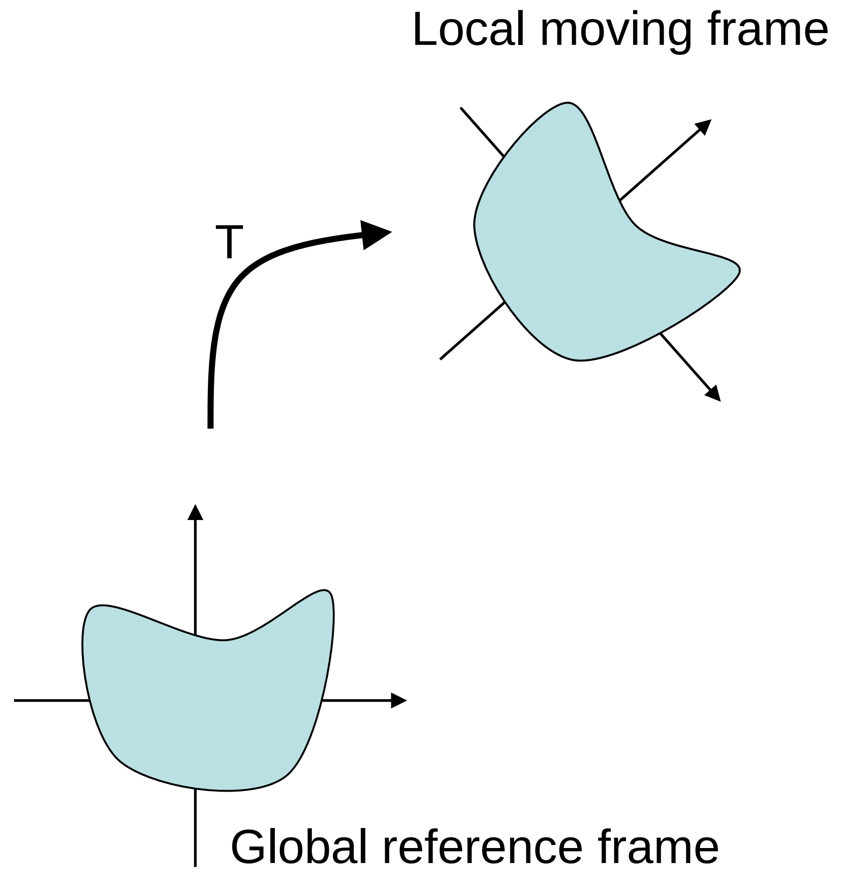
Geometric Transformations

Jehee Lee
Seoul National University

Transformations

- Linear transformations
- Rigid transformations
- Affine transformations
- Projective transformations



Linear Transformations

- A ***linear transformation*** T is a mapping between vector spaces
  - T maps vectors to vectors
  - linear combination is invariant under T

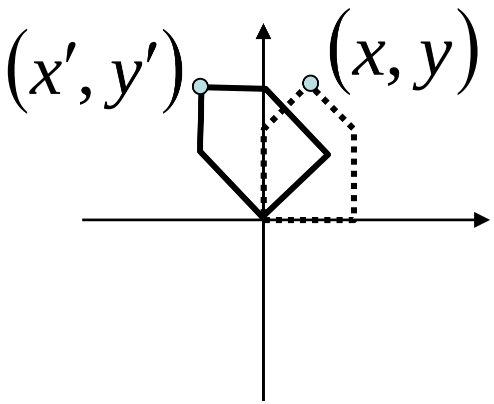
$$T(\sum_{i=0}^{N} c_i \mathbf{v}_i) = c_0 T(\mathbf{v}_0) + c_1 T(\mathbf{v}_1) + \cdots + c_N T(\mathbf{v}_N)$$

- In 3-spaces, T can be represented by a 3x3 matrix

$$\begin{aligned} T(\mathbf{v}) &= \mathbf{M}_{3\times 3}\mathbf{v}_{3\times 1} \quad \text{(Column major)} \\ &= \mathbf{v}_{1\times 3}\mathbf{N}_{3\times 3} \quad \text{(Row major)} \end{aligned}$$

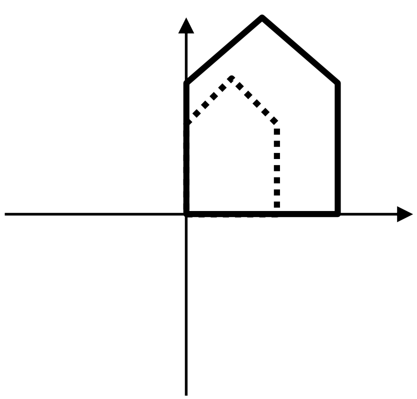
Examples of Linear Transformations

- 2D rotation



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

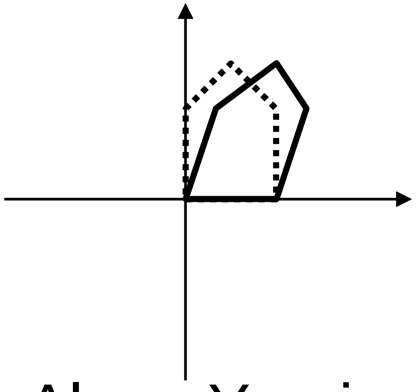
- 2D scaling



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s_x x \\ s_y y \end{pmatrix}$$

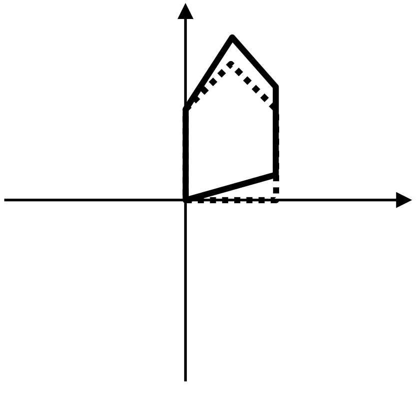
Examples of Linear Transformations

- 2D shear
  - Along X-axis



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + dy \\ y \end{pmatrix}$$

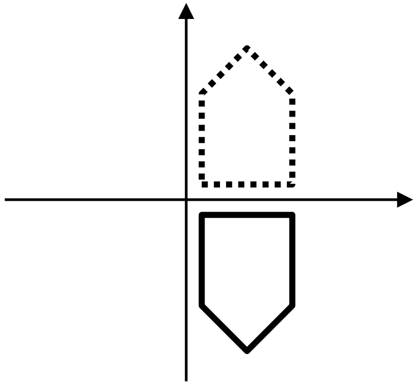
  - Along Y-axis



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + dx \end{pmatrix}$$

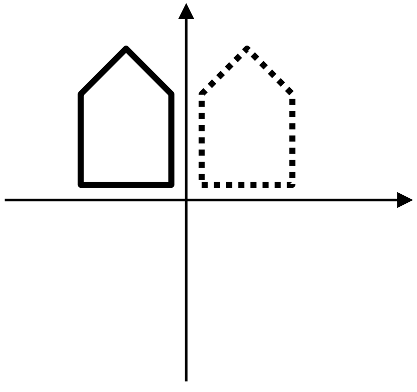
Examples of Linear Transformations

- 2D reflection
  - Along X-axis



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

  - Along Y-axis



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

Properties of Linear Transformations

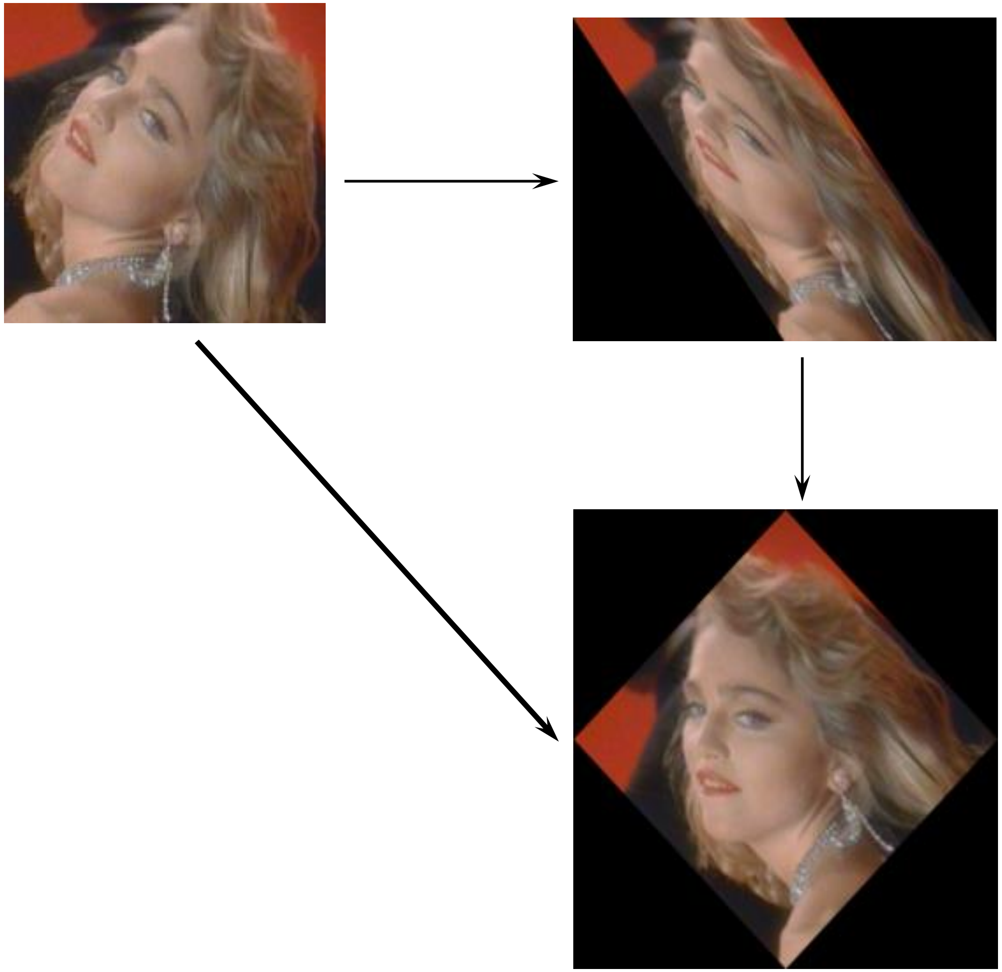
- Any ***linear transformation*** between 3D spaces can be represented by a 3x3 matrix

