My Solutions for “Mathematical Methods of Physics (Second Edition)” by J. Mathews, R. L. Walker

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0 Introduction

This document is an archive of my solutions to J. Mathews and R. L. Walker’s “Mathematical Methods of Physics” textbook. None of the solutions have been verified by anyone other than myself, whom I do not consider a reliable source. Hence, it is strongly advised against to use this solution for any application where accuracy matters, especially for assignments and any academic work. However, I have not yet been able to find any other solutions. Please just use this manual as just a suggestion.

1 Chapter 1

1. We first acknowledge that $y = 0$ is a solution, so we seek solutions that are not identically zero.

Let $y = x \cdot v$.

$$y' = v + xv' \Rightarrow x^2(v + xv') + x^2v^2 = x \cdot xv \cdot (v + xv')$$

$$\Rightarrow v + xv' = xvv' \Rightarrow \left(1 - \frac{1}{v}\right) dv = \frac{1}{x}dx \Rightarrow v - \ln v = \ln x + C$$

$$\therefore \frac{y}{x} - \ln y = C$$

2.

$$\frac{y}{\sqrt{1+y^2}}dy = \frac{x}{\sqrt{1+x^2}}dx$$

$$\therefore \sqrt{1+y^2} = \sqrt{1+x^2} + C$$

3. Let $v := x + y$.

$$v' = 1 + y' \Rightarrow v' - 1 = \frac{a^2}{v^2} \Rightarrow \frac{v^2}{a^2 + v^2} dv = dx$$

$$x + C = \int \left(1 - \frac{a^2}{a^2 + v^2}\right) dv = v - a\tan^{-1}\left(\frac{v}{a}\right) = x + y - a\tan^{-1}\left(\frac{x+y}{a}\right)$$

$$\therefore y - a\tan^{-1}\left(\frac{x+y}{a}\right) = C$$

4. We first seek the complement solutions.

$$y_c' + y_c \cos x = 0 \Rightarrow \frac{dy_c}{y_c} + \cos x dx = 0 \Rightarrow y_c = Ce^{-\sin x}$$

For the particular solution, we first observe that $\frac{1}{2}\sin 2x = \sin x \cos x$. Thus, we shall try the ansatz $y_p = \sin x + A$.

$$\cos x + (\sin x + A)\cos x = \sin x \cos x \Rightarrow A = -1$$

$$\therefore y = Ce^{-\sin x} + \cos x - 1$$

5.

$$\left(1 + x^2\right) y' = xy(y+1) \Rightarrow \frac{dy}{y(y+1)} = \frac{x}{1 - x^2} dx$$

$$\Rightarrow \ln\left(\frac{y}{y+1}\right) = -\frac{1}{2}\ln\left(1 - x^2\right) + C_1$$

$$\Rightarrow \frac{y+1}{y} = e^{-C_1}\sqrt{1 - x^2} = C\sqrt{1 - x^2}$$

$$\therefore y = \frac{1}{C\sqrt{1 - x^2} - 1}$$

6. We first take note that the equation is dimension-consistent with $[y] = [x^{-2}]$. Thus, we define $v := x^2 y$ or $y = \frac{v}{x^2}$.

$$y' = \frac{v'}{x^2} - \frac{2v}{x^3} \Rightarrow 2xv' - 4v = 1 + \sqrt{1 + 4v}$$

$$\Rightarrow \frac{v'}{1 + 4v + \sqrt{1 + 4v}} = \frac{1}{2x}$$

$$\Rightarrow \left(\frac{1}{\sqrt{1 + 4v}} - \frac{1}{\sqrt{1 + 4v} + 1}\right) dv = \frac{dx}{2x}$$

Basic calculus yields $\int \frac{dv}{\sqrt{1+4v}+1} = \frac{1}{2}\left(\sqrt{1 + 4v} - \ln\left(1 + \sqrt{1 + 4v}\right)\right) + C$. Hence,

$$\frac{1}{2}\sqrt{1 + 4v} - \frac{1}{2}\left(\sqrt{1 + 4v} - \ln\left(1 + \sqrt{1 + 4v}\right)\right) = \frac{1}{2}\ln x + C_1$$

$$\Rightarrow 1 + \sqrt{1 + 4v} = e^{2C_1} x = Cx$$

$$\therefore \sqrt{1 + 4x^2 y} = Cx - 1$$

7. Let $v' := y'$.

$$v' + v^2 + 1 = 0 \Rightarrow \frac{dv}{v^2+1} = -1 \Rightarrow v = \tan(C_1 - x)$$

$$\therefore y = \int dx v = \ln\left(\cos(C_1 - x)\right) + C_2$$

8.

