

# Convolutional equivalent layer for magnetic data processing

Diego Takahashi<sup>†</sup>, Vanderlei C. Oliveira Jr.<sup>†\*</sup> and Valéria C. F. Barbosa<sup>†</sup>

<sup>†</sup>*Observatório Nacional, Department of Geophysics, Rio de Janeiro, Brazil*

\* Corresponding author: *vanderlei@on.br*

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## ABSTRACT

A fast equivalent layer for magnetic data processing is presented. Taking advantage of the sensitivity matrix structure of the magnetic kernel, when observation and equivalent sources are aligned on a regular spaced grid with constant height, it is possible to calculate the matrix-vector product in a very fast manner. The structure is called block-Toeplitz Toeplitz-block (BTTB) and this type of matrix is well known in literature to be embedded in a block-Circulant Circulant-block (BCCB), which in turn can have its eigenvalues calculated using only the first column and a 2D fast Fourier transform. We show that, despite this BTTB matrix is not symmetric, by using only the first equivalent source it is possible to calculate all the first columns of the inverse of distance second derivatives that composes the magnetic kernel and reconstruct the first column of the BCCB matrix, saving computational time and system memory. The conjugate gradient iterative method is used to solve the linear system and estimate the physical properties distributed over the equivalent layer. Synthetic tests show a decrease in the order of  $10^4$  in floating-point operations, 25 times in computation runtime with a mid-size  $80 \times 80$  grid, and exponential decrease in memory RAM

usage, allowing to perform this operation with millions of observation points on desktop computers. Synthetic tests with irregular grids also show that this method can work with directional disturbances under certain limits. A real magnetic data of Carajás Province, Brazil, with 1,310,000 observation points in an irregular grid was used to successfully perform a data processing with this method, taking 385.56 seconds to estimate the physical property and 2.64 seconds for the upward-continuation.

## INTRODUCTION

Throughout the years many authors presented solutions to inverse problems in potential methods using magnetic data. The most common approaches are the works that try to parameterize the geometric source (Bott, 1960; Pilkington and Crossley, 1986; Bhattacharyya, 1980) and others tried to estimate the susceptibility or the magnetization as the parameter (Parker and Huestis, 1974; Cribb, 1976; Li and Oldenburg, 1996; Liu et al., 2017). With advances in computation power, the equivalent layer is an inversion technique that is becoming more present in data processing nowadays, despite its use since 1960 in geophysics literature (Danes, 1961; Bott, 1967; Dampney, 1969). The equations deductions of the equivalent layer as a solution of the Laplace's equation in the region above the source was first presented by Kellogg (1929) and detailed explanations can also be found in Blakely (1996). In magnetic data procesing, some authors explored this technique for calculating the first and second vertical derivatives fields (Emilia, 1973), reduction to the pole (Silva, 1986; Oliveira Jr. et al., 2013; Li et al., 2014), upward/downward continuations (Hansen and Miyazaki, 1984; Li and Oldenburg, 2010) and total magnetic induction vector components calculation (Sun et al., 2019).

Together with the rise in computational processing, some works tried new implementations to increase the efficiency of the equivalent layer. In Leão and Silva (1989) the authors used a shifting window over the layer, increasing the number of linear systems to be solved, but decreasing the size of it. In Li and Oldenburg (2010) a method of transforming the full sensitivity matrix into a sparse one by using wavelets was another approach. Oliveira Jr. et al. (2013) also proposed a shifting window method, but instead of directly calculating the equivalent sources physical properties, they estimated the coefficients of a bivariate

polynomial function representing the sources. An iterative method for gravimetric data was proposed by Siqueira et al. (2017), where the mass distribution over the layer is corrected at each iteration by the residuals of observed and estimated data.

In Takahashi et al. (2020), the authors combined the fast equivalent source technique presented by Siqueira et al. (2017) with the concept of symmetric block-Toeplitz Toeplitz-block (BTTB) matrices to introduce the convolutional equivalent layer for gravimetric data technique. When the sensitivity matrix of the linear system, required to solve the gravimetric equivalent layer, is calculated on a regular spaced grid of dataset with constant height and each equivalent source is exactly beneath each observed data point, the BTTB structure appears. This work showed an decrease in the order of  $10^4$  in floating-point operations needed to estimate the equivalent sources.

In potential field methods, the properties of Toeplitz system have been used for downward-continuation (Zhang et al., 2016) and for 3D gravity-data inversion using a 2D multilayer model (Zhang and Wong, 2015). More recently in Hogue et al. (2020) the authors provided an overview on modeling the gravity and magnetic kernels using the BTTB structures and Renaut et al. (2020) used for inversion of both gravity and magnetic data to recover sparse subsurface structures.

A wide variety of applications in mathematics and engineering that fall into Toeplitz systems propelled the development of a large variety of methods for solving them. Direct methods were conceived by Levinson (1946) and by Trench (1964). Currently, the iterative method of conjugate gradient is used in most cases, in Chan and Jin (2007) the authors presented an introduction on the topic for 1D data structures of Toeplitz matrices and also for 2D data structures, which they called block-Toeplitz Toeplitz-block matrices. In both

cases, the solving strategy is to embed the Toeplitz/BTTB matrix into a Circulant/Block-Circulant Circulant-Block matrix, calculate its eigenvalues by a 1D or 2D fast Fourier transform of its first column, respectively and carry the matrix-vector product between kernel and parameters at each iteration of the conjugate gradient method in a very fast manner.

In this work, the convolutional equivalent layer using the block-Toeplitz Toeplitz-block idea, presented in Takahashi et al. (2020), will be used to solve the linear system required to estimate the physical property that produces a magnetic field on regular grids. Despite the non-existence of a fast equivalent source technique solver as with gravity data, it is possible to achieve very fast solutions using a conjugate gradient algorithm combined with the fast Fourier transform. We present a novel method of exploring the symmetric structures of the second derivatives of the inverse of the distance contained in the magnetic kernel, to keep the memory RAM usage to the minimal by using only one equivalent source to carry the calculations of the forward problem. We also show tests of the magnetic convolutional equivalent layer when irregular grids are used. The convergence of the conjugate gradient maintains in an acceptable level even under strong deviations of the data acquisition line. This shows that a fast and robust technique is possible when processing large amounts of magnetic data using the equivalent layer.

## METHODOLOGY

### The total-field anomaly

The Earth's magnetic field is commonly divided in three parts: main field, crustal field and external field. The main field is generated in the outer core in a process of self-sustaining dynamo, the crustal field is generated by magnetic bodies in the lithosphere and the external field is generated by electrical currents in the ionosphere and magnetosphere. For exploration geophysics, the crustal field is the object of study, which makes the separation of this data from the acquisition dataset a very important step.

The combination of the main field and crustal field is known as internal field or total-field. Taking the difference between the main field given by a model (e.g. IGRF) and this internal field, at the same location, we have the total-field anomaly.

Let  $(x_i, y_i, z_i), i = 1, 2, 3, \dots, N$ , be a observed dataset in a region considering a right-handed Cartesian coordinate system with the  $x$ -axis pointing north,  $y$ -axis pointing east and  $z$ -axis pointing downward. The total-field anomaly at the  $i$ th observation can be approximated to:

$$\Delta T(x_i, y_i, z_i) = \hat{\mathbf{F}}^\top \mathbf{B}(x_i, y_i, z_i), \quad (1)$$

where,  $\mathbf{B}(x_i, y_i, z_i)$  is the crustal field,  $\hat{\mathbf{F}}^\top$  is the transposed unitary vector with the main field directions, with  $\hat{\mathbf{F}}$  described as:

$$\hat{\mathbf{F}} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \cos(I_0) \cos(D_0) \\ \cos(I_0) \sin(D_0) \\ \sin(I_0) \end{bmatrix}, \quad (2)$$

where  $I_0$  is the inclination and  $D_0$  the declination of the main field, respectively.

Considering a uniform magnetized body with volume  $v$  and a total magnetization vector  $\mathbf{m}$ , the induced magnetic field is:

$$\mathbf{B}(x_i, y_i, z_i) = c_m \frac{\mu_0}{4\pi} \mathbf{M}(x_i, y_i, z_i) \mathbf{m}, \quad (3)$$

where,  $\mu_0 = 4\pi 10^{-7}$  H/m is the magnetic constant,  $c_m = 10^9$  is a constant that transforms the induced magnetic field from Tesla (T) to nanotesla (nT) and  $\mathbf{M}(x_i, y_i, z_i)$  is a  $3 \times 3$  matrix given by:

$$\mathbf{M}(x_i, y_i, z_i) = \begin{bmatrix} \partial_{xx}\phi(x_i, y_i, z_i) & \partial_{xy}\phi(x_i, y_i, z_i) & \partial_{xz}\phi(x_i, y_i, z_i) \\ \partial_{xy}\phi(x_i, y_i, z_i) & \partial_{yy}\phi(x_i, y_i, z_i) & \partial_{yz}\phi(x_i, y_i, z_i) \\ \partial_{xz}\phi(x_i, y_i, z_i) & \partial_{yz}\phi(x_i, y_i, z_i) & \partial_{zz}\phi(x_i, y_i, z_i) \end{bmatrix}, \quad (4)$$

where,  $\partial_{\alpha\beta}\phi(x_i, y_i, z_i)$  with  $\alpha = x, y, z$  and  $\beta = x, y, z$ , are the second derivatives of the function  $\phi(x_i, y_i, z_i)$  with respect to  $x, y$  and  $z$ :

$$\phi(x_i, y_i, z_i) = \int_v \int \int \frac{1}{r} dv, \quad (5)$$

where the function  $\frac{1}{r}$  is given by:

$$\frac{1}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}} \quad (6)$$

and  $x_j, y_j, z_j$  are the  $j$ th Cartesian coordinates within the volume elements of the magnetized body with volume  $v$ , where the integral (equation 5) is conducted.

Rewriting the equation 1 using the magnetic induction given by equation 3 the total-field anomaly  $\Delta T(x, y, z)$  is given by:

$$\Delta T(x_i, y_i, z_i) = c_m \frac{\mu_0}{4\pi} m \hat{\mathbf{F}}^\top \mathbf{M}(x_i, y_i, z_i) \hat{\mathbf{m}}, \quad (7)$$

where  $m$  is the magnetization intensity and  $\hat{\mathbf{m}}$  is the directional unitary vector.

## Equivalent layer for magnetic data

Grounded on the equivalent layer principle it is possible to calculate the total-field anomaly  $\Delta T(x_i, y_i, z_i)$  (equation 7) with the convolution between the harmonic function and the physical property:

$$\Delta T(x_i, y_i, z_i) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p(x_j, y_j, z_c) \left[ c_m \frac{\mu_0}{4\pi} \hat{\mathbf{F}}^\top \mathbf{H} \hat{\mathbf{h}} \right] dx dy, \quad (8)$$

where  $z_c$  is a constant representing the depth of the equivalent layer with  $z_i < z_c$ . The unitary vector  $\hat{\mathbf{h}}$  is the magnetization directions of the equivalent sources over the layer:

$$\hat{\mathbf{h}} = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \begin{bmatrix} \cos(I) \cos(D) \\ \cos(I) \sin(D) \\ \sin(I) \end{bmatrix}, \quad (9)$$

where  $I$  and  $D$  are, respectively, the inclination and declination of the equivalent sources,  $\mathbf{H}$  is a  $3 \times 3$  matrix that contains the second derivatives in relation to the observed Cartesian coordinates  $x, y, z$  as presented in equation 6:

$$\mathbf{H} = \begin{bmatrix} \partial_{xx} \frac{1}{r} & \partial_{xy} \frac{1}{r} & \partial_{xz} \frac{1}{r} \\ \partial_{xy} \frac{1}{r} & \partial_{yy} \frac{1}{r} & \partial_{yz} \frac{1}{r} \\ \partial_{xz} \frac{1}{r} & \partial_{yz} \frac{1}{r} & \partial_{zz} \frac{1}{r} \end{bmatrix} = \begin{bmatrix} H_{xx} & H_{xy} & H_{xz} \\ H_{xy} & H_{yy} & H_{yz} \\ H_{xz} & H_{yz} & H_{zz} \end{bmatrix}, \quad (10)$$

and the physical property  $p(x_j, y_j, z_c)$  represents the magnetic dipole moment distribution over the layer and has unit of  $Am^2$ .

Discretizing equation 8 we get:

$$\Delta T(x_i, y_i, z_i) = \sum_{j=1}^M p_j a_{ij}, \quad (11)$$

where  $j = 1, 2, 3, \dots, M$  are the equivalent sources discretized and distributed across the layer and  $a_{ij}$  is given by:

$$a_{ij} = c_m \frac{\mu_0}{4\pi} \hat{\mathbf{F}}^\top \mathbf{H}_{ij} \hat{\mathbf{h}}. \quad (12)$$

Equation 11 can be written as:

$$\mathbf{d}(\mathbf{p}) = \mathbf{A}\mathbf{p}, \quad (13)$$

where  $\mathbf{d}(\mathbf{p})$  is the vector of total-field anomaly,  $\mathbf{A}$  is a matrix containing the elements given by equation 12, also known as the sensitivity matrix and  $\mathbf{p}$  is the vector containing the dipole moments of each equivalent source. By using the normal equations:

$$\mathbf{A}^\top \mathbf{A}\mathbf{p} = \mathbf{A}^\top \mathbf{d}^o \quad (14)$$

which is a conventional manner of solving linear systems, the estimative of the dipole moment parameters vector can be calculated from the total-field anomaly as:

$$\hat{\mathbf{p}} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{d}^o \quad (15)$$

where  $\mathbf{d}^o$  is the observed total-field anomaly. Equation 15 will be referenced throughout this work as the classical method for solving the equivalent layer.

### Conjugate Gradient Least Square method (CGLS)

As an alternative from the clasical method of parameter estimative, the conjugate gradient (CG) is a well-known iterative method for solving linear systems with symmetric positive definite matrices. By minimizing the quadratic form:

$$\Phi(\mathbf{p}) = \frac{1}{2} \mathbf{p}^\top \mathbf{A} \mathbf{p} - \mathbf{d}^o \top \mathbf{p}, \quad (16)$$

it is possible to solve the system by constructing a basis of conjugate directions  $c \in R^N$  (Aster et al., 2018). As we are solving a general least square problem and matrix  $\mathbf{A}$  (equation 12) is not symmetric, instead we minimize:

$$\|\mathbf{A} \mathbf{p} - \mathbf{d}^o\|_2, \quad (17)$$

by applying the conjugate gradient to the normal equations (equation 14).

In theory, this method is bound to converge at  $N$  iterations maximum, but in a later part of this work we show with numerical results that the convergence is much faster for the linear system we are solving.

A pseudocode for the CGLS method follows:

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**Algorithm 1** Conjugate Gradient Least Square pseudocode (Aster et al., 2018, p. 166).

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Input:  $\mathbf{A} \in R^{N \times M}$  and  $\mathbf{d}^o \in R^N$ .

Output: Estimative of parameter vector  $\hat{\mathbf{p}}$ .

Let  $it = 0$ ,  $\hat{\mathbf{p}}^{(0)} = \mathbf{0}$ ,  $\mathbf{c}^{(-1)} = \mathbf{0}$ ,  $\beta_0 = 0$ ,  $\mathbf{s}^{(0)} = \mathbf{d}^o - \mathbf{A}\hat{\mathbf{p}}^{(0)}$  and  $\mathbf{r}^{(0)} = \mathbf{A}^\top \mathbf{s}^{(0)}$ .

$$1 - \text{If } it > 0, \text{ let } \beta_{(it)} = \frac{\mathbf{r}^{(it)^\top} \mathbf{r}^{(it)}}{\mathbf{r}^{(it-1)^\top} \mathbf{r}^{(it-1)}}$$

$$2 - \mathbf{c}^{(it)} = \mathbf{r}^{(it)} + \beta_{(it)} \mathbf{c}^{(it-1)}.$$

$$3 - \alpha_{(it)} = \frac{\|\mathbf{r}^{(it)}\|_2^2}{(\mathbf{c}^{(it)^\top} \mathbf{A}^\top)(\mathbf{A} \mathbf{c}^{(it)})}.$$

$$4 - \hat{\mathbf{p}}^{(it+1)} = \hat{\mathbf{p}}^{(it)} + \alpha_{(it)} \mathbf{c}^{(it)}.$$

$$5 - \mathbf{s}^{(it+1)} = \mathbf{s}^{(it)} - \alpha_{(it)} \mathbf{A} \mathbf{c}^{(it)}.$$

$$6 - \mathbf{r}^{(it+1)} = \mathbf{A}^\top \mathbf{s}^{(it+1)}.$$

$$7 - it = it + 1.$$

$$8 - \text{Repeat previous steps until convergence.}$$


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Different from the classical least-square solution (equation 15), the CGLS solution (Algorithm 1) requires neither inverse matrix nor matrix-matrix product. The CGLS method only requires: one matrix-vector multiplication out of the loop and two matrix-vector multiplications, in steps 3 and 6, at each iteration.

In this work, we will reduce the computational cost of the equivalent layer by substituting exactly these two matrix-vector products with a much more efficient algorithm.

## Conjugate Gradient Least Square method convergence criteria

In theory, this method is bound to converge at  $N$  iterations maximum, but in a later part of this work we show with numerical results that the convergence is much faster for the linear system we are solving. Setting a minimum tolerance of the residuals is a good option to carry out this algorithm efficiently and still obtaining very good results. Another possibility is to set an invariance to the euclidian norm of residuals between interactions, which would increase algorithm runtime, but with smaller residuals. We chose the first option, as we achieve satisfactory results.

## Non-symmetric Block-Toeplitz Toeplitz-Block structure of matrix A

Let us consider that the observed total-field anomaly is located on an  $N_x \times N_y$  regular grid of points spaced by  $\Delta_x$  and  $\Delta_y$  along the  $x$ - and  $y$ -directions, respectively. The notation used in this work will be the same as the one presented in Takahashi et al. (2020), where the authors described the structure of the symmetric Block-Toeplitz Toeplitz-Block matrix of the gravimetric equivalent layer. Here, we also establish a relation between the pair of *matrix coordinates*  $(x_i, y_i)$ ,  $i = 1, \dots, N$  or  $(x_j, y_j)$ ,  $j = 1, \dots, M = N$  and a pair of *grid coordinates*  $(x_k, y_l)$  given as:

$$x_i \equiv x_k = x_1 + [k(i) - 1] \Delta x , \quad (18)$$

and

$$y_i \equiv y_l = y_1 + [l(i) - 1] \Delta y . \quad (19)$$

In a *x-oriented grid* the indices  $i$  (or  $j$ ) relate as integer functions of  $k$  and  $l$  by:

$$i(k, l) = (l - 1) N_x + k , \quad (20)$$

$$l(i) = \left\lceil \frac{i}{N_x} \right\rceil \quad (21)$$

and

$$k(i) = i - \left\lceil \frac{i}{N_x} \right\rceil N_x + N_x . \quad (22)$$

For *y-oriented grid* they are given by:

$$i(k, l) = (k - 1) N_y + l , \quad (23)$$

$$k(i) = \left\lceil \frac{i}{N_y} \right\rceil \quad (24)$$

and

$$l(i) = i - \left\lceil \frac{i}{N_y} \right\rceil N_y + N_y , \quad (25)$$

where in equations 21 to 25,  $\lceil \cdot \rceil$  is the ceiling function. Figure 1 shows an example of a grid  $N_x \times N_y$ , where  $N_x = 4$  and  $N_y = 3$  demonstrating the relation between the *matrix coordinates* with  $k(i)$  and  $l(i)$  depending on the orientation of the grid.

The  $N \times M$  sensitivity matrix  $\mathbf{A}$  (equation 12) can be represented as a grid of  $Q \times Q$  blocks  $\mathbf{A}^q$ ,  $q = -Q + 1, \dots, 0, \dots, Q - 1$ . Each block  $\mathbf{A}^q$  has  $P \times P$  elements  $a_p^q$  where  $p = -P + 1, \dots, 0, \dots, P - 1$

In a *x-oriented grid*  $q$  and  $p$  give the number of blocks ( $Q = N_y$ ) and the number of elements of each block ( $P = N_x$ ). They can be defined by the integer functions:

$$q(i, j) = l(i) - l(j) \quad (26)$$

and

$$p(i, j) = k(i) - k(j) , \quad (27)$$

where equations 21 and 22 describe  $l(i)$  and  $l(j)$  and  $k(i)$  and  $k(j)$ , respectively. When using *y-oriented grids*,  $q$  and  $p$  still define the number of block and block elements, but

$Q = N_x$  and  $P = N_y$ . Moreover, the integer functions changes to:

$$q(i, j) = k(i) - k(j) \quad (28)$$

and

$$p(i, j) = l(i) - l(j), \quad (29)$$

where equation 24 now defines  $k(i)$  and  $k(j)$  and equation 25 defines  $l(i)$  and  $l(j)$ . Note that equations 26, 27, 28 and 28 differs from the ones presented in Takahashi et al. (2020) by the absence of the module.

In both  $x$ - or  $y$ -orientation, matrix  $\mathbf{A}$  (equation 12) can be rewritten by the indices  $q = -Q + 1, \dots, 0, \dots, Q - 1$  defining the number of its blocks:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^0 & \mathbf{A}^{-1} & \cdots & \mathbf{A}^{-Q+1} \\ \mathbf{A}^1 & \mathbf{A}^0 & \mathbf{A}^{-1} & \vdots \\ \vdots & \mathbf{A}^1 & \ddots & \mathbf{A}^{-1} \\ \mathbf{A}^{Q-1} & \cdots & \mathbf{A}^1 & \mathbf{A}^0 \end{bmatrix}_{N \times N}, \quad (30)$$

and by indice  $p$ , where each block has  $P \times P$  elements  $a_p^q$ ,  $p = -P + 1, \dots, 0, \dots, P - 1$ :

$$\mathbf{A}^q = \begin{bmatrix} a_0^q & a_{-1}^q & \cdots & a_{-P+1}^q \\ a_1^q & a_0^q & a_{-1}^q & \vdots \\ \vdots & a_1^q & \ddots & a_{-1}^q \\ a_{P-1}^q & \cdots & a_1^q & a_0^q \end{bmatrix}_{P \times P}, \quad (31)$$

In general, matrix  $\mathbf{A}$  (equation 12) is a non-symmetric BTTB, i.e., its blocks are non-symmetric ( $\mathbf{A}^{-Q+1} \neq \mathbf{A}^{Q-1}$ ) and its elements also are non-symmetric ( $a_{-1}^q \neq a_1^q$ ). Depending on specific values of the main field direction and the equivalent sources magnetization directions, matrix  $\mathbf{A}$  can assume other structures, for example, when  $\hat{\mathbf{F}} = [0, 0, 1]$  and

$\hat{\mathbf{h}} = [0, 0, 1]$  it becomes symmetric. In this work, we are considering the more common situation for the matrix  $\mathbf{A}$ .

