My Solutions for "Mathematical Methods of Physics (Second Edition)" by J. Mathews, R. L. Walker

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0 Introduction

This document is an archive of my solutions to J. Mathews and R. L. Walker's "Mathematical Methods of Physics" textbook. None of the solutions have been verified by anyone other than myself, whom I do not consider a reliable source. Hence, it is strongly advised against to use this solution for any application where accuracy matters, especially for assignments and any academic work. However, I have not yet been able to find any other solutions. Please just use this manual as just a suggestion. If you spot any mistakes, please report them to the Github repository. [1]

1 Chapter 1

1. We first acknowledge that $y = 0$ is a solution, so we seek solutions that are not identically zero.

Let $y = xv$.

$$y' = v + xv' \Rightarrow x^2(v + xv') + x^2v^2 = x \cdot xv \cdot (v + xv')$$

$$\Rightarrow v + xv' = xvv' \Rightarrow \left(1 - \frac{1}{v}\right) dv = \frac{1}{x} dx \Rightarrow v - \ln v = \ln x + C$$

$$\therefore \frac{y}{x} - \ln y = C$$

2.

$$\frac{y}{\sqrt{1+y^2}} dy = \frac{x}{\sqrt{1+x^2}} dx$$

$$\therefore \sqrt{1+y^2} = \sqrt{1+x^2} + C$$

[1] `https://github.com/pingpingy1/Mathews-MathPhys-Sol`

3. Let $v = x + y$.

$$v' = 1 + y' \Rightarrow v' - 1 = \frac{a^2}{v^2} \Rightarrow \frac{v^2}{a^2 + v^2} dv = dx$$

$$x + C = \int dv \left(1 - \frac{a^2}{a^2 + v^2}\right) = v - a\tan^{-1}\left(\frac{v}{a}\right) = x + y - a\tan^{-1}\left(\frac{x+y}{a}\right)$$

$$\therefore y - a\tan^{-1}\left(\frac{x+y}{a}\right) = C$$

4. We first seek the complement solutions.

$$y_c' + y_c \cos x = 0 \Rightarrow \frac{dy_c}{y_c} + \cos x dx = 0 \Rightarrow y_c = Ce^{-\sin x}$$

For the particular solution, we first observe that $\frac{1}{2}\sin 2x = \sin x \cos x$. Thus, we shall try the ansatz $y_p = \sin x + A$.

$$\cos x + (\sin x + A)\cos x = \sin x \cos x \Rightarrow A = -1$$

$$\therefore y = Ce^{-\sin x} + \cos x - 1$$

5.

$$\left(1 + x^2\right) y' = xy(y+1) \Rightarrow \frac{dy}{y(y+1)} = \frac{x}{1 - x^2} dx$$

$$\Rightarrow \ln\left(\frac{y}{y+1}\right) = -\frac{1}{2}\ln\left(1 - x^2\right) + C_1$$

$$\Rightarrow \frac{y+1}{y} = e^{-C_1}\sqrt{1 - x^2} = C\sqrt{1 - x^2}$$

$$\therefore y = \frac{1}{C\sqrt{1 - x^2} - 1}$$

6. We first take note that the equation is dimension-consistent with $[y] = [x^{-2}]$. Thus, we define $v = x^2 y$ or $y = \frac{v}{x^2}$.

$$y' = \frac{v'}{x^2} - \frac{2v}{x^3} \Rightarrow 2xv' - 4v = 1 + \sqrt{1 + 4v}$$

$$\Rightarrow \frac{v'}{1 + 4v + \sqrt{1 + 4v}} = \frac{1}{2x}$$

$$\Rightarrow \left(\frac{1}{\sqrt{1 + 4v}} - \frac{1}{\sqrt{1 + 4v} + 1}\right) dv = \frac{dx}{2x}$$

Basic calculus yields $\int \frac{dv}{\sqrt{1+4v}+1} = \frac{1}{2}\left(\sqrt{1 + 4v} - \ln\left(1 + \sqrt{1 + 4v}\right)\right) + C$. Hence,

$$\frac{1}{2}\sqrt{1 + 4v} - \frac{1}{2}\left(\sqrt{1 + 4v} - \ln\left(1 + \sqrt{1 + 4v}\right)\right) = \frac{1}{2}\ln x + C_1$$

$$\Rightarrow 1 + \sqrt{1 + 4v} = e^{2C_1} x = Cx$$

$$\therefore \sqrt{1 + 4x^2 y} = Cx - 1$$

7. Let $v' := y'$.

$$v' + v^2 + 1 = 0 \Rightarrow \frac{dv}{v^2+1} = -1 \Rightarrow v = \tan(C_1 - x)$$

$$\therefore y = \int dx v = \ln\left(\cos(C_1 - x)\right) + C_2$$

8.

$$y'y'' = e^y y' \Rightarrow \int dy' y' = \int dy e^y \Rightarrow \frac{1}{2}y'^2 = e^y + C_1$$

$$\Rightarrow \frac{dy}{\sqrt{e^y + C_1}} = \sqrt{2}dx$$

Basic calculus yields

$$\int \frac{dy}{\sqrt{e^y + C_1}} = \frac{1}{\sqrt{C_1}} \ln\left(\frac{\sqrt{e^y + C_1} - \sqrt{A}}{\sqrt{e^y + C_1} + \sqrt{C_1}}\right) + A.$$

$$\therefore \ln\left(\frac{\sqrt{e^y + C_1} - \sqrt{C_1}}{\sqrt{e^y + C_1} + \sqrt{C_1}}\right) = \sqrt{2C_1}x + C_2$$

9. Notice how $(x(1-x))' = 1 - 2x$.

$$\begin{aligned} 0 &= x(1-x)y'' + 4y' + 2y \\ &= x(1-x)y'' + (x(1-x))'y' + (2x+3)y' + 2y \\ &= (x(1-x)y')' + ((2x+3)y)' \end{aligned}$$

$$\Rightarrow x(1-x)y' + (2x+3)y = A$$

Let us define the integrating factor λ:

$$\begin{aligned} \lambda &= \exp\left(\int dx \frac{2x+3}{x(1-x)}\right) \\ &= \exp\left(\int dx \left(\frac{3}{x} - \frac{5}{x-1}\right)\right) \\ &= \exp\left(3\ln x - 5\ln(x-1)\right) \\ &= \frac{x^3}{(x-1)^5} \end{aligned}$$

$$\Rightarrow (\lambda y)' = \frac{A}{x(1-x)} \cdot \lambda = \frac{Ax^2}{(x-1)^6}$$

$$\begin{aligned} \Rightarrow \lambda y &= \int dx \frac{Ax^2}{(x-1)^6} \\ &= A\int dx \left(\frac{1}{(x-1)^4} + \frac{2}{(x-1)^5} + \frac{1}{(x-1)^6}\right) \\ &= -A\frac{10x^2 - 5x + 1}{30(x-1)^5} + C_2 \end{aligned}$$

$$\therefore y = C_1 \frac{10x^2 - 5x + 1}{x^3} + C_2 \frac{(1-x)^5}{x^3}$$

10.

$$\frac{dy}{y^2} = \frac{1-x}{x^3} dx \Rightarrow -\frac{1}{y} = -\frac{1}{2x^2} + \frac{1}{x} + A$$

$$\therefore y = \frac{2x^2}{Cx^2 + 2x - 1}$$

11. Recall that the Bernoulli equation takes the form $\frac{dy}{dx} + p(x)y = q(x)y^n$. We recognize that this equation is a Bernoulli equation with $p(x) = \frac{1}{x}$, $q(x) = -x^3 e^x$, and $n = 4$. Thus, we make the substitution $v := y^{-3}$.

$$v' = -3y^{-4}y' \Rightarrow -\frac{v'}{3} + \frac{v}{x} = -x^3 e^x \Rightarrow \frac{v'}{x^3} - \frac{3}{x^4} v = 3e^x$$

$$\Rightarrow \left(\frac{v}{x^3}\right)' = 3e^x \Rightarrow v = x^3(3e^x + C)$$

$$\therefore y = \left(x^3(3e^x + C)\right)^{-\frac{1}{3}}$$

12. We define the integrating factor λ:

$$\lambda = \exp\left(\int \frac{dx}{1+x^2}\right) = \exp\left(\tan^{-1} x\right)$$

$$\Rightarrow \left(\exp\left(\tan^{-1} x\right) y\right)' = \frac{1}{1+x^2} \cdot \tan^{-1} x \cdot \exp\left(\tan^{-1} x\right)$$

$$\Rightarrow \exp\left(\tan^{-1} x\right) y = \left(\tan^{-1} x - 1\right) \exp\left(\tan^{-1} x\right) + C$$

$$\therefore y = C \exp\left(-\tan^{-1} x\right) + \tan^{-1} x - 1$$

13. This equation is dimension-consistent with $[y] = [x^{-1}]$. Thus, let us define $v := xy$.

$$y = \frac{v}{x} \Rightarrow y' = \frac{xv' - v}{x^2}$$

$$\begin{aligned} \Rightarrow 0 &= (xv' - v)^2 - 2(v-4)(xv' - v) + v^2 \\ &= x^2 v'^2 - 4(v-2)xv' + 4v(v-2) \end{aligned}$$

$$\Rightarrow v' = \frac{2}{s}\left(v - 2 \pm \sqrt{4 - 2v}\right)$$

If $v = 2$, then we get $y = \frac{2}{x}$, which is the particular solution for this equation. To obtain the general solutions, let $u = 2 - v$.