$$y'y'' = e^y y' \Rightarrow \int dy' y' = \int dy e^y \Rightarrow \frac{1}{2}y'^2 = e^y + A$$

$$\Rightarrow \frac{dy}{\sqrt{e^y + A}} = \sqrt{2}dx$$

Basic calculus yields

$$\int \frac{dy}{\sqrt{e^y+A}} = \frac{1}{\sqrt{A}} \ln\left(\frac{\sqrt{e^y+A} - \sqrt{A}}{\sqrt{e^y+A} + \sqrt{A}}\right) + C_1.$$

$$\therefore \ln\left(\frac{\sqrt{e^y+A} - \sqrt{A}}{\sqrt{e^y+A} + \sqrt{A}}\right) = \sqrt{2A}x + C$$

9. Notice how $(x(1-x))' = 1 - 2x$.

$$\begin{aligned} 0 &= x(1-x)y'' + 4y' + 2y \\ &= x(1-x)y'' + (x(1-x))'y' + (2x+3)y' + 2y \\ &= (x(1-x)y')' + ((2x+3)y)' \end{aligned}$$

$$\Rightarrow x(1-x)y' + (2x+3)y = A$$

Let us define the integrating factor λ:

$$\begin{aligned} \lambda &:= \exp\left(\int dx \frac{2x+3}{x(1-x)}\right) \\ &= \exp\left(\int dx \left(\frac{3}{x} - \frac{5}{x-1}\right)\right) \\ &= \exp\left(3\ln x - 5\ln(x-1)\right) \\ &= \frac{x^3}{(x-1)^5} \end{aligned}$$

$$\Rightarrow (\lambda y)' = \frac{A}{x(1-x)} \cdot \lambda = \frac{Ax^2}{(x-1)^6}$$

$$\Rightarrow \lambda y = \int dx \frac{Ax^2}{(x-1)^6}$$
$$= A \int dx \left(\frac{1}{(x-1)^4} + \frac{2}{(x-1)^5} + \frac{1}{(x-1)^6} \right)$$
$$= -A \frac{10x^2 - 5x + 1}{30(x-1)^5} + C_2$$

$$\therefore y = C_1 \frac{10x^2 - 5x + 1}{x^3} + C_2 \frac{(1-x)^5}{x^3}$$

10.

$$\frac{dy}{y^2} = \frac{1-x}{x^3} dx \Rightarrow -\frac{1}{y} = -\frac{1}{2x^2} + \frac{1}{x} + A$$
$$\therefore y = \frac{2x^2}{Cx^2 + 2x - 1}$$

11. Recall that the Bernoulli equation takes the form $\frac{dy}{dx} + p(x)y = q(x)y^n$. We recognize that this equation is a Bernoulli equation with $p(x) = \frac{1}{x}$, $q(x) = -x^3 e^x$, and $n = 4$. Thus, we make the substitution $v := y^{-3}$.

$$v' = -3y^{-4}y' \Rightarrow -\frac{v'}{3} + \frac{v}{x} = -x^3 e^x \Rightarrow \frac{v'}{x^3} - \frac{3}{x^4} v = 3e^x$$
$$\Rightarrow \left(\frac{v}{x^3} \right)' = 3e^x \Rightarrow v = x^3(3e^x + C)$$
$$\therefore y = \left(x^3(3e^x + C) \right)^{-\frac{1}{3}}$$

12. We define the integrating factor λ:

$$\lambda := \exp\left(\int \frac{dx}{1+x^2} \right) = \exp\left(\tan^{-1} x \right)$$
$$\Rightarrow \left(\exp\left(\tan^{-1} x \right) y \right)' = \frac{1}{1+x^2} \cdot \tan^{-1} x \cdot \exp\left(\tan^{-1} x \right)$$
$$\Rightarrow \exp\left(\tan^{-1} x \right) y = \left(\tan^{-1} x - 1 \right) \exp\left(\tan^{-1} x \right) + C$$
$$\therefore y = C \exp\left(-\tan^{-1} x \right) + \tan^{-1} x - 1$$