Also differently for the symmetric sensitivity matrix described by Takahashi et al. (2020), the non-symmetric BTTB matrix cannot be reconstructed only by its first column. The construction of the matrix  $\mathbf{A}$  (equation 12) needs four columns: the first and last columns of the first column of blocks and the first and last columns of the last column of blocks. This has a physical implication in the equivalent layer which is not possible to use only one equivalent source to reproduce all elements of matrix  $\mathbf{A}$ , such as in the gravity case as demonstrated by Takahashi et al. (2020). Rather, in the magnetic case it takes four equivalent sources positioned at each corner of the equivalent layer. Figure 2 shows the positioning of the equivalent sources in a regular grid  $N_x = 4$   $N_y = 3$  necessary to calculate the four columns capable of recovering the matrix  $\mathbf{A}$ .

In this work, we propose a different approach, by calculating the first column of all six different components of second derivatives matrices from  $\mathbf{H}_{ij}$  (equation 10). These matrices are, in fact, symmetric or skew-symmetric BTTBs, meaning that the first column has all elements of each matrix.

By substituting equations 2, 9 and 10 into equation 12, it is possible to describe each element of the sensitivity matrix by the second derivative components of  $\mathbf{H}_{ij}$ :

$$\begin{aligned} a_{ij} = c_m \frac{\mu_0}{4\pi} & (F_x H_{xx} + F_y H_{xy} + F_z H_{xz}) h_x + \\ & (F_x H_{xy} + F_y H_{yy} + F_z H_{yz}) h_y + \\ & (F_x H_{xz} + F_y H_{yz} + F_z H_{zz}) h_z . \end{aligned} \quad (32)$$

If we consider that  $c_m \frac{\mu_0}{4\pi}$ ,  $\hat{\mathbf{F}}$  (equation 2) and  $\hat{\mathbf{h}}$  (equation 9) are constants multiplying the second derivatives  $\mathbf{H}$  (equation 10), the sensitivity matrix  $\mathbf{A}$  (equation 12) is purely the

sum of the components  $H_{xx} + H_{xy} + H_{xz} + H_{xy} + H_{yy} + H_{yz} + H_{xz} + H_{yz} + H_{zz}$  multiplied by the respective constants of each component. Thus, despite  $\mathbf{A}$  not being a symmetric BTTB matrix, it can be in fact, written by calculating only the first column of these components. In the next few sections we will describe each component  $\mathbf{H}$  as its own matrix.

### Structure of matrices components $\mathbf{H}_{xx}$ , $\mathbf{H}_{yy}$ and $\mathbf{H}_{zz}$

We can describe the elements of  $\mathbf{H}_{xx}$ ,  $\mathbf{H}_{yy}$  and  $\mathbf{H}_{zz}$  by substituting equations 18 and 19 in equation 10 as:

$$h_{ij}^{xx} = \frac{-1}{[(\Delta k_{ij} \Delta x)^2 + (\Delta l_{ij} \Delta y)^2 + (\Delta z)^2]^{\frac{3}{2}}} + \frac{3(\Delta k_{ij} \Delta x)^2}{[(\Delta k_{ij} \Delta x)^2 + (\Delta l_{ij} \Delta y)^2 + (\Delta z)^2]^{\frac{5}{2}}}, \quad (33)$$

$$h_{ij}^{yy} = \frac{-1}{[(\Delta k_{ij} \Delta x)^2 + (\Delta l_{ij} \Delta y)^2 + (\Delta z)^2]^{\frac{3}{2}}} + \frac{3(\Delta l_{ij} \Delta y)^2}{[(\Delta k_{ij} \Delta x)^2 + (\Delta l_{ij} \Delta y)^2 + (\Delta z)^2]^{\frac{5}{2}}} \quad (34)$$

and

$$h_{ij}^{zz} = \frac{-1}{[(\Delta k_{ij} \Delta x)^2 + (\Delta l_{ij} \Delta y)^2 + (\Delta z)^2]^{\frac{3}{2}}} + \frac{3(\Delta z)^2}{[(\Delta k_{ij} \Delta x)^2 + (\Delta l_{ij} \Delta y)^2 + (\Delta z)^2]^{\frac{5}{2}}}, \quad (35)$$

where  $\Delta z = z_j - z_i$ ,  $\Delta k_{ij} = k(i) - k(j)$  (equations 22 or 24) and  $\Delta l_{ij} = l(i) - l(j)$  (equations 21 or 25). The principal components  $\mathbf{H}_{xx}$ ,  $\mathbf{H}_{yy}$  and  $\mathbf{H}_{zz}$  (equations 33, 34 and 35, respectively) of matrix  $\mathbf{H}$  (equation 10) are symmetric-Block-Toeplitz symmetric-Toeplitz-Block matrices. This means that  $\mathbf{H}_{xx}$ ,  $\mathbf{H}_{yy}$  and  $\mathbf{H}_{zz}$  are Toeplitz and symmetric by its blocks and each of the blocks are symmetric Toeplitz matrices. For example,  $\mathbf{H}_{xx}$  can be described by the *block indice*  $q$  that represent the block diagonals of this matrix as a grid of  $Q \times Q$

blocks  $\mathbf{H}_{\mathbf{xx}}^q$ ,  $q = 0, \dots, Q - 1$ :

$$\mathbf{H}_{\mathbf{xx}} = \begin{bmatrix} \mathbf{H}_{\mathbf{xx}}^0 & \mathbf{H}_{\mathbf{xx}}^1 & \dots & \mathbf{H}_{\mathbf{xx}}^{Q-1} \\ \mathbf{H}_{\mathbf{xx}}^1 & \mathbf{H}_{\mathbf{xx}}^0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{H}_{\mathbf{xx}}^1 \\ \mathbf{H}_{\mathbf{xx}}^{Q-1} & \dots & \mathbf{H}_{\mathbf{xx}}^1 & \mathbf{H}_{\mathbf{xx}}^0 \end{bmatrix}_{N \times N}. \quad (36)$$

And each diagonal of the blocks are represented by  $P \times P$  elements  $h_p^q$ ,  $p = 0, \dots, P - 1$ :

$$\mathbf{H}_{\mathbf{xx}}^q = \{h_p^q\} = \begin{bmatrix} h_0^q & h_1^q & \dots & h_{P-1}^q \\ h_1^q & h_0^q & \ddots & \vdots \\ \vdots & \ddots & \ddots & h_1^q \\ h_{P-1}^q & \dots & h_1^q & h_0^q \end{bmatrix}_{P \times P}. \quad (37)$$

In a *x-oriented grid*  $Q = N_x$ ,  $P = N_y$  and  $q$  and  $p$  can be defined by the functions:

$$q(i, j) = |l(i) - l(j)| \quad (38)$$

and

$$p(i, j) = |k(i) - k(j)|, \quad (39)$$

where  $l(i)$  and  $l(j)$  are defined by equation 21 and  $k(i)$  and  $k(j)$  are defined by equation 22. For *y-oriented grids*,  $Q = N_x$ ,  $P = N_y$  and the block indices  $q$  and  $p$  are defined, respectively, by the following integer functions of the matrix indices  $i$  and  $j$ :

$$q(i, j) = |k(i) - k(j)| \quad (40)$$

and

$$p(i, j) = |l(i) - l(j)|, \quad (41)$$

This struture can also describe matrices  $\mathbf{H}_{\mathbf{yy}}$  and  $\mathbf{H}_{\mathbf{zz}}$  in the same manner and they are identical to the structure of the gravity sensitivity matrix from Takahashi et al. (2020).

## Structure of the components matrices $\mathbf{H}_{\mathbf{xy}}$

By substituting equations 18 and 19 in equation 10, we can also describe the elements of  $\mathbf{H}_{\mathbf{xy}}$ , as:

$$h_{ij}^{xy} = \frac{3(\Delta k_{ij} \Delta x)(\Delta l_{ij} \Delta y)}{\left[(\Delta k_{ij} \Delta x)^2 + (\Delta l_{ij} \Delta y)^2 + (\Delta z)^2\right]^{\frac{5}{2}}}, \quad (42)$$

The component  $\mathbf{H}_{\mathbf{xy}}$  (equation 42) of matrix  $\mathbf{H}$  (equation 10) are skew symmetric-Block-Toeplitz skew symmetric-Toeplitz-Block matrices. This means that  $\mathbf{H}_{\mathbf{xy}}$  is Toeplitz and skew symmetric by its blocks and each of the blocks are skew symmetric Toeplitz matrices. This way, matrix  $\mathbf{H}_{\mathbf{xy}}$  can be described by the *block indice*  $q$  that represent the block diagonals of this matrix as a grid of  $Q \times Q$  blocks  $\mathbf{H}_{\mathbf{xy}}^q$ ,  $q = -Q+1, \dots, 0, \dots, Q-1$ :

$$\mathbf{H}_{\mathbf{xy}} = \begin{bmatrix} \mathbf{H}_{\mathbf{xy}}^0 & \mathbf{H}_{\mathbf{xy}}^{-1} & \cdots & \mathbf{H}_{\mathbf{xy}}^{-Q+1} \\ \mathbf{H}_{\mathbf{xy}}^1 & \mathbf{H}_{\mathbf{xy}}^0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{H}_{\mathbf{xy}}^{-1} \\ \mathbf{H}_{\mathbf{xy}}^{Q-1} & \cdots & \mathbf{H}_{\mathbf{xy}}^1 & \mathbf{H}_{\mathbf{xy}}^0 \end{bmatrix}_{N \times N}. \quad (43)$$

And each diagonal of the blocks are represented by  $P \times P$  elements  $h_p^q$ ,  $p = -P+1, \dots, 0, \dots, P-1$ :

$$\mathbf{H}_{\mathbf{xy}}^q = \{h_p^q\} = \begin{bmatrix} h_0^q & h_{-1}^q & \cdots & h_{-P+1}^q \\ h_1^q & h_0^q & \ddots & \vdots \\ \vdots & \ddots & \ddots & h_{-1}^q \\ h_{P-1}^q & \cdots & h_1^q & h_0^q \end{bmatrix}_{P \times P}. \quad (44)$$

In a *x-oriented grid*  $Q = N_x$ ,  $P = N_y$  and  $q$  and  $p$  can be defined by the functions:

$$q(i, j) = l(i) - l(j) \quad (45)$$

and

$$p(i, j) = k(i) - k(j), \quad (46)$$

where  $l(i)$  and  $l(j)$  are defined by equation 21 and  $k(i)$  and  $k(j)$  are defined by equation 22. For  $y$ -oriented grids,  $Q = N_x$ ,  $P = N_y$  and the block indices  $q$  and  $p$  are defined, respectively, by the following integer functions of the matrix indices  $i$  and  $j$ :

$$q(i, j) = k(i) - k(j) \quad (47)$$

and

$$p(i, j) = l(i) - l(j) , \quad (48)$$

Important to clarify that in this case, as a skew symmetric matrix, the values of oposing diagonals have oposing signals, e.g.,  $\mathbf{H}_{\mathbf{xy}}^{-1} = -\mathbf{H}_{\mathbf{xy}}^1$  and  $h_{-1}^q = -h_1^q$ .

### Structure of the components matrices $\mathbf{H}_{xz}$

By substituting equations 18 and 19 in equation 10, the elements of  $\mathbf{H}_{xz}$ , are given by:

$$h_{ij}^{xz} = \frac{3(\Delta k_{ij} \Delta x)(\Delta z)}{\left[(\Delta k_{ij} \Delta x)^2 + (\Delta l_{ij} \Delta y)^2 + (\Delta z)^2\right]^{\frac{5}{2}}} , \quad (49)$$

The component  $\mathbf{H}_{xz}$  (equation 49) of matrix  $\mathbf{H}$  (equation 10) are skew symmetric-Block-Toeplitz symmetric-Toeplitz-Block matrices. This means that  $\mathbf{H}_{xz}$  is Toeplitz and skew symmetric by its blocks and each of the blocks are symmetric Toeplitz matrices. Thus, matrix  $\mathbf{H}_{xz}$  can be described by the *block indice*  $q$  that represent the block diagonals of this matrix as a grid of  $Q \times Q$  blocks  $\mathbf{H}_{xz}^q$ ,  $q = -Q + 1, \dots, 0, \dots, Q - 1$ :

$$\mathbf{H}_{xz} = \begin{bmatrix} \mathbf{H}_{xz}^0 & \mathbf{H}_{xz}^{-1} & \dots & \mathbf{H}_{xz}^{-Q+1} \\ \mathbf{H}_{xz}^1 & \mathbf{H}_{xz}^0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{H}_{xz}^{-1} \\ \mathbf{H}_{xz}^{Q-1} & \dots & \mathbf{H}_{xz}^1 & \mathbf{H}_{xz}^0 \end{bmatrix}_{N \times N} . \quad (50)$$

And each diagonal of the blocks are represented by  $P \times P$  elements  $h_p^q$ ,  $p = 0, \dots, P - 1$ :

$$\mathbf{H}_{xz}^q = \{h_p^q\} = \begin{bmatrix} h_0^q & h_1^q & \cdots & h_{P-1}^q \\ h_1^q & h_0^q & \ddots & \vdots \\ \vdots & \ddots & \ddots & h_1^q \\ h_{P-1}^q & \cdots & h_1^q & h_0^q \end{bmatrix}_{P \times P}. \quad (51)$$

In a *x-oriented grid*  $Q = N_x$ ,  $P = N_y$  and  $q$  and  $p$  can be defined by the functions:

$$q(i, j) = l(i) - l(j) \quad (52)$$

and

$$p(i, j) = |k(i) - k(j)|, \quad (53)$$

where  $l(i)$  and  $l(j)$  are defined by equation 21 and  $k(i)$  and  $k(j)$  are defined by equation 22. For *y-oriented grids*,  $Q = N_x$ ,  $P = N_y$  and the block indices  $q$  and  $p$  are defined, respectively, by the following integer functions of the matrix indices  $i$  and  $j$ :

$$q(i, j) = k(i) - k(j) \quad (54)$$

and

$$p(i, j) = |l(i) - l(j)|, \quad (55)$$

In this case as a skew symmetric matrix by blocks, the values of oposing diagonals blocks have oposing signals, e.g.,  $\mathbf{H}_{xz}^{-1} = -\mathbf{H}_{xz}^1$  but each block is a symmetric matrix.

### Structure of the components matrices $\mathbf{H}_{yz}$

Finally, by substituting equations 18 and 19 in equation 10, we can also describe the elements of  $\mathbf{H}_{yz}$ , as:

$$h_{ij}^{yz} = \frac{3(\Delta l_{ij} \Delta y)(\Delta z)}{\left[(\Delta k_{ij} \Delta x)^2 + (\Delta l_{ij} \Delta y)^2 + (\Delta z)^2\right]^{\frac{5}{2}}}, \quad (56)$$

The component  $\mathbf{H}_{yz}$  (equation 56) of matrix  $\mathbf{H}$  (equation 10) are symmetric-Block-Toeplitz skew symmetric-Toeplitz-Block matrices. This means that  $\mathbf{H}_{yz}$  is Toeplitz and symmetric by its blocks and each of the blocks are skew symmetric Toeplitz matrices. Thus, matrix  $\mathbf{H}_{yz}$  can be described by the *block indice*  $q$  that represent the block diagonals of this matrix as a grid of  $Q \times Q$  blocks  $\mathbf{H}_{yz}^q$ ,  $q = 0, \dots, Q - 1$ :

$$\mathbf{H}_{yz} = \begin{bmatrix} \mathbf{H}_{yz}^0 & \mathbf{H}_{yz}^1 & \cdots & \mathbf{H}_{yz}^{Q-1} \\ \mathbf{H}_{yz}^1 & \mathbf{H}_{yz}^0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{H}_{yz}^1 \\ \mathbf{H}_{yz}^{Q-1} & \cdots & \mathbf{H}_{yz}^1 & \mathbf{H}_{yz}^0 \end{bmatrix}_{N \times N}. \quad (57)$$

And each diagonal of the blocks are represented by  $P \times P$  elements  $h_p^q$ ,  $p = -P + 1, \dots, 0, \dots, P - 1$ :

$$\mathbf{H}_{yz}^q = \{h_p^q\} = \begin{bmatrix} h_0^q & h_{-1}^q & \cdots & h_{-P+1}^q \\ h_1^q & h_0^q & \ddots & \vdots \\ \vdots & \ddots & \ddots & h_{-1}^q \\ h_{P-1}^q & \cdots & h_1^q & h_0^q \end{bmatrix}_{P \times P}. \quad (58)$$

In a *x-oriented grid*  $Q = N_x$ ,  $P = N_y$  and  $q$  and  $p$  can be defined by the functions:

$$q(i, j) = |l(i) - l(j)| \quad (59)$$

and

$$p(i, j) = k(i) - k(j), \quad (60)$$

where  $l(i)$  and  $l(j)$  are defined by equation 21 and  $k(i)$  and  $k(j)$  are defined by equation 22. For *y-oriented grids*,  $Q = N_x$ ,  $P = N_y$  and the block indices  $q$  and  $p$  are defined, respectively, by the following integer functions of the matrix indices  $i$  and  $j$ :

$$q(i, j) = |k(i) - k(j)| \quad (61)$$

and

$$p(i, j) = l(i) - l(j) , \quad (62)$$

Being a symmetric matrix by blocks, the values of  $\mathbf{H}_{yz}$  from oposing diagonals blocks are equal, but each block have skew symmetric oposing diagonals, i.e.,  $h_{-1}^q = -h_1^q$ .

## CGLS matrix-vector substitution

As pointed earlier in this work, the main improvement inside the CGLS method (Algorithm 1) for estimating the parameter vector  $\hat{\mathbf{p}}$  (equation 15) is to substitute the matrix-vector multiplication  $\mathbf{A}^\top \mathbf{s}^{(0)}$  out of the loop and the two matrix-vector multiplications inside the loop at steps 3 an 6,  $\mathbf{A}\mathbf{c}^{(it)}$  and  $\mathbf{A}^\top \mathbf{s}^{(it+1)}$ , that is necessary at each iteration and takes most of its runtime.

Our method consists in calculating the six first columns of the second derivatives of  $\mathbf{H}$  (equation 10) and embed them into the first six columns of the block-circulant circulant-block (BCCB) matrices related to the  $\mathbf{H}$  components. Thus, it is possible to calculate the first column of the BCCB matrix embedded from matrix  $\mathbf{A}$  (equation 12) by multiplying each component with its respective constants and summing as shown in equation 32. In Takahashi et al. (2020), Appendix A, the authors demonstrated in details how to transform a symmetric BTB matrix into a BCCB matrix  $\mathbf{C}$ . The process here is the same and that work can be referenced to achieve the same results.

A new auxilary linear system is constructed to carry the matrix-vector product:

$$\mathbf{w} = \mathbf{C}\mathbf{v} , \quad (63)$$

where

$$\mathbf{w} = \begin{bmatrix} \mathbf{w}_0 \\ \vdots \\ \mathbf{w}_{Q-1} \\ \mathbf{0}_{2N \times 1} \end{bmatrix}_{4N \times 1}, \quad (64)$$

$$\mathbf{w}_q = \begin{bmatrix} \mathbf{d}_q(\mathbf{p}) \\ \mathbf{0}_{P \times 1} \end{bmatrix}_{2P \times 1}, \quad (65)$$

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_{Q-1} \\ \mathbf{0}_{2N \times 1} \end{bmatrix}_{4N \times 1}, \quad (66)$$

and

$$\mathbf{v}_q = \begin{bmatrix} \mathbf{p}_q \\ \mathbf{0}_{P \times 1} \end{bmatrix}_{2P \times 1}, \quad (67)$$

where  $\mathbf{C}$  (equation 63) is a  $4N \times 4N$  non-symmetric (BCCB) resulted from transforming  $\mathbf{A}$  (equation 12). Without having to calculate the whole BCCB matrix, its first column can be used to carry the multiplication of this new system (equation 63). Appendix A and C in Takahashi et al. (2020) shows how to use the 2D-FFT to compute the eigenvalues of matrix  $\mathbf{C}$ , store in a  $2Q \times 2P$  matrix using the *vec*-operator and to carry the matrix-vector product. Denoting matrix  $\mathbf{L}$  as the eigenvalues matrix follows:

$$\mathbf{F}_{2Q}^* [\mathbf{L} \circ (\mathbf{F}_{2Q} \mathbf{V} \mathbf{F}_{2P})] \mathbf{F}_{2P}^* = \mathbf{W}, \quad (68)$$

where the symbol “ $\circ$ ” references the Hadamard product, i.e., a element-wise complex multiplication between the eingenvectors and the 2D-FFT of the matrix rearranged along the rows of the parameters  $\mathbf{V}$  (equation 66) using the same *vec*-operator. The resulting inverse

2D-FFT denoted by  $\mathbf{F}_{2Q}^* \otimes \mathbf{F}_{2P}^*$  is also a  $2Q \times 2P$  matrix ( $\mathbf{W}$ ) that can be rearranged to the predicted data vector  $\mathbf{d}(\hat{\mathbf{p}})$  size  $N$ .

