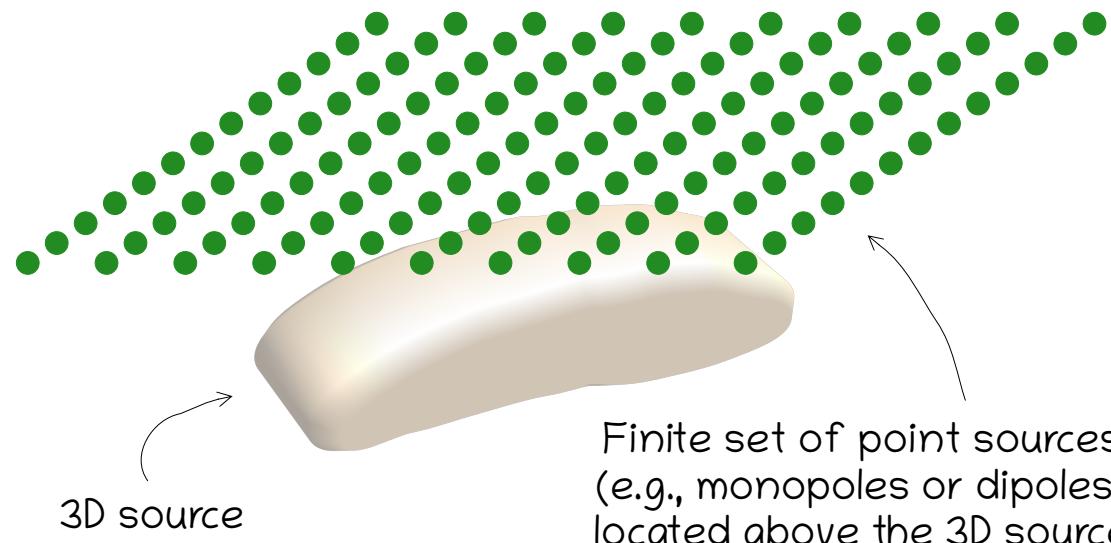


Potential-field data



The Equivalent-Layer (EqL) Technique consists in first estimating the physical-property distribution of these **point sources** so that they satisfactorily reproduce the potential-field data caused by the 3D source at the **observation points**

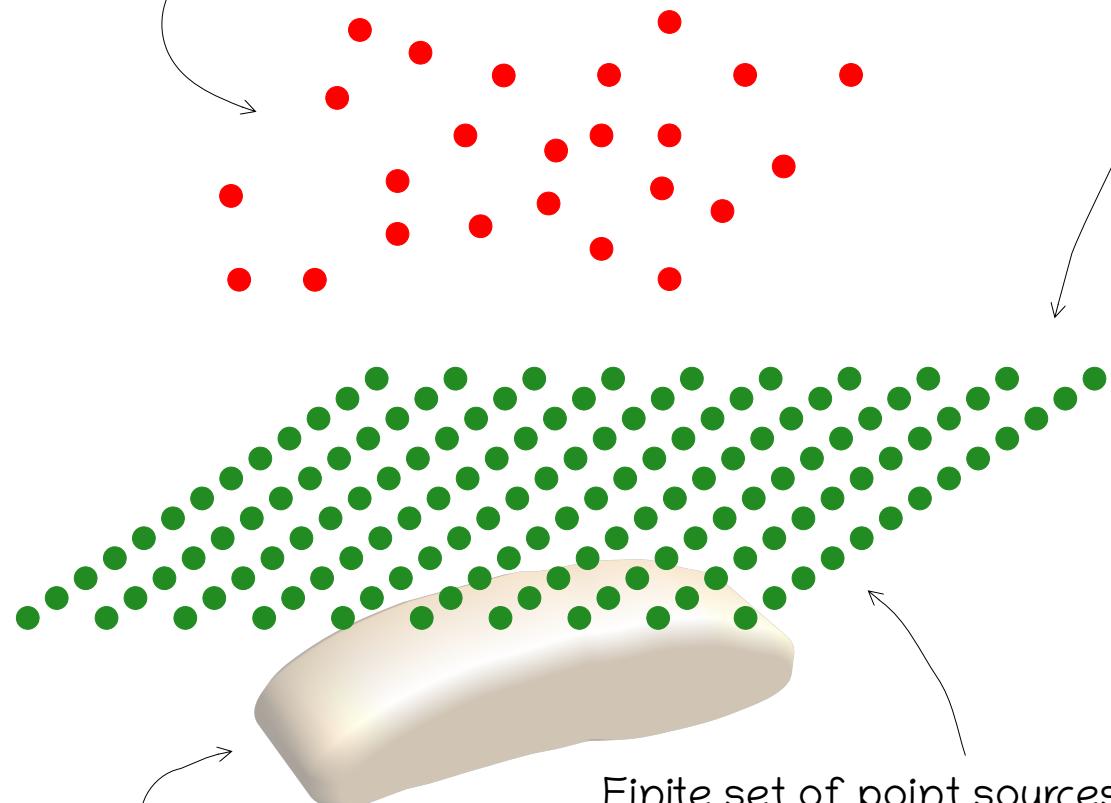
Then, the estimated physical property distribution can be used, for example, to:



- * predict the potential field at points without measurements (interpolation, upward/downward continuation)
- * reduce total-field anomalies to the pole
- * convert gravity disturbance (g_z) into tensor components (g_{xx} , g_{xy} , g_{xz} ...)

Potential-field data

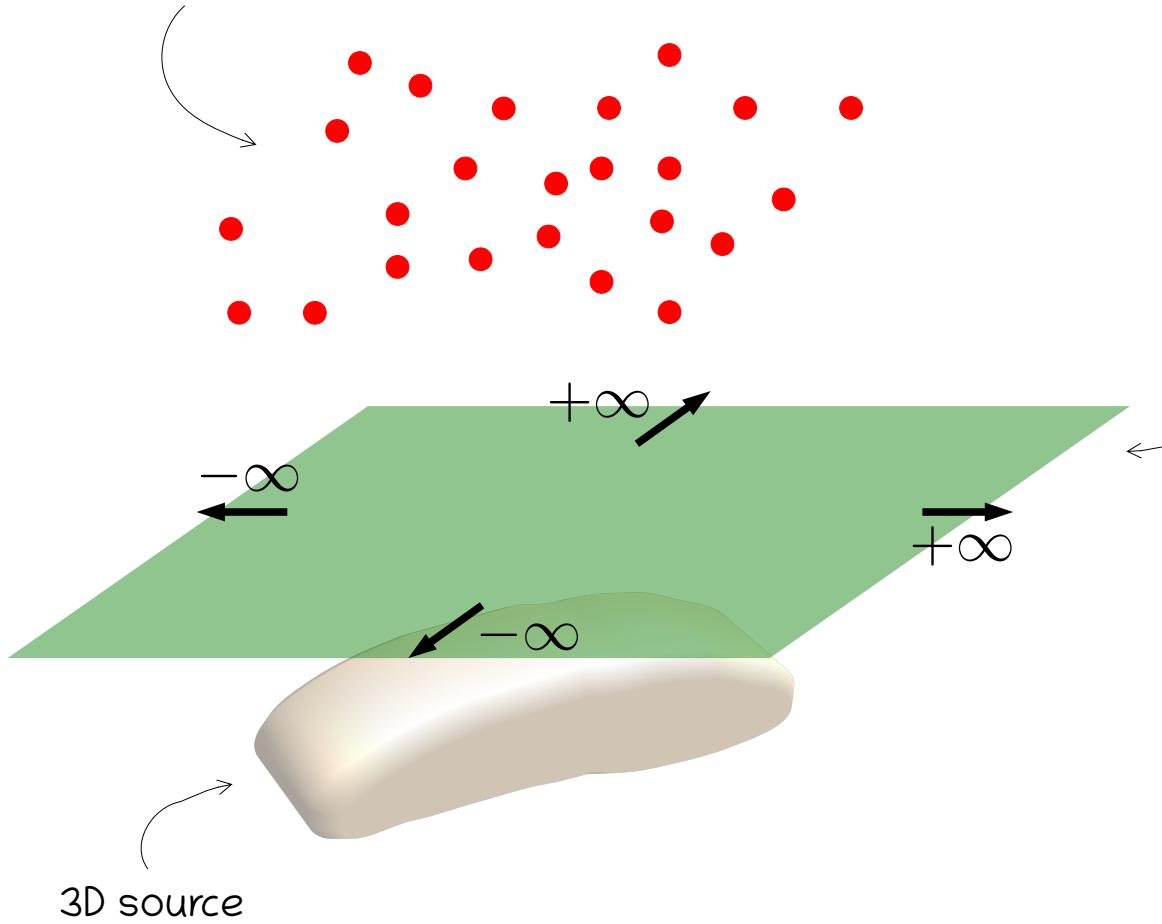
The estimated physical-property distribution of the point sources is a **discrete equivalent layer**



3D source

Finite set of point sources
(e.g., monopoles or dipoles)
located above the 3D source

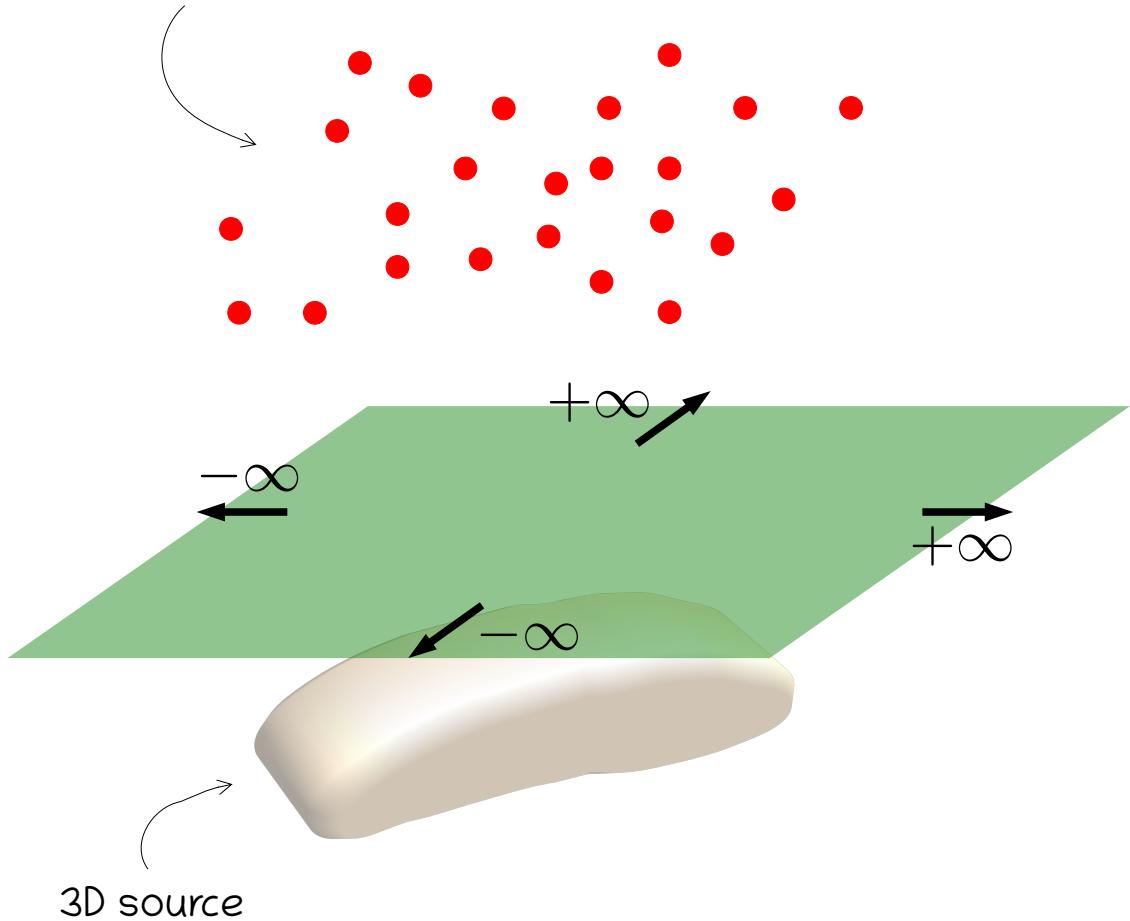
Potential-field data



The estimated physical-property distribution of the point sources is a discrete equivalent layer

A discrete equivalent layer, in turn, is a discrete representation for a **continuous equivalent layer**

Potential-field data

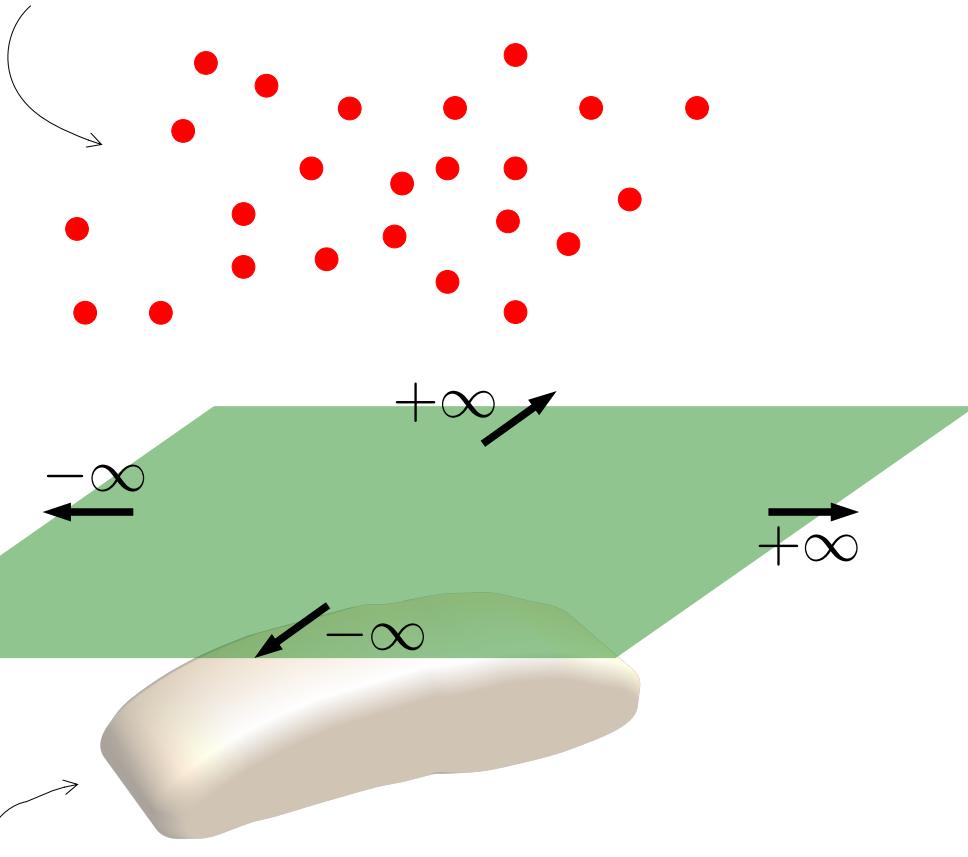


The estimated physical-property distribution of the point sources is a discrete equivalent layer

A discrete equivalent layer, in turn, is a discrete representation for a continuous equivalent layer

Continuous eq. layers can be obtained by solving integral equations for boundary value problems of potential theory, such as the Dirichlet and Neumann's problems

Potential-field data



The estimated physical-property distribution of the point sources is a discrete equivalent layer

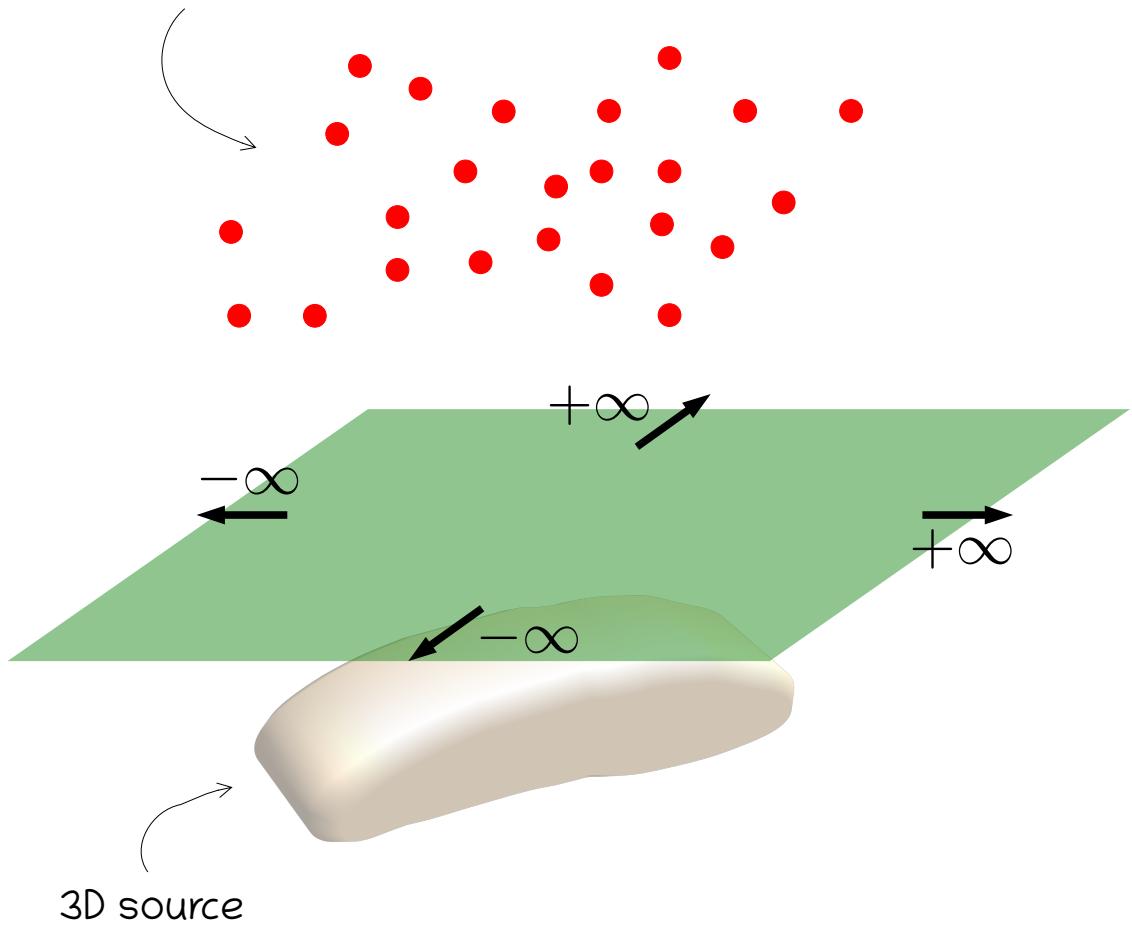
A discrete equivalent layer, in turn, is a discrete representation for a continuous equivalent layer

Continuous eq. layers can be obtained by solving integral equations for boundary value problems of potential theory, such as the Dirichlet and Neumann's problems

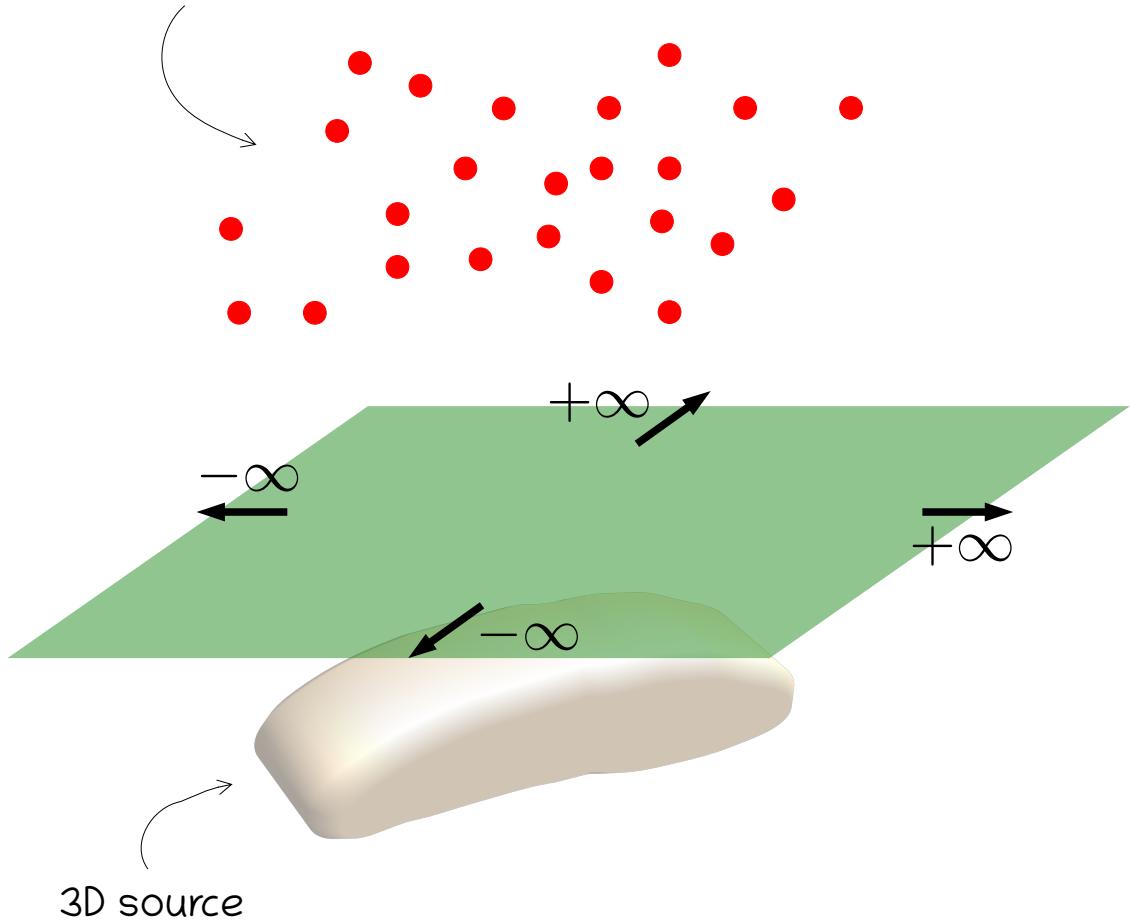
In this case, they are conveniently called here **analytical equivalent layers**

The main goals here are:

Potential-field data



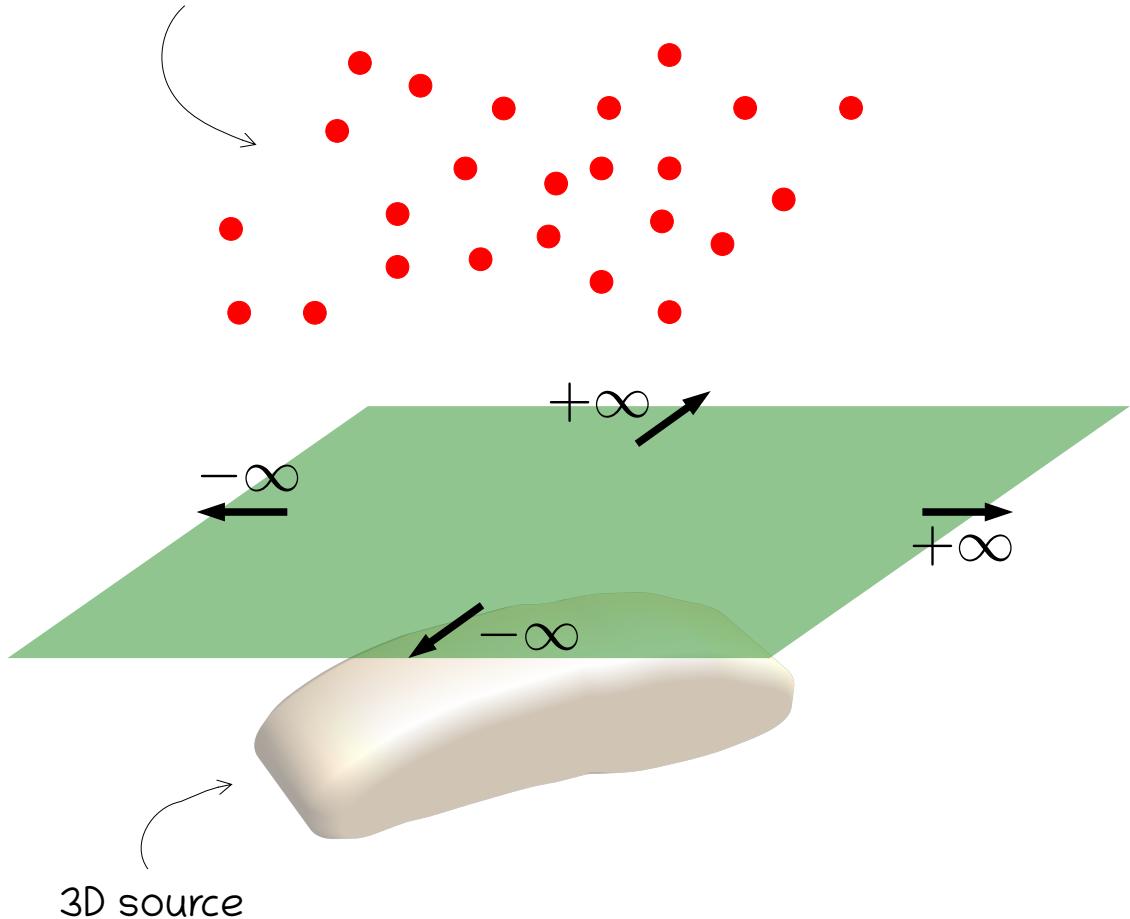
Potential-field data



The main goals here are:

- * deducing analytical equivalent layers of monopoles and dipoles for different potential-field data

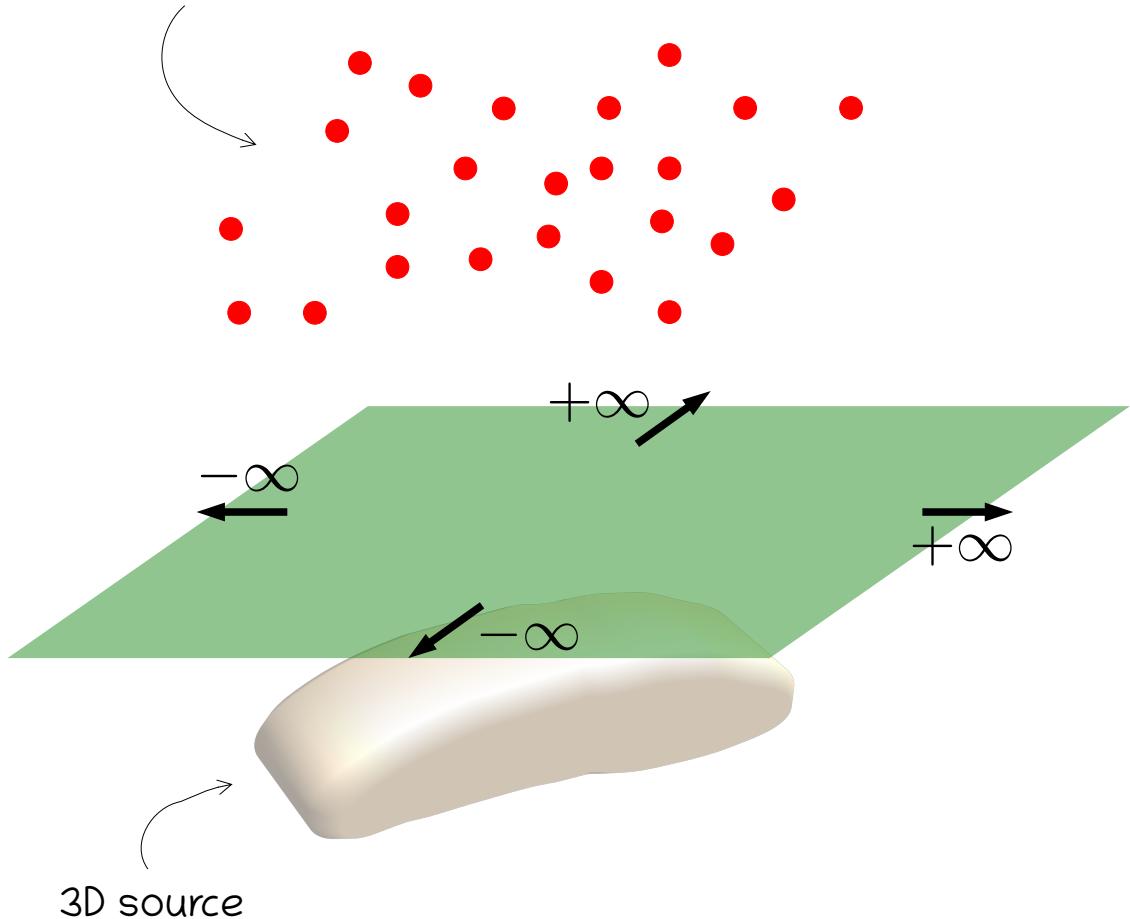
Potential-field data



The main goals here are:

- * deducing analytical equivalent layers of monopoles and dipoles for different potential-field data
- * showing that they fill some theoretical gaps of EqL technique

Potential-field data



The main goals here are:

- * deducing analytical equivalent layers of monopoles and dipoles for different potential-field data
- * showing that they fill some theoretical gaps of EqL technique
- * indicating some theoretical gaps that have not yet been solved

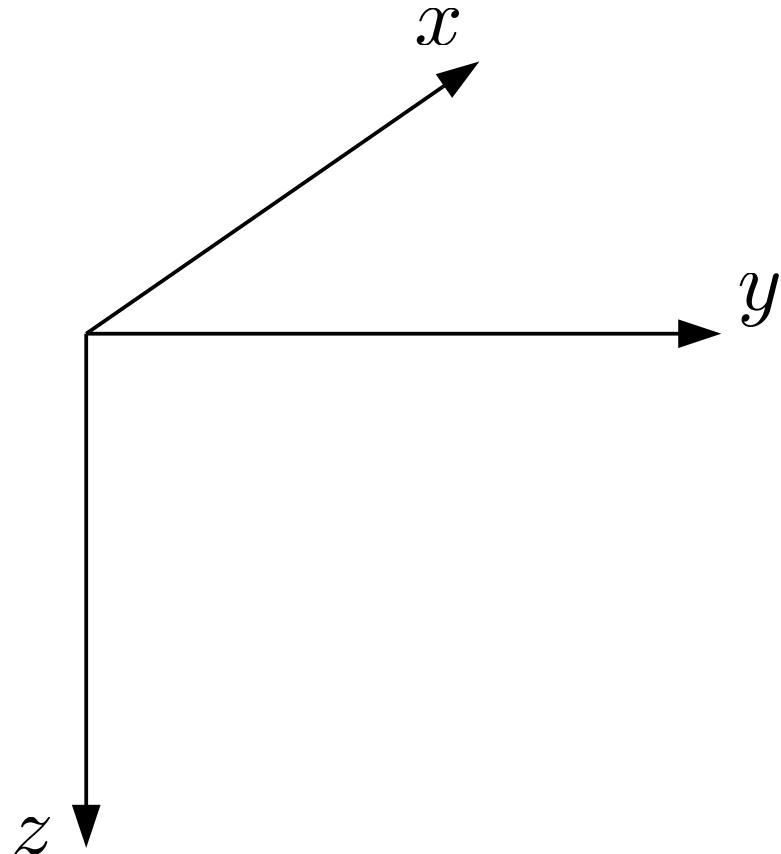
Summary

- **Motivation**
- Potential-field data
- The Equivalent-Layer (EqL) Technique
- Theoretical aspects
- Some open questions

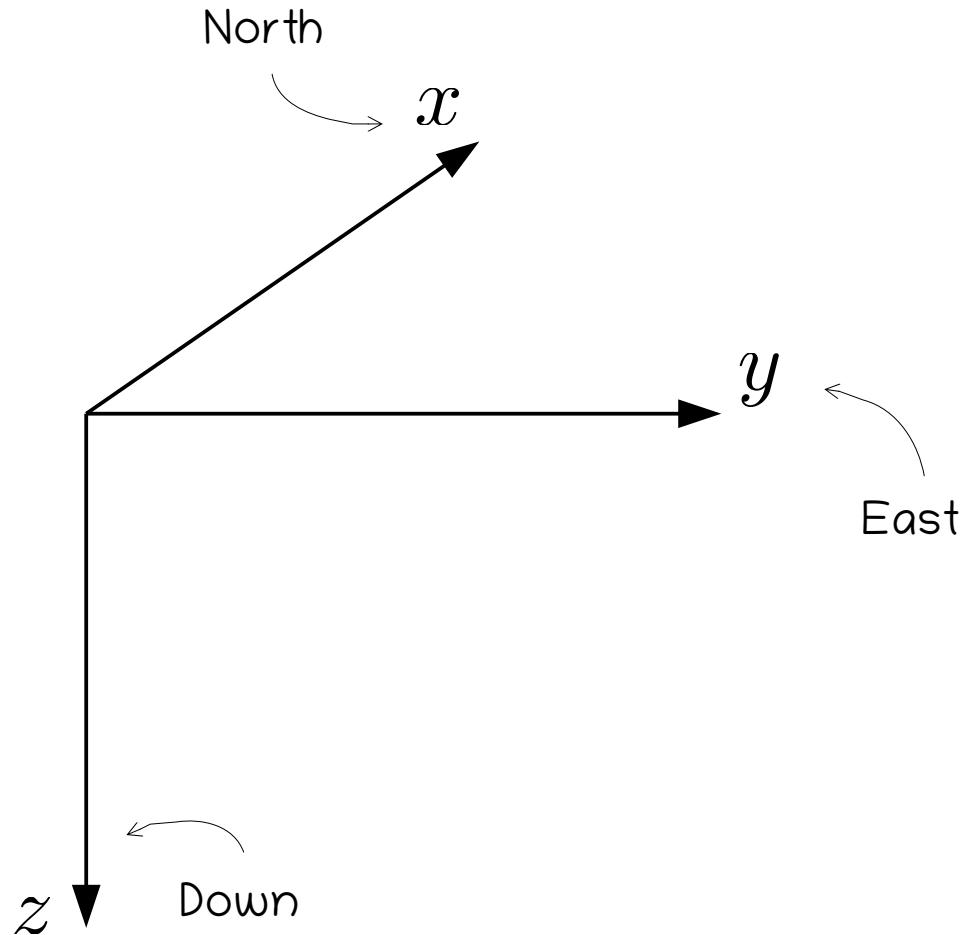
Summary

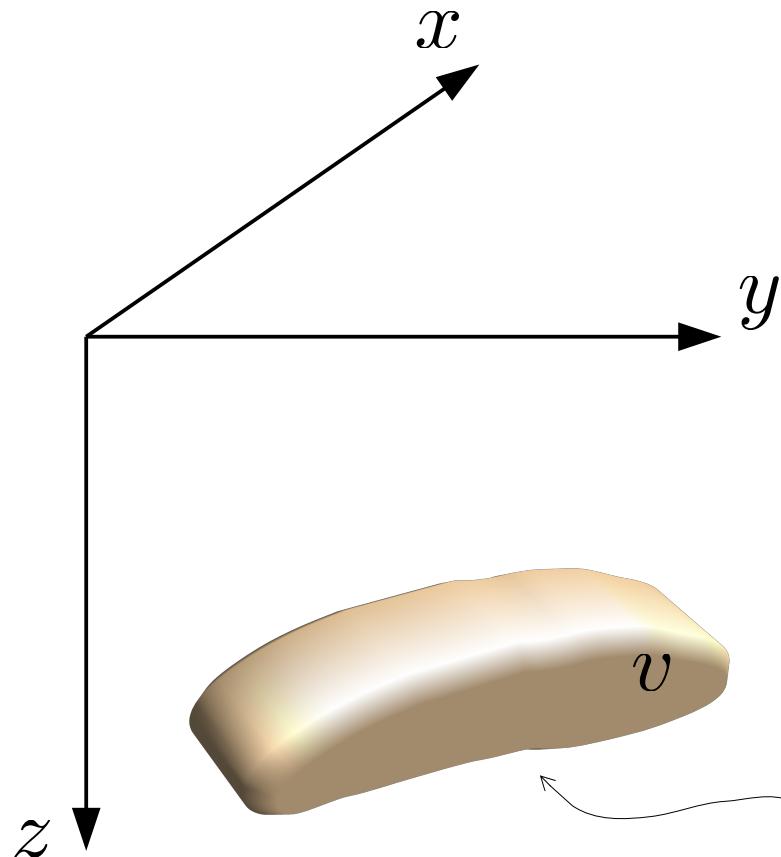
- Motivation
- **Potential-field data**
- The Equivalent-Layer (EqL) Technique
- Theoretical aspects
- Some open questions

Topocentric Cartesian system



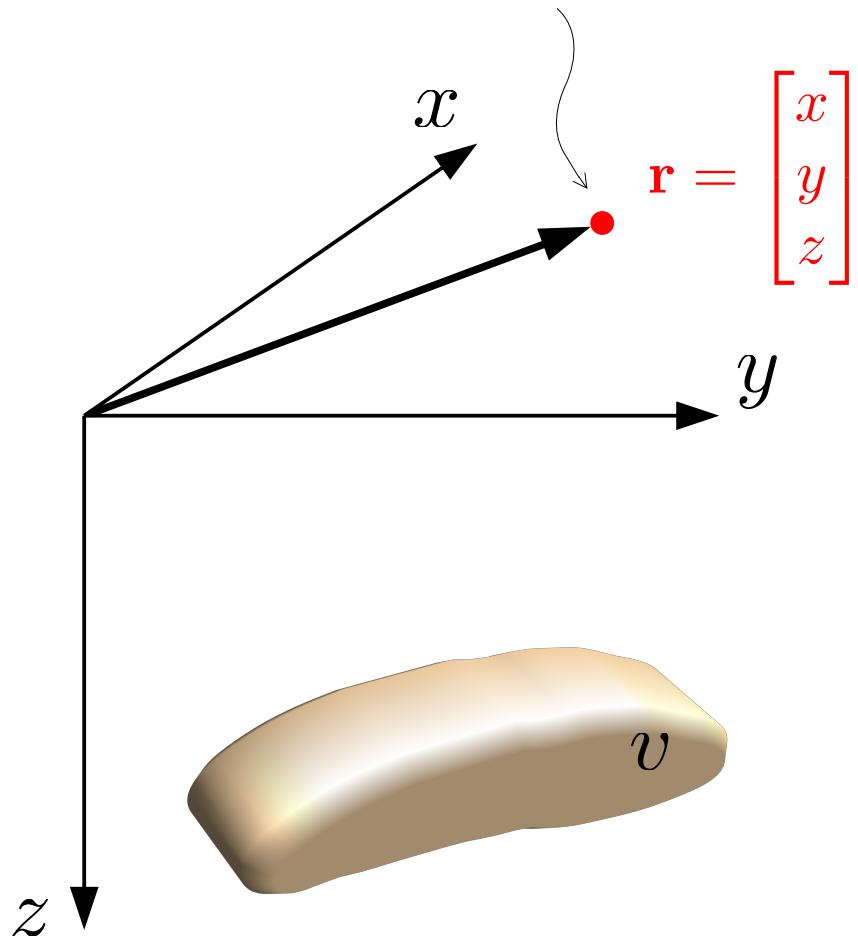
Topocentric Cartesian system



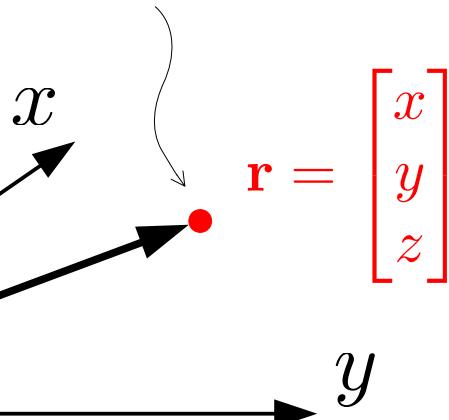


Source with
volume
 v

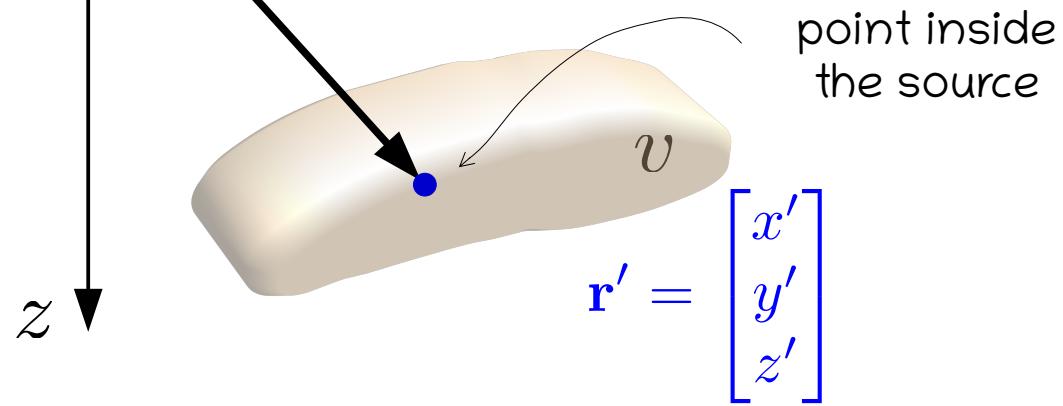
observation point

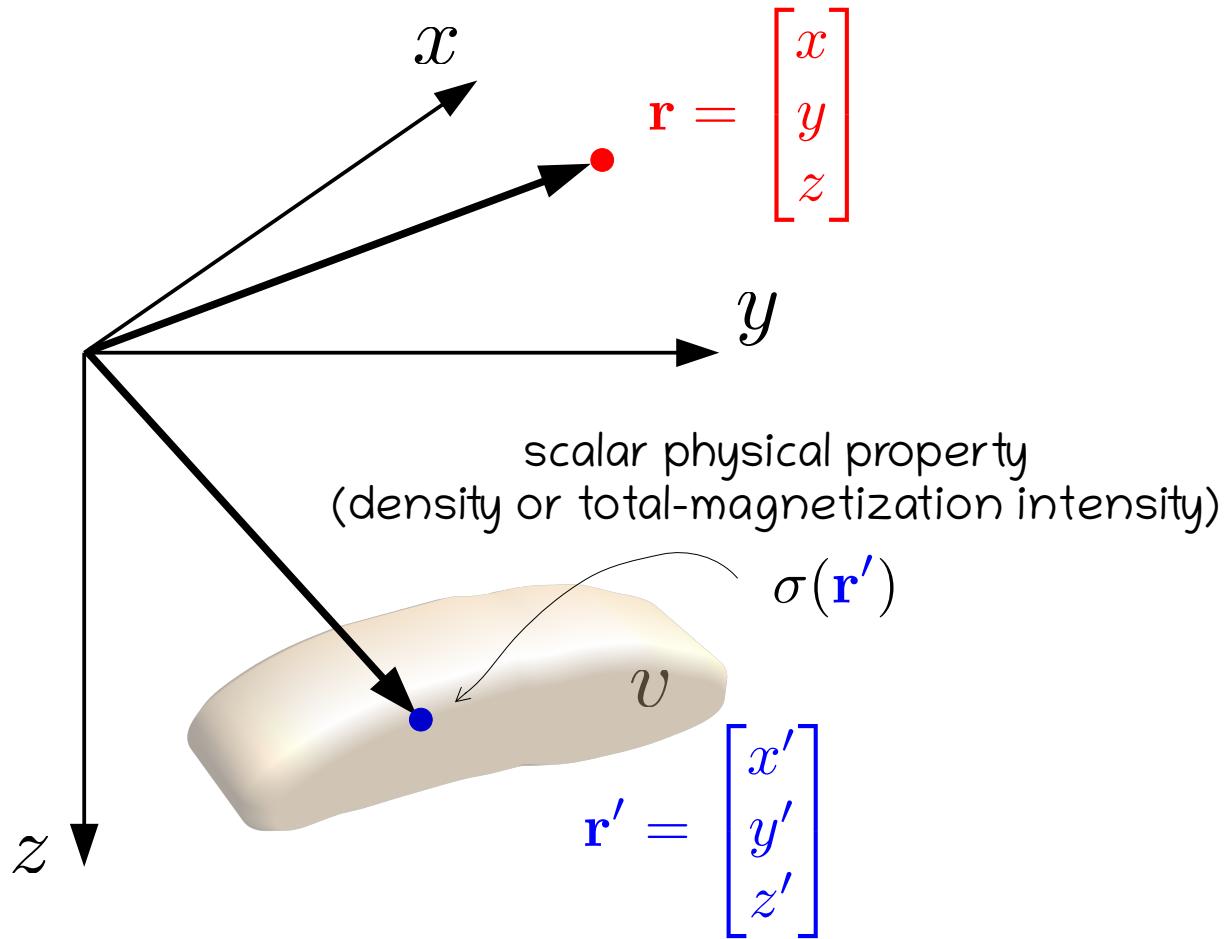


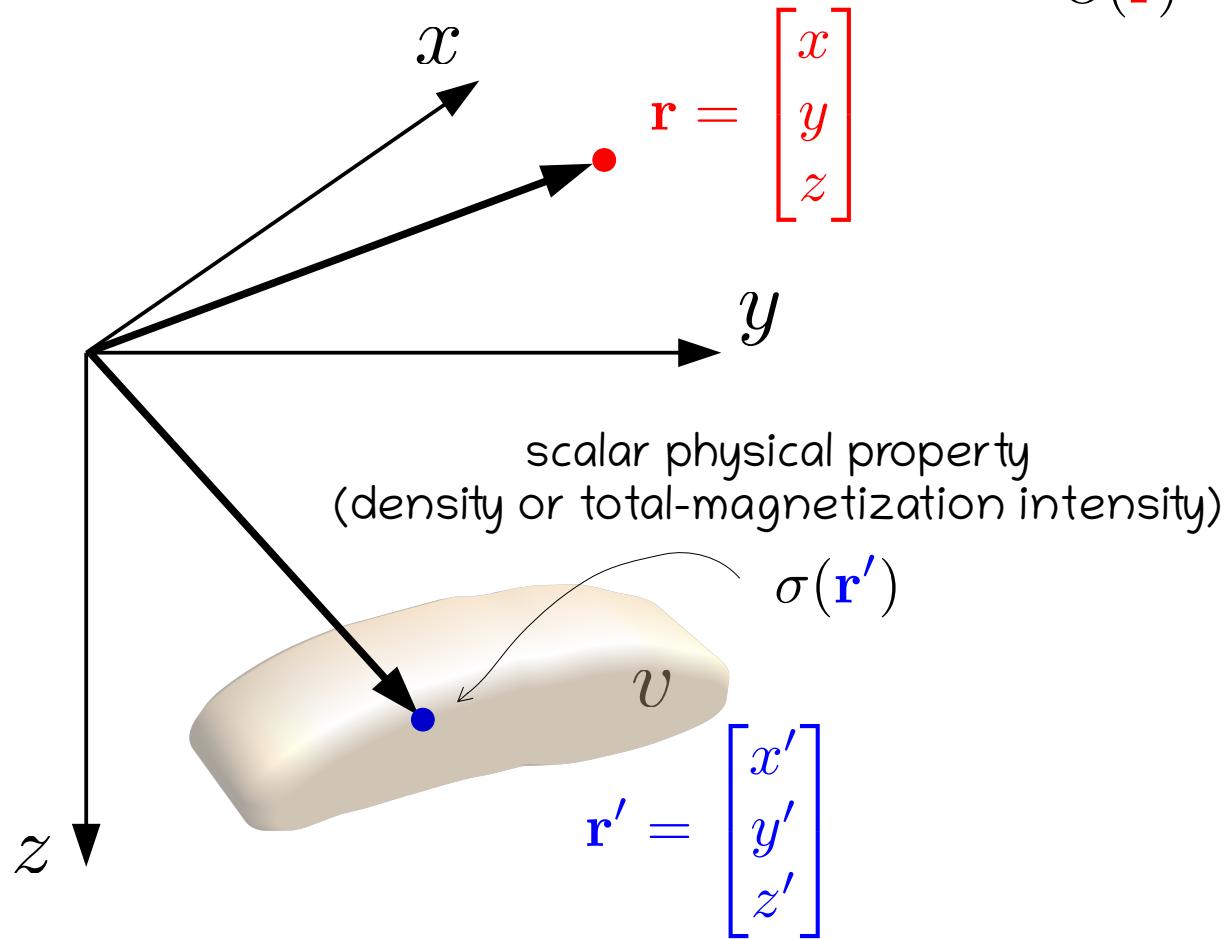
observation point



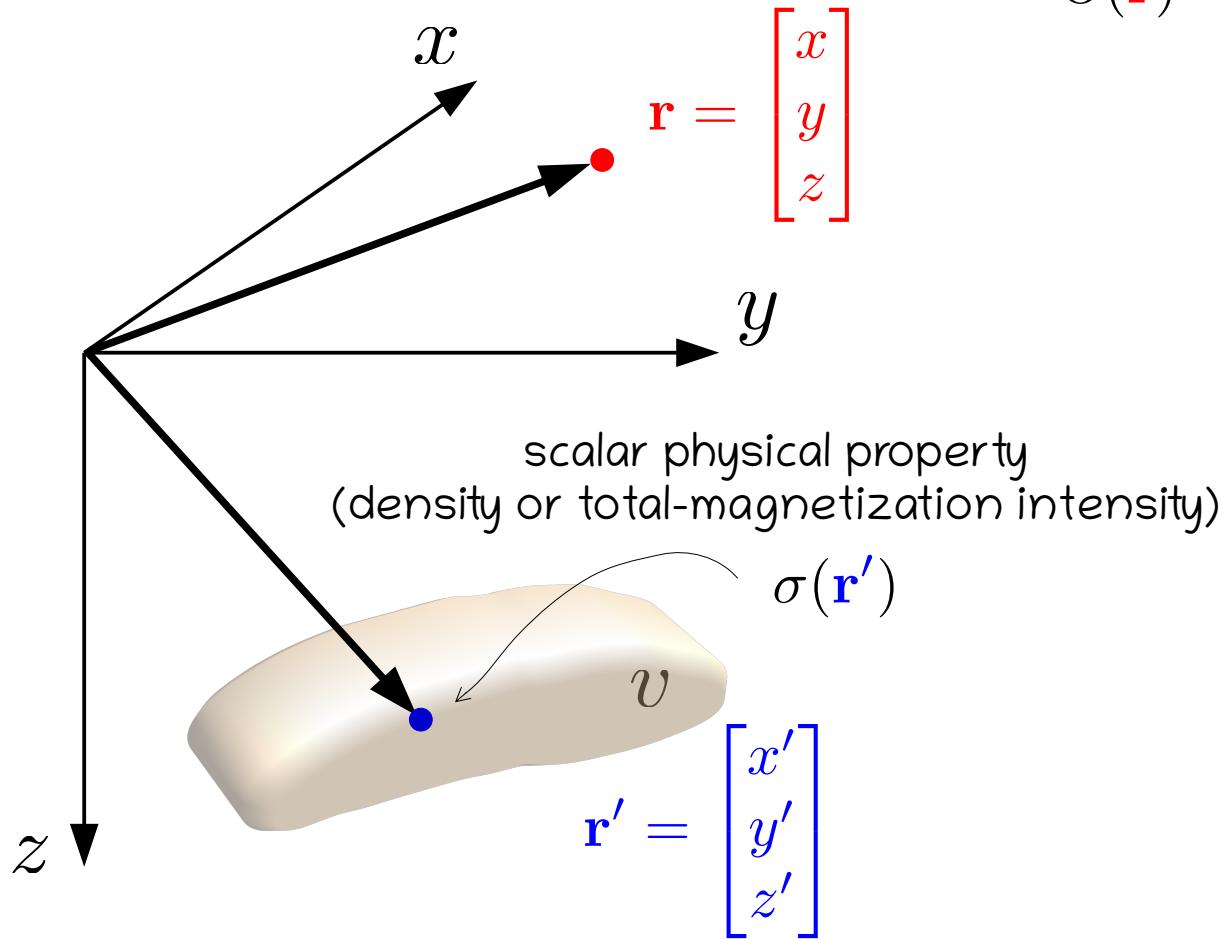
point inside
the source





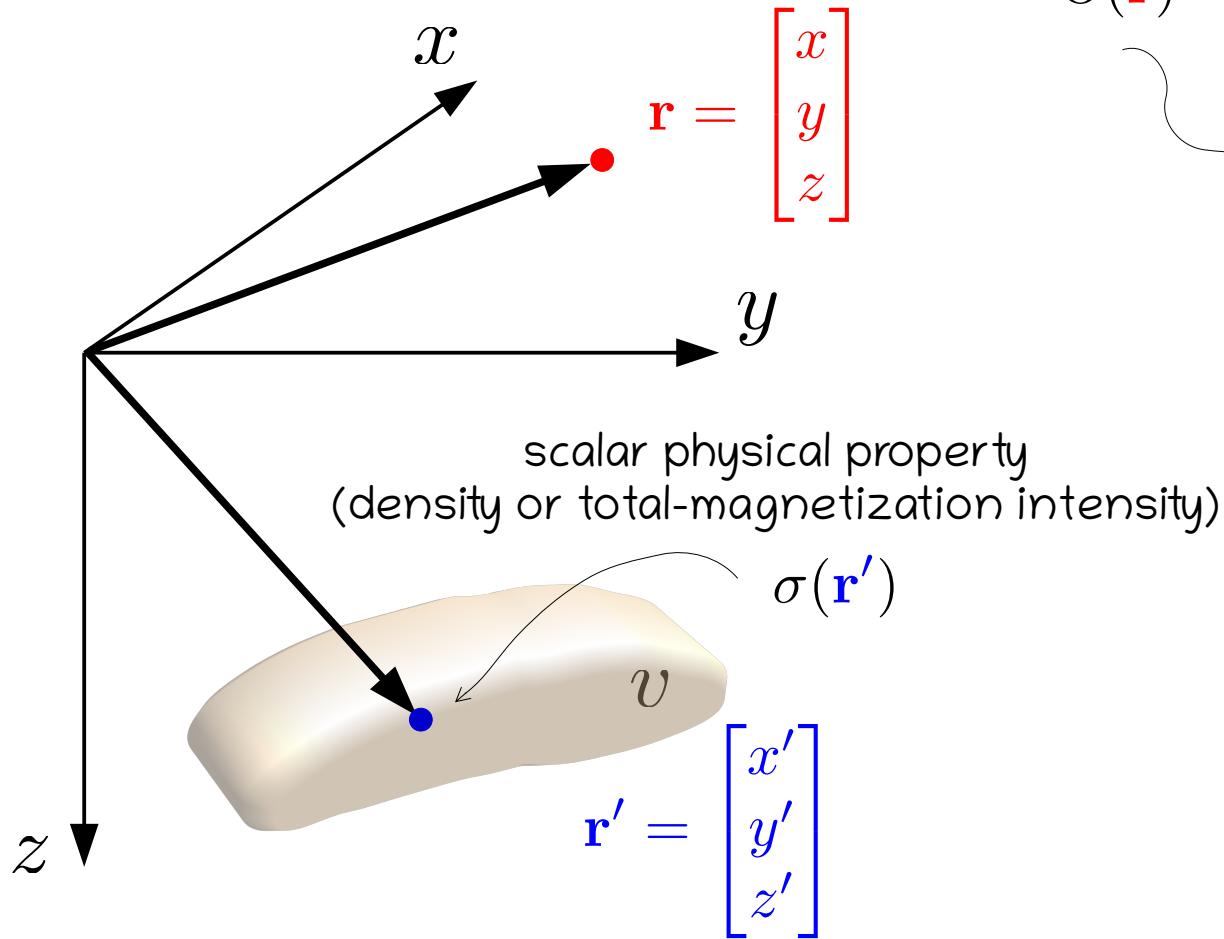


$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$dx' dy' dz'$

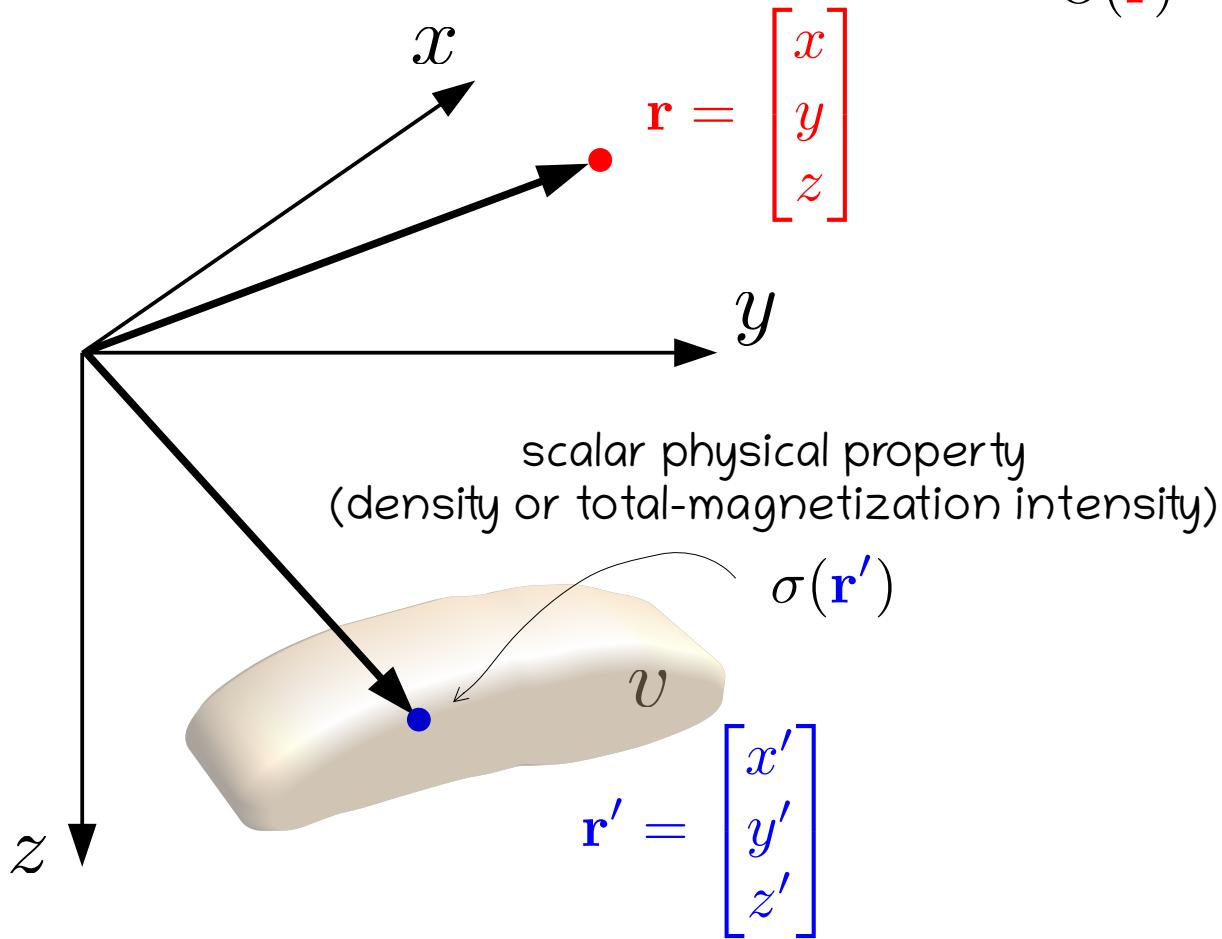


$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

This function approaches 0 as $\|\mathbf{r} - \mathbf{r}'\|$ becomes infinite, for fixed \mathbf{r}' , and it is harmonic at points \mathbf{r} outside v , i.e., it satisfies:

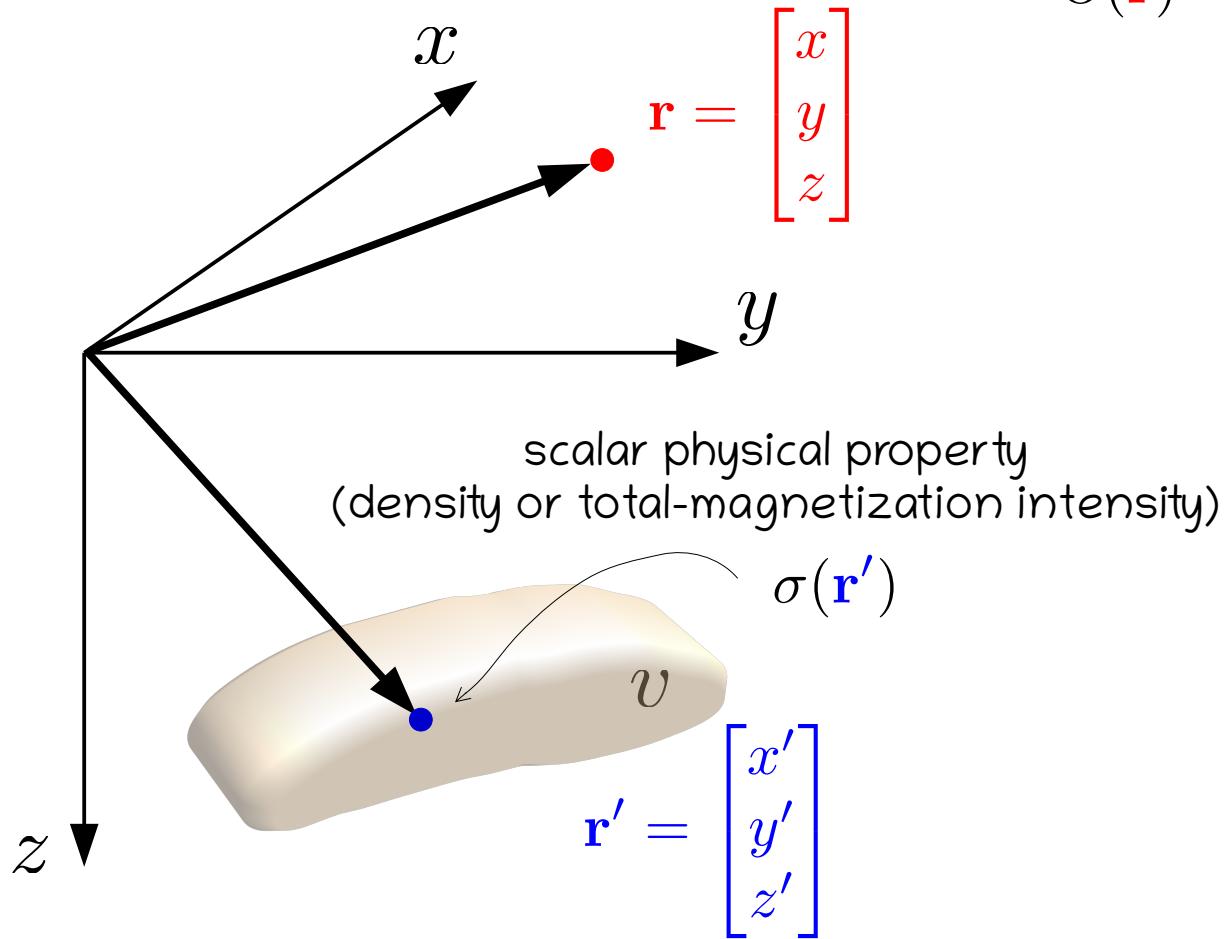
$$\partial_{xx}\Theta(\mathbf{r}) + \partial_{yy}\Theta(\mathbf{r}) + \partial_{zz}\Theta(\mathbf{r}) = 0$$

for $\mathbf{r} \notin v$



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

Some conditions are imposed to the volume v and the scalar function $\sigma(\mathbf{r}')$ in order to guarantee the existence of the first and second derivatives of $\Theta(\mathbf{r})$. For details, see [Kellogg \(1967, Theorem III, p. 156\)](#)