- Any ***linear transformation*** between 3D spaces can be represented as a combination of ***rotation***, ***shear***, and ***scaling***

- ***Rotation*** can be represented as a combination of ***scaling*** and ***shear***

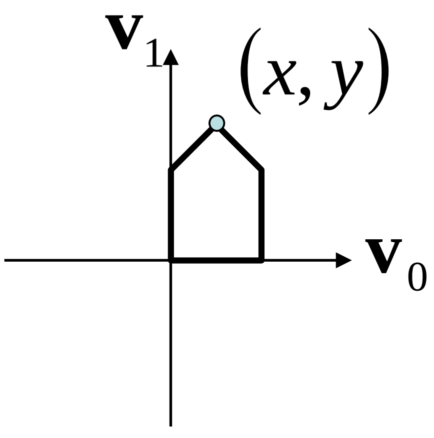
Properties of Linear Transformations

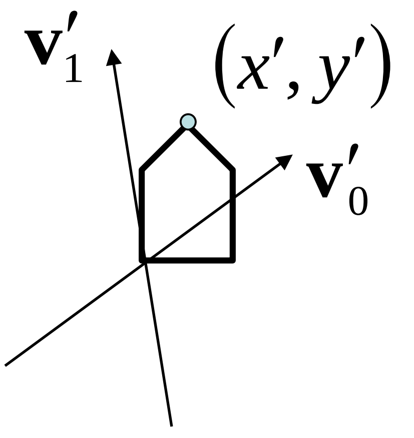
- ***Rotation*** can be computed as two-pass shear transformations



Changing Bases

- Linear transformations as a change of bases



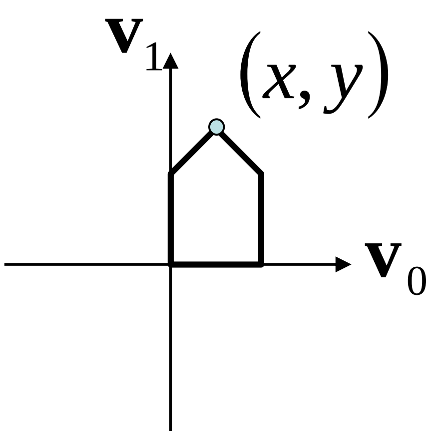


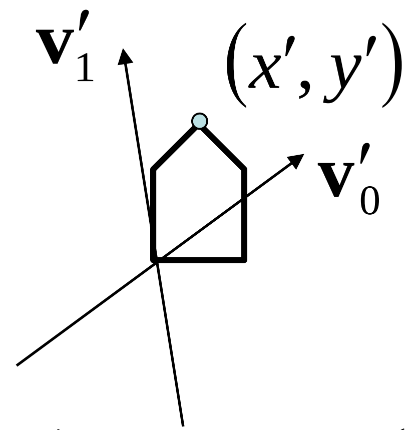
$$x'\mathbf{v}'_0 + y'\mathbf{v}'_1 = x\mathbf{v}_0 + y\mathbf{v}_1$$

$$\begin{pmatrix} \mathbf{v}'_0 & \mathbf{v}'_1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \mathbf{v}_0 & \mathbf{v}_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Changing Bases

- Linear transformations as a change of bases





$$\mathbf{v}_0 = a_0\mathbf{v}'_0 + a_1\mathbf{v}'_1$$

$$\mathbf{v}_1 = b_0\mathbf{v}'_0 + b_1\mathbf{v}'_1$$

$$\begin{pmatrix}\mathbf{v}'_0 & \mathbf{v}'_1\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix} = \begin{pmatrix}\mathbf{v}'_0 & \mathbf{v}'_1\end{pmatrix}\begin{pmatrix}a_0 & b_0\\ a_1 & b_1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{pmatrix}x'\\ y'\end{pmatrix} = \begin{pmatrix}a_0 & b_0\\ a_1 & b_1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$= T\begin{pmatrix}x\\ y\end{pmatrix}$$

Affine Transformations

- An ***affine transformation*** T is an mapping between affine spaces
  - T maps vectors to vectors, and points to points
  - T is a linear transformation on vectors
  - affine combination is invariant under T

$$T\left(\sum_{i=0}^{N} c_i \mathbf{p}_i\right) = c_0 T(\mathbf{p}_0) + c_1 T(\mathbf{p}_1) + \cdots + c_N T(\mathbf{p}_N)$$

- In 3-spaces, T can be represented by a 3x3 matrix together with a 3x1 translation vector

$$T(\mathbf{p}) = \mathbf{M}_{3\times3}\mathbf{p}_{3\times1} + \mathbf{T}_{3\times1}$$

Homogeneous Coordinates

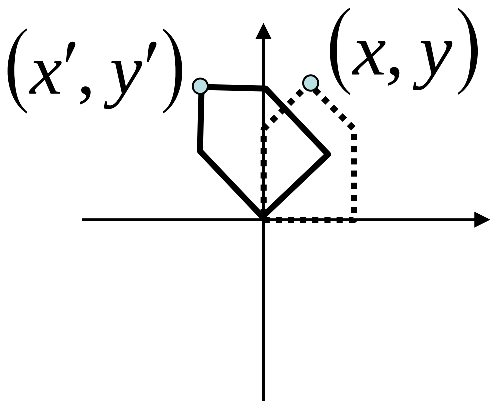
- Any affine transformation between 3D spaces can be represented by a 4x4 matrix

$$T(\mathbf{p}) = \begin{pmatrix} \mathbf{M}_{3\times 3} & \mathbf{T}_{3\times 1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{p}_{3\times 1} \\ 1 \end{pmatrix}$$

- Affine transformation is *linear* in homogeneous coordinates

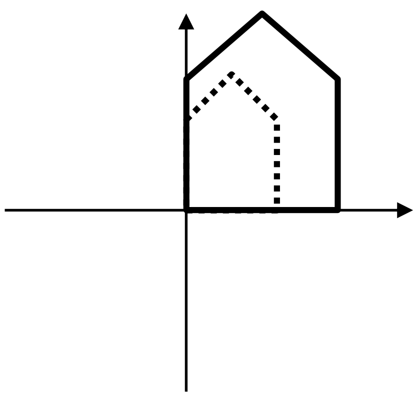
Examples of Affine Transformations

- 2D rotation



$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

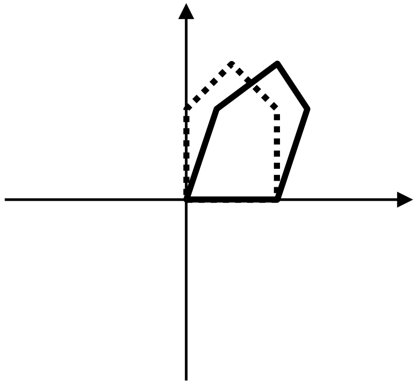
- 2D scaling



$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} s_x x \\ s_y y \\ 1 \end{pmatrix}$$

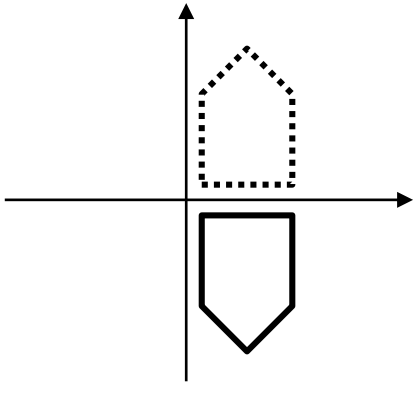
Examples of Affine Transformations

- 2D shear



$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & d & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + dy \\ y \\ 1 \end{pmatrix}$$

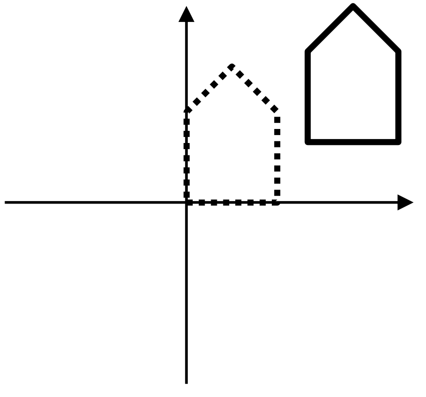
- 2D reflection



$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ -y \\ 1 \end{pmatrix}$$

Examples of Affine Transformations

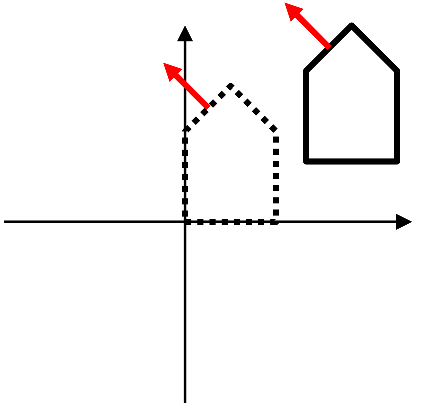
- 2D translation



$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix}$$

Examples of Affine Transformations

- 2D transformation for ***vectors***
  - ***Translation*** is simply ignored

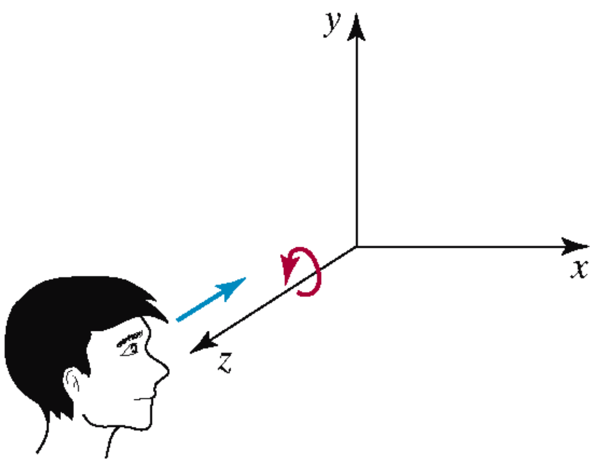


$$\begin{pmatrix} x' \\ y' \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