## Computational performance

To compare the efficiency of our algorithm we will use a numerical approach and calculate the floating-point operations (*flops*), i.e., count the number of mathematical operations necessary to complete the estimative of parameter vector  $\hat{\mathbf{p}}$  of the normal equations (equation 15) and both the CGLS methods (algorithm 1) for calculating the matrix-vector product by its standart way and our approach.

The *flops* needed to solve the linear system in equation 15 using the Cholesky factorization is:

$$f_{classical} = \frac{7}{3}N^3 + 6N^2, \quad (69)$$

where  $N$  is the total number of observation points and also the size of estimated parameter vector  $\hat{\mathbf{p}}$ .

For the more efficient CGLS algorithm the estimative can be done in:

$$f_{cglsl} = 2N^2 + it(4N^2 + 12N). \quad (70)$$

However, our approach reduces further to:

$$f_{ours} = \kappa 16N \log_2(4N) + 24N + it(\kappa 16N \log_2(4N) + 60N), \quad (71)$$

where  $\kappa$  depends on the FFT algorithm. By default, in this work we will use  $\kappa = 5$  for the *radix-2* algorithm (Van Loan, 1992).

Figure 3 shows a comparative between the methods varying the number of observation points up to 1,000,000, where it is possible to observe a reduction of  $10^7$  orders of magnitude

to estimate parameter vector  $\hat{\mathbf{p}}$  in relation to the non-iterative classical method and  $10^3$  orders of magnitude in relation to the standart CGLS algorithm using 50 iterations. A more detailed, step by step, flops count of the classical and CGLS algorithm can be found in Appendix A.

In figure 4 we show the time necessary to construct matrix  $\mathbf{A}$  (equation 12) and solve the linear system up to 10,000 points of observation. With this dataset the classical method takes more than sixty-three seconds, the CGLS more than twelve seconds, while our method takes only half a second. The cpu used for this test was a intel core i7-7700HQ@2.8GHz.

In figure 5 a comparison between the time to complete the task to calculate the first column of the BCCB matrix embedded from the from  $\mathbf{A}$  (equation 12) by using only one equivalent source, i.e., calculating all six first column of the second derivatives matrices from  $\mathbf{H}$  (equation 10) and using four equivalent sources to calculate the four necessary columns from the non-symmetric matrix  $\mathbf{A}$  (equation 12). Although, very similar in time, with one source a small advantage can be observed as the number of data  $N$  increases and goes beyond  $N = 200,000$ . This test was done from  $N = 10,000$  to  $N = 700,000$  with increases of 5,625 observation points.

In Table 1 there is comparison between how much RAM memory is adressed to store the sensitivity matrix for each of the methods. The classical approach and the CGLS have to store the whole matrix  $\mathbf{A}$  (equation 12), this means that a dataset with for example  $N = 10,000$  observation points, the sensitivity matrix has  $N^2 = 100,000,000$  elements and takes approximately 763 Megabytes of memory (8 bytes per element). For our method, it is necessary to store the first six columns of each of the components from matrix  $\mathbf{H}$  (equation 10) embedded into the BCCB matrices. With the same dataset  $N = 10,000$

it needs 1.831 Megabytes. After completing the steps to store the eigenvalues of matrix  $\mathbf{C}$  (equation 63) it takes only 0.6104 Megabytes. Here, we are considering 16 bytes per element as the eigenvalues are complex numbers resulting from the 2D FFT. For a bigger dataset as  $N = 1,000,000$  the amount of RAM necessary goes to 7,629,395, 183.096 and 61.035 Megabytes, respectively, showing the necessity to find improved and efficient methods for the equivalent layer technique as the one presented in this work. We remember that throughout our work we are always considering  $N = M$ .

## APPLICATION TO SYNTHETIC DATA

The synthetic data application of the fast equivalent layer for magnetic data was conducted on a regular grid of  $80 \times 80$  points, totaling  $N = 6,400$  observation points. Three bodies were modeled: two prisms and a sphere with inclination, declination and intensity of  $0^\circ$  and  $45^\circ$  and  $2\sqrt{2}$  A/m, respectively. The main field has inclination and declination of  $10^\circ$  and  $37^\circ$ , respectively. Figure 6a shows the synthetic data created for this test.

Using a classical linear inversion method (equation 15) a predicted data was estimated in figure 7a. The residuals between the observed (6b) and the predicted data (7a), with mean 0.3712 nT and standart deviation of 0.2798 nT is shown in figure 7b. This process took 17.10 seconds.

Using the CGLS method with the fast BTTB matrix-vector product a predicted data was estimated in figure 8a. The residuals between the observed (6b) and the predicted data (8a), with mean 0.5150 nT and standart deviation of 0.4363 nT is shown in figure 8b. This process took 0.18 seconds.

In figure 9 we have the convergence analysis of the CGLS method to estimate the equivalent sources parameter vector  $\mathbf{d}(\hat{\mathbf{p}})$  used for this synthetic test. The squared euclidian norm of residuals decreases as expected, with good results at 50 iterations, stabilizing afterwards. This result shows that it is not necessary to run the conjugate gradient least square method at  $N$  iterations to get a theoretically exactly solution (which in practice would never occur due to roundoff errors), as this could be very demanding to process even with a small layer with  $N = 6,400$  equivalent sources.

## Tests with irregulars grids

As shown in theory, a regular grid of observation points is needed to arise the BTTB matrix. In this section, we show the results when this algorithm is used directly into irregular grids of  $N = 5\,000$  observation points. First, we set up a regular grid of  $100 \times 50$  observation points in the  $x$ - and  $y$ -directions with a grid spacing of  $\Delta x$  of 101.01 m along the  $x$ -axis and  $\Delta y$  of 163.265 m along the  $y$ -axis. Next, the  $x$  and  $y$  coordinates of the observations were also contaminated with pseudorandom Gaussian noise with zero mean and standard deviations of 10%, 30% and 50% of the  $\Delta x$  and  $\Delta y$  spacing.

Figure 10a shows an irregular grid with standart deviations of 10% along both  $x$ - and  $y$ -direction of the observation points. Using the classical approach the residuals between the observed (10b) and the predicted data (11a) has mean 0.3628 nT, standart deviation of 0.2727 nT and is shown in figure 11b. Using the new approach of this work the residuals between the observed (10b) and the predicted data (12a) has mean 0.6024 nT, standart deviation of 0.4998 nT and is shown in figure 12b. In figure 13 we have the convergence analysis of the CGLS method to estimate the equivalent sources used for this synthetic test. The squared euclidian norm decreases as expected, with good results at 50 iterations, stabilizing afterwards.

Figure 14a shows an irregular grid with standart deviations of 20% along both  $x$ - and  $y$ -direction of the observation points. Using the classical approach the residuals between the observed (14b) and the predicted data (15a) has mean 0.3630 nT, standart deviation of 0.2731 nT and is shown in figure 15b. Using the new approach of this work the residuals between the observed (14b) and the predicted data (16a) has mean 0.7147 nT, standart deviation of 0.5622 nT and is shown in figure 16b. In figure 17 we have the convergence

analysis of the CGLS method to estimate the equivalent sources used for this synthetic test.

The squared euclidian norm decreases as expected and stabilizing afterwards.

Figure 18a shows an irregular grid with standart deviations of 30% along both  $x$ - and  $y$ -direction of the observation points. Using the classical approach the residuals between the observed (18b) and the predicted data (19a) has mean 0.3634 nT, standart deviation of 0.2735 nT and is shown in figure 19b. Using the new approach of this work the residuals between the observed (18b) and the predicted data (20a) has mean 0.9788 nT, standart deviation of 0.7462 nT and is shown in figure 20b. In figure 21 we have the convergence analysis of the CGLS method to estimate the equivalent sources used for this synthetic test.

The squared euclidian norm decreases as expected in the begining, but starts increasing and not converging afterwards.

This results show that, when this approach is used in irregular grids, there is a limit of how much deviation the observation points can have before it starts to produce errors in predicted data. This is confirmed with the convergence analysis, when the irregularity is too large, the linear system stops converging.

Another set of tests were also made with the same previous grid configuration, but now with deviations in the  $z$ -*direction*, i.e., the observation points were no longer in a plane. In the previous tests, the total-field anomaly was computed at 900 m height. The  $z$ coordinate of the observations were contaminated with pseudorandom Gaussian noise with zero mean and standard deviations of 5%, 10%, and 20% of the 900 height.

Figure 22a shows a regular grid of  $100 \times 50$  in the  $x$ - and  $y$ -directions located on an uneven surface of observations where the  $z$  coordinates were corrupted with a standard deviation of 5% of the height. Using the classical approach the residuals between the observed (22b) and

the predicted data (23a) has mean 0.3712 nT, standart deviation of 0.2870 nT and is shown in figure 23b. Using the new approach of this work the residuals between the observed (22b) and the predicted data (24a) has mean 0.9542 nT, standart deviation of 0.8943 nT and is shown in figure 24b. In figure 25 we have the convergence analysis of the CGLS method to estimate the equivalent sources used for this synthetic test. The squared euclidian norm decreases as expected, with good results at 50 iterations, stabilizing afterwards.

Figure 26a shows a regular grid of  $100 \times 50$  in the  $x$ - and  $y$ -directions located on an uneven surface of observations where the  $z$  coordinates were corrupted with a standard deviation of 10% of the height. Using the classical approach the residuals between the observed (26b) and the predicted data (27a) has mean 0.3865 nT, standart deviation of 0.3216 nT and is shown in figure 27b. Using the new approach of this work the residuals between the observed (26b) and the predicted data (28a) has mean 1.6105 nT, standart deviation of 1.6231 nT and is shown in figure 28b. In figure 29 we have the convergence analysis of the CGLS method to estimate the equivalent sources used for this synthetic test. The squared euclidian norm decreases as expected, with good results at 50 iterations, stabilizing afterwards.

Figure 30a shows a regular grid of  $100 \times 50$  in the  $x$ - and  $y$ -directions located on an uneven surface of observations where the  $z$  coordinates were corrupted with a standard deviation of 20% of the height. Using the classical approach the residuals between the observed (30b) and the predicted data (31a) has mean 0.4155 nT, standart deviation of 0.4005 nT and is shown in figure 31b. Using the new approach of this work the residuals between the observed (30b) and the predicted data (32a) has mean 6.6220 nT, standart deviation of 5.901 nT and is shown in figure 32b. In figure 33 we have the convergence analysis of the CGLS method to estimate the equivalent sources used for this synthetic

test. The squared euclidian norm decreases slower than previous tests and starts increasing afterwards showing that the convergence is not possible.

Once again, we observe that while the classical linear inversion method can predict the data even with high irregularities in the observation points grid, the method presented in this work stars to create errors in the estimative. Through the convergence graphs it is possible to see the increase of the squared euclidian norm of the residuals, but the sensitivity to uncertainties in the  $z$  coordinates of the observations is higher than the the sensitivity to uncertainties in the  $x$  and  $y$  coordinates of the observations. With an average of 20% of deviation in the  $z$ -*direction* the system stopped converging while only at an average deviation of 30% in the  $x$ -*direction* and 30% in the  $y$ -*direction* that the same convergence problem occured.

## APPLICATION TO FIELD DATA

The field data application was performed with the aeromagnetic data of Carajás, Pará, Brazil, provided by CPRM. The survey is composed of 131 flight lines N-S oriented, spacing  $\Delta y = 3,000$  m. The magnetometer (Scintrex CS-3) was set to a interval between measurements of 0.1 s giving a spacing  $\Delta x = 7.65$  m. The average flight heighth is  $\Delta z = -900$  m. The total number of observation points is  $N = 6,081,345$ . Figure 34 shows the observed magnetic field data of the area.

For the actual data processing, we have made a comparison between an interpolated regular grid of  $10,000 \times 131$  using a nearest neighbour algorithm and a decimated irregular grid, also of  $10,000 \times 131$ , totaling  $N = 1,310,000$  observation points in both cases. The decimated grid was performed by using the regular grid created in the first case as a guide and by finding the nearest real observation point to this regular grid thus, ensuring that the irregular grid is the lesser deviant possible to conduct the BTB scheme. In figure 35a we show the result of the interpolation and in 35b the result of the decimation. With 1,310,000 observation points, it would be necessary 12.49 Terabytes of RAM to store the full sensitivitiy matrix with the classical approach. However, taking advantage that the second derivatives of equation 6 are symmetric or skew-symmetric matrices, it is possible to reconstruct the whole sensitivitiy matrix storing only the first column of each component of equation 10, thus, using only 59.97 Megabytes, allowing desktop computers being able to process this amount of data.

As this area is very large, different values of the magnetic main field can be considered. For this processing, it was considered an approximated mid location of the area (latitude  $-6.5^\circ$  and longitude  $-50.75^\circ$ ) where the declination is  $-19.86^\circ$  for the IGRF model in

1st january, 2014. The inclination was calculated considering the Geocentric axial dipole model ( $\tan I = 2 \times \tan \lambda$ ) and is equal to  $12.84^\circ$ . As the source magnetization is unknown, inclination and declination equals to the main field is being used.

To achieve high efficiency in property estimative of the equivalent sources, the method CGLS for inversion was used, combined with a fast matrix-vector product, only possible because of the BTTB structure of the sensitivity matrix. This fast matrix-vector product was also used for data processing (upward-continuation) in a very efficient way.

Using a equivalent layer at 300 meters above the ground the predicted data and its residual of the interpolated regular grid are shown in figure 36. The mean of 0.07979 nT and the standart deviation of 0.5060 nT of the residual shows the good result of physical property estimative. It was used 200 iterations of the CGLS method taking 390.80 seconds with a Intel core i7 7700HQ@2.8GHz processor in single-processing and single-threading modes. Using the same equivalent layer and CGLS configuration the predicted data and its residual of the decimated irregular grid are shown in figure 37. With a mean of 0.07348 nT, standart deviation of 0.3172 nT and lower residual amplitude compared to the regular grid, we show that the process of decimating the original data, without creating new observation points with interpolation, can be benefical to the method, even if an irregular grid is taken place. It took 385.56 seconds to complete the estimative.

The convergence analysis for the decimated irregular grid up to 2,000 CGLS iterations is in figure 38, showing good convergence rate and guaranteeing that the irregular grid is not disturbing the method.

In figure 39 the upward-continuation transformation using the estimated equivalent layer with the decimated grid was made in a horizontal plane at 5,000 meters and took

2.64 seconds, showing good results without visible errors or border effects problems and accentuating the long wavelenghts.

## CONCLUSIONS

In this work, we were able to develop a fast equivalent layer technique for processing magnetic data with the method of Conjugate Gradient Least Square using the convolutional equivalent layer theory to obtain results of performance more than four orders of magnitude less than the classical equivalent layer. The sensitivity matrix of the magnetic equivalent layer carries the structure of BTTB matrices, which means a very low computational cost matrix-vector product and also the possibility to store only the first column of the matrix BCCB. In this work we propose a novel method to use only one equivalent source and calculating the first six columns of the inverse of distance second derivatives matrices to arrive in the first column of the BCCB matrix embedded from the original magnetic kernel sensitivity matrix.

Synthetic tests showed similar results estimating the physical property using a classical approach to solve a linear system and our method using the CGLS combined with the BTTB matrix-vector product. The difference in time, however, is noticeable: 2.04 seconds using the classical approach and 0.083 seconds using our approach. This difference was obtained with a mid-size mesh of  $80 \times 80$  points, greater results can be obtained if more observation points are used.

Real data test were also conducted in the region of Carajás, Pará, Brazil. With an irregular grid of 1,310,000 observation points, store the full sensitivity matrix it would be necessary 12.49 Terabytes of RAM. However, taking advantage of the symmetric or skew-symmetric matrices structures, it is possible to reconstruct the whole sensitivity matrix using only 59.97 Megabytes. Using 200 iterations of the CGLS method took 385.56 seconds and very good results of property estimative were obtained. Also the upward-continuation

transformation showed good results and took only 2.64 seconds.

## **ACKNOWLEDGEMENTS**

Diego Takahashi was supported by a Phd scholarship from CAPES. Valéria C.F. Barbosa was supported by fellowships from CNPq (grant 307135/2014-4) and FAPERJ (grant 26/202.582/2019). Vanderlei C. Oliveira Jr. was supported by fellowships from CNPq (grant 308945/2017-4) and FAPERJ (grant E-26/202.729/2018). The authors thank the Geological Survey of Brazil (CPRM) for providing the field data.

## APPENDIX A

### FLOPS COMPUTATIONS

#### Classical flops count

The flops count of the classical approach to solve the linear system (equation 15) using the Cholesky factorization is given by equation 69. The step-by-step count follows:

- (1)  $J = \mathbf{A}^\top \mathbf{A}$ :  $\frac{1}{2}N^3$  (lower triangular matrix-matrix product).
- (2)  $\mathbf{C}_f$ :  $\frac{1}{3}N^3$  (one Gaxpy Cholesky factorization of  $J$  (Golub and Loan, 2013,p. 164)).
- (3)  $\mathbf{A}^\top \mathbf{d}^o$ :  $2N^2$  (one matrix-vector product).
- (4)  $\mathbf{C}_f(\mathbf{A}^\top \mathbf{d}^o)$ :  $2N^2$  (one matrix-vector product).
- (5)  $\mathbf{C}_f^\top (\mathbf{C}_f \mathbf{A}^\top \mathbf{d}^o)$ :  $2N^2$  (one matrix-vector product).

Summing all calculations:

$$f_{classical} = \frac{5}{6}N^3 + 6N^2, \quad (A-1)$$

#### CGLS flops count

The flops count of CGLS algorithm 1 can be summarized as:

Out of the loop:

- (1)  $\mathbf{A}^\top \mathbf{s}$ :  $2N^2$  (one matrix-vector product).

Inside the loop:

- (1)  $\frac{\mathbf{r}^{(it)\top} \mathbf{r}^{(it)}}{\mathbf{r}^{(it-1)\top} \mathbf{r}^{(it-1)}}$ :  $4N$  (two vector-vector products).
- (2)  $\mathbf{r}^{it} - \alpha_{it} \beta_{it} \mathbf{c}^{(it-1)}$ :  $2N$  (one scalar-vector product and one vector subtraction).
- (3)  $\frac{\|\mathbf{r}^{(it)}\|_2^2}{(\mathbf{c}^{(it)\top} \mathbf{A}^\top)(\mathbf{A} \mathbf{c}^{(it)})}$ :  $2N^2 + 2N$  (one matrix-vector and one vector-vector product).
- (4)  $\hat{\mathbf{p}}^{it} - \alpha_{it} \mathbf{c}^{(it)}$ :  $2N$  (one vector subtraction).
- (5)  $\mathbf{s}^{it} - \alpha_{it} \mathbf{A} \mathbf{c}^{(it)}$ :  $2N$  (one vector subtraction, the matrix-vector product was calculated in step 3).
- (6)  $\mathbf{A}^\top \mathbf{s}^{(it+1)}$ :  $2N^2$  (one matrix-vector product).

The result of all flops count leads to:

$$f_{cgls} = 2N^2 + it(4N^2 + 12N). \quad (\text{A-2})$$

### Our modified CGLS flops count

All the flops count presented in previous section for the CGLS remains, only substituting the out of the loop matrix-vector product in step 1 and the two matrix-vector products inside the loop in steps 3 and 6. The computations necessary to carry the matrix-vector used in this work are given by:

- (1)  $\mathbf{L}$ :  $\kappa 4N \log_2(4N)$  (one 2D FFT for the eigenvalues calculation of the sensitivity matrix  $\mathbf{A}$  or the transposed sensitivity matrix  $\mathbf{A}^\top$ ).
- (2)  $\mathbf{F}_{2Q} \mathbf{V} \mathbf{F}_{2P}$ :  $\kappa 4N \log_2(4N)$  (one 2D FFT).
- (3)  $\mathbf{L} \circ (\mathbf{F}_{2Q} \mathbf{V} \mathbf{F}_{2P})$ :  $24N$  (one complex Hadamard matrix multiplication).

(4)  $\mathbf{F}_{2Q}^* [\mathbf{L} \circ (\mathbf{F}_{2Q} \mathbf{V} \mathbf{F}_{2P})] \mathbf{F}_{2P}^*$ :  $\kappa 4N \log_2(4N)$  (one inverse 2D FFT).

Matrix-vector product total:  $\kappa 12N \log_2(4N) + 24N$ .

As matrix  $\mathbf{A}$  (equation 12) and its transposed never changes, it is not necessary to calculate the eigenvalues inside the loop at each iteration, we are considering that both are calculated out of the loop. Inside the loop, steps 2 to 4 are repeated two times per iteration. Substituting this result into the CGLS flops count (equation A-2) leads to:

$$f_{ours} = \kappa 16N \log_2(4N) + 24N + it(\kappa 16N \log_2(4N) + 60N). \quad (\text{A-3})$$

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1 This table shows the RAM memory usage (in Megabytes) for storing the whole matrix **A** (equation 12), the sum of all six first columns of the BCCB matrices embedded from the components of the matrix **H** from equation 10 (both need 8 bytes per element) and the matrix **L** containing the eigenvalues complex numbers (16 bytes per element) resulting from the diagonalization of matrix **C** (equation 63). Here we must consider that  $N$  observation points forms a  $N \times N$  matrix.

2 This table shows the means and standard deviations for each residual presented in the synthetic data tests between the observation data in comparison to the classical solution and our solution.

3 This table shows the means and standard deviations for each residual presented in the synthetic data tests section for the upward continuation transformation comparison.

4 This table shows the means and standard deviations for the residuals presented in the real data tests section, for the solution of the regular interpolated grid and the irregular decimated grid.

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1 Schematic representation of an  $N_x \times N_y$  regular grid of points (black dots) defined by  $N_x = 4$  and  $N_y = 3$ . The grids are oriented along the (a)  $x$ -axis and (b)  $y$ -axis. The grid coordinates are  $x_k$  and  $y_l$ , where the  $k = 1, \dots, N_x$  and  $l = 1, \dots, N_y$  are called the grid indices. The insets show the grid indices  $k$  and  $l$ .