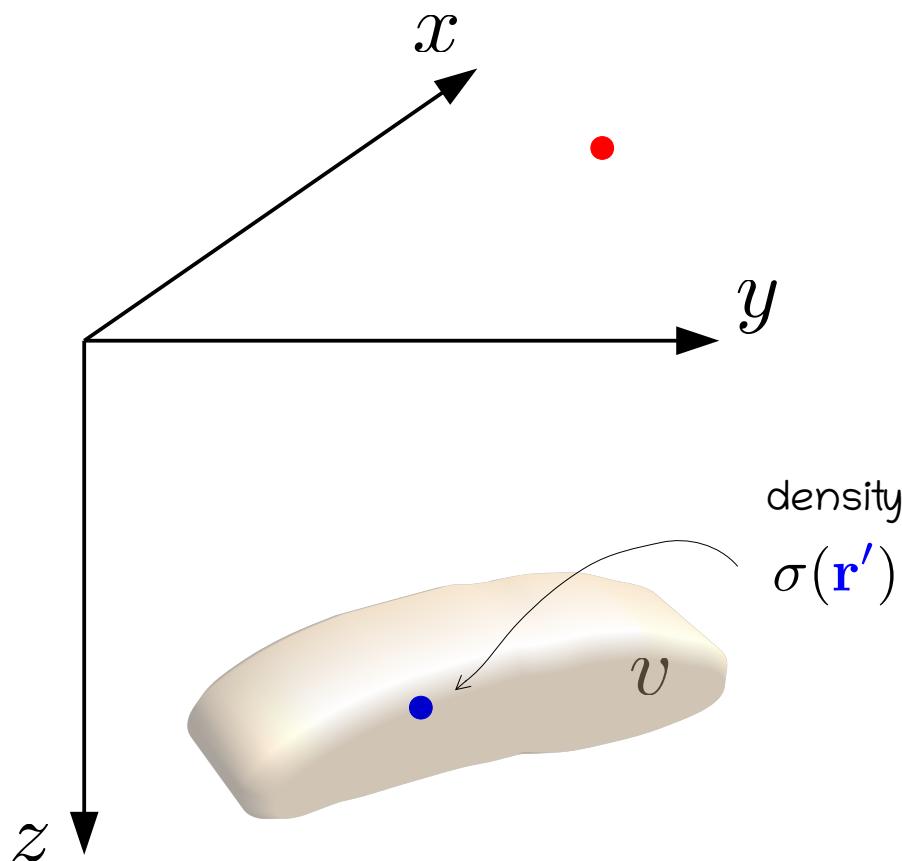


$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

constant defining gravitational or magnetic field

Gravitational field

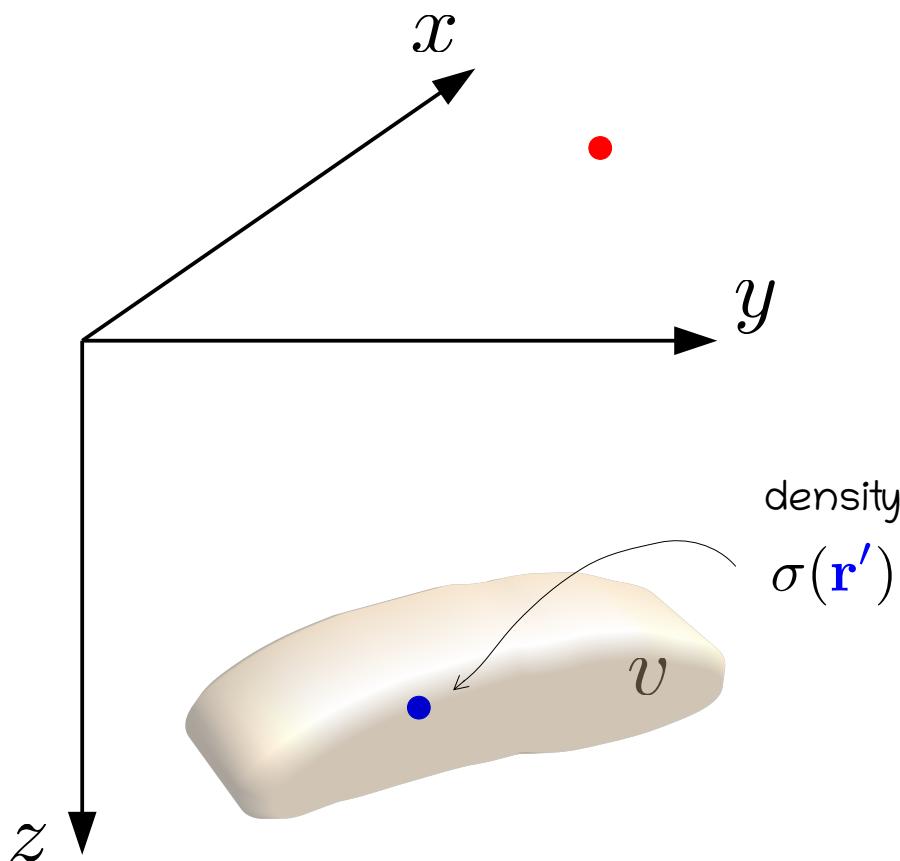
$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$



Gravitational field

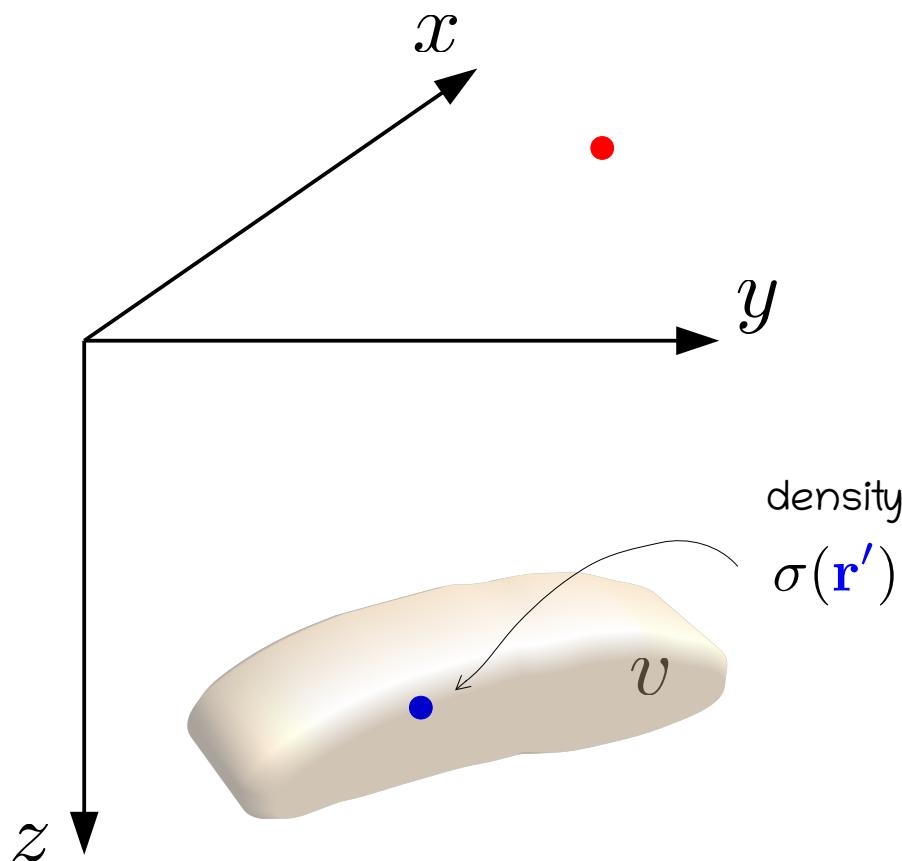
$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

density



$\Theta(\mathbf{r})$ gravitational potential

Gravitational field

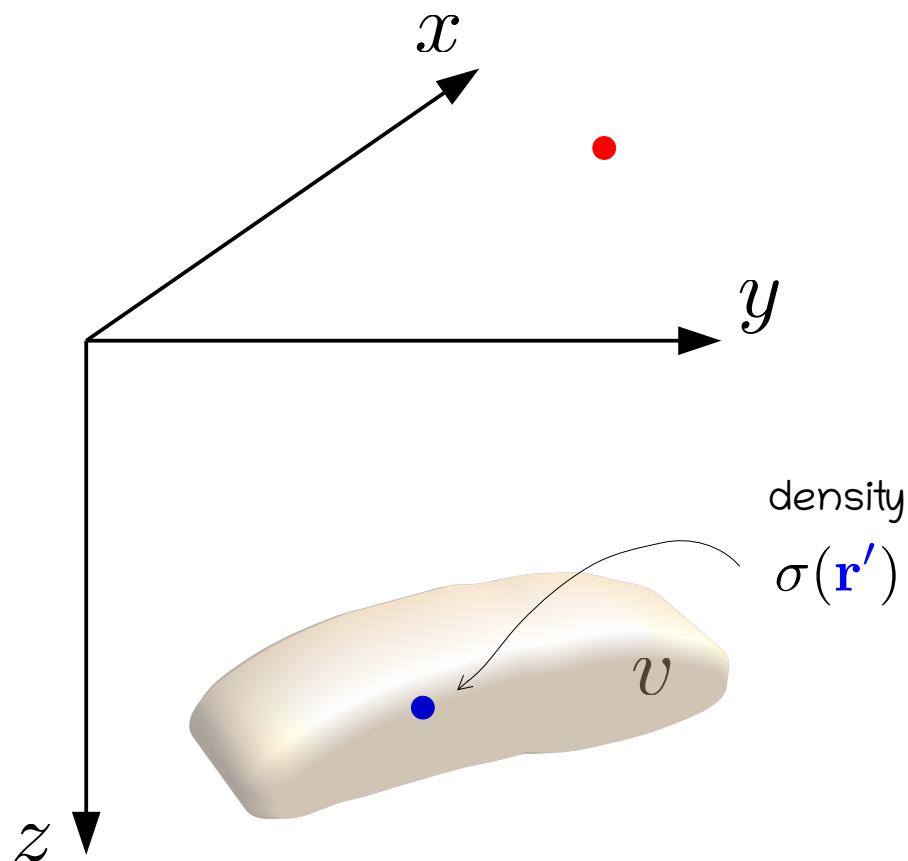


$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$\Theta(\mathbf{r})$ gravitational potential

$\partial_z \Theta(\mathbf{r})$ gravity disturbance

Gravitational field

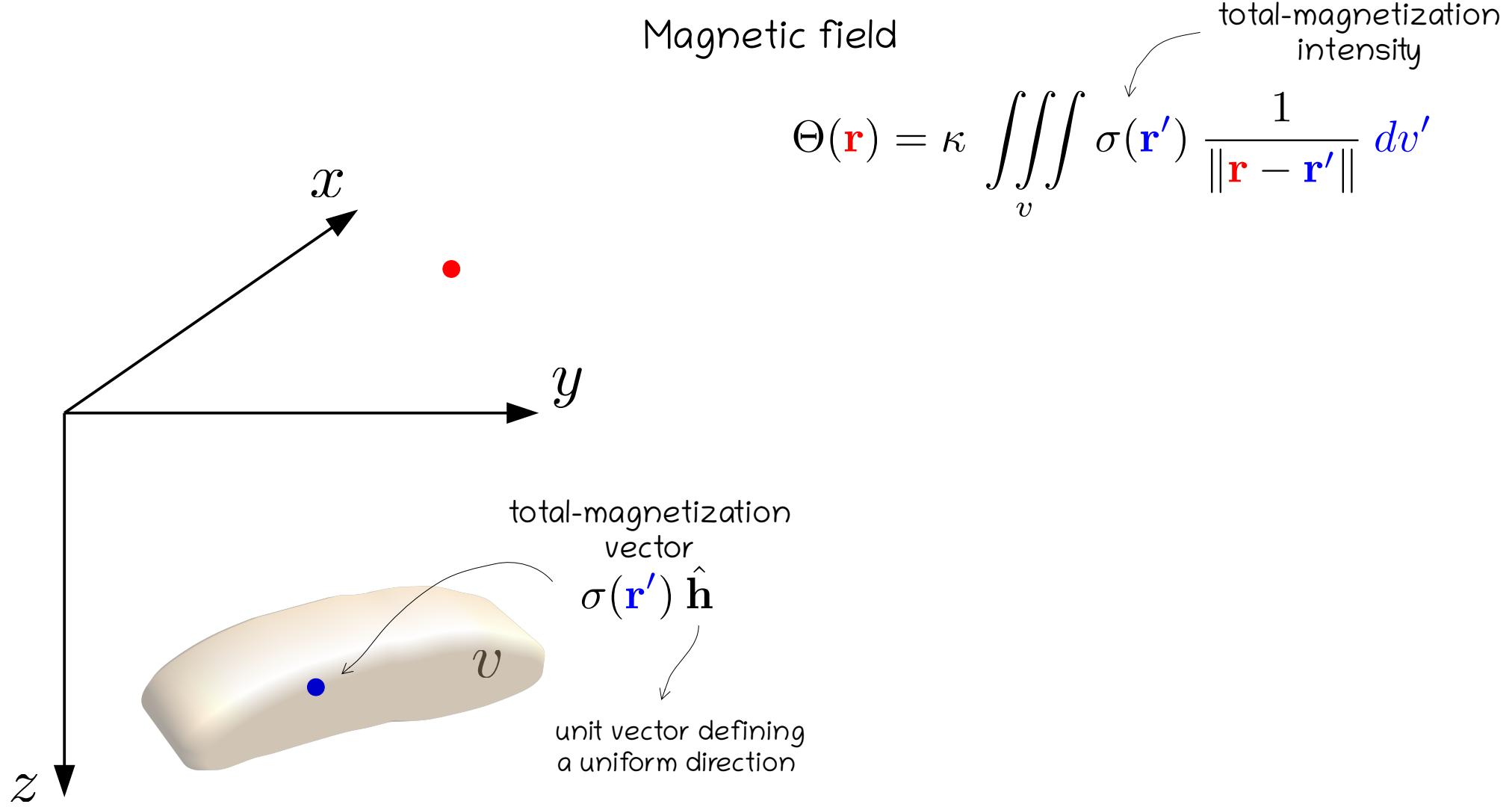


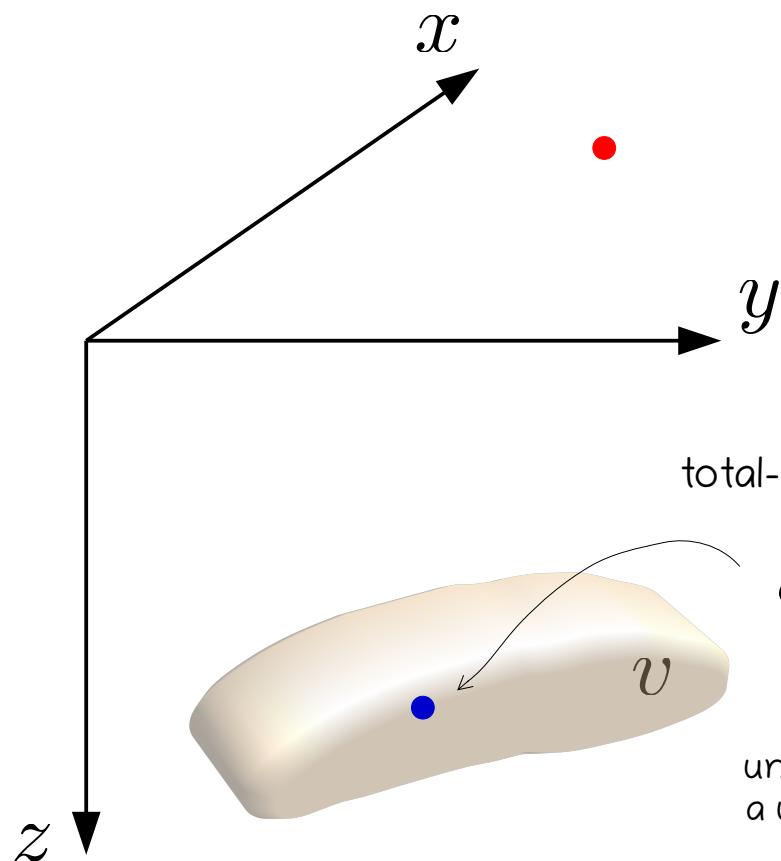
$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$\Theta(\mathbf{r})$ gravitational potential

$\partial_z \Theta(\mathbf{r})$ gravity disturbance

$\partial_{\alpha\beta} \Theta(\mathbf{r})$ gravity gradient tensor
 x, y, z





Magnetic field

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

total-magnetization
intensity

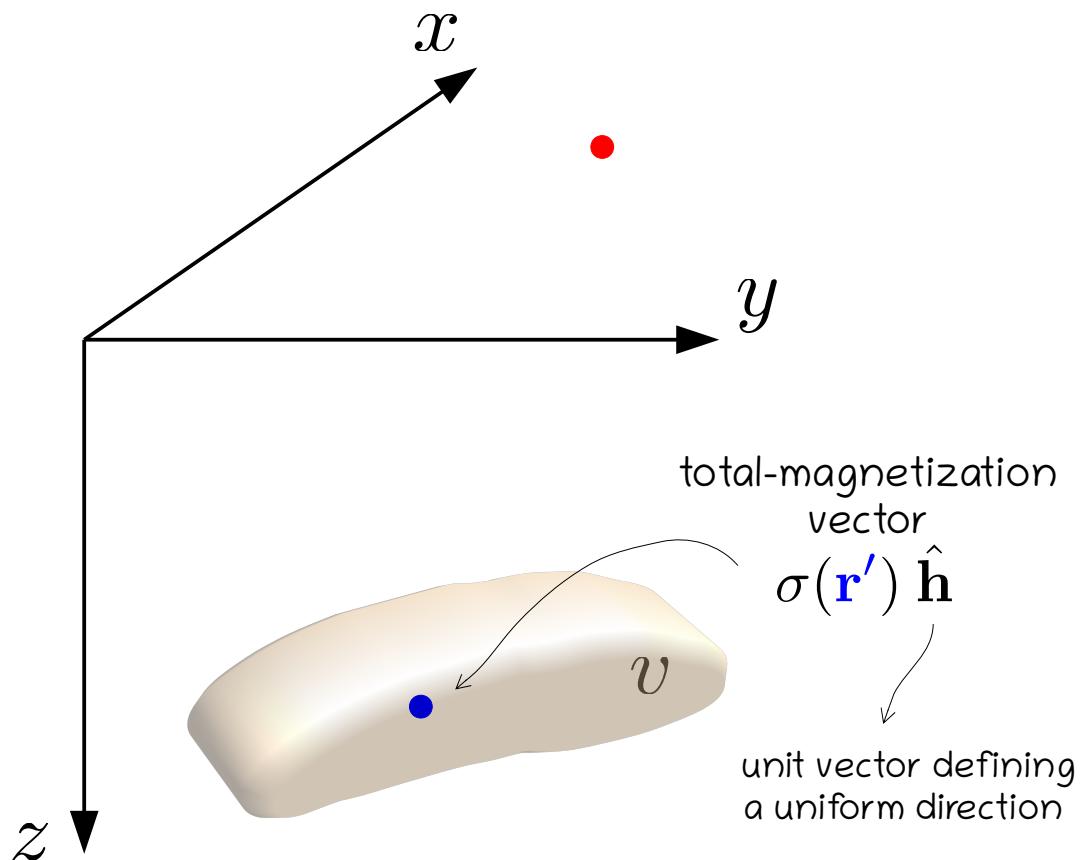
$-\partial_h \Theta(\mathbf{r})$ mag. scalar potential

total-magnetization

vector

$$\sigma(\mathbf{r}') \hat{\mathbf{h}}$$

unit vector defining
a uniform direction



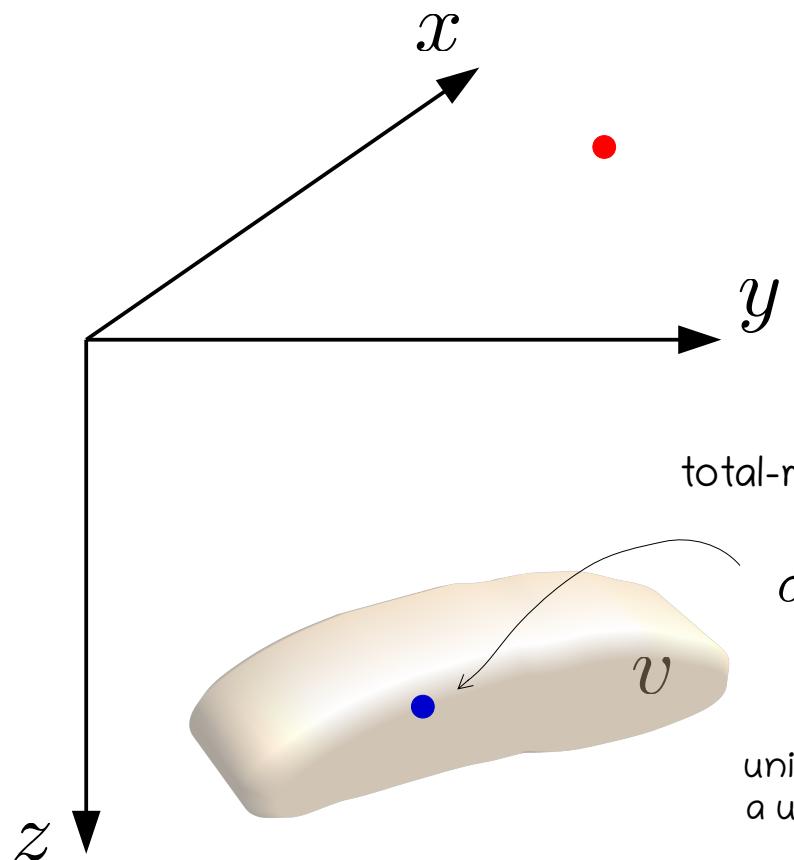
Magnetic field

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

total-magnetization
intensity

$-\partial_h \Theta(\mathbf{r})$ mag. scalar potential

$$\partial_h \Theta(\mathbf{r}) = \nabla \Theta(\mathbf{r})^\top \hat{\mathbf{h}}$$



Magnetic field

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

total-magnetization
intensity

$-\partial_h \Theta(\mathbf{r})$ mag. scalar potential

$\partial_{\alpha h} \Theta(\mathbf{r})$ mag. induction field

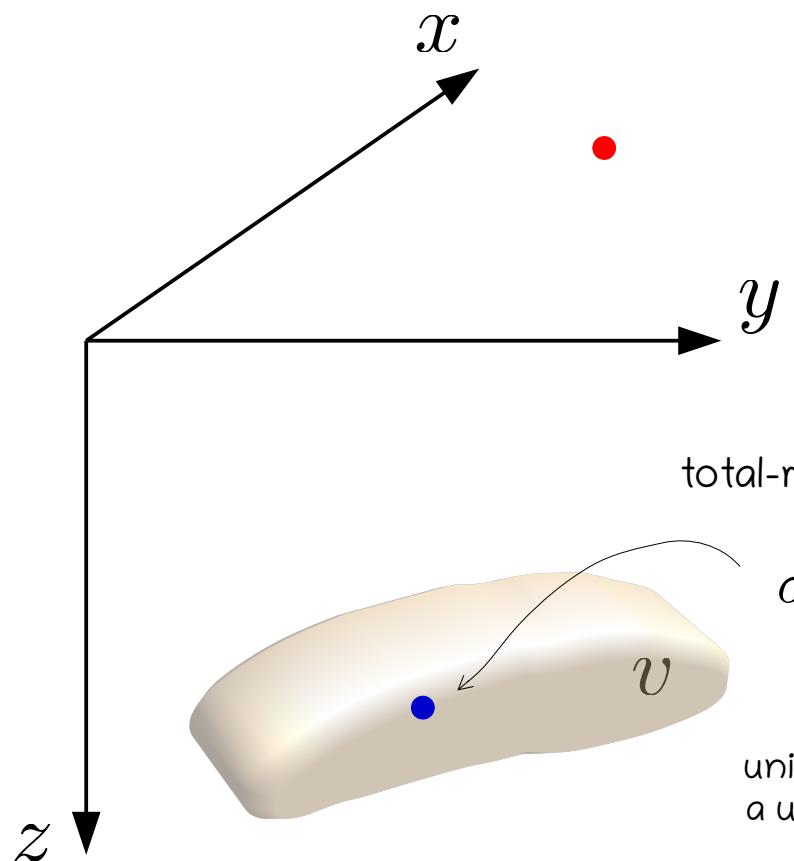
x, y, z

total-magnetization

vector

$\sigma(\mathbf{r}') \hat{\mathbf{h}}$

unit vector defining
a uniform direction



Magnetic field

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

total-magnetization intensity

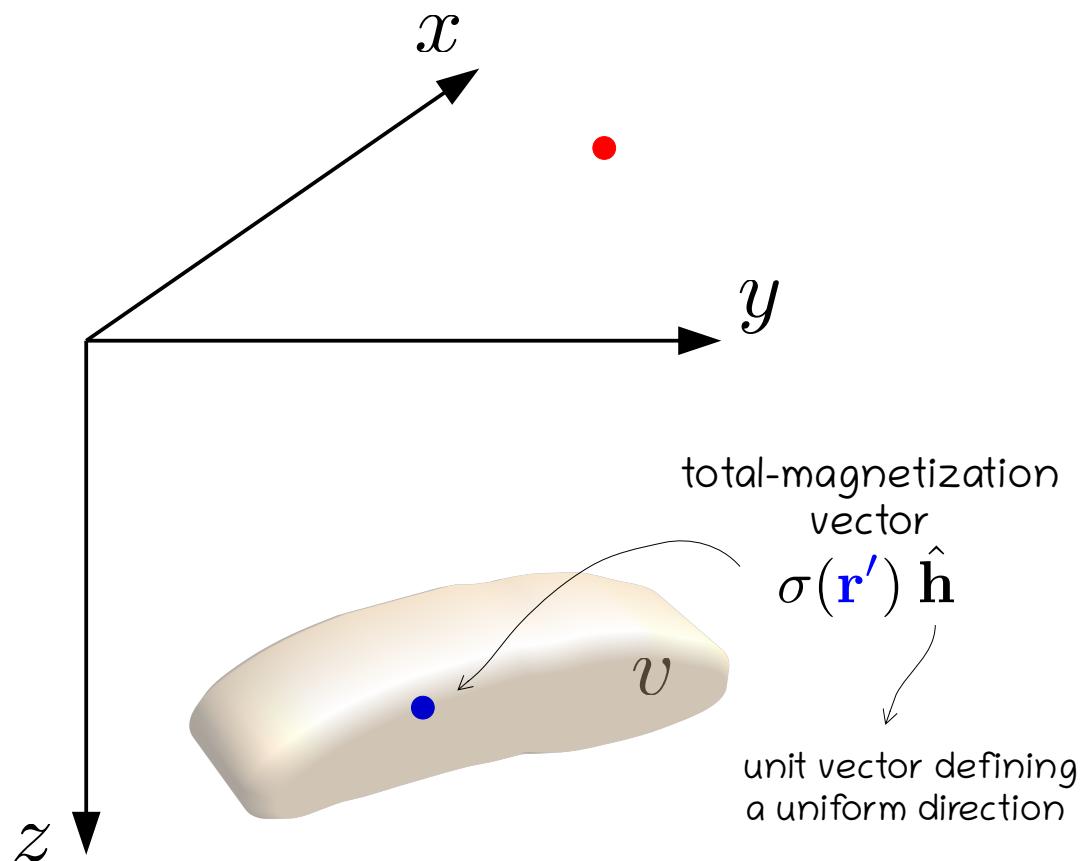
$-\partial_h \Theta(\mathbf{r})$ mag. scalar potential

$\partial_{\alpha h} \Theta(\mathbf{r})$ mag. induction field

x, y, z

$$\partial_{\alpha h} \Theta(\mathbf{r}) = \partial_{\alpha} \nabla \Theta(\mathbf{r})^{\top} \hat{\mathbf{h}}$$

unit vector defining
a uniform direction



Magnetic field

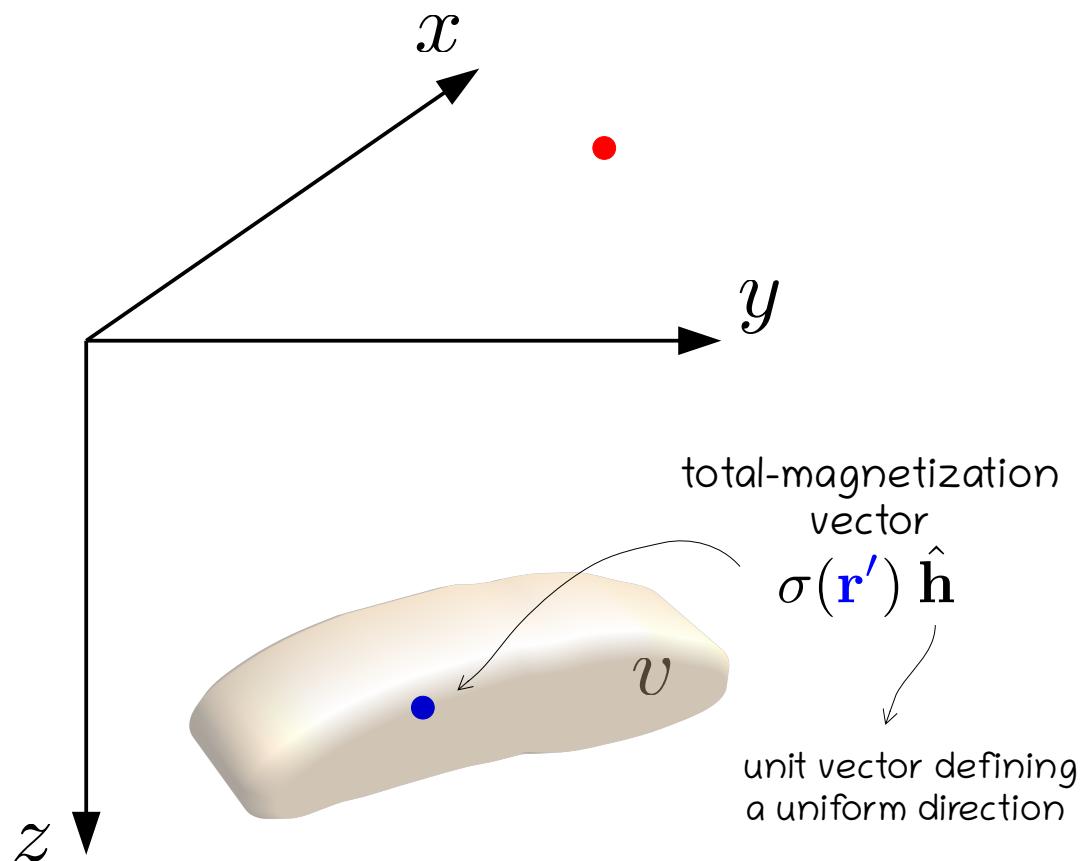
$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

total-magnetization intensity

$-\partial_h \Theta(\mathbf{r})$ mag. scalar potential

$\partial_{\alpha h} \Theta(\mathbf{r})$ mag. induction field

$\partial_{th} \Theta(\mathbf{r})$ approx. total-field anomaly



Magnetic field

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

total-magnetization intensity

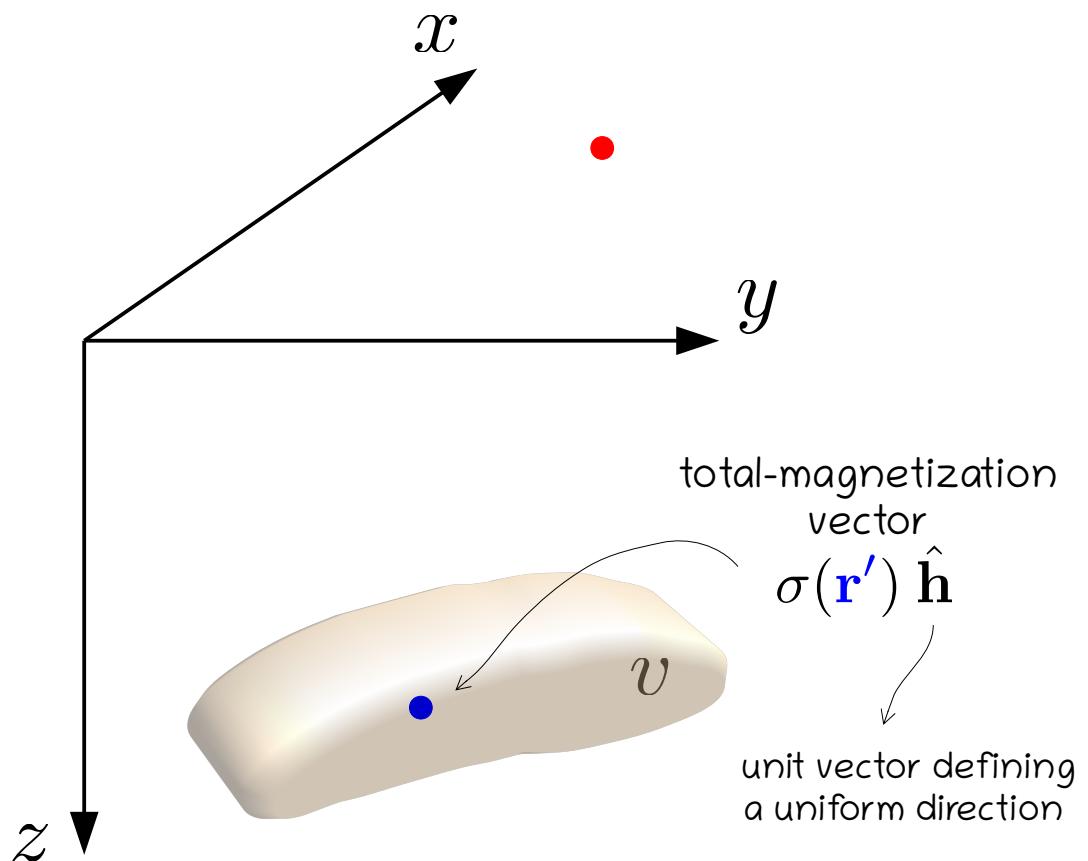
$-\partial_h \Theta(\mathbf{r})$ mag. scalar potential

$\partial_{\alpha h} \Theta(\mathbf{r})$ mag. induction field

$\partial_{th} \Theta(\mathbf{r})$ approx. total-field anomaly

$$\hookrightarrow \partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

unit vector defining
a uniform direction



Magnetic field

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

total-magnetization intensity

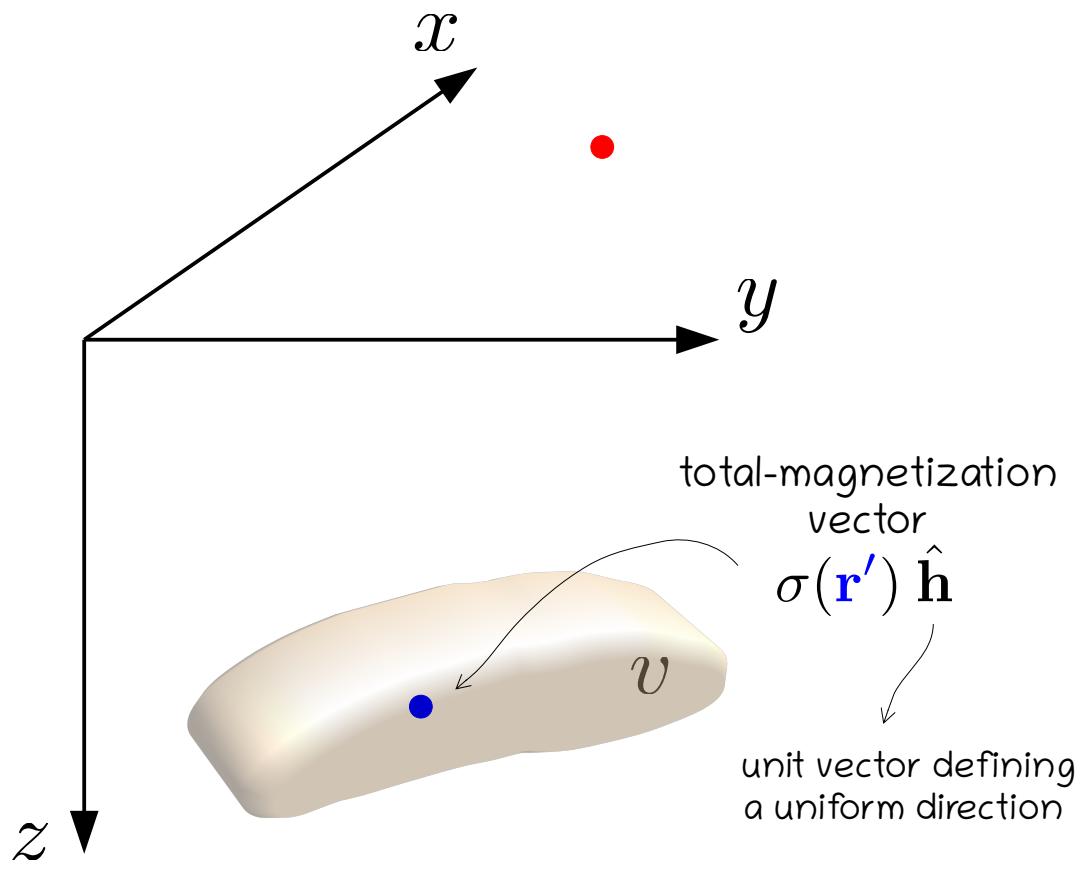
$-\partial_h \Theta(\mathbf{r})$ mag. scalar potential

$\partial_{\alpha h} \Theta(\mathbf{r})$ mag. induction field

$\partial_{th} \Theta(\mathbf{r})$ approx. total-field anomaly

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

unit vector defining a
constant direction for the
main geomagnetic field



Magnetic field

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

total-magnetization intensity

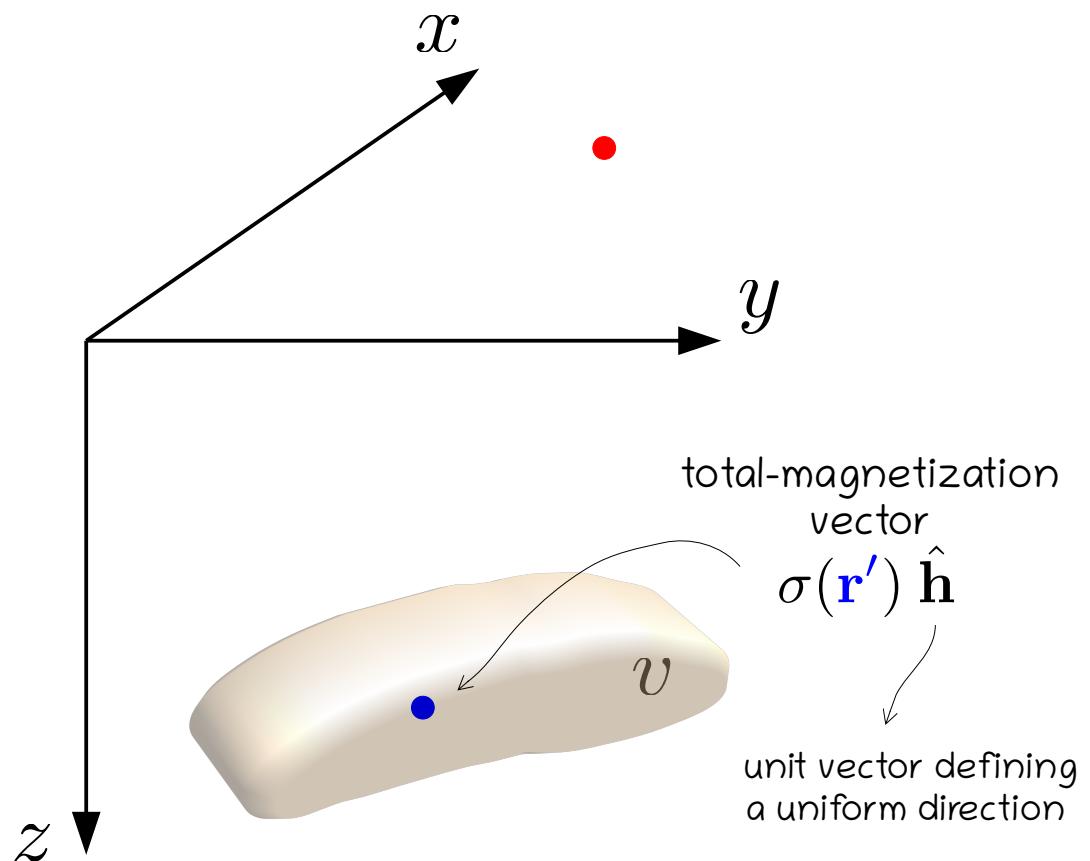
$-\partial_h \Theta(\mathbf{r})$ mag. scalar potential

$\partial_{\alpha h} \Theta(\mathbf{r})$ mag. induction field

$\partial_{th} \Theta(\mathbf{r})$ approx. total-field anomaly

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

$$\mathbf{H}_\Theta(\mathbf{r}) = \begin{bmatrix} \partial_{xx} \Theta(\mathbf{r}) & \partial_{xy} \Theta(\mathbf{r}) & \partial_{xz} \Theta(\mathbf{r}) \\ \partial_{xy} \Theta(\mathbf{r}) & \partial_{yy} \Theta(\mathbf{r}) & \partial_{yz} \Theta(\mathbf{r}) \\ \partial_{xz} \Theta(\mathbf{r}) & \partial_{yz} \Theta(\mathbf{r}) & \partial_{zz} \Theta(\mathbf{r}) \end{bmatrix}$$



Magnetic field

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

total-magnetization intensity

$-\partial_h \Theta(\mathbf{r})$ mag. scalar potential

$\partial_{\alpha h} \Theta(\mathbf{r})$ mag. induction field

$\partial_{th} \Theta(\mathbf{r})$ approx. total-field anomaly

$\partial_{zz} \Theta(\mathbf{r})$ RTP anomaly

} reduced-to-the-pole

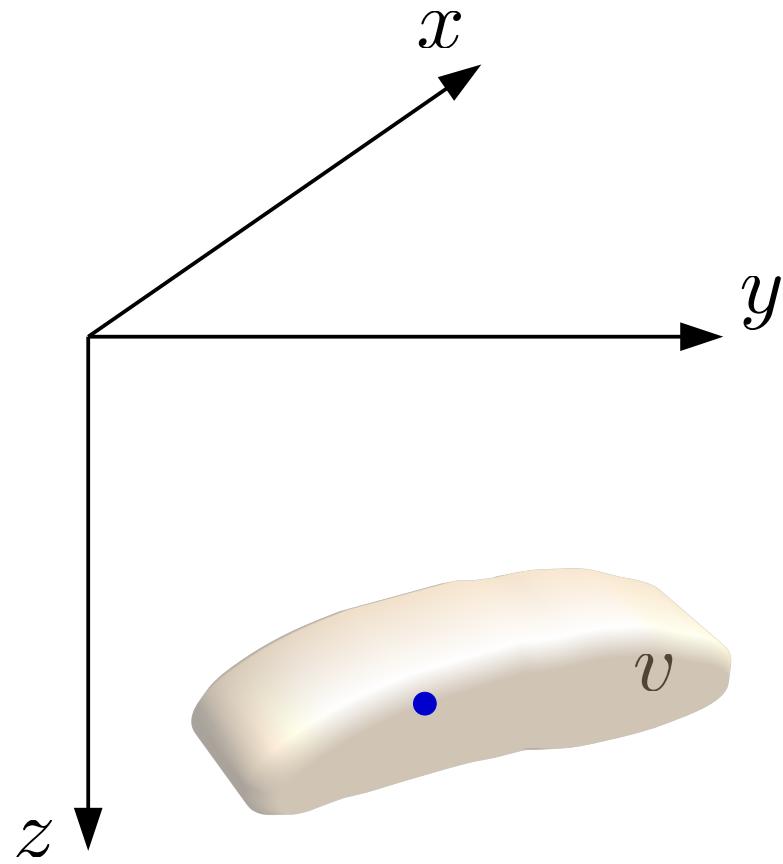
Summary

- Motivation
- **Potential-field data**
- The Equivalent-Layer (EqL) Technique
- Theoretical aspects
- Some open questions

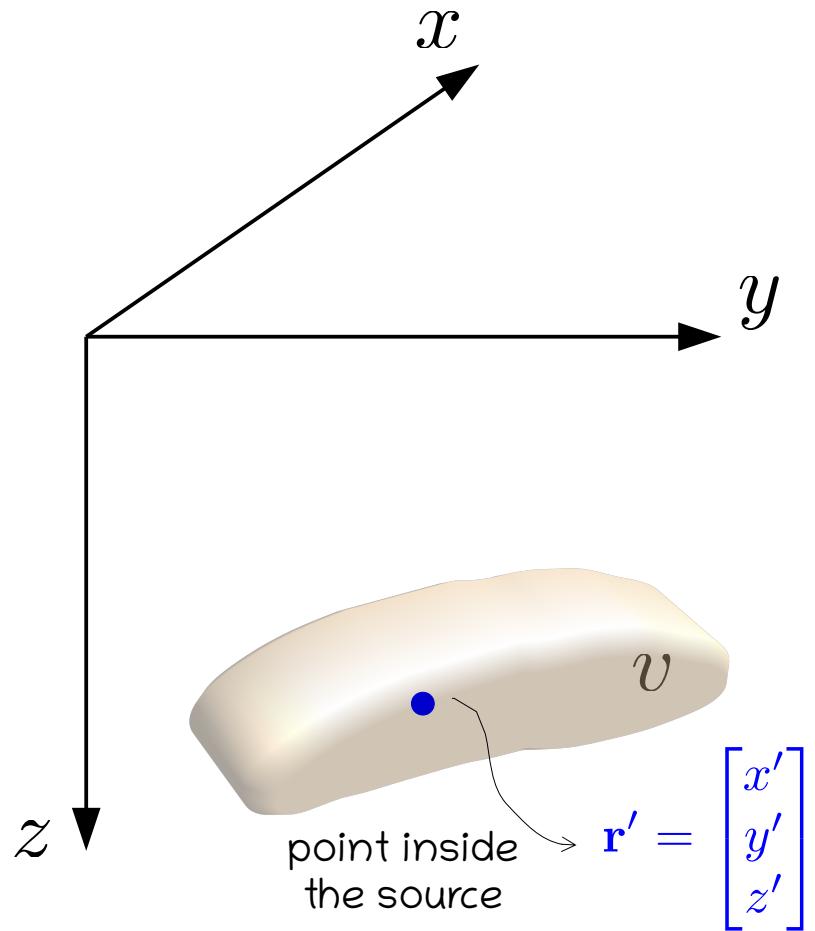
Summary

- Motivation
- Potential-field data
- **The Equivalent-Layer (EqL) Technique**
- Theoretical aspects
- Some open questions

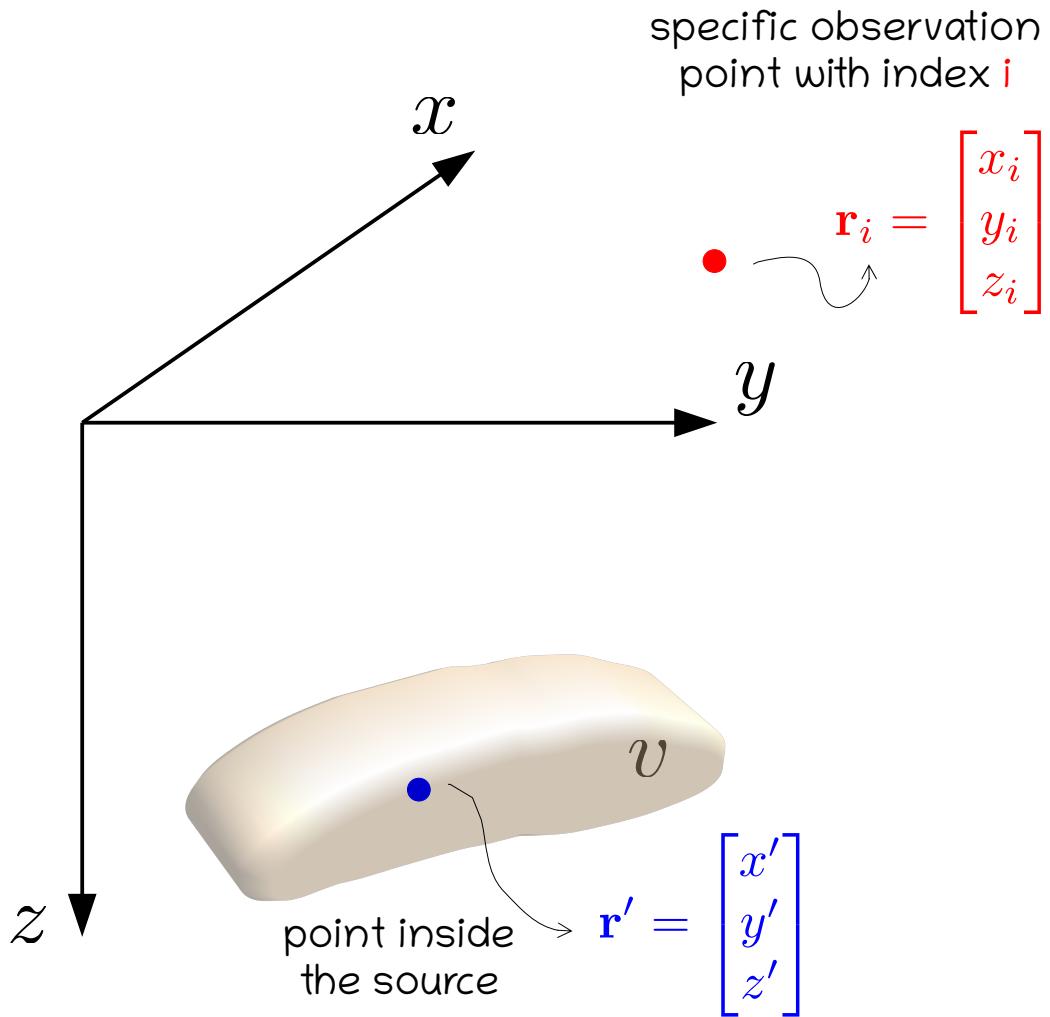
Let us return to our source ...



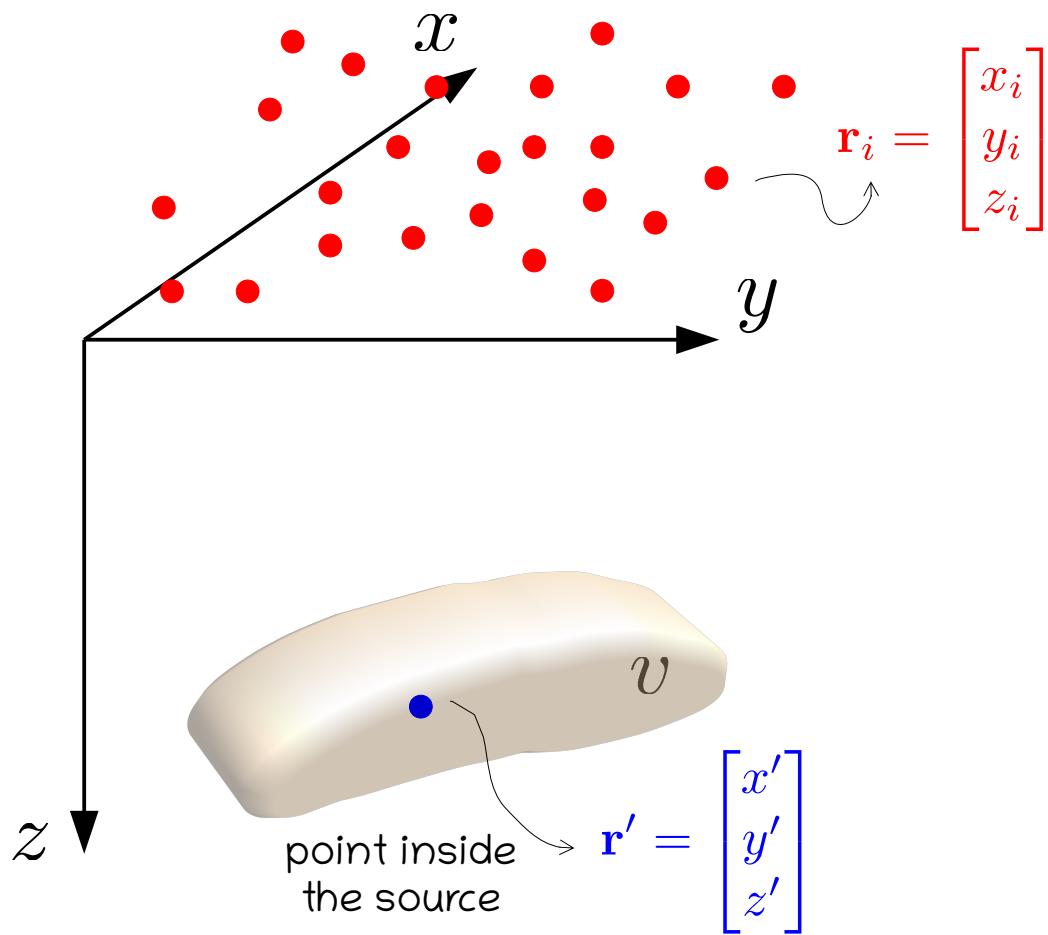
Let us return to our source ...



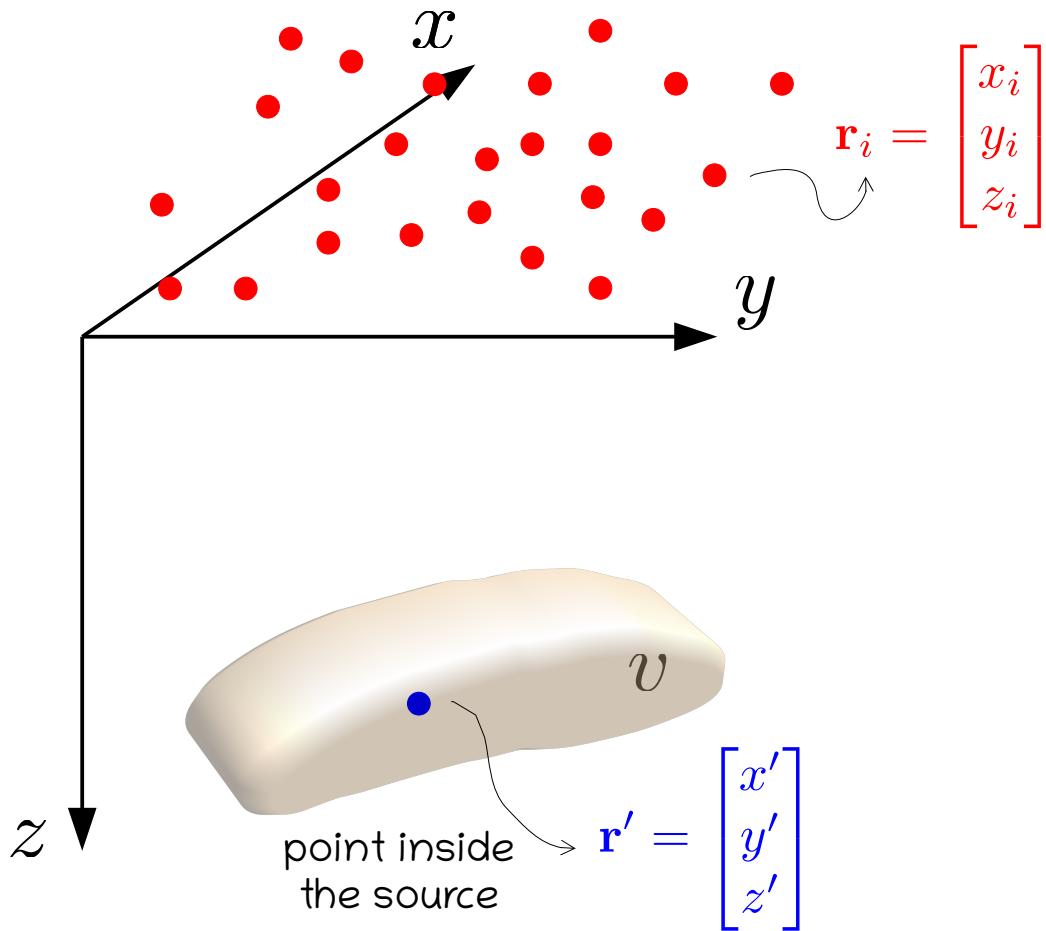
Let us return to our source ...