$$u' = \frac{2}{x}\left(u \pm \sqrt{2u}\right)$$

$$\Rightarrow \frac{du}{u \pm \sqrt{2u}} = \frac{2}{x} dx$$
$$\Rightarrow 2 \ln \left(\sqrt{u} \pm \sqrt{2} \right) = 2 \ln x + A$$
$$\Rightarrow u = \left(Cx^2 \pm \sqrt{2} \right)^2 = 2 - xy$$
$$\therefore y = \frac{2 - \left(Cx^2 \pm \sqrt{2} \right)^2}{x}$$

14.

$$\begin{aligned} 6x &= \frac{y''}{y} - \frac{y'}{y^2} \\ &= \left(\frac{y'}{y} \right)' \end{aligned}$$
$$\Rightarrow \frac{y'}{y} = 3x^2 + C_1$$
$$\Rightarrow \ln y = x^3 + C_1 x + A$$
$$\therefore y = C_2 e^{x^3 + C_1 x}$$

15.

$$\frac{1}{x} = x^3 \left(yy'' + y'^2 \right) + (x^3)' yy' = \left(x^3 yy' \right)'$$
$$\Rightarrow x^3 yy' = \ln x + A \Rightarrow y dy = \frac{\ln x + A}{x^3}$$

Basic calculus yields $\int dx \frac{\ln x}{x^3} = -\frac{2 \ln x + 1}{4x^2}$.

$$\Rightarrow \frac{y^2}{2} = -\frac{2 \ln x + 1}{4x^2} - \frac{A}{2x^2} + B$$
$$\therefore y = \pm \sqrt{C_1 - \frac{\ln x + C_2}{x^2}}$$

16. This equation is dimension-consistent with $[y] = [x]$. Thus, we define $v := \frac{y}{x}$.

$$y = xv \Rightarrow y' = xv' + v \Rightarrow y'' = xv'' + 2v' \Rightarrow v'' + \frac{2}{x} v' - \frac{2}{x^2} v = \frac{1}{x^2}$$

First, observe that $v = -\frac{1}{2}$ is a solution of the equation. As this equation is linear in v, we thus seek the complementary solutions v_c. We shall take the ansatz of $v_c = x^m$.

$$m(m-1)x^{m-2} + 2mx^{m-2} - 2x^{m-2} = 0 \Rightarrow m^2 + m - 2 = 0$$
$$\Rightarrow m = 1, -2 \Rightarrow v_c = C_1 x + \frac{C_2}{x^2} \Rightarrow v = C_1 x + \frac{C_2}{x^2} - \frac{1}{2}$$
$$\therefore y = C_1 x^2 + \frac{C_2}{x} - \frac{x}{2}$$

17. Notice that this equation is linear in y.

(i) Complementary solutions:
Take $y_c = e^{mx}$.

$$m^3 - 2m^2 - m + 2 = 0 \Rightarrow m = \pm 1, 2$$
$$\Rightarrow y_c = C_1 e^{2x} + C_2 e^x + C_3 e^{-x}$$

(ii) Particular solution:
Suppose $y_p = A \sin x + B \cos x$.

$$\begin{aligned}\sin x &= (-A\cos x + B\sin x) - 2(-A\sin x - B\cos x)\\ &\quad - (A\cos x - B\cos x) + 2(A\sin x + B\cos x)\\ &= (4A + 2B)\sin x + (-2A + 4B)\cos x\end{aligned}$$

$$\Rightarrow A = \frac{1}{5}, B = \frac{1}{10}$$

$$\therefore y = \frac{1}{5}\sin x + \frac{1}{10}\cos x + C_1 e^{2x} + C_2 e^x + C_3 e^{-x}$$

18. Again, this equation is linear in y.

(i) Complementary solutions:
Take $y_c = e^{mx}$.

$$m^3 + 2m^2 + 1 = 0 \Rightarrow m = m_1, m_2, m_3$$

(I refuse to write down the exact solutions of this cubic equation.)

$$\Rightarrow y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$$

(ii) Particular solution:
Suppose $y_p = A \sin x + B \cos x$.

$$\begin{aligned}\sin x &= (-A\cos x + B\sin x) + 2(-A\sin x - B\cos x) + (A\sin x + B\cos x)\\ &= (-A + B)\sin x + (-A - B)\cos x\end{aligned}$$

$$\Rightarrow A = B = -\frac{1}{2}$$

$$\therefore y = -\frac{1}{2}\sin x - \frac{1}{2}\cos x + C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$$

19. Following the same steps as before, one can easily find the complementary solutions of

$$y_c = C_1 e^{-x} + C_2 e^{-2x}.$$

Thus, we wish to find one particular solution y_p. Let us try $y_p = f(x) e^{e^x}$.

$$y_p' = \left(\frac{f'}{f} + e^x\right) y_p$$

$$\Rightarrow y_p'' = \left(\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 + e^x\right) y_p + \left(\frac{f'}{f} + e^x\right)^2 y_p$$
$$= \left(\frac{f''}{f} + 2e^x\frac{f'}{f} + e^{2x} + e^x\right) y_p$$

$$\Rightarrow \left(\frac{f''}{f} + (2e^x+3)\frac{f'}{f} + e^{2x} + 5e^x + 2\right) f e^{e^x} = e^{e^x}$$
$$\Rightarrow f'' + (2e^x+3)f' + (e^{2x}+5e^x+2)f = 1$$

One may use an ansatz of $f = ae^{-2x} + be^{-x} + c$ to find that $f = e^{-2x}$ is a possible solution. Thus, we have found a particular solution $y_p = e^{e^x - 2x}$.

$$\therefore y = e^{e^x-2x} + C_1 e^{-x} + C_2 e^{-2x}$$

20.
$$\frac{y''}{(1+y'^2)^{3/2}} = \pm\frac{1}{a} \Rightarrow \frac{y'}{\sqrt{1+y'^2}} = \pm\frac{x+C_1}{a}$$
$$\Rightarrow y'^2 = \left(\frac{x+C_1}{a}\right)^2(1+y'^2) \Rightarrow y' = \pm\frac{\left(\frac{x+C_1}{a}\right)^2}{\sqrt{1-\left(\frac{x+C_1}{a}\right)^2}}$$
$$\therefore y = \pm\frac{1}{2}\left(a\sin^{-1}\left(\frac{x+C_1}{a}\right) - (x+C_1)\sqrt{1-\left(\frac{x+C_1}{a}\right)^2}\right) + C_2$$

21. The complementary solutions are given by

$$\frac{dq_p}{dt} = -\frac{q_p}{RC} \Rightarrow q_p = Ae^{-\frac{t}{RC}}.$$

Suppose a particular solution takes the form $y_p = (at^3 + bt^2 + ct + d)e^{-\frac{t}{\tau}}$.

$$\frac{dq_p}{dt} = \left(-\frac{a}{\tau}t^3 + \left(3a - \frac{b}{\tau}\right)t^2 + \left(2b - \frac{c}{\tau}\right)t + \left(c - \frac{d}{\tau}\right)\right)e^{-\frac{t}{\tau}}$$

$$\begin{aligned}\Rightarrow \frac{V_0}{RC\tau^2}t^2 e^{-\frac{t}{\tau}} = \Bigg(&\left(\frac{1}{RC} - \frac{1}{\tau}\right)at^3\\ &+ \left(3a + \left(\frac{1}{RC} - \frac{1}{\tau}\right)b\right)t^2\\ &+ \left(2b + \left(\frac{1}{RC} - \frac{1}{\tau}\right)c\right)t\\ &+ \left(c + \left(\frac{1}{RC} - \frac{1}{\tau}\right)d\right)\Bigg)e^{-\frac{t}{\tau}}\end{aligned}$$

$$\Rightarrow q_p = \begin{cases} CV_0 \cdot \left(\frac{t^2}{\tau(\tau-RC)} - \frac{RCt}{(\tau-RC)^2} + \frac{\tau(RC)^2}{(\tau-RC)^2} \right) & (\text{if } \tau \neq RC) \\ \frac{1}{3} CV_0 \cdot \left(\frac{t}{\tau}\right)^3 e^{-\frac{t}{\tau}} & (\text{if } \tau = RC) \end{cases}$$

We can then add an appropriate complementary solution so as to satisfy the boundary condition $q(0) = 0$.

$$\therefore q = \begin{cases} CV_0 \cdot \frac{t}{\tau-RC} \left(\frac{t}{\tau} - \frac{RC}{\tau-RC} \right) e^{-\frac{t}{\tau}} & (\text{if } \tau \neq RC) \\ \frac{1}{3} CV_0 \cdot \left(\frac{t}{\tau}\right)^3 e^{-\frac{t}{\tau}} & (\text{if } \tau = RC) \end{cases}$$

22.

$$\frac{dN}{N_s - N} = \lambda dt \Rightarrow \int_0^N \frac{dN}{N_s - N} = \lambda \int_0^t dt \Rightarrow -\ln\left|\frac{N_s - N}{N_s}\right| = \lambda t$$

$$\therefore N = N_s\left(1 - e^{-\lambda t}\right)$$

23. The clever idea here is that we could make a variable change from x to u, where we could choose u so as to make the equation easier to solve.

$$\begin{aligned} 0 &= A\frac{d^2y}{dx^2} + \frac{dA}{dx}\frac{dy}{dx} + \frac{y}{A} \\ &= A\frac{du}{dx}\frac{d}{du}\left(\frac{du}{dx}\frac{dy}{dy}\right) + \frac{dA}{dx}\frac{du}{dx}\frac{dy}{du} + \frac{y}{A} \\ &= A\left(\frac{du}{dx}\right)^2\frac{d^2y}{du^2} + \frac{dy}{du}\frac{d}{dx}\left(A\frac{dy}{dx}\right) \end{aligned}$$