2 Representation of the four equivalent sources (black dots) needed to reconstruct the non-symmetric matrix  $\mathbf{A}$  (equation 12). Each of the equivalent sources are located in the corner of the simulated regular grid of  $M_x = 4$  and  $M_y = 3$  sources. The influence of these sources on each of the observation points (blue dots) in the regular grid of  $N_x = 4$  and  $N_y = 3$  will give the four columns necessary of matrix  $\mathbf{A}$ .

3 Number of flops necessary to estimate the parameter vector  $\hat{\mathbf{p}}$  using the non-iterative classical method (equation 69) the CGLS (equation 70) and our modified CGLS method (equation 71) with  $N^{it} = 50$ . The observation point  $N$  varied from 5,000 to 1,000,000. The radix-2 2D FFT algorithm was considered for our method, with  $\kappa = 5$ .

4 Comparison between the runtime of the equivalent-layer technique using the classical, the CGLS algorithm and our method. The values for the CGLS and our methods were obtained for  $N^{it} = 50$  iterations.

5 Comparison between the runtime to calculate the first column of the BCCB matrix embedded from  $\mathbf{A}$  (equation 12) using only one and using four equivalent sources. Although the time is very similar, with one source a small advantage can be observed as the number of data  $N$  increases. This test was done from  $N = 10,000$  to  $N = 700,000$  with increases of 5,625 observation points.

6 Observed synthetic magnetic field data. A regular grid of  $80 \times 80$  points was used, totaling  $N = 6,400$  observation points. Three bodies were modeled: two prisms and a

sphere with inclination, declination and intensity of  $0^\circ$  and  $45^\circ$  and  $2 \times \sqrt{2}$  A/m, respectively.

7 (a) Predicted data using a classical linear inversion method (equation 15). (b) Residuals between the observed (6) and the predicted data (panel a), with mean 0.3712 nT and standart deviation of 0.2798 nT. This process took 17.6 seconds.

8 (a) Predicted data using the CGLS method with the fast BTTB matrix-vector product. (b) Residuals between the observed (6) and the predicted data (panel a), with mean 0.5150 nT and standart deviation of 0.4363 nT. This process took 0.18 seconds.

9 Convergence analysis of the CGLS method for this synthetic application of the magnetic equivalent layer using the fast BTTB matrix-vector product.

10 (a) Synthetic magnetic field grid visualization. A irregular grid of  $100 \times 50$  points was used, totaling  $N = 5,000$  observation points. Standart deviations of 10% in the  $x$ -direction and 10% in the  $y$ -direction were applied. (b) Observed synthetic magnetic field data using this irregular grid in panel a. Three bodies were modeled: two prisms and a sphere with inclination, declination and intensity of  $0^\circ$  and  $45^\circ$  and  $2 \times \sqrt{2}$  A/m, respectively.

11 (a) Predicted data using a classical linear inversion method (equation 15) for the irregular grid in figure 10a. (b) Residuals between the observed (10b) and the predicted data (panel a), with mean 0.3628 nT and standart deviation of 0.2727 nT.

12 (a) Predicted data using the CGLS method with the fast BTTB matrix-vector product for for the irregular grid in figure 10a. (b) Residuals between the observed (10b) and the predicted data (panel a), with mean 0.6024 nT and standart deviation of 0.4998 nT.

13 Convergence analysis of the CGLS method for the synthetic application of the mag-

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14 (a) Synthetic magnetic field grid visualization. A irregular grid of  $100 \times 50$  points was used, totaling  $N = 5,000$  observation points. Standard deviations of 20% in the *x*-direction and 20% in the *y*-direction were applied. (b) Observed synthetic magnetic field data using this irregular grid in panel a. Three bodies were modeled: two prisms and a sphere with inclination, declination and intensity of  $0^\circ$  and  $45^\circ$  and  $2 \times \sqrt{2}$  A/m, respectively.

15 (a) Predicted data using a classical linear inversion method (equation 15) for the irregular grid in figure 14a. (b) Residuals between the observed (14b) and the predicted data (panel a), with mean 0.3630 nT and standard deviation of 0.2731 nT.

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17 Convergence analysis of the CGLS method for the synthetic application of the magnetic equivalent layer using an irregular grid with 20% of perturbation on the *x*-*direction* and *y*-*direction*.

18 (a) Synthetic magnetic field grid visualization. A irregular grid of  $100 \times 50$  points was used, totaling  $N = 5,000$  observation points. Standard deviations of 30% in the *x*-direction and 30% in the *y*-direction were applied. (b) Observed synthetic magnetic field data using this irregular grid in panel a. Three bodies were modeled: two prisms and a sphere with inclination, declination and intensity of  $0^\circ$  and  $45^\circ$  and  $2 \times \sqrt{2}$  A/m, respectively.

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23 (a) Predicted data using a classical linear inversion method (equation 15) for the irregular grid in figure 22a. (b) Residuals between the observed (22b) and the predicted data (panel a), with mean 0.3712 nT and standart deviation of 0.2870 nT.

24 (a) Predicted data using the CGLS method with the fast BTTB matrix-vector product for for the irregular grid in figure 22a. (b) Residuals between the observed (22b) and the predicted data (panel a), with mean 0.9542 nT and standart deviation of 0.8943 nT.

25 Convergence analysis of the CGLS method for the synthetic application of the magnetic equivalent layer using an irregular grid with 5% of pertubation on the *z*-direction.

26 (a) Undulating surface where the total-field anomaly was computed. A irregular grid of  $100 \times 50$  points was used, totaling  $N = 5,000$  observation points. A standart deviation of 10% in the  $z$ -direction was applied. (b) Observed synthetic magnetic field data using this irregular grid in panel a. Three bodies were modeled: two prisms and a sphere with inclination, declination and intensity of  $0^\circ$  and  $45^\circ$  and  $2 \times \sqrt{2}$  A/m, respectively.

27 (a) Predicted data using a classical linear inversion method (equation 15) for the irregular grid in figure 26a. (b) Residuals between the observed (26b) and the predicted data (panel a), with mean 0.3865 nT and standart deviation of 0.3216 nT.

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29 Convergence analysis of the CGLS method for the synthetic application of the magnetic equivalent layer using an irregular grid with 10% of pertubation on the  $z$ -direction.

30 (a) Undulating surface where the total-field anomaly was computed. A irregular grid of  $100 \times 50$  points was used, totaling  $N = 5,000$  observation points. A standart deviation of 20% in the  $z$ -direction was applied. (b) Observed synthetic magnetic field data using this irregular grid in panel a. Three bodies were modeled: two prisms and a sphere with inclination, declination and intensity of  $0^\circ$  and  $45^\circ$  and  $2 \times \sqrt{2}$  A/m, respectively.

31 (a) Predicted data using a classical linear inversion method (equation 15) for the irregular grid in figure 30a. (b) Residuals between the observed (30b) and the predicted data (panel a), with mean 0.4155 nT and standart deviation of 0.4005 nT.

32 (a) Predicted data using the CGLS method with the fast BTTB matrix-vector

product for for the irregular grid in figure 30a. (b) Residuals between the observed (30b) and the predicted data (panel a), with mean 6.6220 nT and standart deviation of 5.901 nT.

33 Convergence analysis of the CGLS method for the synthetic application of the magnetic equivalent layer using an irregular grid with 20% of pertubation on the *z*-direction.

34 Observed magnetic field data of the Carajás, Brazil area. The aeromagnetic survey was done with 131 N-S lines at approximately  $-900m$  height, totaling  $N = 6,081,345$  observation points.

35 (a) Observed magnetic field data of the Carajás, Brazil area, interpolated for a regular grid of  $10,000 \times 131$ , totaling  $N = 1,310,000$  observation points. (b) Observed magnetic field data of the Carajás, Brazil area, decimated from the flight lines resulting in an irregular grid of  $10,000 \times 131$ , also totaling  $N = 1,310,000$  observation points.

36 (a) Predicted data using our method for the interpolated  $10,000 \times 131$  regular grid.  
(b) Residuals between the observed (35) and the predicted data (panel a), with a mean of 0.07979 nT and standart deviation of 0.5060 nT.

37 (a) Predicted data using our method for the decimated  $10,000 \times 131$  irregular grid.  
(b) Residuals between the observed (35) and the predicted data (panel b), with a mean of 0.07348 nT and standart deviation of 0.3172 nT.

38 Convergence analysis of the CGLS method for the field data of Carajás, Brazil using the magnetic equivalent layer with a decimated irregular grid of  $10,000 \times 131$  obser-  
vation points up to 2,000 iterations.

39 Upward continuation transformation of real data of Carajás, Brazil at 5,000 meter.  
It was necessary 2.64 seconds to complete the process.

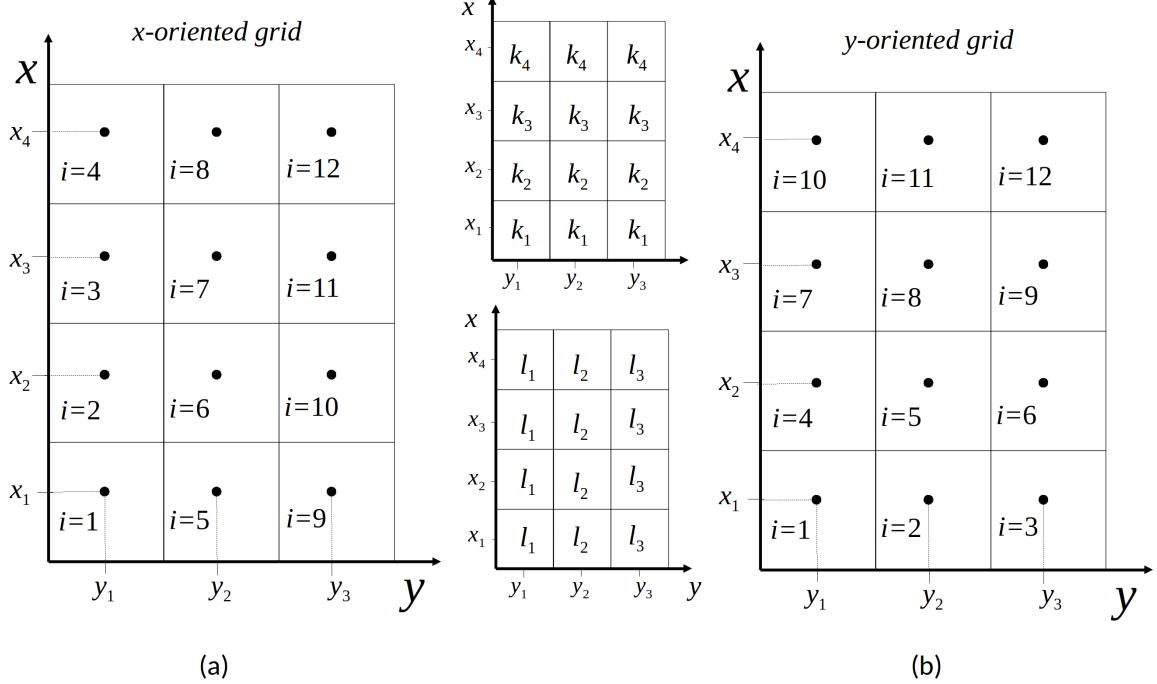


Figure 1: Schematic representation of an  $N_x \times N_y$  regular grid of points (black dots) defined by  $N_x = 4$  and  $N_y = 3$ . The grids are oriented along the (a)  $x$ -axis and (b)  $y$ -axis. The grid coordinates are  $x_k$  and  $y_l$ , where the  $k = 1, \dots, N_x$  and  $l = 1, \dots, N_y$  are called the grid indices. The insets show the grid indices  $k$  and  $l$ .

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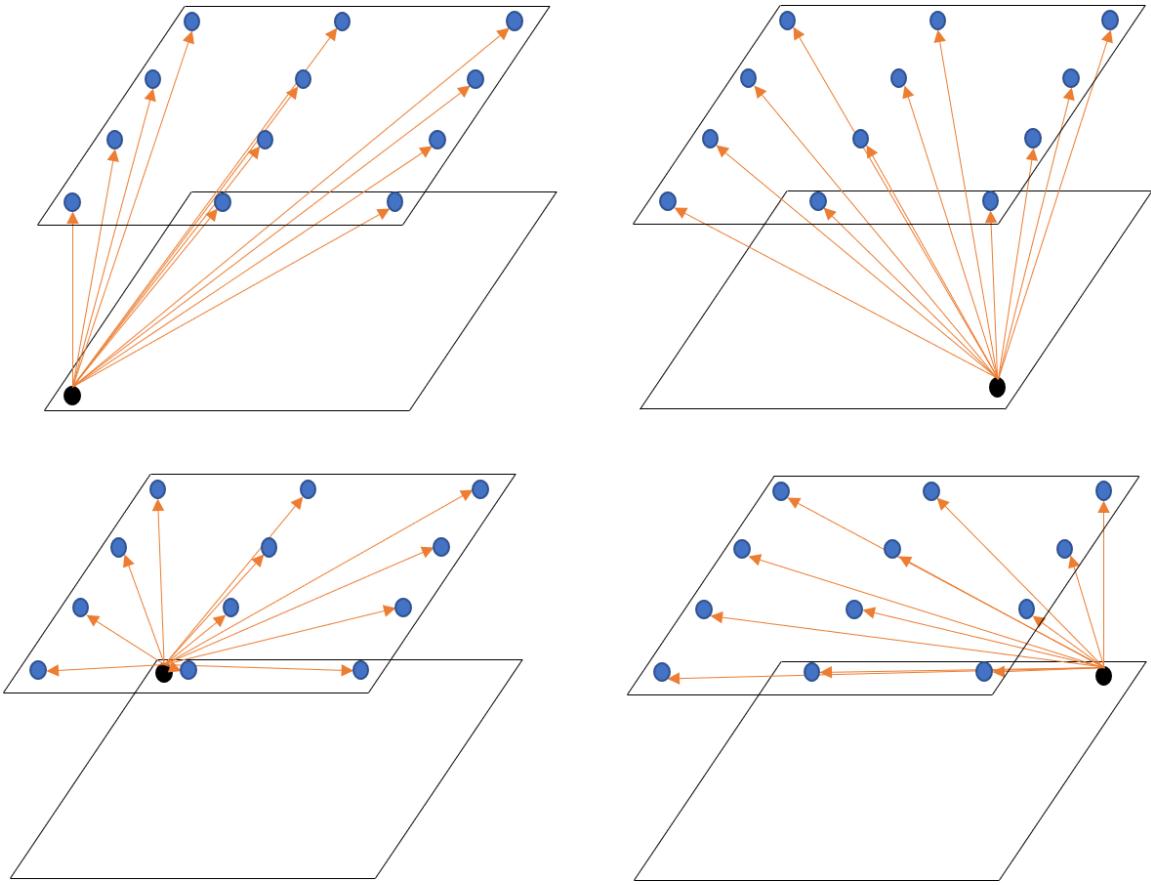


Figure 2: Representation of the four equivalent sources (black dots) needed to reconstruct the non-symmetric matrix  $\mathbf{A}$  (equation 12). Each of the equivalent sources are located in the corner of the simulated regular grid of  $M_x = 4$  and  $M_y = 3$  sources. The influence of these sources on each of the observation points (blue dots) in the regular grid of  $N_x = 4$  and  $N_y = 3$  will give the four columns necessary of matrix  $\mathbf{A}$ .

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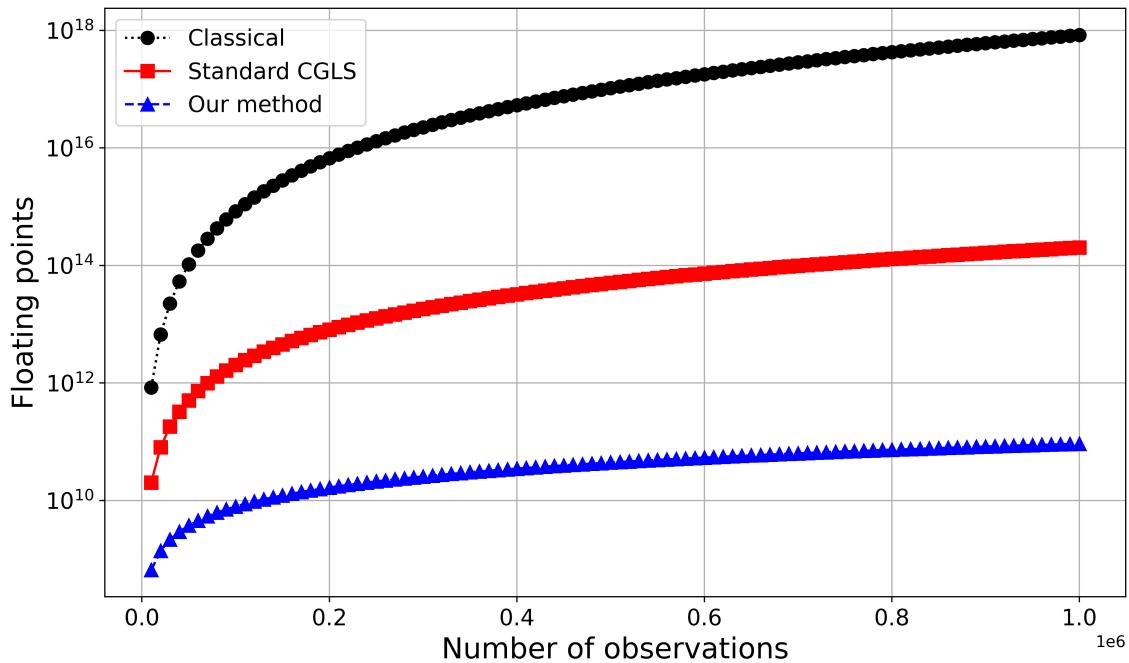


Figure 3: Number of flops necessary to estimate the parameter vector  $\hat{\mathbf{p}}$  using the non-iterative classical method (equation 69) the CGLS (equation 70) and our modified CGLS method (equation 71) with  $N^{it} = 50$ . The observation point  $N$  varied from 5,000 to 1,000,000. The radix-2 2D FFT algorithm was considered for our method, with  $\kappa = 5$ .

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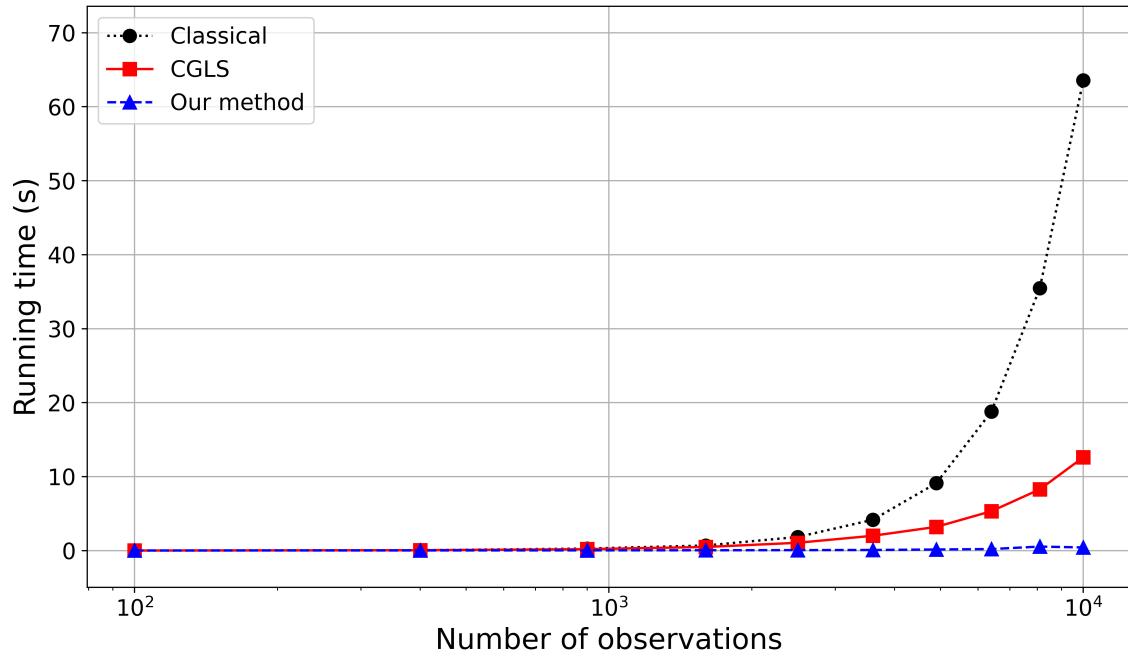


Figure 4: Comparison between the runtime of the equivalent-layer technique using the classical, the CGLS algorithm and our method. The values for the CGLS and our methods were obtained for  $N^{it} = 50$  iterations.