discrete set of N
observation points



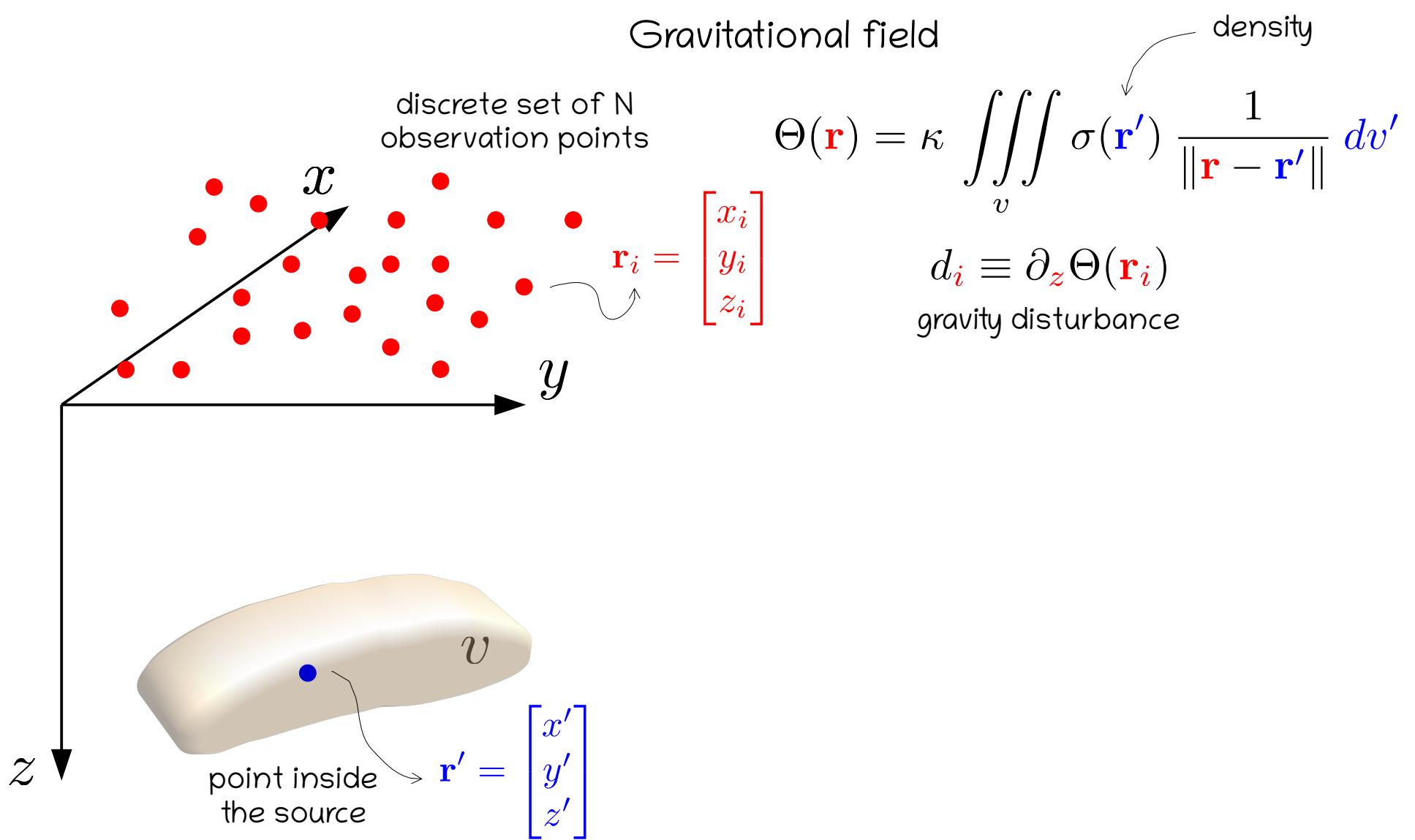
discrete set of N
observation points

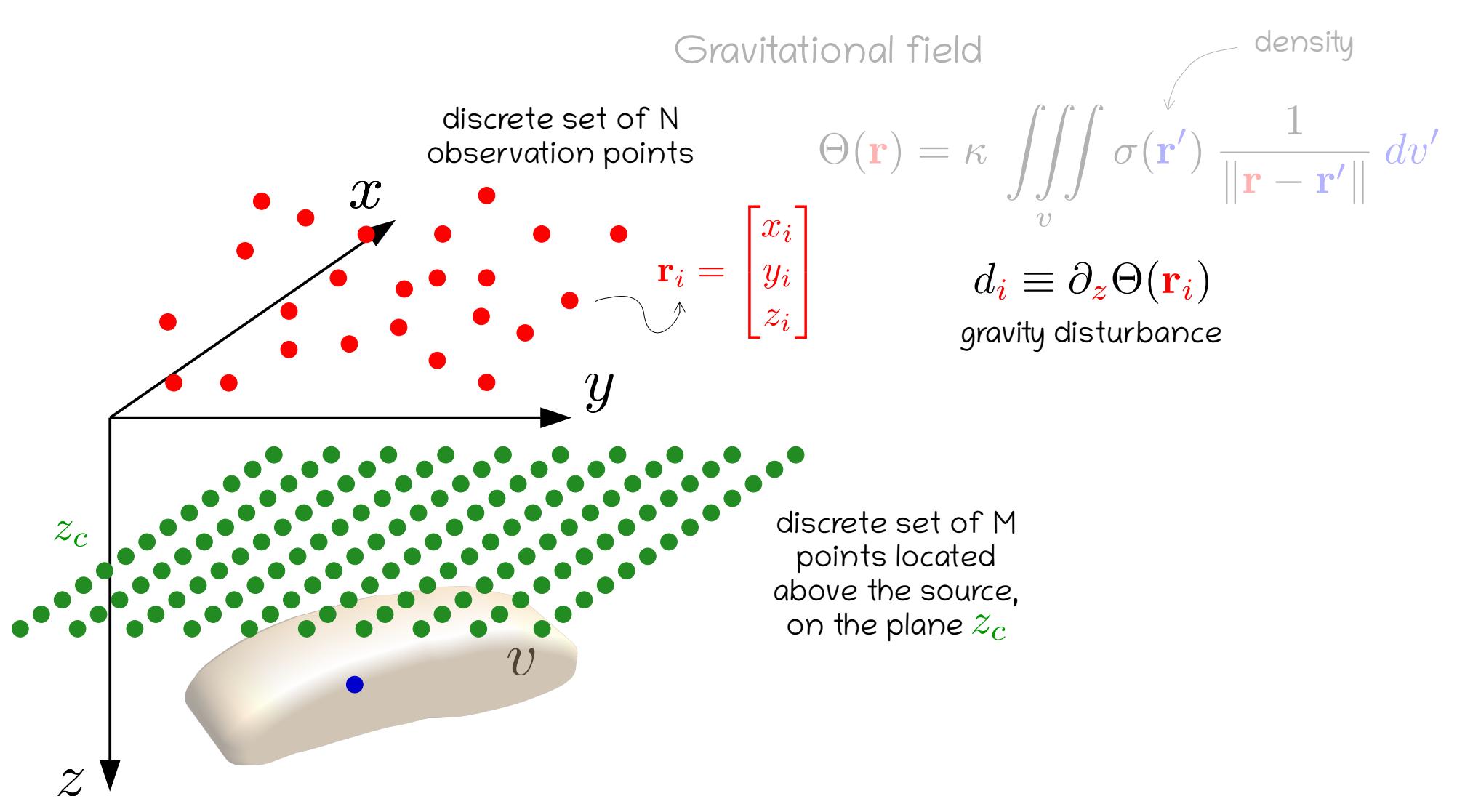


$$d_i \equiv \partial_z \Theta(\mathbf{r}_i)$$

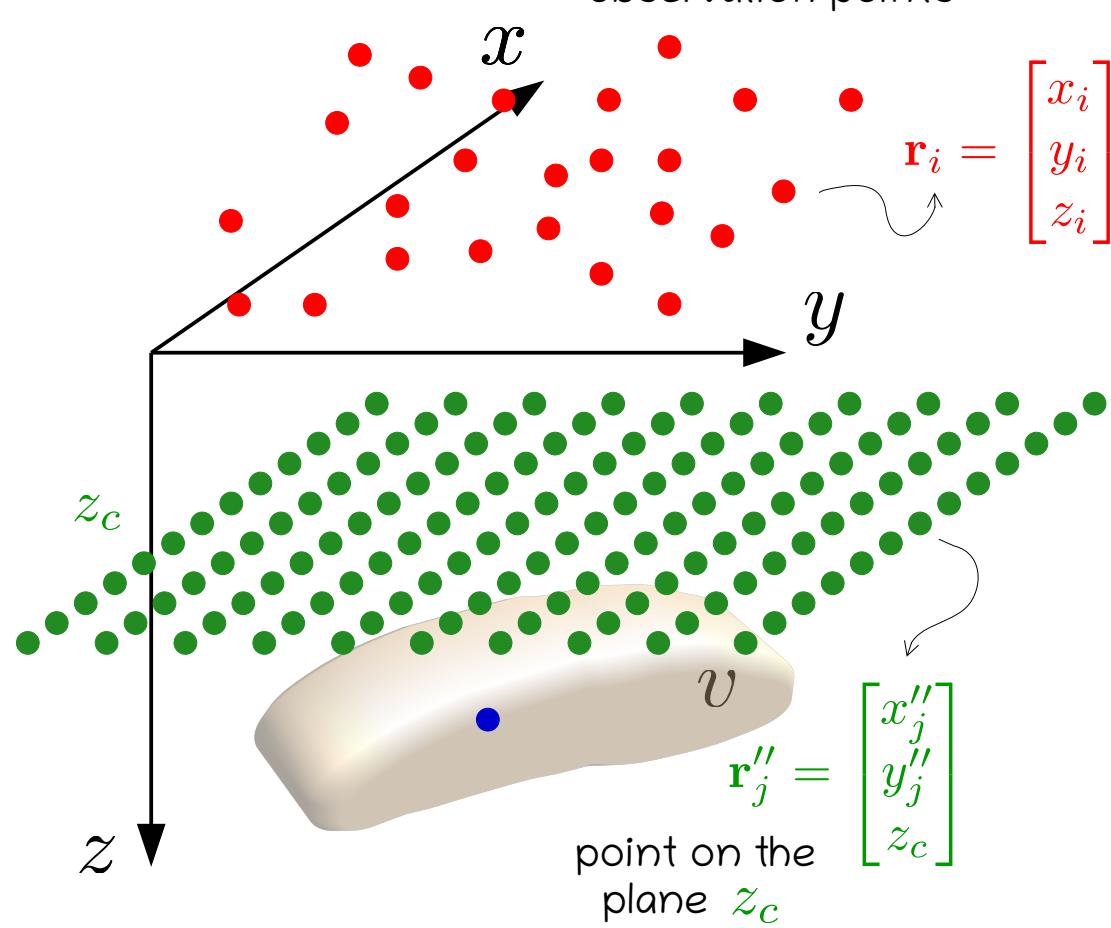
gravity disturbance

Gravitational field





Gravitational field



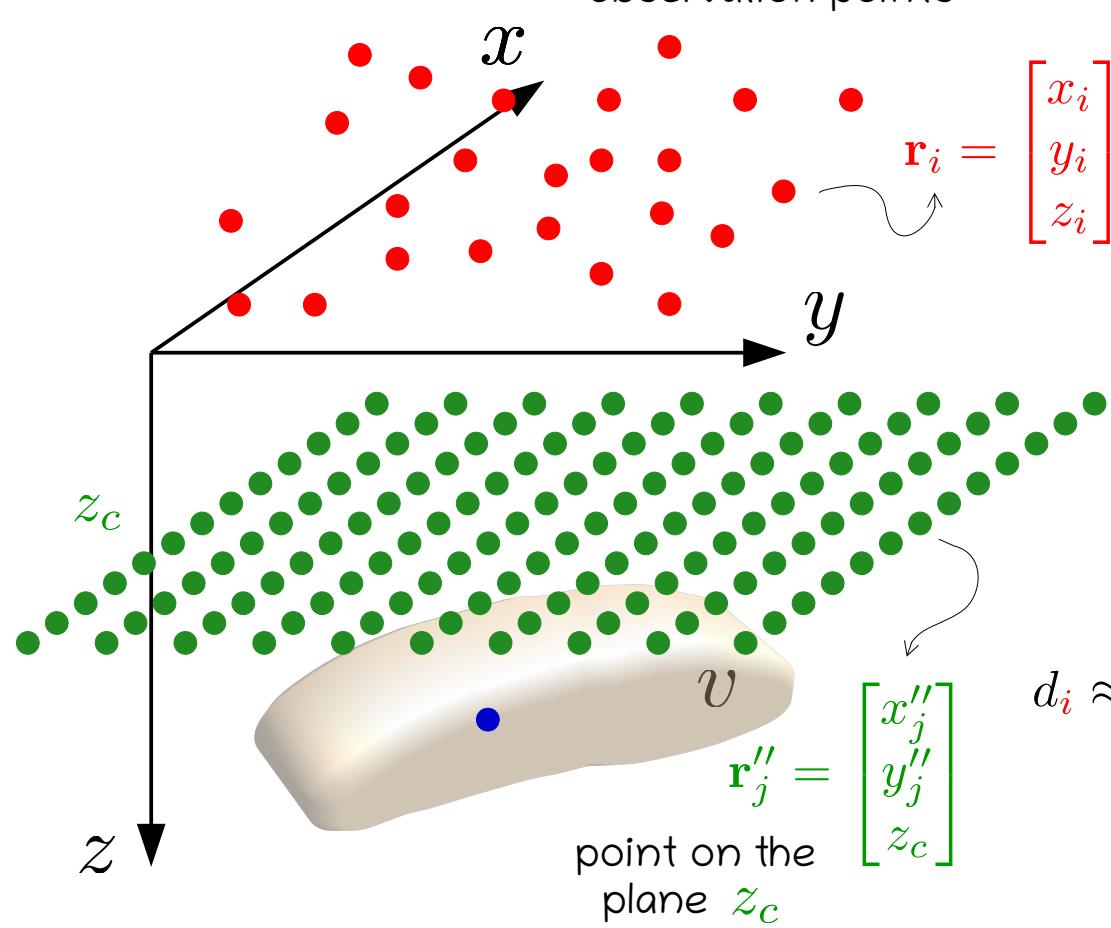
$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

density

$d_i \equiv \partial_z \Theta(\mathbf{r}_i)$

gravity disturbance

Gravitational field



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

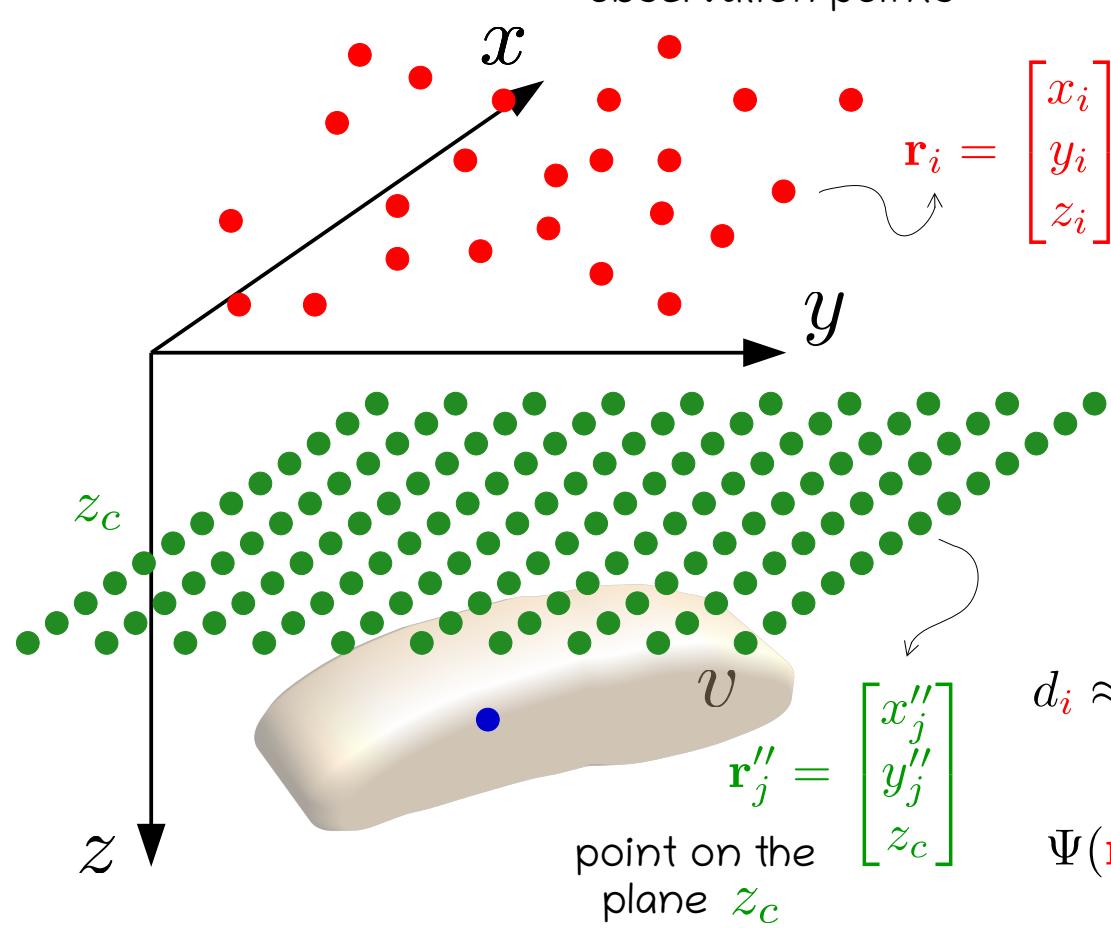
density

$d_i \equiv \partial_z \Theta(\mathbf{r}_i)$

gravity disturbance

$$d_i \approx \sum_{j=1}^M p_j \partial_z \Psi(\mathbf{r}_i, \mathbf{r}''_j)$$

Gravitational field



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

density

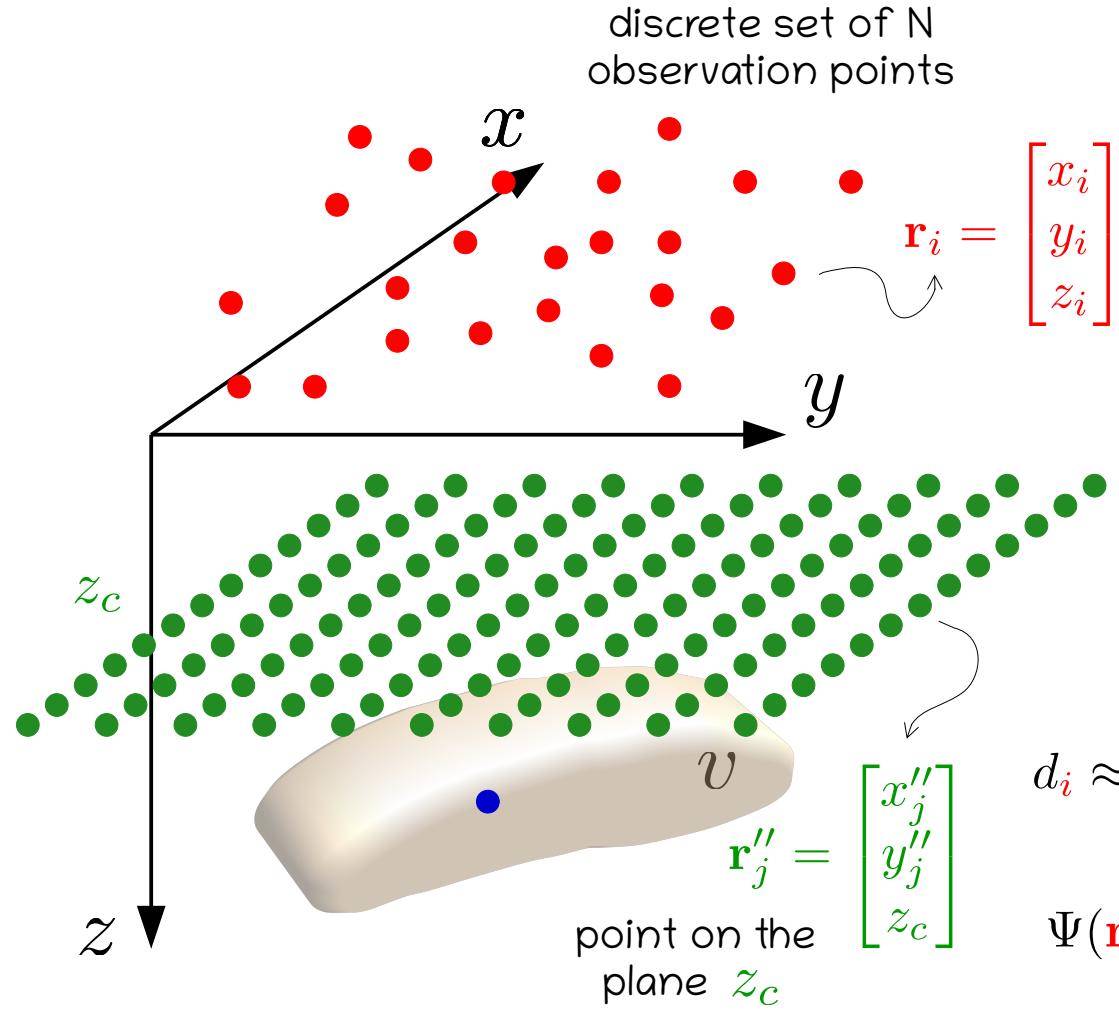
$d_i \equiv \partial_z \Theta(\mathbf{r}_i)$

gravity disturbance

$$d_i \approx \sum_{j=1}^M p_j \partial_z \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

$$\Psi(\mathbf{r}_i, \mathbf{r}_j'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r}_i - \mathbf{r}_j''\|}$$

Gravitational field



density

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

gravity disturbance

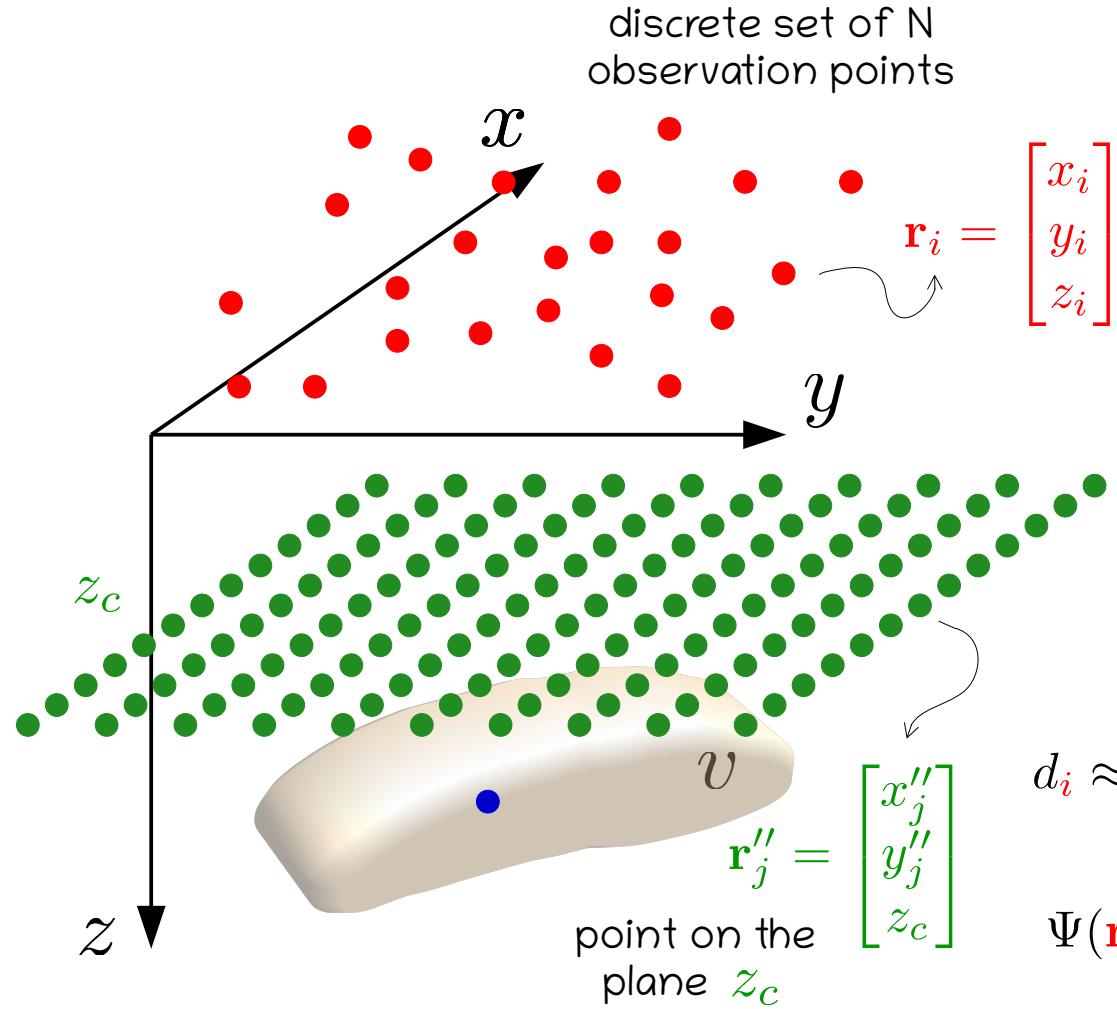
$$d_i \equiv \partial_z \Theta(\mathbf{r}_i)$$

$$d_i \approx \sum_{j=1}^M p_j \partial_z \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

$$\Psi(\mathbf{r}_i, \mathbf{r}_j'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r}_i - \mathbf{r}_j''\|}$$

$p_j \partial_z \Psi(\mathbf{r}_i, \mathbf{r}_j'')$
represents the gravity disturbance produced at the observation point \mathbf{r}_i by a monopole located at \mathbf{r}_j''

Gravitational field



density

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$d_i \equiv \partial_z \Theta(\mathbf{r}_i)$

gravity disturbance

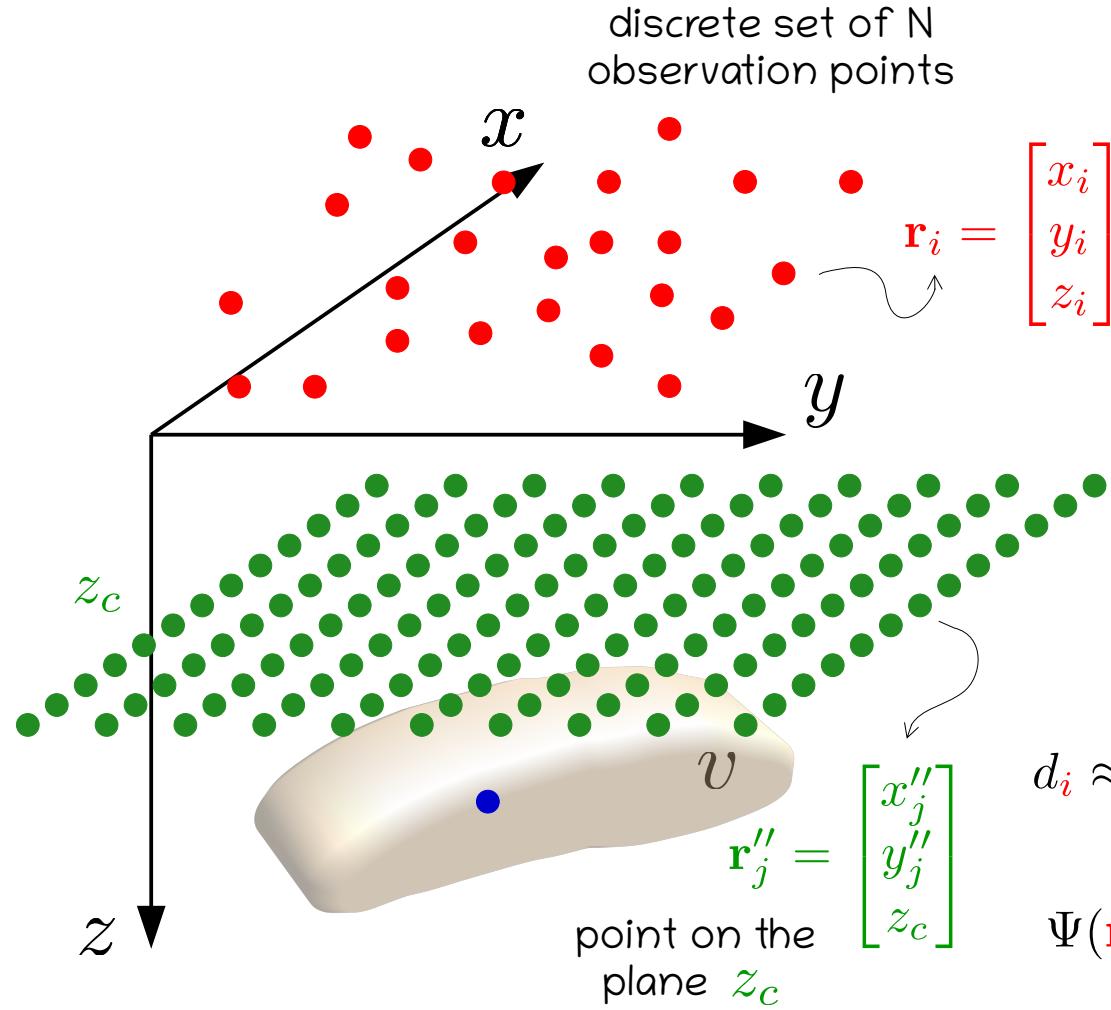
$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

In this case, the EqL Technique consists in solving this linear system for \mathbf{p}

$$d_i \approx \sum_{j=1}^M p_j \partial_z \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

$$\Psi(\mathbf{r}_i, \mathbf{r}_j'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r}_i - \mathbf{r}_j''\|}$$

Gravitational field



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

density

$d_i \equiv \partial_z \Theta(\mathbf{r}_i)$

gravity disturbance

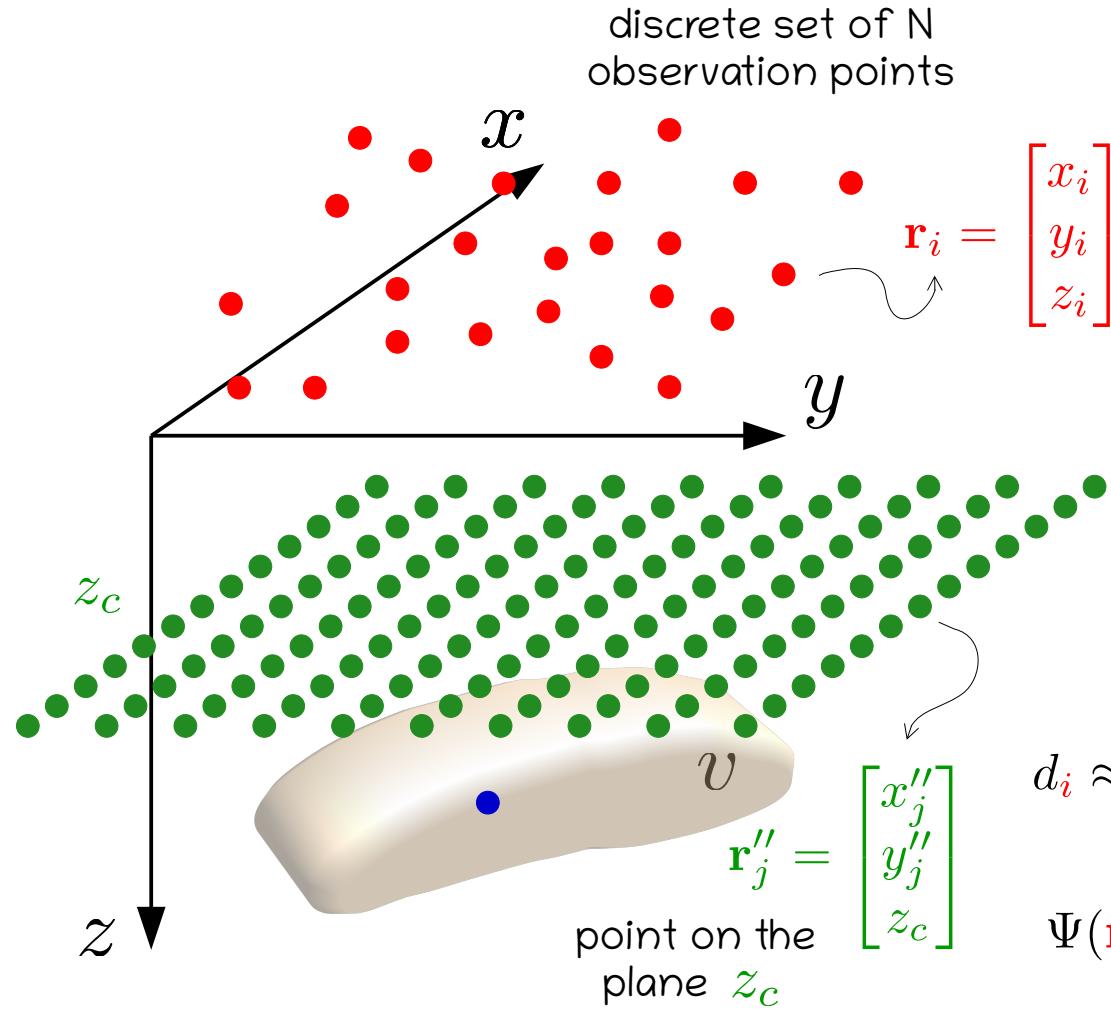
parameter vector
containing the physical-
property distribution
of the monopoles

$$d_i \approx \sum_{j=1}^M p_j \partial_z \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

$$\Psi(\mathbf{r}_i, \mathbf{r}_j'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r}_i - \mathbf{r}_j''\|}$$

$$\mathbf{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_M \end{bmatrix}$$

Gravitational field



density

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$d_i \equiv \partial_z \Theta(\mathbf{r}_i)$

gravity disturbance

data vector containing the observations (in this case, gravity disturbance data)

$$d_i \approx \sum_{j=1}^M p_j \partial_z \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

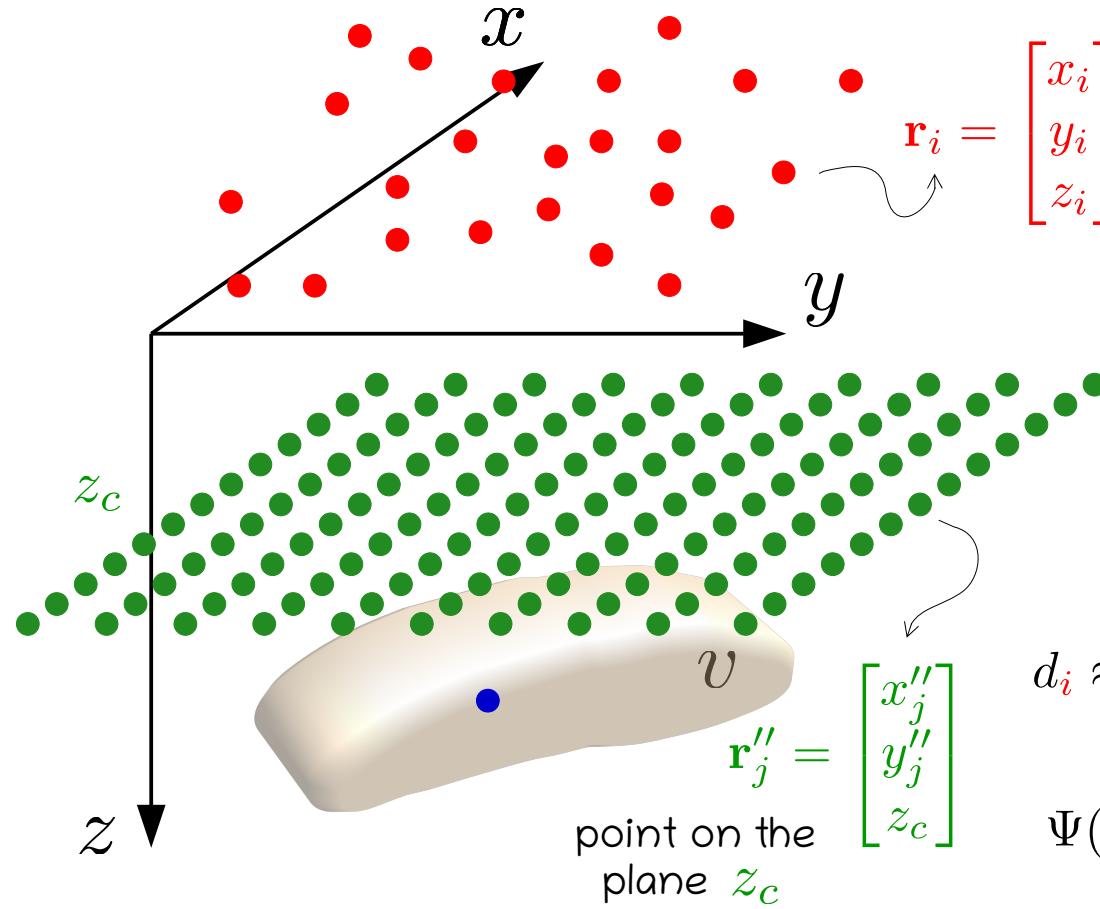
$$\Psi(\mathbf{r}_i, \mathbf{r}_j'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r}_i - \mathbf{r}_j''\|}$$

$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

$$\mathbf{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_N \end{bmatrix}$$

Gravitational field

discrete set of N
observation points



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

density

$$d_i \equiv \partial_z \Theta(\mathbf{r}_i)$$

gravity disturbance

$$d_i \approx \sum_{j=1}^M p_j \partial_z \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

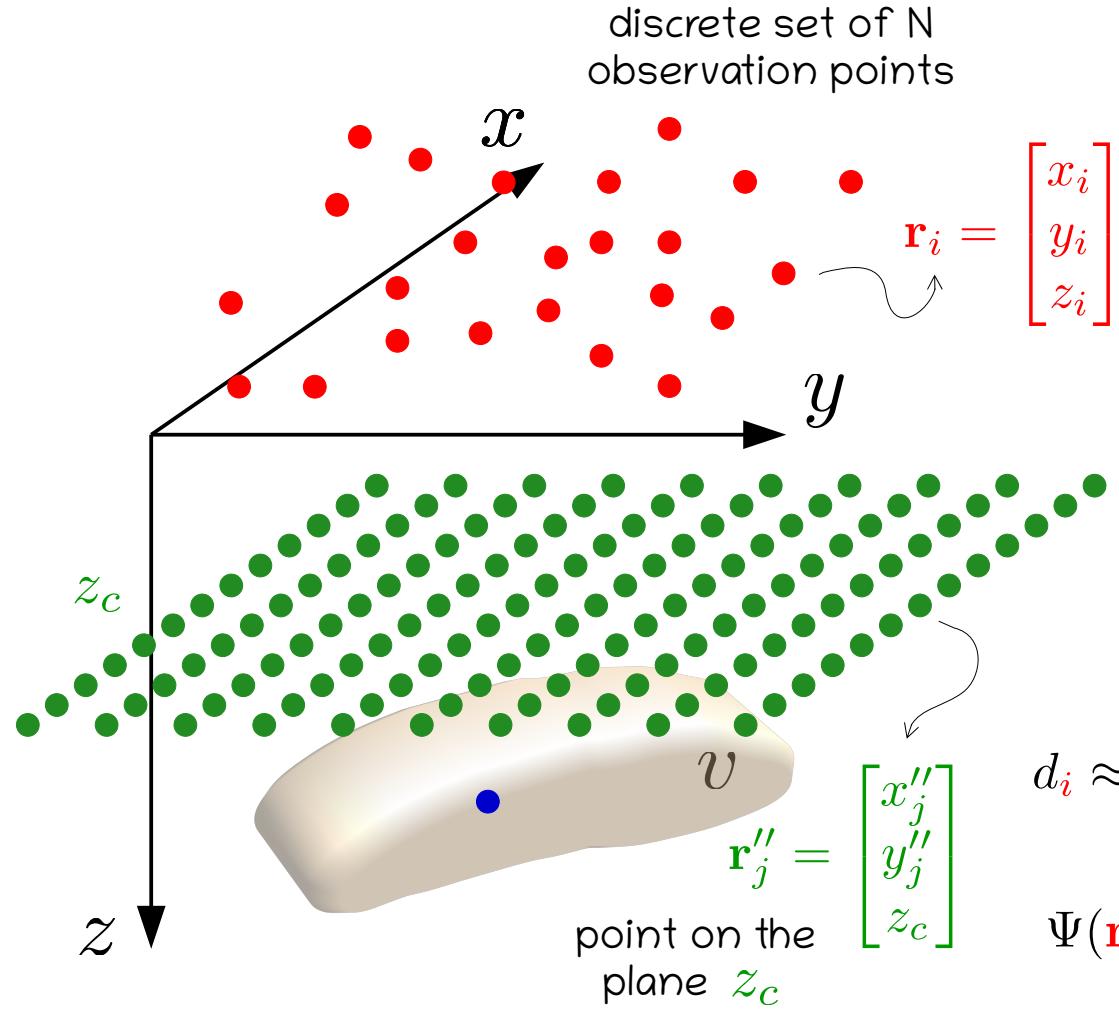
$$\Psi(\mathbf{r}_i, \mathbf{r}_j'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r}_i - \mathbf{r}_j''\|}$$

$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

$$a_{ij} = \partial_z \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

sensitivity matrix

Gravitational field



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

density

$d_i \equiv \partial_z \Theta(\mathbf{r}_i)$

gravity disturbance

The parameter vector \mathbf{p} reproducing the gravity data defines a **discrete equivalent layer**

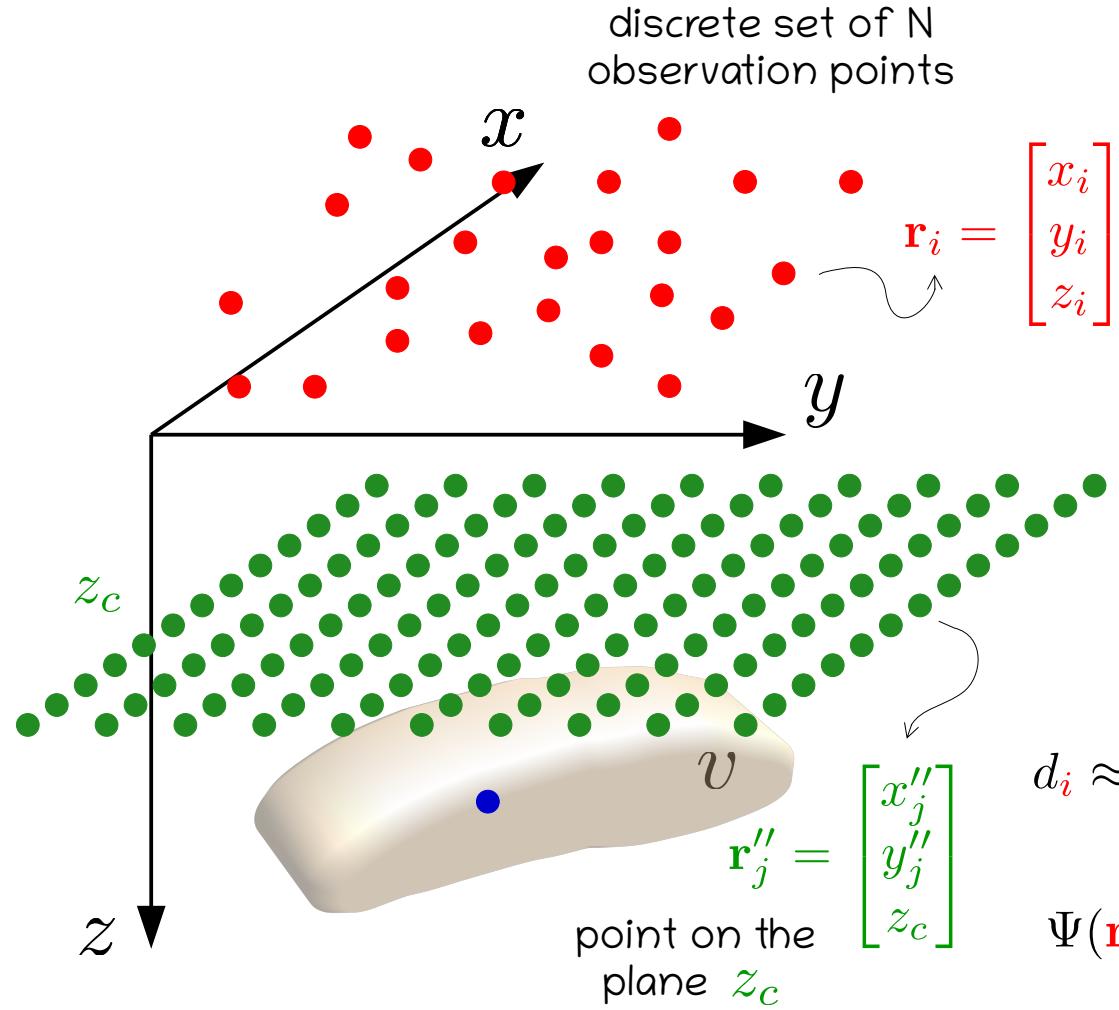
$$d_i \approx \sum_{j=1}^M p_j \partial_z \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

$$\Psi(\mathbf{r}_i, \mathbf{r}_j'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r}_i - \mathbf{r}_j''\|}$$

$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

$a_{ij} = \partial_z \Psi(\mathbf{r}_i, \mathbf{r}_j'')$

Gravitational field



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

density

$$d_i \equiv \partial_z \Theta(\mathbf{r}_i)$$

gravity disturbance

The physical property p_j of a single monopole located at \mathbf{r}_j'' is called **equivalent source**

$$d_i \approx \sum_{j=1}^M p_j \partial_z \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

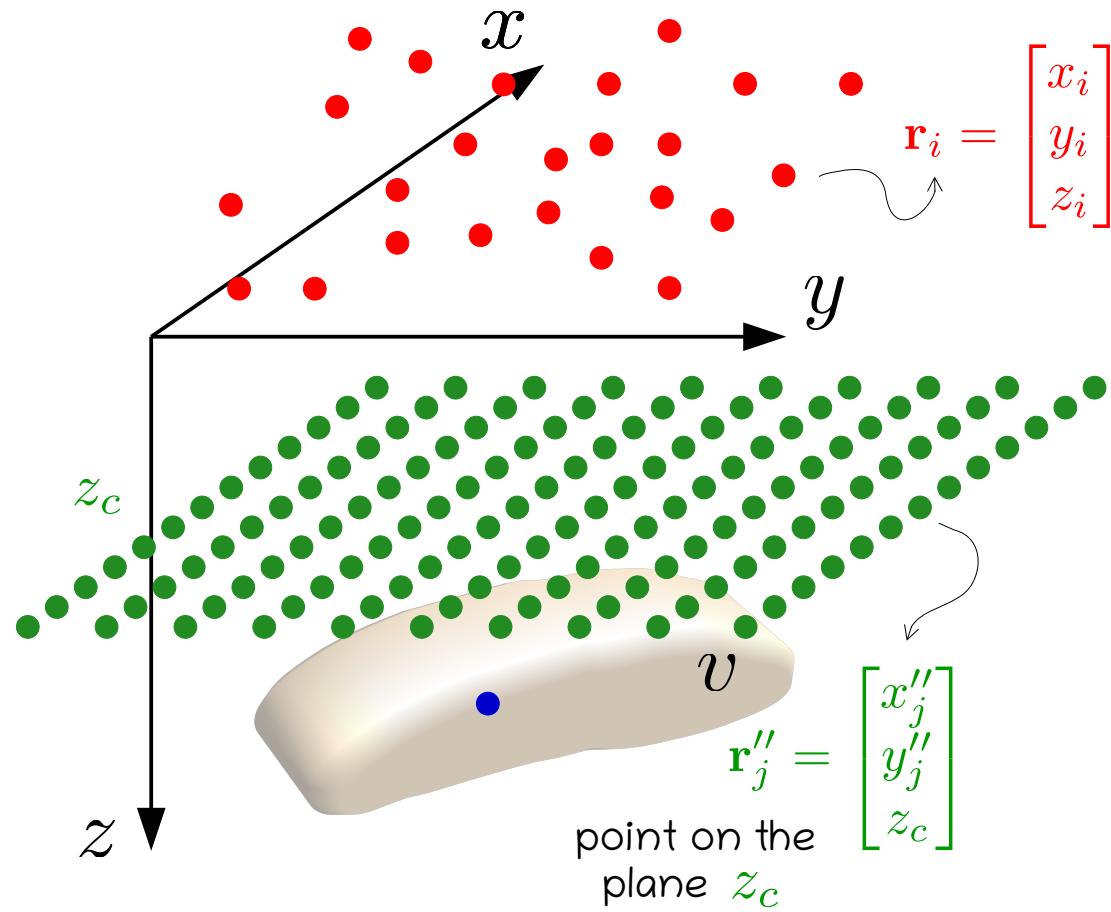
$$\Psi(\mathbf{r}_i, \mathbf{r}_j'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r}_i - \mathbf{r}_j''\|}$$

$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

$a_{ij} = \partial_z \Psi(\mathbf{r}_i, \mathbf{r}_j'')$

Gravitational field

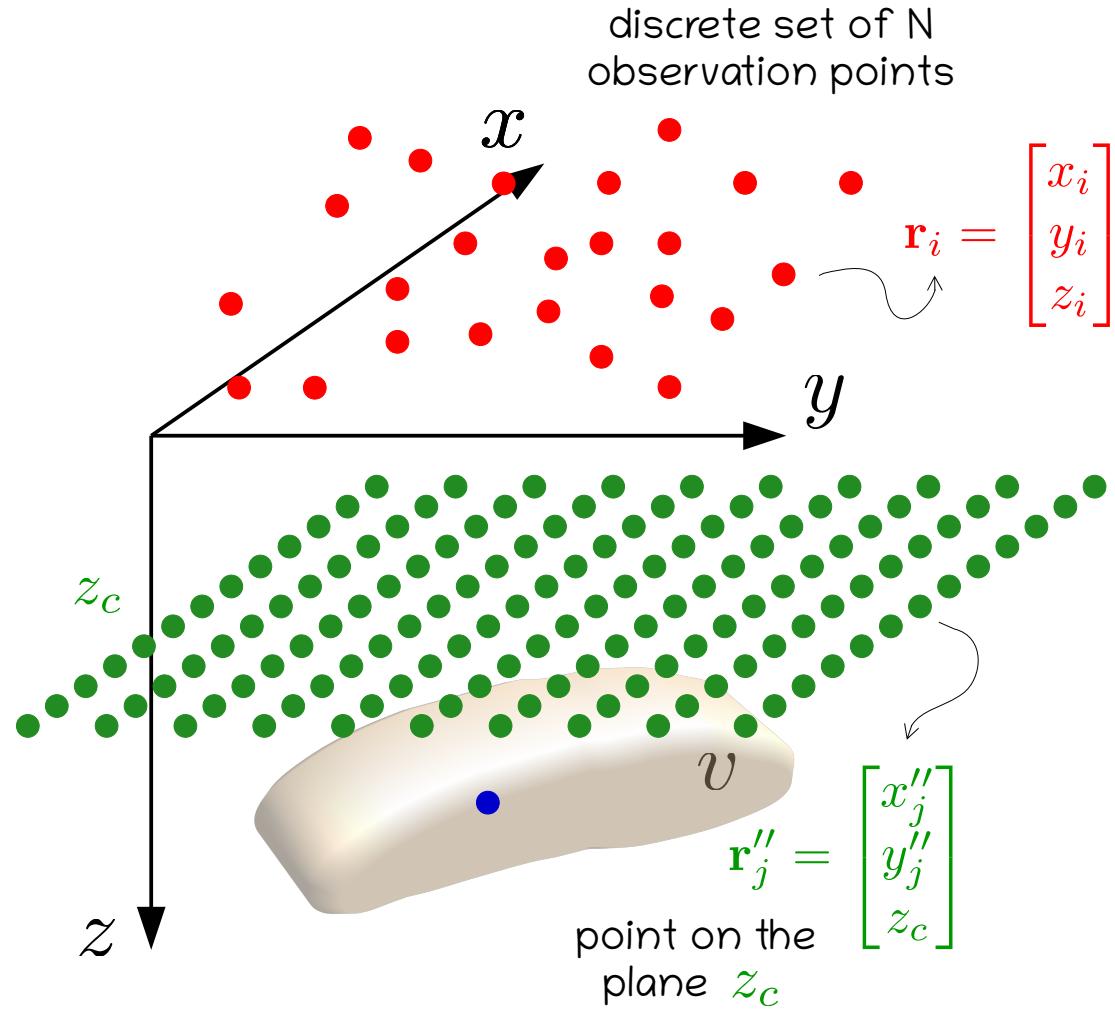
discrete set of N
observation points



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

density

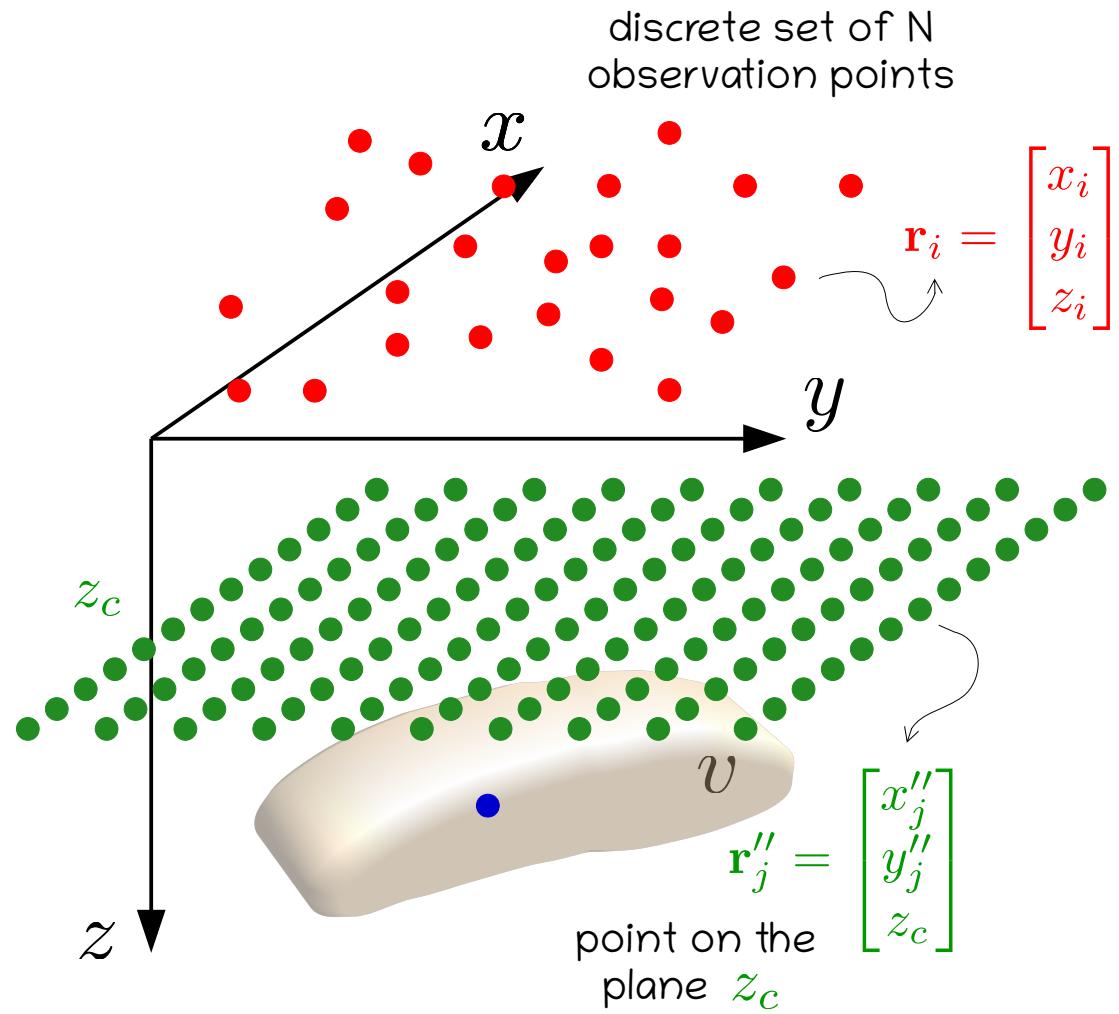
Magnetic field



total-magnetization intensity

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

Magnetic field



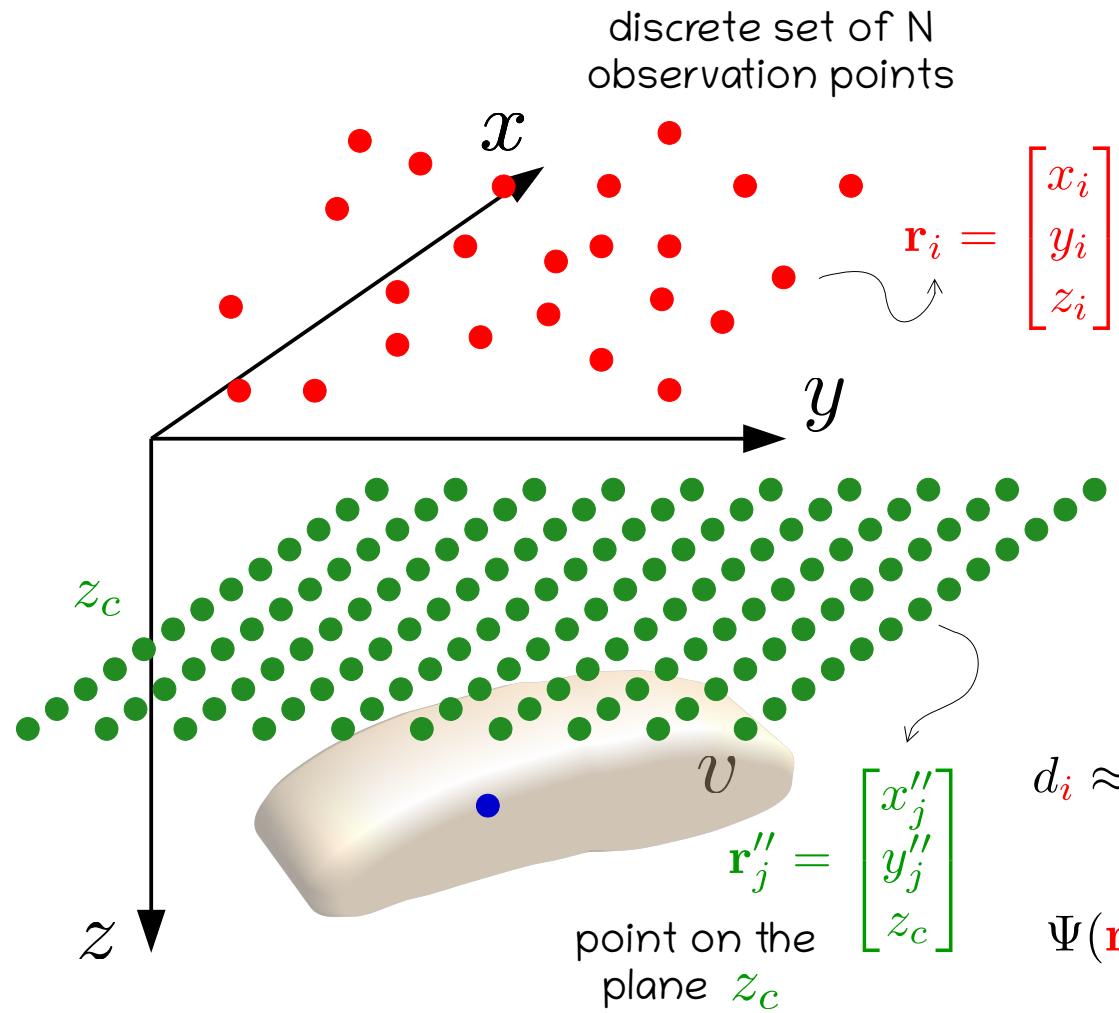
total-magnetization intensity

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$d_i \equiv \partial_{th} \Theta(\mathbf{r}_i)$

approx total-field anomaly

Magnetic field



total-magnetization intensity

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

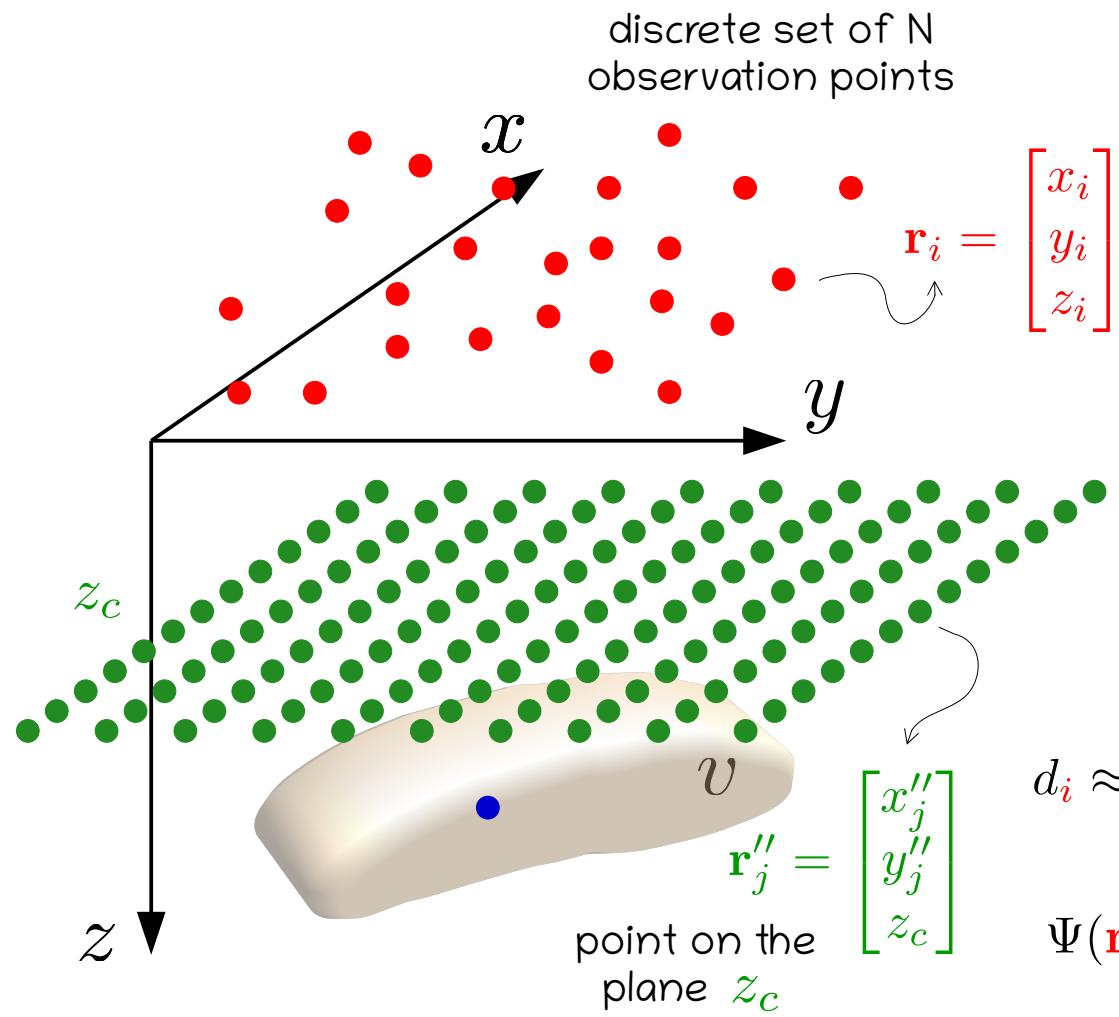
$$d_i \equiv \partial_{th} \Theta(\mathbf{r}_i)$$

approx total-field anomaly

$$d_i \approx \sum_{j=1}^M p_j \partial_{tv} \Psi(\mathbf{r}_i, \mathbf{r}''_j)$$

$$\Psi(\mathbf{r}_i, \mathbf{r}''_j) = \frac{1}{2\pi} \frac{1}{\|\mathbf{r}_i - \mathbf{r}''_j\|}$$

Magnetic field



total-magnetization intensity

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$d_i \equiv \partial_{th} \Theta(\mathbf{r}_i)$$

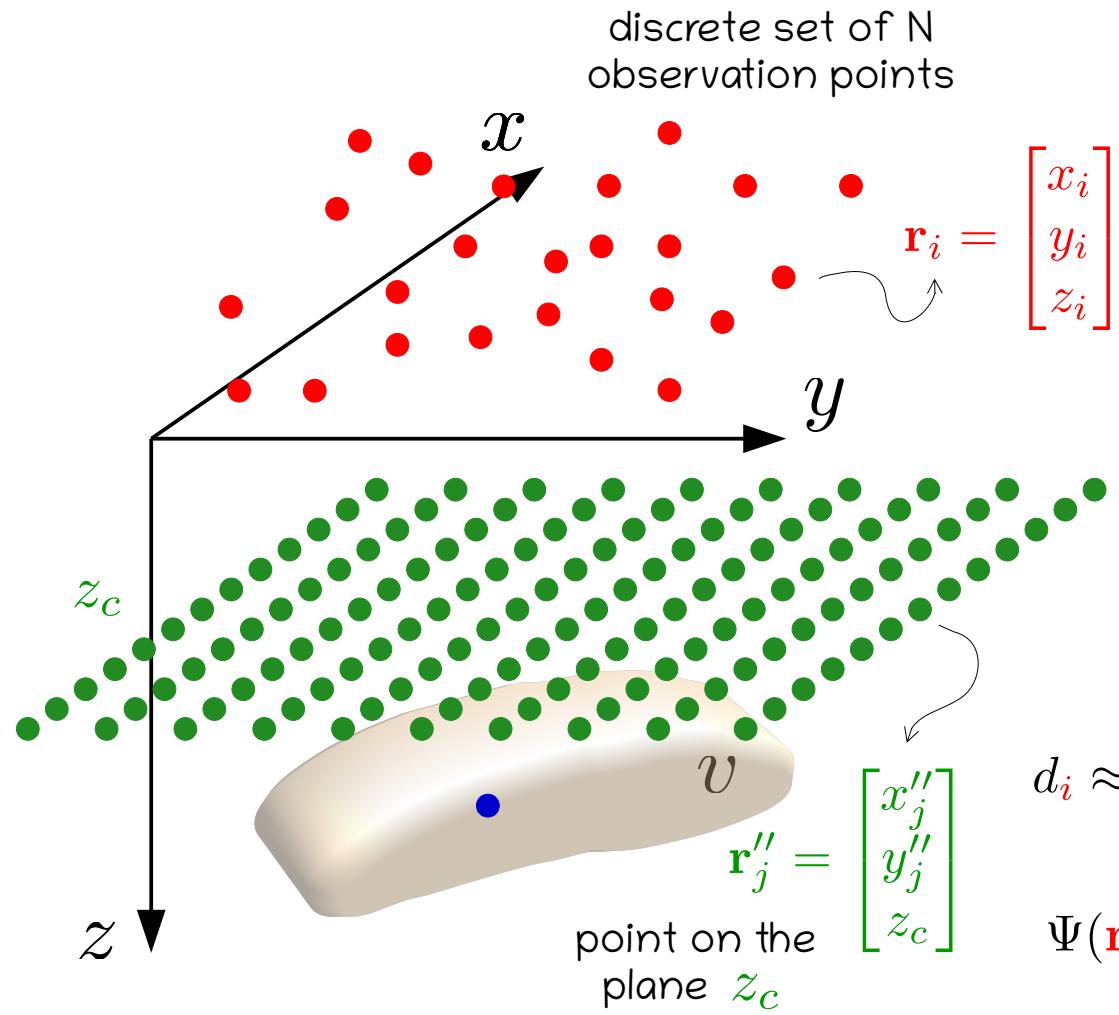
approx total-field anomaly

$$d_i \approx \sum_{j=1}^M p_j \partial_{tv} \Psi(\mathbf{r}_i, \mathbf{r}''_j)$$

$p_j \partial_{tv} \Psi(\mathbf{r}_i, \mathbf{r}''_j)$
represents the approx
total-field anomaly produced
at the observation point
 \mathbf{r}_i by a dipole
located at \mathbf{r}''_j

$$\Psi(\mathbf{r}_i, \mathbf{r}''_j) = \frac{1}{2\pi} \frac{1}{\|\mathbf{r}_i - \mathbf{r}''_j\|}$$

Magnetic field



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

total-magnetization intensity

$$d_i \equiv \partial_{th} \Theta(\mathbf{r}_i)$$

approx total-field anomaly

The physical property p_j of a single dipole located at \mathbf{r}_j'' is an equivalent source

$$d_i \approx \sum_{j=1}^M p_j \partial_{tv} \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

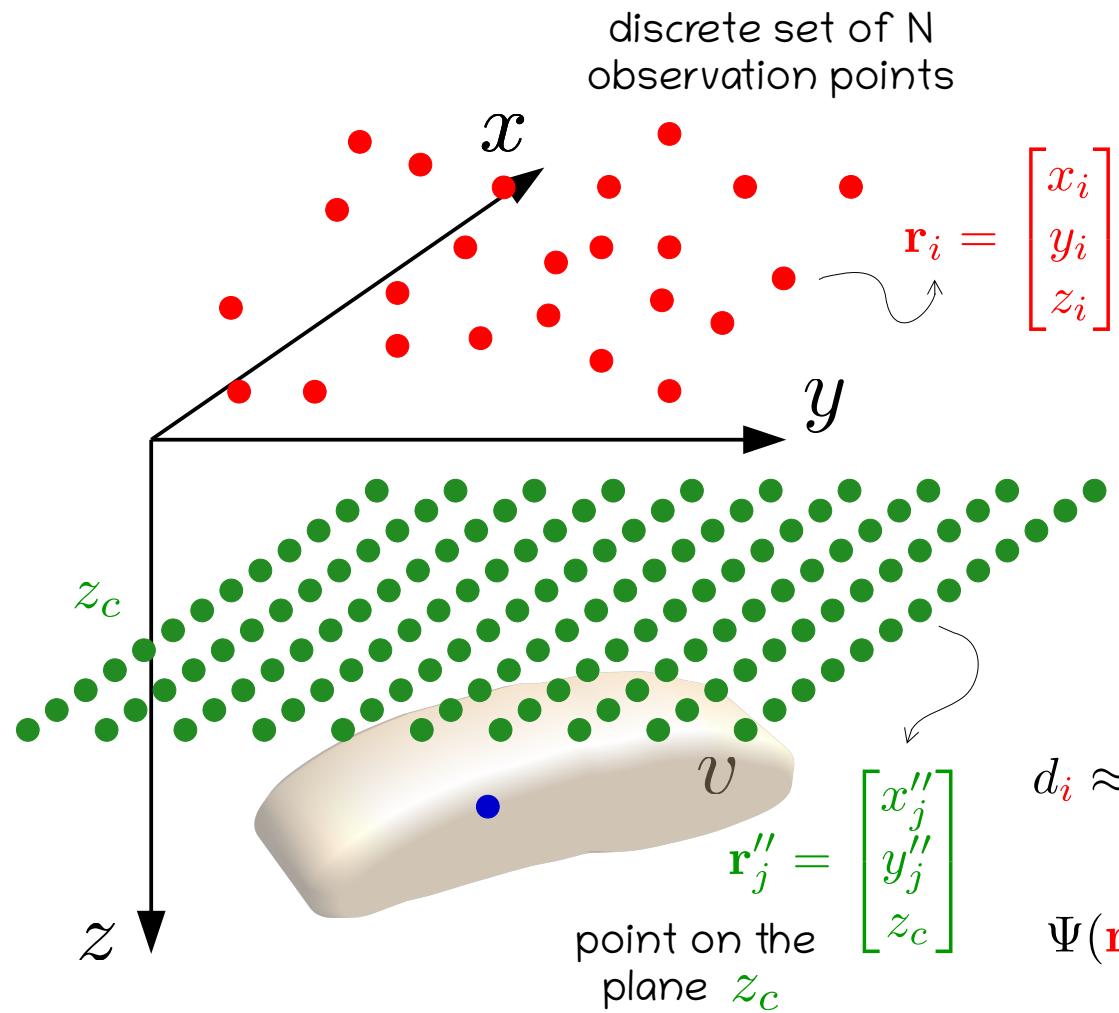
$$p_j \partial_{tv} \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

represents the approx total-field anomaly produced at the observation point

$$\Psi(\mathbf{r}_i, \mathbf{r}_j'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r}_i - \mathbf{r}_j''\|}$$

\mathbf{r}_i by a dipole located at \mathbf{r}_j''

Magnetic field



total-magnetization intensity

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$d_i \equiv \partial_{th} \Theta(\mathbf{r}_i)$$