13. This equation is dimension-consistent with $[y] = [x^{-1}]$. Thus, let us define $v := xy$.

$$y = \frac{v}{x} \Rightarrow y' = \frac{xv' - v}{x^2}$$

$$\Rightarrow 0 = (xv' - v)^2 - 2(v-4)(xv' - v) + v^2$$
$$= x^2 v'^2 - 4(v-2)xv' + 4v(v-2)$$

$$\Rightarrow v' = \frac{2}{s}\left(v - 2 \pm \sqrt{4-2v}\right)$$

If $v = 2$, then we get $y = \frac{2}{x}$, which is the particular solution for this equation. To obtain the general solutions, let $u := 2 - v$.

$$u' = \frac{2}{x}\left(u \pm \sqrt{2u}\right)$$

$$\Rightarrow \frac{du}{u \pm \sqrt{2u}} = \frac{2}{x}dx$$

$$\Rightarrow 2\ln\left(\sqrt{u} \pm \sqrt{2}\right) = 2\ln x + C_1$$

$$\Rightarrow u = \left(Cx^2 \pm \sqrt{2}\right)^2 = 2 - xy$$

$$\therefore y = \frac{2 - \left(Cx^2 \pm \sqrt{2}\right)^2}{x}$$

14.

$$\begin{aligned} 6x &= \frac{y''}{y} - \frac{y'}{y^2} \\ &= \left(\frac{y'}{y}\right)' \end{aligned}$$

$$\Rightarrow \frac{y'}{y} = 3x^2 + C_1$$

$$\Rightarrow \ln y = x^3 + C_1 x + A$$

$$\therefore y = C_2 e^{x^3 + C_1 x}$$

15.

$$\frac{1}{x} = x^3\left(yy'' + y'^2\right) + (x^3)'yy' = \left(x^3yy'\right)'$$

$$\Rightarrow x^3yy' = \ln x + A \Rightarrow ydy = \frac{\ln x + A}{x^3}$$

Basic calculus yields $\int dx \frac{\ln x}{x^3} = -\frac{2\ln x + 1}{4x^2}$.

$$\Rightarrow \frac{y^2}{2} = -\frac{2\ln x + 1}{4x^2} - \frac{A}{2x^2} + B$$

$$\therefore y = \pm\sqrt{C_1 - \frac{\ln x + C_2}{x^2}}$$

16. This equation is dimension-consistent with $[y] = [x]$. Thus, we define $v := \frac{y}{x}$.

$$y = xv \Rightarrow y' = xv' + v \Rightarrow y'' = xv'' + 2v' \Rightarrow v'' + \frac{2}{x}v' - \frac{2}{x^2}v = \frac{1}{x^2}$$

First, observe that $v = -\frac{1}{2}$ is a solution of the equation. As this equation is linear in v, we thus seek the complementary solutions v_c. We shall take the ansatz of $v_c = x^m$.

$$m(m-1)x^{m-2} + 2mx^{m-2} - 2x^{m-2} = 0 \Rightarrow m^2 + m - 2 = 0$$

$$\Rightarrow m = 1, -2 \Rightarrow v_c = C_1 x + \frac{C_2}{x^2} \Rightarrow v = C_1 x + \frac{C_2}{x^2} - \frac{1}{2}$$

$$\therefore y = C_1 x^2 + \frac{C_2}{x} - \frac{x}{2}$$

17. Notice that this equation is linear in y.

(i) Complementary solutions:
Take $y_c = e^{mx}$.

$$m^3 - 2m^2 - m + 2 = 0 \Rightarrow m = \pm 1, 2$$

$$\Rightarrow y_c = C_1 e^{2x} + C_2 e^x + C_3 e^{-x}$$

(ii) Particular solution:
Suppose $y_p = A \sin x + B \cos x$.

$$\begin{aligned} \sin x &= (-A\cos x + B\sin x) - 2(-A\sin x - B\cos x) \\ &\quad - (A\cos x - B\cos x) + 2(A\sin x + B\cos x) \\ &= (4A + 2B)\sin x + (-2A + 4B)\cos x \end{aligned}$$

$$\Rightarrow A = \frac{1}{5}, B = \frac{1}{10}$$

$$\therefore y = \frac{1}{5}\sin x + \frac{1}{10}\cos x + C_1 e^{2x} + C_2 e^x + C_3 e^{-x}$$

18. Again, this equation is linear in y.

(i) Complementary solutions:
Take $y_c = e^{mx}$.

$$m^3 + 2m^2 + 1 = 0 \Rightarrow m = m_1, m_2, m_3$$

(I refuse to write down the exact solutions of this cubic equation.)

$$\Rightarrow y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$$

(ii) Particular solution:
Suppose $y_p = A\sin x + B\cos x$.

