Skew lines midpoint position

Yosef Meller

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The 3D point position of a correspondence tupel (i.e. a number of corresponding 2D points each from a different camera) is obtained by averaging the positions of points in the middle of the shortest-distance line between rays going from each two 3D camera positions into the corresponding image-plane points. Therefore, the most basic calculation here is of the midpoint between the two rays.

Let $\vec{p}_1 = \vec{x}_1 + r_1\vec{u}_1$ and $\vec{p}_2 = \vec{x}_2 + r_2\vec{u}_2$ denote two points along two skew lines (i.e. lines that do not intersect and are not parallel), where u_i are unit vectors. Denote the segment connecting them $\vec{k} = \vec{p}_2 - \vec{p}_1$, and for convenience define $\vec{d} = \vec{x}_2 - \vec{x}_1$. The terminology is sketched in figure 1.

Simple vector arithmetics shows that $\vec{k} = -r_1\vec{u}_1 + \vec{d} + r_2\vec{u}_2$.

The shortest segment connecting two skew lines is necessarily perpendicular to both, hence if p_i are the end points of that line, then $\vec{u}_1 \cdot \vec{k} = \vec{u}_2 \cdot \vec{k} = 0$, yielding the equation system:

$$\begin{aligned} r_1 - (\vec{u}_1 \cdot \vec{u}_2)\, r_2 &= \vec{u}_1 \cdot \vec{d} \\ (\vec{u}_1 \cdot \vec{u}_2)\, r_1 - r_2 &= \vec{u}_2 \cdot \vec{d} \end{aligned}$$

The solution, easily obtained with pen and paper, is

$$r_1 = \frac{(\vec{u}_1 \cdot \vec{u}_2)\left(\vec{u}_2 \cdot \vec{d}\right) - \left(\vec{u}_1 \cdot \vec{d}\right)}{1 - (\vec{u}_1 \cdot \vec{u}_2)^2}$$

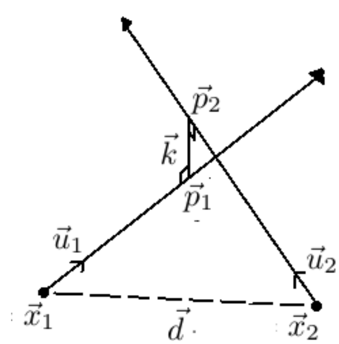

Figure 1: Skew lines

$$r_2 = \frac{(\vec{u}_1 \cdot \vec{u}_2)\left(\vec{u}_1 \cdot \vec{d}\right) - \left(\vec{u}_2 \cdot \vec{d}\right)}{1 - (\vec{u}_1 \cdot \vec{u}_2)^2}$$

To simplify and reduce calculations for the computer implementation, we first define α to be the angle between $\vec{u}_1, \vec{u}_2$. Then we can simplify the denominator using the known trigonometric properties of cross product and dot product[1]:

$$1 - (\vec{u}_1 \cdot \vec{u}_2)^2 = 1 - \cos^2 \alpha = \sin^2 \alpha = \|\vec{u}_1 \times \vec{u}_2\|^2$$

The numerator is simplified using the Binet-Cauchy identity[2]:

$$\begin{aligned} (\vec{u}_1 \cdot \vec{u}_2)\left(\vec{u}_2 \cdot \vec{d}\right) - \left(\vec{u}_1 \cdot \vec{d}\right) &= (\vec{u}_1 \cdot \vec{u}_2)\left(\vec{u}_2 \cdot \vec{d}\right) - (\vec{u}_2 \cdot \vec{u}_2)\left(\vec{u}_1 \cdot \vec{d}\right) = \\ &= (\vec{u}_1 \times \vec{u}_2) \cdot \left(\vec{d} \times \vec{u}_2\right) \end{aligned}$$

and similarly for the r_2 numerator, to get the final form

$$\begin{aligned} r_1 &= \frac{(\vec{u}_1 \times \vec{u}_2) \cdot \left(\vec{d} \times \vec{u}_2\right)}{\|\vec{u}_1 \times \vec{u}_2\|^2} \\ r_2 &= \frac{(\vec{u}_1 \times \vec{u}_2) \cdot \left(\vec{d} \times \vec{u}_1\right)}{\|\vec{u}_1 \times \vec{u}_2\|^2} \end{aligned}$$

Finally, the midpoint is $\frac{1}{2}(\vec{p}_2 + \vec{p}_1) = \frac{1}{2}(\vec{x}_1 + r_1\vec{u}_1 + \vec{x}_2 + r_2\vec{u}_2)$.

[1] https://en.wikipedia.org/wiki/Cross_product
[2] https://en.wikipedia.org/wiki/Cross_product#Lagrange.27s_identity