My Solutions for “Mathematical Methods of Physics (Second Edition)” by J. Mathews, R. L. Walker

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0 Introduction

This document is an archive of my solutions to J. Mathews and R. L. Walker’s “Mathematical Methods of Physics” textbook. None of the solutions have been verified by anyone other than myself, whom I do not consider a reliable source. Hence, it is strongly advised against to use this solution for any application where accuracy matters, especially for assignments and any academic work, not to mention the ethical implications of such actions where it could be considered as cheating. However, I have not yet been able to find any other solutions. Please use this manual as just a suggestion. If you spot any mistakes, please report them to the Github repository. [1]

1 Chapter 1

1. We first acknowledge that $y = 0$ is a solution, so we seek solutions that are not identically zero.

Let $y = xv$.

$$y' = v + xv' \Rightarrow x^2(v + xv') + x^2v^2 = x \cdot xv \cdot (v + xv')$$

$$\Rightarrow v + xv' = xvv' \Rightarrow \left(1 - \frac{1}{v}\right) dv = \frac{1}{x}dx \Rightarrow v - \ln v = \ln x + C$$

$$\therefore \frac{y}{x} - \ln y = C$$

2.

$$\frac{y}{\sqrt{1+y^2}}dy = \frac{x}{\sqrt{1+x^2}}dx$$

$$\therefore \sqrt{1+y^2} = \sqrt{1+x^2} + C$$

[1] https://github.com/pingpingy1/Mathews-MathPhys-Sol

3. Let $v = x + y$.

$$v' = 1 + y' \Rightarrow v' - 1 = \frac{a^2}{v^2} \Rightarrow \frac{v^2}{a^2 + v^2} dv = dx$$

$$x + C = \int dv \left(1 - \frac{a^2}{a^2 + v^2}\right) = v - a\tan^{-1}\left(\frac{v}{a}\right) = x + y - a\tan^{-1}\left(\frac{x+y}{a}\right)$$

$$\therefore y - a\tan^{-1}\left(\frac{x+y}{a}\right) = C$$

4. We first seek the complementary solutions.

$$y_c' + y_c \cos x = 0 \Rightarrow \frac{dy_c}{y_c} + \cos x dx = 0 \Rightarrow y_c = Ce^{-\sin x}$$

For the particular solution, we first observe that $\frac{1}{2}\sin 2x = \sin x \cos x$. Thus, we shall try the ansatz $y_p = \sin x + A$.

$$\cos x + (\sin x + A)\cos x = \sin x \cos x \Rightarrow A = -1$$

$$\therefore y = Ce^{-\sin x} + \cos x - 1$$

5.

$$\left(1 + x^2\right) y' = xy(y+1) \Rightarrow \frac{dy}{y(y+1)} = \frac{x}{1 - x^2} dx$$

$$\Rightarrow \ln\left(\frac{y}{y+1}\right) = -\frac{1}{2}\ln\left(1 - x^2\right) + C_1$$

$$\Rightarrow \frac{y+1}{y} = e^{-C_1}\sqrt{1 - x^2} = C\sqrt{1 - x^2}$$

$$\therefore y = \frac{1}{C\sqrt{1 - x^2} - 1}$$

6. We first take note that the equation is dimension-consistent with $[y] = [x^{-2}]$. Thus, we define $v = x^2 y$ or $y = \frac{v}{x^2}$.

$$y' = \frac{v'}{x^2} - \frac{2v}{x^3} \Rightarrow 2xv' - 4v = 1 + \sqrt{1 + 4v}$$

$$\Rightarrow \frac{v'}{1 + 4v + \sqrt{1 + 4v}} = \frac{1}{2x}$$

$$\Rightarrow \left(\frac{1}{\sqrt{1 + 4v}} - \frac{1}{\sqrt{1 + 4v} + 1}\right) dv = \frac{dx}{2x}$$

Basic calculus yields $\int \frac{dv}{\sqrt{1+4v}+1} = \frac{1}{2}\left(\sqrt{1 + 4v} - \ln\left(1 + \sqrt{1 + 4v}\right)\right) + C$. Hence,

$$\frac{1}{2}\sqrt{1 + 4v} - \frac{1}{2}\left(\sqrt{1 + 4v} - \ln\left(1 + \sqrt{1 + 4v}\right)\right) = \frac{1}{2}\ln x + C_1$$

$$\Rightarrow 1 + \sqrt{1 + 4v} = e^{2C_1} x = Cx$$

$$\therefore \sqrt{1 + 4x^2 y} = Cx - 1$$

7. Let $v' := y'$.

$$v' + v^2 + 1 = 0 \Rightarrow \frac{dv}{v^2+1} = -1 \Rightarrow v = \tan(C_1 - x)$$

$$\therefore y = \int dx v = \ln\left(\cos(C_1 - x)\right) + C_2$$

8.

