My Solutions for “Mathematical Methods of Physics (Second Edition)” by J. Mathews, R. L. Walker

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0 Introduction

This document is an archive of my solutions to J. Mathews and R. L. Walker’s “Mathematical Methods of Physics” textbook. None of the solutions have been verified by anyone other than myself, whom I do not consider a reliable source. Hence, it is strongly advised against to use this solution for any application where accuracy matters, especially for assignments and any academic work. However, I have not yet been able to find any other solutions. Please just use this manual as just a suggestion.

1 Chapter 1

1. We first acknowledge that $y = 0$ is a solution, so we seek solutions that are not identically zero.

Let $y = x \cdot v$.

$$y' = v + xv' \Rightarrow x^2(v + xv') + x^2v^2 = x \cdot xv \cdot (v + xv')$$

$$\Rightarrow v + xv' = xvv' \Rightarrow \left(1 - \frac{1}{v}\right) dv = \frac{1}{x} dx \Rightarrow v - \ln v = \ln x + C$$

$$\therefore \frac{y}{x} - \ln y = C$$

2.

$$\frac{y}{\sqrt{1+y^2}} dy = \frac{x}{\sqrt{1+x^2}} dx$$

$$\therefore \sqrt{1+y^2} = \sqrt{1+x^2} + C$$

3. Let $v := x + y$.

$$v' = 1 + y' \Rightarrow v' - 1 = \frac{a^2}{v^2} \Rightarrow \frac{v^2}{a^2 + v^2} dv = dx$$

$$x + C = \int \left(1 - \frac{a^2}{a^2 + v^2}\right) dv = v - a\tan^{-1}\left(\frac{v}{a}\right) = x + y - a\tan^{-1}\left(\frac{x+y}{a}\right)$$

$$\therefore y - a\tan^{-1}\left(\frac{x+y}{a}\right) = C$$

4. We first seek the complement solutions.

$$y_c' + y_c \cos x = 0 \Rightarrow \frac{dy_c}{y_c} + \cos x dx = 0 \Rightarrow y_c = Ce^{-\sin x}$$

For the particular solution, we first observe that $\frac{1}{2}\sin 2x = \sin x \cos x$. Thus, we shall try the ansatz $y_p = \sin x + A$.

$$\cos x + (\sin x + A)\cos x = \sin x \cos x \Rightarrow A = -1$$

$$\therefore y = Ce^{-\sin x} + \cos x - 1$$

5.

$$\left(1 + x^2\right) y' = xy(y+1) \Rightarrow \frac{dy}{y(y+1)} = \frac{x}{1 - x^2} dx$$

$$\Rightarrow \ln\left(\frac{y}{y+1}\right) = -\frac{1}{2}\ln\left(1 - x^2\right) + C_1$$

$$\Rightarrow \frac{y+1}{y} = e^{-C_1}\sqrt{1 - x^2} = C\sqrt{1 - x^2}$$

$$\therefore y = \frac{1}{C\sqrt{1 - x^2} - 1}$$

6. We first take note that the equation is dimension-consistent with $[y] = [x^{-2}]$. Thus, we define $v := x^2 y$ or $y = \frac{v}{x^2}$.

$$y' = \frac{v'}{x^2} - \frac{2v}{x^3} \Rightarrow 2xv' - 4v = 1 + \sqrt{1 + 4v}$$

$$\Rightarrow \frac{v'}{1 + 4v + \sqrt{1 + 4v}} = \frac{1}{2x}$$

$$\Rightarrow \left(\frac{1}{\sqrt{1 + 4v}} - \frac{1}{\sqrt{1 + 4v} + 1}\right) dv = \frac{dx}{2x}$$

Basic calculus yields $\int \frac{dv}{\sqrt{1+4v}+1} = \frac{1}{2}\left(\sqrt{1 + 4v} - \ln\left(1 + \sqrt{1 + 4v}\right)\right) + C$. Hence,

$$\frac{1}{2}\sqrt{1 + 4v} - \frac{1}{2}\left(\sqrt{1 + 4v} - \ln\left(1 + \sqrt{1 + 4v}\right)\right) = \frac{1}{2}\ln x + C_1$$

$$\Rightarrow 1 + \sqrt{1 + 4v} = e^{2C_1}x = Cx$$

$$\therefore \sqrt{1 + 4x^2 y} = Cx - 1$$

7. Let $v' := y'$.

$$v' + v^2 + 1 = 0 \Rightarrow \frac{dv}{v^2+1} = -1 \Rightarrow v = \tan(C_1 - x)$$

$$\therefore y = \int dx v = \ln\left(\cos(C_1 - x)\right) + C_2$$

8.