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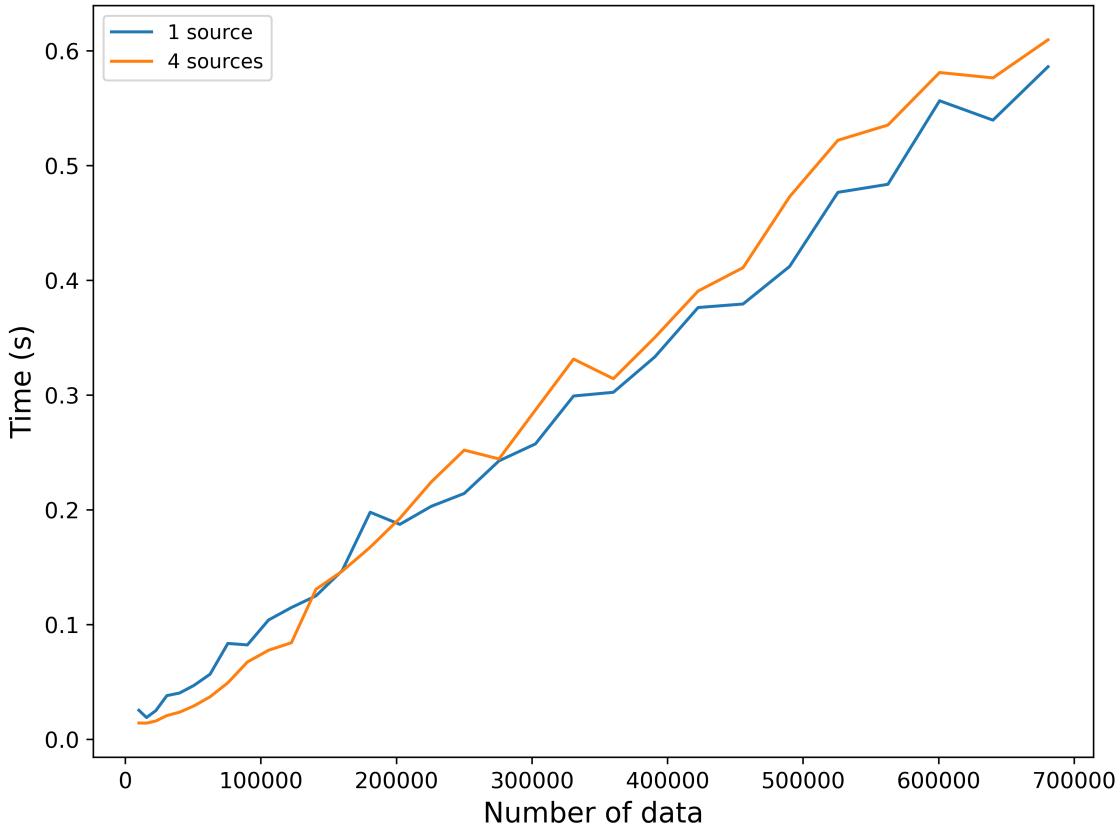


Figure 5: Comparison between the runtime to calculate the first column of the BCCB matrix embedded from  $\mathbf{A}$  (equation 12) using only one and using four equivalent sources. Although the time is very similar, with one source a small advantage can be observed as the number of data  $N$  increases. This test was done from  $N = 10,000$  to  $N = 700,000$  with increases of 5,625 observation points.

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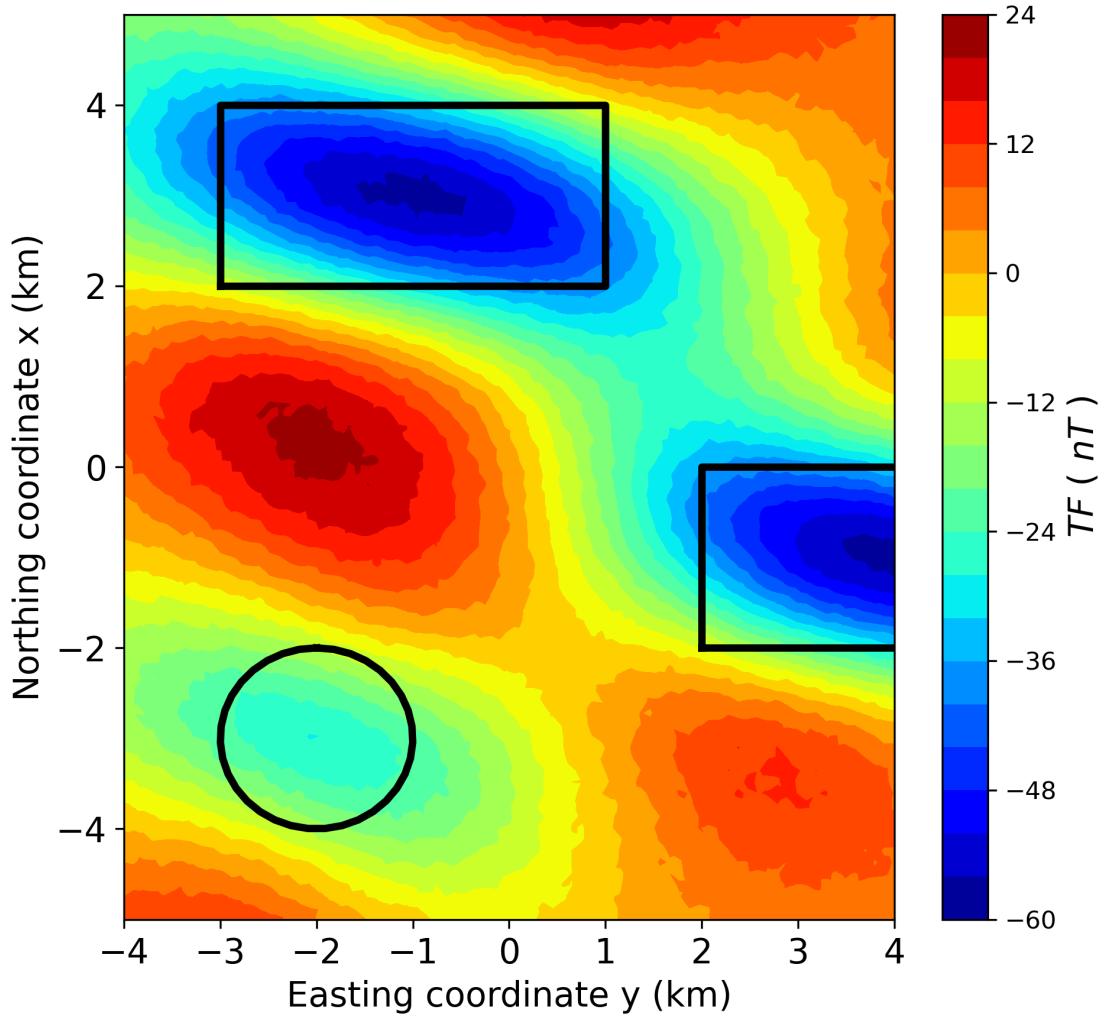


Figure 6: Observed synthetic magnetic field data. A regular grid of  $80 \times 80$  points was used, totaling  $N = 6,400$  observation points. Three bodies were modeled: two prisms and a sphere with inclination, declination and intensity of  $0^\circ$  and  $45^\circ$  and  $2 \times \sqrt{2}$  A/m, respectively.

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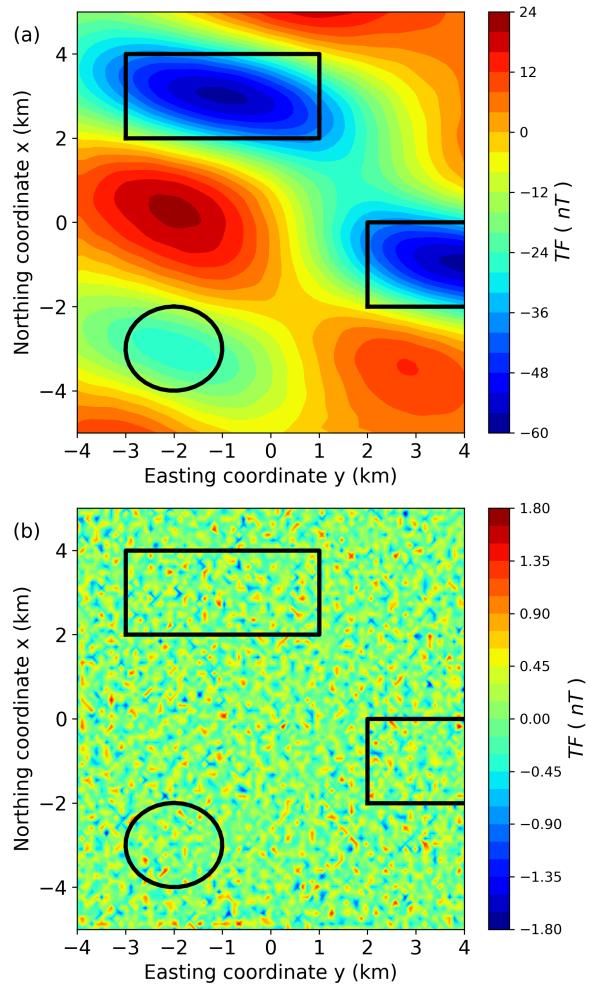


Figure 7: (a) Predicted data using a classical linear inversion method (equation 15). (b) Residuals between the observed (6) and the predicted data (panel a), with mean 0.3712 nT and standard deviation of 0.2798 nT. This process took 17.6 seconds.

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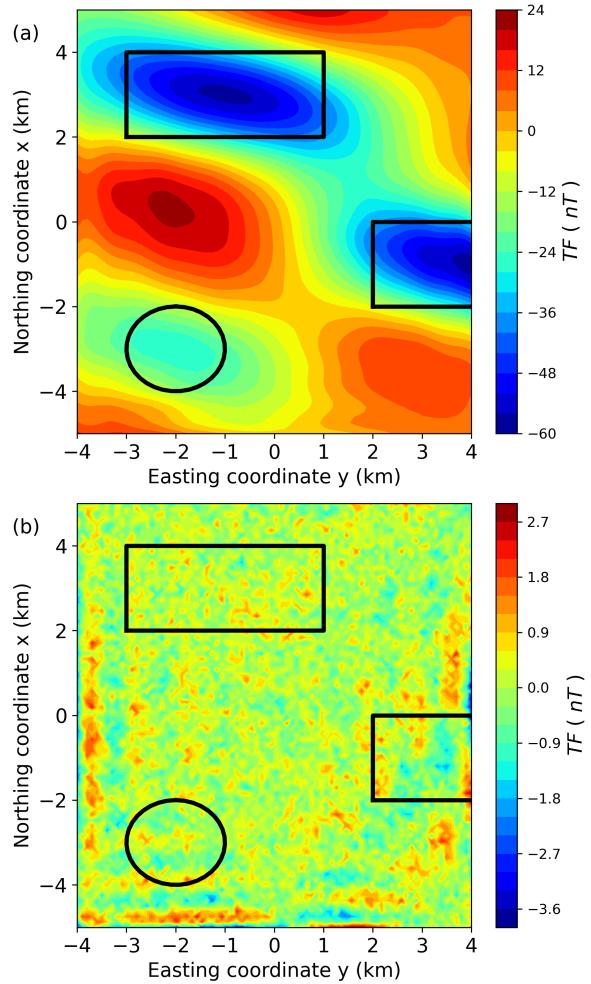


Figure 8: (a) Predicted data using the CGLS method with the fast BTTB matrix-vector product. (b) Residuals between the observed (6) and the predicted data (panel a), with mean 0.5150 nT and standart deviation of 0.4363 nT. This process took 0.18 seconds.

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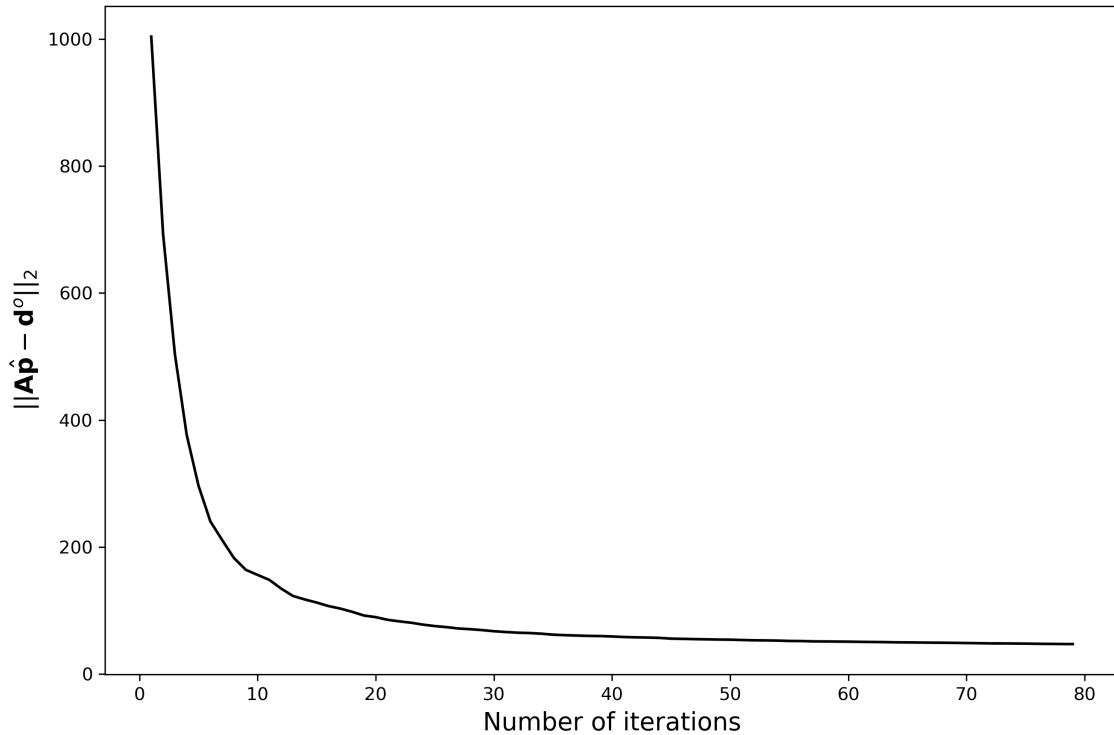


Figure 9: Convergence analysis of the CGLS method for this synthetic application of the magnetic equivalent layer using the fast BTTB matrix-vector product.

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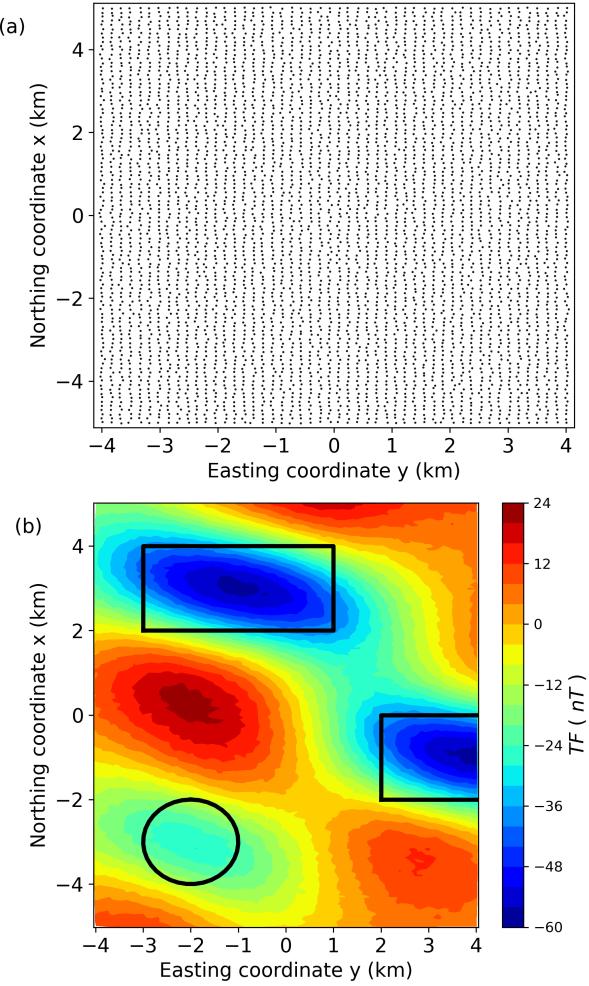


Figure 10: (a) Synthetic magnetic field grid visualization. A irregular grid of  $100 \times 50$  points was used, totaling  $N = 5,000$  observation points. Standard deviations of 10% in the  $x$ -direction and 10% in the  $y$ -direction were applied. (b) Observed synthetic magnetic field data using this irregular grid in panel a. Three bodies were modeled: two prisms and a sphere with inclination, declination and intensity of  $0^\circ$  and  $45^\circ$  and  $2 \times \sqrt{2}$  A/m, respectively.

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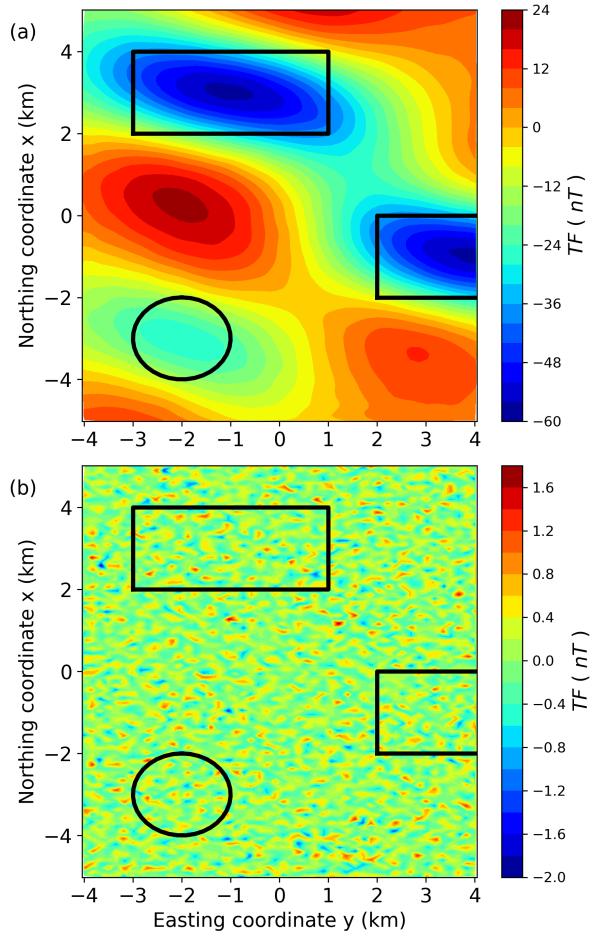


Figure 11: (a) Predicted data using a classical linear inversion method (equation 15) for the irregular grid in figure 10a. (b) Residuals between the observed (10b) and the predicted data (panel a), with mean 0.3628 nT and standart deviation of 0.2727 nT.

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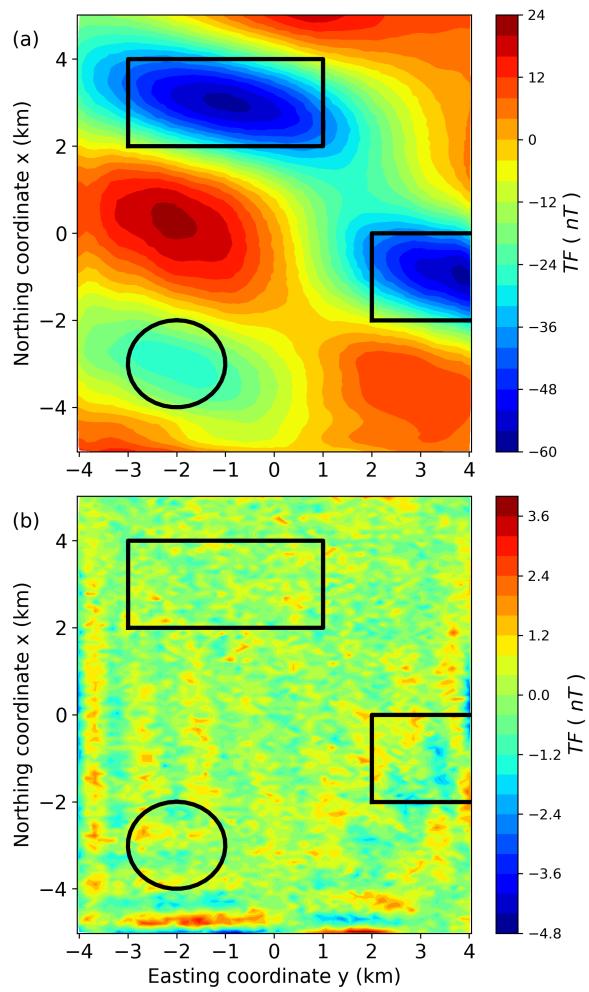


Figure 12: (a) Predicted data using the CGLS method with the fast BTTB matrix-vector product for the irregular grid in figure 10a. (b) Residuals between the observed (10b) and the predicted data (panel a), with mean 0.6024 nT and standard deviation of 0.4998 nT.

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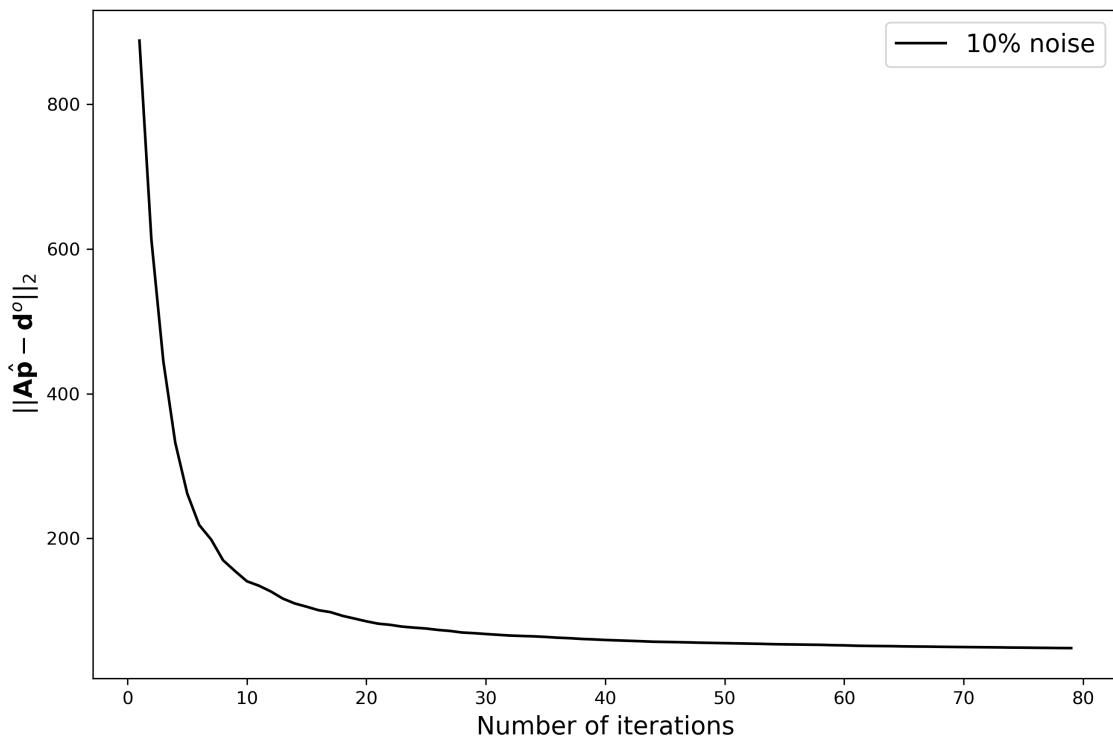


Figure 13: Convergence analysis of the CGLS method for the synthetic application of the magnetic equivalent layer using an irregular grid with 10% of perturbation on the *x-direction* and *y-direction*.