Examples of Affine Transformations

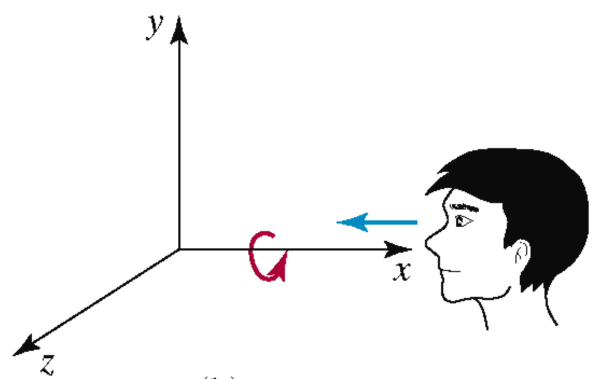
- 3D rotation

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$



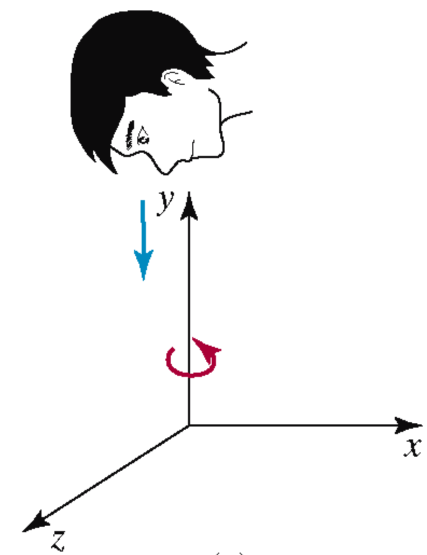
(a)

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$



(b)

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$



(c)

Composite Transformations

- Composite 2D Translation

$$
\begin{aligned}
T &= \mathbf{T}(t_{x1}, t_{y1}) \cdot \mathbf{T}(t_{x2}, t_{y2}) \\
&= \mathbf{T}(t_{x1} + t_{x2}, t_{y1} + t_{y2})
\end{aligned}
$$

$$
\begin{pmatrix} 1 & 0 & t_{x2} \\ 0 & 1 & t_{y2} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & t_{x1} \\ 0 & 1 & t_{y1} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_{x1} + t_{x2} \\ 0 & 1 & t_{y1} + t_{y2} \\ 0 & 0 & 1 \end{pmatrix}
$$

Composite Transformations

- Composite 2D Scaling

$$T = \mathbf{S}(s_{x1}, s_{y1}) \cdot \mathbf{S}(s_{x2}, s_{y2})$$
$$= \mathbf{S}(s_{x1}s_{x2}, s_{y1}s_{y2})$$

$$\begin{pmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} s_{x1} \cdot s_{x2} & 0 & 0 \\ 0 & s_{y1} \cdot s_{y2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Composite Transformations

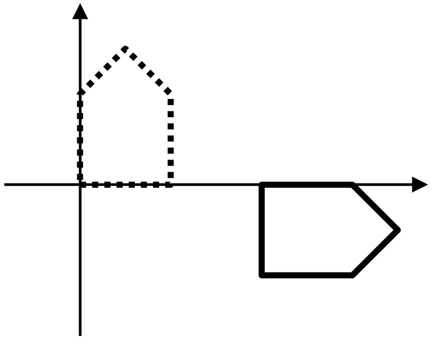
- Composite 2D Rotation

$$T = \mathbf{R}(\theta_2) \cdot \mathbf{R}(\theta_1)$$
$$= \mathbf{R}(\theta_2 + \theta_1)$$

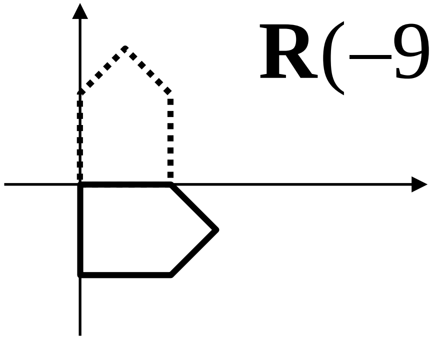
$$\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_2 + \theta_1) & -\sin(\theta_2 + \theta_1) & 0 \\ \sin(\theta_2 + \theta_1) & \cos(\theta_2 + \theta_1) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Composing Transformations

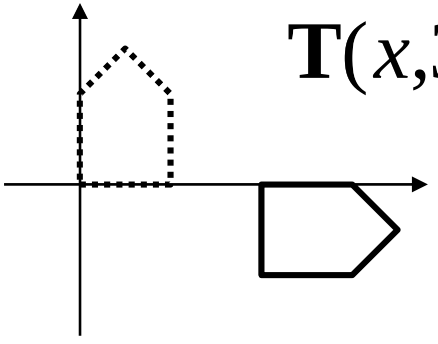
- Suppose we want,



- We have to compose two transformations



$\mathbf{R}(-90°)$



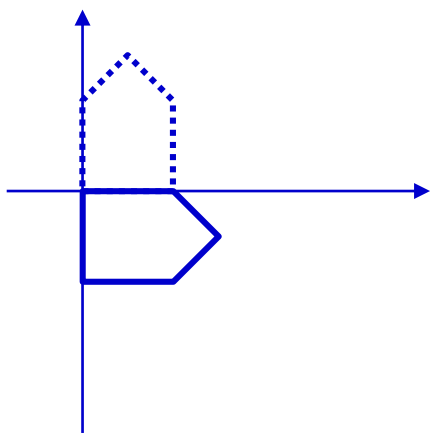
$\mathbf{T}(x,3)$

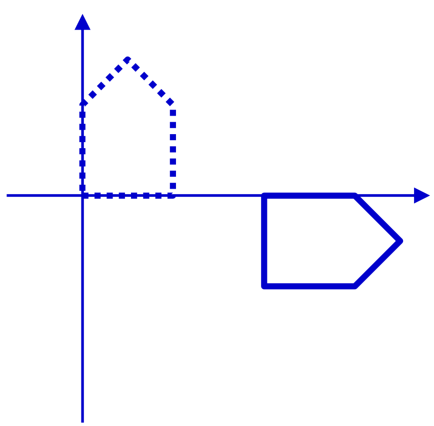
Composing Transformations

- Matrix multiplication is not commutative

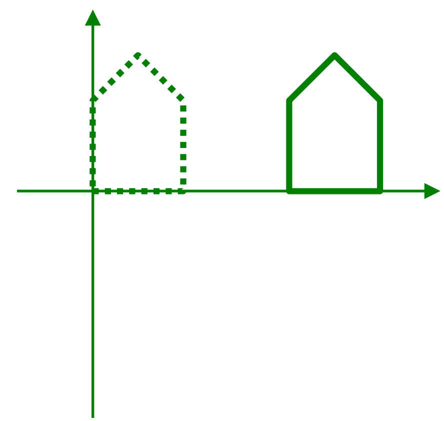
$$\mathbf{T}(x,3)\cdot\mathbf{R}(-90^\circ)\neq\mathbf{R}(-90^\circ)\mathbf{T}(x,3)$$

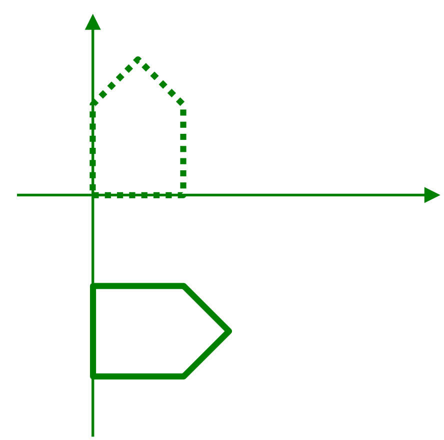
Translation followed by rotation





Translation followed by rotation

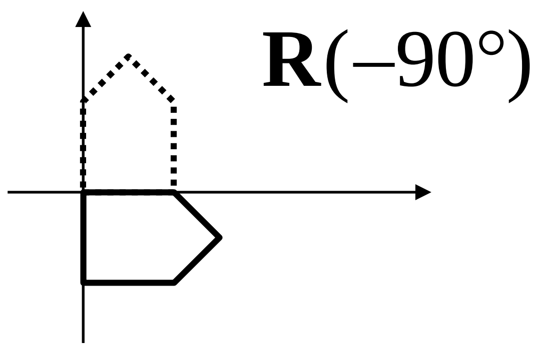


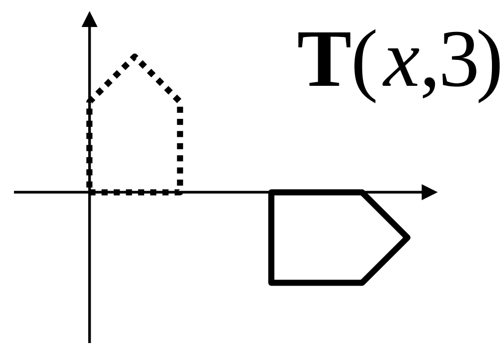


Composing Transformations

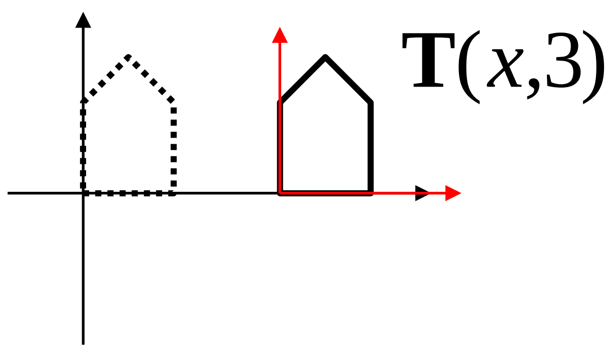
$$T = \mathbf{T}(x,3) \cdot \mathbf{R}(-90°)$$ (Column major convention)

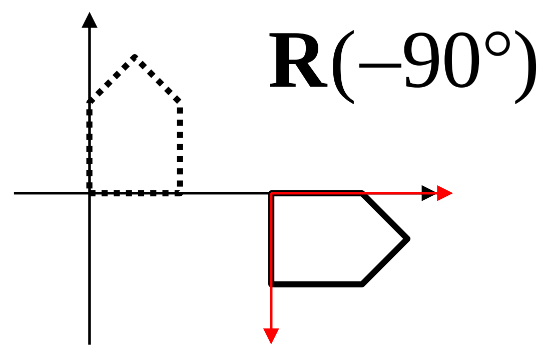
– R-to-L : interpret operations w.r.t. fixed coordinates