$$y'y'' = e^y y' \Rightarrow \int dy'y' = \int dy e^y \Rightarrow \frac{1}{2}y'^2 = e^y + A$$

$$\Rightarrow \frac{dy}{\sqrt{e^y + A}} = \sqrt{2}dx$$

Basic calculus yields

$$\int \frac{dy}{\sqrt{e^y+A}} = \frac{1}{\sqrt{A}} \ln\left(\frac{\sqrt{e^y+A} - \sqrt{A}}{\sqrt{e^y+A} + \sqrt{A}}\right) + C_1.$$

$$\therefore \ln\left(\frac{\sqrt{e^y+A} - \sqrt{A}}{\sqrt{e^y+A} + \sqrt{A}}\right) = \sqrt{2A}x + C$$

9. Notice how $(x(1-x))' = 1 - 2x$.

$$\begin{aligned} 0 &= x(1-x)y'' + 4y' + 2y \\ &= x(1-x)y'' + (x(1-x))'y' + (2x+3)y' + 2y \\ &= (x(1-x)y')' + ((2x+3)y)' \end{aligned}$$

$$\Rightarrow x(1-x)y' + (2x+3)y = A$$

Let us define the integrating factor λ:

$$\begin{aligned} \lambda &:= \exp\left(\int dx \frac{2x+3}{x(1-x)}\right) \\ &= \exp\left(\int dx \left(\frac{3}{x} - \frac{5}{x-1}\right)\right) \\ &= \exp\left(3\ln x - 5\ln(x-1)\right) \\ &= \frac{x^3}{(x-1)^5} \end{aligned}$$

$$\Rightarrow (\lambda y)' = \frac{A}{x(1-x)} \cdot \lambda = \frac{Ax^2}{(x-1)^6}$$

$$\begin{aligned}\Rightarrow \lambda y &= \int dx \frac{Ax^2}{(x-1)^6} \\ &= A\int dx\left(\frac{1}{(x-1)^4}+\frac{2}{(x-1)^5}+\frac{1}{(x-1)^6}\right) \\ &= -A\frac{10x^2-5x+1}{30(x-1)^5}+C_2\end{aligned}$$

$$\therefore y = C_1\frac{10x^2-5x+1}{x^3}+C_2\frac{(1-x)^5}{x^3}$$

10.
$$\frac{dy}{y^2}=\frac{1-x}{x^3}dx \Rightarrow -\frac{1}{y}=-\frac{1}{2x^2}+\frac{1}{x}+A$$

$$\therefore y=\frac{2x^2}{Cx^2+2x-1}$$

11. Recall that the Bernoulli equation takes the form $\frac{dy}{dx}+p(x)y=q(x)y^n$. We recognize that this equation is a Bernoulli equation with $p(x)=\frac{1}{x}$, $q(x)=-x^3e^x$, and $n=4$. Thus, we make the substitution $v:=y^{-3}$.

$$v'=-3y^{-4}y' \Rightarrow -\frac{v'}{3}+\frac{v}{x}=-x^3e^x \Rightarrow \frac{v'}{x^3}-\frac{3}{x^4}v=3e^x$$

$$\Rightarrow \left(\frac{v}{x^3}\right)'=3e^x \Rightarrow v=x^3(3e^x+C)$$

$$\therefore y=\left(x^3(3e^x+C)\right)^{-\frac{1}{3}}$$

12. We define the integrating factor λ:

$$\lambda := \exp\left(\int \frac{dx}{1+x^2}\right)=\exp\left(\tan^{-1}x\right)$$

$$\begin{gathered}\Rightarrow \left(\exp\left(\tan^{-1}x\right)y\right)'=\frac{1}{1+x^2}\cdot\tan^{-1}x\cdot\exp\left(\tan^{-1}x\right) \\ \Rightarrow \exp\left(\tan^{-1}x\right)y=\left(\tan^{-1}x-1\right)\exp\left(\tan^{-1}x\right)+C \\ \therefore y=C\exp\left(-\tan^{-1}x\right)+\tan^{-1}x-1\end{gathered}$$

13. This equation is dimension-consistent with $[y]=[x^{-1}]$. Thus, let us define $v:=xy$.

$$y=\frac{v}{x}\Rightarrow y'=\frac{xv'-v}{x^2}$$

$$\begin{aligned}\Rightarrow 0 &= (xv'-v)^2-2(v-4)(xv'-v)+v^2 \\ &= x^2v'^2-4(v-2)xv'+4v(v-2)\end{aligned}$$

$$\Rightarrow v' = \frac{2}{s}\left(v - 2 \pm \sqrt{4-2v}\right)$$

If $v = 2$, then we get $y = \frac{2}{x}$, which is the particular solution for this equation. To obtain the general solutions, let $u := 2 - v$.

$$u' = \frac{2}{x}\left(u \pm \sqrt{2u}\right)$$

$$\Rightarrow \frac{du}{u \pm \sqrt{2u}} = \frac{2}{x}dx$$

$$\Rightarrow 2\ln\left(\sqrt{u} \pm \sqrt{2}\right) = 2\ln x + C_1$$

$$\Rightarrow u = \left(Cx^2 \pm \sqrt{2}\right)^2 = 2 - xy$$

$$\therefore y = \frac{2 - \left(Cx^2 \pm \sqrt{2}\right)^2}{x}$$

14.