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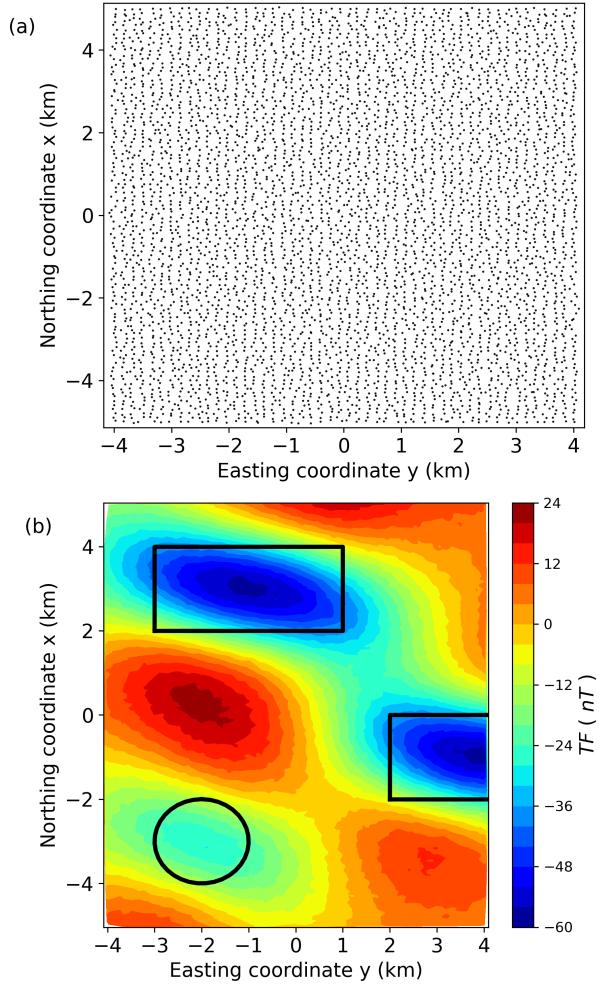


Figure 14: (a) Synthetic magnetic field grid visualization. A irregular grid of  $100 \times 50$  points was used, totaling  $N = 5,000$  observation points. Standart deviations of 20% in the  $x$ -direction and 20% in the  $y$ -direction were applied. (b) Observed synthetic magnetic field data using this irregular grid in panel a. Three bodies were modeled: two prisms and a sphere with inclination, declination and intensity of  $0^\circ$  and  $45^\circ$  and  $2 \times \sqrt{2}$  A/m, respectively.

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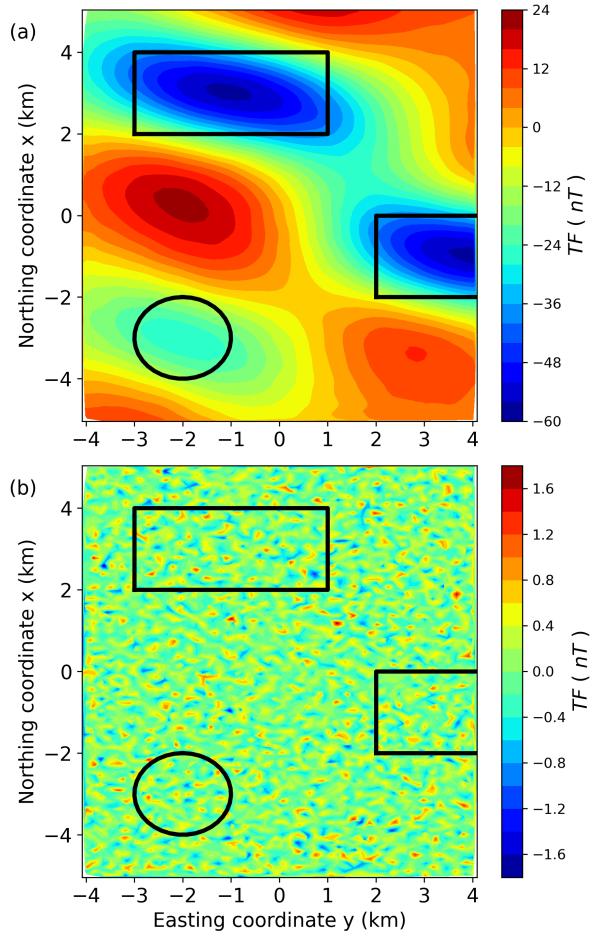


Figure 15: (a) Predicted data using a classical linear inversion method (equation 15) for the irregular grid in figure 14a. (b) Residuals between the observed (14b) and the predicted data (panel a), with mean 0.3630 nT and standart deviation of 0.2731 nT.

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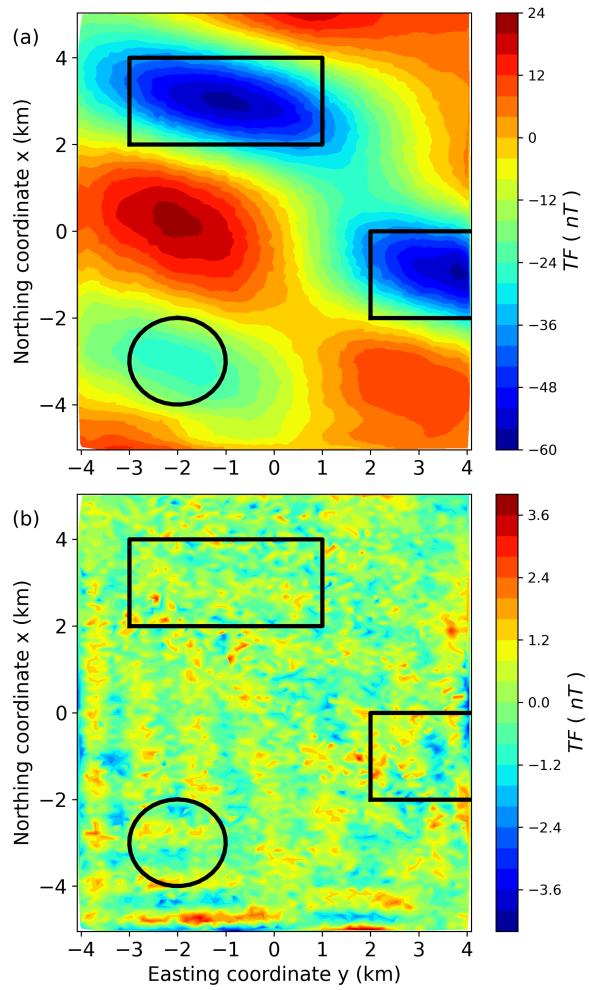


Figure 16: (a) Predicted data using the CGLS method with the fast BTTB matrix-vector product for the irregular grid in figure 14a. (b) Residuals between the observed (14b) and the predicted data (panel a), with mean 0.7147 nT and standard deviation of 0.5622 nT.

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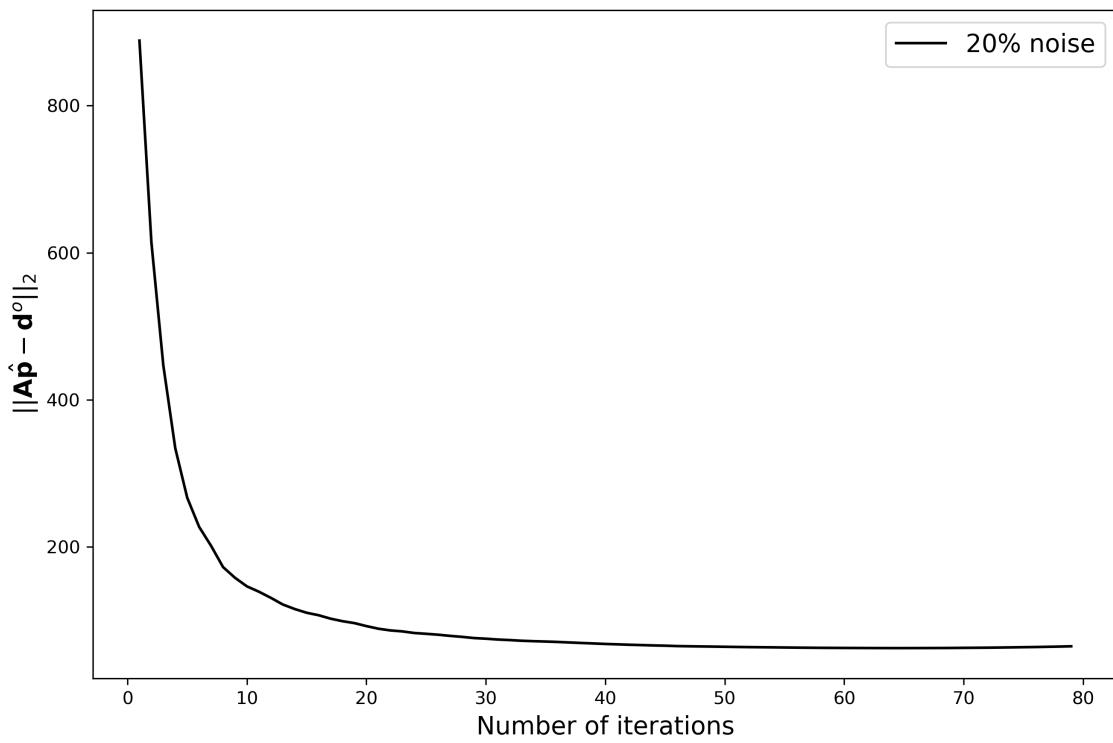


Figure 17: Convergence analysis of the CGLS method for the synthetic application of the magnetic equivalent layer using an irregular grid with 20% of perturbation on the *x-direction* and *y-direction*.

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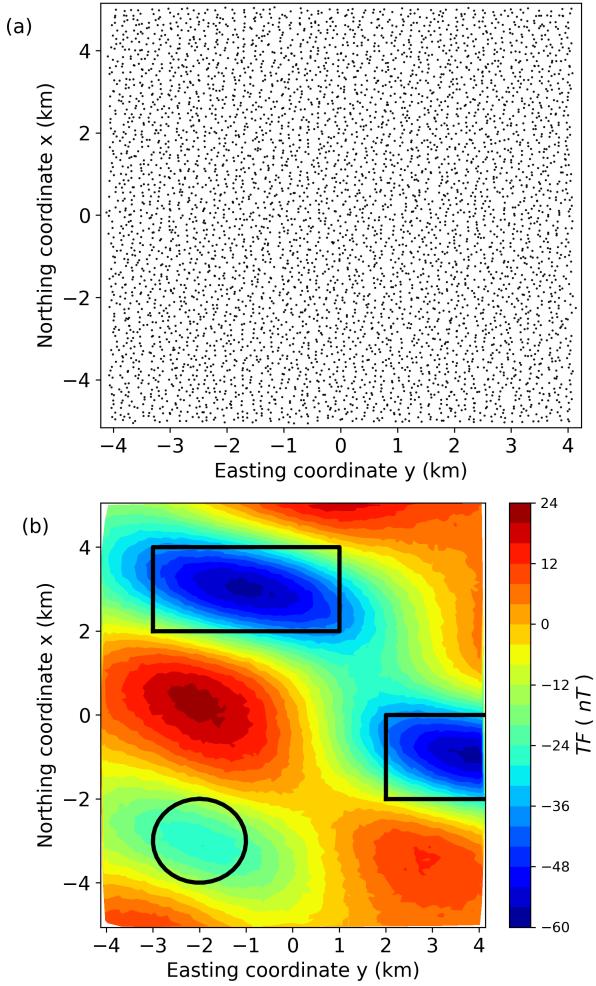


Figure 18: (a) Synthetic magnetic field grid visualization. A irregular grid of  $100 \times 50$  points was used, totaling  $N = 5,000$  observation points. Standart deviations of 30% in the  $x$ -direction and 30% in the  $y$ -direction were applied. (b) Observed synthetic magnetic field data using this irregular grid in panel a. Three bodies were modeled: two prisms and a sphere with inclination, declination and intensity of  $0^\circ$  and  $45^\circ$  and  $2 \times \sqrt{2}$  A/m, respectively.

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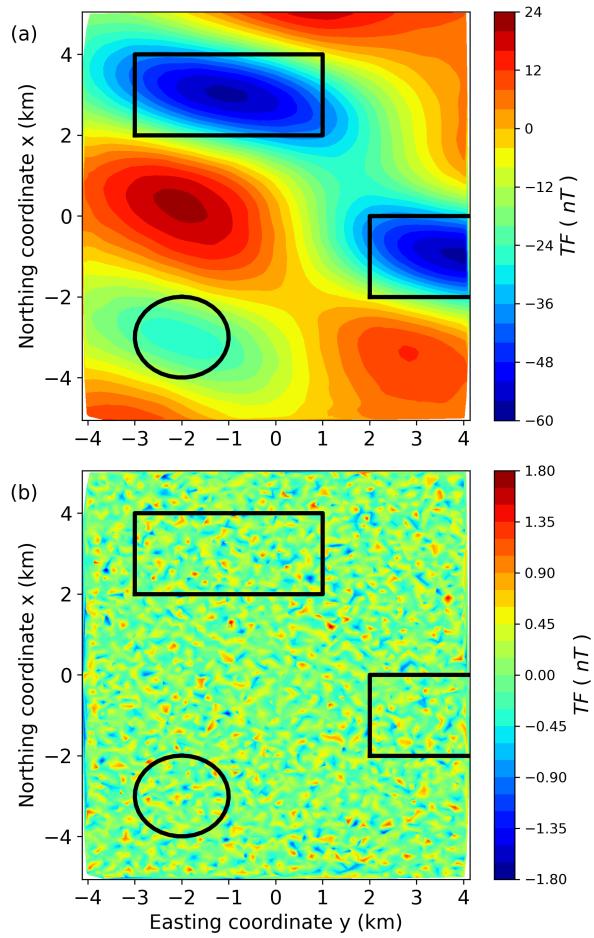


Figure 19: (a) Predicted data using a classical linear inversion method (equation 15) for the irregular grid in figure 18a. (b) Residuals between the observed (18b) and the predicted data (panel a), with mean 0.3634 nT and standart deviation of 0.2735 nT.

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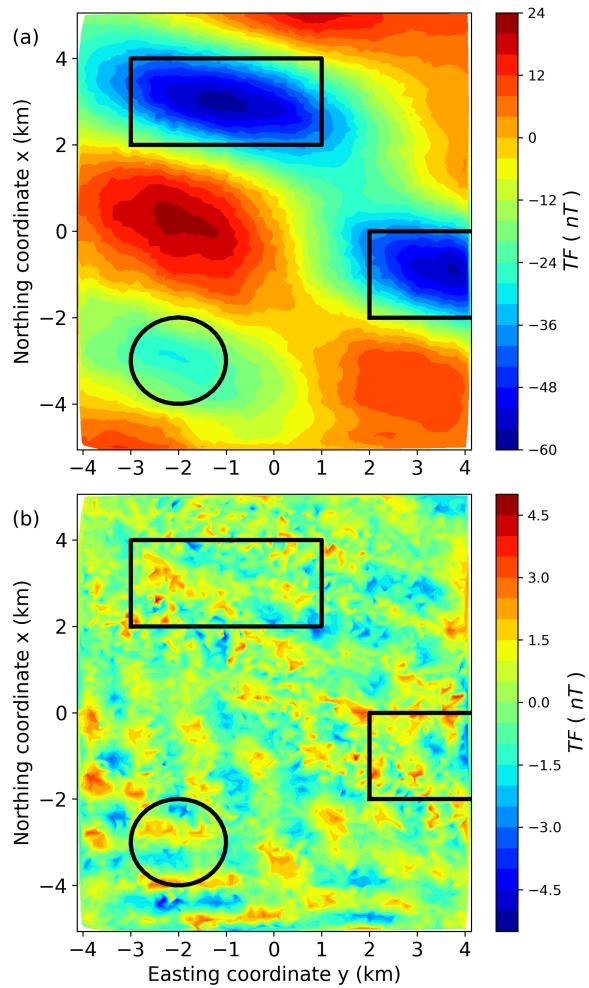


Figure 20: (a) Predicted data using the CGLS method with the fast BTTB matrix-vector product for the irregular grid in figure 18a. (b) Residuals between the observed (18b) and the predicted data (panel a), with mean 0.9788 nT and standard deviation of 0.7462 nT.

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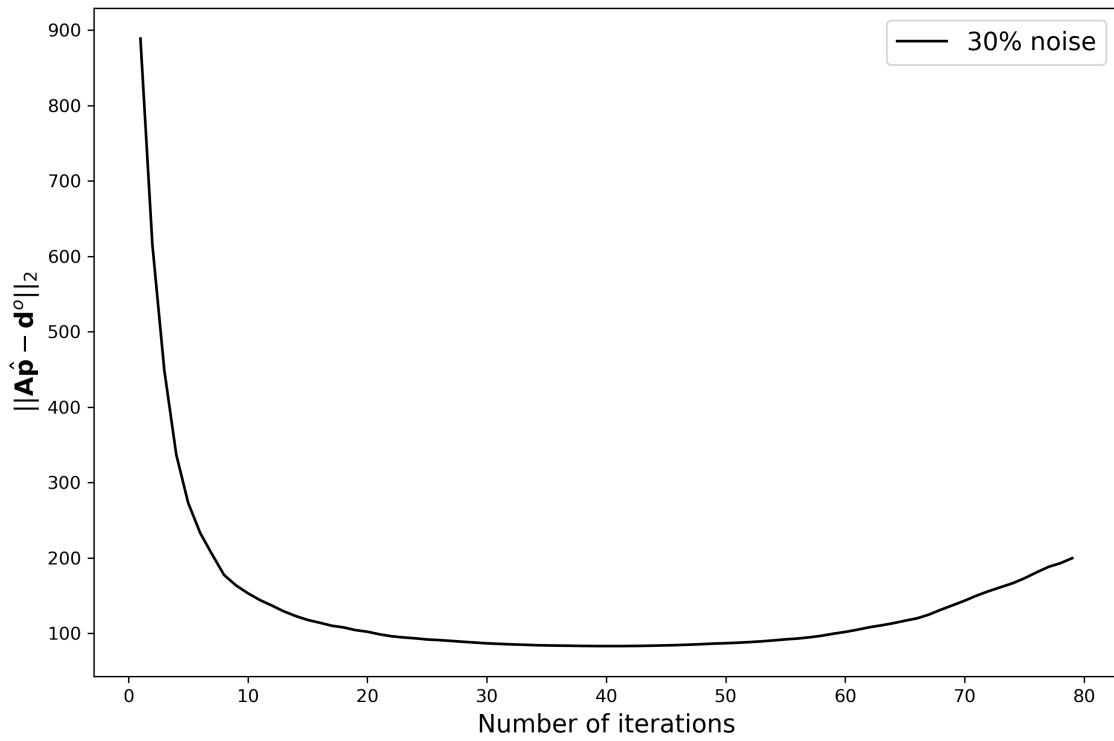


Figure 21: Convergence analysis of the CGLS method for the synthetic application of the magnetic equivalent layer using an irregular grid with 30% of perturbation on the *x-direction* and *y-direction*.

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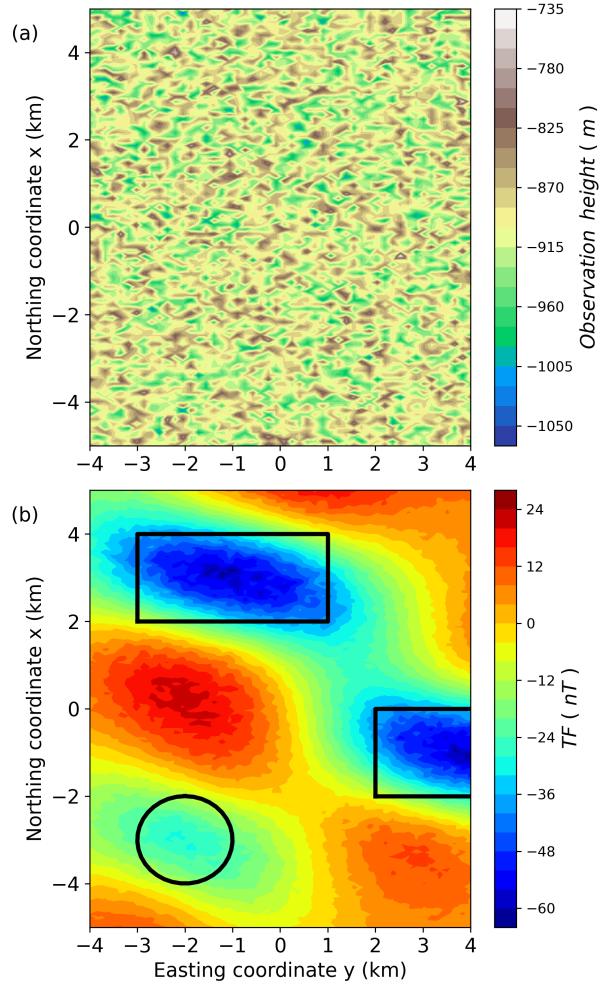


Figure 22: (a) Undulating surface where the total-field anomaly was computed. A irregular grid of  $100 \times 50$  points was used, totaling  $N = 5,000$  observation points. A standart deviation of 5% in the  $z$ -direction was applied. (b) Observed synthetic magnetic field data using this irregular grid in panel a. Three bodies were modeled: two prisms and a sphere with inclination, declination and intensity of  $0^\circ$  and  $45^\circ$  and  $2 \times \sqrt{2}$  A/m, respectively.