– L-to-R : interpret operations w.r.t local coordinates





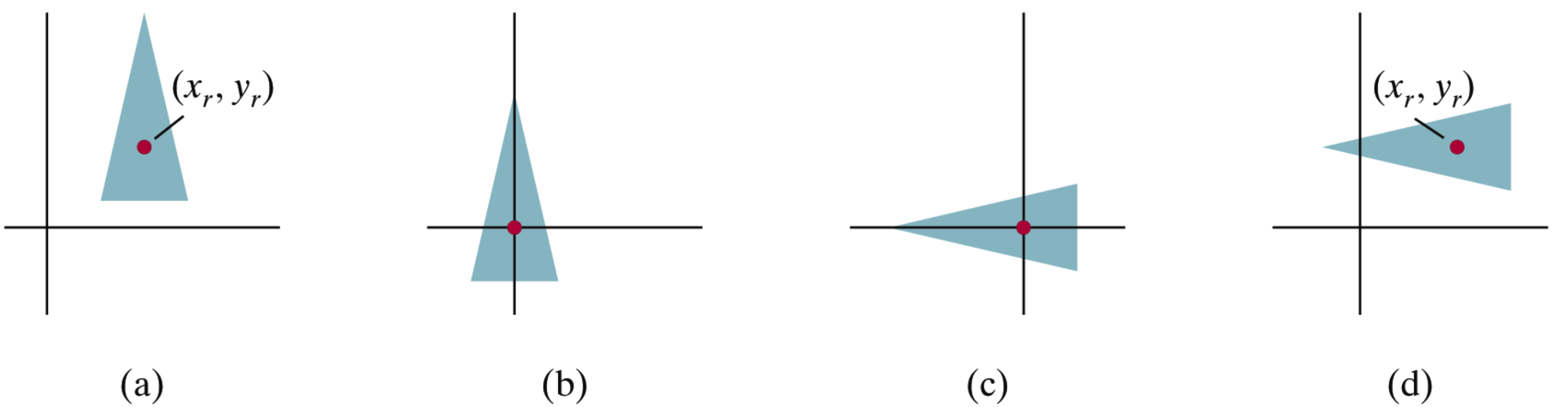
Pivot-Point Rotation

- Rotation with respect to a pivot point (x,y)

$$T(x,y)\cdot R(\theta)\cdot T(-x,-y)$$

$$=\begin{pmatrix}1 & 0 & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & -x\\ 0 & 1 & -y\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}\cos\theta & -\sin\theta & x(1-\cos\theta)+y\sin\theta\\ \sin\theta & \cos\theta & y(1-\cos\theta)-x\sin\theta\\ 0 & 0 & 1\end{pmatrix}$$



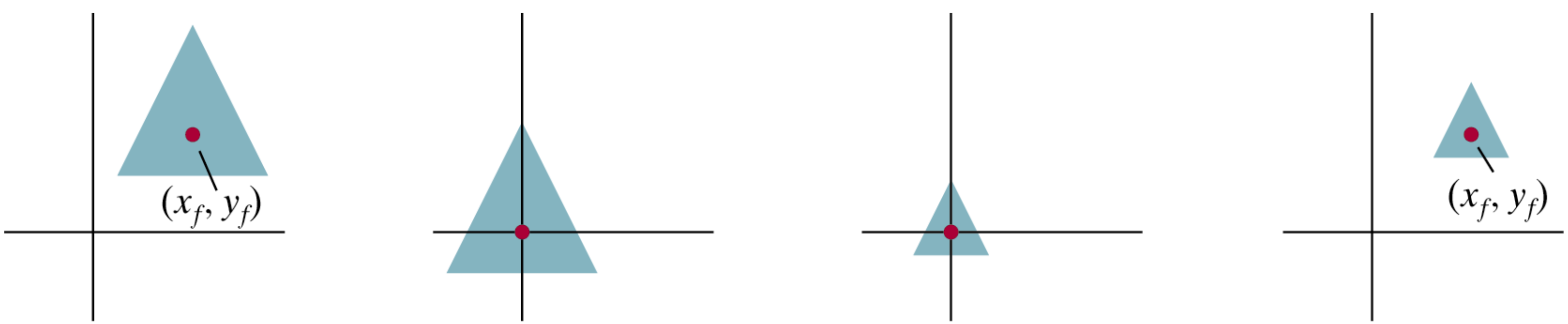
Fixed-Point Scaling

- Scaling with respect to a fixed point (x,y)

$$T(x,y)\cdot S(s_x,s_y)\cdot T(-x,-y)$$

$$=\begin{pmatrix}1 & 0 & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}s_x & 0 & 0\\ 0 & s_y & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & -x\\ 0 & 1 & -y\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}s_x & 0 & (1-s_x)\cdot x\\ 0 & s_y & (1-s_y)\cdot y\\ 0 & 0 & 1\end{pmatrix}$$

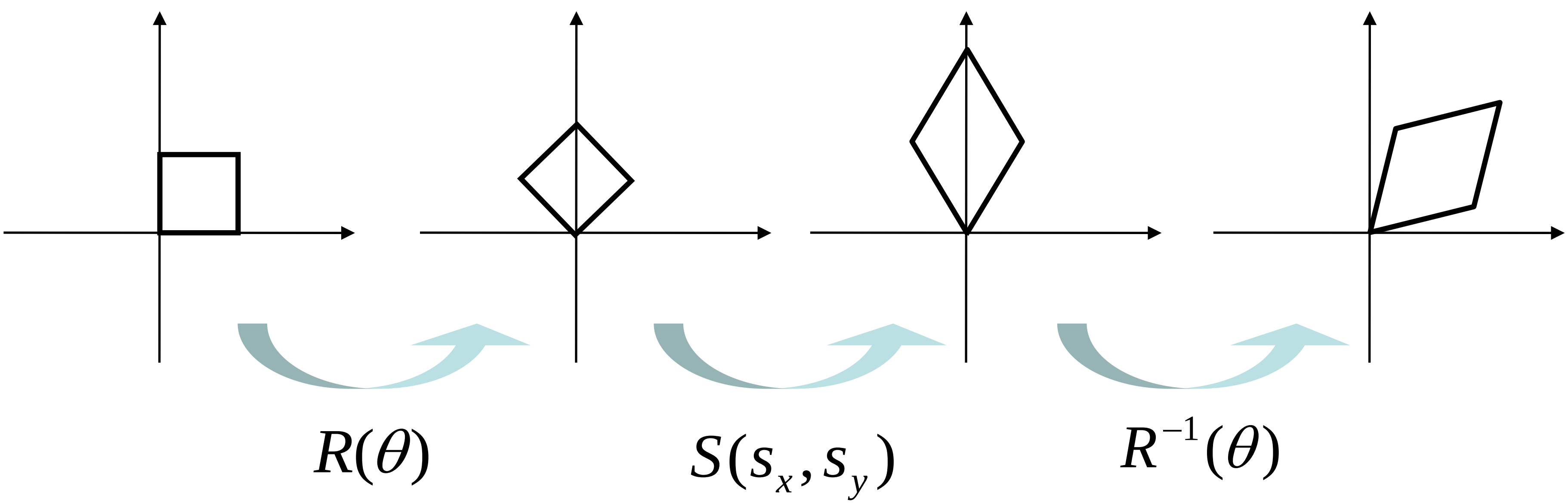


(a) (b) (c) (d)

Scaling Direction

- Scaling along an arbitrary axis

$$R^{-1}(\theta) \cdot S(s_x, s_y) \cdot R(\theta)$$



Properties of Affine Transformations

- Any ***affine transformation*** between 3D spaces can be represented as a combination of a ***linear transformation*** followed by ***translation***
- An affine transf. maps ***lines*** to ***lines***
- An affine transf. maps ***parallel lines*** to ***parallel lines***
- An affine transf. preserves ***ratios of distance*** along a line
- An affine transf. does not preserve absolute distances and angles

Review of Affine Frames

- A ***frame*** is defined as a set of vectors $\{\mathbf{v}_i \mid i=1, \ldots, N\}$ and a point $\mathbf{o}$
  - Set of vectors $\{\mathbf{v}_i\}$ are bases of the associate vector space
  - $\mathbf{o}$ is an origin of the frame
  - N is the dimension of the affine space
  - Any point $\mathbf{p}$ can be written as

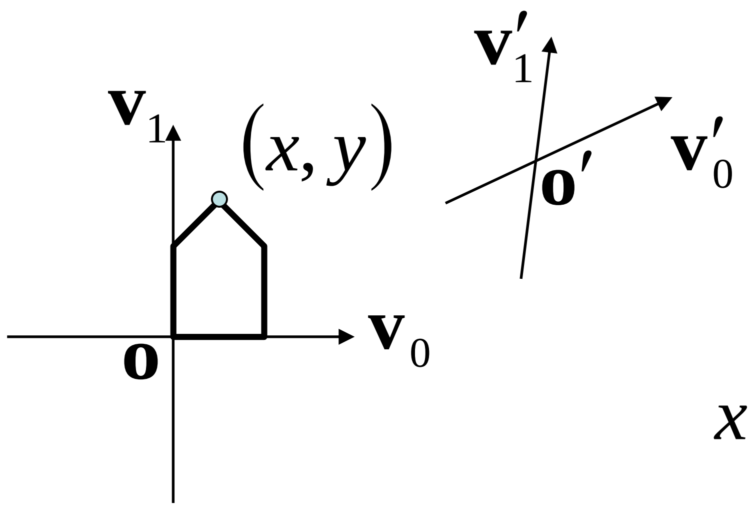
$$\mathbf{p} = \mathbf{o} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_N\mathbf{v}_N$$

  - Any vector $\mathbf{v}$ can be written as

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_N\mathbf{v}_N$$

Changing Frames

- Affine transformations as a change of frame

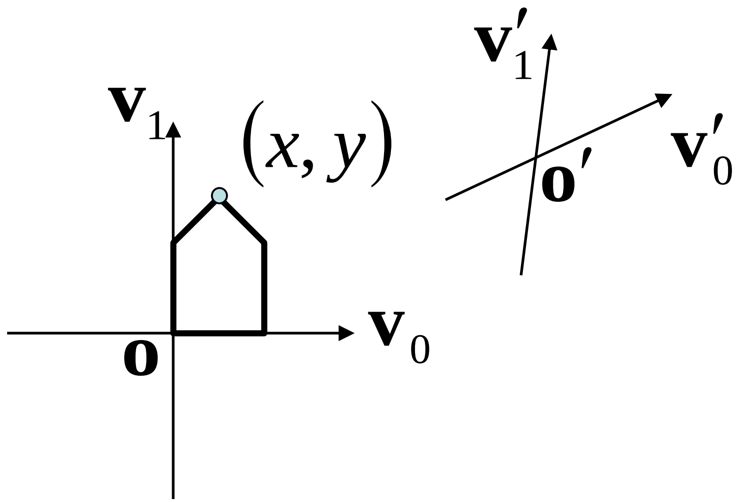


$$x'\mathbf{v}_0' + y'\mathbf{v}_1' + \mathbf{o}' = x\mathbf{v}_0 + y\mathbf{v}_1 + \mathbf{o}$$

$$\begin{pmatrix} \mathbf{v}_0' & \mathbf{v}_1' & \mathbf{o}' \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{o} \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Changing Frames