Thus, we could define $\frac{du}{dx} = \frac{1}{A(x)}$ or $u = \int \frac{dx}{A(x)}$ to obtain

$$\frac{d^2y}{du^2} = -y \Rightarrow y(u) = C_1 \cos(u + C_2)$$

$$\therefore y(x) = C_1 \cos\left(\int \frac{dx}{A(x)} + C_2\right)$$

24. Suppose we may write $y = \lambda v$ where λ and v are to be determined later.

$$y' = \lambda v' + \lambda' v \Rightarrow y'' = \lambda v'' + 2\lambda' v' + \lambda'' v$$

$$\Rightarrow \lambda v'' + 2\left(\lambda' + \frac{\lambda}{x}\right)v' + \left(\lambda'' + \frac{2\lambda'}{x} + n^2\lambda\right)v = \frac{\sin\omega x}{x}$$

We may choose λ so as to make the coefficient of v' vanish, i.e.,

$$\lambda' = -\frac{\lambda}{x} \Rightarrow \lambda = \frac{1}{x}.$$

$$\Rightarrow v'' + n^2 v = \sin\omega x$$

(i) $n \neq \omega$:
The complementary solutions are

$$v_c = C_1 \cos(nx + C_2).$$

Using the ansatz $v_p = A \sin \omega x$ for a particular solution, one can easily find $A = \frac{1}{n^2-\omega^2}$.

$$\Rightarrow v = \frac{\sin \omega x}{n^2 - \omega^2} + C_1 \sin(nx + C_2)$$

$$\therefore y = \frac{1}{n^2 - \omega^2} \frac{\sin \omega x}{x} + C_1 \frac{\sin(nx + C_2)}{x}$$

(ii) $n = \omega$:
The complementary solutions are

$$v_c = C_1 \cos(\omega x + C_2).$$

Using the ansatz $v_p = Ax \sin \omega x + Bx \cos \omega x$ for a particular solution, one can easily find $A = -\frac{1}{2\omega}$ and $B = 0$.

$$\Rightarrow v = -\frac{1}{2\omega} x \cos \omega x + C_1 \cos(\omega x + C_2)$$

$$\therefore y = -\frac{1}{2\omega} \cos \omega x + C_1 \frac{\sin(\omega x + C_2)}{x}$$

25. From the hint, we may consider $y = xv$, where $v = 1$ is a solution if the right hand side were zero.

$$y' = xv' + v \Rightarrow y'' = xv'' + 2v'$$

$$\Rightarrow x(1-x)v'' + \left(1 + (1-x)^2\right) v' = (1-x)^2$$

$$\Rightarrow v'' + \left(\frac{2}{x} + \frac{1}{1-x} - 1\right) v' = \frac{1}{x} - 1$$

We can thus define the integration factor as

$$\lambda := \exp\left(\int dx \left(\frac{2}{x} + \frac{1}{1-x} - 1\right)\right) = \frac{x^2 e^{-x}}{1-x}.$$

$$\Rightarrow (\lambda v')' = \lambda \left(\frac{1}{x} - 1\right) = xe^{-x}$$

$$\Rightarrow \lambda v' = -(x+1)e^{-x} + A$$

$$\Rightarrow v' = 1 - \frac{1}{x^2} + A \frac{1-x}{x^2} e^x$$

$$\Rightarrow v = x + \frac{1}{x} - A \frac{e^x}{x} + B$$

$$\therefore y = x^2 + 1 + C_1 e^x + C_2 x$$

26. While the use of Dirac delta "function" must be dealt with caution, we are physicists so we shall simply accept that they exist.

$$\begin{aligned}
f(x) = y'' + py' + q &= \int_a^b dx' f(x') \left(\frac{\partial^2}{\partial x^2} + p(x)\frac{\partial}{\partial x} + q(x) \right) G(x,x') \\
&= \int_a^b dx' f(x') \delta(x-x') \\
\Rightarrow \left(\frac{\partial^2}{\partial x^2} + p(x)\frac{\partial}{\partial x} + q(x) \right) G(x,x') &= \delta(x-x')
\end{aligned}$$

From this, we see that $G(x,x')$ must separately equal zero for both $x < x'$ and $x > x'$. We thus write

$$G(x,x') = \begin{cases} y_1(x)\alpha(x') + y_2(x)\beta(x') & (x < x') \\ y_1(x)\gamma(x') + y_2(x)\zeta(x') & (x > x') \end{cases}.$$

From the definition of delta functions, we find that:

$$\begin{aligned}
1 &= \int_{x'-\epsilon}^{x'+\epsilon} dx \delta(x-x') \\
&= \int_{x'-\epsilon}^{x'+\epsilon} dx \left(\frac{\partial^2}{\partial x^2} + p(x)\frac{\partial}{\partial x} + q(x) \right) G(x,x') \\
&\approx \left[\left(\frac{\partial}{\partial x} + p(x) \right) G(x,x') \right]_{x'-\epsilon}^{x'+\epsilon} - \int_{x'-\epsilon}^{x'+\epsilon} dx p'(x) G(x,x') \\
&\approx \left[\frac{\partial}{\partial x} G(x,x') \right]_{x'-\epsilon}^{x'+\epsilon} \\
&\approx \lim_{x\to x'+} \frac{\partial}{\partial x} G(x,x') - \lim_{x\to x'-} \frac{\partial}{\partial x} G(x,x')
\end{aligned}$$

where we take infinitesimally small ϵ.

The boundary conditions for y imply

$$y(a) = y(b) = 0$$

$$\Rightarrow G(a,x') = G(b,x') = 0$$

$$\Rightarrow y_1(a)\alpha(x') + y_2(a)\beta(x') = y_1(b)\alpha(x') + y_2(b)\beta(x') = 0$$

$$\Rightarrow \beta(x') = \gamma(x') = 0.$$

The derivative condition, in turn, yields

$$y_2'(x')\zeta(x') - y_1'(x')\alpha(x') = 1.$$

As any such choice of $\zeta(x')$ and $\alpha(x')$ leads to a valid Green's function, we may arbitrarily choose

$$\alpha(x') = y_2'(x'),\ \zeta(x') = y_1'(x') + \frac{1}{y_2'(x')}$$

$$\Rightarrow G(x,x') = \begin{cases} y_1(x)y_2'(x') & (x < x') \\ y_2(x)y_1'(x') + \frac{y_2(x)}{y_2'(x')} & (x > x') \end{cases}.$$

$$\therefore y(x) = y_1(x)\int_x^b dx' f(x')y_2'(x') + y_2(x)\int_a^x f(x')\left(y_1'(x') + \frac{1}{y_2'(x')}\right)$$

For the given example $y'' + k^2 y = f(x)$, we may take

$$y_1(x) = \sin k(x-a),\ y_2(x) = \sin k(b-x)$$

provided that $b-a$ is not an integer multiple of the period $\frac{2\pi}{k}$. This leads to

$$G(x,x') = \begin{cases} -k\sin k(x-a)\cos k(b-x) & (x < x') \\ \sin k(b-x)\left(k\cos k(x'-a) - \frac{1}{k\cos k(b-x')}\right) & (x > x') \end{cases}$$

$$\begin{aligned} \Rightarrow y = &- k\sin k(x-a)\int_x^b dx' f(x')\cos k(b-x') \\ &+ \sin k(b-x)\int_a^x dx' f(x')\left(k\cos k(x'-a) - \frac{1}{k\cos k(b-x')}\right). \end{aligned}$$

One may verify that this solution satisfies both the boundary conditions and the given differential equation.

27. The complementary solutions satisfy

$$y_c'' + \frac{3}{x^2}y_c = 0.$$

Suppose $y_c = x^m$.

$$m^2 - m + 3 = 0 \Rightarrow \left(m - \frac{1}{2}\right)^2 = -\frac{11}{4} \Rightarrow m = \frac{1 \pm \sqrt{11}i}{2}$$

$$\begin{aligned} \Rightarrow y_c &= Ax^{\frac{1+\sqrt{11}i}{2}} + Bx^{\frac{1-\sqrt{11}i}{2}} \\ &= C_1\sqrt{x}\cos\left(\frac{\sqrt{11}}{2}\ln x\right) + C_2\sqrt{x}\sin\left(\frac{\sqrt{11}}{2}\ln x\right) \end{aligned}$$

We then look for two separate particular solutions:

$$y_{p1}'' + \frac{3}{x^2}y_{p1} = x^2 \text{ and } y_{p2}'' + \frac{3}{x^2}y_{p2} = \frac{1}{x}.$$

Using the monomial ansatz separately, that is, assuming $y_p = Ax^m$, one easily finds

$$y_{p1} = \frac{x^4}{15} \text{ and } y_{p2} = \frac{x}{3}.$$

$$\therefore y = \frac{x^4}{15} + \frac{x}{3} + C_1\sqrt{x}\cos\left(\frac{\sqrt{11}}{2}\ln x\right) + C_2\sqrt{x}\sin\left(\frac{\sqrt{11}}{2}\ln x\right)$$

28. Let us first look for asymptotic behaviors.

(i) $x \to 0+$ Suppose $y = O(x^m)$.

$$m(m-1) + 2m - l(l+1) = 0 \Rightarrow m = l \text{ or } -l-1$$

To vanish near the origin, we must have $y = O(x^l)$.

(ii) $x \to \infty$

$$\frac{d^2y}{dx^2} \approx -Ky \Rightarrow y \approx e^{\pm\sqrt{-K}x}$$

To vanish at infinity, we must have $y = x^{-\sqrt{-K}x}$ with $K < 0$.