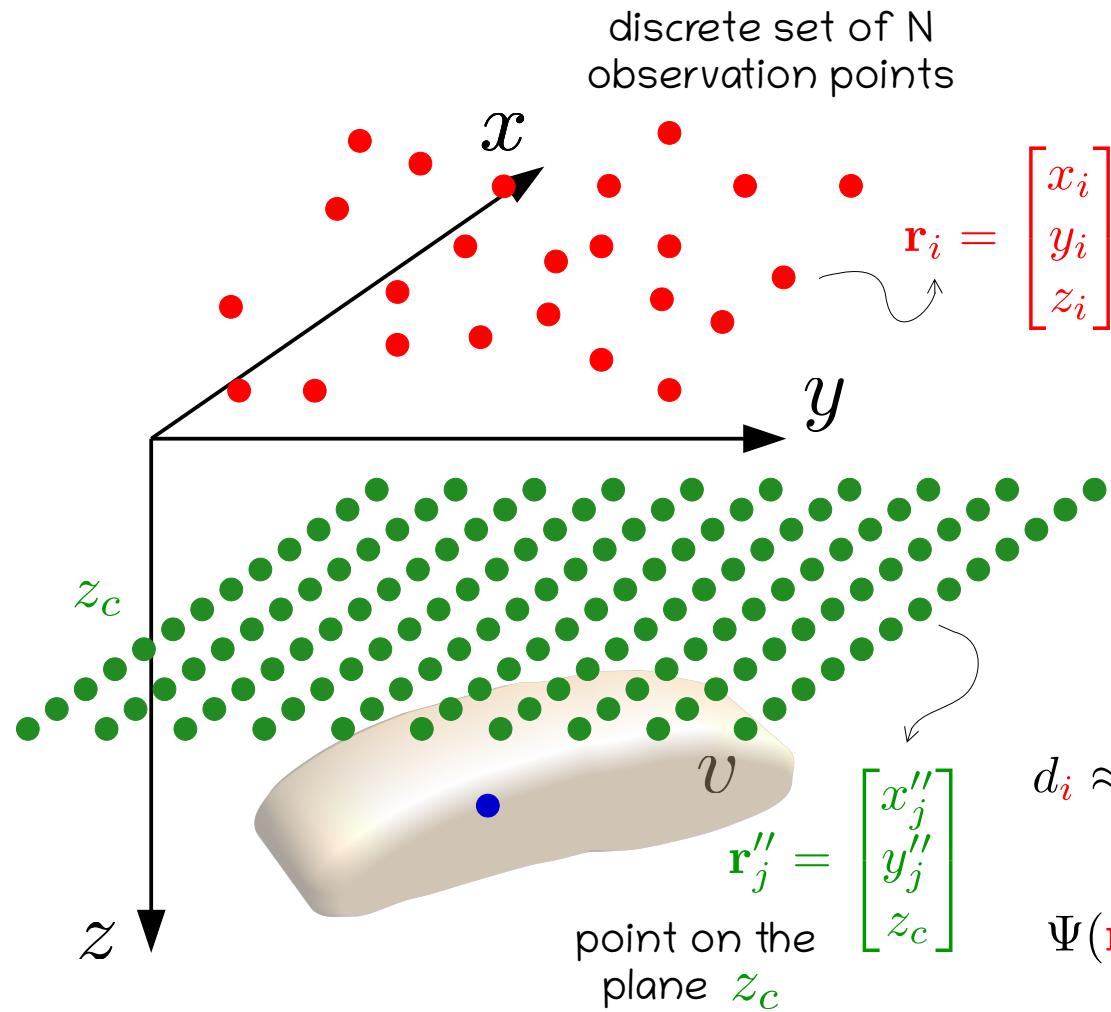
approx total-field anomaly

$$d_i \approx \sum_{j=1}^M p_j \partial_{tv} \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

$$\partial_{tv} \Psi(\mathbf{r}, \mathbf{r}'') = \hat{\mathbf{t}}^\top \mathbf{H}_\Psi(\mathbf{r}, \mathbf{r}'') \hat{\mathbf{v}}$$

$$\Psi(\mathbf{r}_i, \mathbf{r}_j'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r}_i - \mathbf{r}_j''\|}$$

Magnetic field



total-magnetization intensity

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$d_i \equiv \partial_{th} \Theta(\mathbf{r}_i)$$

approx total-field anomaly

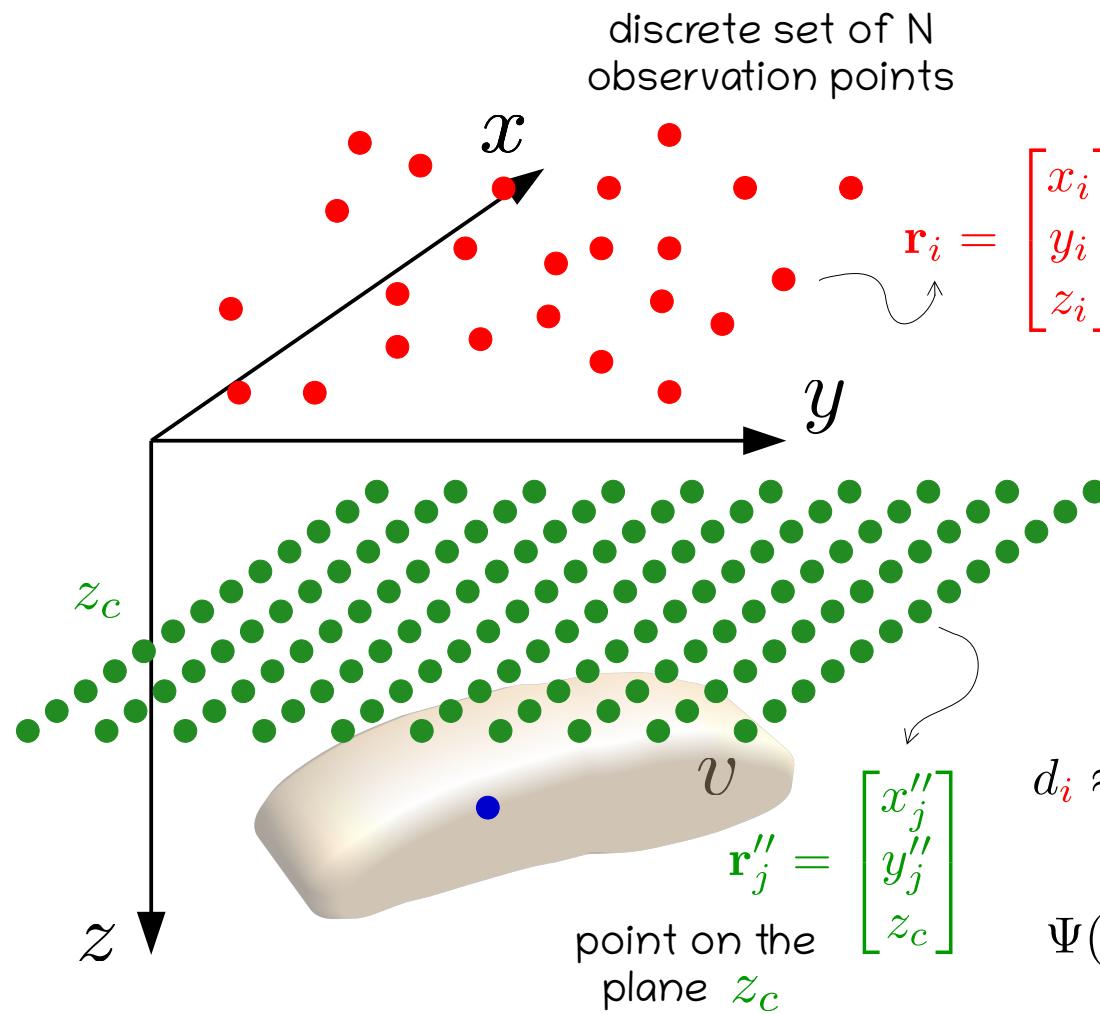
unit vector defining
a constant direction for the
main geomagnetic field

$$d_i \approx \sum_{j=1}^M p_j \partial_{tv} \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

$$\partial_{tv} \Psi(\mathbf{r}, \mathbf{r}'') = \hat{\mathbf{t}}^\top \mathbf{H}_\Psi(\mathbf{r}, \mathbf{r}'') \hat{\mathbf{v}}$$

$$\Psi(\mathbf{r}_i, \mathbf{r}_j'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r}_i - \mathbf{r}_j''\|}$$

Magnetic field



total-magnetization intensity

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$d_i \equiv \partial_{th} \Theta(\mathbf{r}_i)$

approx total-field anomaly

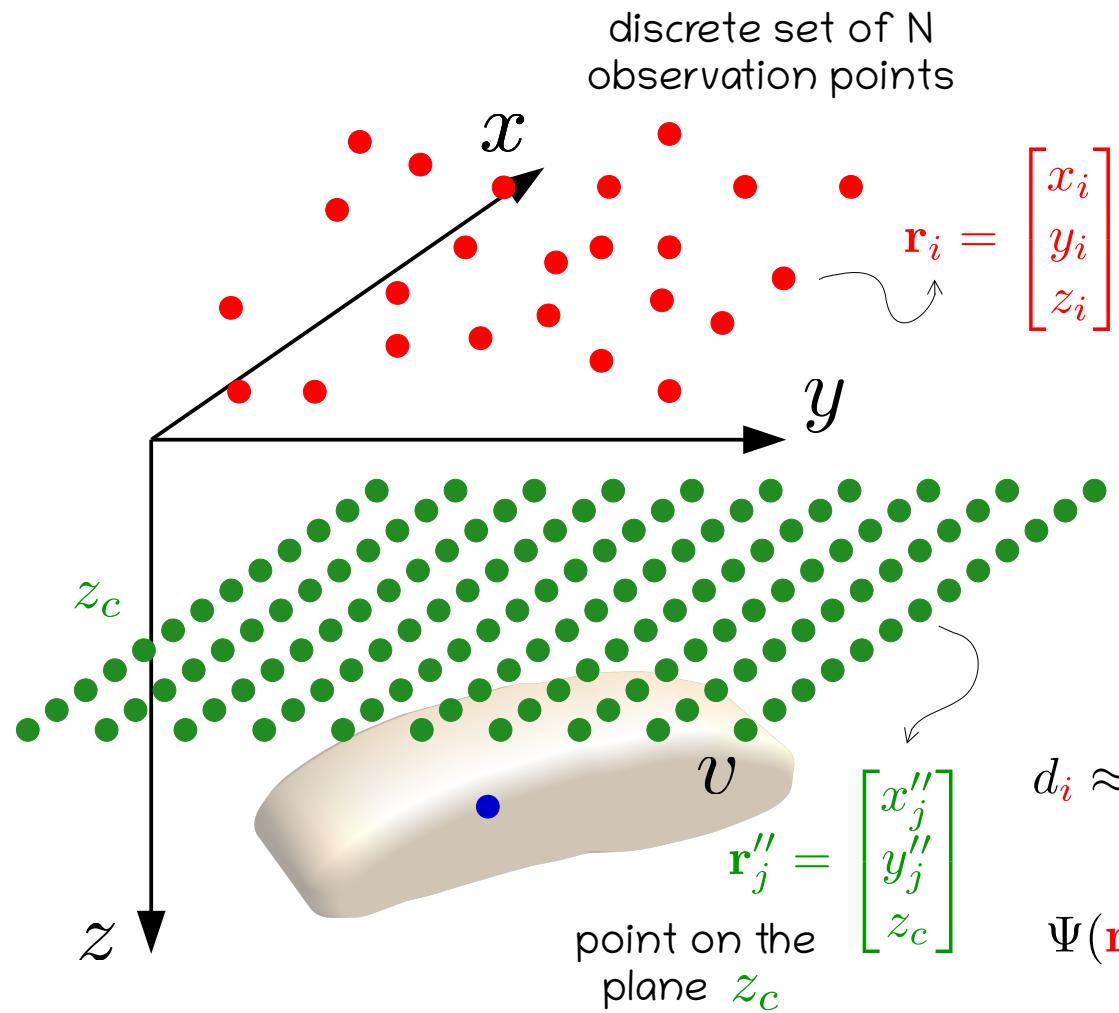
unit vector defining
a uniform magnetization
direction
for the dipoles

$$d_i \approx \sum_{j=1}^M p_j \partial_{tv} \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

$$\Psi(\mathbf{r}_i, \mathbf{r}_j'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r}_i - \mathbf{r}_j''\|}$$

$$\partial_{tv} \Psi(\mathbf{r}, \mathbf{r}'') = \hat{\mathbf{t}}^\top \mathbf{H}_\Psi(\mathbf{r}, \mathbf{r}'') \hat{\mathbf{v}}$$

Magnetic field



total-magnetization intensity

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$d_i \equiv \partial_{th} \Theta(\mathbf{r}_i)$$

approx total-field anomaly

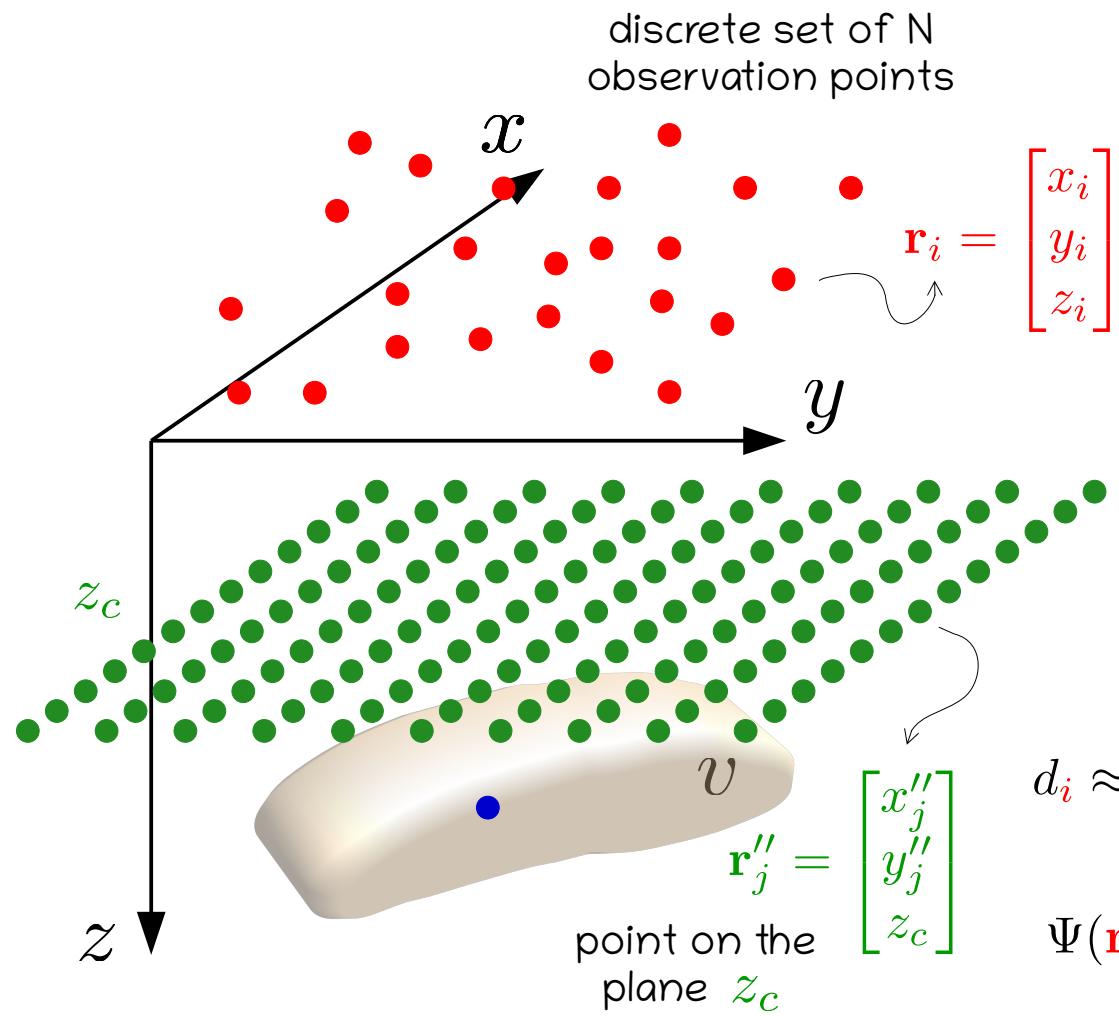
this direction may
be arbitrary

$$d_i \approx \sum_{j=1}^M p_j \partial_{tv} \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

$$\partial_{tv} \Psi(\mathbf{r}, \mathbf{r}'') = \hat{\mathbf{t}}^\top \mathbf{H}_\Psi(\mathbf{r}, \mathbf{r}'') \hat{\mathbf{v}}$$

$$\Psi(\mathbf{r}_i, \mathbf{r}_j'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r}_i - \mathbf{r}_j''\|}$$

Magnetic field



total-magnetization intensity

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$d_i \equiv \partial_{th} \Theta(\mathbf{r}_i)$$

approx total-field anomaly

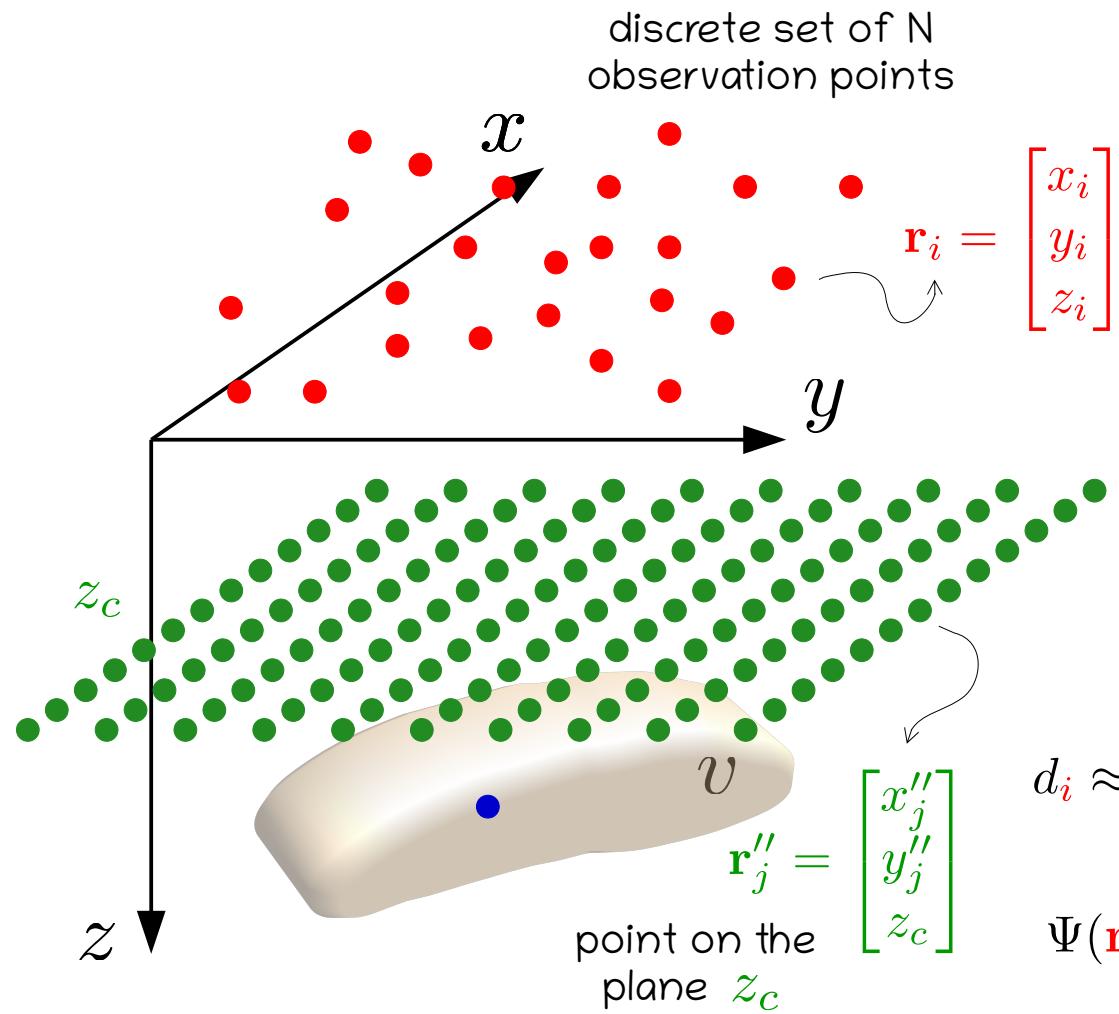
$$\mathbf{H}_\Psi(\mathbf{r}, \mathbf{r}'') = \begin{bmatrix} \partial_{xx} \Psi(\mathbf{r}, \mathbf{r}'') & \partial_{xy} \Psi(\mathbf{r}, \mathbf{r}'') & \partial_{xz} \Psi(\mathbf{r}, \mathbf{r}'') \\ \partial_{xy} \Psi(\mathbf{r}, \mathbf{r}'') & \partial_{yy} \Psi(\mathbf{r}, \mathbf{r}'') & \partial_{yz} \Psi(\mathbf{r}, \mathbf{r}'') \\ \partial_{xz} \Psi(\mathbf{r}, \mathbf{r}'') & \partial_{yz} \Psi(\mathbf{r}, \mathbf{r}'') & \partial_{zz} \Psi(\mathbf{r}, \mathbf{r}'') \end{bmatrix}$$

$$d_i \approx \sum_{j=1}^M p_j \partial_{tv} \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

$$\partial_{tv} \Psi(\mathbf{r}, \mathbf{r}'') = \hat{\mathbf{t}}^\top \mathbf{H}_\Psi(\mathbf{r}, \mathbf{r}'') \hat{\mathbf{v}}$$

$$\Psi(\mathbf{r}_i, \mathbf{r}_j'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r}_i - \mathbf{r}_j''\|}$$

Magnetic field



total-magnetization intensity

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$d_i \equiv \partial_{th} \Theta(\mathbf{r}_i)$$

approx total-field anomaly

$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

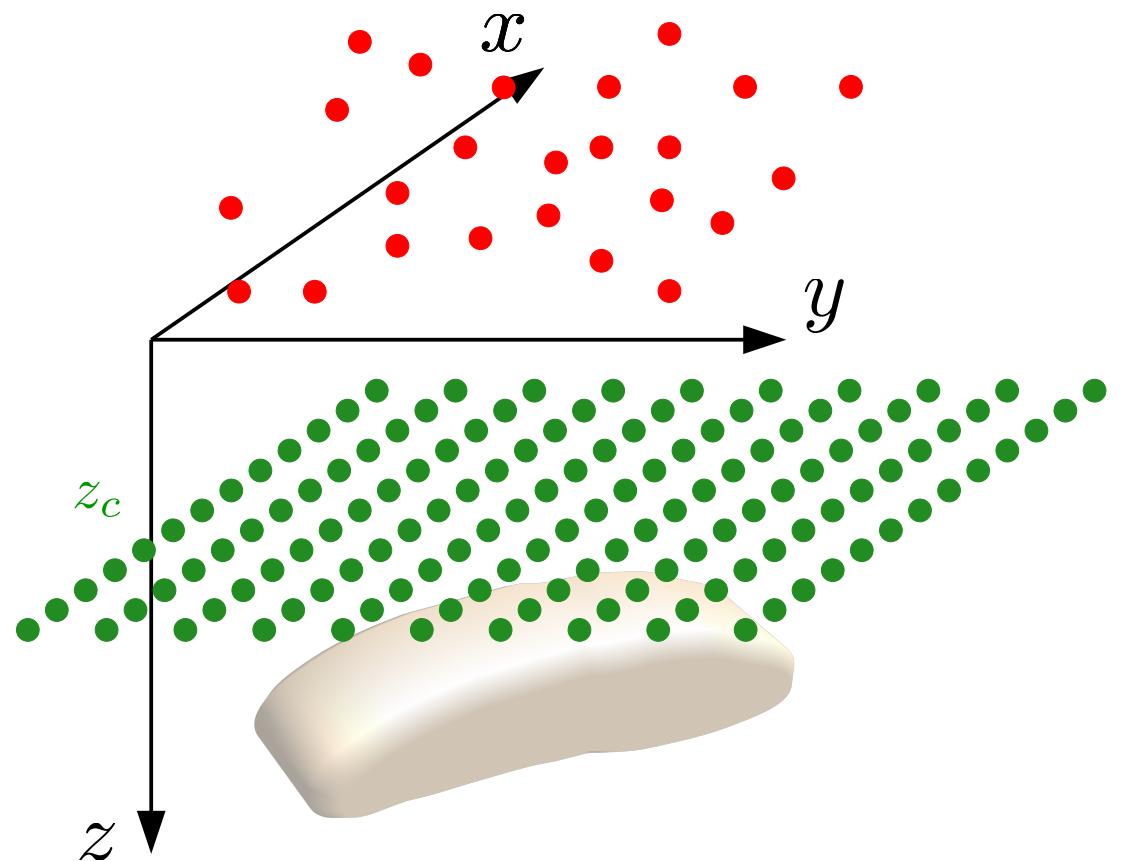
Again, the EqL Technique consists in solving this linear system for \mathbf{p}

$$d_i \approx \sum_{j=1}^M p_j \partial_{tv} \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

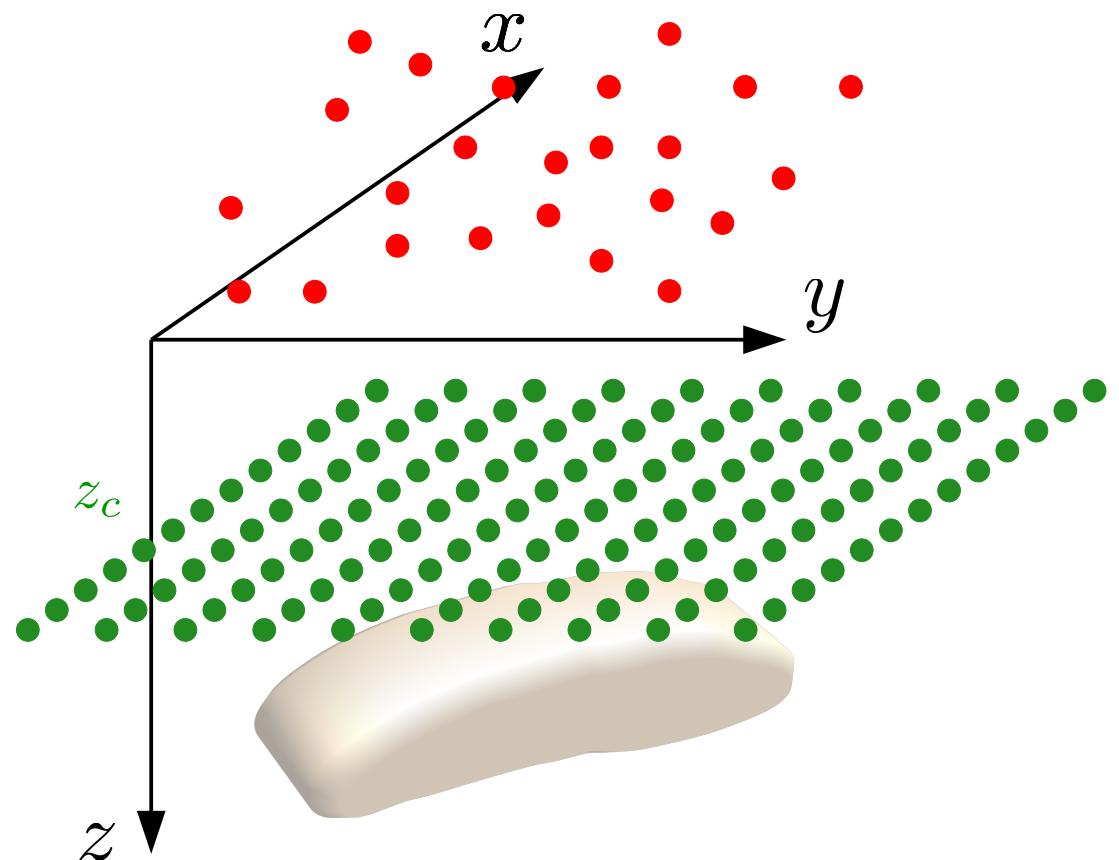
$$\Psi(\mathbf{r}_i, \mathbf{r}_j'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r}_i - \mathbf{r}_j''\|}$$

Classical EqL Technique

$$d \approx A p$$



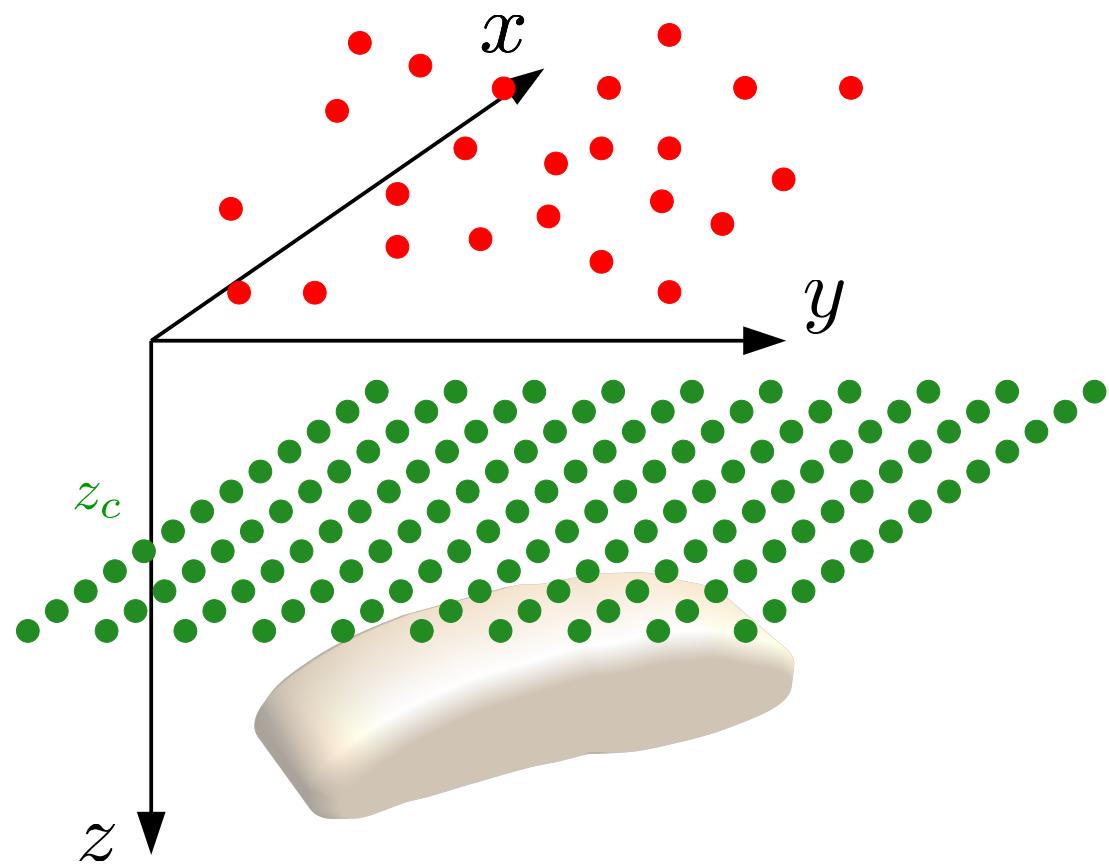
Classical EqL Technique



$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

solving a linear
system for \mathbf{p}

Classical EqL Technique



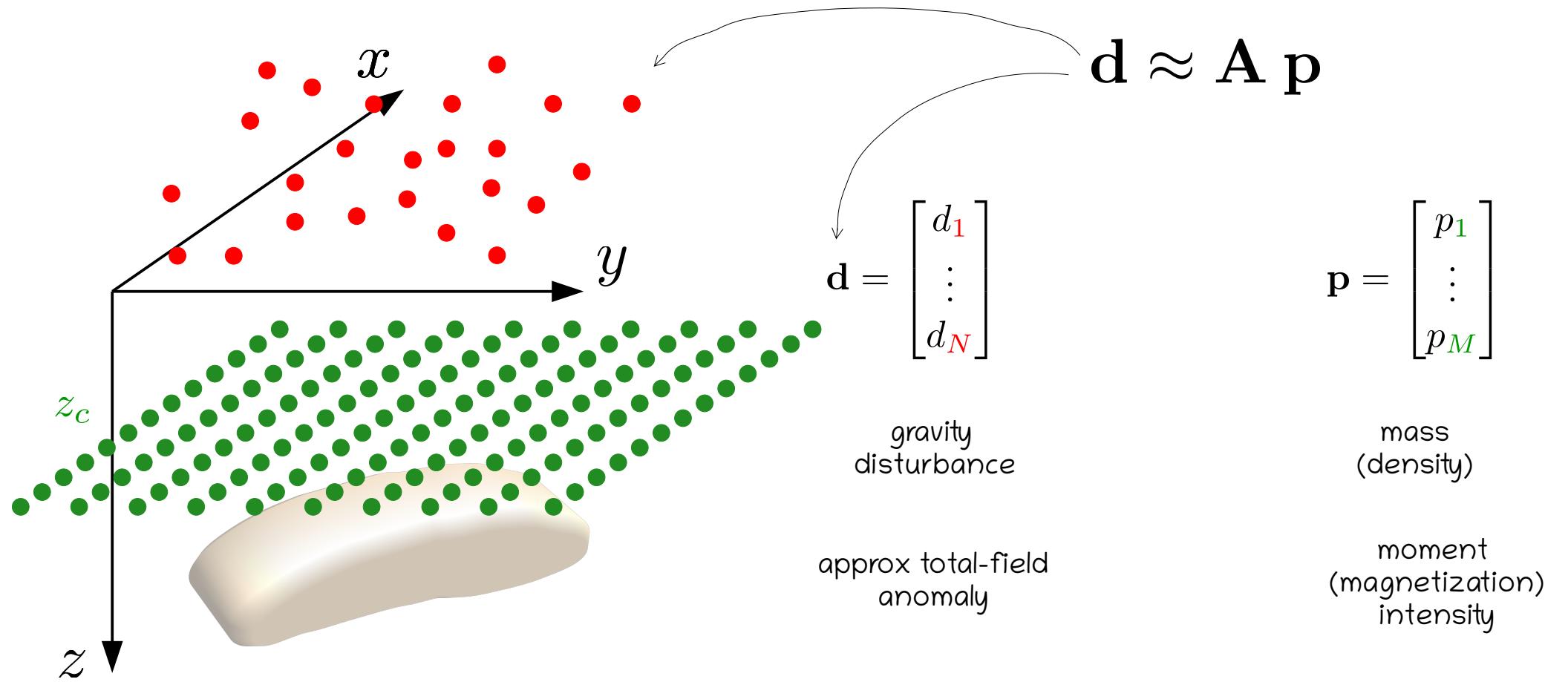
$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

$\mathbf{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_M \end{bmatrix}$

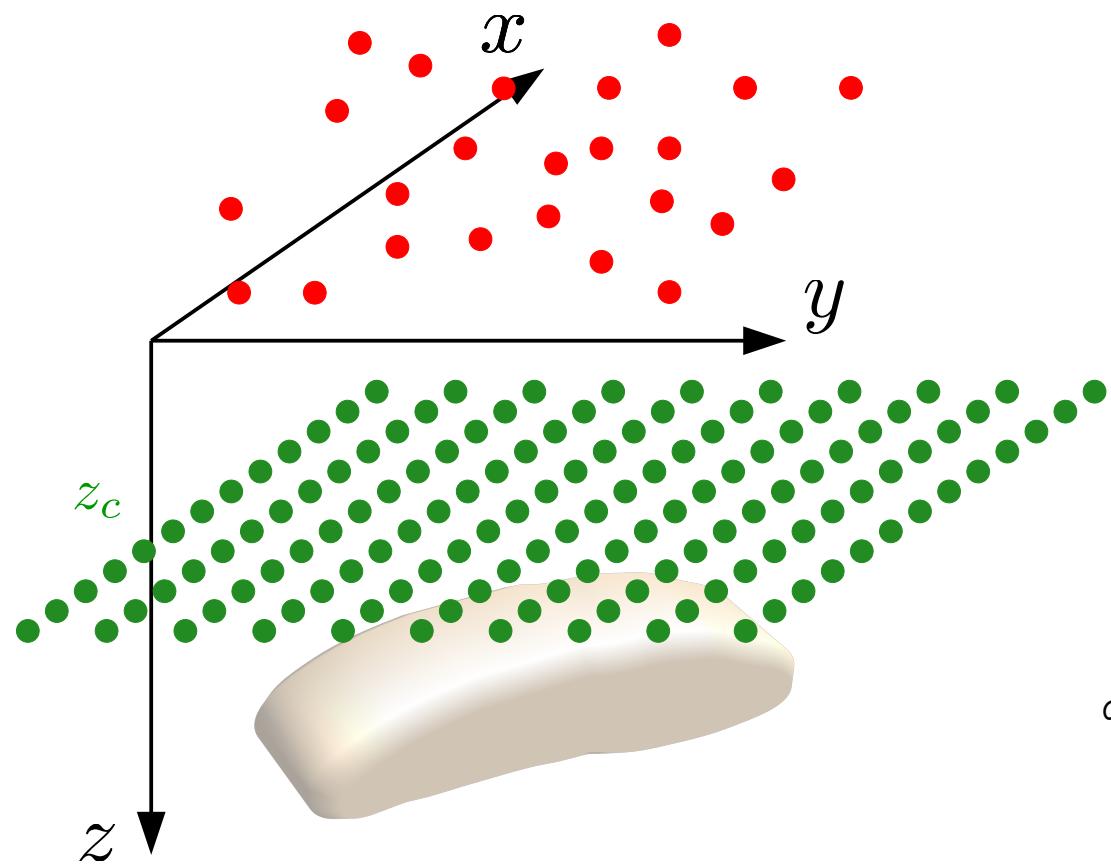
mass
(density)

moment
(magnetization)
intensity

Classical EqL Technique



Classical EqL Technique



$$\mathbf{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_N \end{bmatrix}$$

gravity
disturbance

approx total-field
anomaly

$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

$$a_{ij}$$

monopole

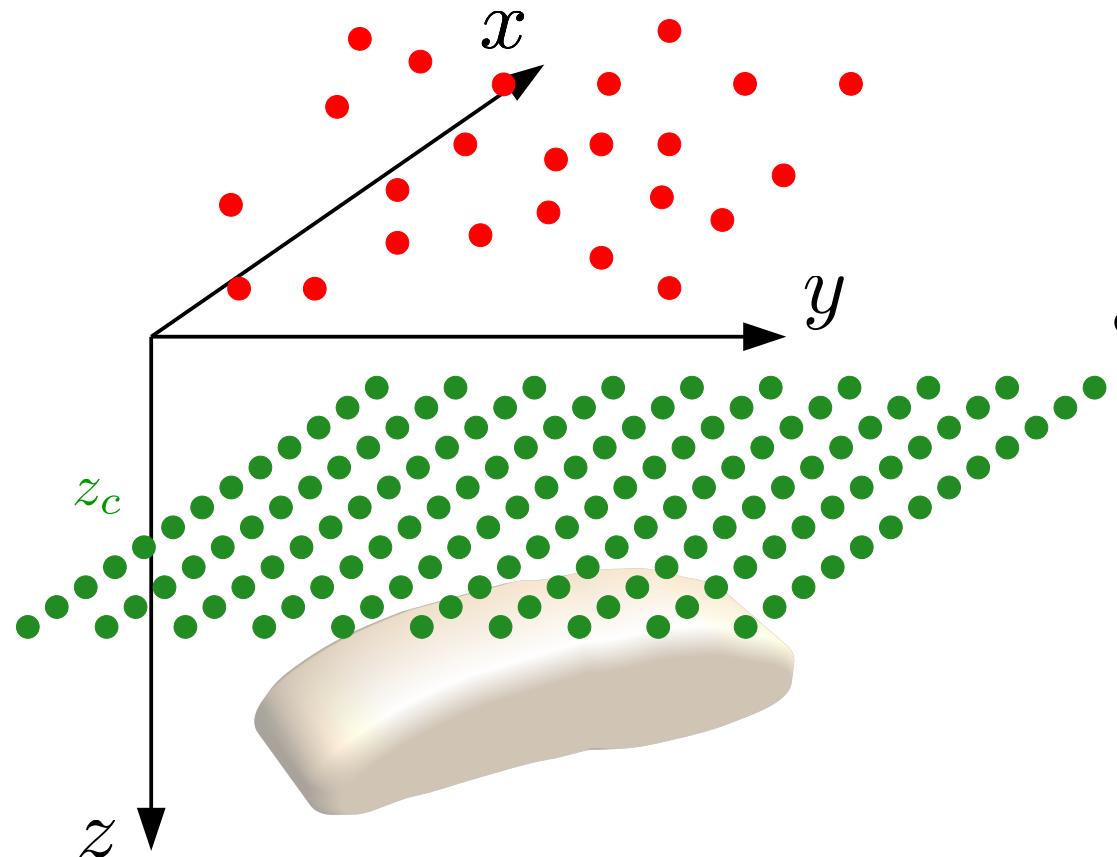
$$\mathbf{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_M \end{bmatrix}$$

mass
(density)

moment
(magnetization)
intensity

dipole

Classical EqL Technique



$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

$$\mathbf{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_N \end{bmatrix}$$

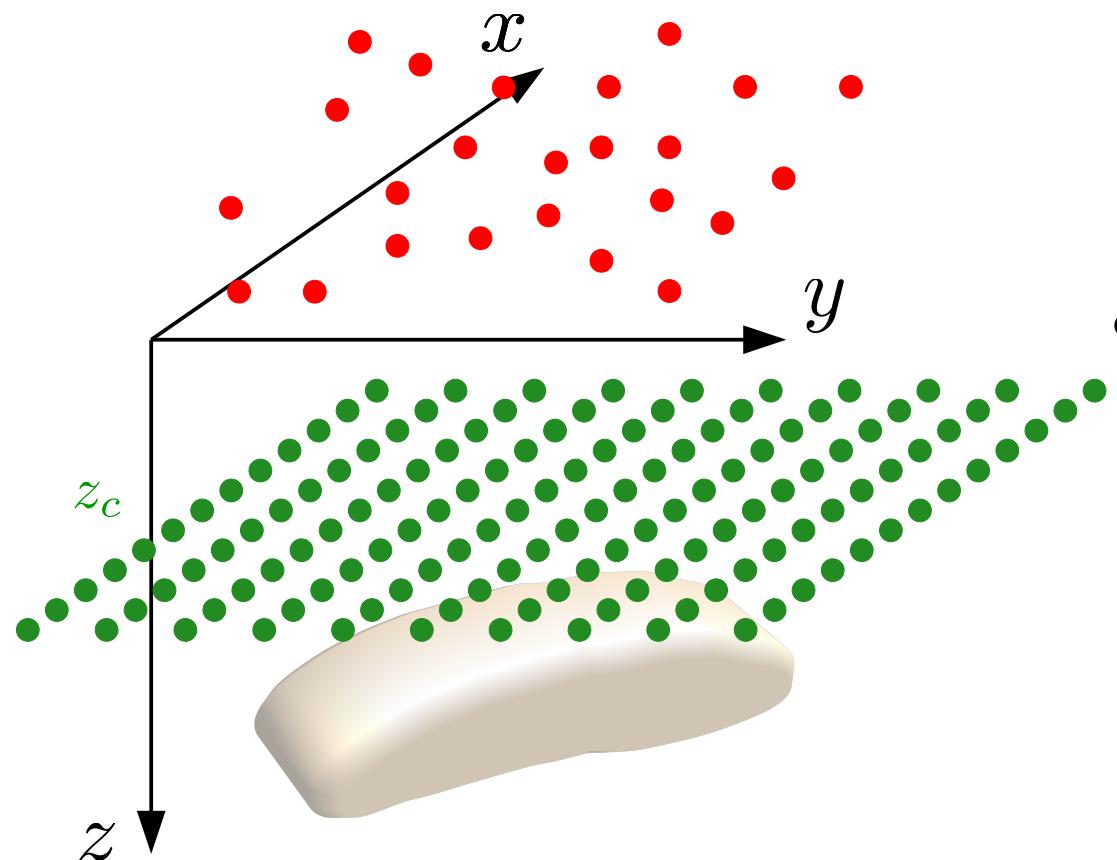
$$a_{ij}$$

$$\mathbf{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_M \end{bmatrix}$$

$$\mathbf{w} = \mathbf{T} \tilde{\mathbf{p}}$$

potential-field transformation

Classical EqL Technique



$$\mathbf{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_N \end{bmatrix}$$

$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

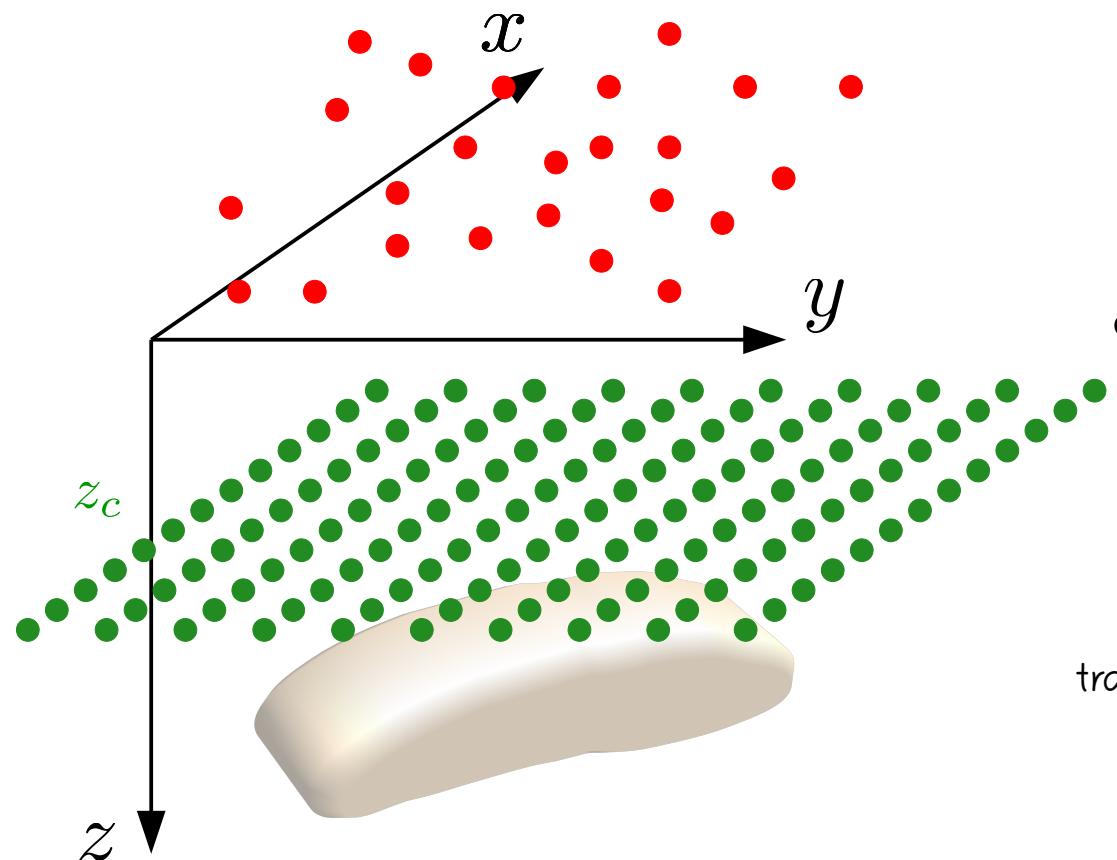
a_{ij}

$$\mathbf{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_M \end{bmatrix}$$

$$\mathbf{w} = \mathbf{T} \tilde{\mathbf{p}}$$

estimated
parameter vector
(equivalent layer)

Classical EqL Technique



$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

$$\mathbf{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_N \end{bmatrix}$$

a_{ij}

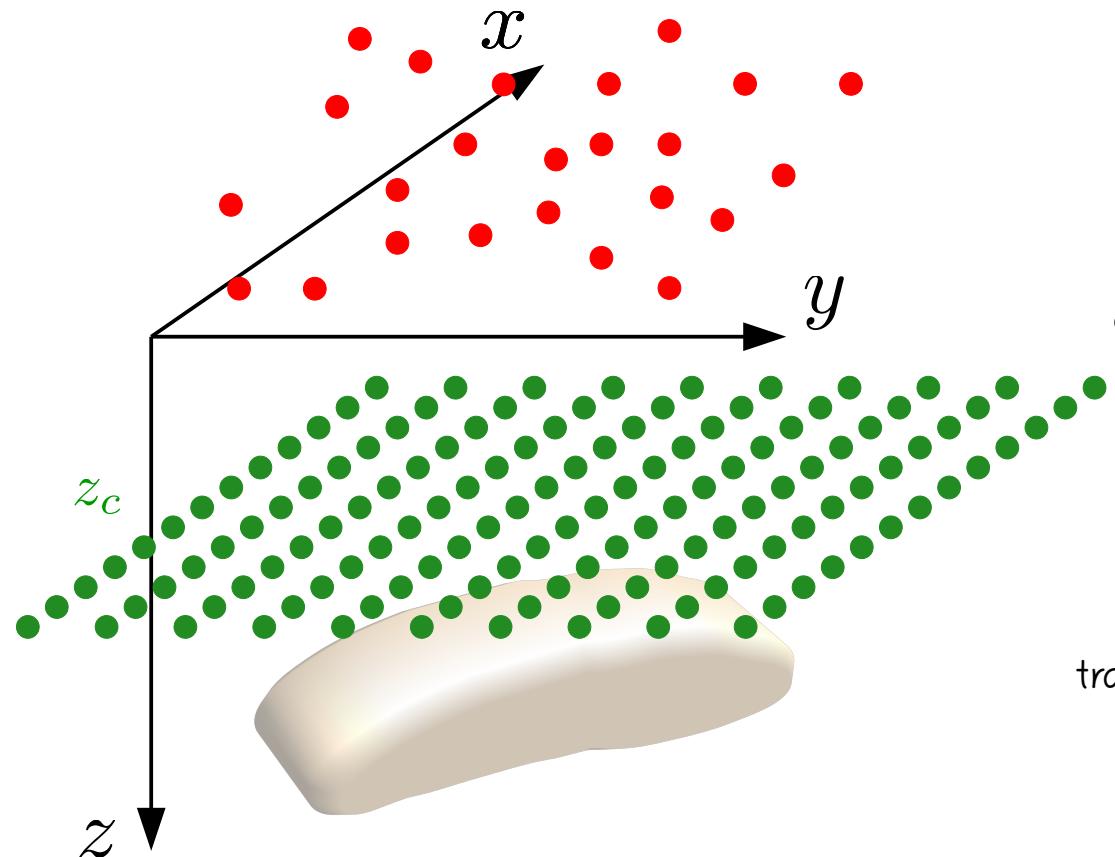
$$\mathbf{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_M \end{bmatrix}$$

$$\mathbf{w} = \mathbf{T} \tilde{\mathbf{p}}$$

transformed
data

estimated
parameter vector
(equivalent layer)

Classical EqL Technique



$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

$$\mathbf{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_N \end{bmatrix}$$

a_{ij}

$$\mathbf{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_M \end{bmatrix}$$

$$\mathbf{w} = \mathbf{T} \tilde{\mathbf{p}}$$

transformed
data

t_{kj}

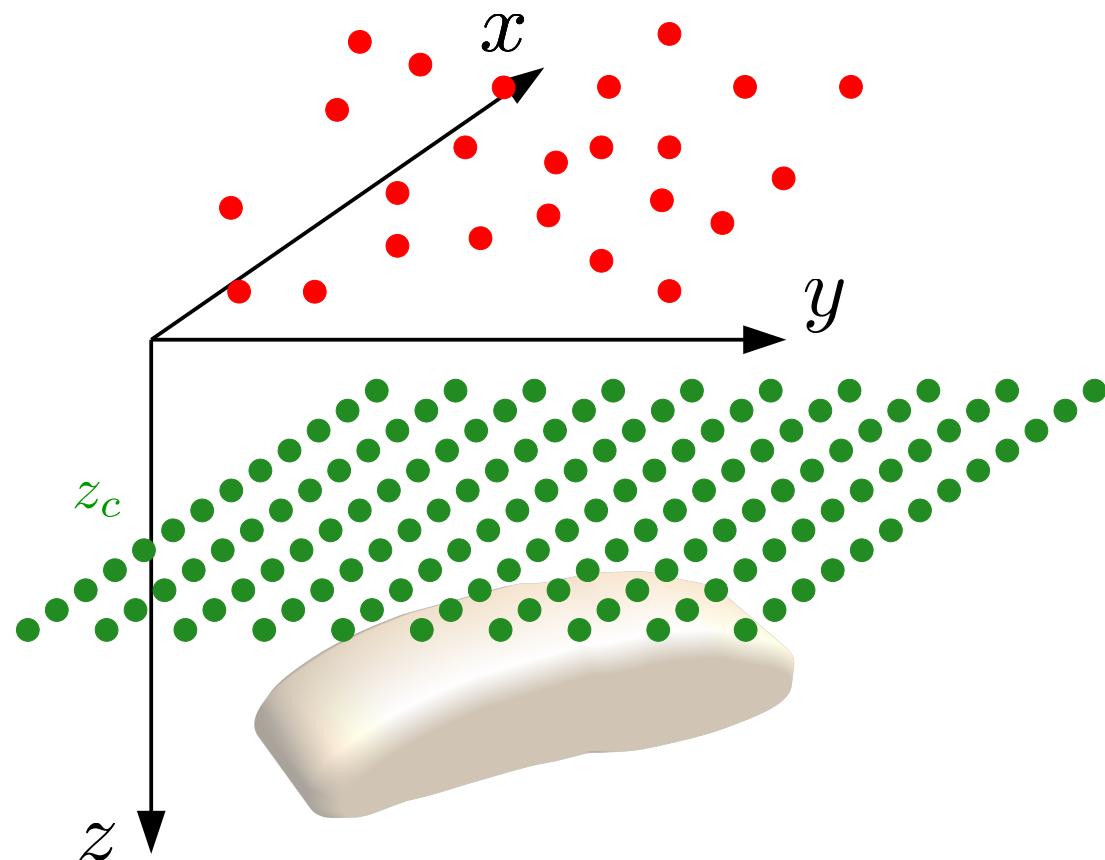
interpolation,
upward continuation, RTP,
field component conversion, ...

estimated
parameter vector
(equivalent layer)

Classical EqL Technique

$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

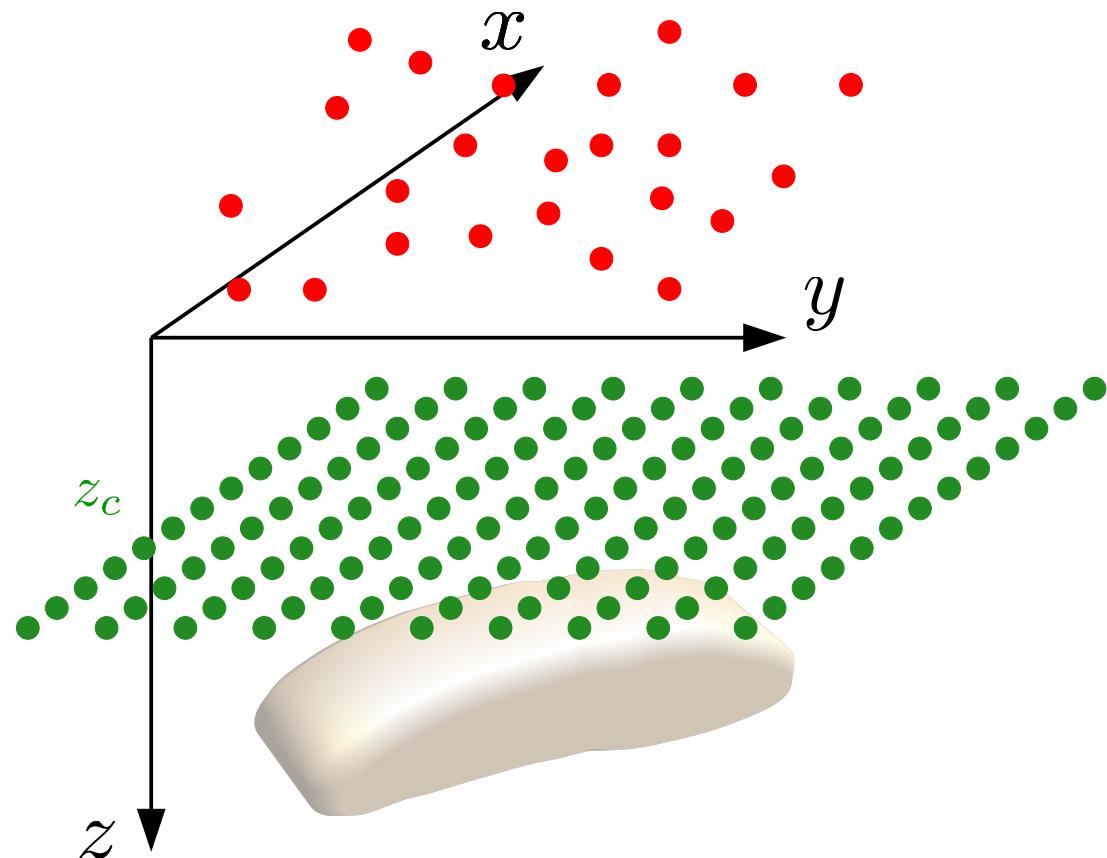
A considerable amount of literature has been published on numerical methods for efficiently solving this linear system (e.g., [Leão and Silva, 1989](#); [Cordell, 1992](#); [Mendonça and Silva, 1994](#); [Guspí and Novara, 2009](#); [Li and Oldenburg, 2010](#); [Barnes and Lumley, 2011](#); [Oliveira Jr. et al., 2013](#); [Siqueira et al., 2017](#); [Mendonça, 2020](#); [Takahashi et al., 2020](#); [Soler and Uieda, 2021](#))



Classical EqL Technique

$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

A considerable amount of literature has been published on numerical methods for efficiently solving this linear system (e.g., [Leão and Silva, 1989](#); [Cordell, 1992](#); [Mendonça and Silva, 1994](#); [Guspí and Novara, 2009](#); [Li and Oldenburg, 2010](#); [Barnes and Lumley, 2011](#); [Oliveira Jr. et al., 2013](#); [Siqueira et al., 2017](#); [Mendonça, 2020](#); [Takahashi et al., 2020](#); [Soler and Uieda, 2021](#))



In this presentation, however, I will explore theoretical aspects of the EqL technique

Summary

- Motivation
- Potential-field data
- **The Equivalent-Layer (EqL) Technique**
- Theoretical aspects
- Some open questions

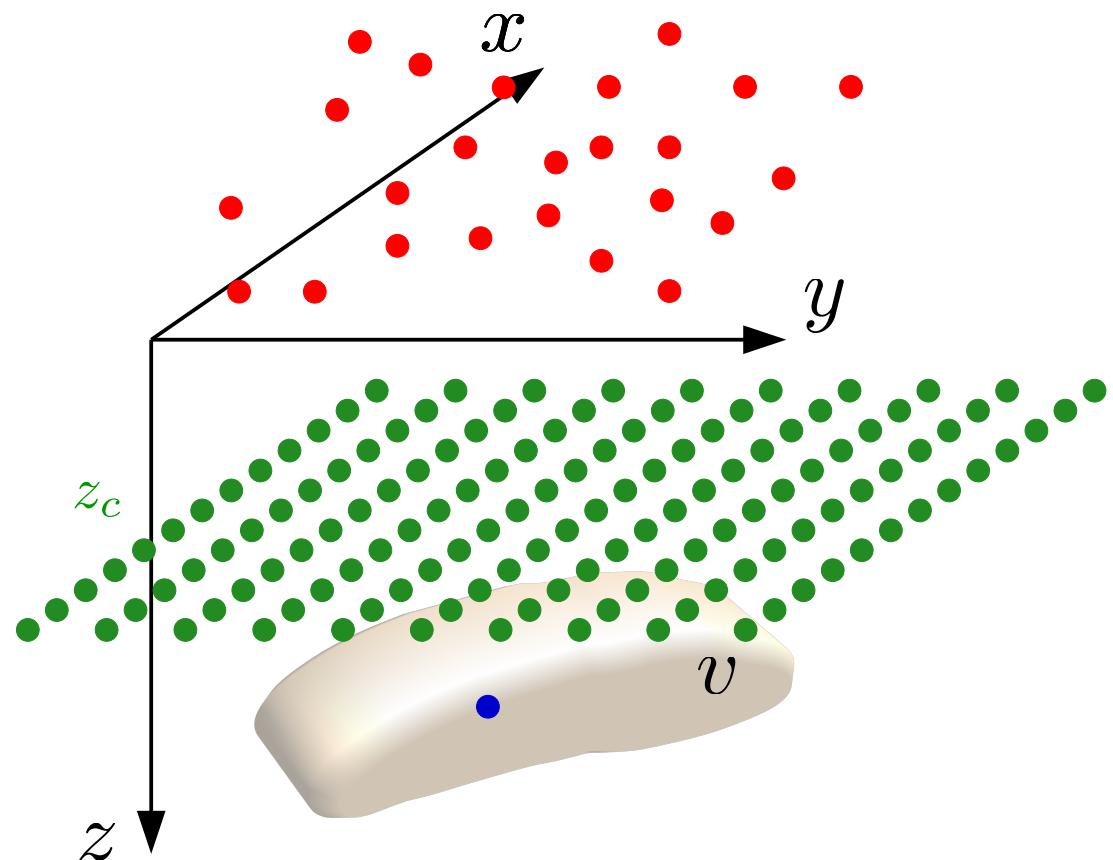
Summary

- Motivation
- Potential-field data
- The Equivalent-Layer (EqL) Technique
- **Theoretical aspects**
- Some open questions

We have briefly
discussed how the EqL
technique works, ...

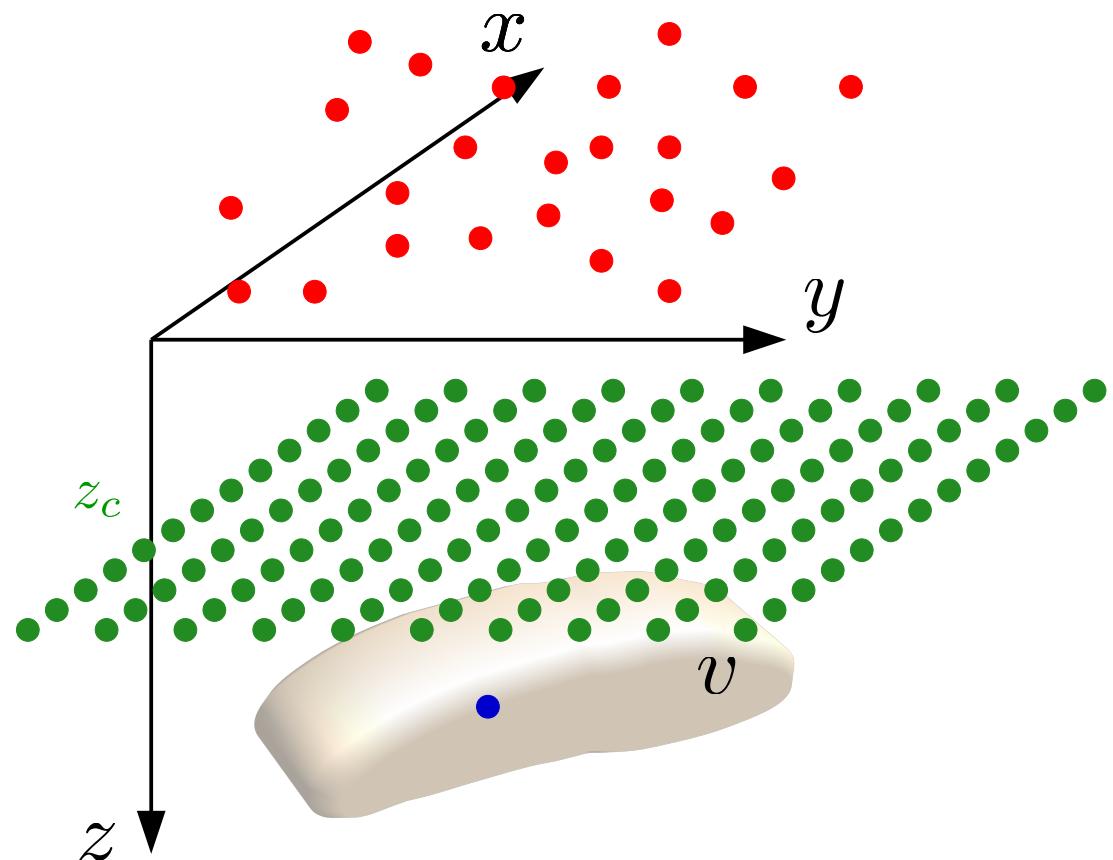
We have briefly
discussed how the EqL
technique works, ...