$$\begin{aligned} 6x &= \frac{y''}{y} - \frac{y'}{y^2} \\ &= \left(\frac{y'}{y}\right)' \end{aligned}$$

$$\Rightarrow \frac{y'}{y} = 3x^2 + C_1$$

$$\Rightarrow \ln y = x^3 + C_1 x + A$$

$$\therefore y = C_2 e^{x^3 + C_1 x}$$

15.

$$\frac{1}{x} = x^3\left(yy'' + y'^2\right) + (x^3)'yy' = \left(x^3 yy'\right)'$$

$$\Rightarrow x^3 yy' = \ln x + A \Rightarrow ydy = \frac{\ln x + A}{x^3}$$

Basic calculus yields $\int dx \frac{\ln x}{x^3} = -\frac{2\ln x + 1}{4x^2}$.

$$\Rightarrow \frac{y^2}{2} = -\frac{2\ln x + 1}{4x^2} - \frac{A}{2x^2} + B$$

$$\therefore y = \pm\sqrt{C_1 - \frac{\ln x + C_2}{x^2}}$$

16. This equation is dimension-consistent with $[y] = [x]$. Thus, we define $v := \frac{y}{x}$.

$$y = xv \Rightarrow y' = xv' + v \Rightarrow y'' = xv'' + 2v' \Rightarrow v'' + \frac{2}{x}v' - \frac{2}{x^2}v = \frac{1}{x^2}$$

First, observe that $v = -\frac{1}{2}$ is a solution of the equation. As this equation is linear in v, we thus seek the complementary solutions v_c. We shall take the ansatz of $v_c = x^m$.

$$m(m-1)x^{m-2} + 2mx^{m-2} - 2x^{m-2} = 0 \Rightarrow m^2 + m - 2 = 0$$

$$\Rightarrow m = 1, -2 \Rightarrow v_c = C_1 x + \frac{C_2}{x^2} \Rightarrow v = C_1 x + \frac{C_2}{x^2} - \frac{1}{2}$$

$$\therefore y = C_1 x^2 + \frac{C_2}{x} - \frac{x}{2}$$

17. Notice that this equation is linear in y.

(i) Complementary solutions:
Take $y_c = e^{mx}$.

$$m^3 - 2m^2 - m + 2 = 0 \Rightarrow m = \pm 1, 2$$

$$\Rightarrow y_c = C_1 e^{2x} + C_2 e^x + C_3 e^{-x}$$

(ii) Particular solution:
Suppose $y_p = A \sin x + B \cos x$.

$$\begin{aligned} \sin x &= (-A\cos x + B\sin x) - 2(-A\sin x - B\cos x) \\ &\quad - (A\cos x - B\cos x) + 2(A\sin x + B\cos x) \\ &= (4A + 2B)\sin x + (-2A + 4B)\cos x \end{aligned}$$

$$\Rightarrow A = \frac{1}{5}, B = \frac{1}{10}$$

$$\therefore y = \frac{1}{5}\sin x + \frac{1}{10}\cos x + C_1 e^{2x} + C_2 e^x + C_3 e^{-x}$$

18. Again, this equation is linear in y.

(i) Complementary solutions:
Take $y_c = e^{mx}$.

$$m^3 + 2m^2 + 1 = 0 \Rightarrow m = m_1, m_2, m_3$$

(I refuse to write down the exact solutions of this cubic equation.)

$$\Rightarrow y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$$

(ii) Particular solution:
Suppose $y_p = A\sin x + B\cos x$.

$$\begin{aligned}\sin x &= (-A\cos x + B\sin x) + 2(-A\sin x - B\cos x) + (A\sin x + B\cos x)\\ &= (-A+B)\sin x + (-A-B)\cos x\end{aligned}$$

$$\Rightarrow A = B = -\frac{1}{2}$$

$$\therefore y = -\frac{1}{2}\sin x - \frac{1}{2}\cos x + C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$$

19. Following the same steps as before, one can easily find the complementary solutions of

$$y_c = C_1 e^{-x} + C_2 e^{-2x}.$$

Thus, we wish to find one particular solution y_p. Let us try $y_p = f(x)e^{e^x}$.

$$y_p' = \left(\frac{f'}{f} + e^x\right) y_p$$

$$\begin{aligned}\Rightarrow y_p'' &= \left(\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 + e^x\right) y_p + \left(\frac{f'}{f} + e^x\right)^2 y_p\\ &= \left(\frac{f''}{f} + 2e^x\frac{f'}{f} + e^{2x} + e^x\right) y_p\end{aligned}$$

$$\Rightarrow \left(\frac{f''}{f} + (2e^x+3)\frac{f'}{f} + e^{2x} + 5e^x + 2\right) f e^{e^x} = e^{e^x}$$

$$\Rightarrow f'' + (2e^x+3)f' + (e^{2x} + 5e^x + 2)f = 1$$

One may use an ansatz of $f = ae^{-2x} + be^{-x} + c$ to find that $f = e^{-2x}$ is a possible solution. Thus, we have found a particular solution $y_p = e^{e^x - 2x}$.