We can thus write $y = f(x)x^l e^{-\sqrt{-K}x}$ with f regular everywhere.

$$y' = \left(\frac{f'}{f} + \frac{l}{x} - \sqrt{-K}\right)y$$

$$\begin{aligned}\Rightarrow y'' &= \left(\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 - \frac{l}{x^2}\right)y + \left(\frac{f'}{f} + \frac{l}{x} - \sqrt{-K}\right)^2 y \\ &= \left(\frac{f''}{f} + 2\left(\frac{l}{x} - \sqrt{-K}\right)\frac{f'}{f} + \frac{l(l-1)}{x^2} - \frac{2l\sqrt{-K}}{x} - K\right)y\end{aligned}$$

Tedious algebra yields

$$xf'' + 2(l+1-\sqrt{-K})f' - 2((l+1)\sqrt{-K} - 1)f = 0.$$

Let us write $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Substitution and tedious algebra yield

$$\sum_{n=0}^{\infty}((n+1)(n+2l+2)a_{n+1} - (\sqrt{-K}n + 2((l+1)\sqrt{-K} - 1))a_n)x^n = 0$$

$$\Rightarrow (\forall n)\ (n+1)(n+2l+2)a_{n+1} = (\sqrt{-K}n + 2((l+1)\sqrt{-K} - 1))a_n.$$

If $\{a_n\}$ never terminated, this would mean that $a_n \approx \frac{(\sqrt{-K})^n}{n!}$ for large values of n, leading to $f(x) \approx e^{\sqrt{-K}x}$ and y diverging to infinity as x increases.

$$\Rightarrow \sqrt{-K_{nl}}n + 2((l+1)\sqrt{-K_{nl}} - 1) = 0$$

$$\therefore K_{nl} = -\frac{4}{(n+2l+2)^2}\ (n = 0, 1, 2, \dots)$$

29. For very large x,

$$y'' \approx \frac{y}{4} \Rightarrow y \approx e^{\pm\frac{x}{2}}.$$

Thus, for y to vanish at infinity, we must have $y \approx e^{-x/2}$. As y must also vanish at the origin, let us write $y = f(x)xe^{-x/2}$ with $f(x)$ regular everywhere.

$$y' = \left(\frac{f'}{f} + \frac{1}{x} - \frac{1}{2}\right) y$$

$$\begin{aligned}
\Rightarrow y'' &= \left(\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 - \frac{1}{x^2}\right) y + \left(\frac{f'}{f} + \frac{1}{x} - \frac{1}{2}\right)^2 y \\
&= \left(\frac{f''}{f} + \left(\frac{2}{x} - 1\right)\frac{f'}{f} - \frac{1}{x} + \frac{1}{4}\right) y \\
&\Rightarrow \frac{f''}{f} + \left(\frac{2}{x} - 1\right)\frac{f'}{f} - \frac{K+1}{x} = 0
\end{aligned}$$

Let us write $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Substitution and tedious algebra yield

$$2a_1 - (K+1)a_0 + \sum_{n=1}^{\infty}((n+1)(n+2)a_{n+1} - (n+K+1)a_n)x^n = 0$$

$$\Rightarrow (\forall n)\ (n+1)(n+2)a_{n+1} = (n+K+1)a_n.$$

If $\{a_n\}$ never terminated, this would mean that $a_n \approx \frac{1}{n!}$ for large values of n, leading to $f(x) \approx e^x$ and y diverging to infinity as x increases.

$$\therefore K_n = -n\ (n = 1, 2, 3, \dots)$$

30. I assume that there must exist a "nontrivial" solution, since $y = 0$ is clearly a solution of the equation for any value of k. For large values of x,

$$y'' - 2y' - 3y \approx 0 \Rightarrow y \approx Ae^{-x} + Be^{3x}.$$

Thus, to be bounded everywhere, we must have $y \approx e^{-x}$. We then write $y = f(x)e^{-x}$ with f analytic everywhere, including the origin.

$$y' = \left(\frac{f'}{f} - 1\right) y \Rightarrow y'' = \left(\frac{f''}{f} - \frac{f'}{f} + \left(\frac{f'}{f} - 1\right)^2\right) y = \left(\frac{f''}{f} - 2\frac{f'}{f} + 1\right) y$$

Substitution yields

$$xf'' - 4xf' + kf = 0.$$

Let us write $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Substitution and tedious algebra yield

$$\sum_{n=0}^{\infty}(n(n+1)a_{n+1} - (4n-k)a_n)x^n = 0$$

$$\Rightarrow (\forall n)\; n(n+1)a_{n+1} = (4n-k)a_n.$$

If $\{a_n\}$ never terminated, this would mean that $a_n \approx \frac{4^n}{n!}$ for large values of n, leading to $f(x) \approx e^{4x}$ and y diverging to infinity as x increases.

$$\therefore k_n = 4n \; (n = 0, 1, 2, \dots)$$

31. For large values of x,

$$y'' \approx y \Rightarrow y \approx e^{\pm x}.$$

Thus, to be bounded everywhere, we must have $y \approx e^{-x}$. We then write $y = f(x)e^{-x}$ with f analytic everywhere, with $f(0) = 1$.

$$y' = \left(\frac{f'}{f} - 1\right) y \Rightarrow y'' = \left(\frac{f''}{f} - \frac{f'}{f} + \left(\frac{f'}{f} - 1\right)^2\right) y = \left(\frac{f''}{f} - 2\frac{f'}{f} + 1\right) y$$

Substitution yields

$$xf'' - 2(x-1)f' + (E-2)f = 0.$$

Let us write $f(x) = \sum_{n=0}^{\infty} a_n x^n$, with $a_0 = 1$. Substitution and tedious algebra yield

$$\sum_{n=0}^{\infty} ((n+1)(n+2)a_{n+1} - (2n+2-E)a_n)x^n = 0$$

$$\Rightarrow (\forall n)\; (n+1)(n+2)a_{n+1} = (2n+2-E)a_n.$$

If $\{a_n\}$ never terminated, this would mean that $a_n \approx \frac{2^n}{n!}$ for large values of n, leading to $f(x) \approx e^{2x}$ and y diverging to infinity as x increases.

$$\therefore E_n = 2n \; (n = 1, 2, 3, \dots)$$

32. Recall that

$$\frac{c_{r+2}}{c_r} = \frac{(r+m-n)(r+m+n+1)}{(r+1)(r+2)} \text{ and } v(x) = \sum_{r=0}^{\infty} c_r x^r$$

where the fraction is well-defined since we are assuming that $\{c_r\}$ never terminates. Notice how

$$\frac{c_{r+2}}{c_r} \approx \frac{r+2m+1}{r+3}$$

for large values of r.

On the other hand, the definition of binomial coefficients

$$\binom{-m}{r} := \frac{-m \cdot (-m-1) \cdot \dots \cdot (-m-r+1)}{r!}$$

naturally yields

$$\frac{\binom{-m}{r+2}}{\binom{-m}{r}} = \frac{(-m-r)(-m-r-1)}{(r+1)(r+2)}$$
$$= \frac{(r+m)(r+m+1)}{(r+1)(r+2)}$$
$$\approx \frac{r+2m+1}{r+3}$$

for large values of r.

Therefore, c_r behaves like $\binom{-m}{r}$ as r grows without bound, and consequently,

$$v(x) \approx \left(1-x^2\right)^{-m}.$$

33. Suppose $y = J_0(x)\ln x + \sum_{n=0}^{\infty} a_n x^n$ as prompted by the problem. Substitution and tedious algebra yield

$$2xJ_0'(x) + a_1 x + \sum_{n=2}^{\infty}(n^2 a_n + a_{n-2}).$$

Let us write $J_0(x) = \sum_{n=0}^{\infty} b_{2n}x^{2n}$ since we know that J_0 is even.

$$xJ_0'(x) = \sum_{n=0}^{\infty} 2nb_{2n}x^{2n}$$

$$\Rightarrow a_1 x + \sum_{n=1}^{\infty}\Bigg(\left(4nb_{2n} + 4n^2 a_{2n} + a_{2n-2}\right)x^{2n} + \left((2n+1)^2 a_{2n+1} + a_{2n-1}\right)x^{2n+1} \Bigg) = 0$$

This immediately tells us that all odd coefficients $a_1 = a_3 = \cdots = 0$.

Bessel's equation is linear, so addition of any multiple of J_0 to y also yields a valid solution. In other words, any variant $a_{2n}' := a_{2n} + \lambda b_{2n}$ also satisfies the above equation, for real parameter λ. Thus, we may choose $a_0 = 0$.

Using $b_0 = 1$, $b_2 = -\frac{1}{4}$, $b_4 = \frac{1}{64}$, $b_6 = -\frac{b_4}{36} = -\frac{1}{2304}$, and the recursion relation $a_2 n = -\frac{a_{2n-2}}{4n^2} - \frac{b_{2n}}{n}$ from above, we can iteratively calculate the values of a_{2n}.

$$\therefore y = J_0(x)\ln x + \frac{x^2}{4} - \frac{3}{128}x^4 + \frac{5}{4608}x^6 - \cdots$$

34. (i) $y(0) = 1$
We may write $y = 1 + \sum_{n=1}^{\infty} a_n x^n$. Substitution and tedious algebra yields

$$2a_1 - 2 + \sum_{n=1}^{\infty}(n+2)((n+1)a_{n+1} - a_n)x^n = 0.$$

This directly leads to $a_n = \frac{1}{n!}$, and thus,

$$y = e^x.$$

(ii) $y = \frac{1}{x} + A\ln x + B$
Substitution of the above form into the differential equation and patiently performing algebraic manipulations yields

$$-\frac{1}{x} + (xA'' + (2-x)A' - 2A)\ln x + 2A' + \left(\frac{1}{x} - 1\right)A + xB''(2-x)B' - 2B = 0.$$