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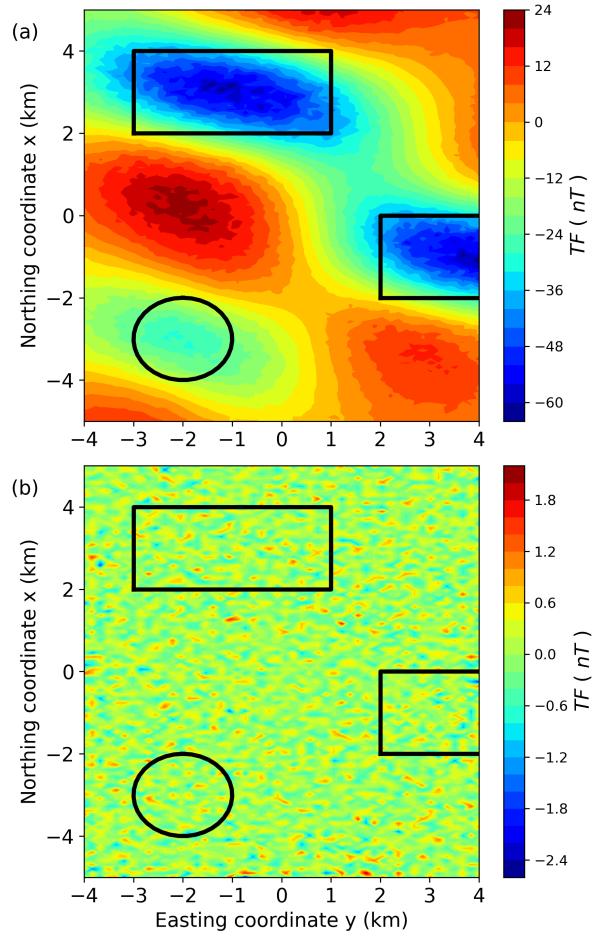


Figure 23: (a) Predicted data using a classical linear inversion method (equation 15) for the irregular grid in figure 22a. (b) Residuals between the observed (22b) and the predicted data (panel a), with mean 0.3712 nT and standart deviation of 0.2870 nT.

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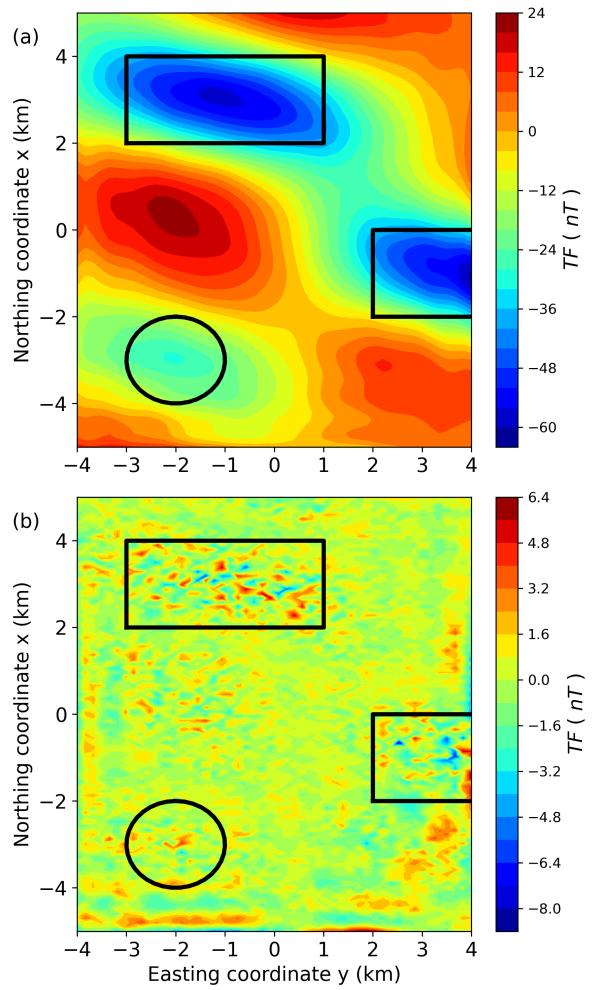


Figure 24: (a) Predicted data using the CGLS method with the fast BTTB matrix-vector product for the irregular grid in figure 22a. (b) Residuals between the observed (22b) and the predicted data (panel a), with mean 0.9542 nT and standard deviation of 0.8943 nT.

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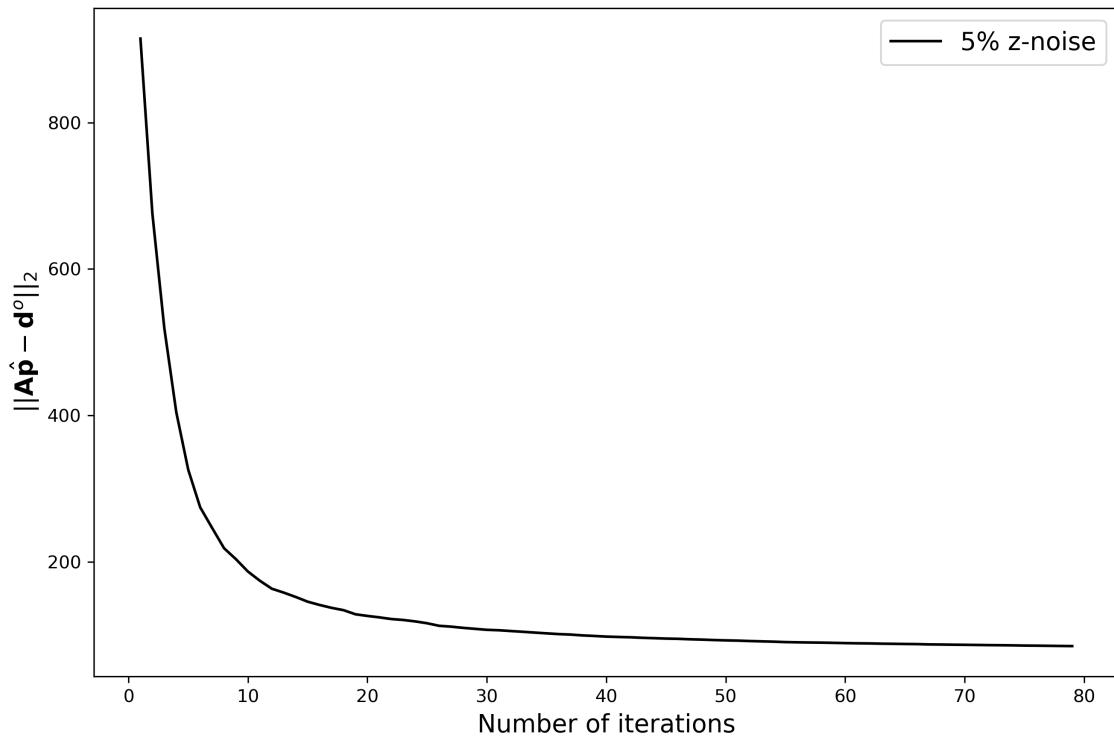


Figure 25: Convergence analysis of the CGLS method for the synthetic application of the magnetic equivalent layer using an irregular grid with 5% of perturbation on the *z-direction*.

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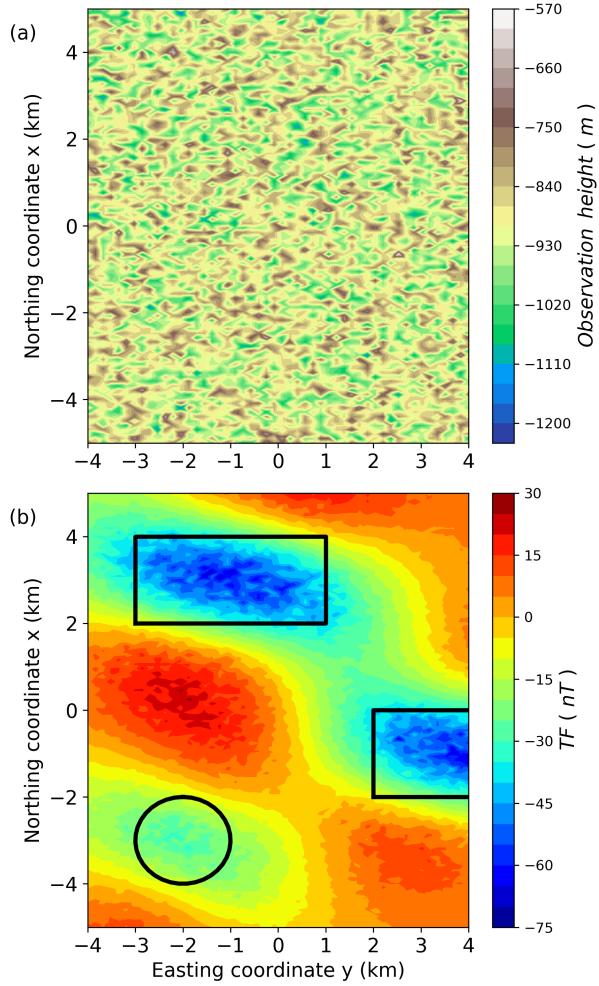


Figure 26: (a) Undulating surface where the total-field anomaly was computed. A irregular grid of  $100 \times 50$  points was used, totaling  $N = 5,000$  observation points. A standart deviation of 10% in the  $z$ -direction was applied. (b) Observed synthetic magnetic field data using this irregular grid in panel a. Three bodies were modeled: two prisms and a sphere with inclination, declination and intensity of  $0^\circ$  and  $45^\circ$  and  $2 \times \sqrt{2} \text{ A/m}$ , respectively.

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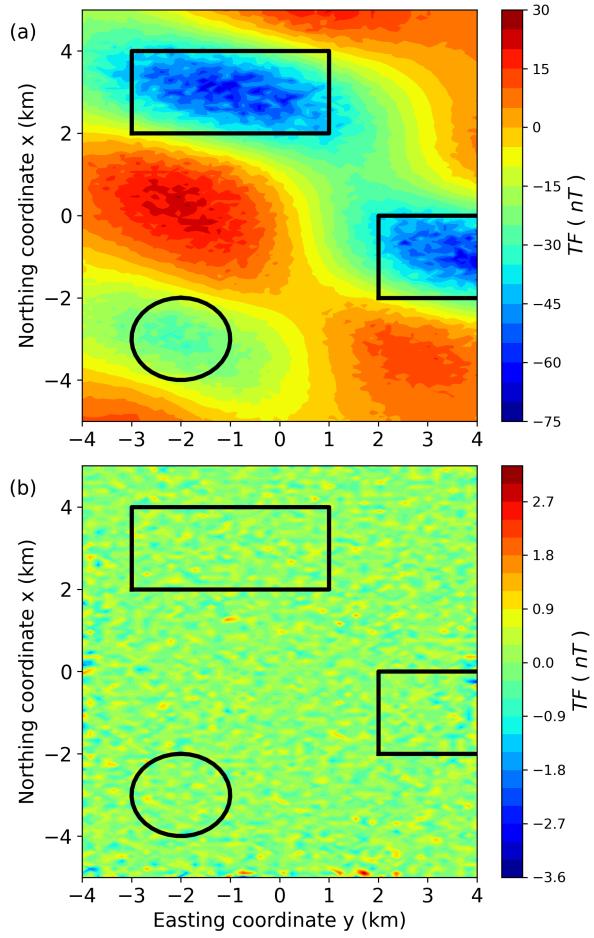


Figure 27: (a) Predicted data using a classical linear inversion method (equation 15) for the irregular grid in figure 26a. (b) Residuals between the observed (26b) and the predicted data (panel a), with mean 0.3865 nT and standart deviation of 0.3216 nT.

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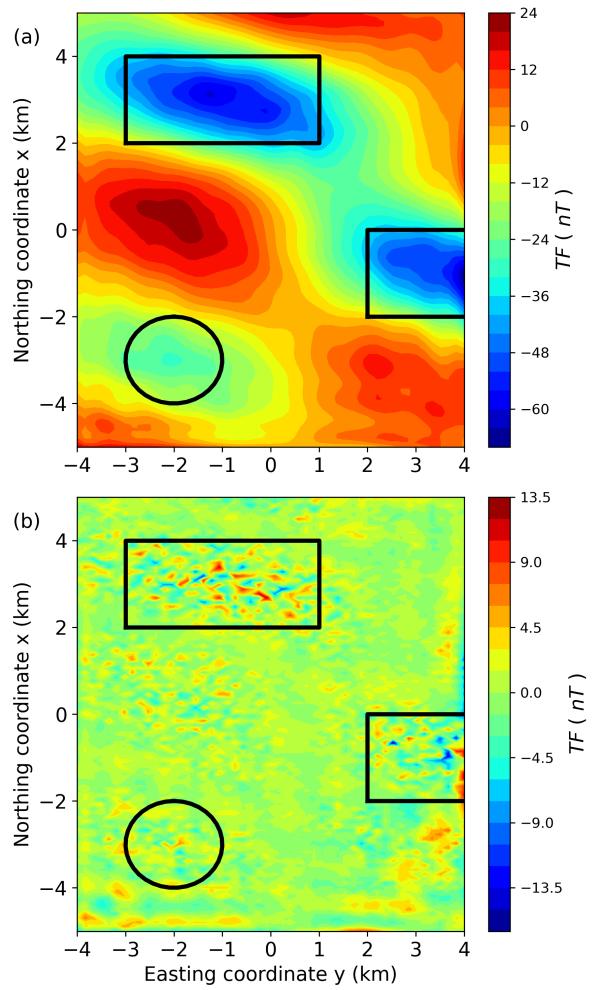


Figure 28: (a) Predicted data using the CGLS method with the fast BTTB matrix-vector product for the irregular grid in figure 26a. (b) Residuals between the observed (26b) and the predicted data (panel a), with mean 1.6105 nT and standard deviation of 1.6231 nT.

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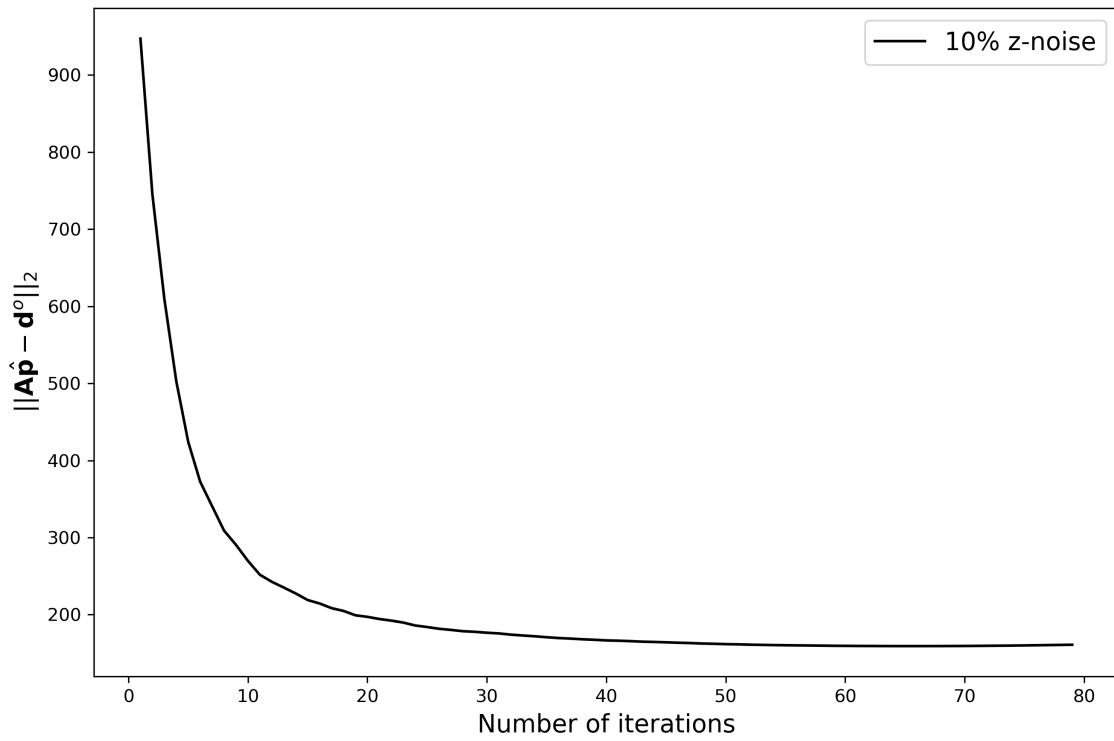


Figure 29: Convergence analysis of the CGLS method for the synthetic application of the magnetic equivalent layer using an irregular grid with 10% of perturbation on the *z-direction*.

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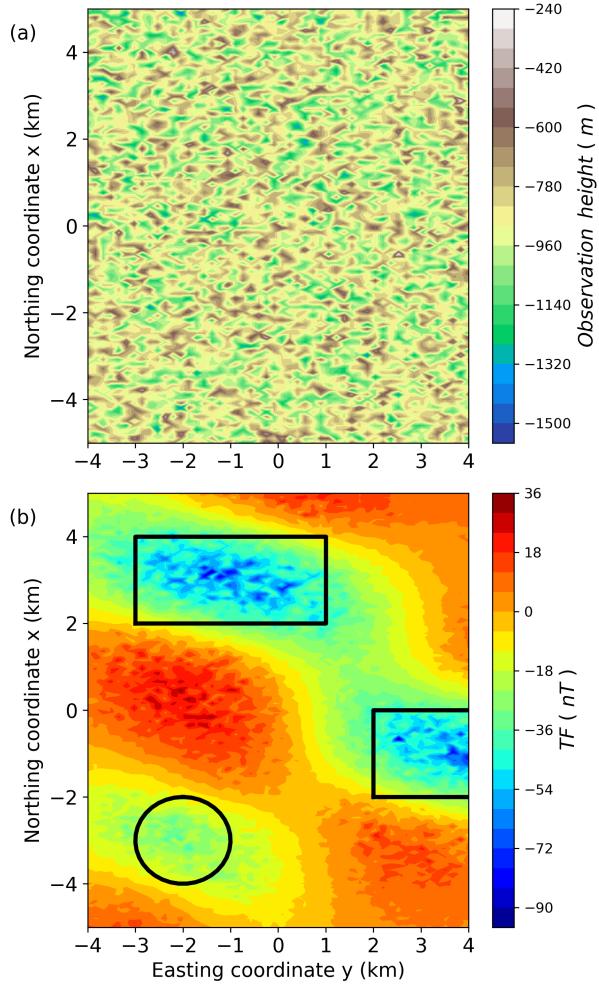


Figure 30: (a) Undulating surface where the total-field anomaly was computed. A irregular grid of  $100 \times 50$  points was used, totaling  $N = 5,000$  observation points. A standart deviation of 20% in the  $z$ -direction was applied. (b) Observed synthetic magnetic field data using this irregular grid in panel a. Three bodies were modeled: two prisms and a sphere with inclination, declination and intensity of  $0^\circ$  and  $45^\circ$  and  $2 \times \sqrt{2}$  A/m, respectively.

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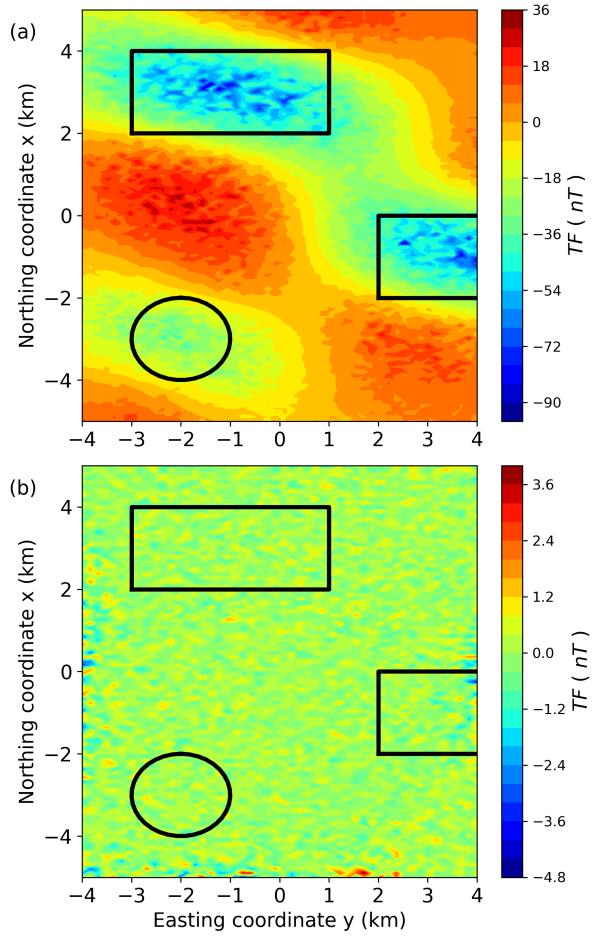


Figure 31: (a) Predicted data using a classical linear inversion method (equation 15) for the irregular grid in figure 30a. (b) Residuals between the observed (30b) and the predicted data (panel a), with mean 0.4155 nT and standart deviation of 0.4005 nT.