$$\therefore y = e^{e^x - 2x} + C_1 e^{-x} + C_2 e^{-2x}$$

20.

$$\frac{y''}{(1+y'^2)^{3/2}} = \pm\frac{1}{a} \Rightarrow \frac{y'}{\sqrt{1+y'^2}} = \pm\frac{x+C_1}{a}$$

$$\Rightarrow y'^2 = \left(\frac{x+C_1}{a}\right)^2 (1+y'^2) \Rightarrow y' = \pm\frac{\left(\frac{x+C_1}{a}\right)^2}{\sqrt{1-\left(\frac{x+C_1}{a}\right)^2}}$$

$$\therefore y = \pm\frac{1}{2}\left(a\sin^{-1}\left(\frac{x+C_1}{a}\right) - (x+C_1)\sqrt{1-\left(\frac{x+C_1}{a}\right)^2}\right) + C_2$$

21. The complementary solutions are given by

$$\frac{dq_p}{dt} = -\frac{q_p}{RC} \Rightarrow q_p = Ae^{-\frac{t}{RC}}.$$

Suppose a particular solution takes the form $y_p = (at^3 + bt^2 + ct + d)e^{-\frac{t}{\tau}}$.

$$\frac{dq_p}{dt} = \left(-\frac{a}{\tau}t^3 + \left(3a - \frac{b}{\tau}\right)t^2 + \left(2b - \frac{c}{\tau}\right)t + \left(c - \frac{d}{\tau}\right)\right)e^{-\frac{t}{\tau}}$$

$$\begin{aligned}\Rightarrow \frac{V_0}{RC\tau^2}t^2 e^{-\frac{t}{\tau}} = \Bigg(&\left(\frac{1}{RC} - \frac{1}{\tau}\right)at^3 \\ &+ \left(3a + \left(\frac{1}{RC} - \frac{1}{\tau}\right)b\right)t^2 \\ &+ \left(2b + \left(\frac{1}{RC} - \frac{1}{\tau}\right)c\right)t \\ &+ \left(c + \left(\frac{1}{RC} - \frac{1}{\tau}\right)d\right)\Bigg)e^{-\frac{t}{\tau}}\end{aligned}$$

$$\Rightarrow q_p = \begin{cases} CV_0 \cdot \left(\frac{t^2}{\tau(\tau - RC)} - \frac{RCt}{(\tau - RC)^2} + \frac{\tau(RC)^2}{(\tau - RC)^2}\right) & (\text{if } \tau \neq RC) \\ \frac{1}{3}CV_0 \cdot \left(\frac{t}{\tau}\right)^3 e^{-\frac{t}{\tau}} & (\text{if } \tau = RC) \end{cases}$$

We can then add an appropriate complementary solution so as to satisfy the boundary condition $q(0) = 0$.

$$\therefore q = \begin{cases} CV_0 \cdot \frac{t}{\tau - RC}\left(\frac{t}{\tau} - \frac{RC}{\tau - RC}\right)e^{-\frac{t}{\tau}} & (\text{if } \tau \neq RC) \\ \frac{1}{3}CV_0 \cdot \left(\frac{t}{\tau}\right)^3 e^{-\frac{t}{\tau}} & (\text{if } \tau = RC) \end{cases}$$

22.

$$\frac{dN}{N_s - N} = \lambda dt \Rightarrow \int_0^N \frac{dN}{N_s - N} = \lambda \int_0^t dt \Rightarrow -\ln\left|\frac{N_s - N}{N_s}\right| = \lambda t$$

$$\therefore N = N_s\left(1 - e^{-\lambda t}\right)$$

23. The clever idea here is that we could make a variable change from x to u, where we could choose u so as to make the equation easier to solve.

$$\begin{aligned}0 &= A\frac{d^2y}{dx^2} + \frac{dA}{dx}\frac{dy}{dx} + \frac{y}{A} \\ &= A\frac{du}{dx}\frac{d}{du}\left(\frac{du}{dx}\frac{dy}{dy}\right) + \frac{dA}{dx}\frac{du}{dx}\frac{dy}{du} + \frac{y}{A} \\ &= A\left(\frac{du}{dx}\right)^2\frac{d^2y}{du^2} + \frac{dy}{du}\frac{d}{dx}\left(A\frac{dy}{dx}\right)\end{aligned}$$

Thus, we could define $\frac{du}{dx} = \frac{1}{A(x)}$ or $u = \int^x \frac{dt}{A(t)}$ to obtain

$$\frac{d^2y}{du^2} = -y \Rightarrow y(u) = C_1 \cos(u + C_2)$$

$$\therefore y(x) = C_1 \cos\left(\int^x \frac{dt}{A(t)} + C_2\right)$$

24. Suppose we may write $y = \lambda v$ where λ and v are to be determined later.

$$y' = \lambda v' + \lambda' v \Rightarrow y'' = \lambda v'' + 2\lambda' v' + \lambda'' v$$

$$\Rightarrow \lambda v'' + 2\left(\lambda' + \frac{\lambda}{x}\right) v' + \left(\lambda'' + \frac{2\lambda'}{x} + n^2\lambda\right) v = \frac{\sin \omega x}{x}$$