- Affine transformations as a change of frame



$$\mathbf{v}_0 = a_0\mathbf{v}'_0 + a_1\mathbf{v}'_1$$

$$\mathbf{v}_1 = b_0\mathbf{v}'_0 + b_1\mathbf{v}'_1$$

$$\mathbf{o} = c_0\mathbf{v}'_0 + c_1\mathbf{v}' + \mathbf{o}'$$

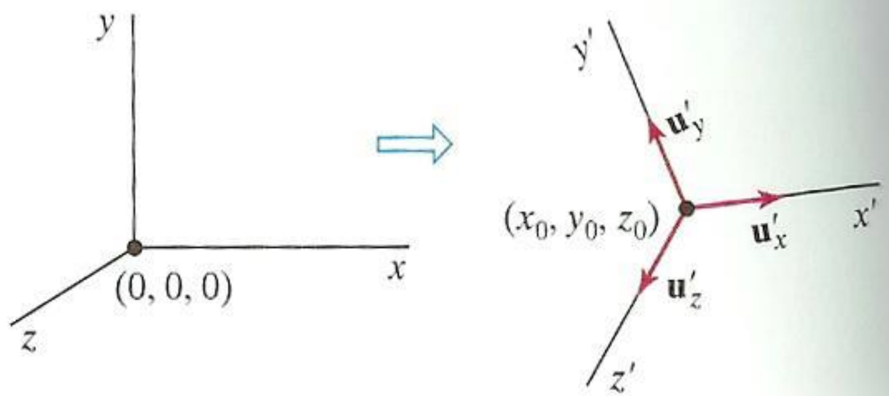
$$\begin{pmatrix} \mathbf{v}'_0 & \mathbf{v}'_1 & \mathbf{o} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{v}'_0 & \mathbf{v}'_1 & \mathbf{o} \end{pmatrix} \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Changing Frames

- In case the xyz system has standard bases

FIGURE 5-54 An $x'y'z'$ coordinate system defined within an xyz system. A scene description is transferred to the new coordinate reference using a transformation sequence that superimposes the $x'y'z'$ frame on the xyz axes.



Rigid Transformations

- A ***rigid transformation*** T is a mapping between affine spaces
  - T maps vectors to vectors, and points to points
  - T preserves distances between all points
  - T preserves cross product for all vectors (to avoid reflection)
- In 3-spaces, T can be represented as

$$T(\mathbf{p}) = \mathbf{R}_{3\times3}\mathbf{p}_{3\times1} + \mathbf{T}_{3\times1}, \quad \text{where}$$

$$\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I} \quad \text{and} \quad \det\mathbf{R} = 1$$

Rigid Body Rotation

- Rigid body transformations allow only rotation and translation

- Rotation matrices form SO(3)
  - Special orthogonal group

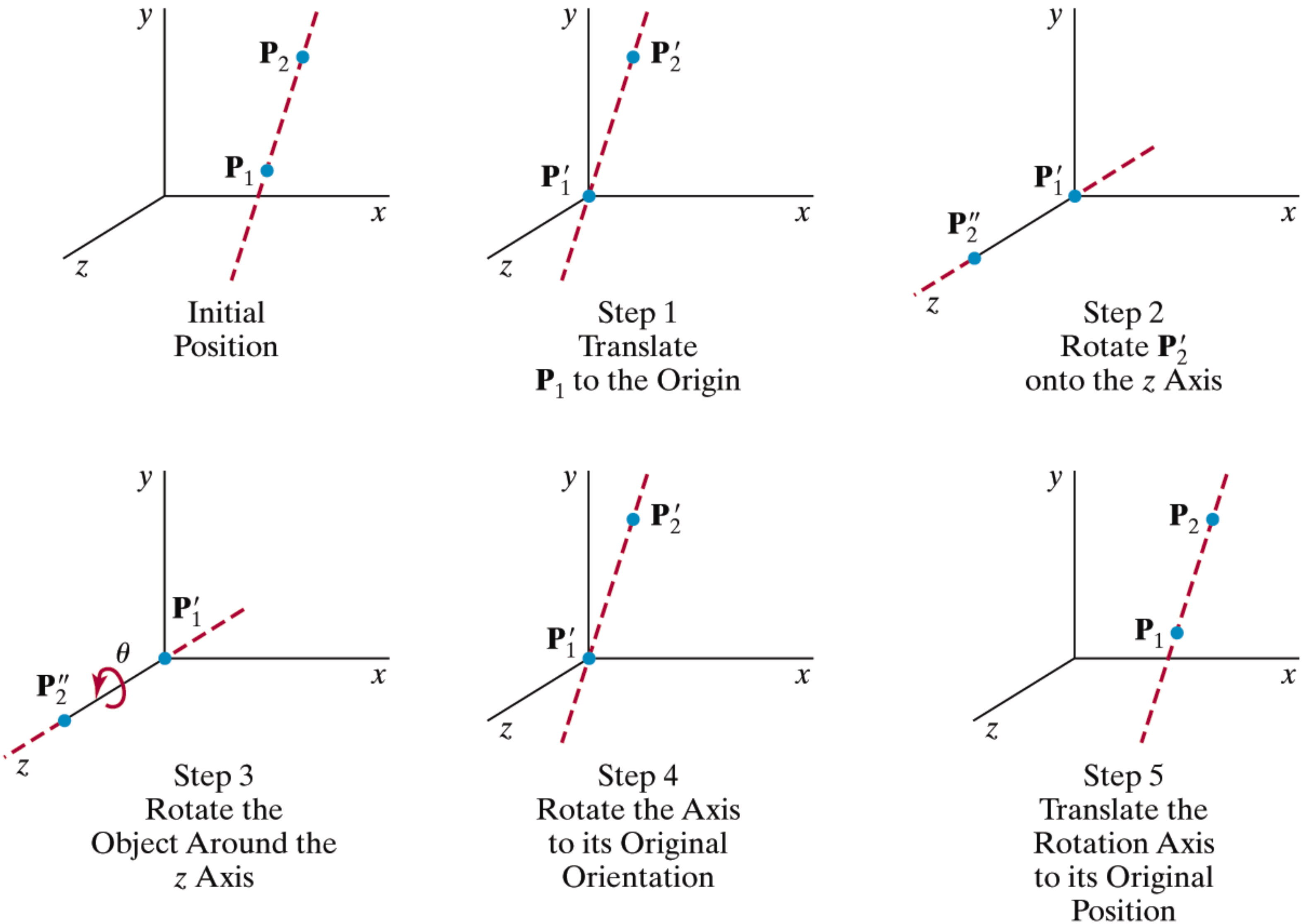
$$\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}$$ (Distance preserving)

$$\det \mathbf{R} = 1$$ (No reflection)

Rigid Body Rotation

- R is normalized
  - The squares of the elements in any row or column sum to 1
- R is orthogonal $\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}$
  - The dot product of any pair of rows or any pair columns is 0
- The rows (columns) of R correspond to the vectors of the principle axes of the rotated coordinate frame

3D Rotation About Arbitrary Axis



3D Rotation About Arbitrary Axis

1. Translation : rotation axis passes through the origin $$T(-x_1, -y_1, -z_1)$$

2. Make the rotation axis on the z-axis
$$R_x(\alpha) \cdot R_y(\beta)$$

3. Do rotation $R_z(\theta)$

4. Rotation & translation
$$T^{-1} \cdot R_y^{-1}(\beta) \cdot R_x^{-1}(\alpha)$$

3D Rotation About Arbitrary Axis

- Rotate **u** onto the *z*-axis

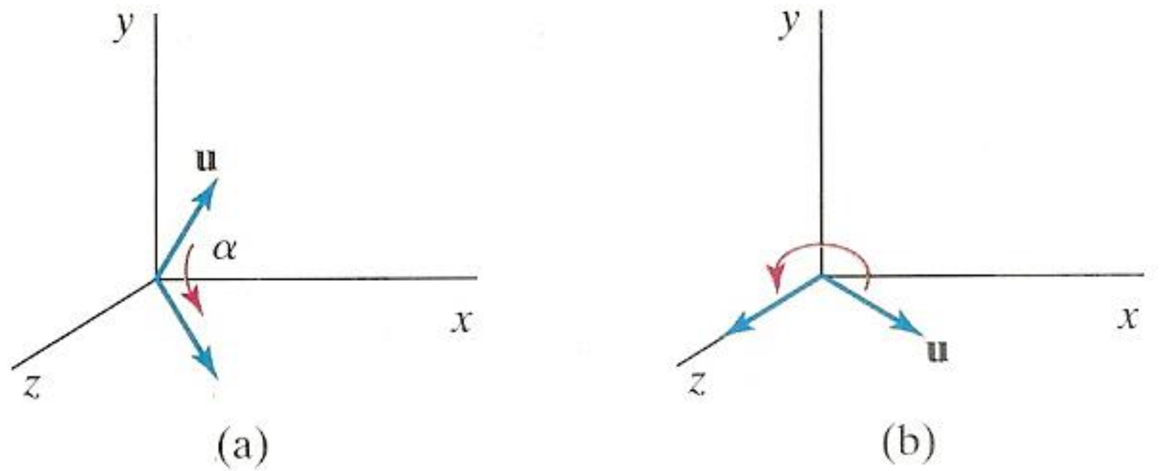


FIGURE 5-45 Unit vector **u** is rotated about the *x* axis to bring it into the *xz* plane (a), then it is rotated around the *y* axis to align it with the *z* axis (b).