... now we need to
discuss why
the EqL technique works



$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

Let us return to the linear system for estimating the discrete equivalent layer



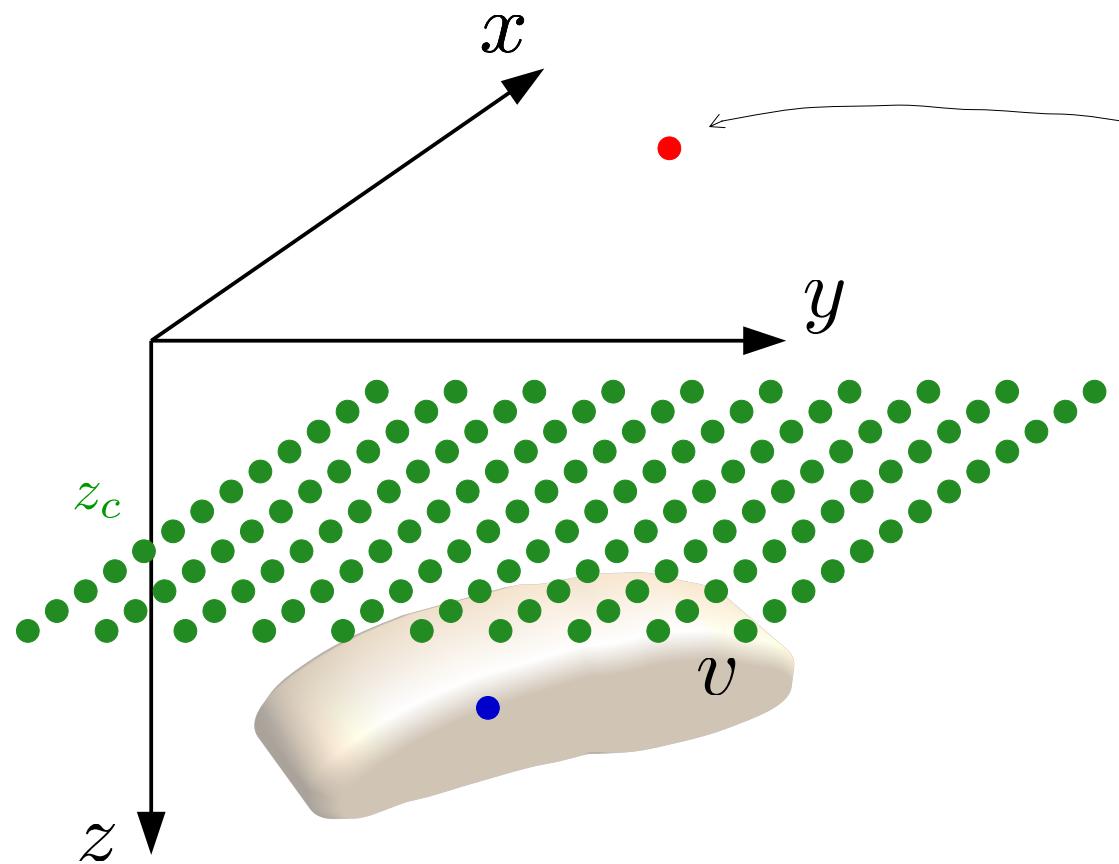
Discrete
equivalent layer

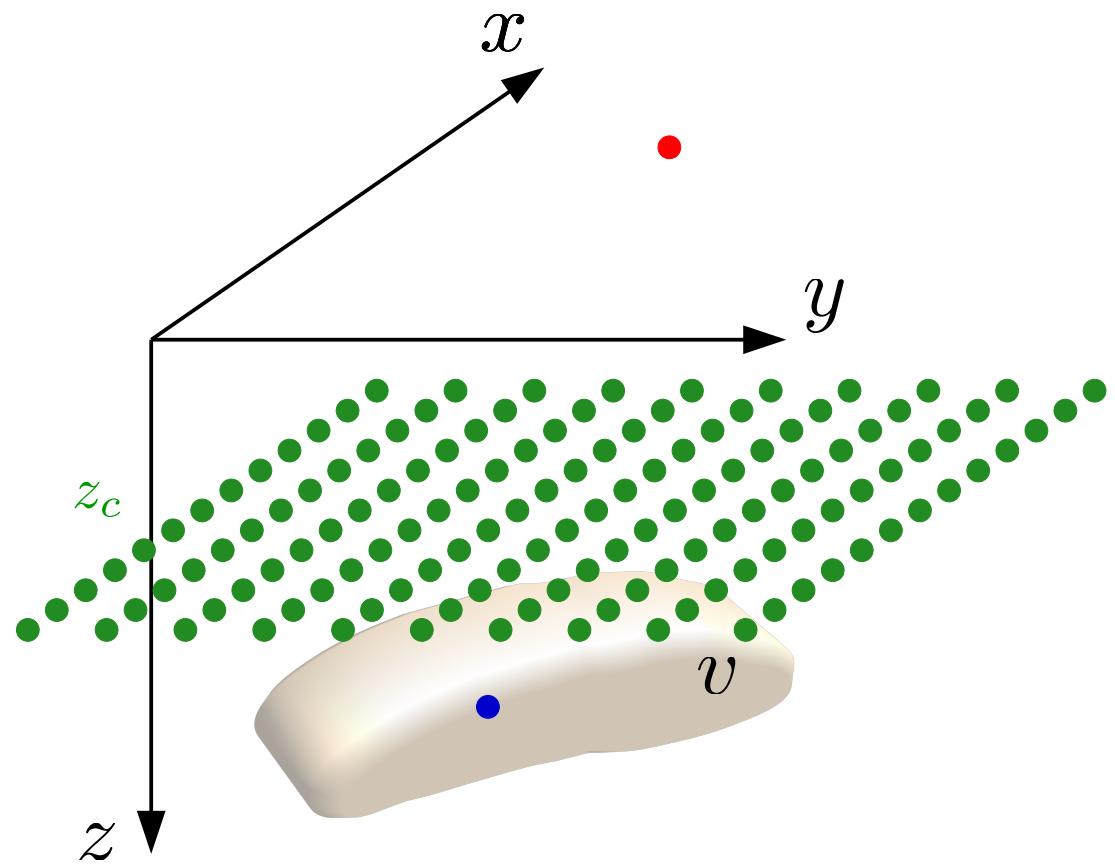
$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

Let us return to the linear system for estimating the discrete equivalent layer

$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

$$d_{\textcolor{red}{i}} \approx \sum_{j=1}^M p_{\textcolor{green}{j}} a_{\textcolor{red}{i}j}$$

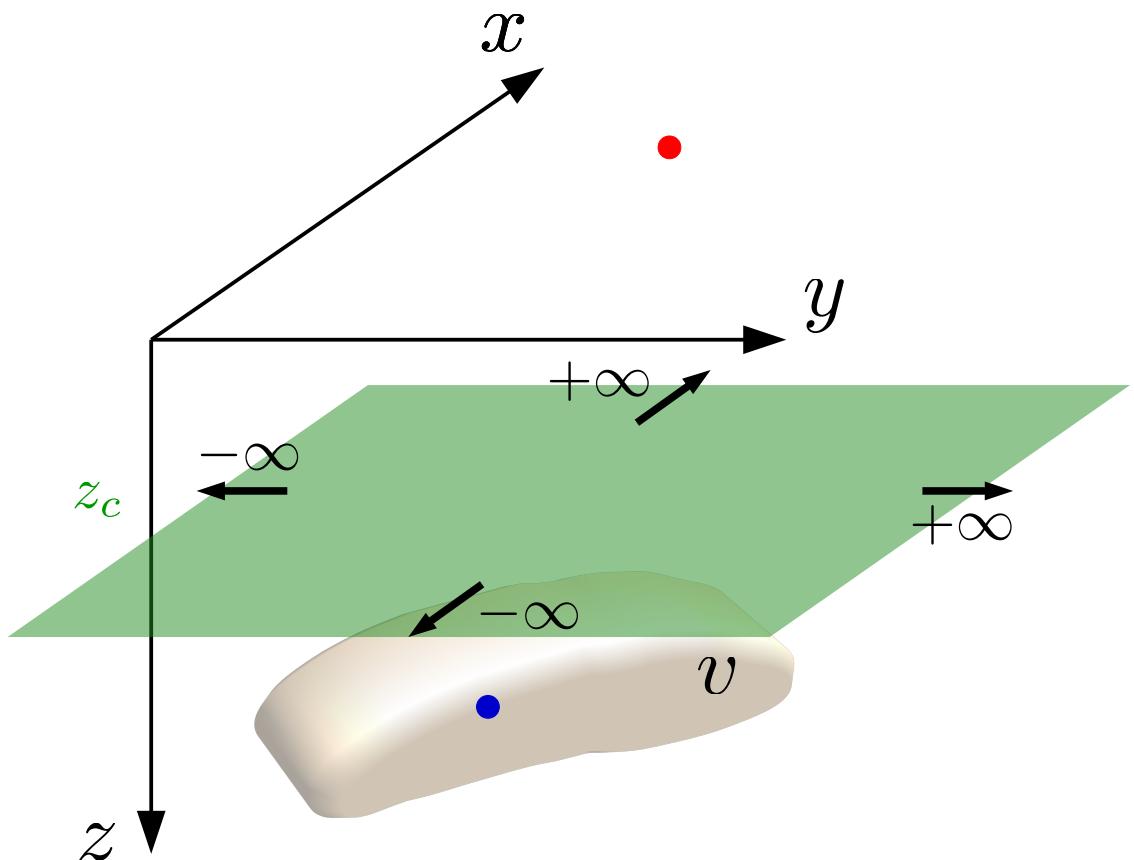




$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

$$d_{\textcolor{red}{i}} \approx \sum_{j=1}^M p_{\textcolor{green}{j}} a_{\textcolor{red}{i}j}$$

Finite set of points defining
a discrete equivalent layer



$$\mathbf{d} \approx \mathbf{A} \mathbf{p}$$

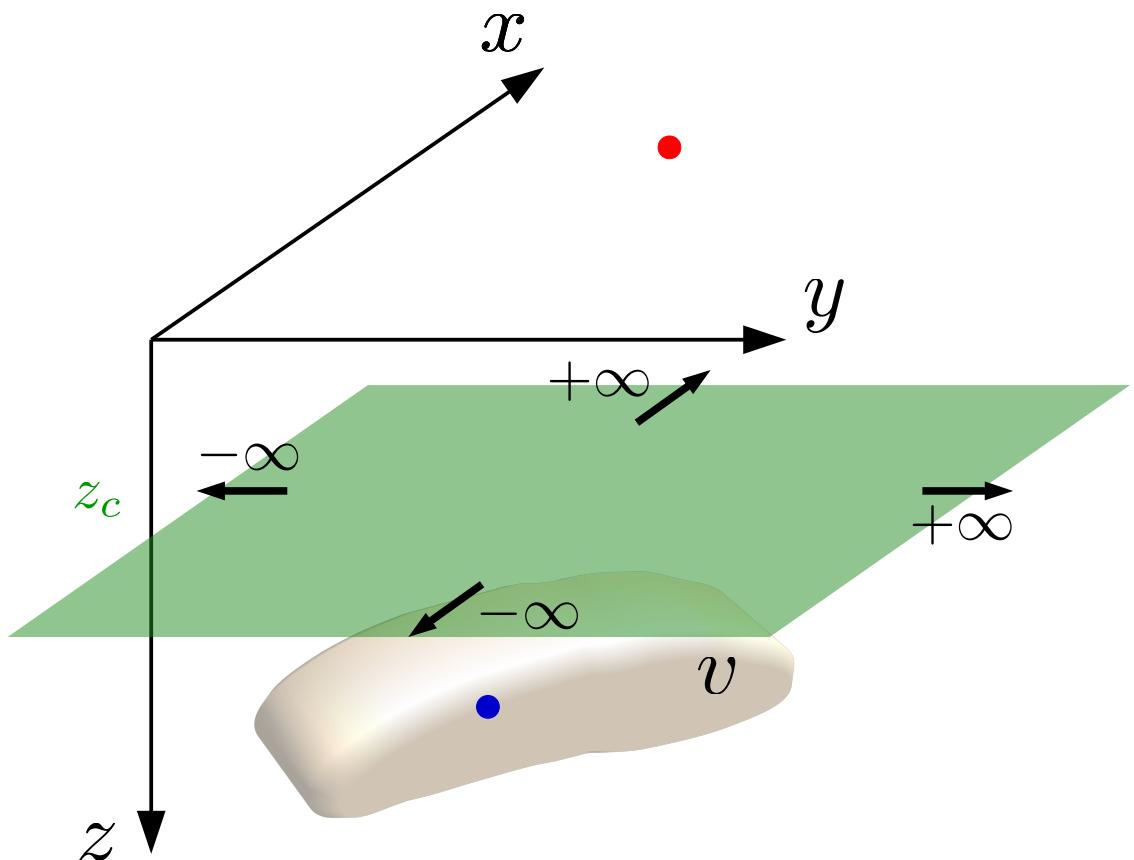
$$d_{\textcolor{red}{i}} \approx \sum_{j=1}^M p_{\textcolor{green}{j}} a_{\textcolor{red}{i}\textcolor{green}{j}}$$

Finite set of points defining
a discrete equivalent layer

$$d(\mathbf{r}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\mathbf{r}'') a(\mathbf{r}, \mathbf{r}'') dS''$$

Infinite horizontal plane defining
a continuous equivalent layer

$\curvearrowright dx'' dy''$

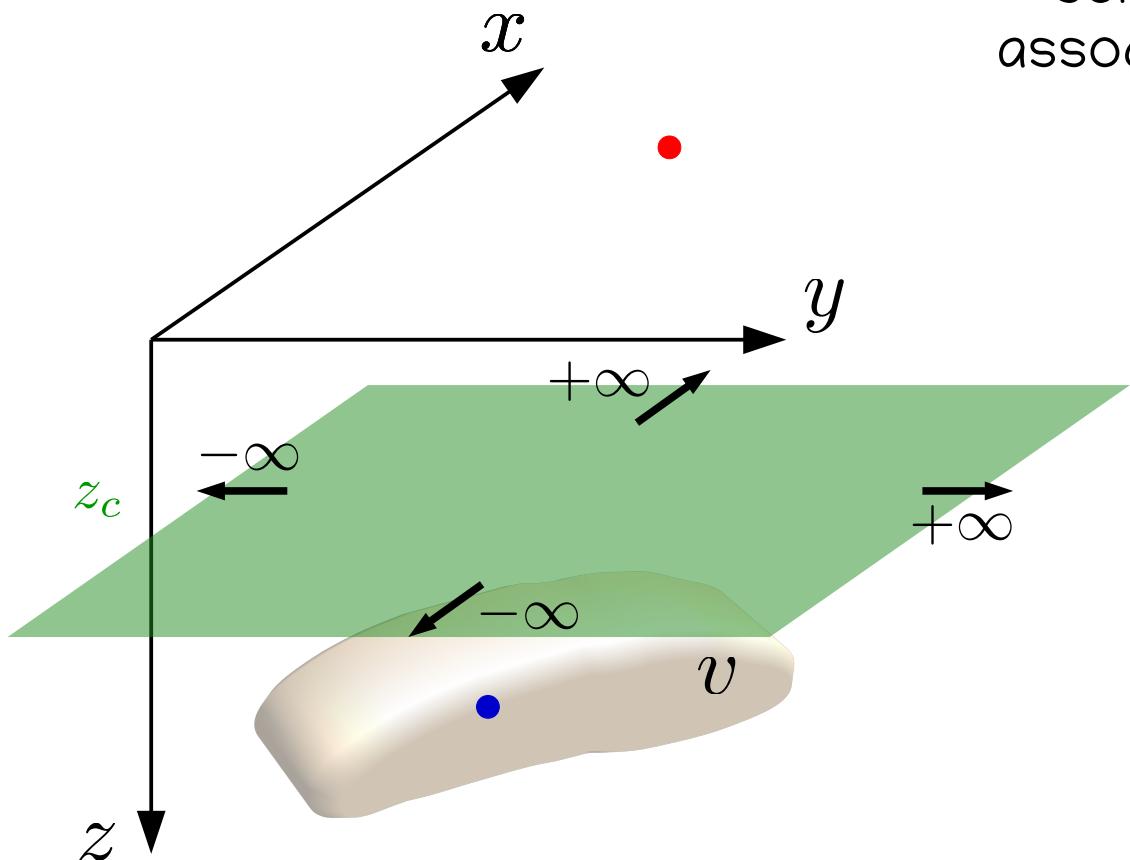


Integral equation for the continuous physical-property distribution (equivalent layer) $p(\mathbf{r}'')$

$$d(\mathbf{r}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\mathbf{r}'') a(\mathbf{r}, \mathbf{r}'') dS''$$

$dx'' dy''$

The following slides present analytical solutions for the integral equations associated with different potential-field data

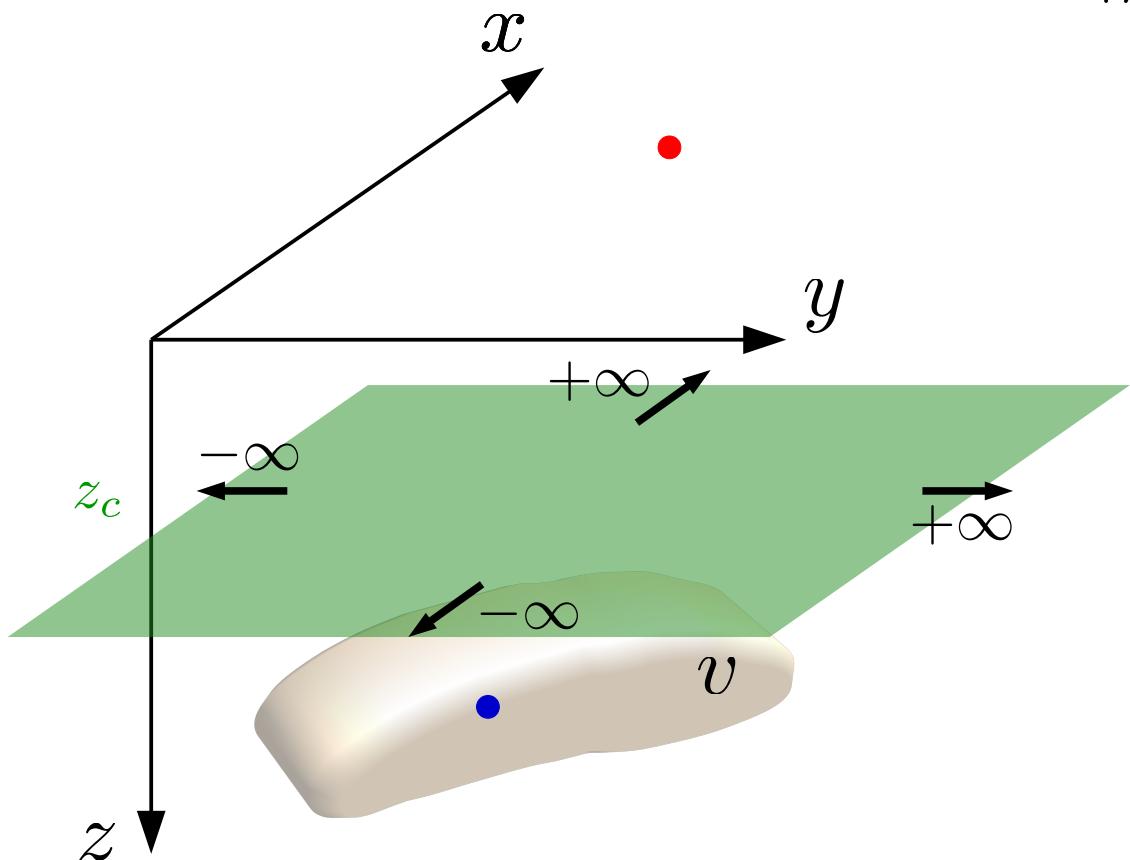


Integral equation for the continuous physical-property distribution (equivalent layer) $p(\mathbf{r}'')$

$$d(\mathbf{r}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\mathbf{r}'') a(\mathbf{r}, \mathbf{r}'') dS''$$

$\curvearrowright dx'' dy''$

These solutions are conveniently called here **analytical equivalent layers**

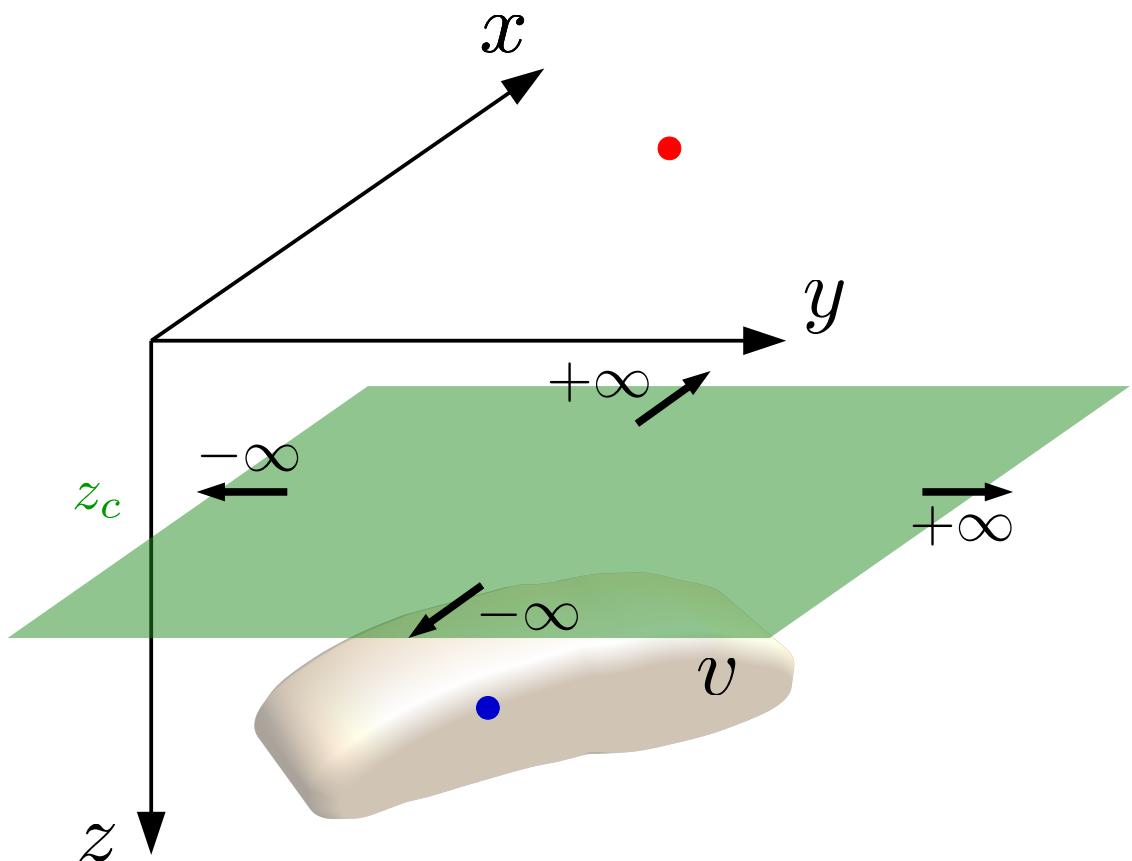


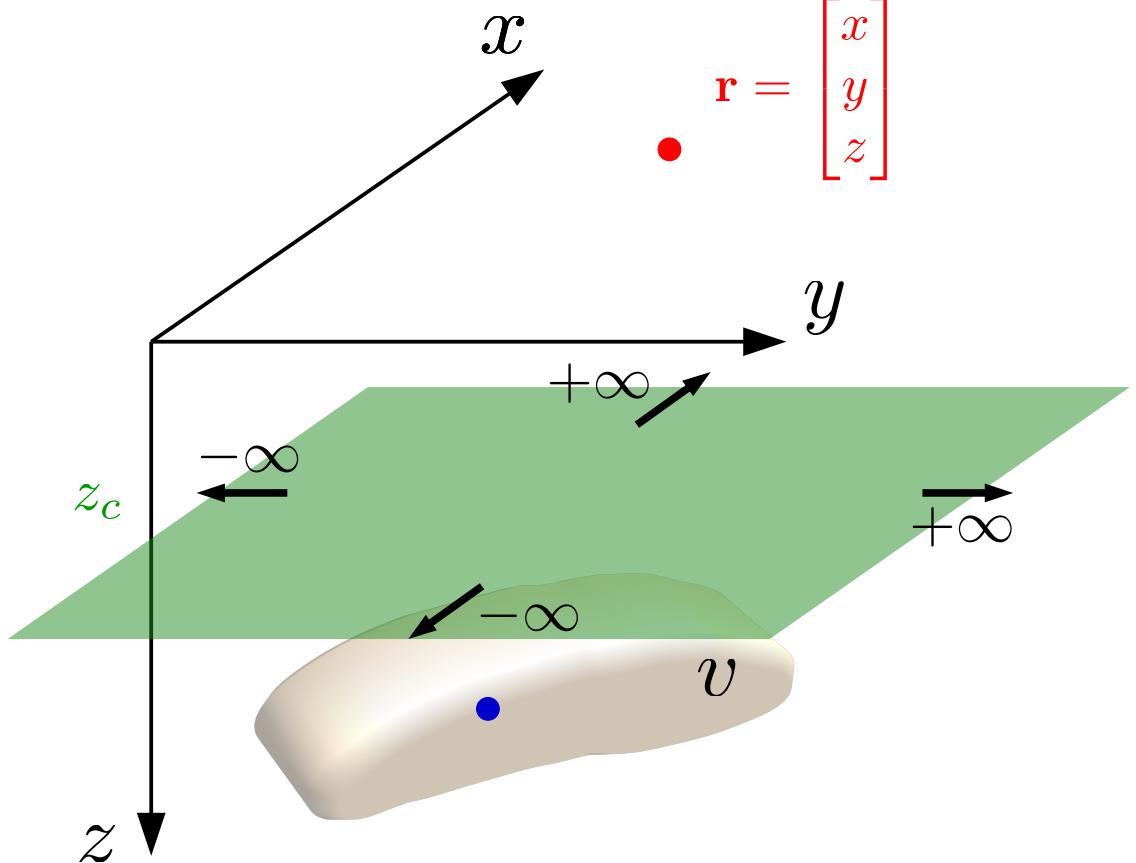
Integral equation for the continuous physical-property distribution (equivalent layer) $p(\mathbf{r}'')$

$$d(\mathbf{r}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\mathbf{r}'') a(\mathbf{r}, \mathbf{r}'') dS''$$

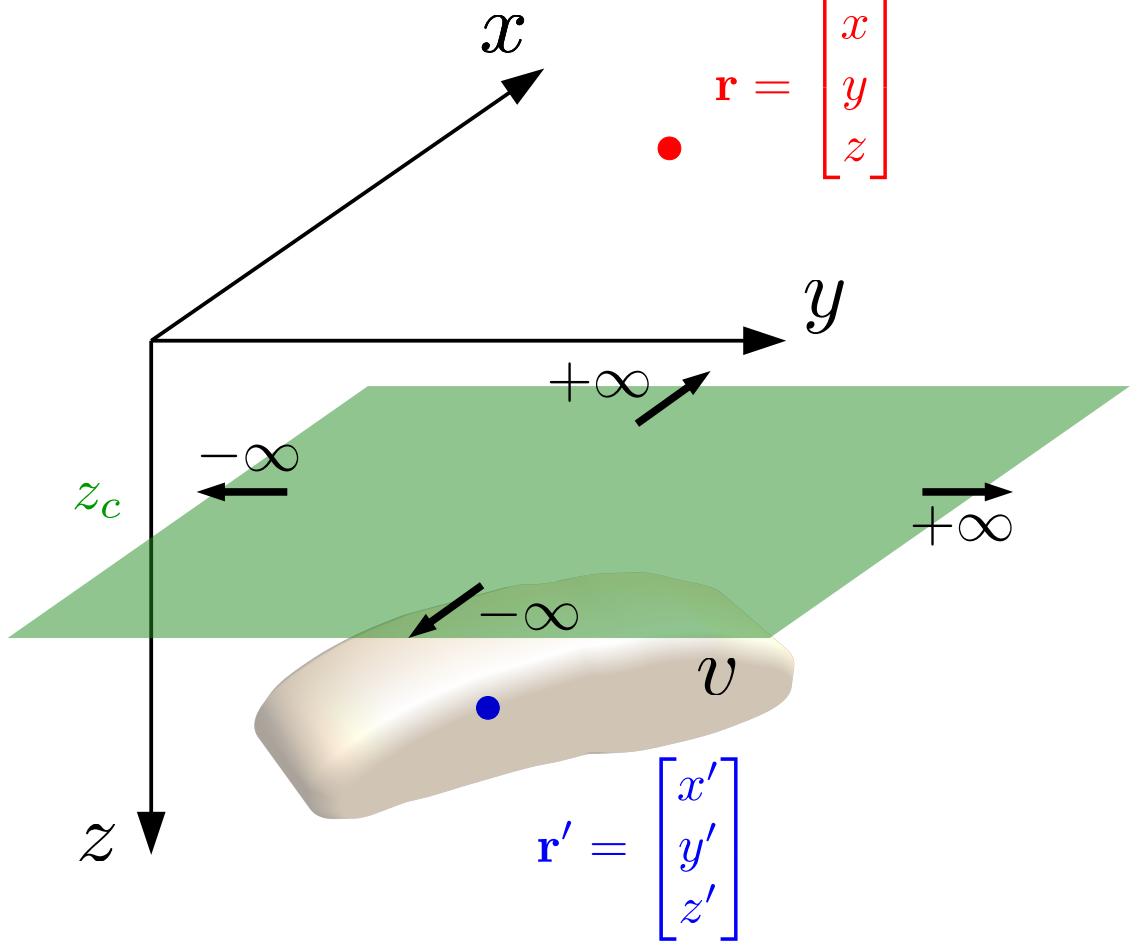
Let us turn our attention
back to this function

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

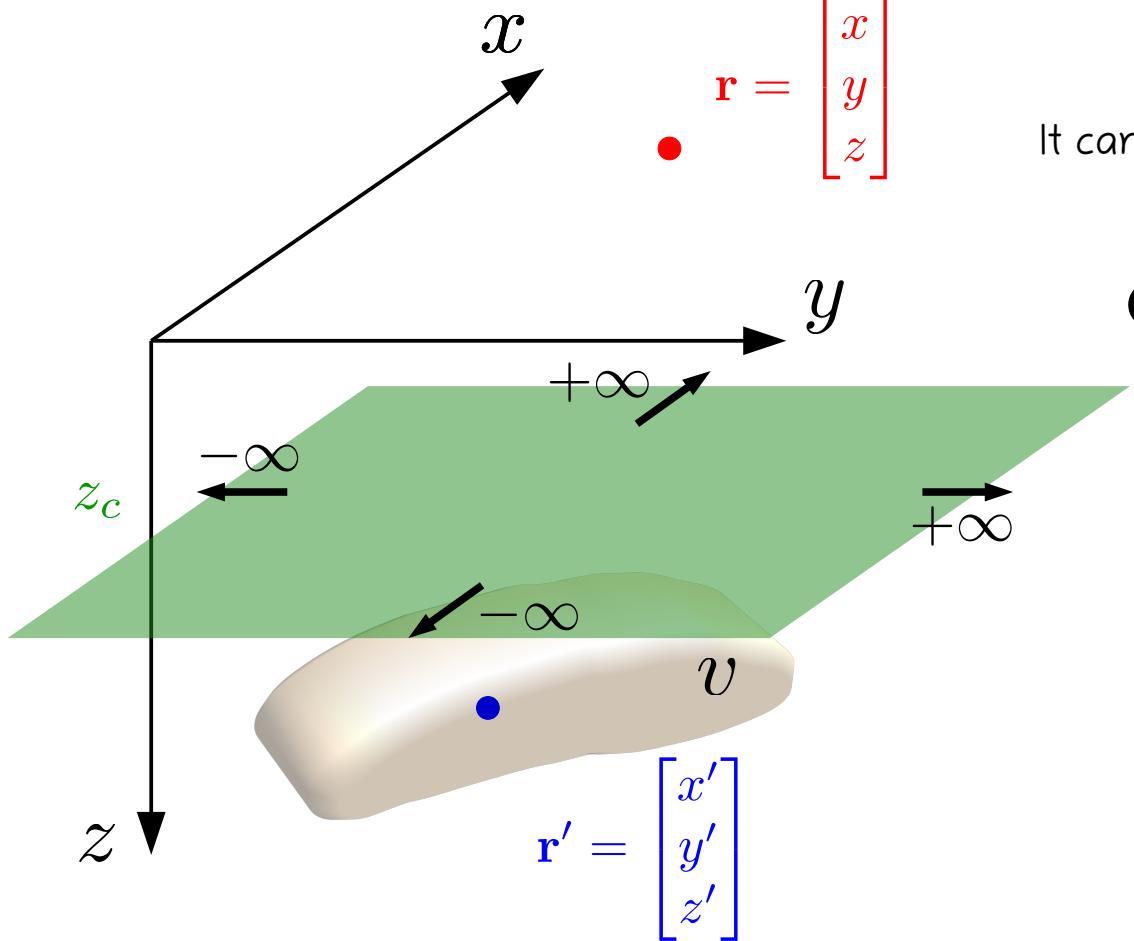




$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

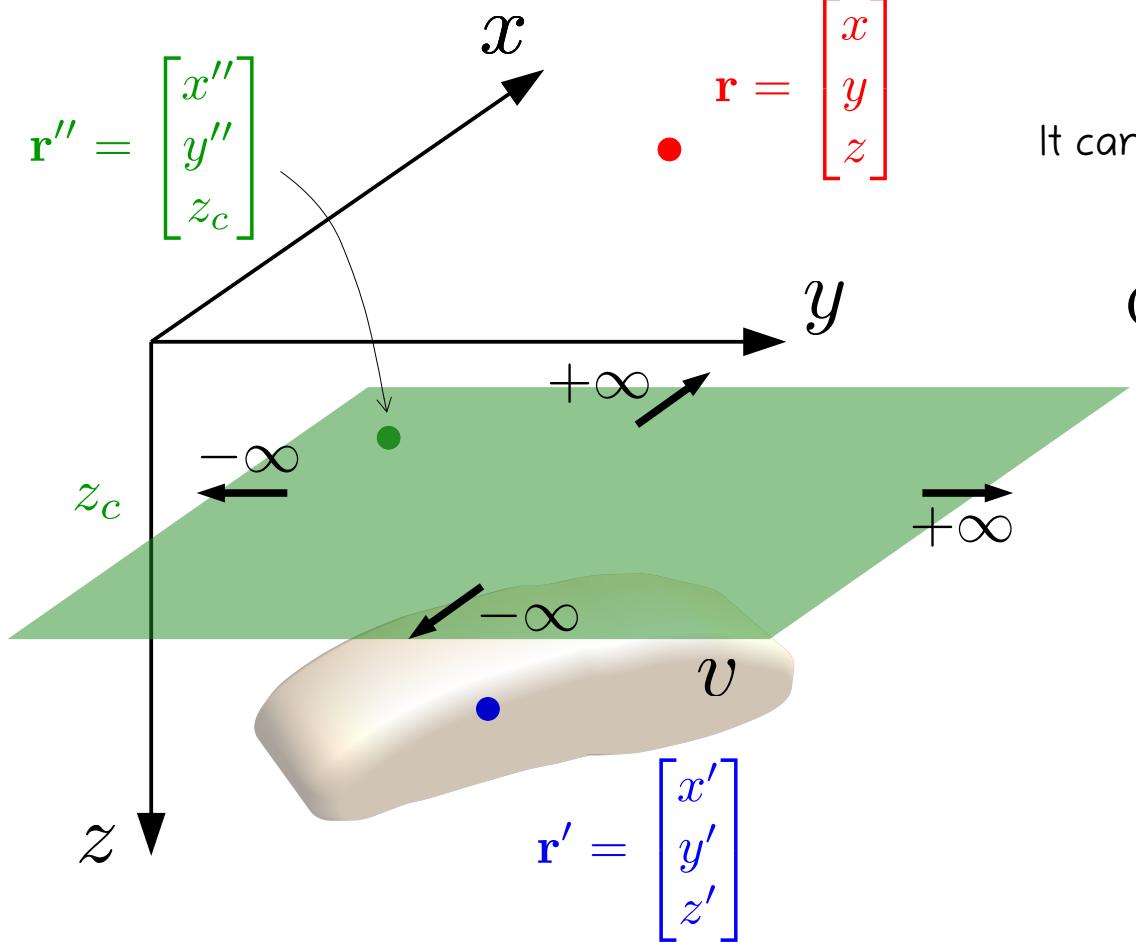


$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

It can be shown that (e.g., [Roy, 1962](#); [Reis et al., 2020](#)):

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

which represents the solution of the **Neumann's problem** or the **second boundary value problem of potential theory** ([Kellogg, 1967, p. 246](#)) on a plane

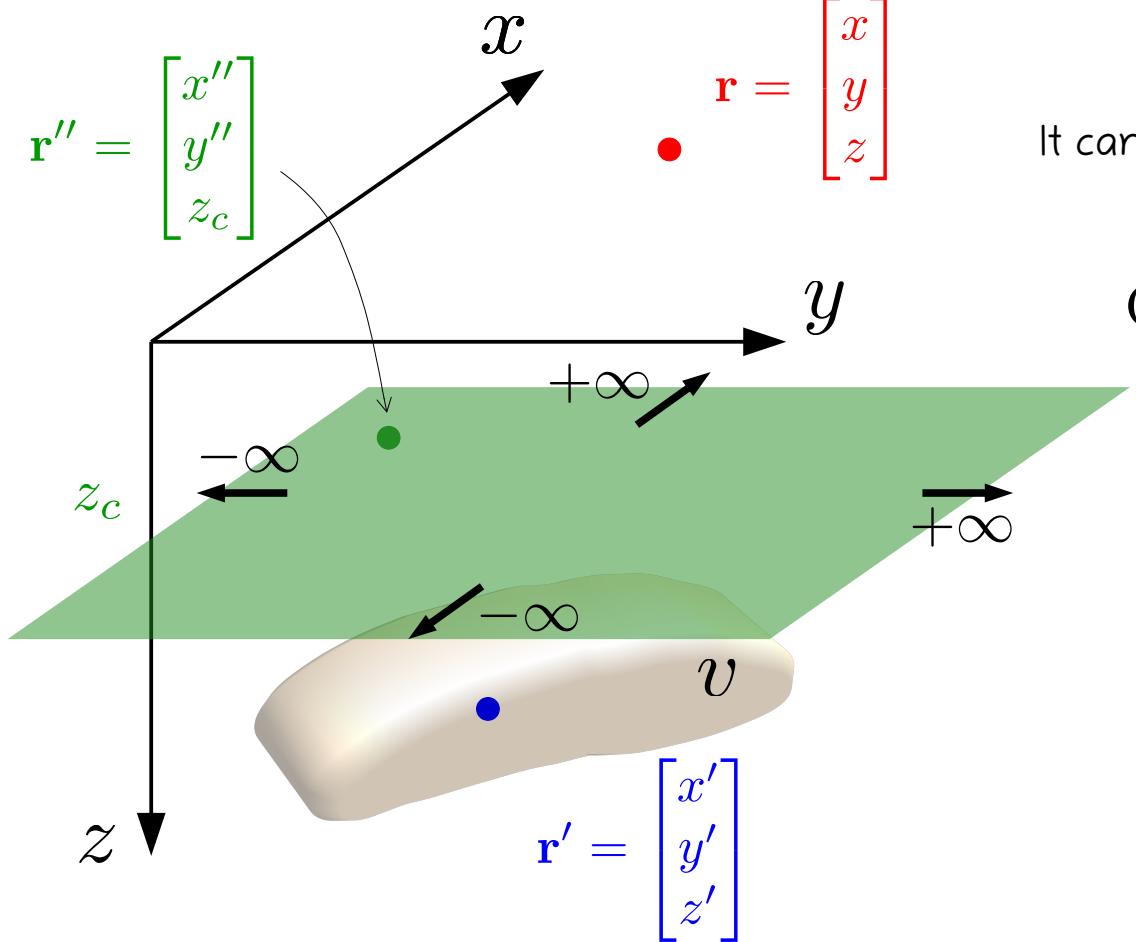


$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

It can be shown that (e.g., [Roy, 1962](#); [Reis et al., 2020](#)):

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

which represents the solution of the **Neumann's problem** or the **second boundary value problem of potential theory** ([Kellogg, 1967, p. 246](#)) on a plane



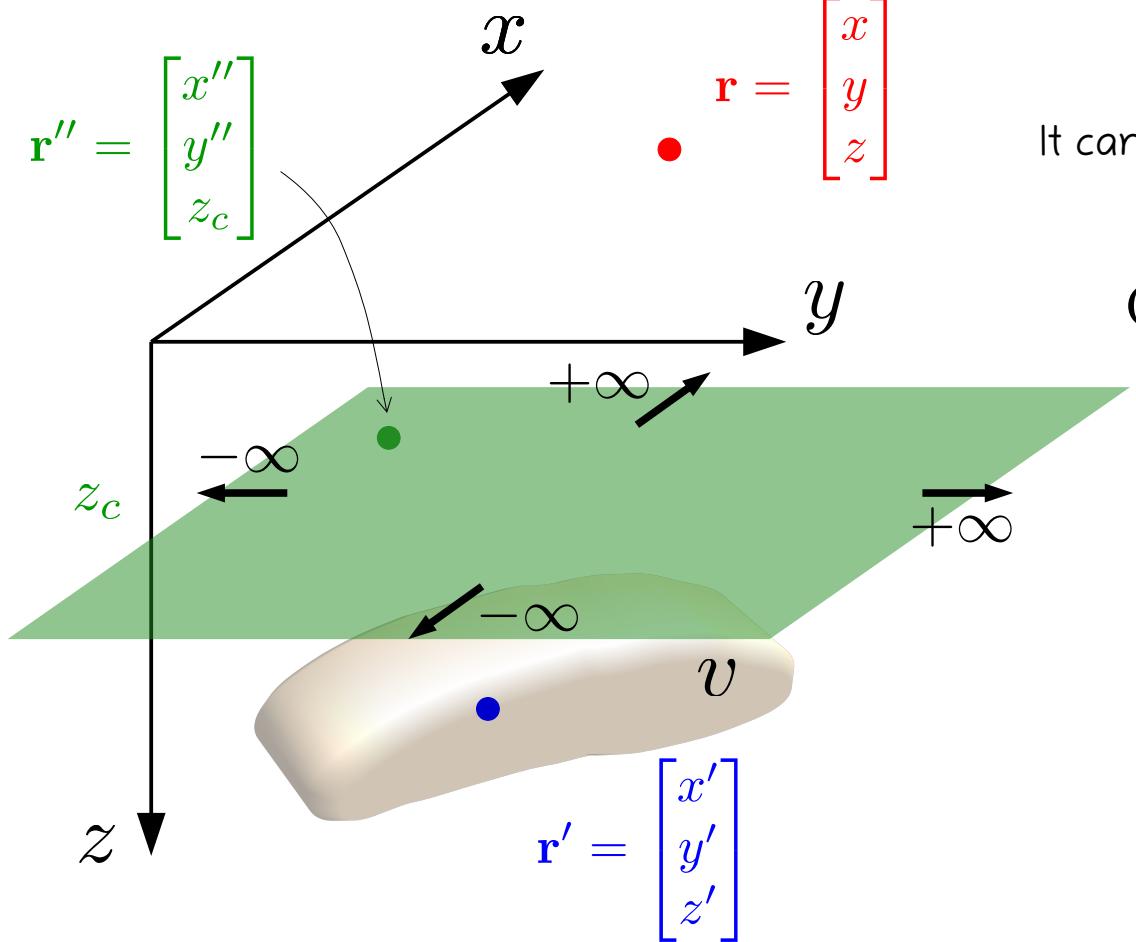
$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

It can be shown that (e.g., [Roy, 1962](#); [Reis et al., 2020](#)):

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

which represents the solution of the **Neumann's problem** or the **second boundary value problem of potential theory** ([Kellogg, 1967, p. 246](#)) on a plane

$$\partial_z \Theta(\mathbf{r}'') = \kappa \iiint_v \sigma(\mathbf{r}') \partial_z \frac{1}{\|\mathbf{r}'' - \mathbf{r}'\|} dv'$$



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

It can be shown that (e.g., [Roy, 1962](#); [Reis et al., 2020](#)):

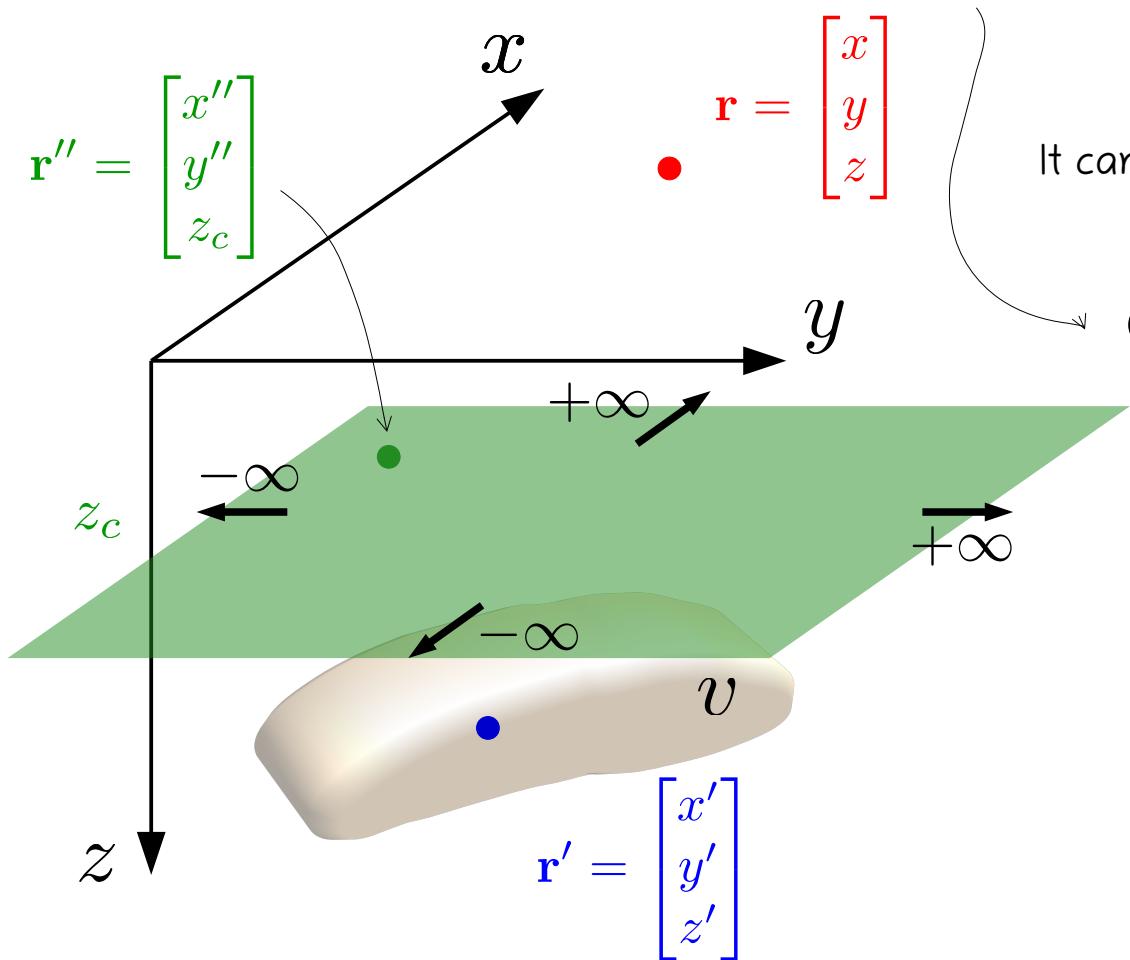
$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

which represents the solution of the **Neumann's problem** or the **second boundary value problem of potential theory** ([Kellogg, 1967, p. 246](#)) on a plane

$$\partial_z \Theta(\mathbf{r}'') = \kappa \iiint_v \sigma(\mathbf{r}') \partial_z \frac{1}{\|\mathbf{r}'' - \mathbf{r}'\|} dv'$$

$$\Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Most of what is known or assumed to be true without any proof about the EqL technique can be deduced from this equation



It can be shown that (e.g., [Roy, 1962](#); [Reis et al., 2020](#)):

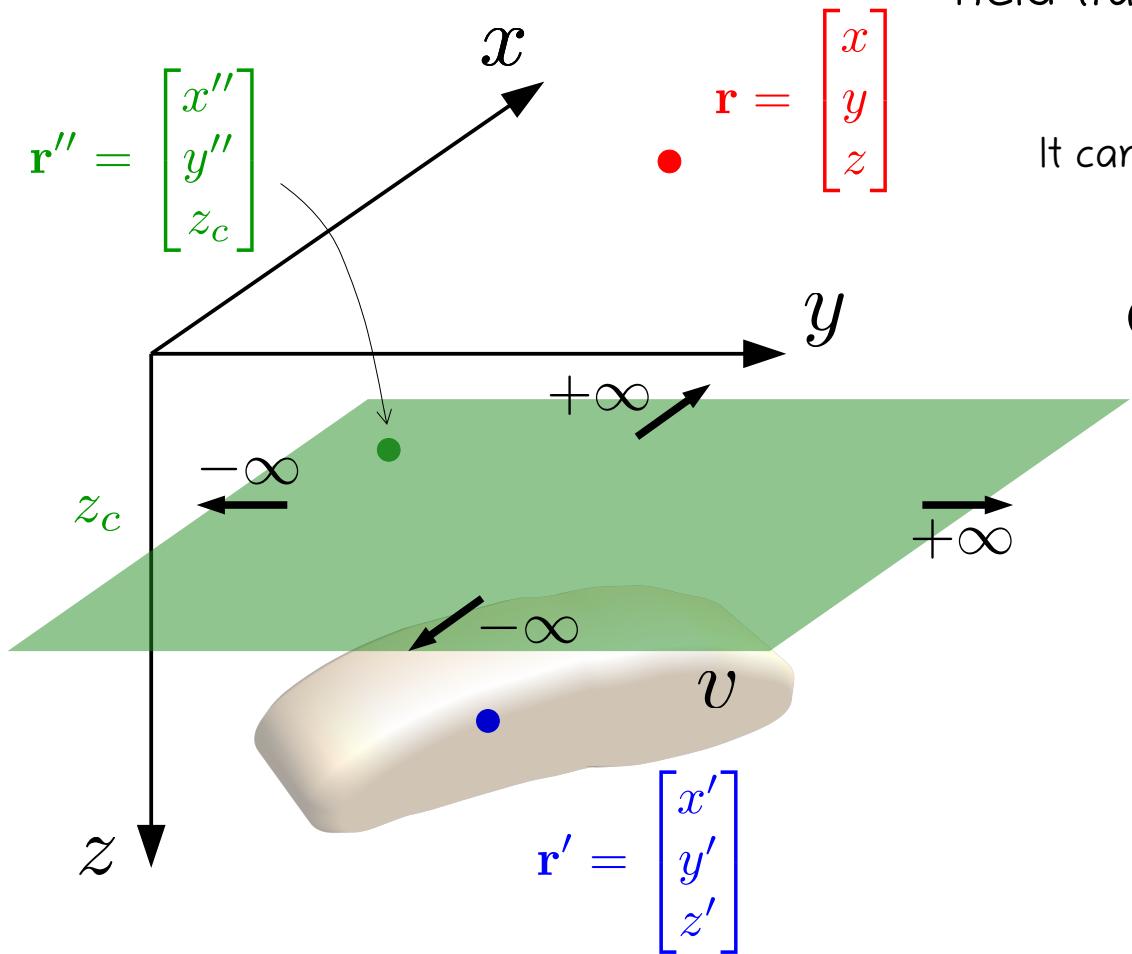
$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

which represents the solution of the **Neumann's problem** or the **second boundary value problem of potential theory** ([Kellogg, 1967, p. 246](#)) on a plane

$$\partial_z \Theta(\mathbf{r}'') = \kappa \iiint_v \sigma(\mathbf{r}') \partial_z \frac{1}{\|\mathbf{r}'' - \mathbf{r}'\|} dv'$$

$$\Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

We are going to see four theoretical results that support most of the common potential-field transformations made via EqL technique



It can be shown that (e.g., [Roy, 1962](#); [Reis et al., 2020](#)):

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

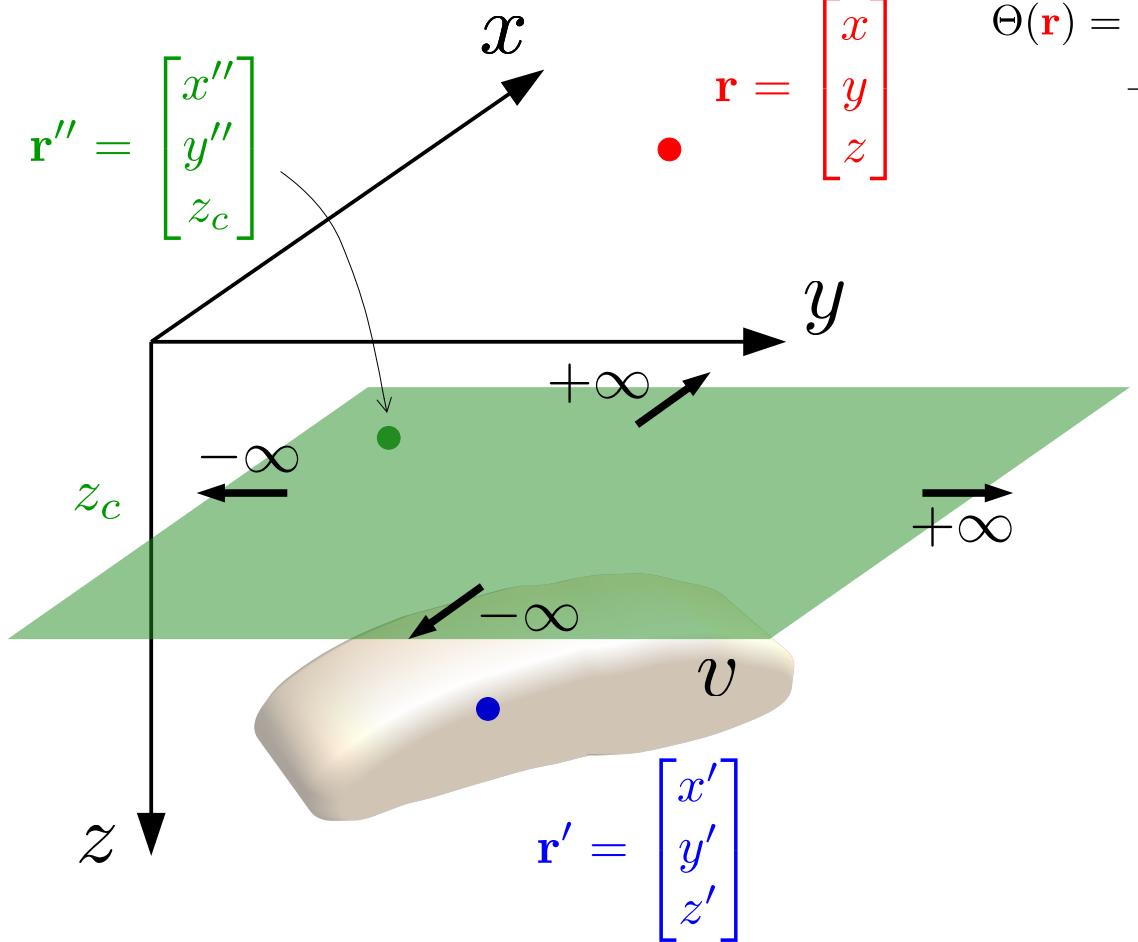
which represents the solution of the **Neumann's problem** or the **second boundary value problem of potential theory** ([Kellogg, 1967, p. 246](#)) on a plane

$$\partial_z \Theta(\mathbf{r}'') = \kappa \iiint_v \sigma(\mathbf{r}') \partial_z \frac{1}{\|\mathbf{r}'' - \mathbf{r}'\|} dv'$$

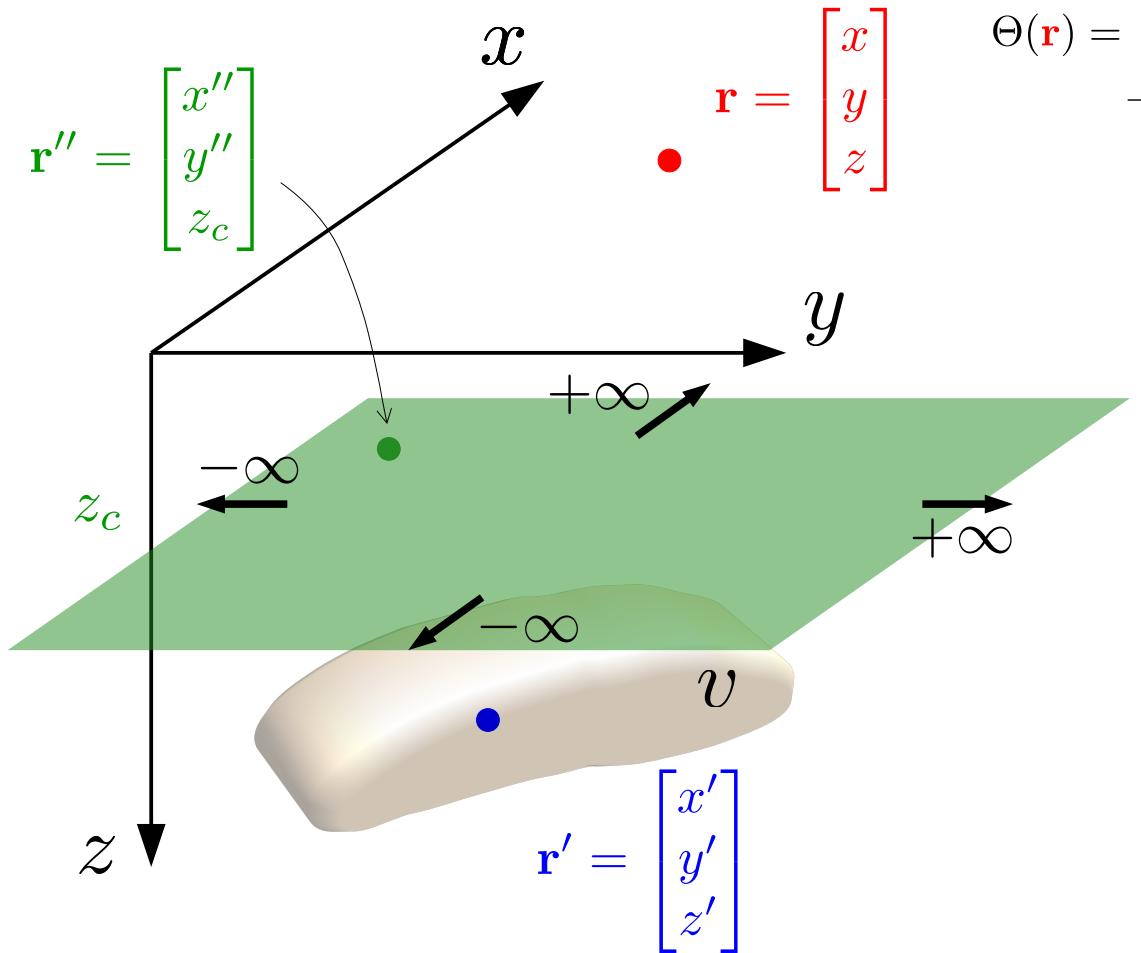
$$\Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$



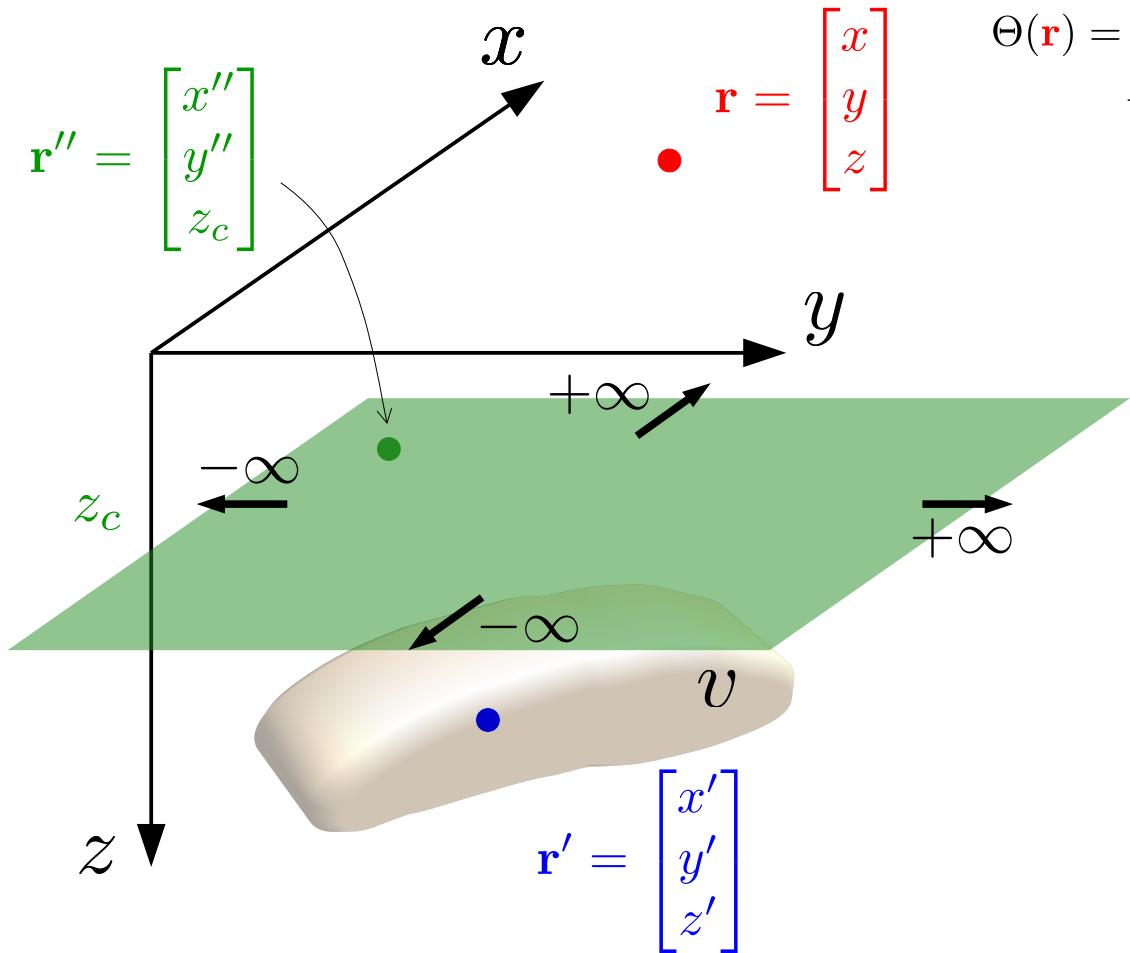
Result 1/4: The gravity disturbance can be exactly reproduced by an analytical eq. layer of monopoles



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Result 1/4: The gravity disturbance can be exactly reproduced by an analytical eq. layer of monopoles

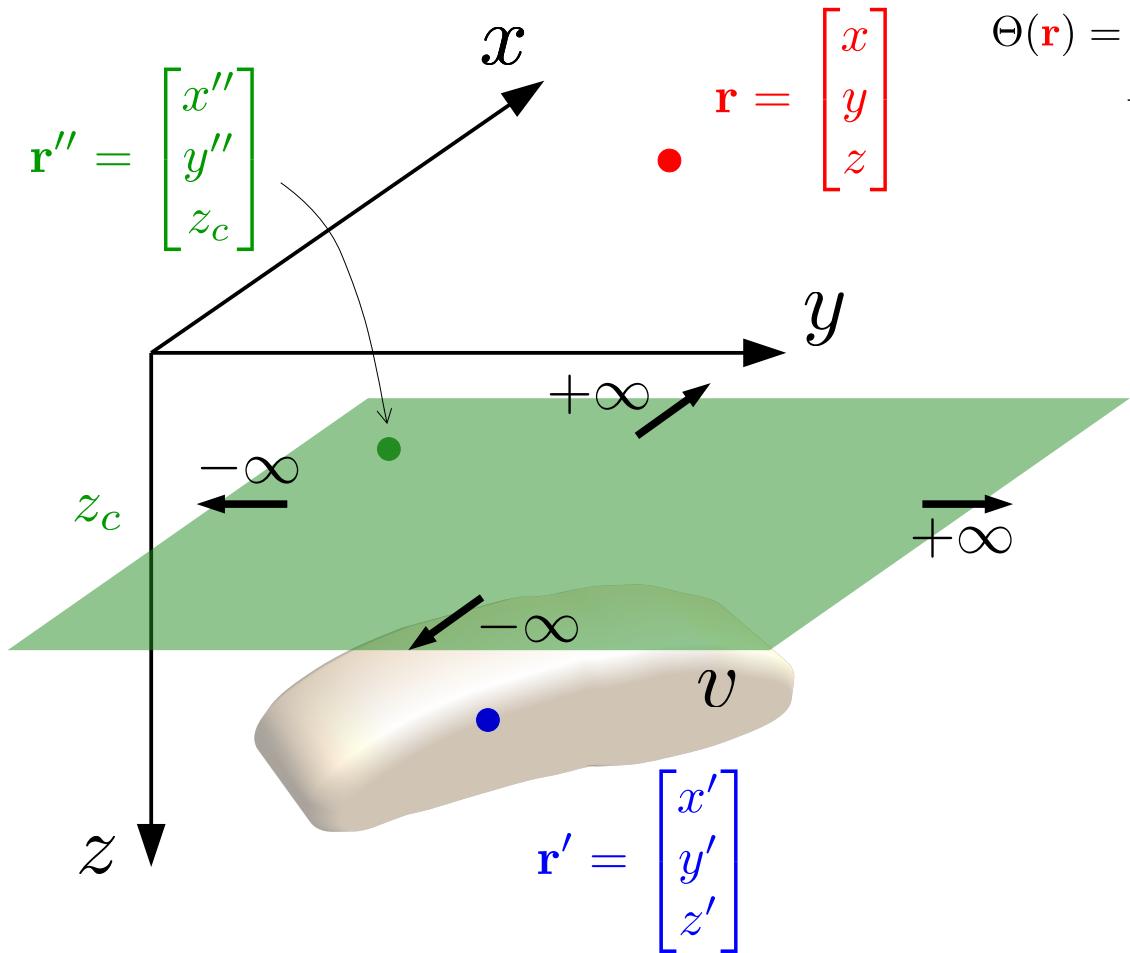


$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Dampney (1969) was the pioneer in using this result to interpolate gravity data.