We may choose λ so as to make the coefficient of v' vanish, i.e.,

$$\lambda' = -\frac{\lambda}{x} \Rightarrow \lambda = \frac{1}{x}.$$

$$\Rightarrow v'' + n^2 v = \sin \omega x$$

(i) $n \neq \omega$:
The complementary solutions are

$$v_c = C_1 \cos(nx + C_2).$$

Using the ansatz $v_p = A \sin \omega x$ for a particular solution, one can easily find $A = \frac{1}{n^2 - \omega^2}$.

$$\Rightarrow v = \frac{\sin \omega x}{n^2 - \omega^2} + C_1 \sin(nx + C_2)$$

$$\therefore y = \frac{1}{n^2 - \omega^2} \frac{\sin \omega x}{x} + C_1 \frac{\sin(nx + C_2)}{x}$$

(ii) $n = \omega$:
The complementary solutions are

$$v_c = C_1 \cos(\omega x + C_2).$$

Using the ansatz $v_p = Ax \sin \omega x + Bx \cos \omega x$ for a particular solution, one can easily find $A = -\frac{1}{2\omega}$ and $B = 0$.

$$\Rightarrow v = -\frac{1}{2\omega} x \cos \omega x + C_1 \cos(\omega x + C_2)$$

$$\therefore y = -\frac{1}{2\omega} \cos \omega x + C_1 \frac{\sin(\omega x + C_2)}{x}$$

25. From the hint, we may consider $y = xv$, where $v = 1$ is a solution if the right hand side were zero.

$$y' = xv' + v \Rightarrow y'' = xv'' + 2v'$$
$$\Rightarrow x(1-x)v'' + \left(1 + (1-x)^2\right) v' = (1-x)^2$$
$$\Rightarrow v'' + \left(\frac{2}{x} + \frac{1}{1-x} - 1\right) v' = \frac{1}{x} - 1$$

We can thus define the integration factor as

$$\lambda := \exp\left(\int dx \left(\frac{2}{x} + \frac{1}{1-x} - 1\right)\right) = \frac{x^2 e^{-x}}{1-x}.$$

$$\Rightarrow (\lambda v')' = \lambda \left(\frac{1}{x} - 1\right) = xe^{-x}$$
$$\Rightarrow \lambda v' = -(x+1)e^{-x} + A$$
$$\Rightarrow v' = 1 - \frac{1}{x^2} + A\frac{1-x}{x^2}e^x$$
$$\Rightarrow v = x + \frac{1}{x} - A\frac{e^x}{x} + B$$
$$\therefore y = x^2 + 1 + C_1 e^x + C_2 x$$

26. While the use of Dirac delta "function" must be dealt with caution, we are physicists so we shall simply accept that they exist.

$$\begin{aligned} f(x) = y'' + py' + q &= \int_a^b dx' f(x') \left(\frac{\partial^2}{\partial x^2} + p(x)\frac{\partial}{\partial x} + q(x)\right) G(x, x') \\ &= \int_a^b dx' f(x') \delta(x - x') \end{aligned}$$
$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + p(x)\frac{\partial}{\partial x} + q(x)\right) G(x, x') = \delta(x - x')$$

From this, we see that $G(x, x')$ must separately equal zero for both $x < x'$ and $x > x'$. We thus write

$$G(x, x') = \begin{cases} y_1(x)\alpha(x') + y_2(x)\beta(x') & (x < x') \\ y_1(x)\gamma(x') + y_2(x)\zeta(x') & (x > x') \end{cases}.$$

From the definition of delta functions, we find that:

$$\begin{aligned}
1 &= \int_{x'-\epsilon}^{x'+\epsilon} dx\delta(x-x') \\
&= \int_{x'-\epsilon}^{x'+\epsilon} dx \left(\frac{\partial^2}{\partial x^2} + p(x)\frac{\partial}{\partial x} + q(x)\right) G(x,x') \\
&\approx \left[\left(\frac{\partial}{\partial x} + p(x)\right) G(x,x')\right]_{x'-\epsilon}^{x'+\epsilon} - \int_{x-\epsilon}^{x+\epsilon} dxp'(x)G(x,x') \\
&\approx \left[\frac{\partial}{\partial x}G(x,x')\right]_{x'-\epsilon}^{x'+\epsilon} \\
&\approx \lim_{x\to x'+}\frac{\partial}{\partial x}G(x,x') - \lim_{x\to x'-}\frac{\partial}{\partial x}G(x,x')
\end{aligned}$$

where we take infinitesimally small ϵ.