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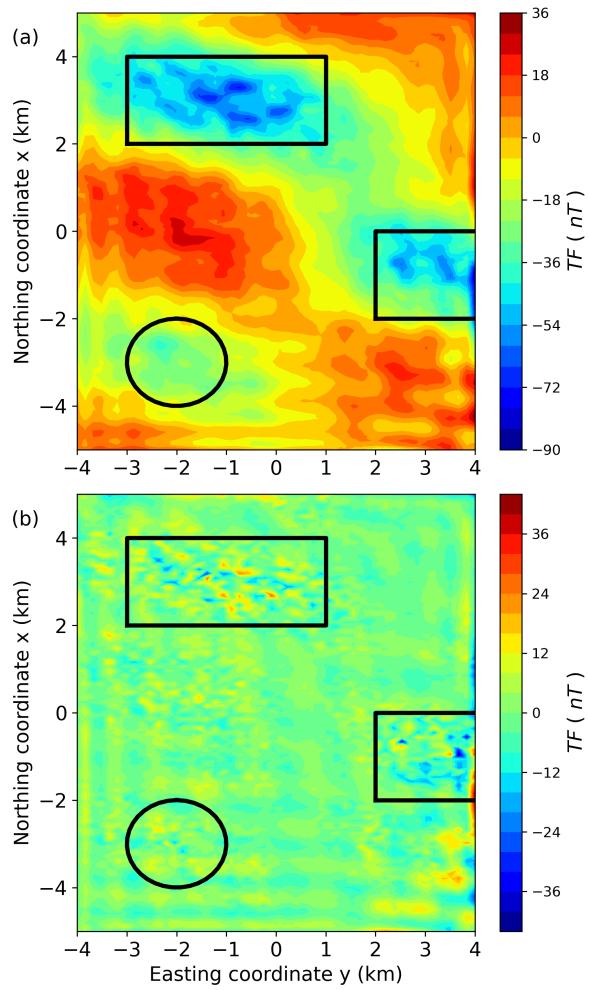


Figure 32: (a) Predicted data using the CGLS method with the fast BTTB matrix-vector product for the irregular grid in figure 30a. (b) Residuals between the observed (30b) and the predicted data (panel a), with mean 6.6220 nT and standard deviation of 5.901 nT.

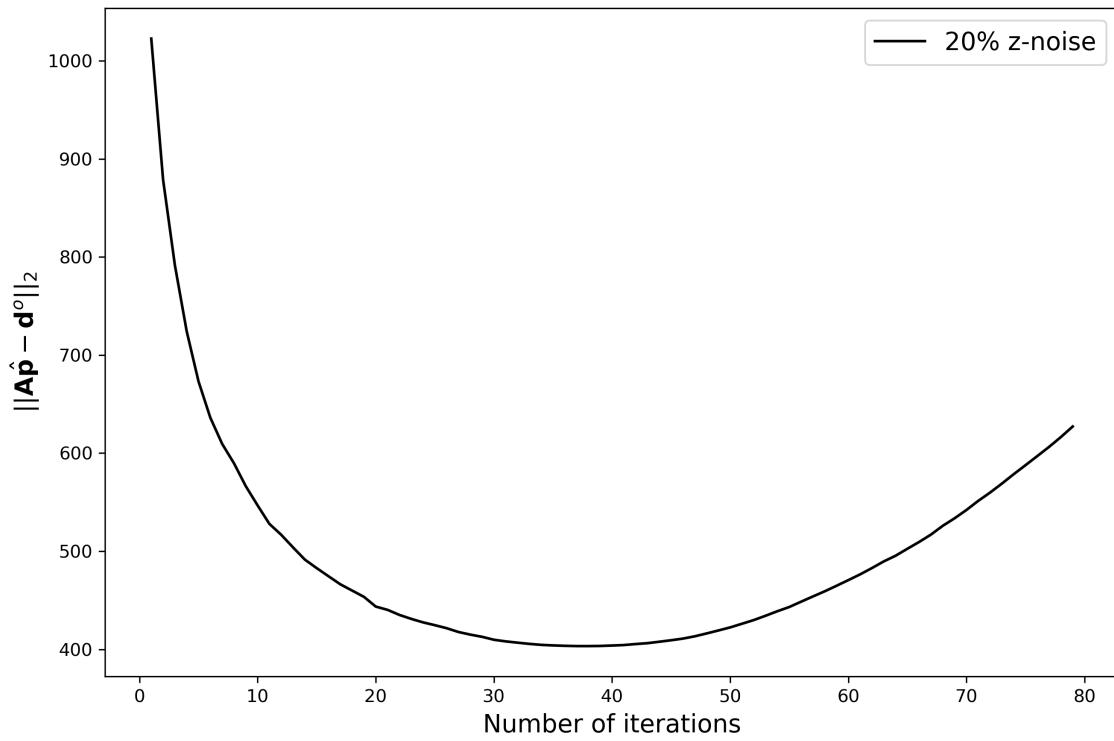


Figure 33: Convergence analysis of the CGLS method for the synthetic application of the magnetic equivalent layer using an irregular grid with 20% of perturbation on the *z-direction*.

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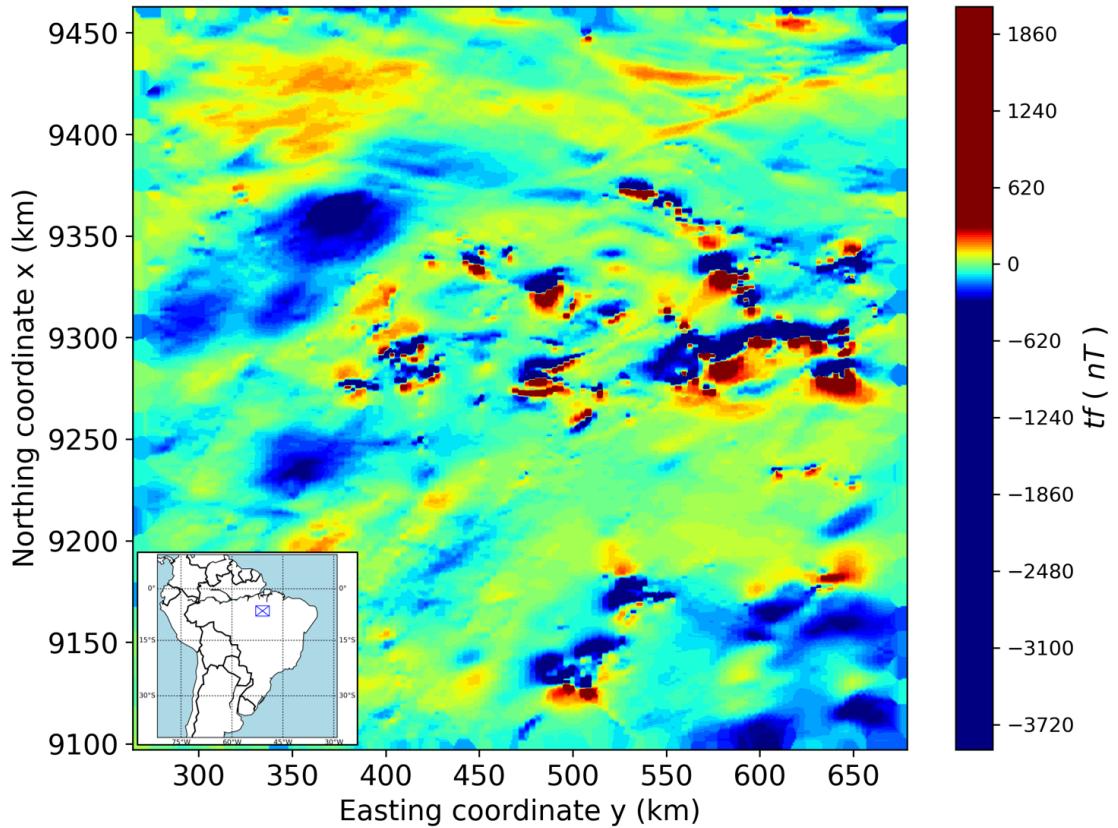


Figure 34: Observed magnetic field data of the Carajás, Brazil area. The aeromagnetic survey was done with 131 N-S lines at approximately  $-900m$  height, totaling  $N = 6,081,345$  observation points.

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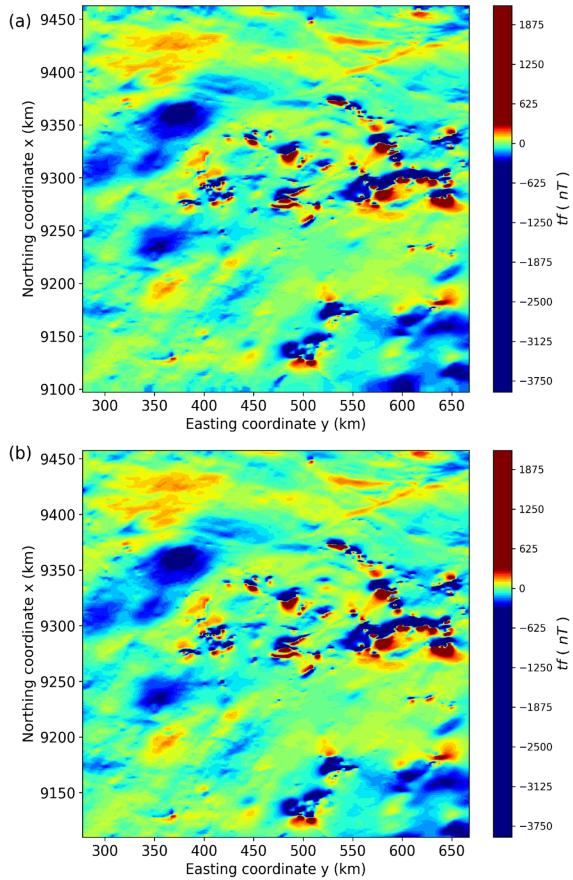


Figure 35: (a) Observed magnetic field data of the Carajás, Brazil area, interpolated for a regular grid of  $10,000 \times 131$ , totaling  $N = 1,310,000$  observation points. (b) Observed magnetic field data of the Carajás, Brazil area, decimated from the flight lines resulting in an irregular grid of  $10,000 \times 131$ , also totaling  $N = 1,310,000$  observation points.

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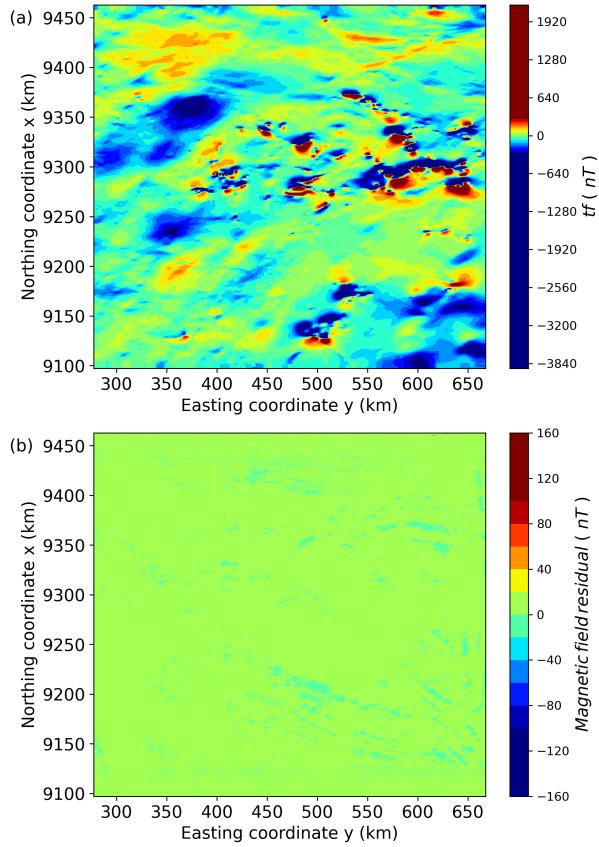


Figure 36: (a) Predicted data using our method for the interpolated  $10,000 \times 131$  regular grid. (b) Residuals between the observed (35) and the predicted data (panel a), with a mean of 0.07979 nT and standart deviation of 0.5060 nT.

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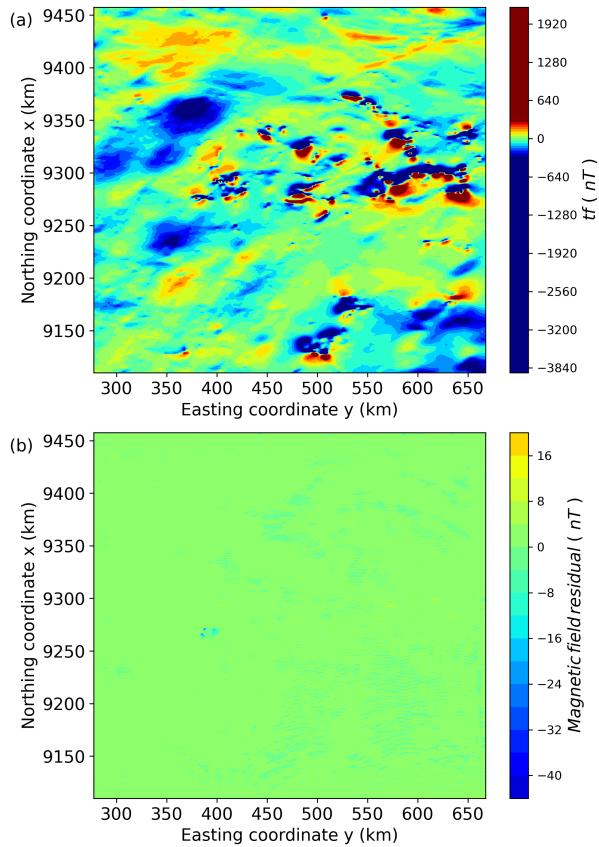


Figure 37: (a) Predicted data using our method for the decimated  $10,000 \times 131$  irregular grid. (b) Residuals between the observed (35) and the predicted data (panel b), with a mean of 0.07348 nT and standart deviation of 0.3172 nT.

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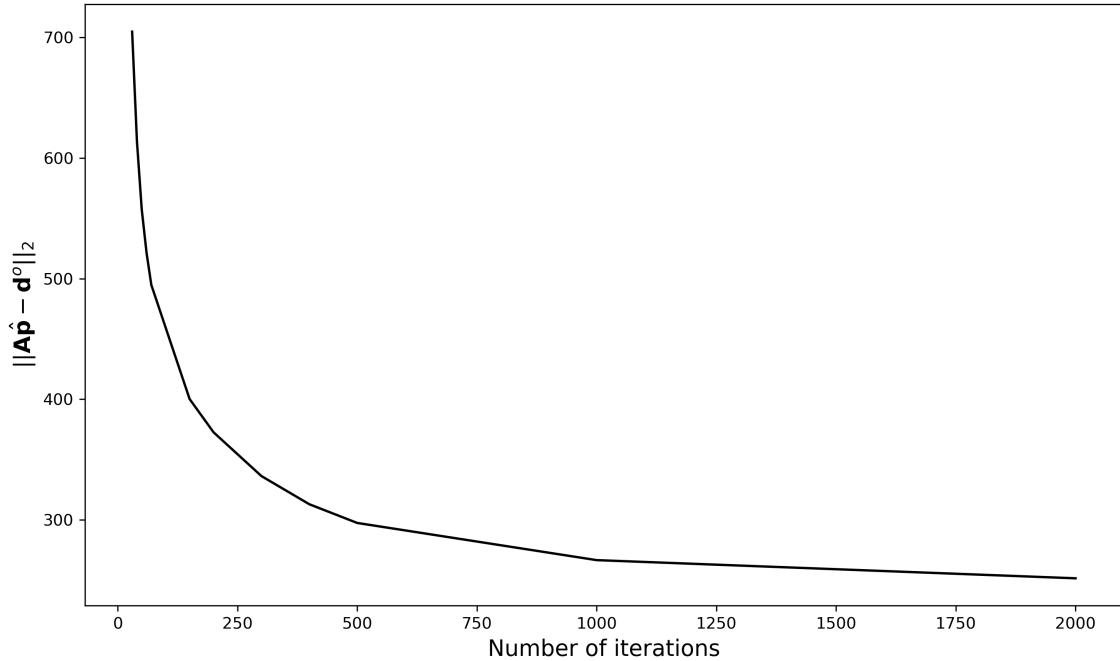


Figure 38: Convergence analysis of the CGLS method for the field data of Carajás, Brazil using the magnetic equivalent layer with a decimated irregular grid of  $10,000 \times 131$  observation points up to 2,000 iterations.

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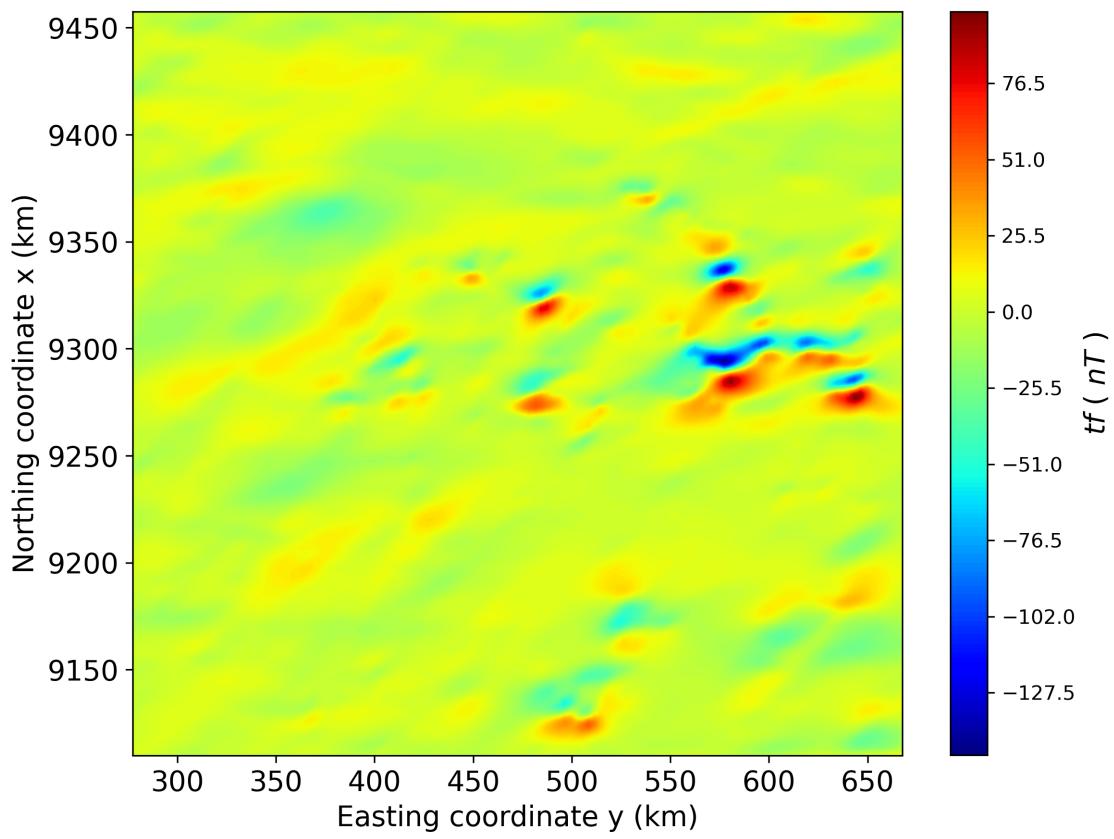


Figure 39: Upward continuation transformation of real data of Carajás, Brazil at 5,000 meter. It was necessary 2.64 seconds to complete the process.

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$N$	Matrix <b>A</b>	All six first columns of BCCB matrices	Matrix <b>L</b>
100	0.0763	0.0183	0.00610
400	1.22	0.0744	0.0248
2,500	48	0.458	0.1528
10,000	763	1.831	0.6104
40,000	12,207	7.32	2.4416
250,000	476,837	45.768	15.3
500,000	1,907,349	91.56	30.518
1,000,000	7,629,395	183.096	61.035

Table 1: This table shows the RAM memory usage (in Megabytes) for storing the whole matrix **A** (equation 12), the sum of all six first columns of the BCCB matrices embedded from the components of the matrix **H** from equation 10 (both need 8 bytes per element) and the matrix **L** containing the eigenvalues complex numbers (16 bytes per element) resulting from the diagonalization of matrix **C** (equation 63). Here we must consider that  $N$  observation points forms a  $N \times N$  matrix.

Residual	Mean ( $nT$ )	Std ( $nT$ )
Noiseless grid classical solution	0.3627	0.2724
Noiseless grid our solution	0.5223	0.4323
X, Y - 20% noise grid classical solution	0.3630	0.2731
X, Y - 20% noise grid our solution	0.7147	0.5622
X, Y - 30% noise grid classical solution	0.3634	0.2735
X, Y - 30% noise grid our solution	0.9788	0.7462
Z - 05% noise surface classical solution	0.3712	0.2870
Z - 05% noise surface our solution	0.9542	0.8943
Z - 10% noise surface classical solution	0.3865	0.3216
Z - 10% noise surface our solution	1.6109	1.6231

Table 2: This table shows the means and standard deviations for each residual presented in the synthetic data tests between the observation data in comparison to the classical solution and our solution.

Residual	Mean ( $nT$ )	Std ( $nT$ )
Noiseless grid upward continuation classical solution	0.5130	0.6533
Noiseless grid upward continuation our solution	0.6127	0.8121
Noiseless grid upward continuation Fourier solution	2.1120	2.2870
X, Y - 20% noise grid upward continuation classical solution	0.5060	0.6427
X, Y - 20% noise grid upward continuation our solution	0.6764	0.8147
X, Y - 20% noise grid upward continuation Fourier solution	2.1025	2.2774
X, Y - 30% noise grid upward continuation classical solution	0.4988	0.6324
X, Y - 30% noise grid upward continuation our solution	0.7477	0.7796
X, Y - 30% noise grid upward continuation Fourier solution	2.1116	2.2763
Z - 05% noise surface upward continuation classical solution	0.4979	0.6380
Z - 05% noise surface upward continuation our solution	1.5656	0.9857
Z - 05% noise surface upward continuation Fourier solution	2.0925	2.2907
Z - 10% noise surface upward continuation classical solution	0.4792	0.6234
Z - 10% noise surface upward continuation our solution	4.2127	2.3897
Z - 10% noise surface upward continuation Fourier solution	2.0728	2.2924

Table 3: This table shows the means and standard deviations for each residual presented in the synthetic data tests section for the upward continuation transformation comparison.

Residual	Mean ( $nT$ )	Std ( $nT$ )
Regular interpolated grid	0.0798	0.5060
Irregular decimated grid	0.0735	0.3172

Table 4: This table shows the means and standard deviations for the residuals presented in the real data tests section, for the solution of the regular interpolated grid and the irregular decimated grid.