Result 1/4: The gravity disturbance can be exactly reproduced by an analytical eq. layer of monopoles



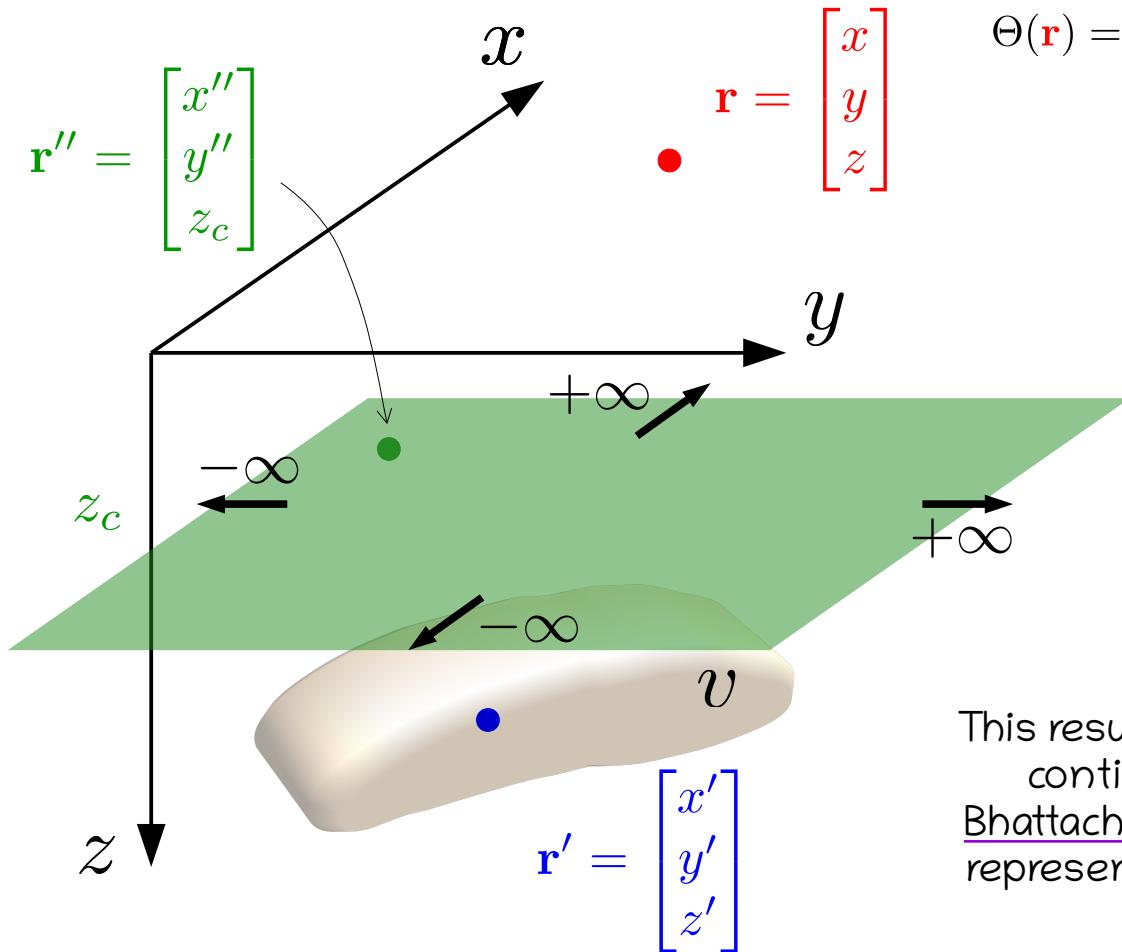
$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Dampney (1969) was the pioneer in using this result to interpolate gravity data.

A considerable amount of literature has been published on this topic since then

Result 1/4: The gravity disturbance can be exactly reproduced by an analytical eq. layer of monopoles



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

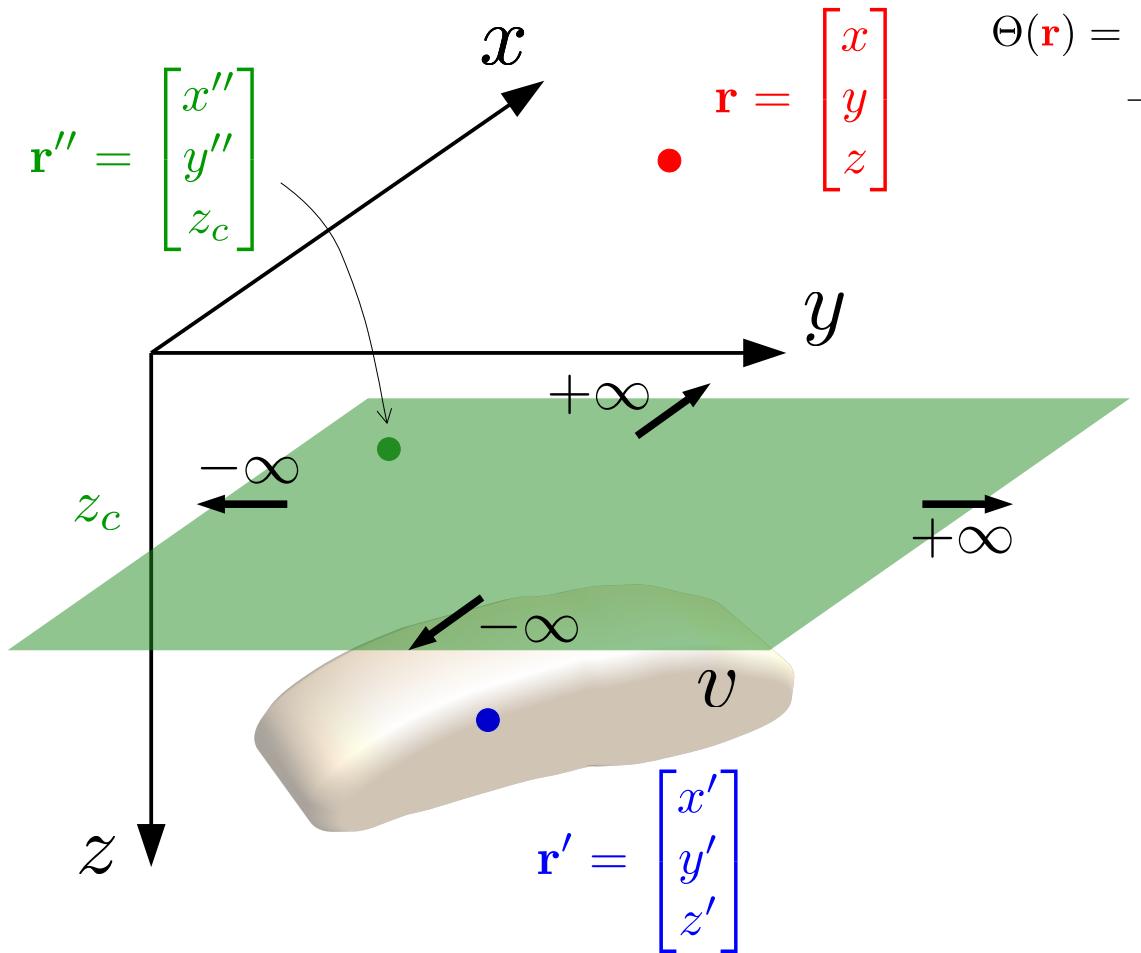
$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Dampney (1969) was the pioneer in using this result to interpolate gravity data.

A considerable amount of literature has been published on this topic since then

This result can also be directly deduced from the upward-continuation integral (e.g., Peters, 1949; Roy, 1962; Bhattacharyya, 1967; Henderson, 1970, Gunn, 1975), which represents the solution of Dirichlet's problem on a plane (Kellogg, 1967, p. 236)

Result 1/4: The gravity disturbance can be exactly reproduced by an analytical eq. layer of monopoles



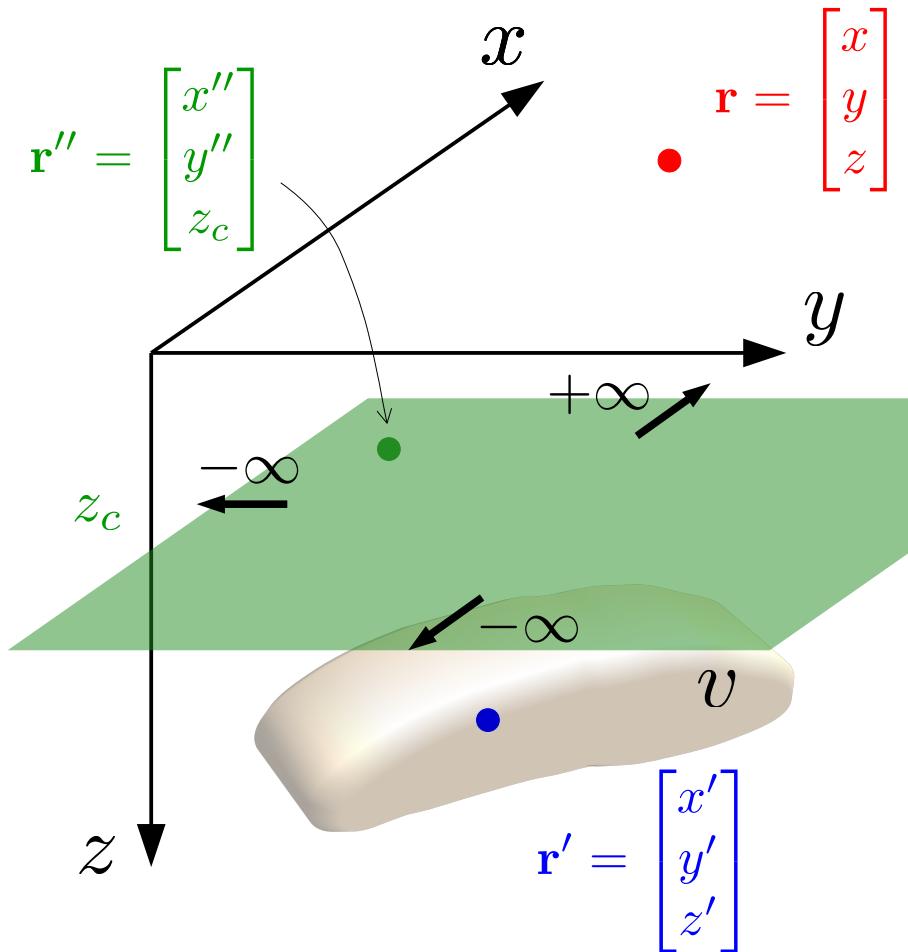
$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

$\partial_z \Theta(\mathbf{r})$ gravity disturbance data

Result 1/4: The gravity disturbance can be exactly reproduced by an analytical eq. layer of monopoles



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

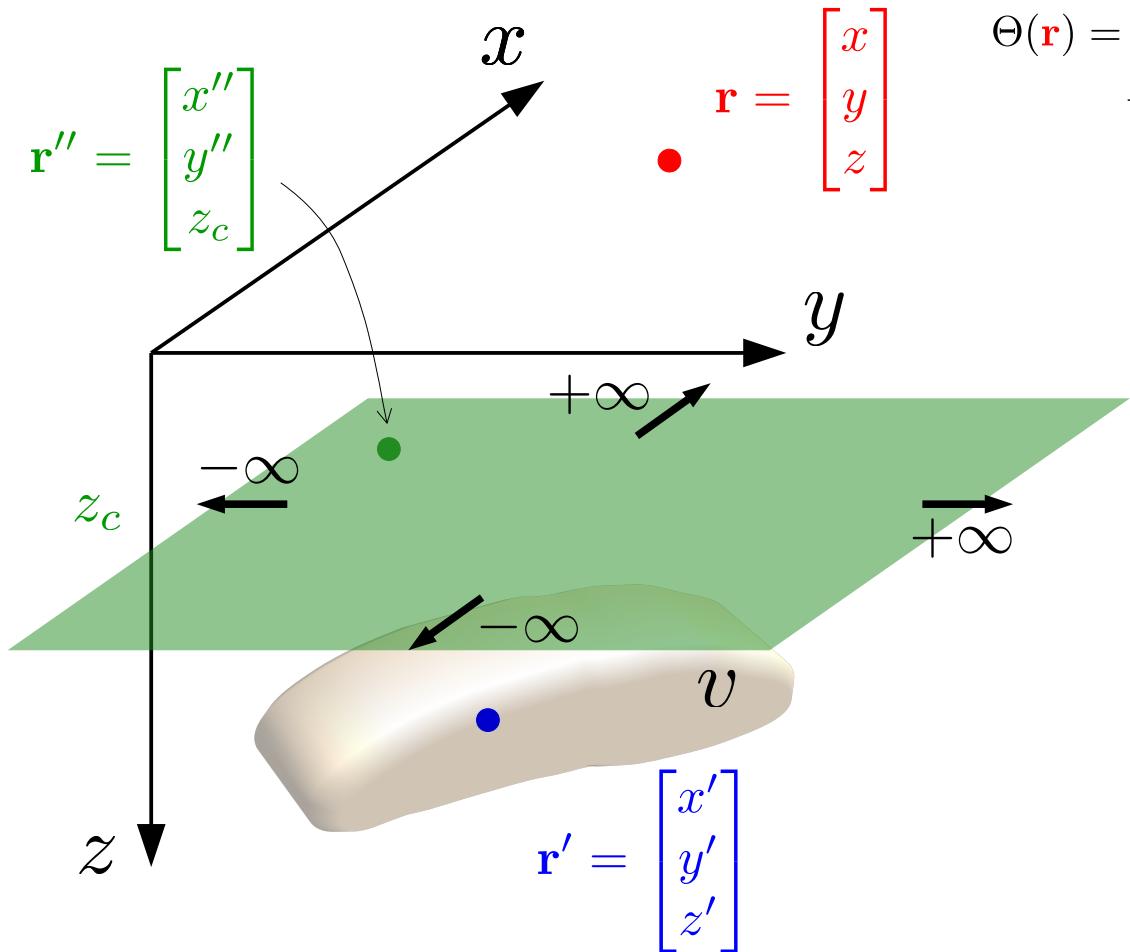
$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

$\partial_z \Theta(\mathbf{r})$ gravity disturbance data

$$\partial_z \Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \partial_z \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

Result 1/4: The gravity disturbance can be exactly reproduced by an analytical eq. layer of monopoles



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

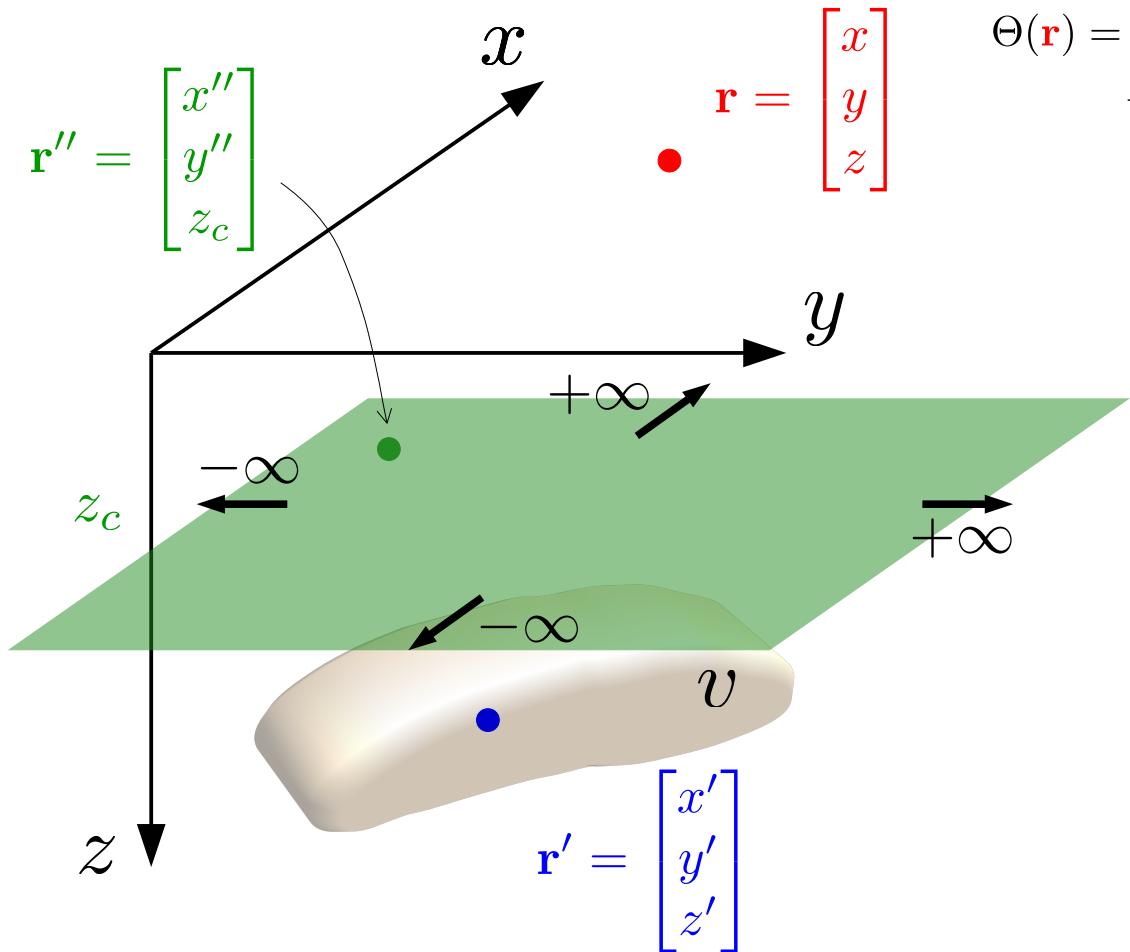
Deduction:

$\partial_z \Theta(\mathbf{r})$ gravity disturbance data

$$\partial_z \Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \underbrace{\partial_z \Theta(\mathbf{r}'')} \underbrace{\partial_z \Psi(\mathbf{r}, \mathbf{r}'')} dS''$$

This term represents the gravity disturbance produced at \mathbf{r} by a monopole located at \mathbf{r}''

Result 1/4: The gravity disturbance can be exactly reproduced by an analytical eq. layer of monopoles



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

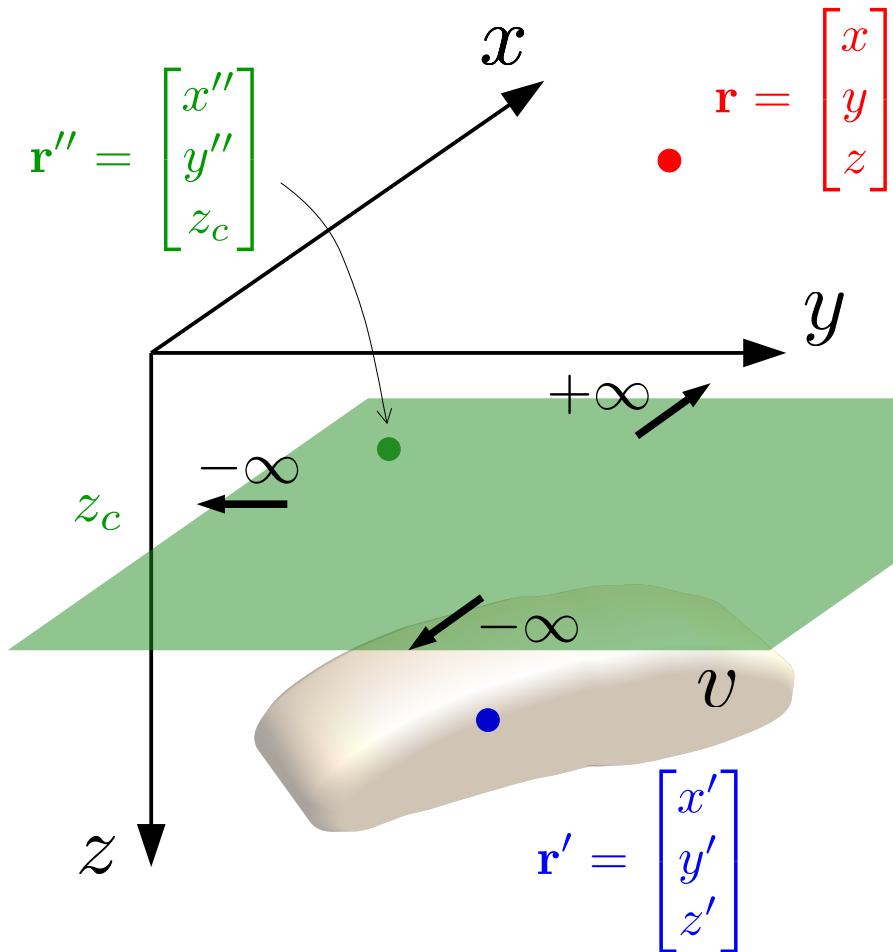
Deduction:

$\partial_z \Theta(\mathbf{r})$ gravity disturbance data

$$\partial_z \Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \partial_z \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

This equation is the upward-continuation integral applied to the vertical derivative of $\Theta(\mathbf{r})$

Result 1/4: The gravity disturbance can be exactly reproduced by an analytical eq. layer of monopoles



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

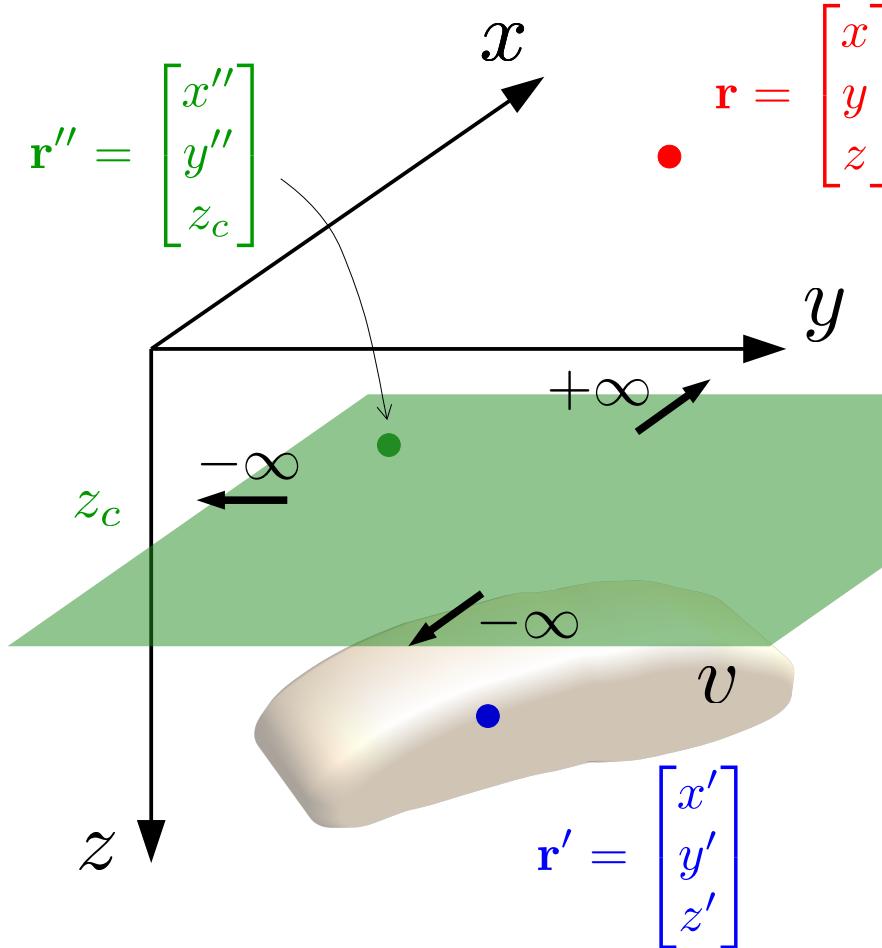
$\partial_z \Theta(\mathbf{r})$ gravity disturbance data

$$\partial_z \Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \partial_z \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

Analytical eq. layer given by:

$$\partial_z \Theta(\mathbf{r}'') = \kappa \iiint_v \sigma(\mathbf{r}') \partial_z \frac{1}{\|\mathbf{r}'' - \mathbf{r}'\|} dv'$$

Result 1/4: The gravity disturbance can be exactly reproduced by an analytical eq. layer of monopoles



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

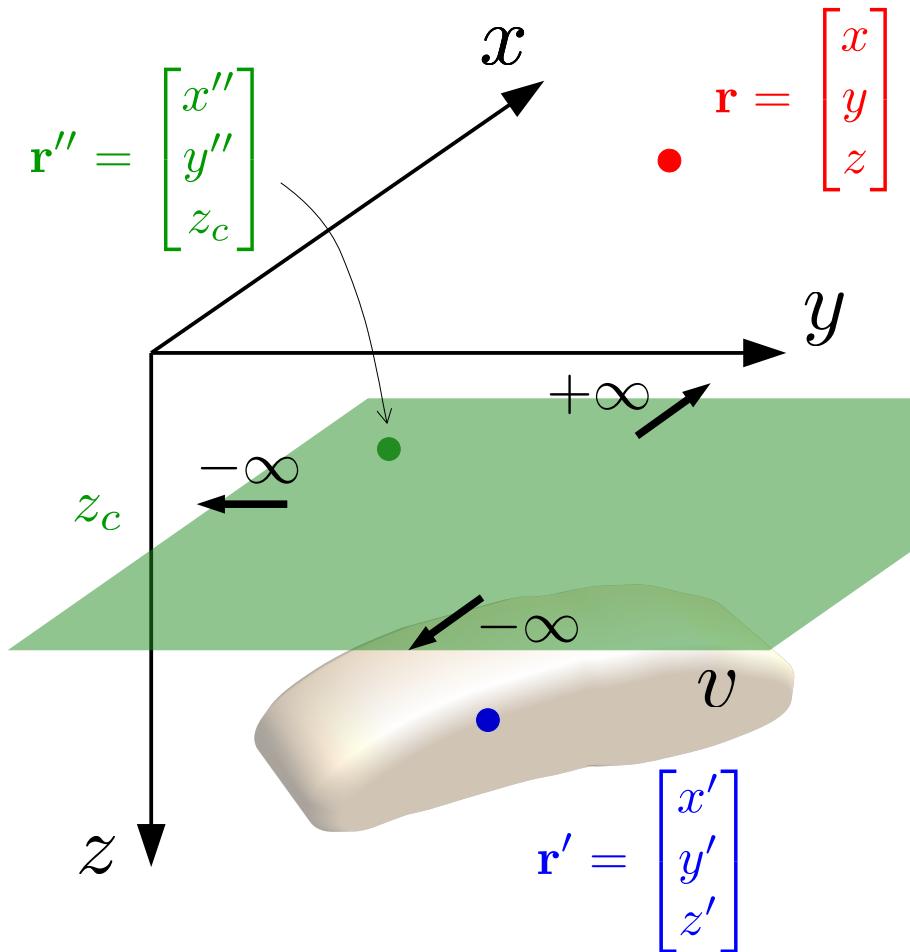
$$\partial_z \Theta(\mathbf{r}) \quad \text{gravity disturbance data}$$

$$\partial_z \Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \partial_z \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

This analytical eq. layer represents the grav. disturbance produced by the true sources on the plane \mathbf{z}_c

$$\partial_z \Theta(\mathbf{r}'') = \kappa \iiint_v \sigma(\mathbf{r}') \partial_z \frac{1}{\|\mathbf{r}'' - \mathbf{r}'\|} dv'$$

Result 1/4: The gravity disturbance can be exactly reproduced by an analytical eq. layer of monopoles



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

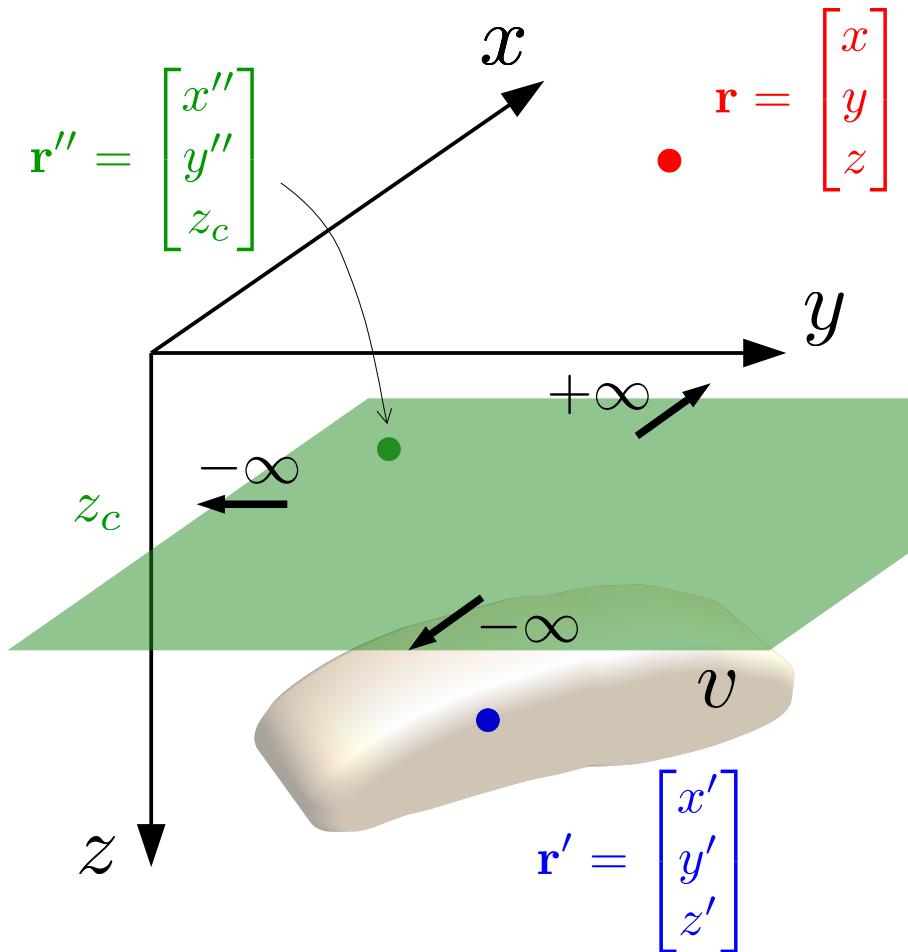
Deduction:

$$\partial_z \Theta(\mathbf{r}) \quad \text{gravity disturbance data}$$

$$\partial_z \Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \partial_z \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$$d_i \approx \sum_{j=1}^M p_j \partial_z \Psi(\mathbf{r}_i, \mathbf{r}'_j)$$

Result 1/4: The gravity disturbance can be exactly reproduced by an analytical eq. layer of monopoles



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

$\partial_z \Theta(\mathbf{r})$ gravity disturbance data

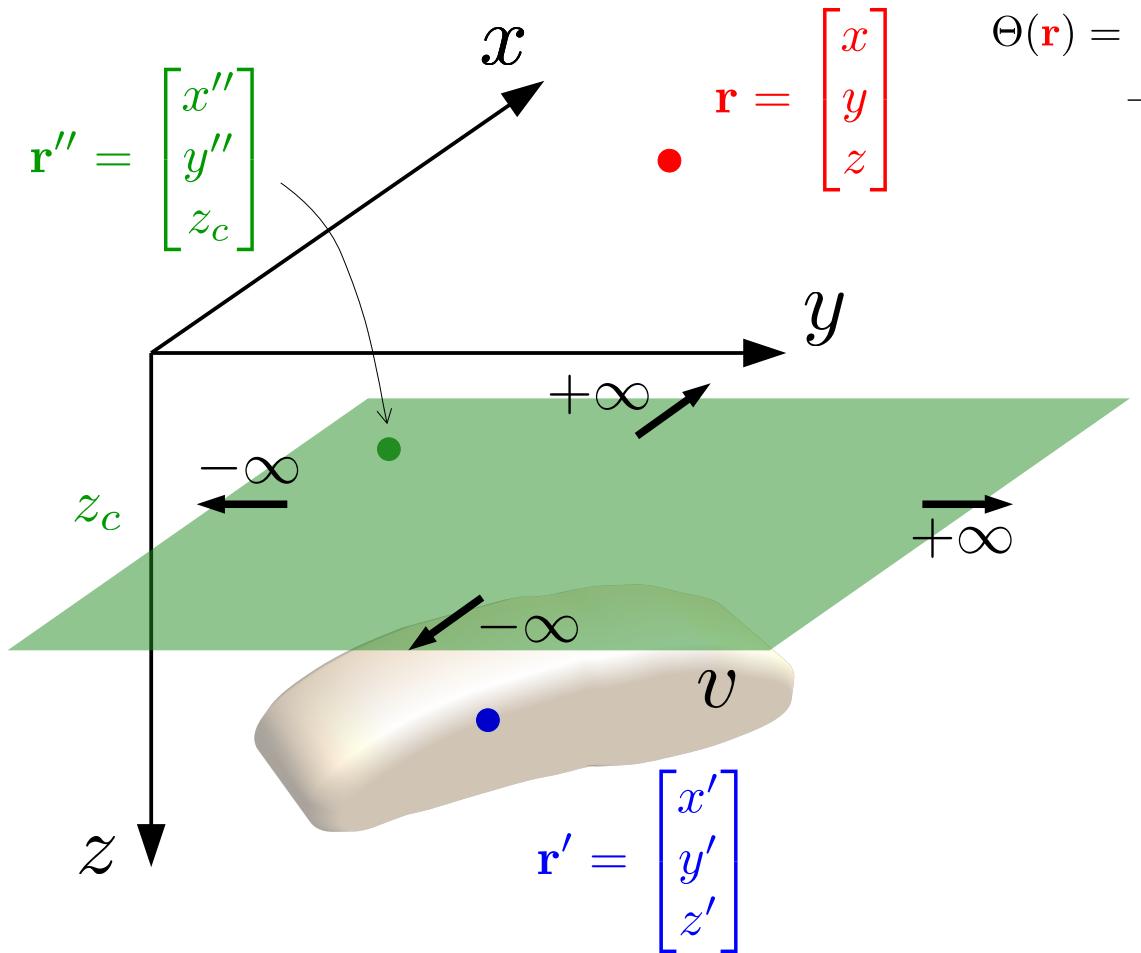
$$\partial_z \Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \partial_z \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$$d_i \approx \sum_{j=1}^M p_j \partial_z \Psi(\mathbf{r}_i, \mathbf{r}'_j)$$

\curvearrowright

$$\partial_z \Theta(\mathbf{r}_i)$$

Result 1/4: The gravity disturbance can be exactly reproduced by an analytical eq. layer of monopoles



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

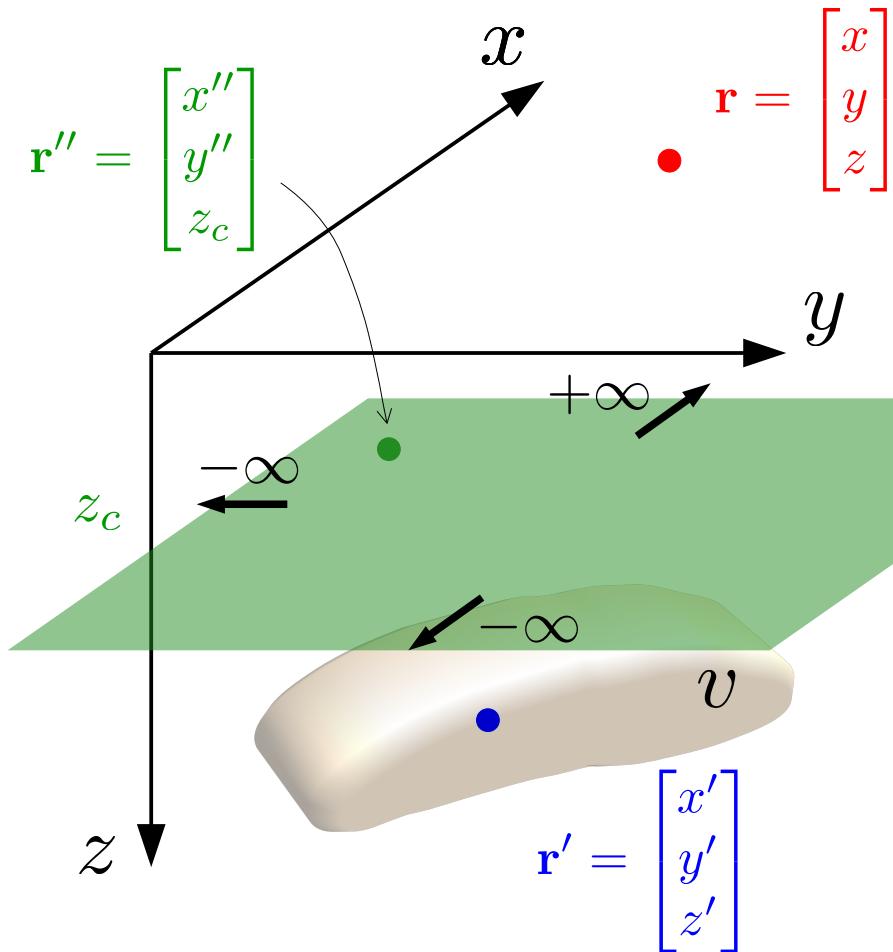
Deduction:

$\partial_z \Theta(\mathbf{r})$ gravity disturbance data

$$\partial_z \Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \partial_z \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$$d_i \approx \sum_{j=1}^M p_j \partial_z \Psi(\mathbf{r}_i, \mathbf{r}_j'') \approx \partial_z \Theta(\mathbf{r}_j'')$$

Result 1/4: The gravity disturbance can be exactly reproduced by an analytical eq. layer of monopoles



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

$\partial_z \Theta(\mathbf{r})$ gravity disturbance data

$$\partial_z \Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \partial_z \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

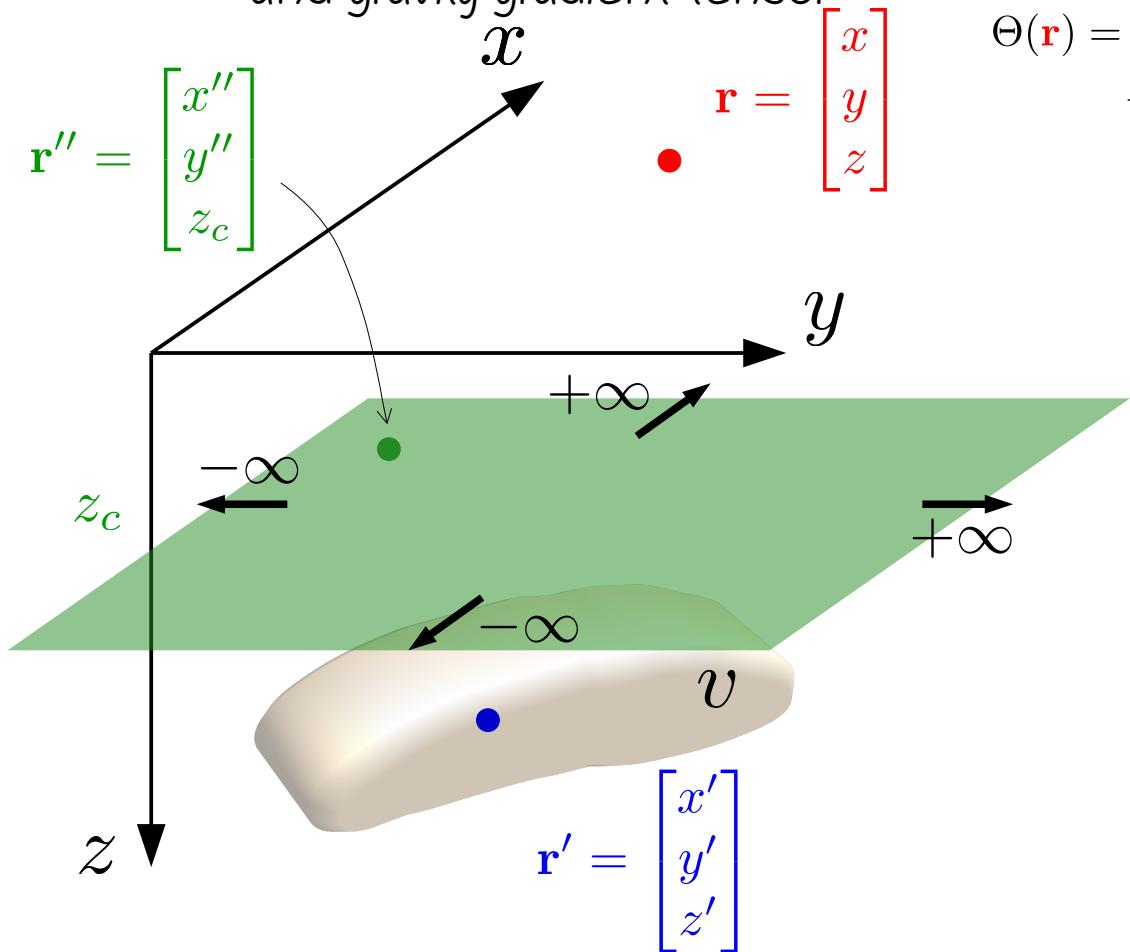
$$d_i \approx \sum_{j=1}^M p_j \partial_z \Psi(\mathbf{r}_i, \mathbf{r}'_j)$$

classical Eql technique applied to gravity data

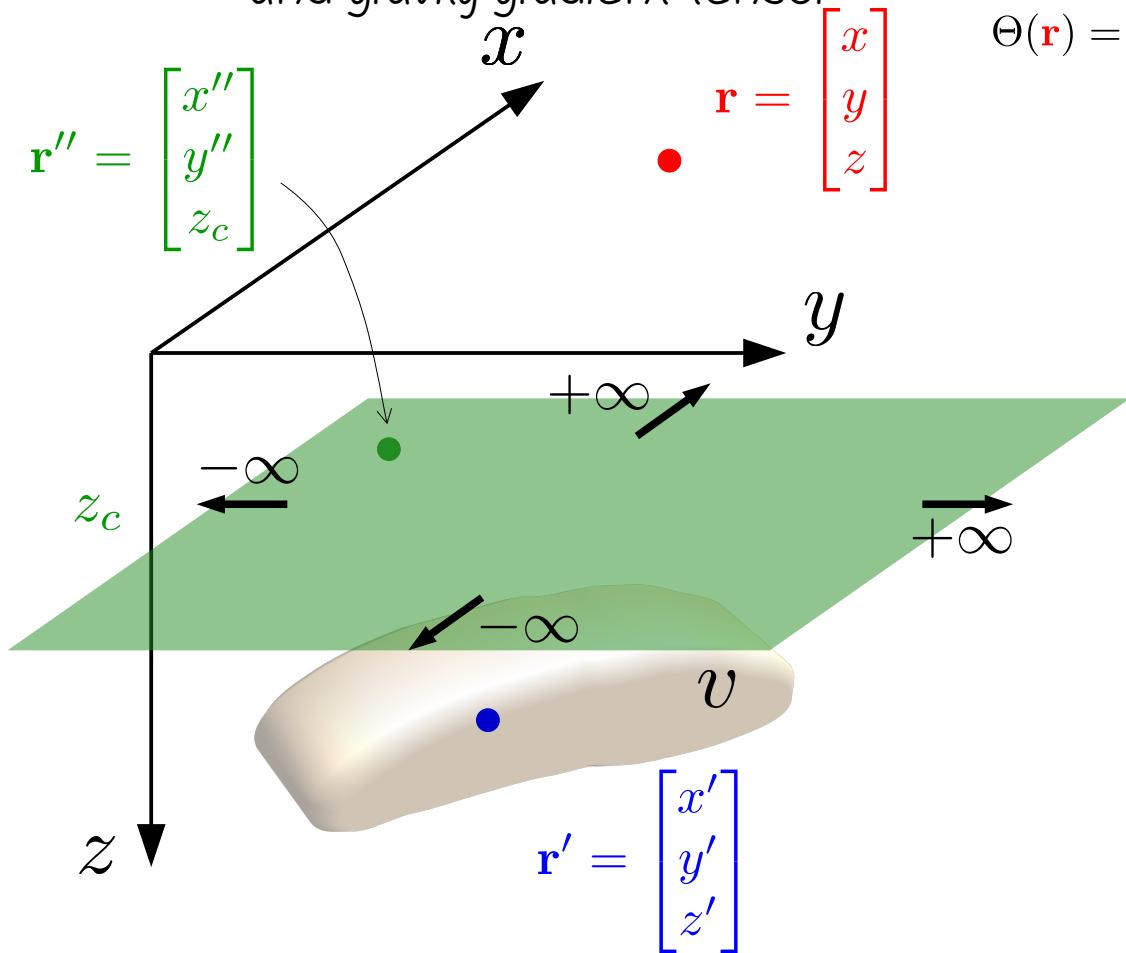
Result 2/4: The same analytical eq. layer of monopoles reproducing the gravity disturbance can also reproduce the gravitational potential and gravity gradient tensor

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$



Result 2/4: The same analytical eq. layer of monopoles reproducing the gravity disturbance can also reproduce the gravitational potential and gravity gradient tensor

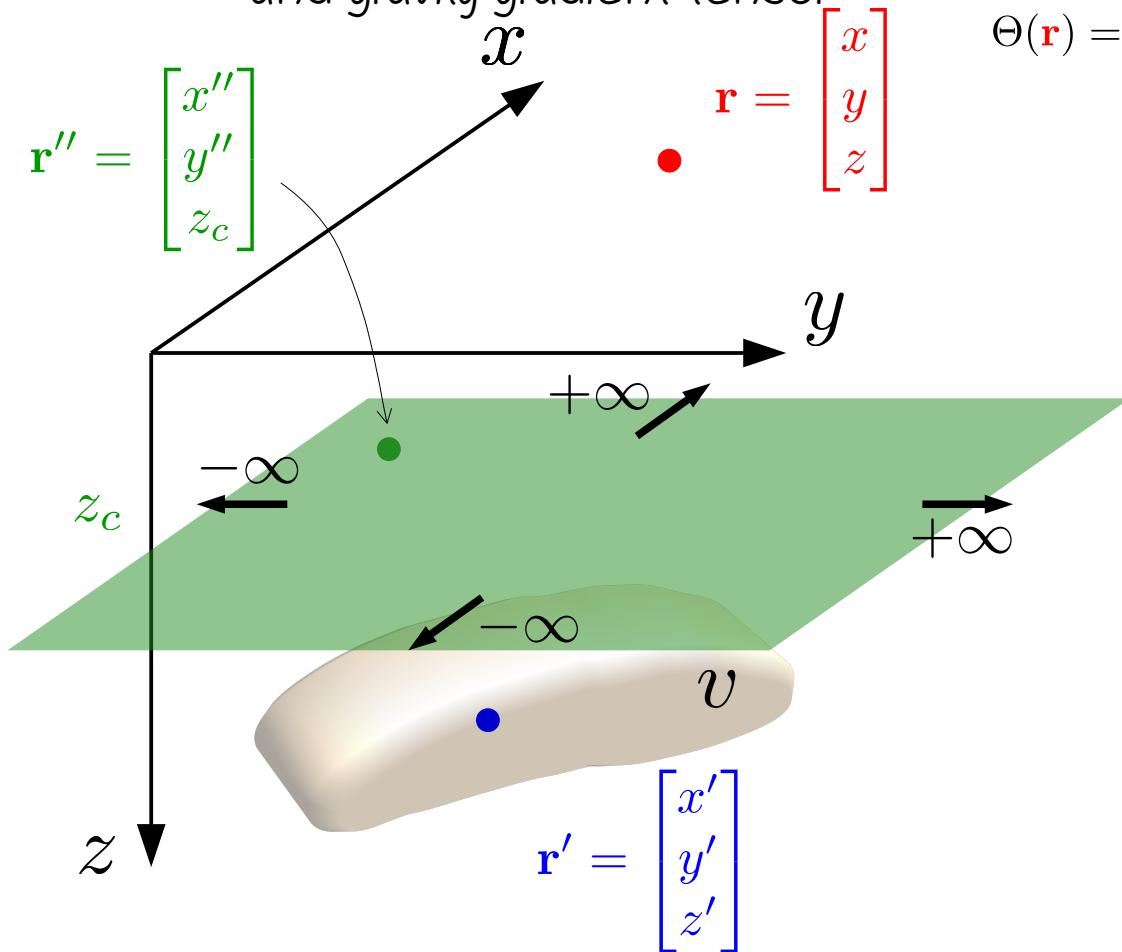


$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

This result has been used by the scientific community for processing gravity-gradient data (e.g., [Barnes and Lumley, 2011](#)) or converting gravity disturbance into the gravity-gradient tensor (e.g., [Piauillino et al., 2019](#)), for example

Result 2/4: The same analytical eq. layer of monopoles reproducing the gravity disturbance can also reproduce the gravitational potential and gravity gradient tensor



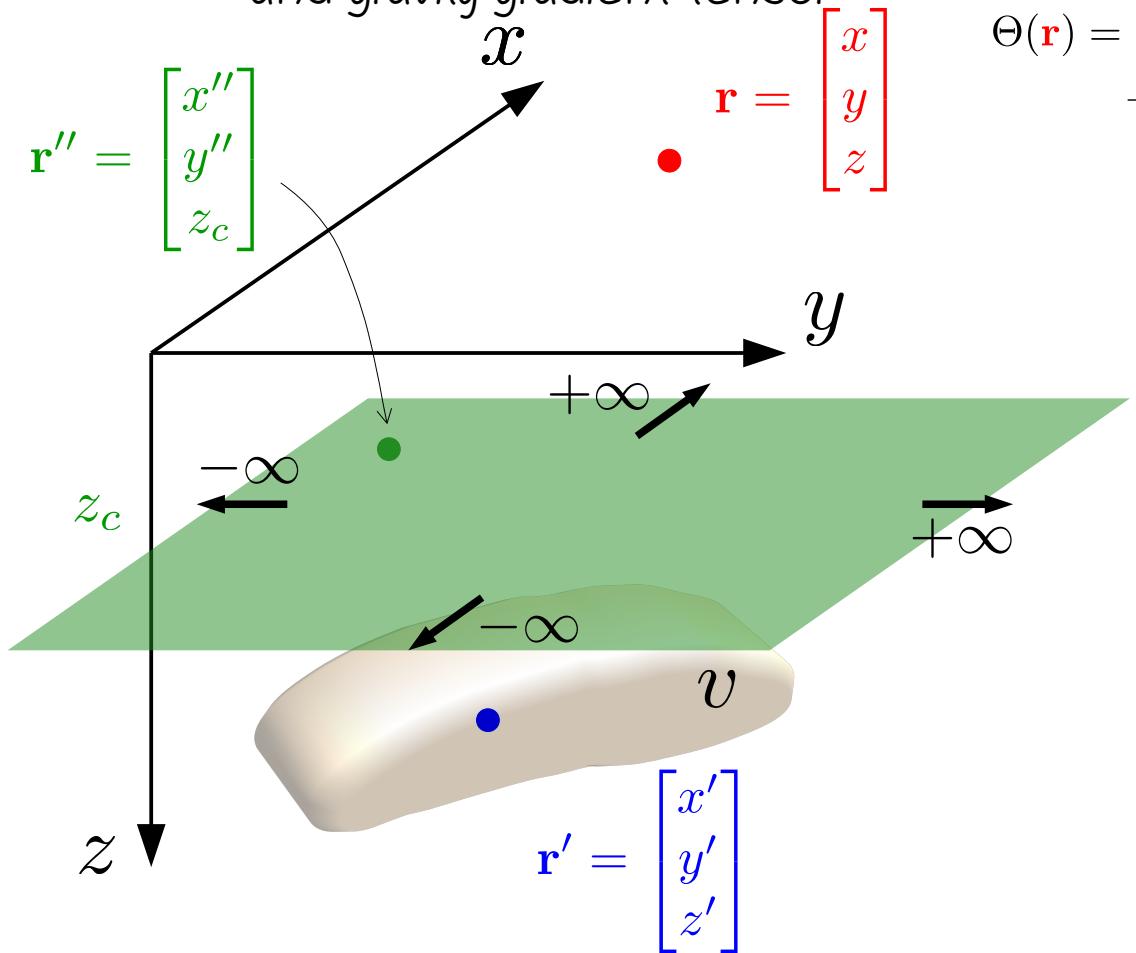
$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

This result has been used by the scientific community for processing gravity-gradient data (e.g., [Barnes and Lumley, 2011](#)) or converting gravity disturbance into the gravity-gradient tensor (e.g., [Piauillino et al., 2019](#)), for example

However, researchers have not treated this topic in much detail and it is usually assumed to be true without any proof

Result 2/4: The same analytical eq. layer of monopoles reproducing the gravity disturbance can also reproduce the gravitational potential and gravity gradient tensor



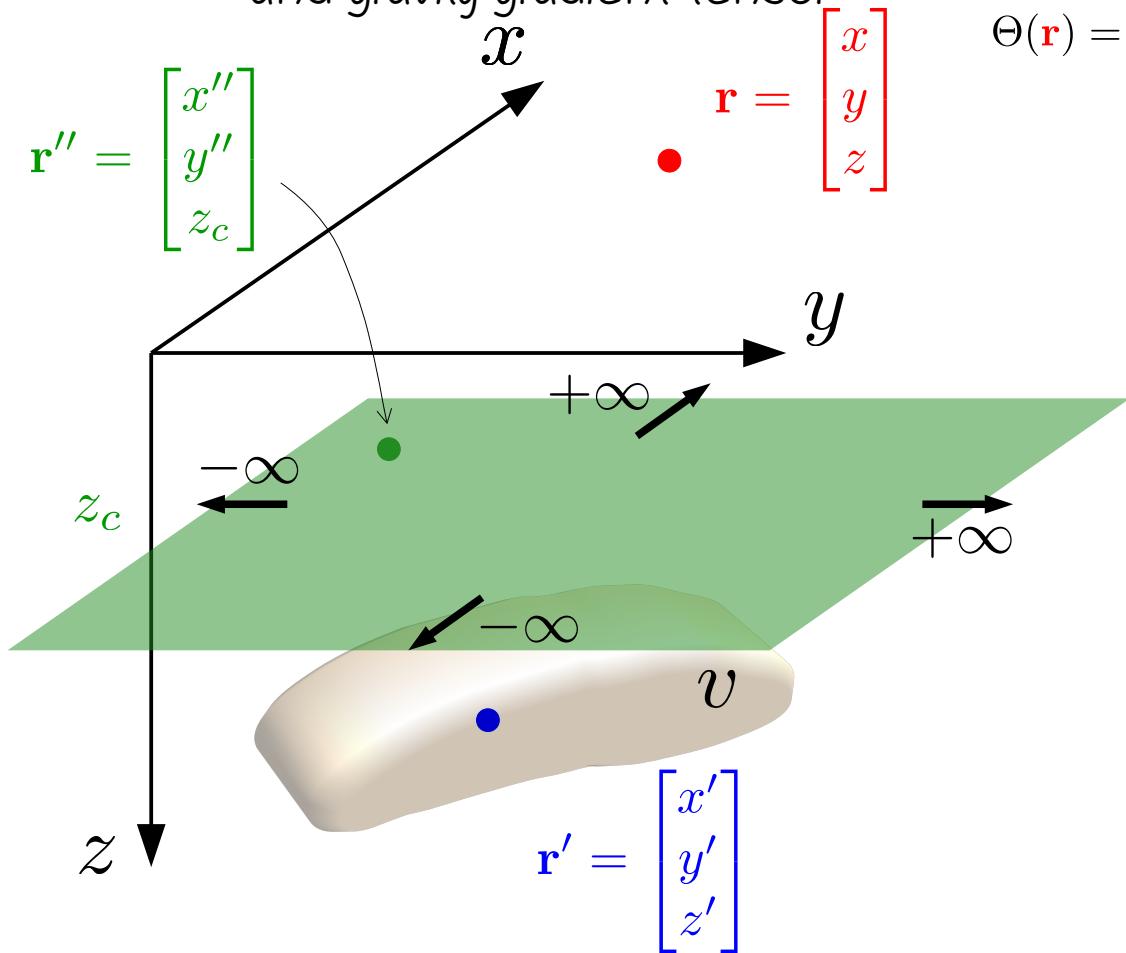
$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

$\Theta(\mathbf{r})$ gravitational potential

Result 2/4: The same analytical eq. layer of monopoles reproducing the gravity disturbance can also reproduce the gravitational potential and gravity gradient tensor



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

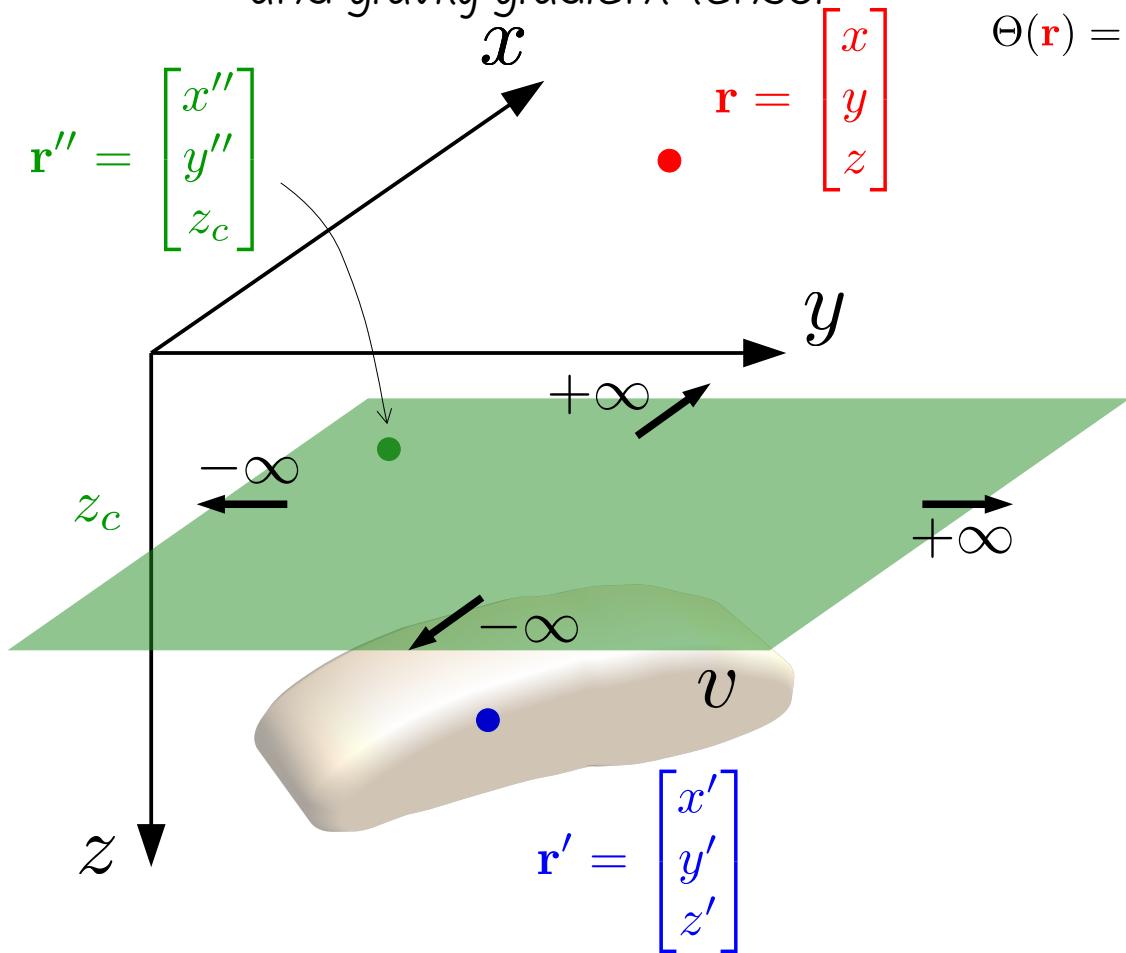
$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

$\Theta(\mathbf{r})$ gravitational potential

$$\Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

Result 2/4: The same analytical eq. layer of monopoles reproducing the gravity disturbance can also reproduce the gravitational potential and gravity gradient tensor



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

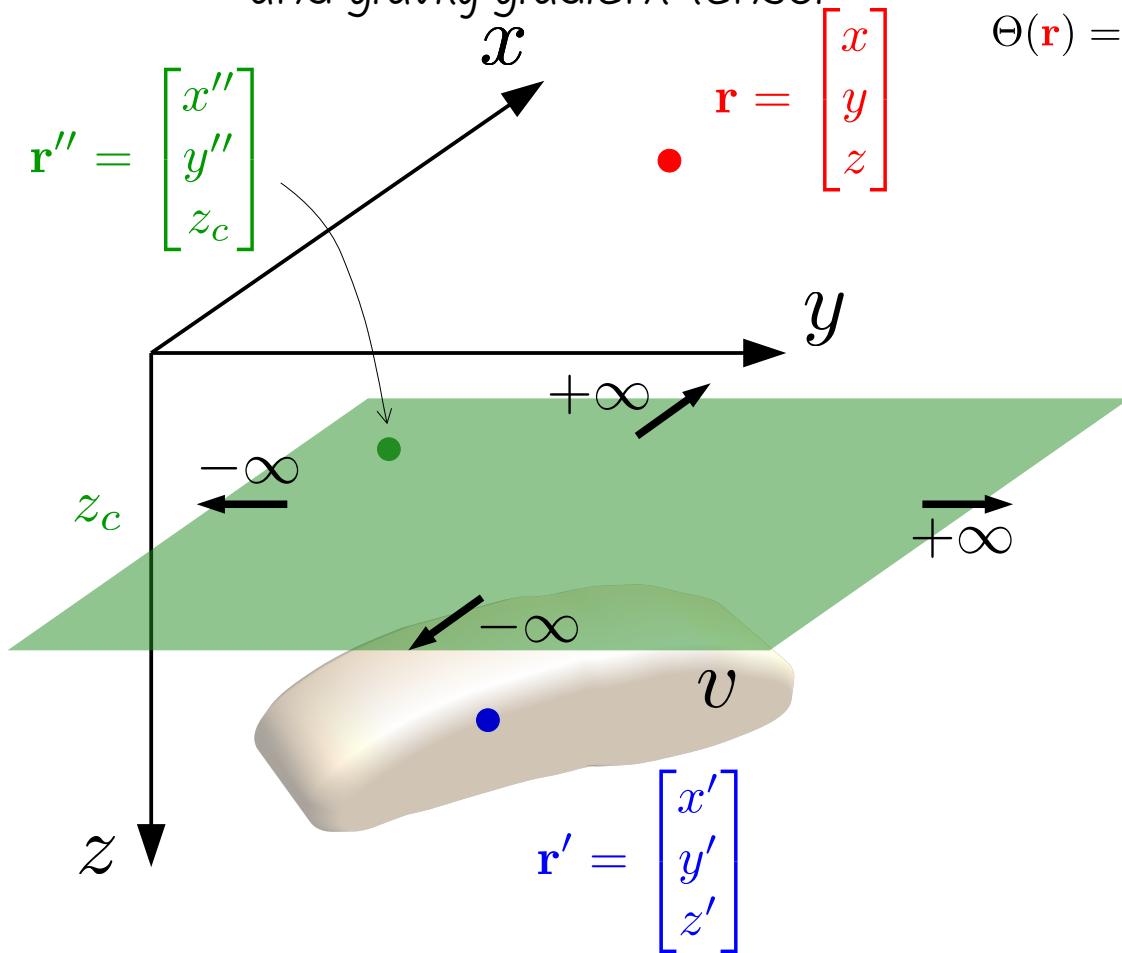
$\Theta(\mathbf{r})$ gravitational potential

$$\Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$\partial_z \Theta(\mathbf{r})$ gravity disturbance

$$\partial_z \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_z \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