$$\begin{aligned}\sin x &= (-A\cos x + B\sin x) + 2(-A\sin x - B\cos x) + (A\sin x + B\cos x)\\ &= (-A+B)\sin x + (-A-B)\cos x\end{aligned}$$

$$\Rightarrow A = B = -\frac{1}{2}$$

$$\therefore y = -\frac{1}{2}\sin x - \frac{1}{2}\cos x + C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$$

19. Following the same steps as before, one can easily find the complementary solutions of

$$y_c = C_1 e^{-x} + C_2 e^{-2x}.$$

Thus, we wish to find one particular solution y_p. Let us try $y_p = f(x)e^{e^x}$.

$$y_p' = \left(\frac{f'}{f} + e^x\right) y_p$$

$$\begin{aligned}\Rightarrow y_p'' &= \left(\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 + e^x\right) y_p + \left(\frac{f'}{f} + e^x\right)^2 y_p\\ &= \left(\frac{f''}{f} + 2e^x\frac{f'}{f} + e^{2x} + e^x\right) y_p\end{aligned}$$

$$\begin{aligned}\Rightarrow \left(\frac{f''}{f} + (2e^x+3)\frac{f'}{f} + e^{2x} + 5e^x + 2\right) f e^{e^x} &= e^{e^x}\\ \Rightarrow f'' + (2e^x+3)f' + (e^{2x} + 5e^x + 2)f &= 1\end{aligned}$$

One may use an ansatz of $f = ae^{-2x} + be^{-x} + c$ to find that $f = e^{-2x}$ is a possible solution. Thus, we have found a particular solution $y_p = e^{e^x - 2x}$.

$$\therefore y = e^{e^x - 2x} + C_1 e^{-x} + C_2 e^{-2x}$$

20.

$$\frac{y''}{(1+y'^2)^{3/2}} = \pm\frac{1}{a} \Rightarrow \frac{y'}{\sqrt{1+y'^2}} = \pm\frac{x+C_1}{a}$$

$$\Rightarrow y'^2 = \left(\frac{x+C_1}{a}\right)^2 (1+y'^2) \Rightarrow y' = \pm\frac{\left(\frac{x+C_1}{a}\right)^2}{\sqrt{1-\left(\frac{x+C_1}{a}\right)^2}}$$

$$\therefore y = \pm\frac{1}{2}\left(a\sin^{-1}\left(\frac{x+C_1}{a}\right) - (x+C_1)\sqrt{1-\left(\frac{x+C_1}{a}\right)^2}\right) + C_2$$

21. The complementary solutions are given by

$$\frac{dq_p}{dt} = -\frac{q_p}{RC} \Rightarrow q_p = Ae^{-\frac{t}{RC}}.$$

Suppose a particular solution takes the form $y_p = (at^3 + bt^2 + ct + d)e^{-\frac{t}{\tau}}$.

$$\frac{dq_p}{dt} = \left(-\frac{a}{\tau}t^3 + \left(3a - \frac{b}{\tau}\right)t^2 + \left(2b - \frac{c}{\tau}\right)t + \left(c - \frac{d}{\tau}\right)\right)e^{-\frac{t}{\tau}}$$

$$\begin{aligned}\Rightarrow \frac{V_0}{RC\tau^2}t^2 e^{-\frac{t}{\tau}} = \Bigg(&\left(\frac{1}{RC} - \frac{1}{\tau}\right)at^3 \\ &+ \left(3a + \left(\frac{1}{RC} - \frac{1}{\tau}\right)b\right)t^2 \\ &+ \left(2b + \left(\frac{1}{RC} - \frac{1}{\tau}\right)c\right)t \\ &+ \left(c + \left(\frac{1}{RC} - \frac{1}{\tau}\right)d\right)\Bigg)e^{-\frac{t}{\tau}}\end{aligned}$$

$$\Rightarrow q_p = \begin{cases} CV_0 \cdot \left(\frac{t^2}{\tau(\tau-RC)} - \frac{RCt}{(\tau-RC)^2} + \frac{\tau(RC)^2}{(\tau-RC)^2}\right) & (\text{if } \tau \neq RC) \\ \frac{1}{3}CV_0 \cdot \left(\frac{t}{\tau}\right)^3 e^{-\frac{t}{\tau}} & (\text{if } \tau = RC) \end{cases}$$

We can then add an appropriate complementary solution so as to satisfy the boundary condition $q(0) = 0$.