Read the problem carefully: we only need to "give" two solutions, not find a general form for all solutions. Thus, we could choose the oddly specific solution of $A = e^x$ to make the coefficient of the $\ln x$ term vanish. Substituting this into the above equation yields

$$xB'' + (2-x)B' - 2B + e^x + \frac{e^x - 1}{x} = 0.$$

Expanding $B = \sum_{n=0}^{\infty} b_n x^n$, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, and $\frac{e^x-1}{x} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$, one can write

$$\sum_{n=0}^{\infty}\left((n+1)(n+2)b_{n+1} - (n+2)b_n + \frac{n+2}{(n+1)!}\right) = 0$$

$$\Rightarrow b_{n+1} = \frac{b_n}{n+1} - \frac{1}{(n+1)\cdot(n+1)!}.$$

Again, B may arbitrarily chosen, so let us choose $b_0 = 0$.

$$\therefore y = \frac{1}{x} + e^x \ln x - x - \frac{3}{4}x^2 - \frac{11}{36}x^3 - \cdots$$

35. Let $u := \frac{1}{x}$.

$$\frac{d}{dx} = \frac{du}{dx}\frac{d}{du} = -u^2\frac{d}{du}$$

$$\Rightarrow \frac{d^2}{dx^2} = -u^2\frac{d}{du}\left(-u^2\frac{d}{du}\right) = u^4\frac{d^2}{du^2} + 2u^3\frac{d}{du}$$

$$\Rightarrow \frac{d^2y}{du^2} + \frac{2}{u}\frac{dy}{du} = \frac{1}{\left(1+u^2y^2\right)^2}$$

As $u \to 0+$, $y \to 0$. Thus,

$$\frac{2}{u}\frac{dy}{du} \approx 1 \Rightarrow y \approx u^2$$

Hence, we write $y = f(u)u^2$ with f analytic everywhere.

$$\begin{aligned} u^2 f''(u) + 6uf'(u) + 6f(u) &= \frac{1}{\left(1+u^6 f(u)\right)^2} \\ &= 1 - 2u^6 f(u)^2 + 3u^{12} f(u)^4 + O\left(u^{18}\right) \end{aligned}$$

We write $f(u) = \sum_{n=0}^{\infty} a_n u^n$. The left-hand side (after tedious algebra) evaluates to

$$\sum_{n=0}^{\infty} \left(n^2 + 5n + 5\right) a_n u^n.$$

Evaluation of the right-hand side is quite more technical. We make use of the Cauchy multiplication formula:

$$f(u)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) u^n.$$

Substituting this into the equation, we get

$$\begin{aligned} &(6a_0 - 1) + \sum_{n=1}^{5} \left(n^2 + 5n + 6\right) a_n u^n \\ &+ \sum_{n=6}^{\infty} \left(\left(n^2 + 5n + 6\right) a_n + 2 \sum_{k=0}^{n-6} a_k a_{n-6-k} \right) u^n = 3u^{12} f(u)^4 + O(u^{18}). \end{aligned}$$

Let us compare each power of u separately.

$$a_0 = \frac{1}{6},\ a_1 = \cdots = a_5 = 0$$

$$O(u^6):\ 72a_6 + 2a_0^2 = 0 \Rightarrow a_6 = -\frac{1}{1296}$$

$$a_7 = \cdots = a_{11} = 0$$

$$O(u^{12}): 210a_{12} + 2\left(a_0 a_6 + a_6 a_0\right) = 3a_0^2 \Rightarrow a_{12} = \frac{11}{816480}$$

$$\Rightarrow f(u) = \frac{1}{6} - \frac{u^6}{1296} + \frac{11}{816480} u^{12} - \dots$$

$$\therefore y = \frac{1}{6x^2} - \frac{1}{1296x^8} + \frac{11}{816480} \frac{1}{x^{14}} + \dots$$

36. (a) Assume $y = \sum_{n=0}^{\infty} c_n x^n$. Using

$$y'' = y\left(x^2 - y^2\right),\ x^2 = \sum_{n=0}^{\infty} \delta_{n2} x^n,$$

and

$$y^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} c_k c_{n-k} \right) x^n,$$

one can find (after algebraic tedium) that

$$y\left(x^2 - y^2\right) = -c_0^2 - 3c_1 c_0^2 x + \sum_{n=2}^{\infty} \left(c_{n-2} - \sum_{k=0}^{n} \sum_{l=0}^{k} c_l c_{k-l} c_{n-k} \right) x^n$$

$$\Rightarrow \left(2c_2 + c_0^2\right) + \left(6c_3 + 3c_1c_0^2\right)x$$
$$+ \sum_{n=2}^{\infty}\left((n+2)(n+1)c_{n+2} - c_{n-2} + \sum_{k=0}^{n}\sum_{l=0}^{k} c_l c_{k-l} c_{n-k}\right) x^n = 0.$$

We thus find two degrees of freedom, namely $c_0 = A$ and $c_1 = B$.

$$c_2 = -\frac{A^2}{2},\ c_3 = -\frac{A^2B}{2},\ c_{n+2} = -\frac{1}{(n+2)(n+1)}\left(c_{n-2} - \sum_{k=0}^{n}\sum_{l=0}^{k} c_l c_{k-l} c_{n-k}\right)$$

We could calculate a few terms using the recurrence relation:

$$c_4 = \frac{1}{4\cdot 3}\left(c_0 - 3c_2c_0^2 - 3c_1^2c_0\right) = \frac{A}{12}\left(1 + \frac{3}{2}A^3 - 3B^2\right),$$

$$\begin{aligned} c_5 &= \frac{1}{5\cdot 4}\left(c_1 - 3c_3c_0^2 - 6c_2c_1c_0 - 3c_2c_1\right) \\ &= \frac{B}{20}\left(1 + 3A^3\left(\frac{A}{4} + 1\right)\right) \end{aligned}$$

$$\begin{aligned} \therefore y =& A + Bx - \frac{A^2}{2}x^2 - \frac{A^2B}{2}x^3 \\ &+ \frac{A}{12}\left(1 + \frac{3}{2}A^3 - 3B^2\right)x^4 + \frac{B}{20}\left(1 + 3A^3\left(\frac{A}{4} + 1\right)\right)x^5 + \cdots \end{aligned}$$

(b) We iteratively approximate the particular nonoscillating solution, for which the second derivative would stay relatively small. That is, we use the following iterative scheme:

$$-xy^{(0)} + {y^{(0)}}^3 = 0,\ y^{(n+1)} = \sqrt[3]{xy^{(n)} - {y^{(n)}}''}$$

We thus get

$$y^{(0)} = \sqrt{x} = x^{1/2}$$

$$y^{(1)} = \sqrt[3]{x^{3/2} + \frac{1}{4x^{3/2}}} \approx x^{1/2} + \frac{1}{12}x^{-5/2}$$

$$\begin{aligned} y^{(2)} &= \sqrt[3]{x^{3/2} + \frac{1}{3}x^{-3/2} - \frac{35}{48}x^{-9/2}} \\ &\approx x^{1/2} + \frac{1}{9}x^{-5/2} - \frac{35}{144}x^{-11/2} \end{aligned}$$

$$\begin{aligned} y^{(3)} &= \sqrt[3]{x^{3/2} + \frac{13}{36}x^{-3/2} - \frac{175}{144}x^{-9/2} + \frac{5005}{576}x^{-15/2}} \\ &\approx x^{1/2} + \frac{13}{108}x^{-5/2} - \frac{175}{432}x^{-11/2} + \frac{5005}{1728}x^{-17/2} \end{aligned}$$

$$\therefore y \approx x^{1/2} + \frac{13}{108}x^{-5/2} - \frac{175}{432}x^{-11/2} + \frac{5005}{1728}x^{-17/2}$$

(Note: One may find more accurate approximations by (i) iterating more times, or (ii) expanding to more terms for each binomial expansion above.)

37. We first take note that $y = x + \alpha$ is a trivial solution to the equation, as is easily verifiable. Let $z := y - x$. We directly have

$$z'' = z^2 - e^{2z}.$$

We multiply each side by z' and integrate each side to obtain

$$\frac{z'^2}{2} = \frac{1}{3}z^3 - \frac{1}{2}e^{2z} + E.$$

I find it illuminating to consider z as the position of a particle of mass 1. The above equation could then be interpreted as the energy conservation of this particle as it moves under the influence of the potential

$$V(z) = -\frac{1}{3}x^3 + \frac{1}{2}e^{2z}$$

and retains its total energy E. Thus, we could consider the infinitesimal oscillation about $z = \alpha$ as a harmonic oscillator.

$$\left.\frac{d^2V}{dz^2}\right|_{z=\alpha} = -2\alpha + 2e^{2\alpha} = 2\alpha(\alpha - 1) = m\omega^2 = \omega^2$$

$$\Rightarrow \omega = \sqrt{2\alpha(\alpha-1)}$$

$$\Rightarrow z \approx C_1 \sin\left(\sqrt{2\alpha(\alpha-1)x} + C_2\right)$$

$$\therefore y \approx x + C_1 \sin\left(\sqrt{2\alpha(\alpha-1)x} + C_2\right)$$

(Note: More accurate expressions may be obtained using the higher-order terms of the potential energy and perturbation methods.)