$$y'y'' = e^y y' \Rightarrow \int dy' y' = \int dy e^y \Rightarrow \frac{1}{2}y'^2 = e^y + C_1$$

$$\Rightarrow \frac{dy}{\sqrt{e^y + C_1}} = \sqrt{2}dx$$

Basic calculus yields

$$\int \frac{dy}{\sqrt{e^y + C_1}} = \frac{1}{\sqrt{C_1}} \ln\left(\frac{\sqrt{e^y + C_1} - \sqrt{A}}{\sqrt{e^y + C_1} + \sqrt{C_1}}\right) + A.$$

$$\therefore \ln\left(\frac{\sqrt{e^y + C_1} - \sqrt{C_1}}{\sqrt{e^y + C_1} + \sqrt{C_1}}\right) = \sqrt{2C_1}x + C_2$$

9. Notice how $(x(1-x))' = 1 - 2x$.

$$\begin{aligned}
0 &= x(1-x)y'' + 4y' + 2y \\
&= x(1-x)y'' + (x(1-x))'y' + (2x+3)y' + 2y \\
&= (x(1-x)y')' + ((2x+3)y)' \\
&\Rightarrow x(1-x)y' + (2x+3)y = A
\end{aligned}$$

Let us define the integrating factor λ:

$$\begin{aligned}
\lambda &= \exp\left(\int dx \frac{2x+3}{x(1-x)}\right) \\
&= \exp\left(\int dx \left(\frac{3}{x} - \frac{5}{x-1}\right)\right) \\
&= \exp\left(3\ln x - 5\ln(x-1)\right) \\
&= \frac{x^3}{(x-1)^5}
\end{aligned}$$

$$\Rightarrow (\lambda y)' = \frac{A}{x(1-x)} \cdot \lambda = \frac{Ax^2}{(x-1)^6}$$

$$\begin{aligned}
\Rightarrow \lambda y &= \int dx \frac{Ax^2}{(x-1)^6} \\
&= A\int dx \left(\frac{1}{(x-1)^4} + \frac{2}{(x-1)^5} + \frac{1}{(x-1)^6}\right) \\
&= -A\frac{10x^2 - 5x + 1}{30(x-1)^5} + C_2
\end{aligned}$$

$$\therefore y = C_1 \frac{10x^2 - 5x + 1}{x^3} + C_2 \frac{(1-x)^5}{x^3}$$

10.

$$\frac{dy}{y^2} = \frac{1-x}{x^3} dx \Rightarrow -\frac{1}{y} = -\frac{1}{2x^2} + \frac{1}{x} + A$$

$$\therefore y = \frac{2x^2}{Cx^2 + 2x - 1}$$

11. Recall that the Bernoulli equation takes the form $\frac{dy}{dx} + p(x)y = q(x)y^n$. We recognize that this equation is a Bernoulli equation with $p(x) = \frac{1}{x}$, $q(x) = -x^3 e^x$, and $n = 4$. Thus, we make the substitution $v := y^{-3}$.

$$v' = -3y^{-4}y' \Rightarrow -\frac{v'}{3} + \frac{v}{x} = -x^3 e^x \Rightarrow \frac{v'}{x^3} - \frac{3}{x^4} v = 3e^x$$

$$\Rightarrow \left(\frac{v}{x^3}\right)' = 3e^x \Rightarrow v = x^3(3e^x + C)$$

$$\therefore y = \left(x^3(3e^x + C)\right)^{-\frac{1}{3}}$$

12. We define the integrating factor λ:

$$\lambda = \exp\left(\int \frac{dx}{1+x^2}\right) = \exp\left(\tan^{-1} x\right)$$

$$\Rightarrow \left(\exp\left(\tan^{-1} x\right) y\right)' = \frac{1}{1+x^2} \cdot \tan^{-1} x \cdot \exp\left(\tan^{-1} x\right)$$

$$\Rightarrow \exp\left(\tan^{-1} x\right) y = \left(\tan^{-1} x - 1\right) \exp\left(\tan^{-1} x\right) + C$$

$$\therefore y = C \exp\left(-\tan^{-1} x\right) + \tan^{-1} x - 1$$

13. This equation is dimension-consistent with $[y] = [x^{-1}]$. Thus, let us define $v := xy$.

$$y = \frac{v}{x} \Rightarrow y' = \frac{xv' - v}{x^2}$$

$$\begin{aligned} \Rightarrow 0 &= (xv' - v)^2 - 2(v-4)(xv' - v) + v^2 \\ &= x^2 v'^2 - 4(v-2)xv' + 4v(v-2) \end{aligned}$$

$$\Rightarrow v' = \frac{2}{s}\left(v - 2 \pm \sqrt{4 - 2v}\right)$$

If $v = 2$, then we get $y = \frac{2}{x}$, which is the particular solution for this equation. To obtain the general solutions, let $u = 2 - v$.