$$\therefore q = \begin{cases} CV_0 \cdot \frac{t}{\tau-RC}\left(\frac{t}{\tau} - \frac{RC}{\tau-RC}\right)e^{-\frac{t}{\tau}} & (\text{if } \tau \neq RC) \\ \frac{1}{3}CV_0 \cdot \left(\frac{t}{\tau}\right)^3 e^{-\frac{t}{\tau}} & (\text{if } \tau = RC) \end{cases}$$

22.

$$\frac{dN}{N_s - N} = \lambda dt \Rightarrow \int_0^N \frac{dN}{N_s - N} = \lambda \int_0^t dt \Rightarrow -\ln\left|\frac{N_s - N}{N_s}\right| = \lambda t$$

$$\therefore N = N_s\left(1 - e^{-\lambda t}\right)$$

23. The clever idea here is that we could make a variable change from x to u, where we could choose u so as to make the equation easier to solve.

$$\begin{aligned}0 &= A\frac{d^2y}{dx^2} + \frac{dA}{dx}\frac{dy}{dx} + \frac{y}{A} \\ &= A\frac{du}{dx}\frac{d}{du}\left(\frac{du}{dx}\frac{dy}{dy}\right) + \frac{dA}{dx}\frac{du}{dx}\frac{dy}{du} + \frac{y}{A} \\ &= A\left(\frac{du}{dx}\right)^2\frac{d^2y}{du^2} + \frac{dy}{du}\frac{d}{dx}\left(A\frac{dy}{dx}\right)\end{aligned}$$

Thus, we could define $\frac{du}{dx} = \frac{1}{A(x)}$ or $u = \int^x \frac{dt}{A(t)}$ to obtain

$$\frac{d^2y}{du^2} = -y \Rightarrow y(u) = C_1 \cos(u + C_2)$$

$$\therefore y(x) = C_1 \cos\left(\int^x \frac{dt}{A(t)} + C_2\right)$$

24. Suppose we may write $y = \lambda v$ where λ and v are to be determined later.

$$y' = \lambda v' + \lambda' v \Rightarrow y'' = \lambda v'' + 2\lambda' v' + \lambda'' v$$

$$\Rightarrow \lambda v'' + 2\left(\lambda' + \frac{\lambda}{x}\right) v' + \left(\lambda'' + \frac{2\lambda'}{x} + n^2\lambda\right) v = \frac{\sin\omega x}{x}$$

We may choose λ so as to make the coefficient of v' vanish, i.e.,

$$\lambda' = -\frac{\lambda}{x} \Rightarrow \lambda = \frac{1}{x}.$$

$$\Rightarrow v'' + n^2 v = \sin\omega x$$

(i) $n \neq \omega$:
The complementary solutions are

$$v_c = C_1 \cos(nx + C_2).$$

Using the ansatz $v_p = A\sin\omega x$ for a particular solution, one can easily find $A = \frac{1}{n^2-\omega^2}$.

$$\Rightarrow v = \frac{\sin\omega x}{n^2 - \omega^2} + C_1 \sin(nx + C_2)$$

$$\therefore y = \frac{1}{n^2 - \omega^2}\frac{\sin\omega x}{x} + C_1 \frac{\sin(nx + C_2)}{x}$$

(ii) $n = \omega$:
The complementary solutions are

$$v_c = C_1 \cos(\omega x + C_2).$$

Using the ansatz $v_p = Ax\sin\omega x + Bx\cos\omega x$ for a particular solution, one can easily find $A = -\frac{1}{2\omega}$ and $B = 0$.

$$\Rightarrow v = -\frac{1}{2\omega} x\cos\omega x + C_1 \cos(\omega x + C_2)$$

$$\therefore y = -\frac{1}{2\omega}\cos\omega x + C_1 \frac{\sin(\omega x + C_2)}{x}$$

25. From the hint, we may consider $y = xv$, where $v = 1$ is a solution if the right hand side were zero.

$$y' = xv' + v \Rightarrow y'' = xv'' + 2v'$$

$$\Rightarrow x(1-x)v'' + \left(1 + (1-x)^2\right) v' = (1-x)^2$$

$$\Rightarrow v'' + \left(\frac{2}{x} + \frac{1}{1-x} - 1\right) v' = \frac{1}{x} - 1$$