38. (a) Let $y := \sum_{n=1}^{\infty} c_n (x-1)^n$.

$$\begin{aligned}
e^{y/x} &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{y}{x}\right)^n = \sum_{n=0}^{\infty} \frac{y^n}{n!}(1 + (x-1))^{-n} \\
&= \sum_{n=0}^{\infty} \frac{y^n}{n!}\left(\sum_{m=0}^{\infty} \binom{-n}{m}(x-1)^m\right) \\
&= 1 + \sum_{m=0}^{2} \binom{-1}{m}(x-1)^m y + \sum_{m=0}^{1} \binom{-2}{m}(x-1)^m \frac{y^2}{2} + \frac{y^3}{6} + O\left((x-1)^4\right) \\
&= 1 + c_1(x-1) + \left(c_2 - c_1 + \frac{c_1^2}{2}\right)(x-1)^2 \\
&\quad + \left(c_3 - c_2 + c_1 - c_1c_2 - c_1^2 + \frac{c_1^3}{6}\right)(x-1)^3 + O\left((x-1)^4\right)
\end{aligned}$$

Comparing this with

$$y' = c_1 + 2c_2(x-1) + 3c_3(x-1)^2 + O\left((x-1)^3\right),$$

we can see that

$$c_1 = 1,\ c_2 = \frac{c_1}{2} = \frac{1}{2},\ c_3 = \frac{1}{3}\left(c_2 - c_1 + \frac{c_1^2}{2}\right) = 0$$

$$\therefore y = (x-1) + \frac{1}{2}(x-1)^2 + O\left((x-1)^4\right)$$

(b) Let $y = xv$.

$$xv' + v = e^v \Rightarrow \frac{dv}{e^v - v} = \frac{dx}{x} \Rightarrow \int_1^{y/x} \frac{dv}{e^v - v} = \ln x + C_1$$

$$\ln x_0 + C_1 = \int_1^{\infty} \frac{dv}{e^v - v} =: \alpha \Rightarrow C_1 = \alpha - \ln x_0$$

$$\int_1^{y/x} \frac{dv}{e^v - v} = \ln\frac{x}{x_0} + \alpha \Rightarrow \int_{y/x}^{\infty} \frac{dv}{e^v - v} = -\ln\frac{x}{x_0}$$

For values of x slightly less than x_0, we have $\frac{y}{x} >> 1$.

$$\Rightarrow -\ln\frac{x}{x_0} \approx \int_{y/x}^{\infty} dv e^{-v} = e^{-y/x}$$

$$\therefore y \approx -x \ln\ln\frac{x_0}{x}$$

39. From the WKB method followed the Bohr-Sommerfeld quantization condition:

$$\int_{x_1}^{x_2} dx\sqrt{E - V(x)} = \left(n + \frac{1}{2}\right)\pi.$$

($\hbar = 1$ in this unit system.)

If the particle has a total energy of $E < 0$, it then has classical turning points at $\pm a\left(1 + \frac{E}{V_0}\right)$. Hence,

$$\left(n + \frac{1}{2}\right)\pi = 2\int_0^{a\left(1+\frac{E}{V_0}\right)} dx\sqrt{V_0 + E - \frac{V_0 x}{a}} = \frac{4a}{3V_0}(V_0 + E)^{3/2}$$

$$\Rightarrow (V_0 + E_n)^{3/2} = \left(n + \frac{1}{2}\right)\frac{3\pi V_0}{4a}$$

$$\therefore E_n = -\left(V_0 - \left(\left(n + \frac{1}{2}\right)\frac{3\pi V_0}{4a}\right)^{2/3}\right)$$

Here, n takes nonnegative integers as its value such that $E_n < 0$.

40. To use the WKB method, we want to make a substitution $y = up$ for some known function p such that the first derivative term vanishes. Substituting this into the equation yields

$$pu'' + \left(2p' - \frac{3}{x}p\right)u' + \left(p'' - \frac{3}{x}p' + \left(\frac{15}{4x^2} + x^{\frac{1}{2}}\right)\right)u = 0.$$

Thus, we choose $p = x^{\frac{3}{2}}$ to obtain

$$u'' + \left(x^{\frac{1}{2}} + \frac{15}{4}x^{-2} - \frac{21}{4}x^{-2}\right)u = 0.$$

Let

$$f(x) := x^{\frac{1}{2}} + \frac{15}{4}x^{-2} - \frac{21}{4}x^{-2} = x^{\frac{1}{2}}\left(1 + \frac{15}{4}x^{-\frac{5}{2}} - \frac{21}{4}x^{-\frac{7}{2}}\right).$$

This means that, using Eq. (1–90) from the textbook, one gets

$$u \approx \frac{1}{f^{1/4}}\left(c_+ \exp\left(i\int dx\sqrt{f}\right) + c_- \exp\left(-i\int dx\sqrt{f}\right)\right).$$

Evaluating these can be done using the binomial expansion for large values of x.

$$\begin{aligned}
\sqrt{f} &= \sqrt{x^{\frac{1}{2}}\left(1 + \frac{15}{4}x^{-\frac{5}{2}} - \frac{21}{4}x^{-\frac{7}{2}}\right)} \\
&\approx x^{\frac{1}{4}}\left(1 + \frac{15}{8}x^{-\frac{5}{2}} - \frac{21}{8}x^{-\frac{7}{2}}\right) \\
&= x^{\frac{1}{4}} + \frac{15}{8}x^{-\frac{9}{4}} - \frac{21}{8}x^{-\frac{13}{4}} \\
\Rightarrow \int dx\sqrt{f} &= \frac{4}{5}x^{\frac{5}{4}} - \frac{3}{2}x^{-\frac{5}{4}} + \frac{7}{6}x^{-\frac{9}{4}}
\end{aligned}$$

$$\begin{aligned}
f^{-\frac{1}{4}} &= \left(x^{\frac{1}{2}}\left(1 + \frac{15}{4}x^{-\frac{5}{2}} - \frac{21}{4}x^{-\frac{7}{2}}\right)\right)^{-\frac{1}{4}} \\
&\approx x^{-\frac{1}{8}}\left(1 - \frac{15}{16}x^{-\frac{5}{2}} + \frac{21}{16}x^{-\frac{7}{2}}\right)
\end{aligned}$$

$$\Rightarrow u \approx x^{-\frac{1}{8}}\left(1 - \frac{15}{16}x^{-\frac{5}{2}} + \frac{21}{16}x^{-\frac{7}{2}}\right)\cdot C_1\cos\left(\frac{4}{5}x^{\frac{5}{4}} - \frac{3}{2}x^{-\frac{5}{4}} + \frac{7}{6}x^{-\frac{9}{4}} + C_2\right)$$

$$\therefore y \approx x^{\frac{11}{8}}\left(1 - \frac{15}{16}x^{-\frac{5}{2}} + \frac{21}{16}x^{-\frac{7}{2}}\right)\cdot C_1\cos\left(\frac{4}{5}x^{\frac{5}{4}} - \frac{3}{2}x^{-\frac{5}{4}} + \frac{7}{6}x^{-\frac{9}{4}} + C_2\right)$$

41. We again make the substitution $y = up$ to eliminate the first derivative term. Substituting this into the Bessel equation

$$x^2y'' + xy' + \left(x^2 - m^2\right)y = 0$$

yields

$$x^2pu'' + \left(2x^2p' + xp\right)u' + \left(x^2p'' + xp' + \left(x^2 - m^2\right)p\right)u = 0.$$

Thus, we choose $p = x^{-\frac{1}{2}}$ to obtain

$$u'' + \left(1 - \left(m^2 - \frac{1}{4}\right)x^{-2}\right)u = 0.$$

Let

$$f(x) := 1 - \left(m^2 - \frac{1}{4}\right)x^{-2}.$$

Now, we wish to find the asymptotic form of J_m, Bessel's function of the first kind. J_m is distinguished from that of the second kind, Y_m by its analyticity at the origin. Notice how f is a monotone increasing function with a single zero at $\sqrt{m^2 - \frac{1}{4}}$. Therefore, for $m >> \frac{1}{2}$, one can approximate the analyticity condition as the function being bounded for $x << \sqrt{m^2 - \frac{1}{4}}$. This leads us to use the connection formula Eq. (1–113).

$$\sqrt{f} = \sqrt{1 - \left(m^2 - \frac{1}{4}\right)x^{-2}} \approx 1 - \frac{1}{2}\left(m^2 - \frac{1}{4}\right)x^{-2}$$

$$\begin{aligned}
\Rightarrow \int_{\sqrt{m^2-\frac{1}{4}}}^{x} dx\sqrt{f} &\approx x - \sqrt{m^2 - \frac{1}{4}} + \frac{1}{2}\left(m^2 - \frac{1}{4}\right)\left(\frac{1}{x} - \frac{1}{\sqrt{m^2 - \frac{1}{4}}}\right) \\
&= x + \frac{m^2 - \frac{1}{4}}{2x} - \frac{1}{2}\sqrt{m^2 - \frac{1}{4}} \\
f^{-\frac{1}{4}} &\approx 1 + \frac{1}{4}\left(m^2 - \frac{1}{4}\right)x^{-2}
\end{aligned}$$

$$\begin{aligned}
\therefore J_m &\approx 2x^{-\frac{1}{2}}f^{-\frac{1}{4}}\cos\left(\int_{\sqrt{m^2-\frac{1}{4}}}^{x} dx\sqrt{f} - \frac{\pi}{4}\right) \\
&\approx \left(2x^{-\frac{1}{2}} + \left(\frac{m^2}{2} - \frac{1}{8}\right)x^{-\frac{5}{2}}\right)\cos\left(x + \frac{m^2 - \frac{1}{4}}{2x} - \frac{1}{2}\sqrt{m^2 - \frac{1}{4}} - \frac{\pi}{4}\right)
\end{aligned}$$

42. This problem is a simple application of the connection formulae Eq. (1–113) and (1–122) with $f(x) = x$.

(a) Here, we use Eq. (1–122) with $\phi = 0$.

$$\therefore y \sim \frac{1}{\sqrt{2}(-x)^{-\frac{1}{4}}}\exp\left(\frac{2}{3}(-x)^{\frac{3}{2}}\right)$$

(b) Here, we use Eq. (1–113).

$$\therefore y \sim \frac{2}{x^{\frac{1}{4}}} \cos\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{\pi}{4}\right)$$

(c) This problem is not defined, as explained after Eq. (1–122). In short, the large exponential alone cannot determine the phase.