3D Rotation About Arbitrary Axis

- Rotate **u** onto the z-axis
  - **u'**: Project **u** onto the yz-plane to compute angle α
  - **u''**: Rotate **u** about the x-axis by angle α
  - Rotate **u''** onto the z-axis

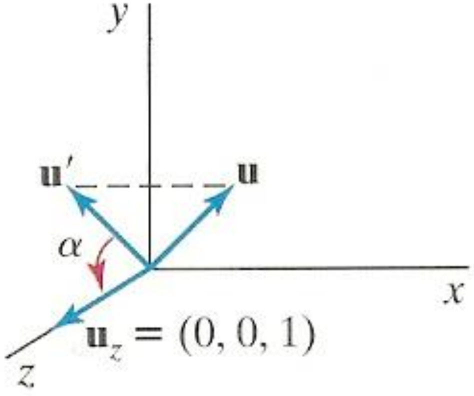


FIGURE 5-46 Rotation of $\mathbf{u}$ around the x axis into the xz plane is accomplished by rotating $\mathbf{u}'$ (the projection of $\mathbf{u}$ in the yz plane) through angle α onto the z axis.

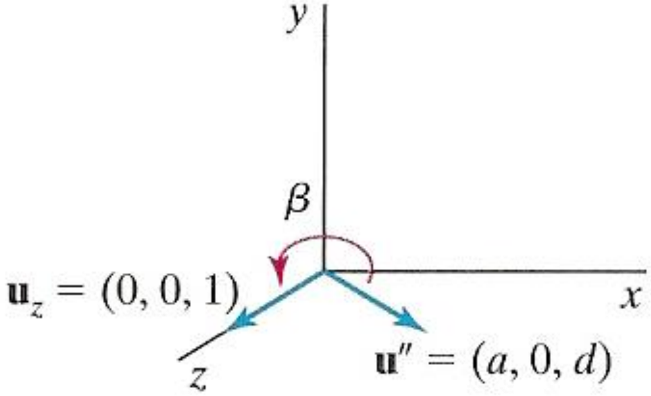
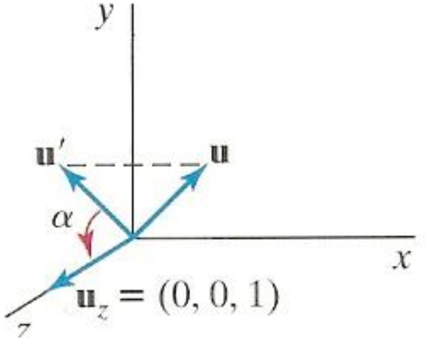


FIGURE 5-47 Rotation of unit vector $\mathbf{u}''$ (vector $\mathbf{u}$ after rotation into the xz plane) about the y axis. Positive rotation angle β aligns $\mathbf{u}''$ with vector $\mathbf{u}_z$.

3D Rotation About Arbitrary Axis

- Rotate **u'** about the x-axis onto the *z*-axis
  - Let **u**=(a,b,c) and thus **u'**=(0,b,c)
  - Let $\mathbf{u}_z$=(0,0,1)



$$\cos\alpha = \frac{\mathbf{u}' \cdot \mathbf{u}_z}{\|\mathbf{u}'\| \|\mathbf{u}_z\|} = \frac{c}{\sqrt{b^2 + c^2}}$$

$$\begin{aligned} \mathbf{u}' \times \mathbf{u}_z &= \mathbf{u}_x \|\mathbf{u}'\| \|\mathbf{u}_z\| \sin\alpha \\ &= \mathbf{u}_x \cdot b \end{aligned} \quad \Rightarrow \quad \sin\alpha = \frac{b}{\|\mathbf{u}'\| \|\mathbf{u}_z\|} = \frac{b}{\sqrt{b^2 + c^2}}$$

3D Rotation About Arbitrary Axis

- Rotate **u'** about the x-axis onto the *z*-axis
  - Since we know both cos α and sin α, the rotation matrix can be obtained

$$\mathbf{R}_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{c}{\sqrt{b^2+c^2}} & \frac{-b}{\sqrt{b^2+c^2}} & 0 \\ 0 & \frac{b}{\sqrt{b^2+c^2}} & \frac{c}{\sqrt{b^2+c^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  - Or, we can compute the signed angle α

$$\operatorname{atan2}\left(\frac{c}{\sqrt{b^2+c^2}}, \frac{b}{\sqrt{b^2+c^2}}\right)$$

  - Do not use acos() since its domain is limited to [-1,1]

Euler angles

- Arbitrary orientation can be represented by three rotation along x,y,z axis

$$R_{XYZ}(\gamma,\beta,\alpha) = R_z(\alpha)R_y(\beta)R_x(\gamma)$$

$$= \begin{bmatrix} \mathrm{C}\alpha\mathrm{C}\beta & \mathrm{C}\alpha\mathrm{S}\beta\mathrm{S}\gamma - \mathrm{S}\alpha\mathrm{C}\gamma & C\alpha S\beta C\gamma + S\alpha S\gamma & 0 \\ S\alpha C\beta & S\alpha S\beta S\gamma + C\alpha C\gamma & S\alpha S\beta C\gamma - C\alpha S\gamma & 0 \\ -S\beta & C\beta S\gamma & C\beta C\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Gimble

- Hardware implementation of Euler angles
- Aircraft, Camera

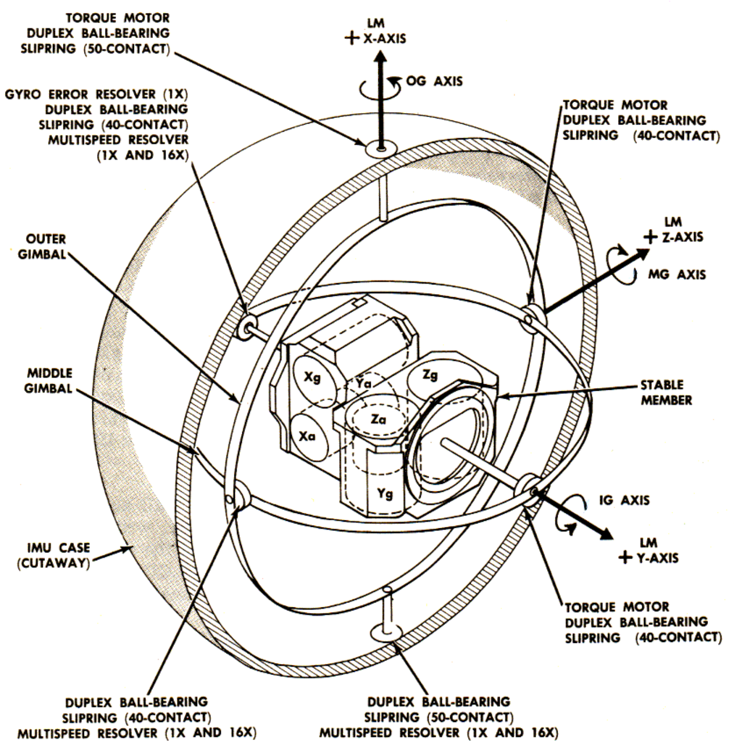
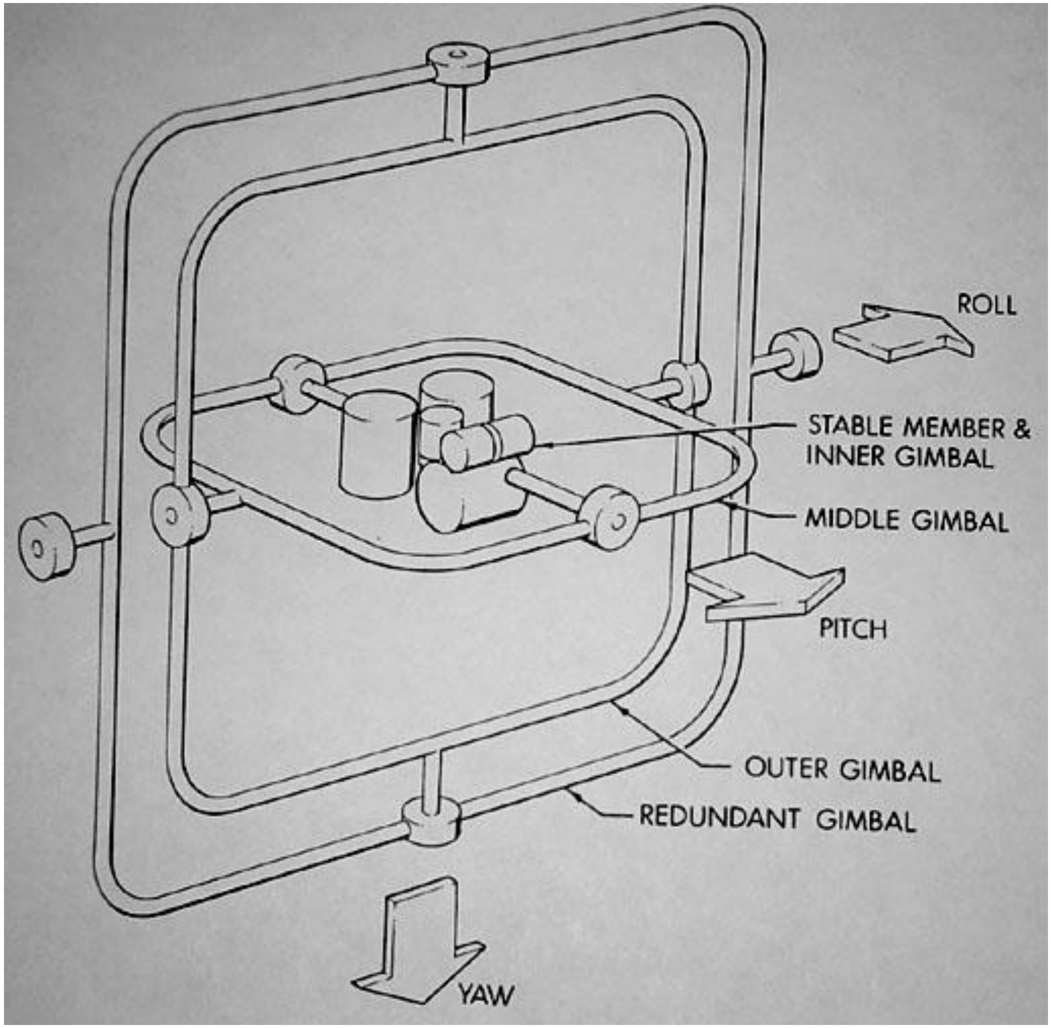


Figure 2.1-24. IMU Gimbal Assembly

Euler Angles

- Rotation about three orthogonal axes
  - 12 combinations
    - XYZ, XYX, XZY, XZX
    - YZX, YZY, YXZ, YXY
    - ZXY, ZXZ, ZYX, ZYZ

- ***Gimble lock***
  - Coincidence of inner most and outmost gimbles’ rotation axes
  - Loss of degree of freedom



Euler Angles

- Euler angles are ambiguous
  - Two different Euler angles can represent the same orientation

$$R_1 = (r_x, r_y, r_z) = (\theta, \frac{\pi}{2}, 0) \quad \text{and} \quad R_2 = (0, \frac{\pi}{2}, -\theta)$$

  - This ambiguity brings unexpected results of animation where frames are generated by interpolation.