The boundary conditions for y imply

$$\begin{gathered}
y(a) = y(b) = 0 \\
\Rightarrow G(a,x') = G(b,x') = 0 \\
\Rightarrow y_1(a)\alpha(x') + y_2(a)\beta(x') = y_1(b)\alpha(x') + y_2(b)\beta(x') = 0 \\
\Rightarrow \beta(x') = \gamma(x') = 0.
\end{gathered}$$

The derivative condition, in turn, yields

$$y_2'(x')\zeta(x') - y_1'(x')\alpha(x') = 1.$$

As any such choice of $\zeta(x')$ and $\alpha(x')$ leads to a valid Green's function, we may arbitrarily choose

$$\alpha(x') = y_2'(x'),\ \zeta(x') = y_1'(x') + \frac{1}{y_2'(x')}$$

$$\Rightarrow G(x,x') = \begin{cases} y_1(x)y_2'(x') & (x<x') \\ y_2(x)y_1'(x') + \frac{y_2(x)}{y_2'(x')} & (x>x') \end{cases}.$$

$$\therefore y(x) = y_1(x)\int_x^b dx' f(x')y_2'(x') + y_2(x)\int_a^x f(x')\left(y_1'(x') + \frac{1}{y_2'(x')}\right)$$

For the given example $y'' + k^2y = f(x)$, we may take

$$y_1(x) = \sin k(x-a),\ y_2(x) = \sin k(b-x)$$

provided that $b-a$ is not an integer multiple of the period $\frac{2\pi}{k}$. This leads to

$$G(x,x') = \begin{cases} -k\sin k(x-a)\cos k(b-x) & (x<x') \\ \sin k(b-x)\left(k\cos k(x'-a) - \frac{1}{k\cos k(b-x')}\right) & (x>x') \end{cases}$$

$$\Rightarrow y = -\,k \sin k(x-a) \int_x^b dx' f(x') \cos k(b-x')$$
$$+ \sin k(b-x) \int_a^x dx' f(x') \left(k \cos k(x'-a) - \frac{1}{k \cos k(b-x')} \right).$$

One may verify that this solution satisfies both the boundary conditions and the given differential equation.

27. The complementary solutions satisfy

$$y_c'' + \frac{3}{x^2} y_c = 0.$$

Suppose $y_c = x^m$.

$$m^2 - m + 3 = 0 \Rightarrow \left(m - \frac{1}{2}\right)^2 = -\frac{11}{4} \Rightarrow m = \frac{1 \pm \sqrt{11} i}{2}$$

$$\begin{aligned} \Rightarrow y_c &= A x^{\frac{1+\sqrt{11}i}{2}} + B x^{\frac{1-\sqrt{11}i}{2}} \\ &= C_1 \sqrt{x} \cos\left(\frac{\sqrt{11}}{2} \ln x\right) + C_2 \sqrt{x} \sin\left(\frac{\sqrt{11}}{2} \ln x\right) \end{aligned}$$

We then look for two separate particular solutions:

$$y_{p1}'' + \frac{3}{x^2} y_{p1} = x^2 \text{ and } y_{p2}'' + \frac{3}{x^2} y_{p2} = \frac{1}{x}.$$

Using the monomial ansatz separately, that is, assuming $y_p = Ax^m$, one easily finds

$$y_{p1} = \frac{x^4}{15} \text{ and } y_{p2} = \frac{x}{3}.$$

$$\therefore y = \frac{x^4}{15} + \frac{x}{3} + C_1 \sqrt{x} \cos\left(\frac{\sqrt{11}}{2} \ln x\right) + C_2 \sqrt{x} \sin\left(\frac{\sqrt{11}}{2} \ln x\right)$$

28. Let us first look for asymptotic behaviors.

(i) $x \to 0+$ Suppose $y = O(x^m)$.

$$m(m-1) + 2m - l(l+1) = 0 \Rightarrow m = l \text{ or } -l-1$$

To vanish near the origin, we must have $y = O(x^l)$.

(ii) $x \to \infty$

$$\frac{d^2 y}{dx^2} \approx -Ky \Rightarrow y \approx e^{\pm\sqrt{-K}x}$$

To vanish at infinity, we must have $y = x^{-\sqrt{-K}x}$ with $K < 0$.

We can thus write $y = f(x)x^l e^{-\sqrt{-K}x}$ with f regular everywhere.

$$y' = \left(\frac{f'}{f} + \frac{l}{x} - \sqrt{-K}\right) y$$

$$\begin{aligned}\Rightarrow y'' &= \left(\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 - \frac{l}{x^2}\right) y + \left(\frac{f'}{f} + \frac{l}{x} - \sqrt{-K}\right)^2 y \\ &= \left(\frac{f''}{f} + 2\left(\frac{l}{x} - \sqrt{-K}\right)\frac{f'}{f} + \frac{l(l-1)}{x^2} - \frac{2l\sqrt{-K}}{x} - K\right) y\end{aligned}$$

Tedious algebra yields

$$xf'' + 2(l + 1 - \sqrt{-K})f' - 2((l+1)\sqrt{-K} - 1)f = 0.$$

Let us write $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Substitution and tedious algebra yield

$$\sum_{n=0}^{\infty}((n+1)(n+2l+2)a_{n+1} - (\sqrt{-K}n + 2((l+1)\sqrt{-K} - 1))a_n)x^n = 0$$

$$\Rightarrow (\forall n)\ (n+1)(n+2l+2)a_{n+1} = (\sqrt{-K}n + 2((l+1)\sqrt{-K} - 1))a_n.$$