We can thus define the integration factor as

$$\lambda := \exp\left(\int dx \left(\frac{2}{x} + \frac{1}{1-x} - 1\right)\right) = \frac{x^2 e^{-x}}{1-x}.$$

$$\Rightarrow (\lambda v')' = \lambda \left(\frac{1}{x} - 1\right) = xe^{-x}$$

$$\Rightarrow \lambda v' = -(x+1)e^{-x} + A$$

$$\Rightarrow v' = 1 - \frac{1}{x^2} + A\frac{1-x}{x^2}e^x$$

$$\Rightarrow v = x + \frac{1}{x} - A\frac{e^x}{x} + B$$

$$\therefore y = x^2 + 1 + C_1 e^x + C_2 x$$

26. While the use of Dirac delta "function" must be dealt with caution, we are physicists so we shall simply accept that they exist.

$$\begin{aligned} f(x) = y'' + py' + q &= \int_a^b dx' f(x') \left(\frac{\partial^2}{\partial x^2} + p(x)\frac{\partial}{\partial x} + q(x)\right) G(x, x') \\ &= \int_a^b dx' f(x') \delta(x - x') \end{aligned}$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + p(x)\frac{\partial}{\partial x} + q(x)\right) G(x, x') = \delta(x - x')$$

From this, we see that $G(x, x')$ must separately equal zero for both $x < x'$ and $x > x'$. We thus write

$$G(x, x') = \begin{cases} y_1(x)\alpha(x') + y_2(x)\beta(x') & (x < x') \\ y_1(x)\gamma(x') + y_2(x)\zeta(x') & (x > x') \end{cases}.$$

From the definition of delta functions, we find that:

$$\begin{aligned}
1 &= \int_{x'-\epsilon}^{x'+\epsilon} dx\delta(x-x') \\
&= \int_{x'-\epsilon}^{x'+\epsilon} dx \left(\frac{\partial^2}{\partial x^2} + p(x)\frac{\partial}{\partial x} + q(x) \right) G(x,x') \\
&\approx \left[\left(\frac{\partial}{\partial x} + p(x) \right) G(x,x') \right]_{x'-\epsilon}^{x'+\epsilon} - \int_{x-\epsilon}^{x+\epsilon} dx p'(x) G(x,x') \\
&\approx \left[\frac{\partial}{\partial x} G(x,x') \right]_{x'-\epsilon}^{x'+\epsilon} \\
&\approx \lim_{x\to x'+} \frac{\partial}{\partial x} G(x,x') - \lim_{x\to x'-} \frac{\partial}{\partial x} G(x,x')
\end{aligned}$$

where we take infinitesimally small ϵ.

The boundary conditions for y imply

$$\begin{gathered}
y(a) = y(b) = 0 \\
\Rightarrow G(a,x') = G(b,x') = 0 \\
\Rightarrow y_1(a)\alpha(x') + y_2(a)\beta(x') = y_1(b)\alpha(x') + y_2(b)\beta(x') = 0 \\
\Rightarrow \beta(x') = \gamma(x') = 0.
\end{gathered}$$

The derivative condition, in turn, yields

$$y_2'(x')\zeta(x') - y_1'(x')\alpha(x') = 1.$$

As any such choice of $\zeta(x')$ and $\alpha(x')$ leads to a valid Green's function, we may arbitrarily choose

$$\alpha(x') = y_2'(x'),\ \zeta(x') = y_1'(x') + \frac{1}{y_2'(x')}$$

$$\Rightarrow G(x,x') = \begin{cases} y_1(x)y_2'(x') & (x < x') \\ y_2(x)y_1'(x') + \frac{y_2(x)}{y_2'(x')} & (x > x') \end{cases}.$$

$$\therefore y(x) = y_1(x)\int_x^b dx' f(x')y_2'(x') + y_2(x)\int_a^x f(x')\left(y_1'(x') + \frac{1}{y_2'(x')} \right)$$

For the given example $y'' + k^2 y = f(x)$, we may take

$$y_1(x) = \sin k(x-a),\ y_2(x) = \sin k(b-x)$$

provided that $b-a$ is not an integer multiple of the period $\frac{2\pi}{k}$. This leads to

$$G(x,x') = \begin{cases} -k\sin k(x-a)\cos k(b-x) & (x < x') \\ \sin k(b-x)\left(k\cos k(x'-a) - \frac{1}{k\cos k(b-x')} \right) & (x > x') \end{cases}$$

$$
\begin{aligned}
\Rightarrow y = &- k \sin k(x-a) \int_x^b dx' f(x') \cos k(b-x') \\
&+ \sin k(b-x) \int_a^x dx' f(x') \left(k \cos k(x'-a) - \frac{1}{k \cos k(b-x')} \right).
\end{aligned}
$$

One may verify that this solution satisfies both the boundary conditions and the given differential equation.