43. We again use the WKB approximation with $f(x) = x^2$.

$$W_\pm := \frac{1}{f^{\frac{1}{4}}} \exp\left(\pm i \int_0^x dx \sqrt{f}\right) = x^{-\frac{1}{2}} \exp\left(\pm \frac{i}{2} x^2\right)$$

$$\Rightarrow y \approx x^{-\frac{1}{2}} \left(A \cos \frac{x^2}{2} + B \sin \frac{x^2}{2}\right)$$

Solving for A and B using the initial conditions $y(10) = 0$ and $y'(10) = 1$, we get

$$y \approx \frac{1}{\sqrt{10x}} \sin\left(\frac{x^2}{2} - 50\right).$$

(a)

$$\frac{x^2}{2} - 50 \approx \pi \Rightarrow x \approx \sqrt{100 + 2\pi} \approx 10.309$$

(b)

$$\frac{x^2}{2} - 50 \approx \frac{\pi}{2} \Rightarrow x \approx \sqrt{100 + \pi} \Rightarrow y \approx \frac{1}{\sqrt{10\sqrt{100 + \pi}}} \approx 0.0992$$

44. The error estimation for this solution follows the example following Eq. (1–103). I personally am not confident that this solution is correct, but it is the best I could come up with.

The WKB approximation of y_1 is given by

$$y_1 \approx \frac{C_1}{\sqrt{5x}} \sin\left(\frac{x^2 - 25}{2}\right)$$

which can be obtained in the same way as in Problem 1–43. The 25th zero beyond $x = 5$ is then given by

$$\frac{x^2 - 25}{2} \approx 25\pi \Rightarrow x \approx 5\sqrt{2\pi + 1} \approx 13.494.$$

We now begin the error analysis. The following notation follows that of the aforementioned example. Let

$$y_1 = \alpha_+ x^{-\frac{1}{2}} \exp\left(\frac{i}{2}x^2\right) + \alpha_- x^{-\frac{1}{2}} \exp\left(-\frac{i}{2}x^2\right).$$

Near $x = 5$, we then obtain

$$\alpha_+ \approx \frac{A}{2i} e^{-\frac{25}{2}i}, \ \alpha_- \approx -\frac{A}{2i} e^{\frac{25}{2}i}.$$

Let

$$g = \frac{1}{4}\frac{f''}{f} - \frac{5}{16}\left(\frac{f'}{f}\right)^2 = -\frac{3}{4x^2}$$

$$\begin{aligned}
\Rightarrow \alpha'_\pm &\approx \mp \frac{i}{2}\frac{g}{\sqrt{f}}\left(\alpha_\pm + \alpha_\mp \exp\left(\mp 2i \int dx \sqrt{f}\right)\right) \\
&= \pm \frac{3i}{8x^3}\left(\alpha_\pm + \alpha_\mp \exp\left(\mp i x^2\right)\right) \\
&\approx \frac{3iA}{32x^3} e^{\mp\frac{25}{2}i}\left(1 - e^{\pm 25i \mp ix^2}\right)
\end{aligned}$$

Now, upon integrating this from $x = 5$ to $5\sqrt{2\pi + 1}$, we may ignore the $e^{\pm 25i \mp ix^2}$ term as it rotates rapidly, thus making little contribution to the result.

$$\begin{aligned}
\Rightarrow |\Delta y_1| &\approx |\Delta \alpha_\pm| \\
&\approx \left| \int_5^{5\sqrt{2\pi+1}} dx \alpha'_\pm \right| \\
&\approx \left| \frac{3iA}{32x^3} e^{\mp\frac{25}{2}i} \int_5^{5\sqrt{2\pi+1}} \frac{dx}{x^3} \right| \\
&= \frac{3\pi A}{800(2\pi + 1)}
\end{aligned}$$

We also have

$$\begin{aligned}
y'_1|_{x=5\sqrt{2\pi+1}} &\approx \frac{A}{\sqrt{5}}\left(-\frac{1}{2}x^{-\frac{3}{2}} \sin\left(\frac{x^2 - 25}{2}\right) + x^{\frac{1}{2}} \cos\left(\frac{x^2 - 25}{2}\right)\right)\Bigg|_{x=5\sqrt{2\pi+1}} \\
&\approx -A(2\pi + 1)^{\frac{1}{4}}.
\end{aligned}$$

$$\therefore |\Delta x| \approx \left|\frac{\Delta y}{y'}\right| \approx \frac{3\pi}{800(2\pi + 1)^{\frac{1}{4}}} \approx 3.488 \times 10^{-4}$$

2 Chapter 2

1. A useful trick when dealing with complex sign patterns like this is to represent them using an exponential. In this case, we acknowledge that

$$\left\{\Re\left\{e^{i\left(\frac{n}{2} - \frac{1}{4}\right)\pi}\right\}\right\} = \frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}}, \ -\frac{1}{\sqrt{2}}, \ -\frac{1}{\sqrt{2}}, \cdots$$

and write the series as

$$
\begin{aligned}
\text{(Given series)} &= \sum_{n=0}^{\infty} \frac{1}{4^n} \cdot \sqrt{2}\Re\left\{e^{i\left(\frac{n}{2}-\frac{1}{4}\right)\pi}\right\} \\
&= \Re\left\{\sqrt{2}e^{-i\frac{\pi}{4}} \sum_{n=0}^{\infty} \left(\frac{e^{i\frac{\pi}{4}}}{2}\right)^n\right\} \\
&= \Re\left\{(1-i) \sum_{n=0}^{\infty} \left(\frac{i}{4}\right)^n\right\} \\
&= \Re\left\{(1-i) \cdot \frac{1}{1-\frac{i}{4}}\right\} \\
&= \frac{16}{17}\Re\left\{(1-i)\left(1+\frac{i}{4}\right)\right\} \\
&= \frac{16}{17}\left(1+\frac{1}{4}\right) \\
&= \frac{20}{17}.
\end{aligned}
$$

For series problems, it is always a good idea to verify the results numerically with a calculator, provided that the series converges quickly enough. In this case, evaluating upto $\frac{1}{1024}$ yields about 1.1768 while the correct answer is about 1.1765.

2.

$$
\begin{aligned}
\text{(Given series)} &= \sum_{n=0}^{\infty} \frac{1}{n(n+2)} \\
&= \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) \\
&= \frac{1}{2}\left(1+\frac{1}{2}\right) \\
&= \frac{3}{4}
\end{aligned}
$$

3. To describe the sign pattern, we shall use

$$
\left\{\Re\left\{e^{i\left(\frac{n}{3}+\frac{1}{6}\right)\pi}\right\}\right\} = \frac{\sqrt{3}}{2},\ 0,\ -\frac{\sqrt{3}}{2},\ -\frac{\sqrt{3}}{2},\ 0,\ \frac{\sqrt{3}}{2},\ \cdots.
$$

$$
\begin{aligned}
\text{(Given series)} &= \frac{1}{1\cdot 3^0} + \frac{0}{3\cdot 3^1} + \frac{-1}{5\cdot 3^2} + \frac{-1}{7\cdot 3^3} + \frac{0}{9\cdot 3^4} + \frac{1}{11\cdot 3^5} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)3^n} \cdot \frac{2}{\sqrt{3}} \Re\left\{ e^{i\left(\frac{n}{3}+\frac{1}{6}\right)\pi} \right\} \\
&= \Re\left\{ \frac{2}{\sqrt{3}} e^{i\frac{\pi}{6}} \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{1}{3} e^{i\frac{\pi}{3}} \right)^n \right\} \\
&= \Re\left\{ 2 \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{1}{\sqrt{3}} e^{i\frac{\pi}{6}} \right)^{2n+1} \right\} \\
&= \Re\left\{ 2\tanh^{-1}\left(\frac{1}{\sqrt{3}} e^{i\frac{\pi}{6}} \right) \right\} \\
&= \Re\left\{ \ln\left(\frac{1+\frac{1}{\sqrt{3}} e^{i\frac{\pi}{6}}}{1-\frac{1}{\sqrt{3}} e^{i\frac{\pi}{6}}} \right) \right\} \\
&= \ln\left| \frac{1+\frac{1}{\sqrt{3}} e^{i\frac{\pi}{6}}}{1-\frac{1}{\sqrt{3}} e^{i\frac{\pi}{6}}} \right| \\
&= \ln\left| \frac{5-\sqrt{3}i}{2} \right| \\
&= \frac{1}{2}\ln 7
\end{aligned}
$$

4.

$$
\begin{aligned}
\text{(Given series)} &= \sum_{n=0}^{\infty} \frac{n+1}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!} \Bigg|_{x=1} \\
&= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} \Bigg|_{x=1} \\
&= \frac{d}{dx}\left(xe^x\right)\Bigg|_{x=1} \\
&= \left((x+1)e^x\right)\Big|_{x=1} \\
&= 2e
\end{aligned}
$$

5.

$$
\begin{aligned}
\text{(Given series)} &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\
&= \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= \frac{3}{4} \cdot \frac{\pi^2}{6} \\
&= \frac{\pi^2}{8}
\end{aligned}
$$

6.

$$
\begin{aligned}
\text{(Given series)} &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^4} - \sum_{n=1}^{\infty} \frac{1}{(2n)^4} \\
&= \frac{15}{16} \sum_{n=1}^{\infty} \frac{1}{n^4} \\
&= \frac{15}{16} \cdot \frac{\pi^4}{90} \\
&= \frac{\pi^4}{96}
\end{aligned}
$$

7.

$$
\begin{aligned}
\text{(Given series)} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} - 2\sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n^2} \\
&= \frac{1}{2} \cdot \frac{\pi^2}{6} \\
&= \frac{\pi^2}{12}
\end{aligned}
$$

8.