If $\{a_n\}$ never terminated, this would mean that $a_n \approx \frac{(\sqrt{-K})^n}{n!}$ for large values of n, leading to $f(x) \approx e^{\sqrt{-K}x}$ and y diverging to infinity as x increases.

$$\Rightarrow \sqrt{-K_{nl}}n + 2((l+1)\sqrt{-K_{nl}} - 1) = 0$$

$$\therefore K_{nl} = -\frac{4}{(n+2l+2)^2}\ (n = 0, 1, 2, \dots)$$

29. For very large x,

$$y'' \approx \frac{y}{4} \Rightarrow y \approx e^{\pm\frac{x}{2}}.$$

Thus, for y to vanish at infinity, we must have $y \approx e^{-x/2}$. As y must also vanish at the origin, let us write $y = f(x)xe^{-x/2}$ with $f(x)$ regular everywhere.

$$y' = \left(\frac{f'}{f} + \frac{1}{x} - \frac{1}{2}\right) y$$

$$\begin{aligned}\Rightarrow y'' &= \left(\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 - \frac{1}{x^2}\right) y + \left(\frac{f'}{f} + \frac{1}{x} - \frac{1}{2}\right)^2 y \\ &= \left(\frac{f''}{f} + \left(\frac{2}{x} - 1\right)\frac{f'}{f} - \frac{1}{x} + \frac{1}{4}\right) y\end{aligned}$$

$$\Rightarrow \frac{f''}{f} + \left(\frac{2}{x} - 1\right)\frac{f'}{f} - \frac{K+1}{x} = 0$$

Let us write $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Substitution and tedious algebra yield

$$2a_1 - (K+1)a_0 + \sum_{n=1}^{\infty}((n+1)(n+2)a_{n+1} - (n+K+1)a_n)x^n = 0$$

$$\Rightarrow (\forall n)\ (n+1)(n+2)a_{n+1} = (n+K+1)a_n.$$

If $\{a_n\}$ never terminated, this would mean that $a_n \approx \frac{1}{n!}$ for large values of n, leading to $f(x) \approx e^x$ and y diverging to infinity as x increases.

$$\therefore K_n = -n\ (n = 1, 2, 3, \dots)$$

30. I assume that there must exist a "nontrivial" solution, since $y = 0$ is clearly a solution of the equation for any value of k. For large values of x,

$$y'' - 2y' - 3y \approx 0 \Rightarrow y \approx Ae^{-x} + Be^{3x}.$$

Thus, to be bounded everywhere, we must have $y \approx e^{-x}$. We then write $y = f(x)e^{-x}$ with f analytic everywhere, including the origin.

$$y' = \left(\frac{f'}{f} - 1\right) y \Rightarrow y'' = \left(\frac{f''}{f} - \frac{f'}{f} + \left(\frac{f'}{f} - 1\right)^2\right) y = \left(\frac{f''}{f} - 2\frac{f'}{f} + 1\right) y$$

Substitution yields

$$xf'' - 4xf' + kf = 0.$$

Let us write $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Substitution and tedious algebra yield

$$\sum_{n=0}^{\infty}(n(n+1)a_{n+1} - (4n-k)a_n)x^n = 0$$

$$\Rightarrow (\forall n)\ n(n+1)a_{n+1} = (4n-k)a_n.$$

If $\{a_n\}$ never terminated, this would mean that $a_n \approx \frac{4^n}{n!}$ for large values of n, leading to $f(x) \approx e^{4x}$ and y diverging to infinity as x increases.

$$\therefore k_n = 4n\ (n = 0, 1, 2, \dots)$$

31. For large values of x,

$$y'' \approx y \Rightarrow y \approx e^{\pm x}.$$

Thus, to be bounded everywhere, we must have $y \approx e^{-x}$. We then write $y = f(x)e^{-x}$ with f analytic everywhere, with $f(0) = 1$.

$$y' = \left(\frac{f'}{f} - 1\right) y \Rightarrow y'' = \left(\frac{f''}{f} - \frac{f'}{f} + \left(\frac{f'}{f} - 1\right)^2\right) y = \left(\frac{f''}{f} - 2\frac{f'}{f} + 1\right) y$$

Substitution yields

$$xf'' - 2(x-1)f' + (E-2)f = 0.$$

Let us write $f(x) = \sum_{n=0}^{\infty} a_n x^n$, with $a_0 = 1$. Substitution and tedious algebra yield

$$\sum_{n=0}^{\infty}((n+1)(n+2)a_{n+1} - (2n+2-E)a_n)x^n = 0$$

$$\Rightarrow (\forall n)\ (n+1)(n+2)a_{n+1} = (2n+2-E)a_n.$$

If $\{a_n\}$ never terminated, this would mean that $a_n \approx \frac{2^n}{n!}$ for large values of n, leading to $f(x) \approx e^{2x}$ and y diverging to infinity as x increases.

$$\therefore E_n = 2n\ (n = 1, 2, 3, \dots)$$

32. Recall that

$$\frac{c_{r+2}}{c_r} = \frac{(r+m-n)(r+m+n+1)}{(r+1)(r+2)} \text{ and } v(x) = \sum_{r=0}^{\infty} c_r x^r$$

where the fraction is well-defined since we are assuming that $\{c_r\}$ never terminates. Notice how

$$\frac{c_{r+2}}{c_r} \approx \frac{r+2m+1}{r+3}$$

for large values of r.