Taxonomy of Transformations

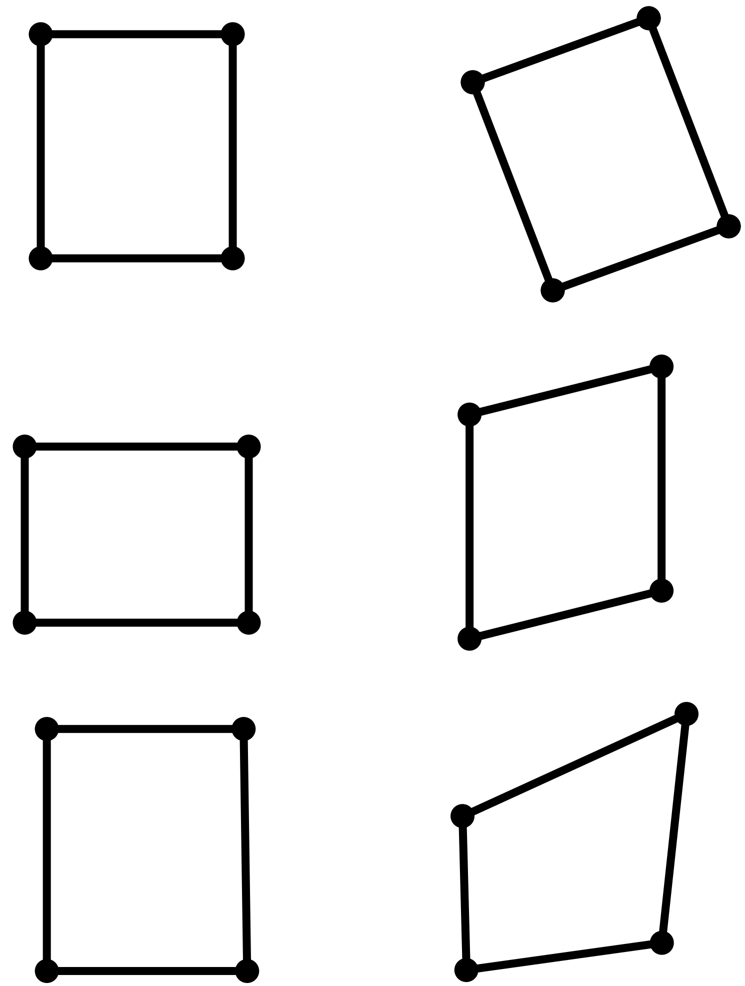
- **Linear** transformations
  - 3x3 matrix
  - Rotation + scaling + shear
- **Rigid** transformations
  - SO(3) for rotation
  - 3D vector for translation
- **Affine** transformation
  - 3x3 matrix + 3D vector or 4x4 homogenous matrix
  - Linear transformation + translation
- **Projective** transformation
  - 4x4 matrix
  - Affine transformation + perspective projection

Taxonomy of Transformations

Rigid

Affine

Projective

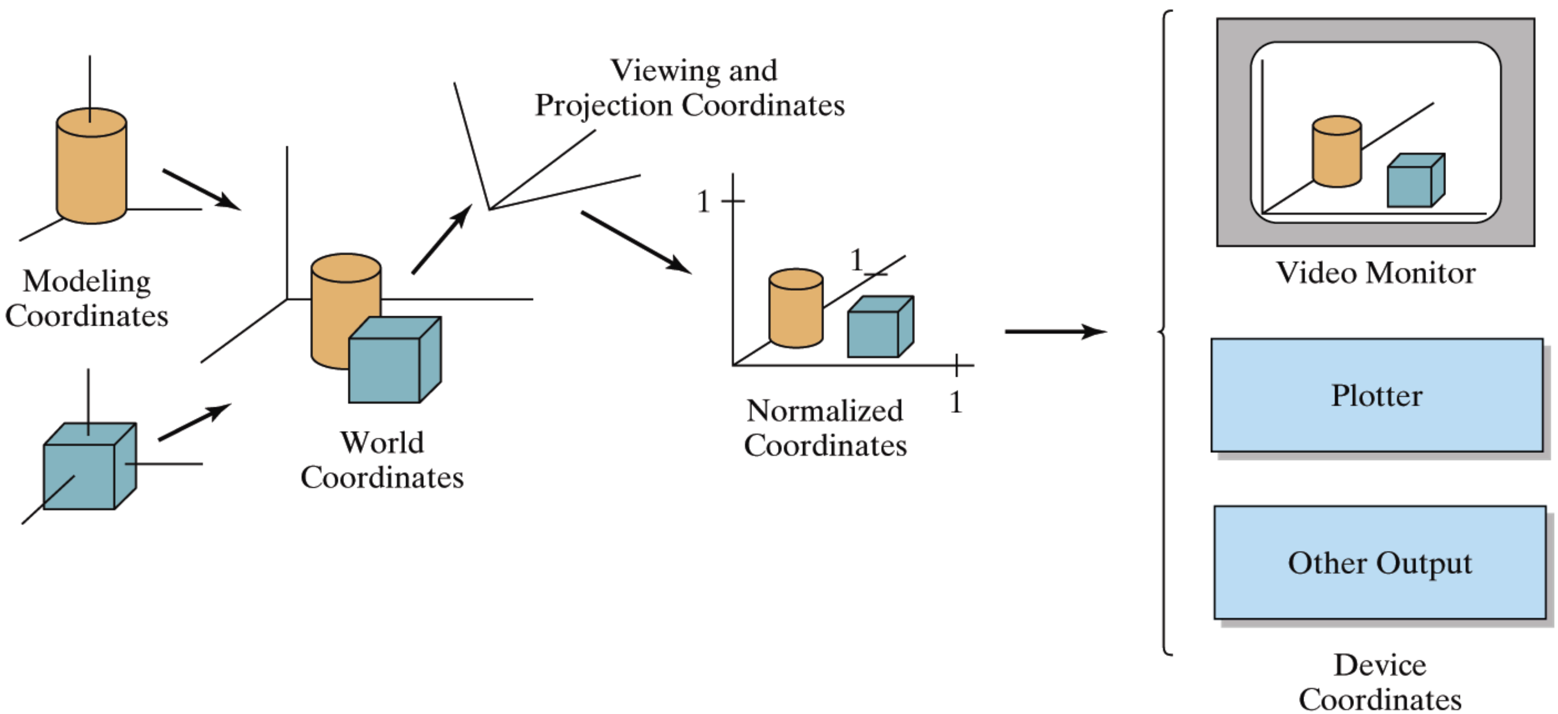


Composition of Transforms

- What is the composition of linear/affine/rigid transformations ?
- What is the linear (or affine) combination of linear (or affine) transformations ?
- What is the linear (or affine) combination of rigid transformations ?

OpenGL Geometric Transformations

- **`glMatrixMode(GL_MODELVIEW);`**

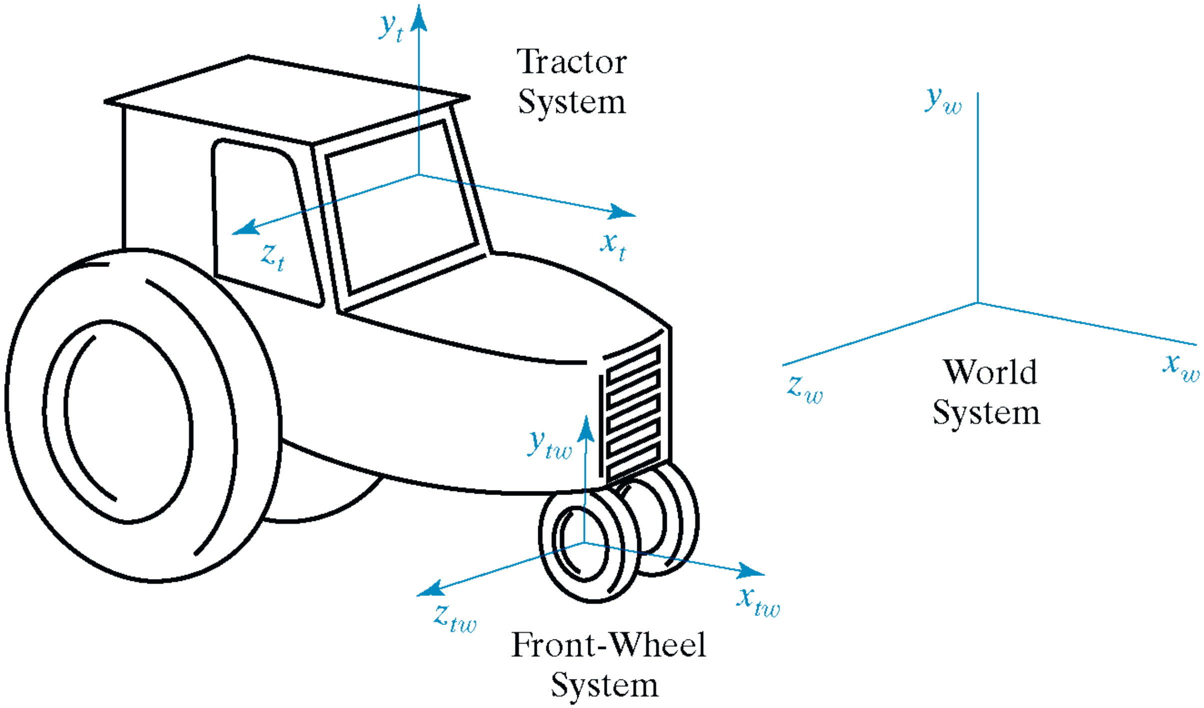


OpenGL Geometric Transformations

- Construction
  - **`glLoadIdentity();`**
  - **`glTranslatef(tx, ty, tz);`**
  - **`glRotatef(theta, vx, vy, vz);`**
    - **`(vx, vy, vz)`** is automatically normalized
  - **`glScalef(sx, sy, sz);`**
  - **`glLoadMatrixf(Glfloat elems[16]);`**
- Multiplication
  - **`glMultMatrixf(Glfloat elems[16]);`**
  - The current matrix is postmultiplied by the matrix
  - Row major