$$
\begin{aligned}
f(\theta) &= \sum_{n=0}^{\infty} \frac{\sin(n+1)\theta}{2n+1} \\
&= \Im\left\{\sum_{n=0}^{\infty} \frac{1}{2n+1} e^{i(n+1)\theta}\right\} \\
&= \Im\left\{e^{i\frac{\theta}{2}} \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(e^{i\frac{\theta}{2}}\right)^{2n+1}\right\} \\
&= \Im\left\{e^{i\frac{\theta}{2}} \tanh^{-1}\left(e^{i\frac{\theta}{2}}\right)\right\} \\
&= \frac{1}{2}\Im\left\{e^{i\frac{\theta}{2}} \ln\left(\frac{1+e^{i\frac{\theta}{2}}}{1-e^{i\frac{\theta}{2}}}\right)\right\} \\
&= \frac{1}{2}\Im\left\{e^{i\frac{\theta}{2}} \ln\left(i\cot\frac{\theta}{4}\right)\right\} \\
&= \frac{1}{2}\Im\left\{\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\left(\ln\left(\cot\frac{\theta}{4}\right) + i\frac{\pi}{2}\right)\right\} \\
&= \frac{\pi}{4}\cos\frac{\theta}{2} + \frac{1}{2}\ln\left(\cot\frac{\theta}{4}\right)\sin\frac{\theta}{2}
\end{aligned}
$$

9. Half of the challenge of this problem is expressing the nth term in closed form. Let us denote the first term as a_1.

$$
\begin{aligned}
a_n &= \frac{\left(\prod_{j=1}^{n-1} \frac{2j-1}{2}\right) \cdot \left(\frac{2n-1}{2}\right)^2}{(2n+7)(2n+5)(2n+3)^2 n!} \\
&= \frac{(2n-1)!!(2n-1)}{2^{n+1}(2n+7)(2n+5)(2n+3)^2 n!} \\
&= \frac{\frac{(2n)!}{2^n n!} \cdot (2n-1)}{2^{n+1}(2n+7)(2n+5)(2n+3)^2 n!} \\
&= \frac{(2n-1) \cdot (2n)!}{2^{2n+1}(2n+7)(2n+5)(2n+3)(n!)^2}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \frac{a_{n+1}}{a_n} &= \frac{(2n+1)^2(2n+2)(2n+3)^2}{(2n+9)(2n+5)(n+1)^2(2n-1)} \\
&\xrightarrow{n\to\infty} 4 > 1
\end{aligned}
$$

Therefore, the given series diverges.

10. Again, we shall denote the first term as a_1.

$$\begin{aligned}
a_n &= \frac{\left((2n+1)!!\right)^2}{4^{n-1}\cdot n\cdot (n!)^2} \\
&= \frac{\left(\dfrac{(2n+1)!}{2^n n!}\right)^2}{4^{n-1}\cdot n\cdot (n!)^2} \\
&= \frac{\left((2n+1)!\right)^2}{4^{2n+1}\cdot n\cdot (n!)^4}
\end{aligned}$$

To deduce the limit of this sequence, we employ Stirling's approximation:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \text{ (for large } n\text{).}$$

$$\begin{aligned}
\Rightarrow a_n &\approx \frac{2\pi(2n+1)\left(\dfrac{2n+1}{e}\right)^{2n+1}}{4^{2n+1}\cdot n\cdot 4\pi^2 n^2\left(\dfrac{n}{e}\right)^n} \\
&= \frac{(2n+1)^{4n+3}}{\pi e^2 2^{8n+3} n^{4n+3}} \\
&= \frac{4}{\pi e}\left(1+\frac{1}{2n}\right)^{4n+3} \\
&= \frac{4}{\pi e}\left(\left(1+\frac{1}{2n}\right)^{2n}\right)^{\frac{4n+3}{2n}} \\
&\xrightarrow{n\to\infty} \frac{4}{\pi} \neq 0
\end{aligned}$$

Therefore, the given series diverges.

11. We want to evaluate

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)^2 x^{2n+1}}{(2n+1)!}.$$

To this end, we want to express $(n+1)^2 = n^2 + 2n + 1$ in terms of $2n+2$ and $(2n+1)(2n+3)$ as they can easily be obtained by differentiation. We employ Horner's schema to do so.

$$\begin{array}{r|ccc}
 & 1 & 2 & 1 \\
-1 & & -1 & -1 \\
\hline
 & 1 & 1 & \multicolumn{1}{|c}{0} \\
-\frac{3}{2} & & -\frac{3}{2} & \\
\hline
 & \multicolumn{1}{c|}{1} & -\frac{1}{2} &
\end{array}$$

$$\begin{aligned}
\Rightarrow n^2+2n+1 &= \left(\left(n+\frac{3}{2}\right)-\frac{1}{2}\right)(n+1) \\
&= \frac{1}{4}(2n+3)(2n+2)-\frac{1}{4}(2n+2)
\end{aligned}$$

$$\begin{aligned}
\therefore f(x) &= \frac{1}{4}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2n+3)(2n+2)x^{2n+1}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2n+2)x^{2n+1}\right) \\
&= \frac{1}{4}\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+3}\right)-\frac{1}{4}\frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+2}\right) \\
&= \frac{1}{4}\frac{d^2}{dx^2}\left(x^2\sin x\right)-\frac{1}{4}\frac{d}{dx}\left(x\sin x\right) \\
&= \frac{1}{4}\left(2\sin x+4x\cos x-x^2\sin x\right)-\frac{1}{4}\left(\sin x+x\cos x\right) \\
&= \frac{1-x^2}{4}\sin x+\frac{3}{4}x\cos x
\end{aligned}$$

12. (a) We should first formally state what the problem is asking of us.

$$\begin{aligned}
(E+1)^k+(E-1)^k &= \sum_{i=0}^{k}\binom{k}{i}E_i+\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}E_i \\
&= \sum_{i=0}^{k}\left(1+(-1)^i\right)\binom{k}{i}E_i \\
&= 2\sum_{i=0}^{k/2}\binom{k}{2i}E_{2i}
\end{aligned}$$

Therefore, we need to show that

$$\sum_{i=0}^{l}\binom{2l}{2i}E_{2i}=0$$

for all integers l.

$$\begin{aligned}
1 &= \sec z\cdot\cos z \\
&= \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}E_{2n}z^{2n}\right)\cdot\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}\right) \\
&= \sum_{l=0}^{\infty}\left(\sum_{i=0}^{l}\frac{(-1)^i}{(2i)!}E_{2i}\cdot\frac{(-1)^{2l-i}}{(k-2i)!}\right)z^{2l} \\
&= \sum_{l=0}^{\infty}\left(\frac{1}{(2l)!}\sum_{i=0}^{l}\binom{2l}{2i}E_{2i}\right)z^{2l}
\end{aligned}$$

$$\therefore (\forall l \geq 1) \sum_{i=0}^{l} \binom{2l}{2i} E_{2i} = 0$$

We could use this formula to determine successive terms of this sequence.

$$l = 1: \; E_2 + E_0 = 0 \Rightarrow E_2 = -1$$

$$l = 2: \; E_4 + 6E_2 + E_0 = 0 \Rightarrow E_4 = 5$$

$$l = 3: \; E_6 + 15E_4 + 15E_2 + E_0 = 0 \Rightarrow E_6 = -61$$

$$l = 4: \; E_8 + 28E_6 + 70E_4 + 28E_2 + E_0 = 0 \Rightarrow E_9 = 1385$$

(b) It is sometimes a good idea to start with the more general formula then apply it to the more concrete problem, as this could lead to less work.

$$\begin{aligned}
\sec \pi x &= \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i (2i+1)}{(2i+1)^2 - 4x^2} \\
&= \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \left(1 - \left(\frac{2x}{2i+1}\right)^2\right)^{-1} \\
&= \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \sum_{n=0}^{\infty} \left(\frac{2x}{2i+1}\right)^{2n} \\
&= \frac{4}{\pi} \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} (-1)^i \frac{4^n}{2i+1^{2n+1}} x^{2n} \\
&= \sum_{n=0}^{\infty} \left(\frac{4^{n+1}}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^{2n+1}}\right) x^{2n} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} (\pi x)^{2n}
\end{aligned}$$

We thus compare the last two series term-by-term to obtain

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^{2n+1}} = \frac{\pi^{2n+1} (-1)^n E_{2n}}{4^{n+1} (2n)!}$$

as the answer to part (b).

Substituting $n = 1$ for part (a), we get

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^3} = \frac{\pi^3 (-1)^1 E_2}{4^2 \cdot 2!} = \frac{\pi^3}{32}.$$

3 Chapter 3

1. Let us define

$$I(a,b) := \int_0^\infty dy \frac{e^{-ay} - e^{-by}}{y}.$$

$$\frac{\partial I}{\partial a} = -\int_0^\infty dy e^{-ay} = -\frac{1}{a}, \; \frac{\partial I}{\partial b} = \int_0^\infty dy e^{-by} = \frac{1}{b}$$

$$\Rightarrow I(a,b) = \ln\left(\frac{b}{a}\right) + C$$

Now, notice that

$$C = I(a,a) = \int_0^\infty dy 0 = 0.$$

$$\therefore I(a,b) = \ln\left(\frac{b}{a}\right)$$

2. (If your knee-jerk reaction to this problem is any form of skepticism, then you are probably correct; if not, you are a perfect fit for a physicist in my opinion.)

$$\begin{aligned}
\int_0^\infty dx \sin bx &= \lim_{a\to 0+} \int_0^\infty dx e^{-ax} \sin bx \\
&= \lim_{a\to 0+} \Im\left\{\int_0^\infty dx e^{-ax} e^{ibx}\right\} \\
&= \lim_{a\to 0+} \Im\left\{\frac{1}{a - ib}\right\} \\
&= \lim_{a\to 0+} \frac{b}{a^2 + b^2} \\
&= \frac{1}{b}
\end{aligned}$$