Result 2/4: The same analytical eq. layer of monopoles reproducing the gravity disturbance can also reproduce the gravitational potential and gravity gradient tensor



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

$\Theta(\mathbf{r})$ gravitational potential

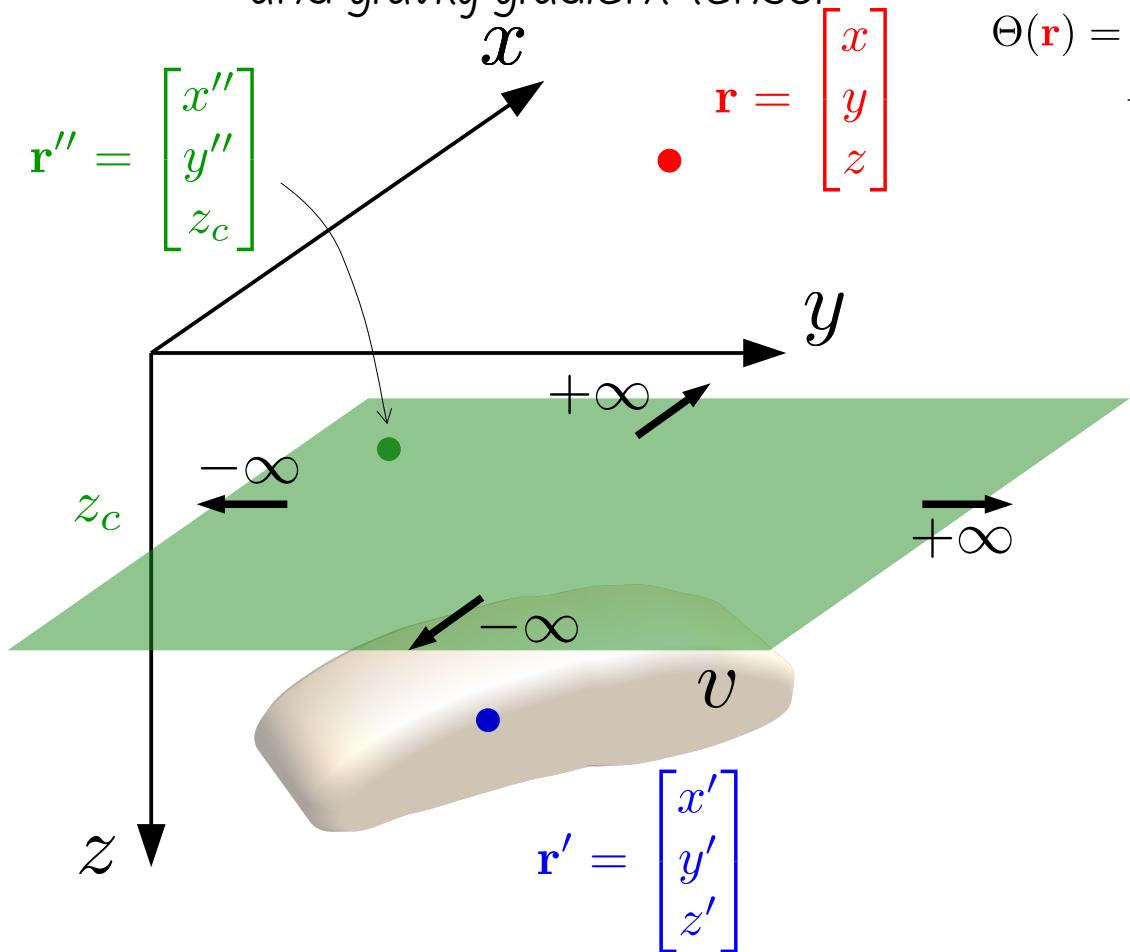
$$\Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$\partial_z \Theta(\mathbf{r})$ gravity disturbance

$$\partial_z \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_z \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

Note that this term is not affected by the derivative

Result 2/4: The same analytical eq. layer of monopoles reproducing the gravity disturbance can also reproduce the gravitational potential and gravity gradient tensor



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

$\Theta(\mathbf{r})$ gravitational potential

$$\Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

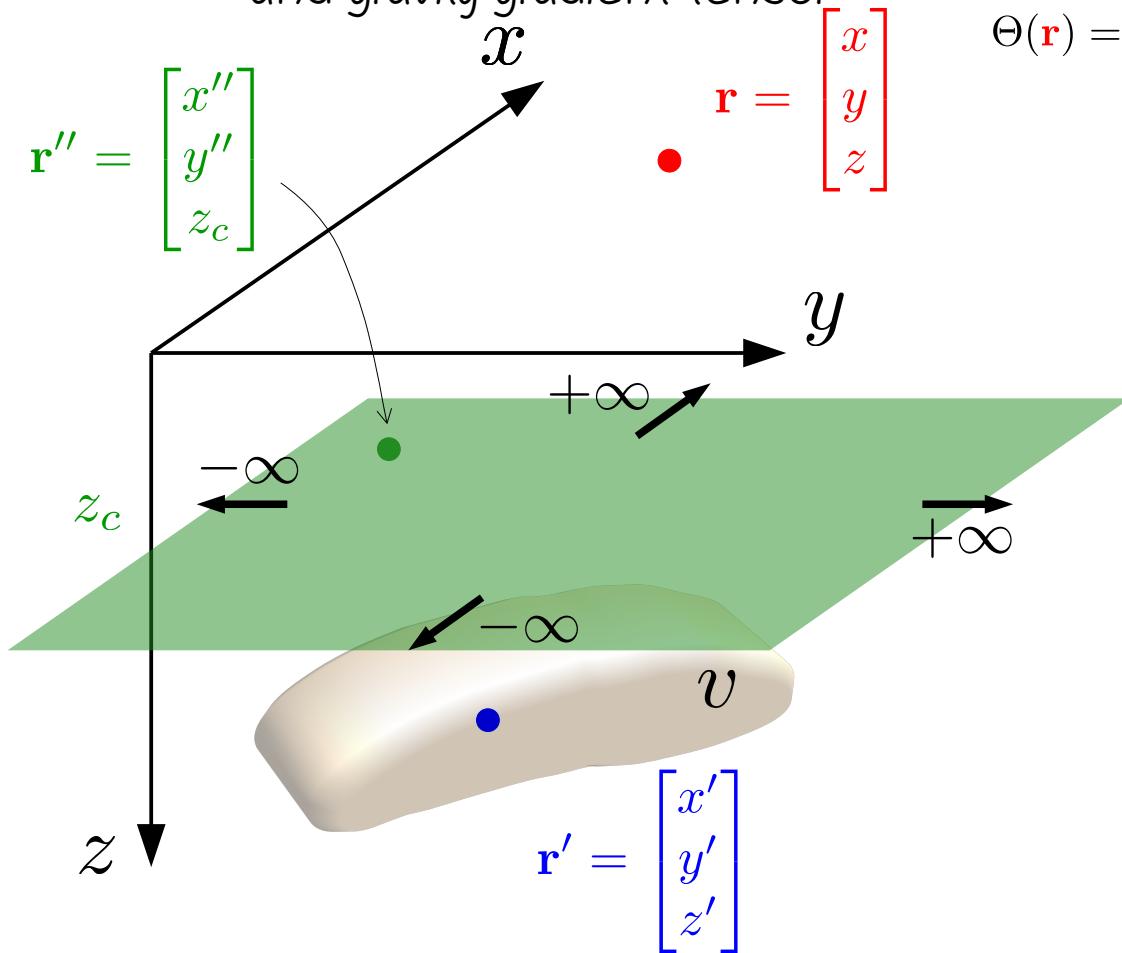
$\partial_z \Theta(\mathbf{r})$ gravity disturbance

$$\partial_z \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_z \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$\partial_{\alpha\beta} \Theta(\mathbf{r})$ gravity-gradient tensor component

$$\partial_{\alpha\beta} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{\alpha\beta} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

Result 2/4: The same analytical eq. layer of monopoles reproducing the gravity disturbance can also reproduce the gravitational potential and gravity gradient tensor



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

$\Theta(\mathbf{r})$ gravitational potential

$$\Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$\partial_z \Theta(\mathbf{r})$ gravity disturbance

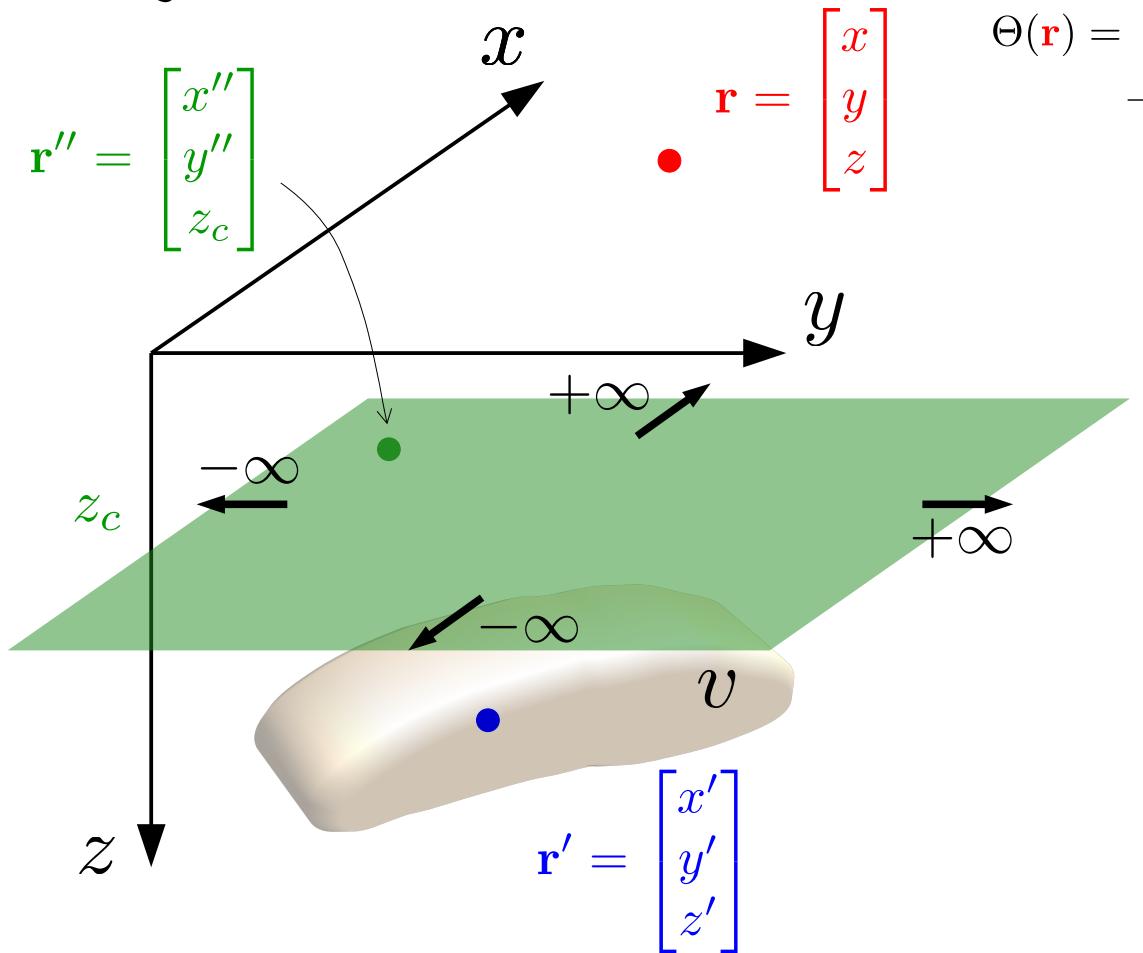
$$\partial_z \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_z \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$\partial_{\alpha\beta} \Theta(\mathbf{r})$ gravity-gradient tensor component

$$\partial_{\alpha\beta} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{\alpha\beta} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources

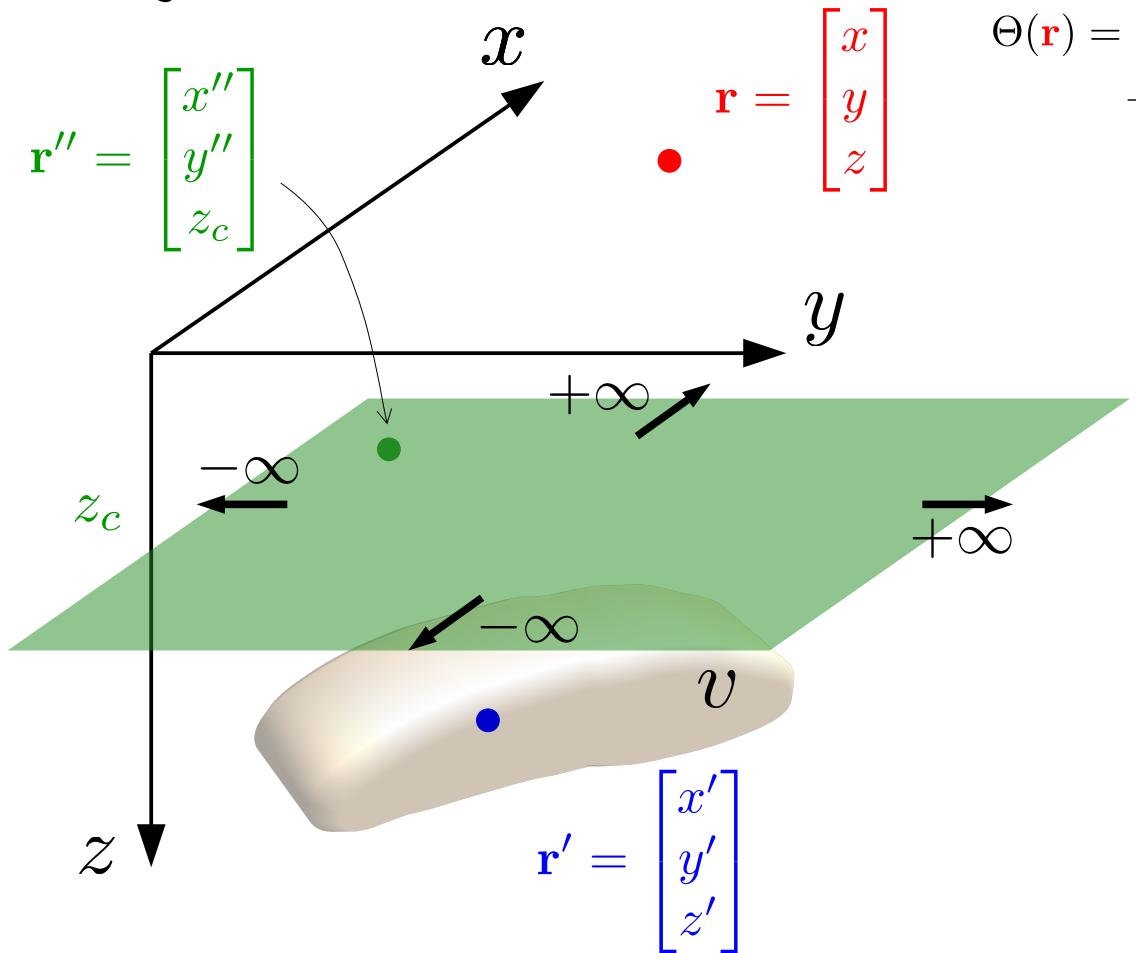
$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$



$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources

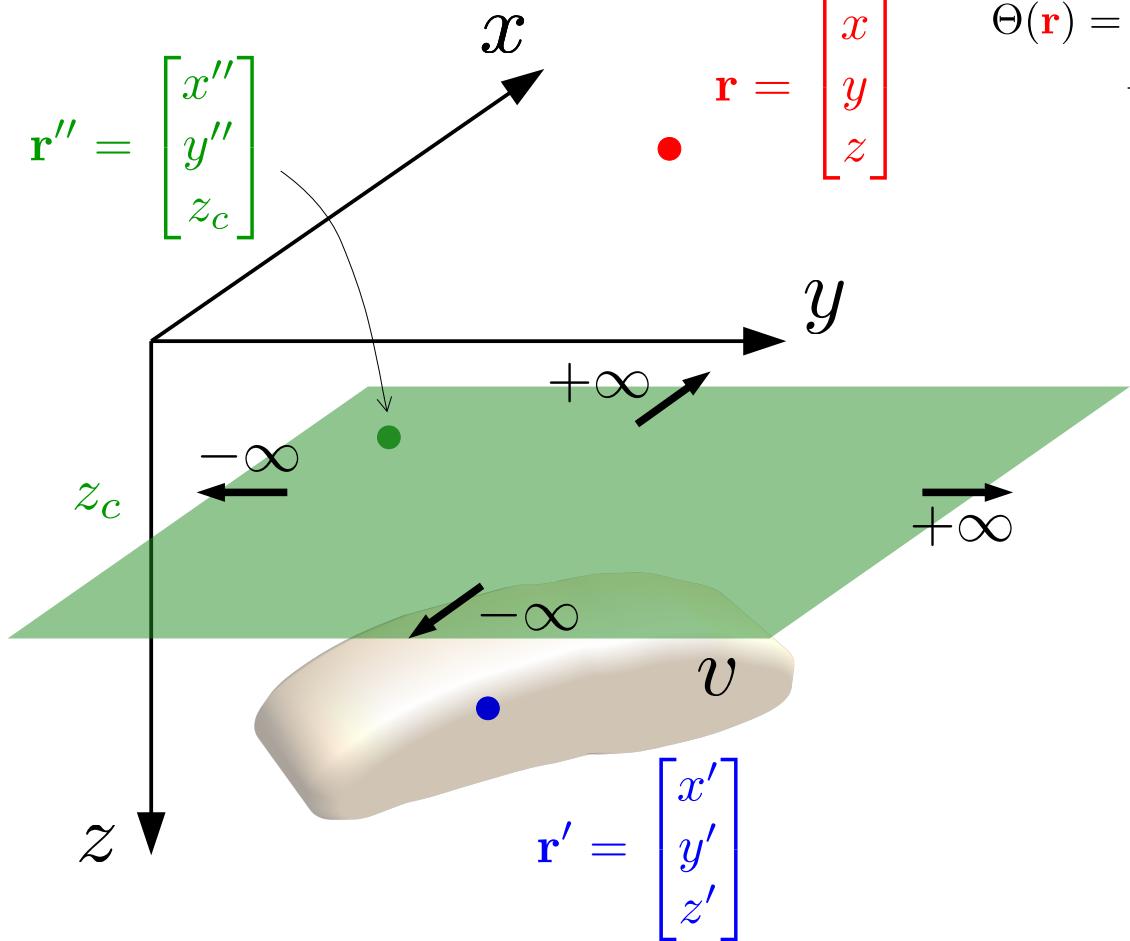
$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$



$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Most previous published studies are limited to the empirical use of a planar eq. layer of dipoles for processing total-field anomaly data

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



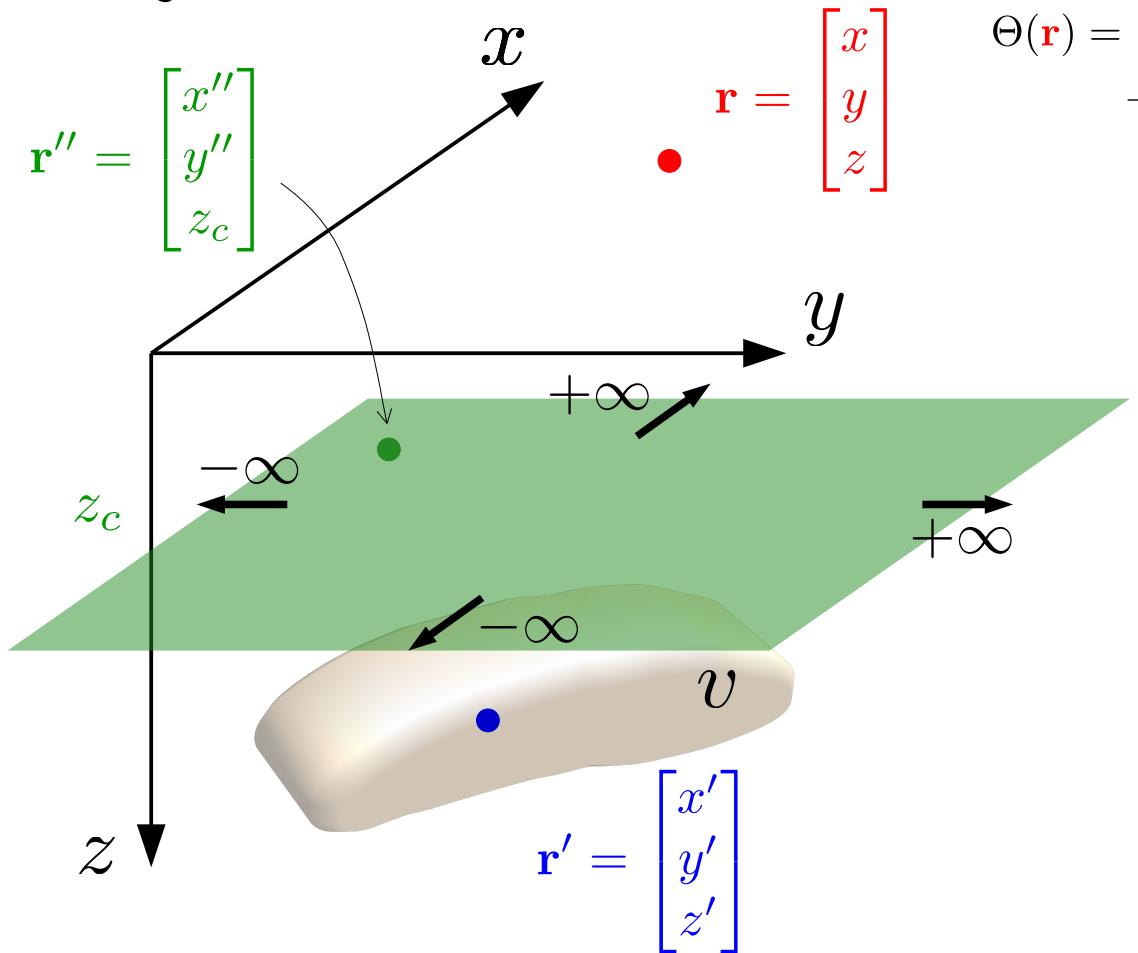
$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Most previous published studies are limited to the empirical use of a planar eq. layer of dipoles for processing total-field anomaly data

Few studies (e.g., [Pedersen, 1991](#); [Li et al., 2014](#); [Reis et al., 2020](#)) have drawn attention to the problem of proving the existence of a planar eq. layer of dipoles that exactly reproduces the approx total-field anomaly

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

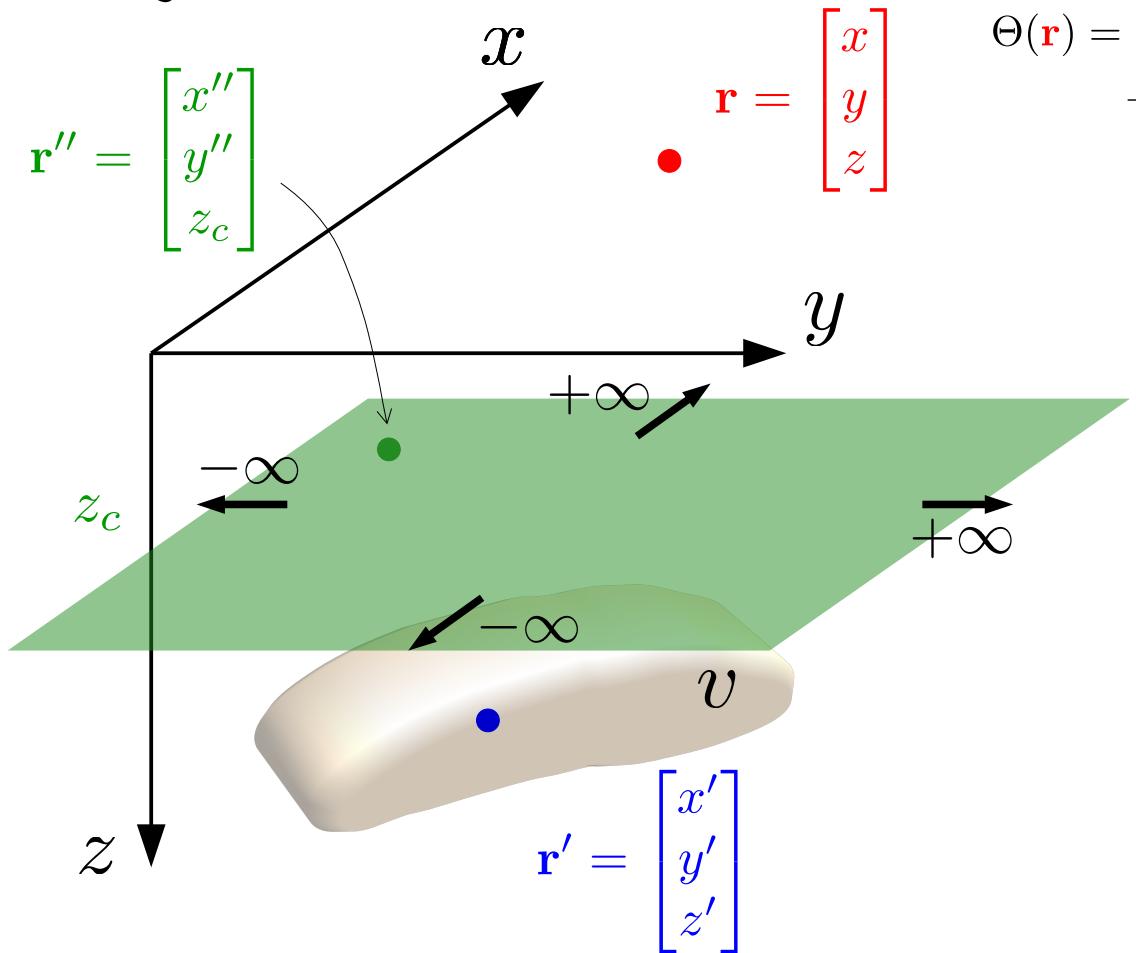
$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

approx total-field anomaly

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

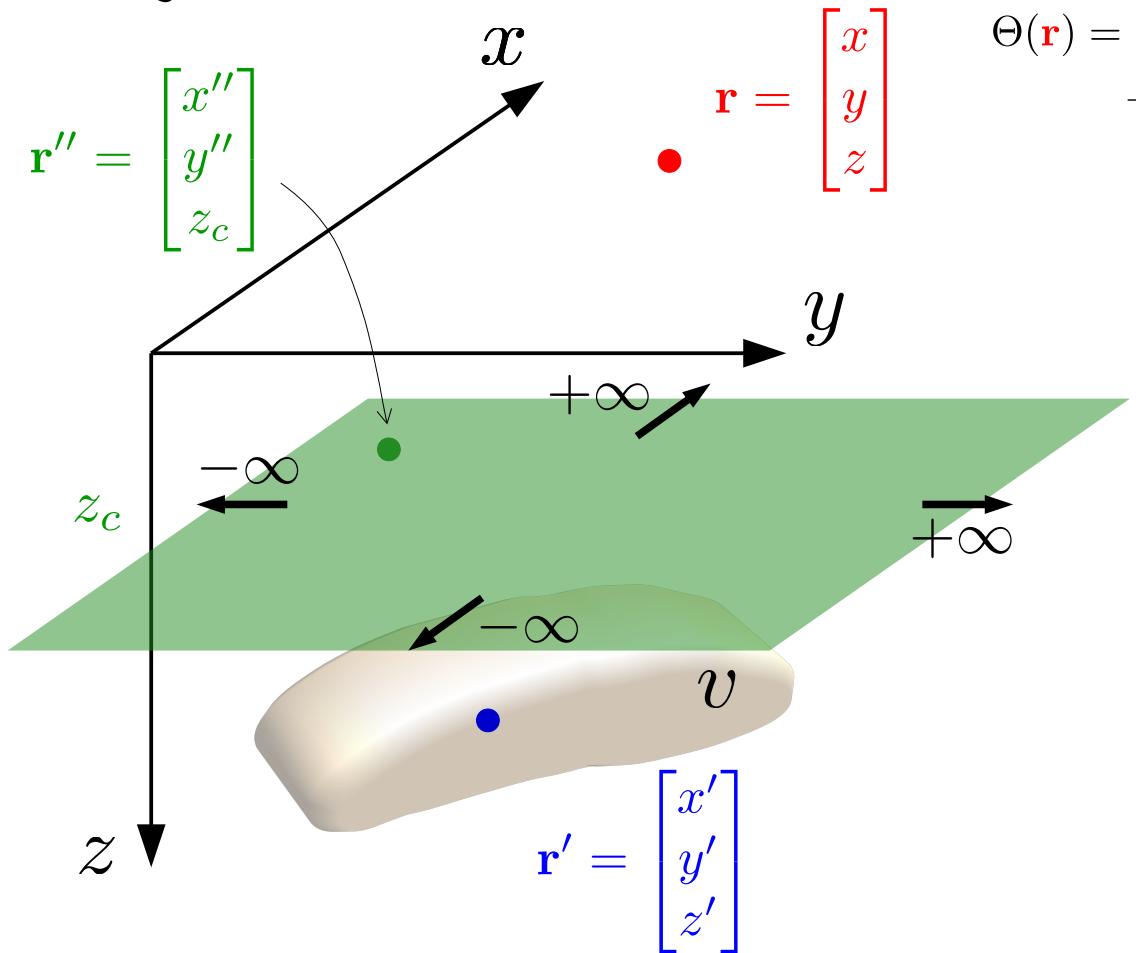
Deduction:

approx total-field anomaly

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

Unit vector defining
the constant direction
of the main
geomagnetic field

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

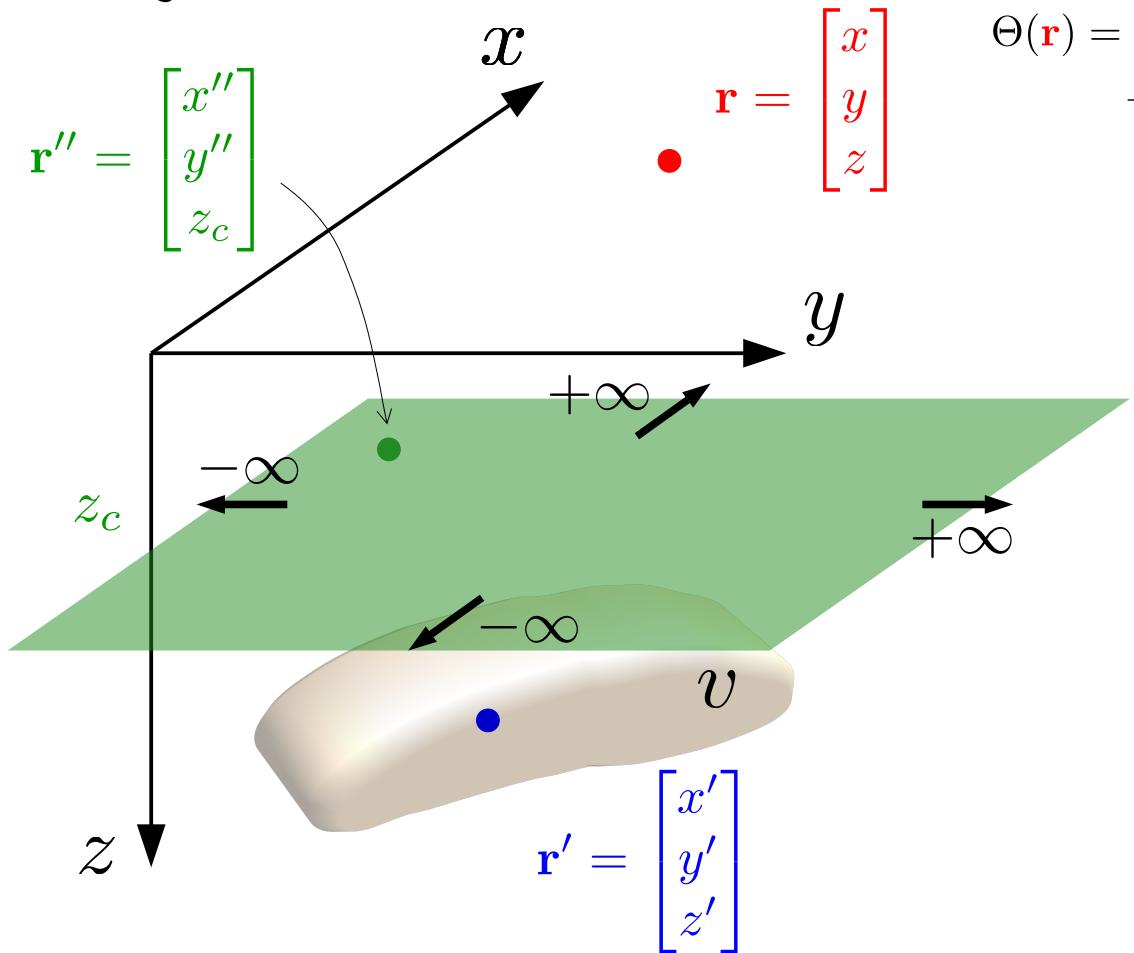
Deduction:

approx total-field anomaly

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

Unit vector defining the uniform total-magnetization direction of the true sources

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

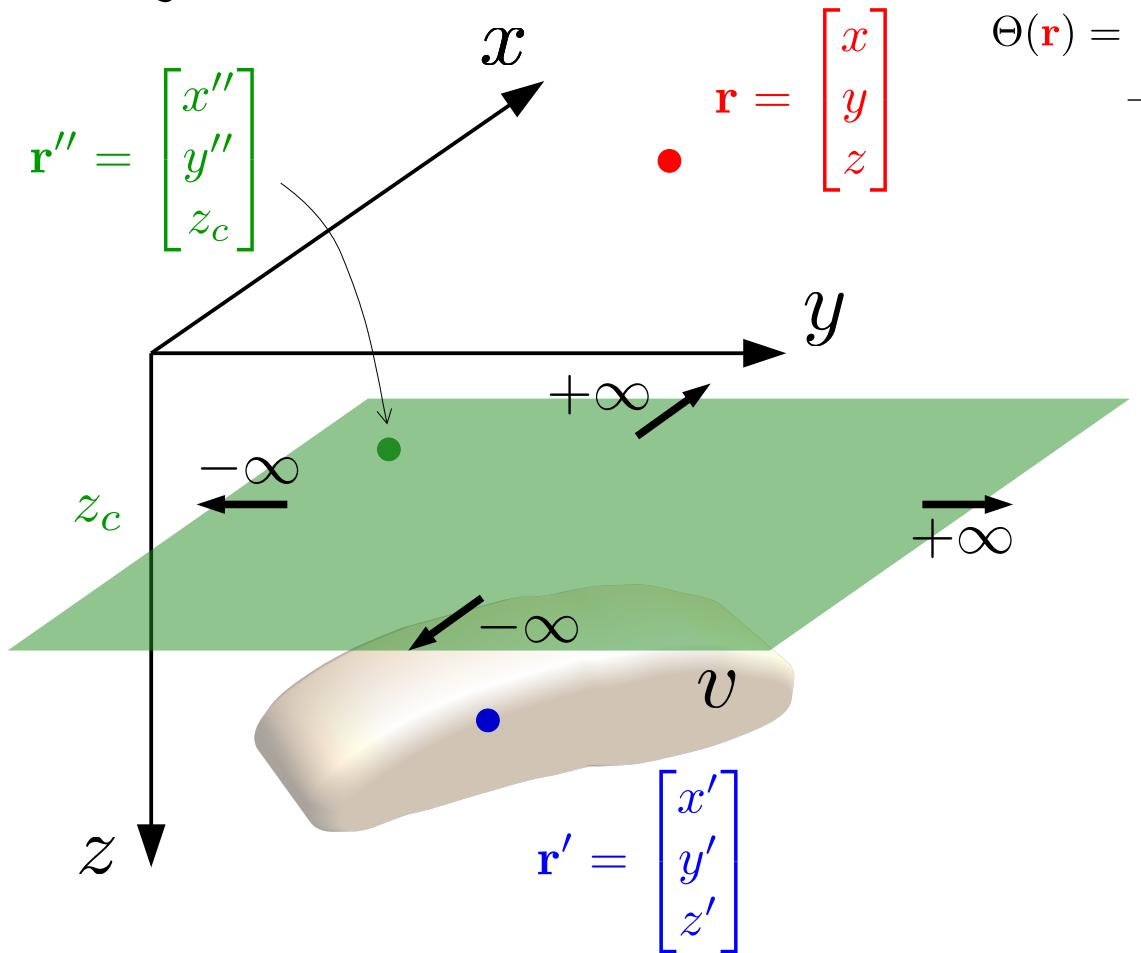
Deduction:

approx total-field anomaly

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

$$\mathbf{H}_\Theta(\mathbf{r}) = \begin{bmatrix} \partial_{xx} \Theta(\mathbf{r}) & \partial_{xy} \Theta(\mathbf{r}) & \partial_{xz} \Theta(\mathbf{r}) \\ \partial_{xy} \Theta(\mathbf{r}) & \partial_{yy} \Theta(\mathbf{r}) & \partial_{yz} \Theta(\mathbf{r}) \\ \partial_{xz} \Theta(\mathbf{r}) & \partial_{yz} \Theta(\mathbf{r}) & \partial_{zz} \Theta(\mathbf{r}) \end{bmatrix}$$

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

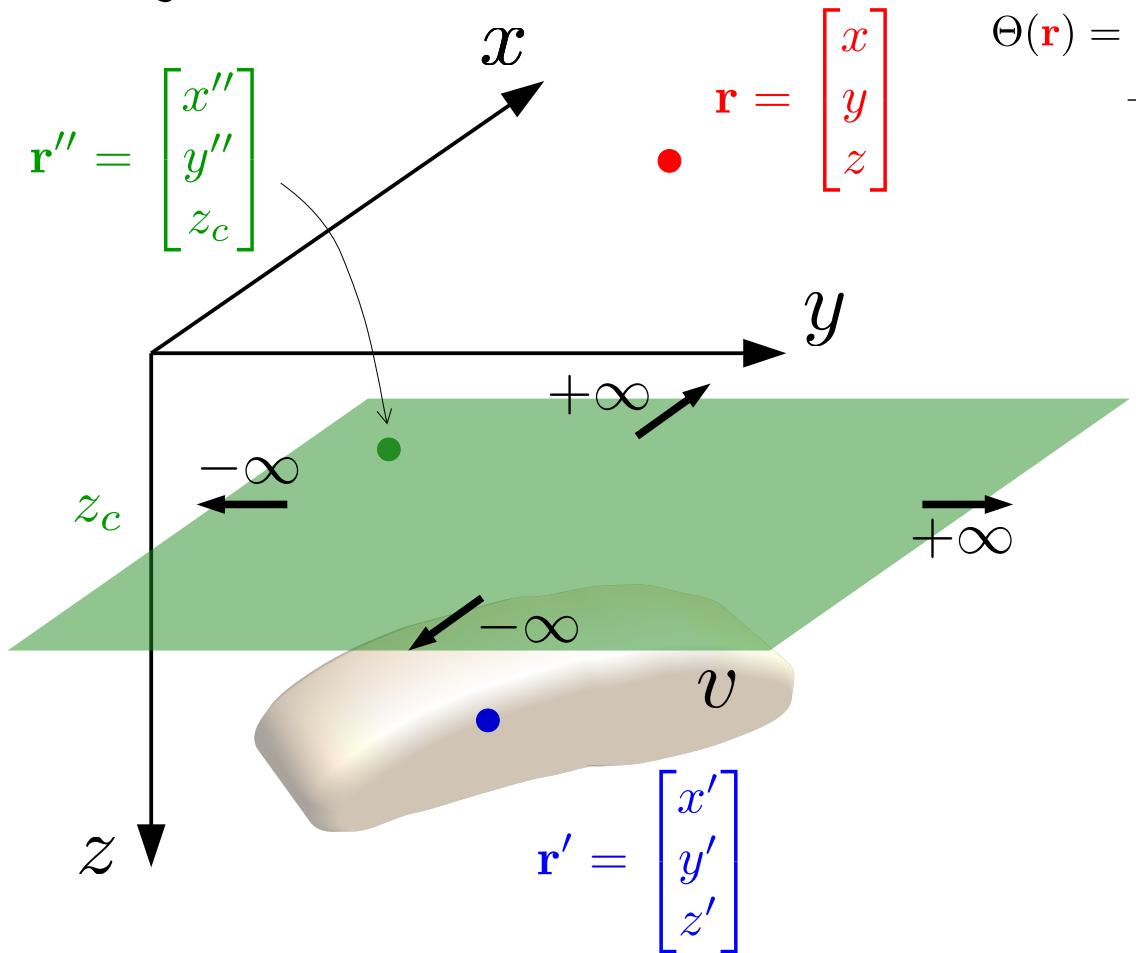
Deduction:

approx total-field anomaly

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

$$\partial_{th} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

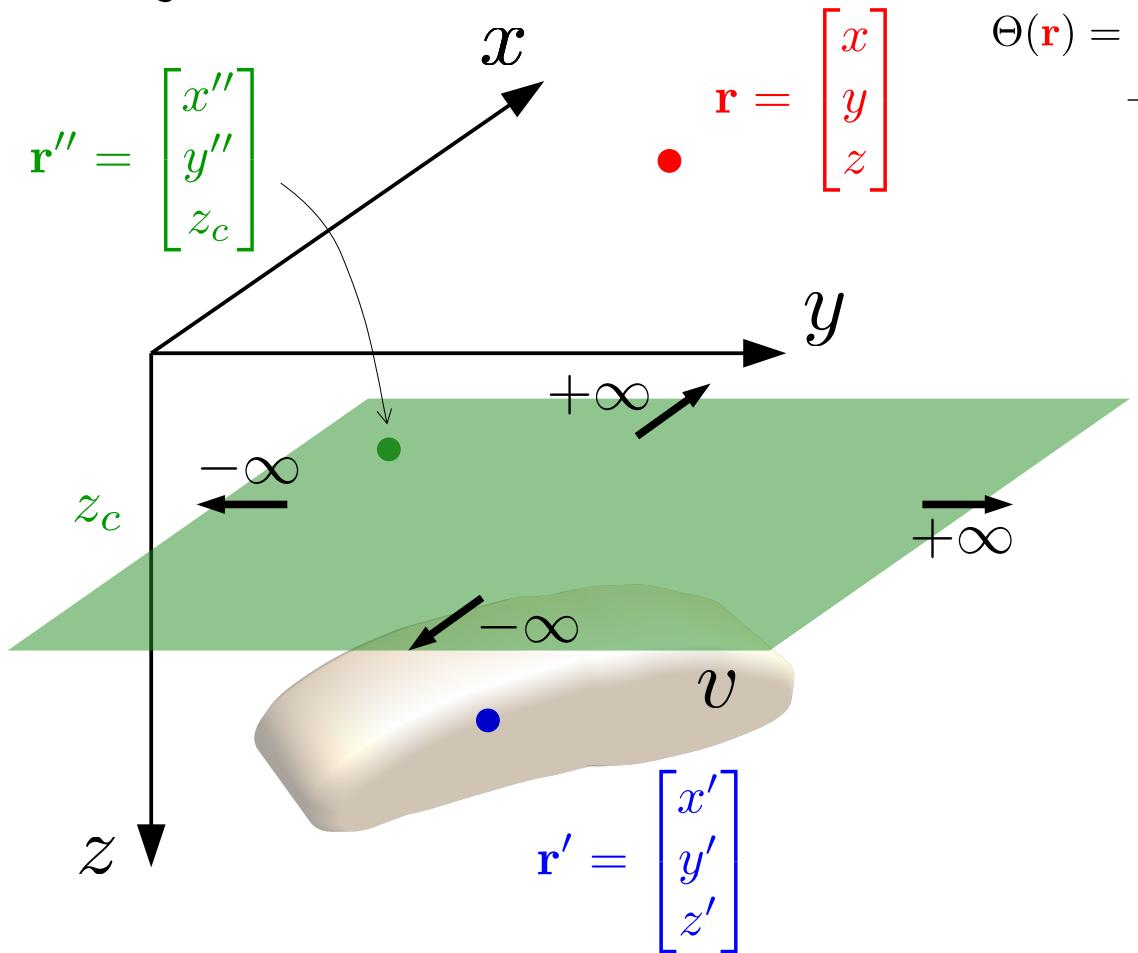
approx total-field anomaly

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

$$\partial_{th} \Theta(\mathbf{r}) = \iint \underbrace{\partial_z \Theta(\mathbf{r}'')}_{\partial_{th} \Psi(\mathbf{r}, \mathbf{r}'')} \partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

This term represents the approx total-field anomaly produced at \mathbf{r} by a dipole located at \mathbf{r}''

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

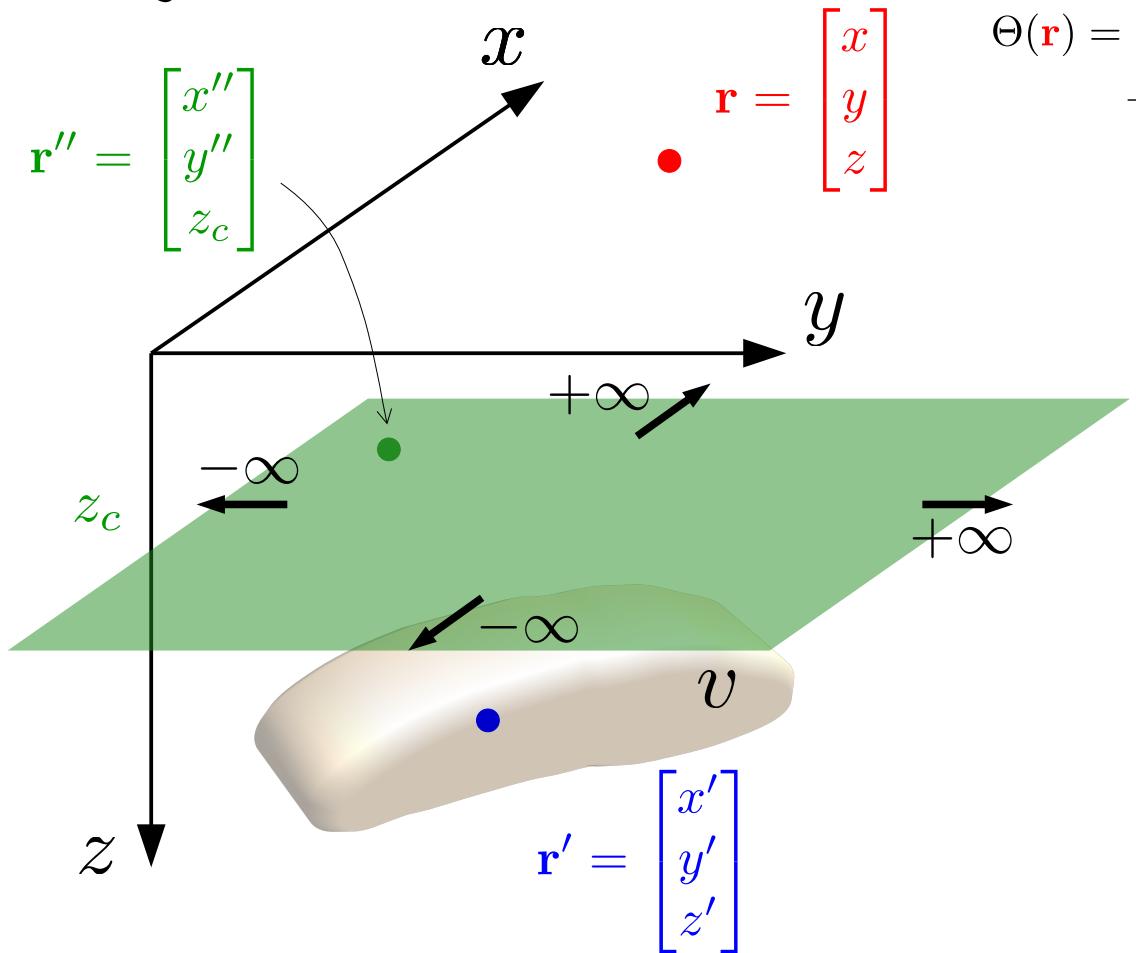
approx total-field anomaly

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

$$\partial_{th} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$$\partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') = \hat{\mathbf{t}}^\top \mathbf{H}_\Psi(\mathbf{r}, \mathbf{r}'') \hat{\mathbf{h}}$$

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

approx total-field anomaly

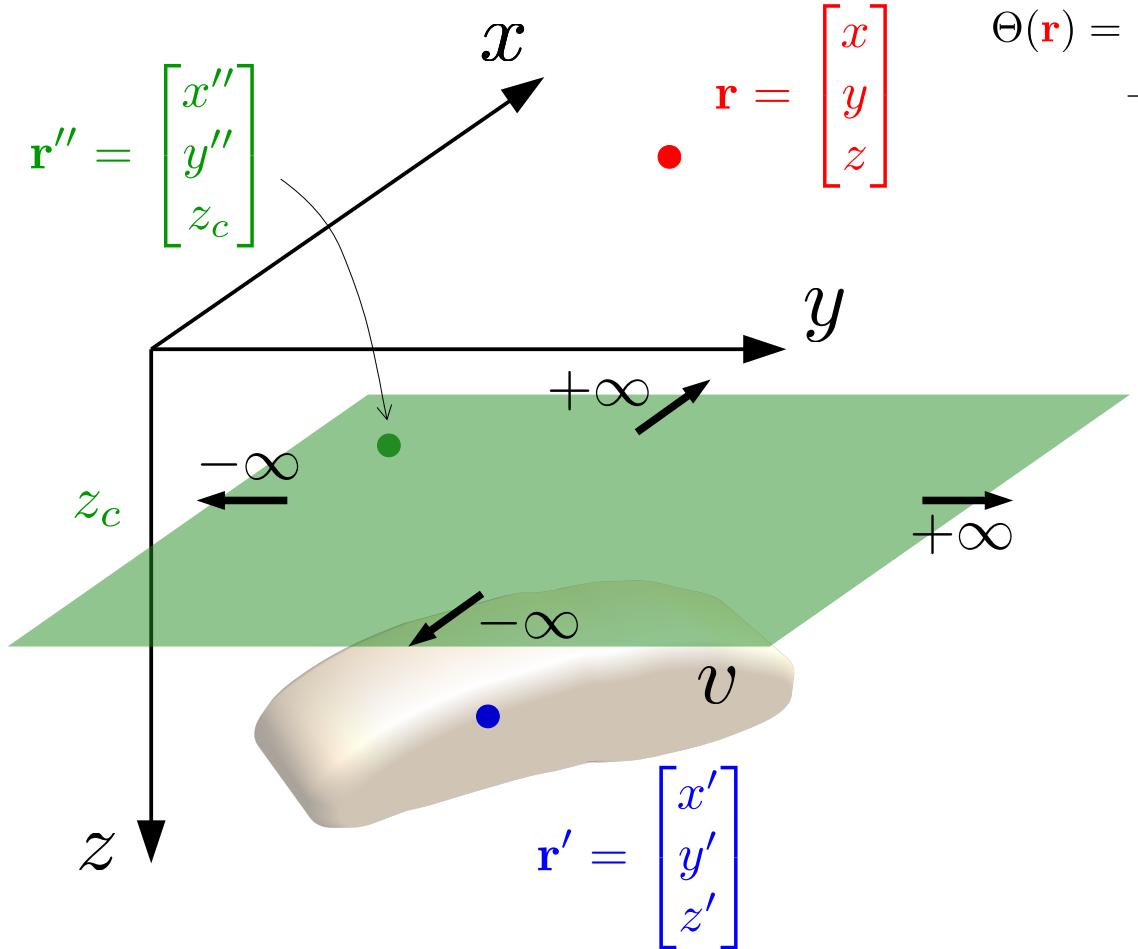
$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

$$\partial_{th} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$$\partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') = \hat{\mathbf{t}}^\top \mathbf{H}_\Psi(\mathbf{r}, \mathbf{r}'') \hat{\mathbf{h}}$$

Unit vector defining the same uniform total-magnetization direction of the true sources

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

approx total-field anomaly

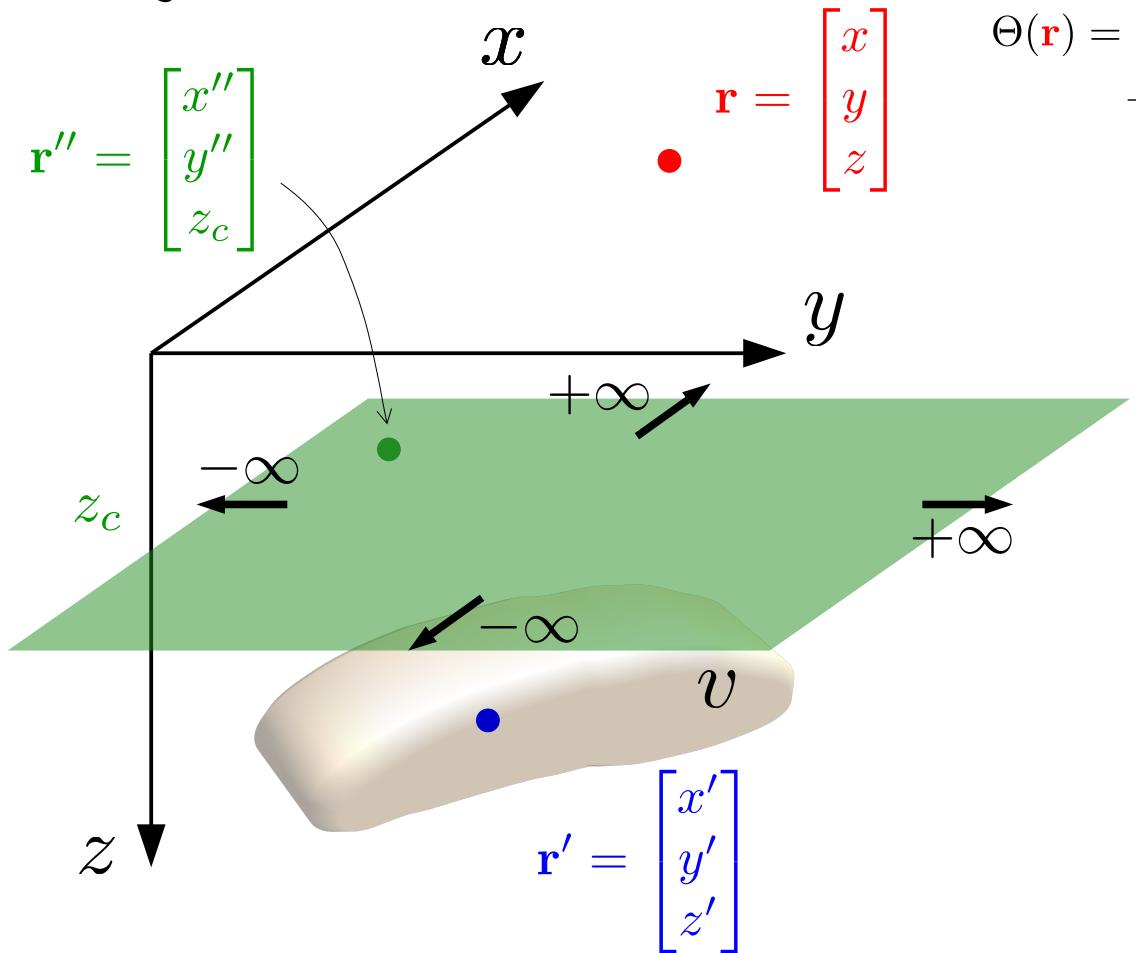
$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

$$\partial_{th} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$$\partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') = \hat{\mathbf{t}}^\top \mathbf{H}_\Psi(\mathbf{r}, \mathbf{r}'') \hat{\mathbf{h}}$$

$$\mathbf{H}_\Psi(\mathbf{r}, \mathbf{r}'') = \begin{bmatrix} \partial_{xx} \Psi(\mathbf{r}, \mathbf{r}'') & \partial_{xy} \Psi(\mathbf{r}, \mathbf{r}'') & \partial_{xz} \Psi(\mathbf{r}, \mathbf{r}'') \\ \partial_{xy} \Psi(\mathbf{r}, \mathbf{r}'') & \partial_{yy} \Psi(\mathbf{r}, \mathbf{r}'') & \partial_{yz} \Psi(\mathbf{r}, \mathbf{r}'') \\ \partial_{xz} \Psi(\mathbf{r}, \mathbf{r}'') & \partial_{yz} \Psi(\mathbf{r}, \mathbf{r}'') & \partial_{zz} \Psi(\mathbf{r}, \mathbf{r}'') \end{bmatrix}$$

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

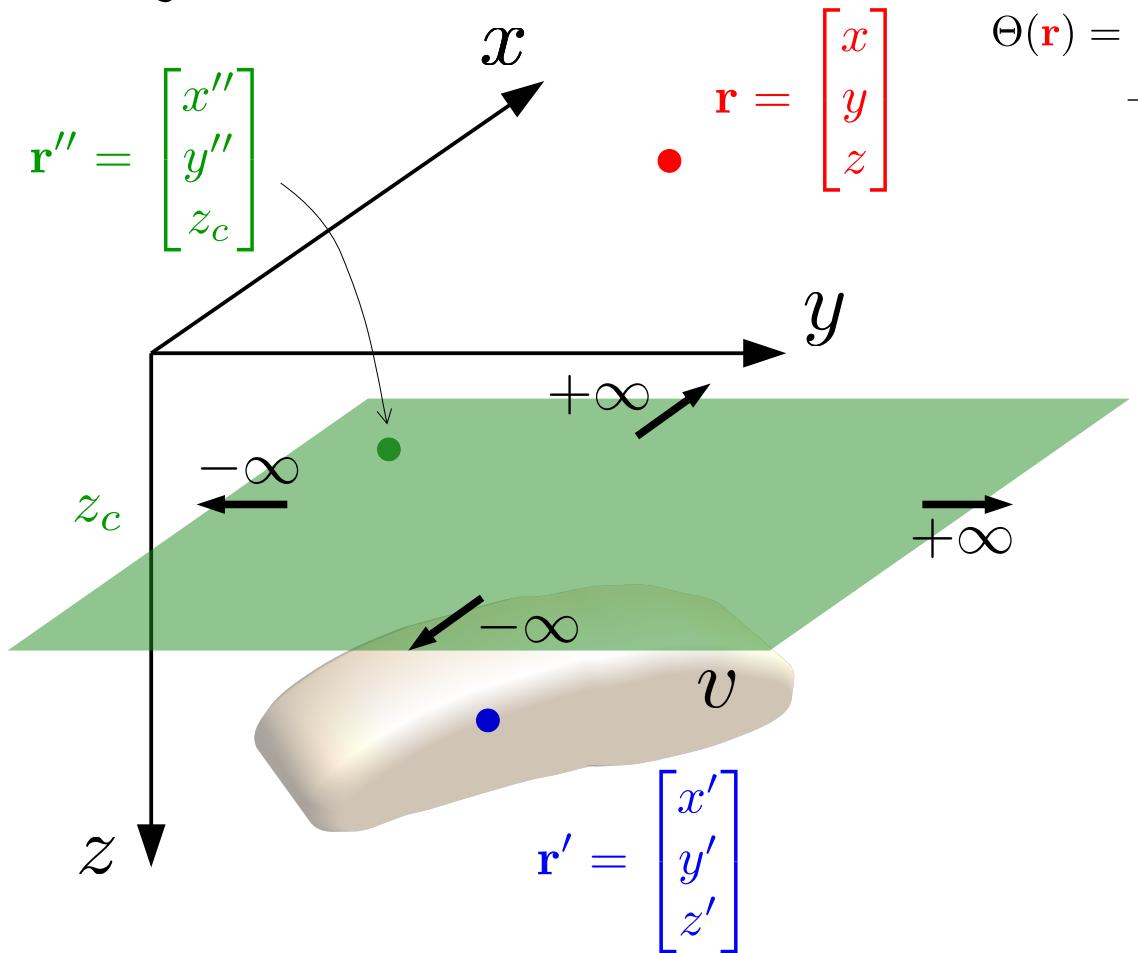
approx total-field anomaly

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

$$\partial_{th} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

Reis et al., 2020 were the first to deduce this analytical eq. layer by following a slightly different approach

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

approx total-field anomaly

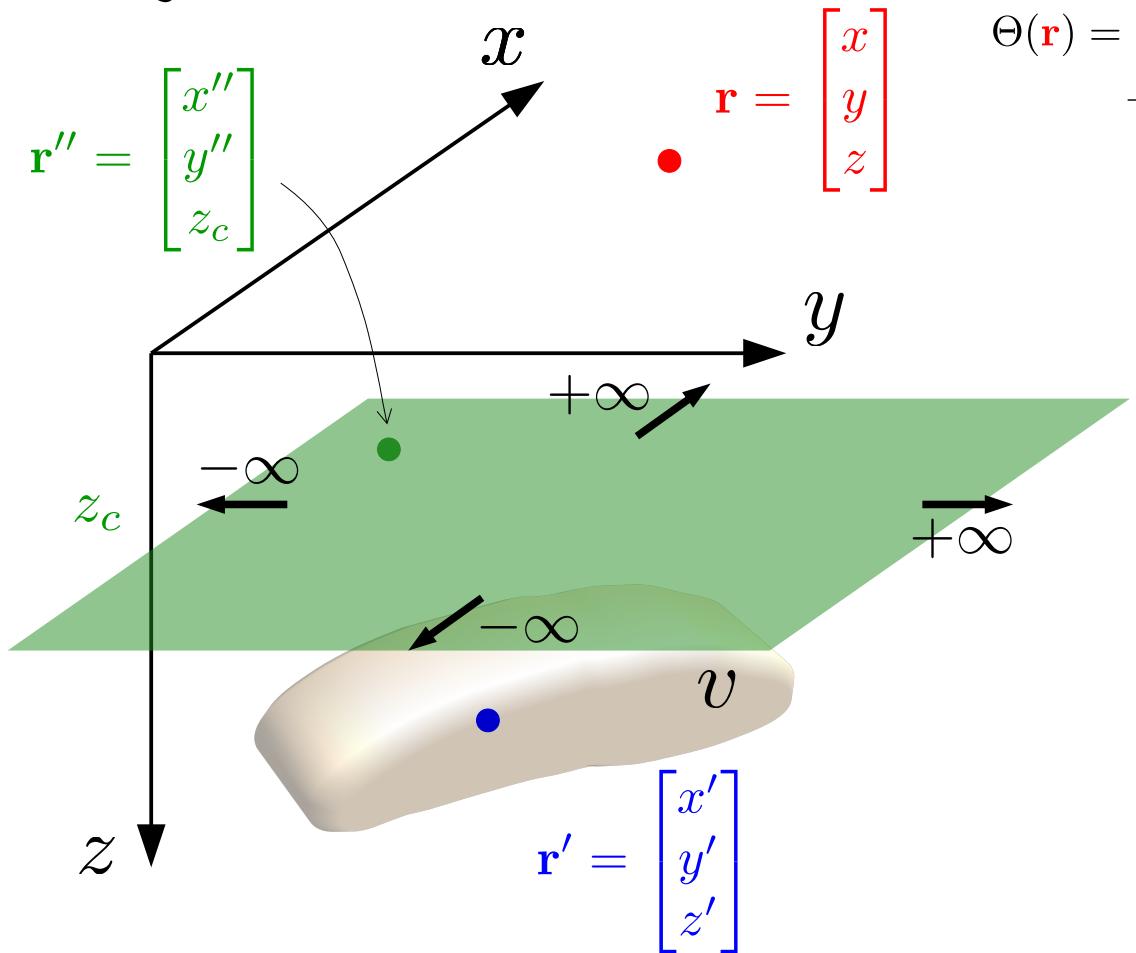
$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

$$\partial_{th} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

Total-magnetization intensity distribution within the true sources

$$\partial_z \Theta(\mathbf{r}'') = \kappa \iiint_v \sigma(\mathbf{r}') \partial_z \frac{1}{\|\mathbf{r}'' - \mathbf{r}'\|} dv'$$