Hierarchical Modeling

- A hierarchical model is created by nesting the descriptions of subparts into one another to form a tree organization



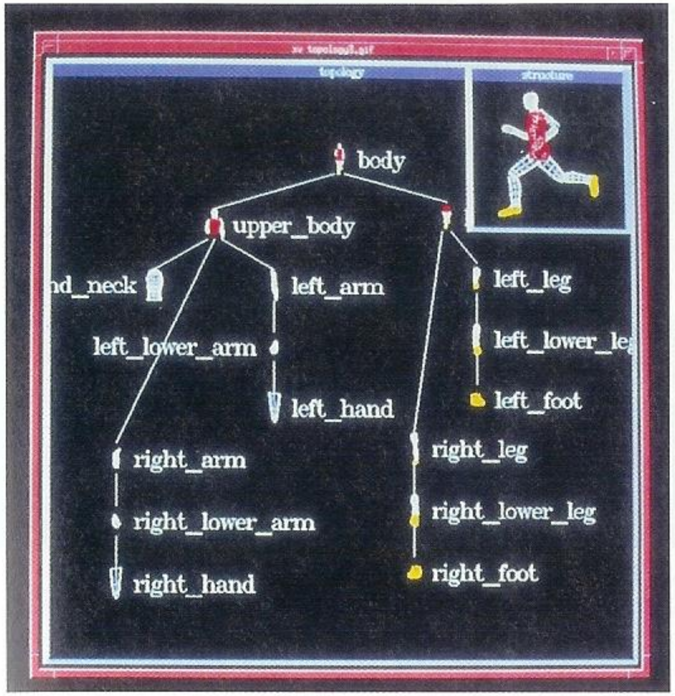


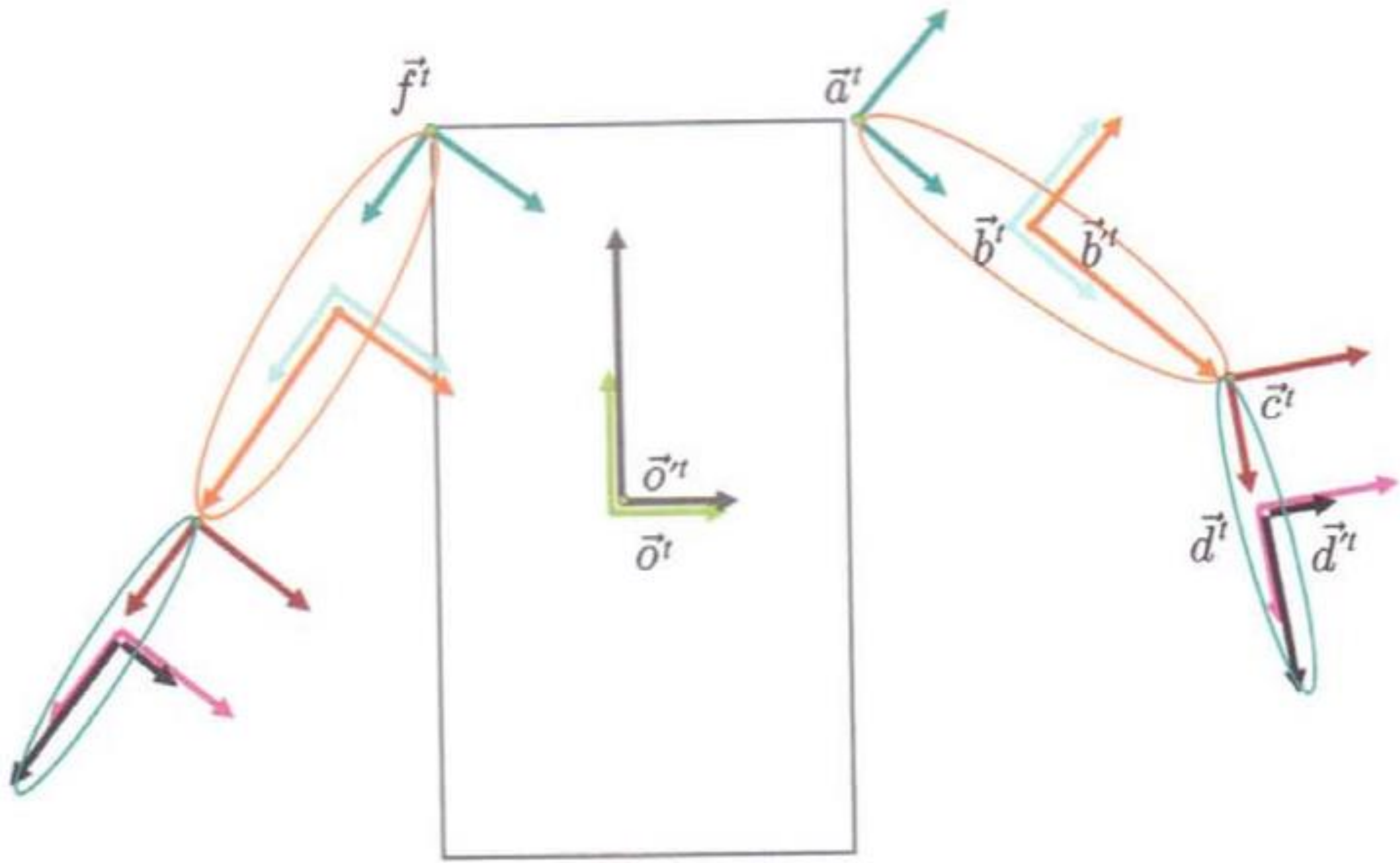
FIGURE 14-4 An object hierarchy generated using the PHIGS Toolkit package developed at the University of Manchester. The displayed object tree is itself a PHIGS structure. (*Courtesy of T. L. J. Howard, J. G. Williams, and W. T. Hewitt, Department of Computer Science, University of Manchester, United Kingdom.*)

OpenGL Matrix Stacks

- Four matrix modes
  - Modelview, projection, texture, and color
  - **`glGetIntegerv(GL_MAX_MODELVIEW_STACK_DEPTH, stackSize);`**

- Stack processing
  - The top of the stack is the “current” matrix
  - **`glPushMatrix();`** *//* Duplicate the current matrix at the top
  - **`glPopMatrix();`** *//* Remove the matrix at the top

OpenGL Matrix Stacks



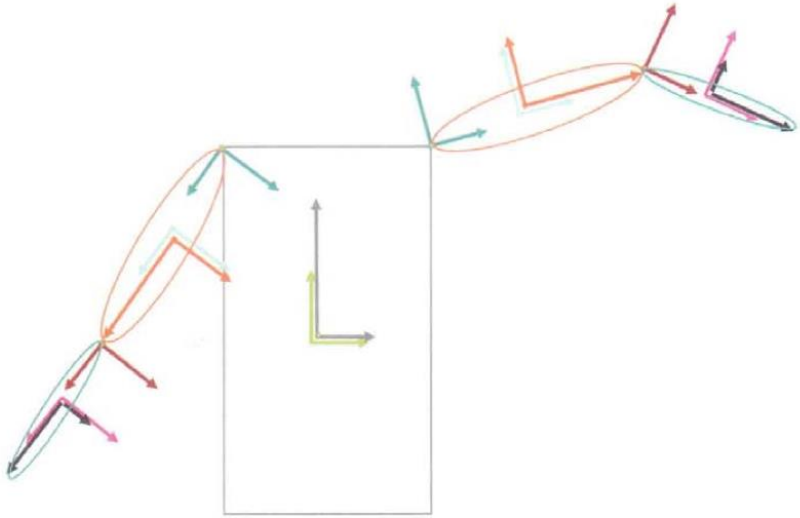


Figure 5.6
To bend the arm at the shoulder, we update the A matrix.

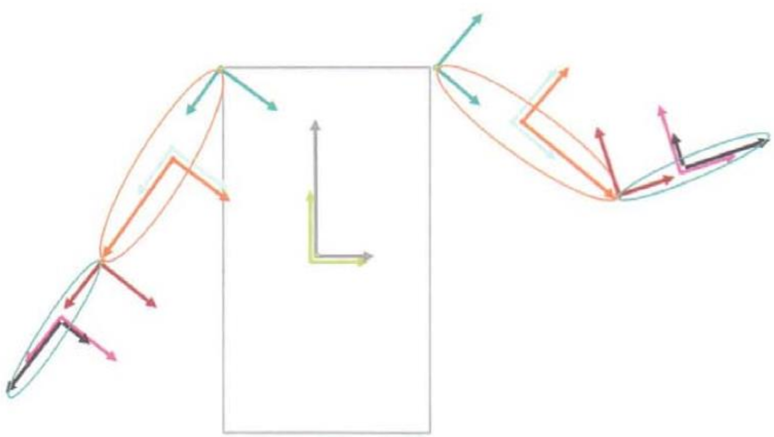


Figure 5.7
To bend the elbow robot, we update its C matrix.

```
...
matrixStack.initialize(inv(E));
matrixStack.push(O);
  matrixStack.push(O');
    draw(matrixStack.top(), cube); \\body
  matrixStack.pop(); \\O'

  matrixStack.push(A);
    matrixStack.push(B);

  matrixStack.push(B');
    draw(matrixStack.top(),sphere); \\upper arm
  matrixStack.pop(); \\B'

  matrixStack.push(C);
    matrixStack.push(C');
      draw(matrixStack.top(),sphere); \\lower arm
    matrixStack.pop(); \\C'
  matrixStack.pop(); \\C

matrixStack.pop();  \\B
matrixStack.pop();     \\A
```

Programming Assignment #1

- Create a hierarchical model using matrix stacks
- The model should consists of three-dimensional primitives such as polygons, boxes, cylinders, spheres and quadrics.
- The model should have a hierarchy of at least three levels
- Animate the model to show the hierarchical structure
  - Eg) a car driving with rotating wheels
  - Eg) a runner with arms and legs swing
- Make it aesthetically pleasing or technically illustrative