On the other hand, the definition of binomial coefficients

$$\binom{-m}{r} := \frac{-m \cdot (-m-1) \cdot \dots \cdot (-m-r+1)}{r!}$$

naturally yields

$$\begin{aligned}\frac{\binom{-m}{r+2}}{\binom{-m}{r}} &= \frac{(-m-r)(-m-r-1)}{(r+1)(r+2)} \\ &= \frac{(r+m)(r+m+1)}{(r+1)(r+2)} \\ &\approx \frac{r+2m+1}{r+3}\end{aligned}$$

for large values of r.

Therefore, c_r behaves like $\binom{-m}{r}$ as r grows without bound, and consequently,

$$v(x) \approx \left(1 - x^2\right)^{-m}.$$

33. Suppose $y = J_0(x) \ln x + \sum_{n=0}^{\infty} a_n x^n$ as prompted by the problem. Substitution and tedious algebra yield

$$2xJ_0'(x) + a_1 x + \sum_{n=2}^{\infty}(n^2 a_n + a_{n-2}).$$

Let us write $J_0(x) = \sum_{n=0}^{\infty} b_{2n}x^{2n}$ since we know that J_0 is even.

$$xJ_0'(x) = \sum_{n=0}^{\infty} 2nb_{2n}x^{2n}$$

$$\Rightarrow a_1 x + \sum_{n=1}^{\infty} \Big(\left(4nb_{2n} + 4n^2 a_{2n} + a_{2n-2}\right) x^{2n} + \left((2n+1)^2 a_{2n+1} + a_{2n-1}\right) x^{2n+1} \Big) = 0$$

This immediately tells us that all odd coefficients $a_1 = a_3 = \cdots = 0$.

Bessel's equation is linear, so addition of any multiple of J_0 to y also yields a valid solution. In other words, any variant $a'_{2n} := a_{2n} + \lambda b_{2n}$ also satisfies the above equation, for real parameter λ. Thus, we may choose $a_0 = 0$.

Using $b_0 = 1,\ b_2 = -\frac{1}{4},\ b_4 = \frac{1}{64},\ b_6 = -\frac{b_4}{36} = -\frac{1}{2304}$, and the recursion relation $a_2 n = -\frac{a_{2n-2}}{4n^2} - \frac{b_{2n}}{n}$ from above, we can iteratively calculate the values of a_{2n}.

$$\therefore y = J_0(x)\ln x + \frac{x^2}{4} - \frac{3}{128}x^4 + \frac{5}{4608}x^6 - \cdots$$

34. (i) $y(0) = 1$
We may write $y = 1 + \sum_{n=1}^{\infty} a_n x^n$. Substitution and tedious algebra yields

$$2a_1 - 2 + \sum_{n=1}^{\infty}(n+2)((n+1)a_{n+1} - a_n)x^n = 0.$$

This directly leads to $a_n = \frac{1}{n!}$, and thus,

$$y = e^x.$$

(ii) $y = \frac{1}{x} + A\ln x + B$
Substitution of the above form into the differential equation and patiently performing algebraic manipulations yields

$$-\frac{1}{x} + (xA'' + (2-x)A' - 2A)\ln x + 2A' + \left(\frac{1}{x} - 1\right)A + xB''(2-x)B' - 2B = 0.$$

Read the problem carefully: we only need to "give" two solutions, not find a general form for all solutions. Thus, we could choose the oddly specific solution of $A = e^x$ to make the coefficient of the $\ln x$ term vanish. Substituting this into the above equation yields

$$xB'' + (2-x)B' - 2B + e^x + \frac{e^x - 1}{x} = 0.$$

Expanding $B = \sum_{n=0}^{\infty} b_n x^n$, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, and $\frac{e^x-1}{x} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$, one can write

$$\sum_{n=0}^{\infty} \left((n+1)(n+2)b_{n+1} - (n+2)b_n + \frac{n+2}{(n+1)!} \right) = 0$$

$$\Rightarrow b_{n+1} = \frac{b_n}{n+1} - \frac{1}{(n+1)\cdot(n+1)!}.$$

Again, B may arbitrarily chosen, so let us choose $b_0 = 0$.

$$\therefore y = \frac{1}{x} + e^x \ln x - x - \frac{3}{4}x^2 - \frac{11}{36}x^3 - \cdots$$