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

approx total-field anomaly

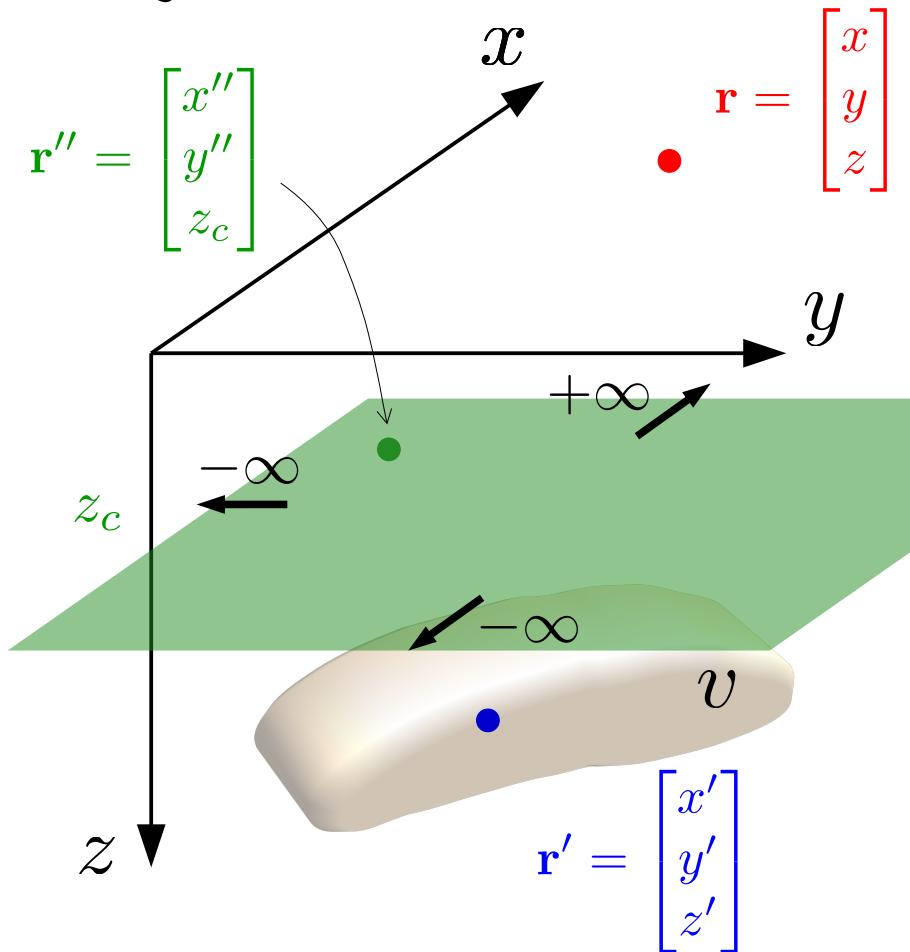
$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

$$\partial_{th} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

Note that this function is ≥ 0 at all points \mathbf{r}' within the sources

$$\partial_z \Theta(\mathbf{r}'') = \kappa \iiint_v \sigma(\mathbf{r}') \partial_z \frac{1}{\|\mathbf{r}'' - \mathbf{r}'\|} dv'$$

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

approx total-field anomaly

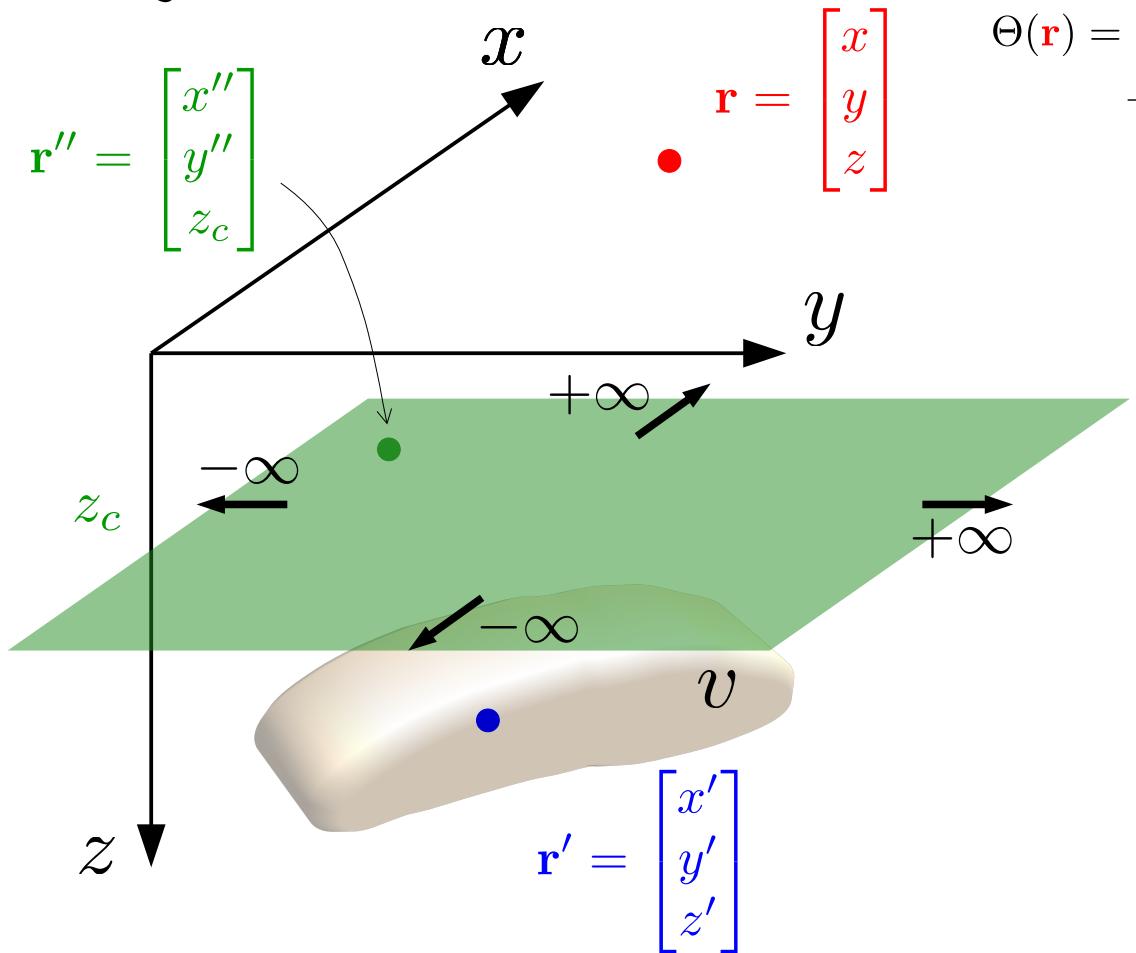
$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

$$\partial_{th} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

This analytical eq. layer represents the gravity disturbance that would be produced by the true sources on the plane \mathbf{z}_c if they had a density distribution proportional to the total-magnetization intensity $\sigma(\mathbf{r}')$

$$\partial_z \Theta(\mathbf{r}'') = \kappa \iiint_v \sigma(\mathbf{r}') \partial_z \frac{1}{\|\mathbf{r}'' - \mathbf{r}'\|} dv'$$

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

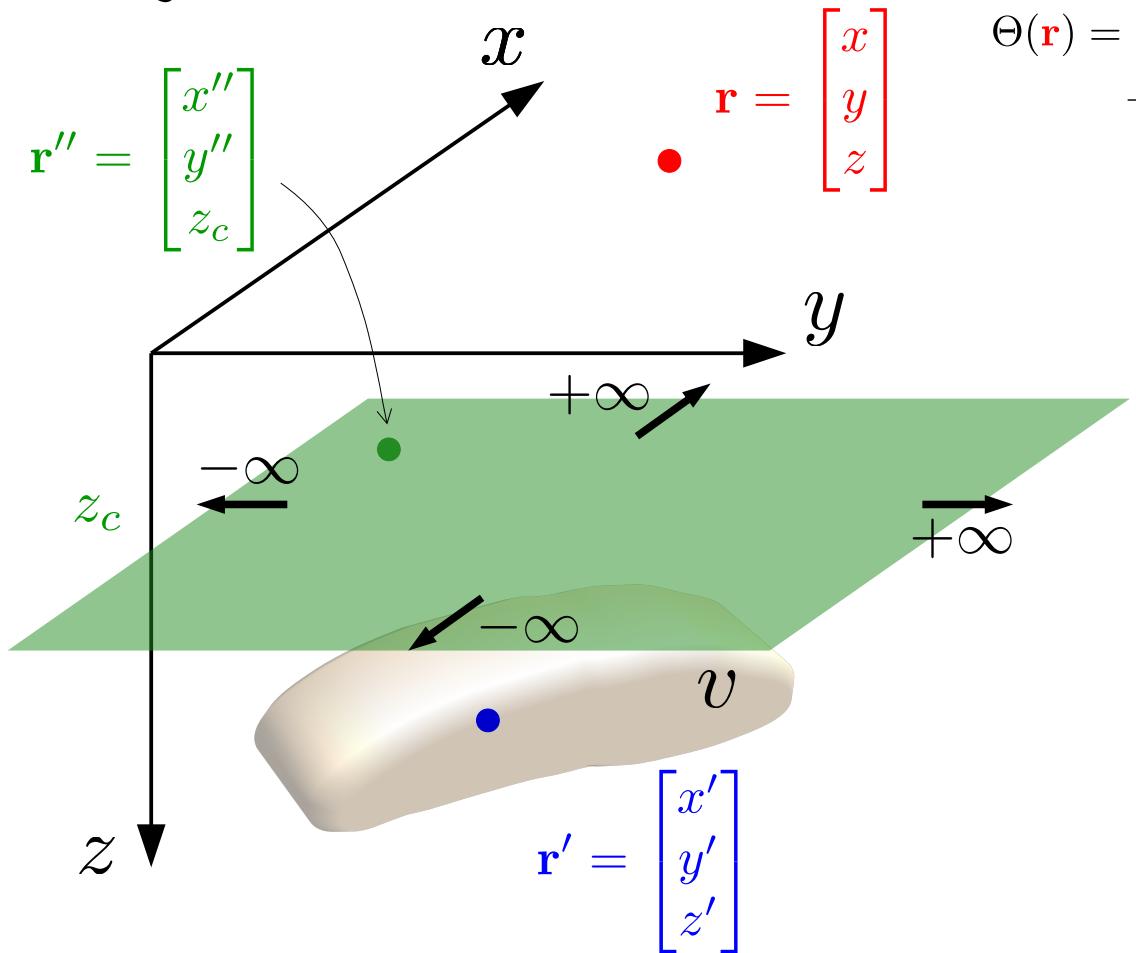
approx total-field anomaly

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

$$\partial_{th} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$$d_i \approx \sum_{j=1}^M p_j \partial_{th} \Psi(\mathbf{r}_i, \mathbf{r}'_j)$$

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

approx total-field anomaly

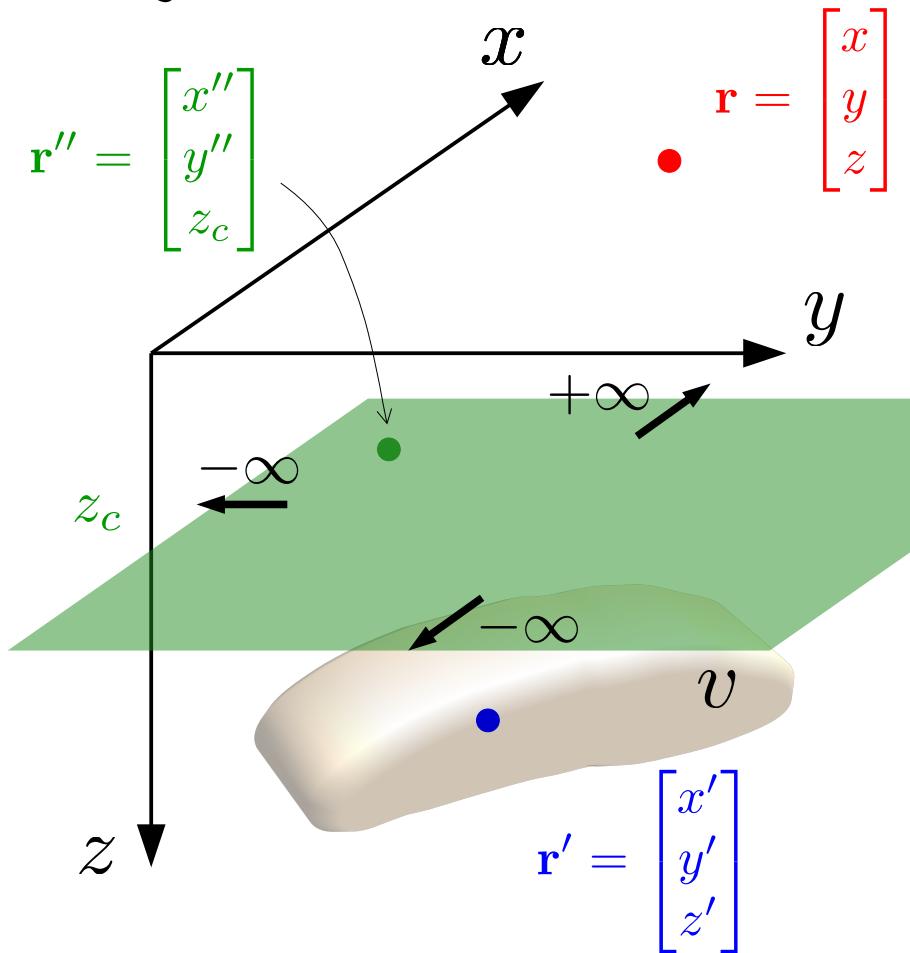
$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

$$\partial_{th} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$$d_i \approx \sum_{j=1}^M p_j \partial_{th} \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

$$\approx \partial_z \Theta(\mathbf{r}_j'')$$

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

approx total-field anomaly

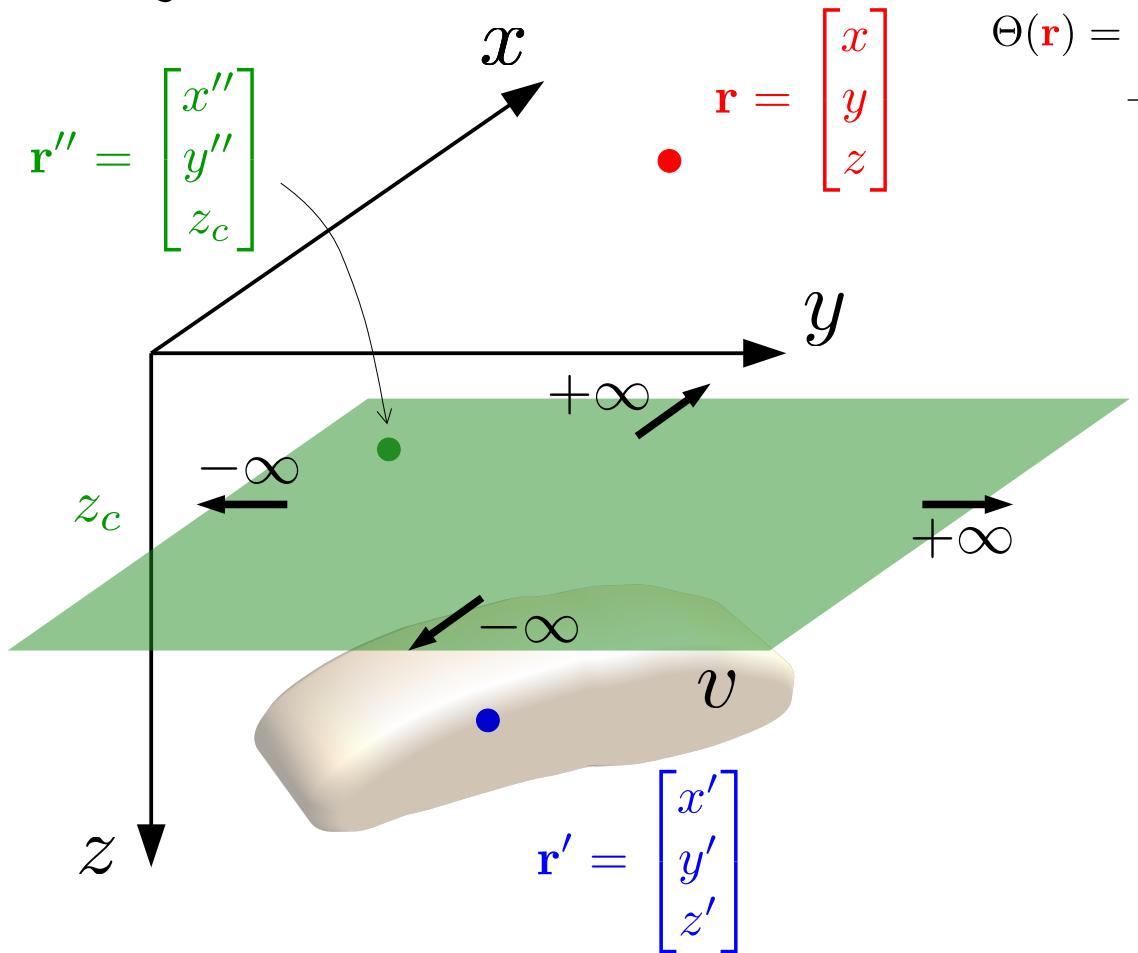
$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

$$\partial_{th} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$$d_i \approx \sum_{j=1}^M p_j \partial_{th} \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

$$\partial_{th} \Theta(\mathbf{r}_i) \approx \partial_z \Theta(\mathbf{r}_j'')$$

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-magnetization direction of the true sources



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

approx total-field anomaly

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

$$\partial_{th} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$$d_i \approx \sum_{j=1}^M p_j \partial_{th} \Psi(\mathbf{r}_i, \mathbf{r}_j'')$$

classical EqI technique
applied to magnetic data

Result 4/4: The same analytical eq. layer of dipoles reproducing the approx total-field anomaly with the same uniform total-mag. direction of the true sources can also reproduce the mag. scalar potential, field components and

$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

RTP anomaly

x

y

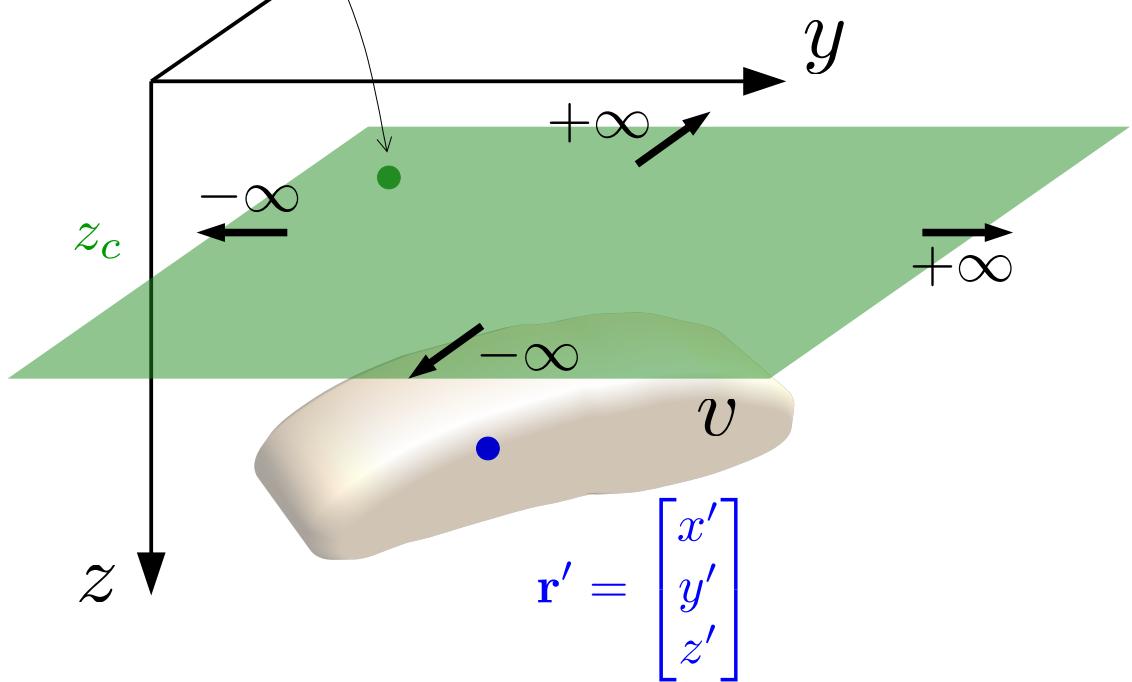
z

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\mathbf{r}'' = \begin{bmatrix} x'' \\ y'' \\ z_c \end{bmatrix}$$

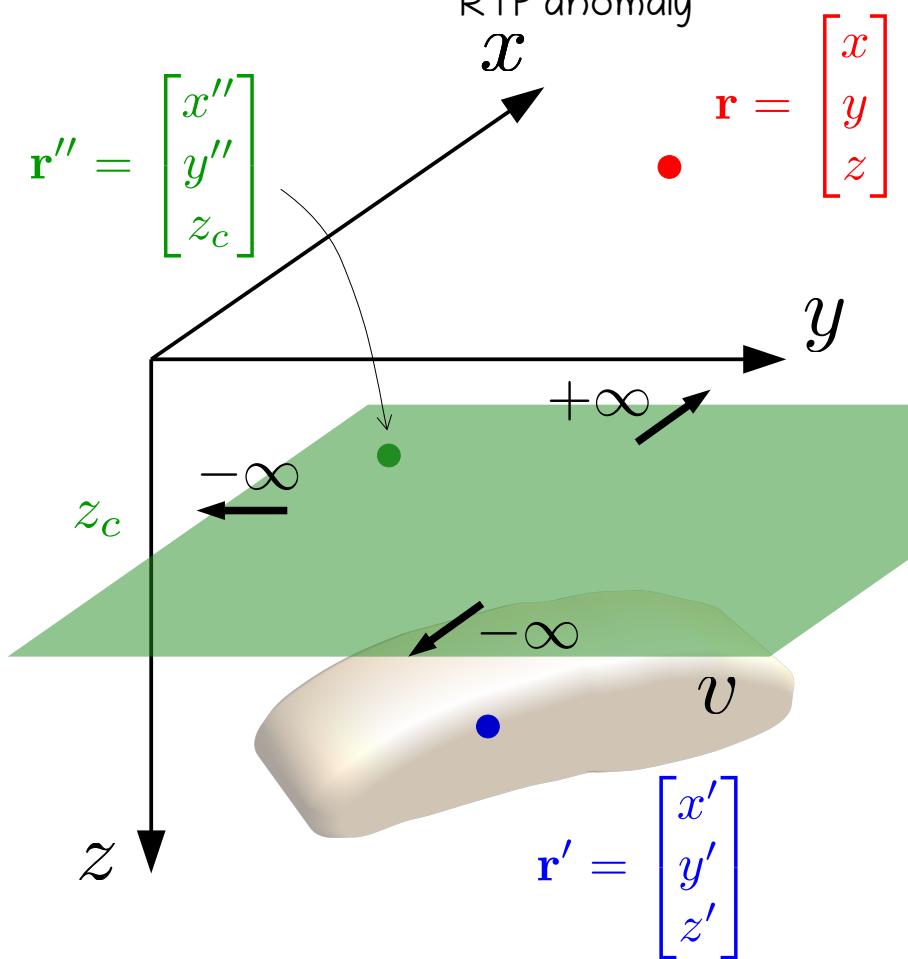
$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$$\Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$



Result 4/4: The same analytical eq. layer of dipoles reproducing the approx total-field anomaly with the same uniform total-mag. direction of the true sources can also reproduce the mag. scalar potential, field components and

RTP anomaly



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

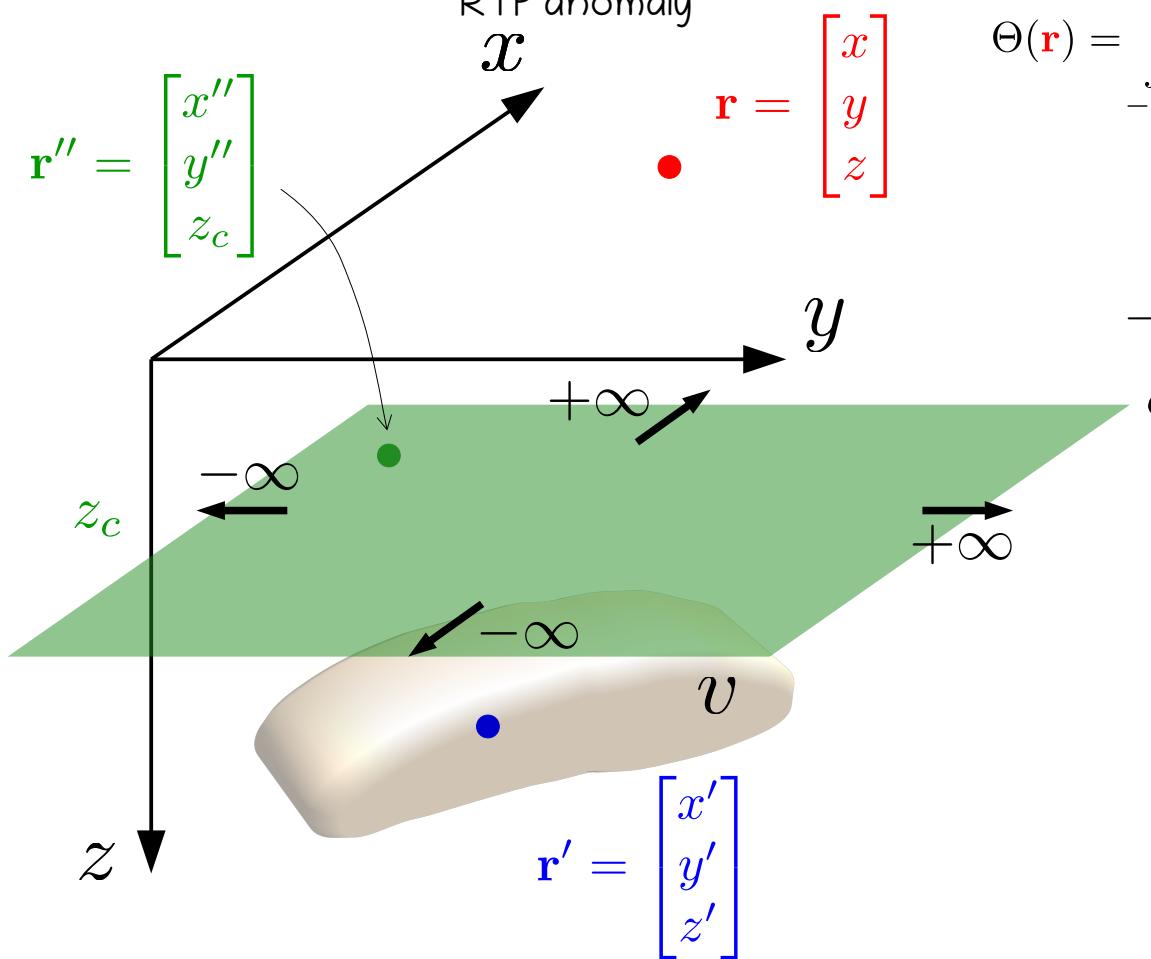
$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

$$-\partial_h \Theta(\mathbf{r}) = -\nabla \Theta(\mathbf{r})^\top \hat{\mathbf{h}} \text{ mag scalar potential}$$

Result 4/4: The same analytical eq. layer of dipoles reproducing the approx total-field anomaly with the same uniform total-mag. direction of the true sources can also reproduce the mag. scalar potential, field components and

RTP anomaly



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

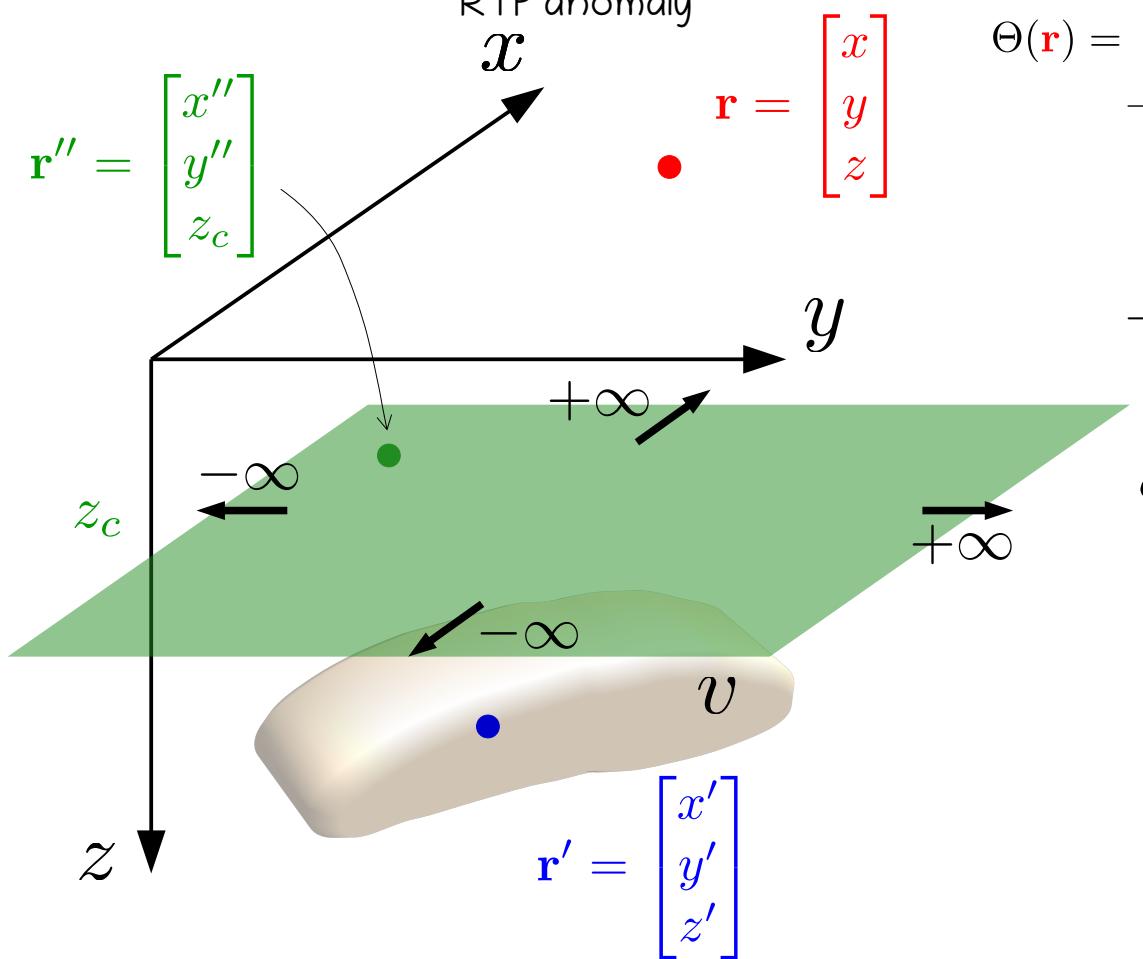
Deduction:

$$-\partial_h \Theta(\mathbf{r}) = -\nabla \Theta(\mathbf{r})^\top \hat{\mathbf{h}} \quad \text{mag scalar potential}$$

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}} \quad \text{approx total-field an}$$

Result 4/4: The same analytical eq. layer of dipoles reproducing the approx total-field anomaly with the same uniform total-mag. direction of the true sources can also reproduce the mag. scalar potential, field components and

RTP anomaly



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

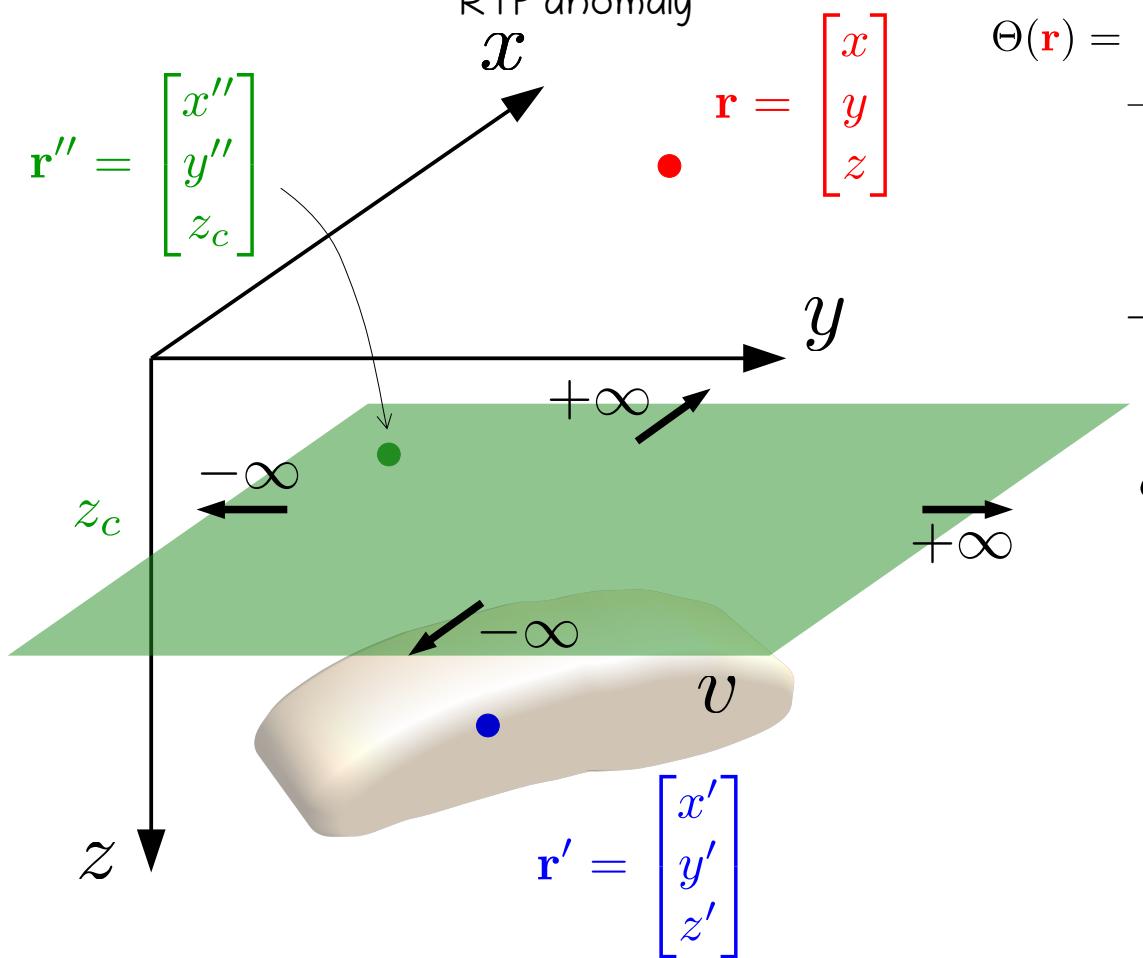
$$-\partial_h \Theta(\mathbf{r}) = -\nabla \Theta(\mathbf{r})^\top \hat{\mathbf{h}} \quad \text{mag scalar potential}$$

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}} \quad \text{approx total-field an}$$

$$\partial_{\alpha h} \Theta(\mathbf{r}) = \partial_\alpha \nabla \Theta(\mathbf{r})^\top \hat{\mathbf{h}} \quad \text{mag field component}$$

Result 4/4: The same analytical eq. layer of dipoles reproducing the approx total-field anomaly with the same uniform total-mag. direction of the true sources can also reproduce the mag. scalar potential, field components and

RTP anomaly



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

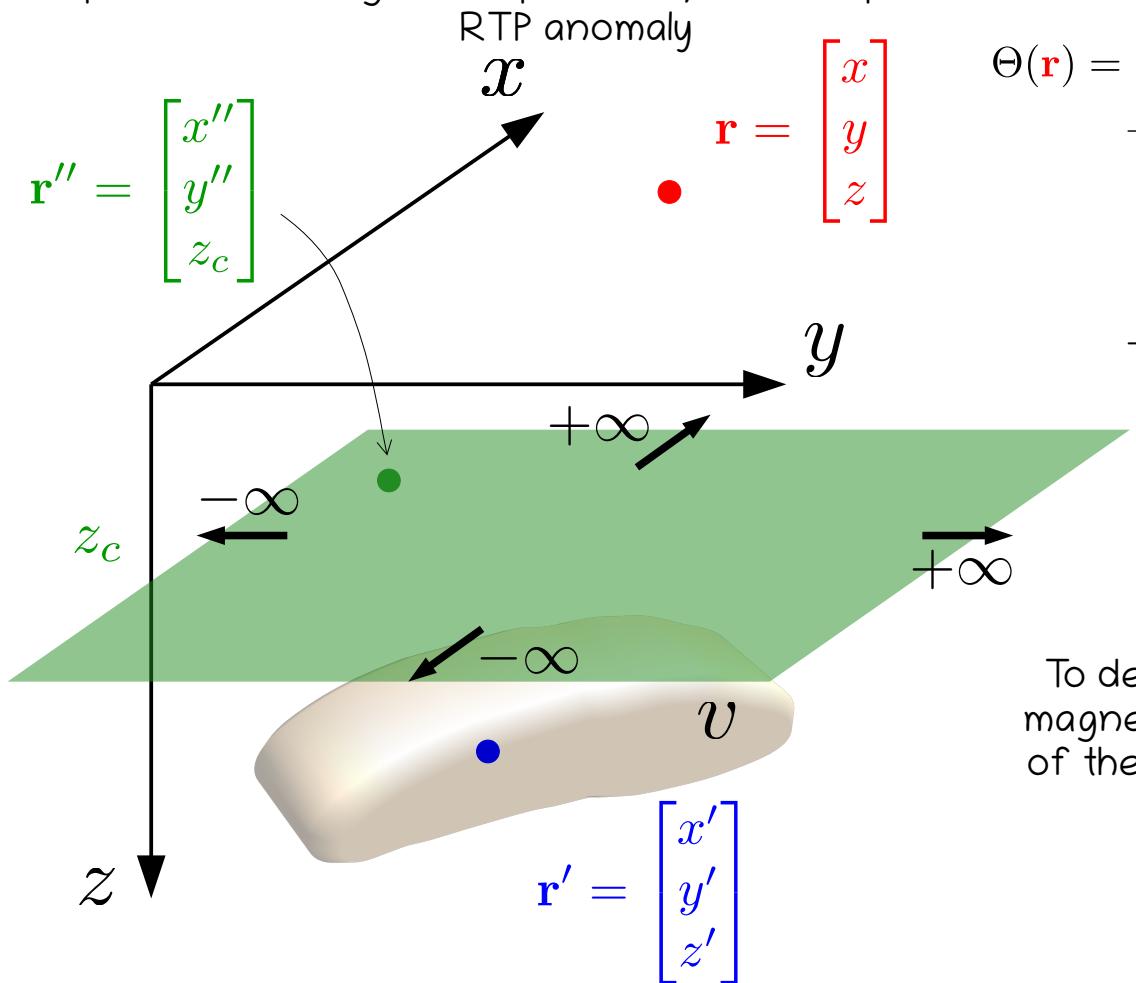
$$-\partial_h \Theta(\mathbf{r}) = -\nabla \Theta(\mathbf{r})^\top \hat{\mathbf{h}} \quad \text{mag scalar potential}$$

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}} \quad \text{approx total-field an}$$

$$\partial_{\alpha h} \Theta(\mathbf{r}) = \partial_\alpha \nabla \Theta(\mathbf{r})^\top \hat{\mathbf{h}} \quad \text{mag field component}$$

$$\partial_{zz} \Theta(\mathbf{r}) \quad \text{RTP anomaly}$$

Result 4/4: The same analytical eq. layer of dipoles reproducing the approx total-field anomaly with the same uniform total-mag. direction of the true sources can also reproduce the mag. scalar potential, field components and RTP anomaly



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

$$-\partial_h \Theta(\mathbf{r}) = -\nabla \Theta(\mathbf{r})^\top \hat{\mathbf{h}} \quad \text{mag scalar potential}$$

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}} \quad \text{approx total-field an}$$

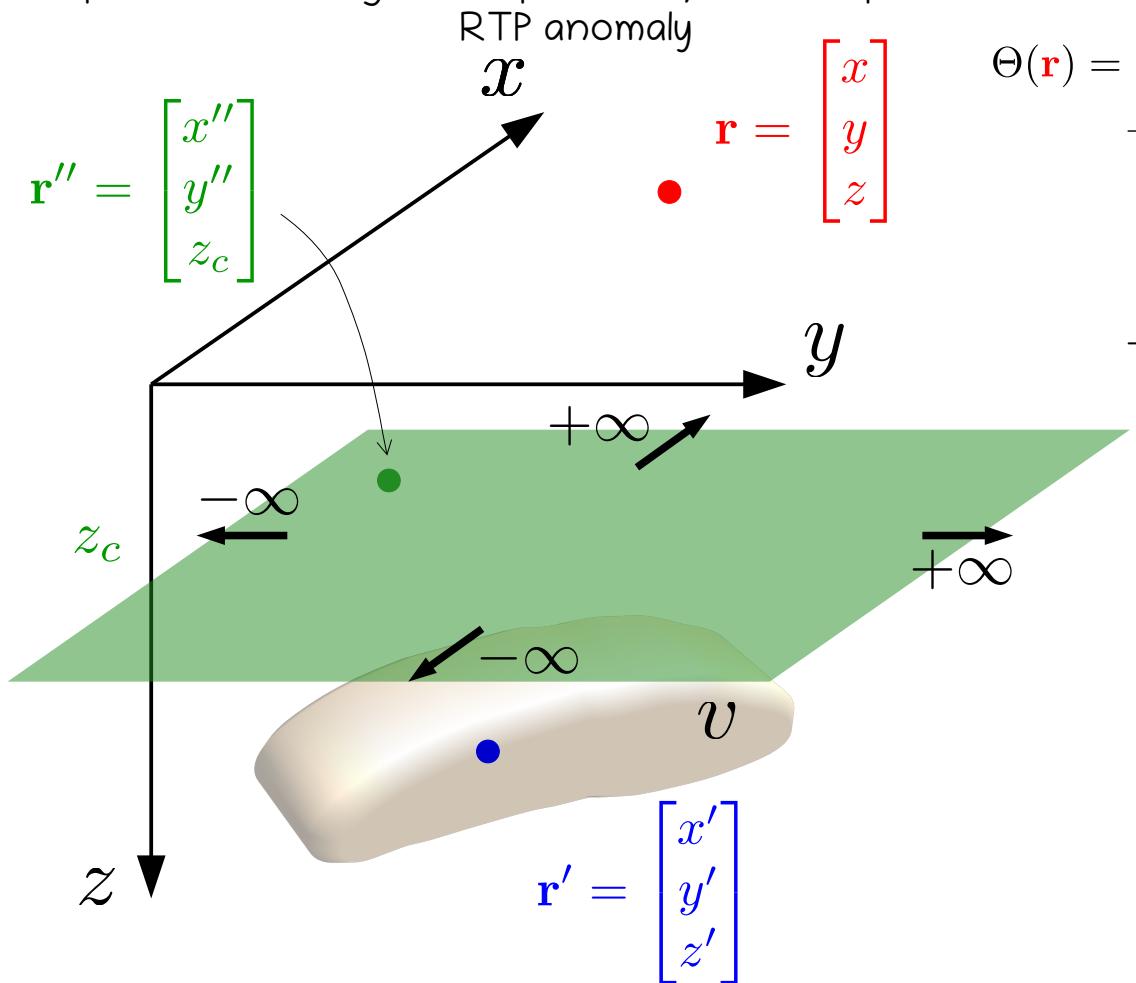
$$\partial_{\alpha h} \Theta(\mathbf{r}) = \partial_\alpha \nabla \Theta(\mathbf{r})^\top \hat{\mathbf{h}} \quad \text{mag field component}$$

$$\partial_{zz} \Theta(\mathbf{r}) \quad \text{RTP anomaly}$$

To deduce the analytical eq. layer associated with these magnetic field quantities, we have to compute derivatives of the equation below with respect to the coordinates of the observation point \mathbf{r}

$$\Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

Result 4/4: The same analytical eq. layer of dipoles reproducing the approx total-field anomaly with the same uniform total-mag. direction of the true sources can also reproduce the mag. scalar potential, field components and RTP anomaly



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

$$-\partial_h \Theta(\mathbf{r}) = -\nabla \Theta(\mathbf{r})^\top \hat{\mathbf{h}} \quad \text{mag scalar potential}$$

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}} \quad \text{approx total-field an}$$

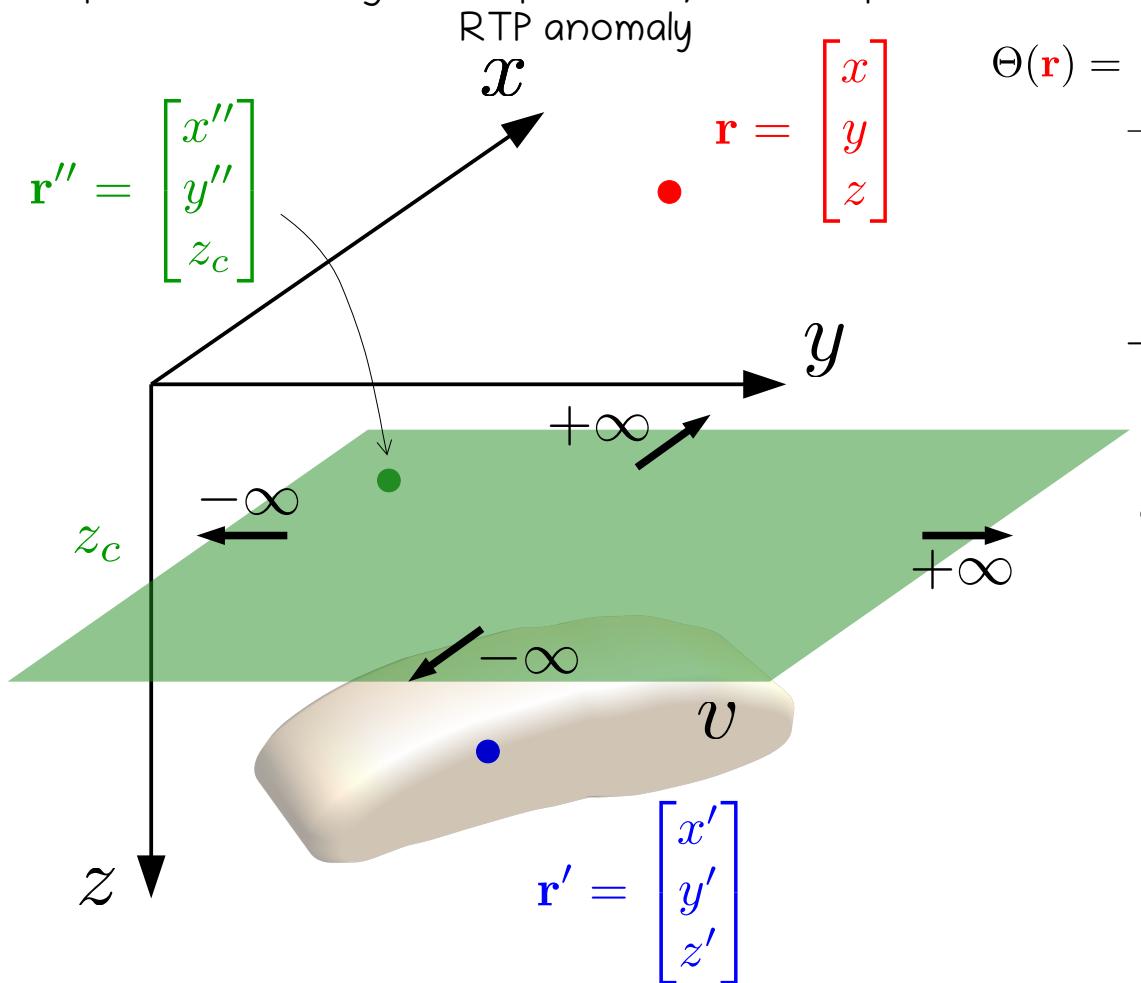
$$\partial_{\alpha h} \Theta(\mathbf{r}) = \partial_\alpha \nabla \Theta(\mathbf{r})^\top \hat{\mathbf{h}} \quad \text{mag field component}$$

$$\partial_{zz} \Theta(\mathbf{r}) \quad \text{RTP anomaly}$$

These derivatives, however, affect only this term

$$\Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

Result 4/4: The same analytical eq. layer of dipoles reproducing the approx total-field anomaly with the same uniform total-mag. direction of the true sources can also reproduce the mag. scalar potential, field components and RTP anomaly



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

$$-\partial_h \Theta(\mathbf{r}) = -\nabla \Theta(\mathbf{r})^\top \hat{\mathbf{h}} \quad \text{mag scalar potential}$$

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}} \quad \text{approx total-field an}$$

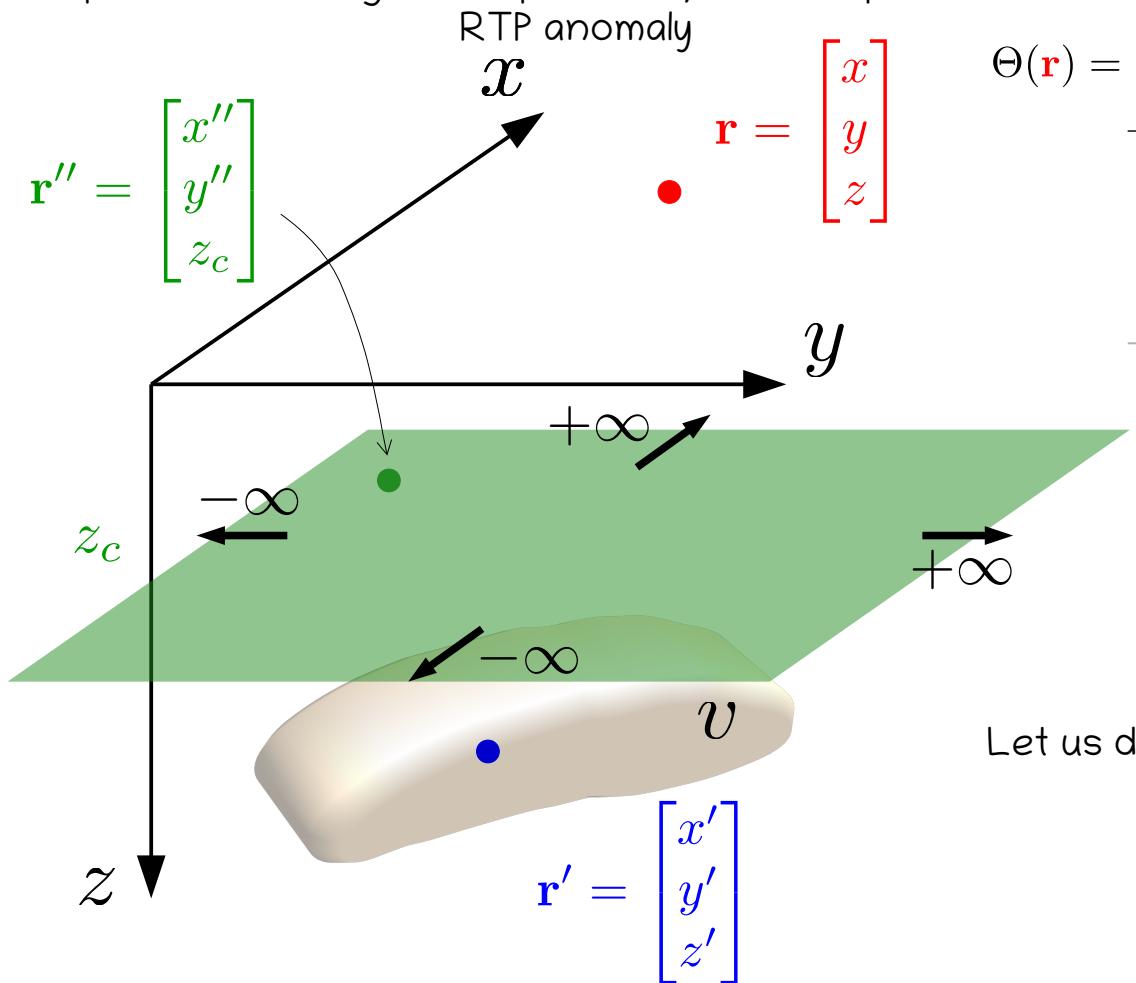
$$\partial_{\alpha h} \Theta(\mathbf{r}) = \partial_\alpha \nabla \Theta(\mathbf{r})^\top \hat{\mathbf{h}} \quad \text{mag field component}$$

$$\partial_{zz} \Theta(\mathbf{r}) \quad \text{RTP anomaly}$$

Hence, all these quantities are defined in terms of the same eq. layer

$$\Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

Result 4/4: The same analytical eq. layer of dipoles reproducing the approx total-field anomaly with the same uniform total-mag. direction of the true sources can also reproduce the mag. scalar potential, field components and RTP anomaly



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

$$-\partial_h \Theta(\mathbf{r}) = -\nabla \Theta(\mathbf{r})^\top \hat{\mathbf{h}} \quad \text{mag scalar potential}$$

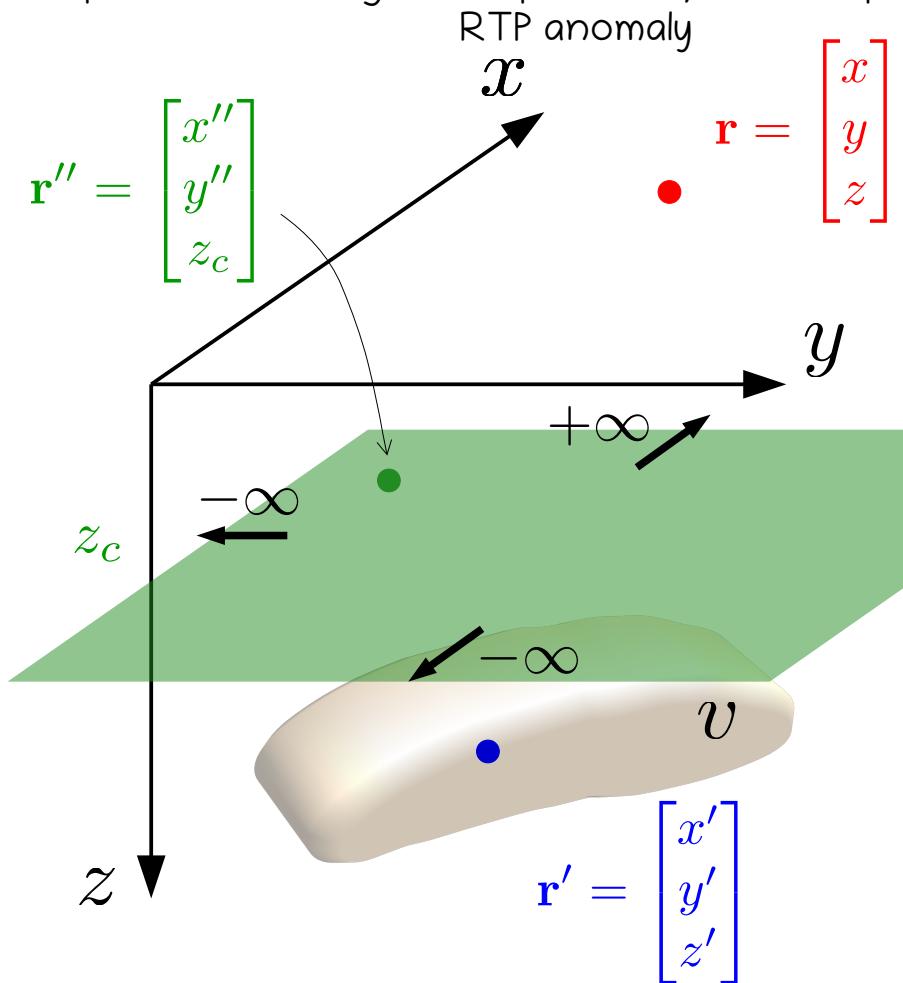
$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}} \quad \text{approx total-field an}$$

$$\partial_{\alpha h} \Theta(\mathbf{r}) = \partial_\alpha \nabla \Theta(\mathbf{r})^\top \hat{\mathbf{h}} \quad \text{mag field component}$$

$$\partial_{zz} \Theta(\mathbf{r}) \quad \text{RTP anomaly}$$

Let us draw our attention to these two field quantities

Result 4/4: The same analytical eq. layer of dipoles reproducing the approx total-field anomaly with the same uniform total-mag. direction of the true sources can also reproduce the mag. scalar potential, field components and RTP anomaly



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

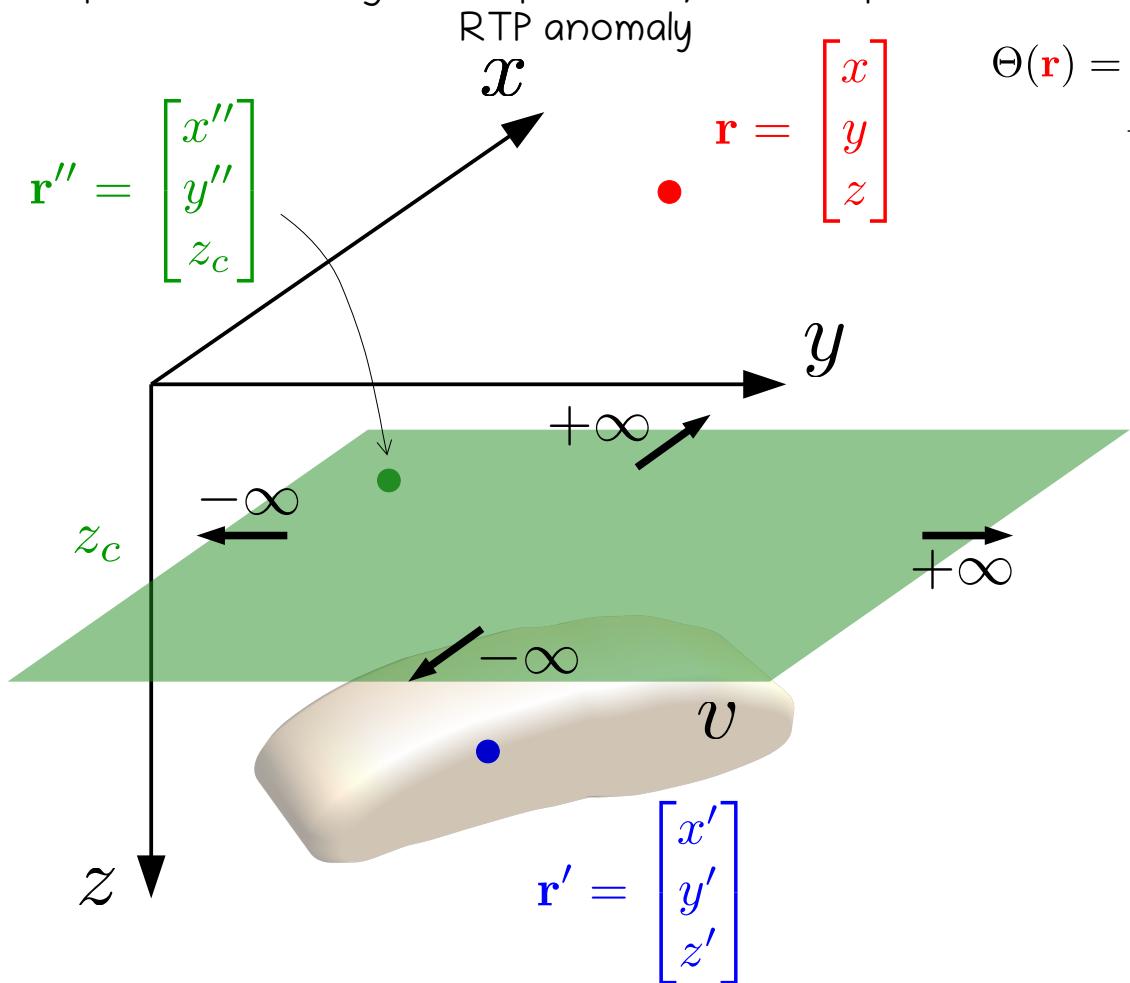
Deduction:

These equations represent the theoretical basis for computing the RTP in space domain, from total-field anomaly data, via EqL technique, according to the pioneer work of Silva (1986)

$$\partial_{th} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$$\partial_{zz} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{zz} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

Result 4/4: The same analytical eq. layer of dipoles reproducing the approx total-field anomaly with the same uniform total-mag. direction of the true sources can also reproduce the mag. scalar potential, field components and RTP anomaly



$$\Theta(\mathbf{r}) = \kappa \iiint_v \sigma(\mathbf{r}') \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} dv'$$

$$\Theta(\mathbf{r}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \partial_z \Theta(\mathbf{r}'') \Psi(\mathbf{r}, \mathbf{r}'') dS'' \quad \Psi(\mathbf{r}, \mathbf{r}'') = \frac{1}{2\pi} \frac{1}{\|\mathbf{r} - \mathbf{r}''\|}$$

Deduction:

However, these equations have never been clearly defined in pioneer work of [Silva \(1986\)](#), nor in the succeeding studies (e.g., [Leão and Silva, 1989](#); [Guspí and Novara, 2009](#); [Oliveira Jr. et al., 2013](#); [Li et al., 2014](#); [Reis et al., 2020](#); [Takahashi et al., 2020](#)) proposed to compute the RTP in space-domain, via EqL technique

$$\partial_{th} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

$$\partial_{zz} \Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{zz} \Psi(\mathbf{r}, \mathbf{r}'') dS''$$

List of theoretical results

Result 1/4: The gravity disturbance can be exactly reproduced by an analytical eq. layer of monopoles

Result 2/4: The same analytical eq. layer of monopoles reproducing the gravity disturbance can also reproduce the gravitational potential and gravity gradient tensor

Result 3/4: The approx total-field anomaly can be exactly reproduced by an analytical eq. layer of dipoles having the same uniform total-mag. direction of the true sources

Result 4/4: The same analytical eq. layer of dipoles reproducing the approx total-field anomaly with the same uniform total-mag. direction of the true sources can also reproduce the mag. scalar potential, field components and RTP anomaly

Summary

- Motivation
- Potential-field data
- The Equivalent-Layer (EqL) Technique
- **Theoretical aspects**
- Some open questions

Summary

- Motivation
- Potential-field data
- The Equivalent-Layer (EqL) Technique
- Theoretical aspects
- **Some open questions**

- Is there an analytical eq. layer of dipoles with arbitrary uniform magnetization direction?

- Is there an analytical eq. layer of dipoles with arbitrary uniform magnetization direction?

$$\partial_{th} \Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

approx total-field anomaly

total-magnetization direction
of the true sources

- Is there an analytical eq. layer of dipoles with arbitrary uniform magnetization direction?

approx total-field anomaly total-magnetization direction
 of the true sources

$$\partial_{th}\Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

$\partial_{th}\Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \partial_{th} \Psi(\mathbf{r}, \mathbf{r}'') dS''$

analytical equivalent
layer



- Is there an analytical eq. layer of dipoles with arbitrary uniform magnetization direction?

approx total-field anomaly

$\partial_{th}\Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$

total-magnetization direction
of the true sources

$\partial_{th}\Theta(\mathbf{r}) = \iint \partial_z \Theta(\mathbf{r}'') \underbrace{\partial_{th} \Psi(\mathbf{r}, \mathbf{r}'')}_{\text{analytical equivalent layer}} dS''$

$\hat{\mathbf{t}}^\top \mathbf{H}_\Psi(\mathbf{r}, \mathbf{r}'') \hat{\mathbf{h}}$

total-magnetization direction
of the true sources

- Is there an analytical eq. layer of dipoles with arbitrary uniform magnetization direction?

$$\partial_{th}\Theta(\mathbf{r}) = \hat{\mathbf{t}}^\top \mathbf{H}_\Theta(\mathbf{r}) \hat{\mathbf{h}}$$

approx total-field anomaly total-magnetization direction
of the true sources

$$\partial_{th}\Theta(\mathbf{r}) = \iint p(\mathbf{r}'') \underbrace{\partial_{th}\Psi(\mathbf{r}, \mathbf{r}'')}_{\hat{\mathbf{t}}^\top \mathbf{H}_\Psi(\mathbf{r}, \mathbf{r}'') \hat{\mathbf{v}}} dS''$$

arbitrary direction

?

- Is there an analytical eq. layer of dipoles with arbitrary uniform magnetization direction?
- Is it possible to determine an analytical eq. layer of dipoles for the case in which the sources have a nonuniform total-magnetization direction?

- Is there an analytical eq. layer of dipoles with arbitrary uniform magnetization direction?
- Is it possible to determine an analytical eq. layer of dipoles for the case in which the sources have a nonuniform total-magnetization direction?
- What parameters of the true volumetric sources can be obtained from an analytical equivalent layer?

- Is there an analytical eq. layer of dipoles with arbitrary uniform magnetization direction?
- Is it possible to determine an analytical eq. layer of dipoles for the case in which the sources have a nonuniform total-magnetization direction?
- What parameters of the true volumetric sources can be obtained from an analytical equivalent layer?
- Can we determine an analytical eq. layer that crosses the sources?

- Is there an analytical eq. layer of dipoles with arbitrary uniform magnetization direction?
- Is it possible to determine an analytical eq. layer of dipoles for the case in which the sources have a nonuniform total-magnetization direction?
- What parameters of the true volumetric sources can be obtained from an analytical equivalent layer?
- Can we determine an analytical eq. layer that crosses the sources?
- How to generalize the approach presented here to deduce analytical eq. layers in spherical coordinates?