$$u' = \frac{2}{x}\left(u \pm \sqrt{2u}\right)$$

$$\Rightarrow \frac{du}{u \pm \sqrt{2u}} = \frac{2}{x}dx$$
$$\Rightarrow 2\ln\left(\sqrt{u} \pm \sqrt{2}\right) = 2\ln x + A$$
$$\Rightarrow u = \left(Cx^2 \pm \sqrt{2}\right)^2 = 2 - xy$$
$$\therefore y = \frac{2 - \left(Cx^2 \pm \sqrt{2}\right)^2}{x}$$

14.

$$6x = \frac{y''}{y} - \frac{y'}{y^2}$$
$$= \left(\frac{y'}{y}\right)'$$
$$\Rightarrow \frac{y'}{y} = 3x^2 + C_1$$
$$\Rightarrow \ln y = x^3 + C_1 x + A$$
$$\therefore y = C_2 e^{x^3 + C_1 x}$$

15.

$$\frac{1}{x} = x^3\left(yy'' + y'^2\right) + (x^3)'yy' = \left(x^3 yy'\right)'$$
$$\Rightarrow x^3 yy' = \ln x + A \Rightarrow ydy = \frac{\ln x + A}{x^3}$$

Basic calculus yields $\int dx \frac{\ln x}{x^3} = -\frac{2\ln x + 1}{4x^2}$.

$$\Rightarrow \frac{y^2}{2} = -\frac{2\ln x + 1}{4x^2} - \frac{A}{2x^2} + B$$
$$\therefore y = \pm\sqrt{C_1 - \frac{\ln x + C_2}{x^2}}$$

16. This equation is dimension-consistent with $[y] = [x]$. Thus, we define $v := \frac{y}{x}$.

$$y = xv \Rightarrow y' = xv' + v \Rightarrow y'' = xv'' + 2v' \Rightarrow v'' + \frac{2}{x}v' - \frac{2}{x^2}v = \frac{1}{x^2}$$

First, observe that $v = -\frac{1}{2}$ is a solution of the equation. As this equation is linear in v, we thus seek the complementary solutions v_c. We shall take the ansatz of $v_c = x^m$.

$$m(m-1)x^{m-2} + 2mx^{m-2} - 2x^{m-2} = 0 \Rightarrow m^2 + m - 2 = 0$$
$$\Rightarrow m = 1, -2 \Rightarrow v_c = C_1 x + \frac{C_2}{x^2} \Rightarrow v = C_1 x + \frac{C_2}{x^2} - \frac{1}{2}$$
$$\therefore y = C_1 x^2 + \frac{C_2}{x} - \frac{x}{2}$$

17. Notice that this equation is linear in y.

(i) Complementary solutions:
Take $y_c = e^{mx}$.

$$m^3 - 2m^2 - m + 2 = 0 \Rightarrow m = \pm 1, 2$$

$$\Rightarrow y_c = C_1 e^{2x} + C_2 e^{x} + C_3 e^{-x}$$

(ii) Particular solution:
Suppose $y_p = A\sin x + B\cos x$.

$$\begin{aligned} \sin x &= (-A\cos x + B\sin x) - 2(-A\sin x - B\cos x) \\ &\quad - (A\cos x - B\cos x) + 2(A\sin x + B\cos x) \\ &= (4A + 2B)\sin x + (-2A + 4B)\cos x \end{aligned}$$

$$\Rightarrow A = \frac{1}{5}, B = \frac{1}{10}$$

$$\therefore y = \frac{1}{5}\sin x + \frac{1}{10}\cos x + C_1 e^{2x} + C_2 e^{x} + C_3 e^{-x}$$

18. Again, this equation is linear in y.

(i) Complementary solutions:
Take $y_c = e^{mx}$.

$$m^3 + 2m^2 + 1 = 0 \Rightarrow m = m_1, m_2, m_3$$

(I refuse to write down the exact solutions of this cubic equation.)

$$\Rightarrow y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$$

(ii) Particular solution:
Suppose $y_p = A\sin x + B\cos x$.

$$\begin{aligned} \sin x &= (-A\cos x + B\sin x) + 2(-A\sin x - B\cos x) + (A\sin x + B\cos x) \\ &= (-A + B)\sin x + (-A - B)\cos x \end{aligned}$$

$$\Rightarrow A = B = -\frac{1}{2}$$

$$\therefore y = -\frac{1}{2}\sin x - \frac{1}{2}\cos x + C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$$

19. Following the same steps as before, one can easily find the complementary solutions of

$$y_c = C_1 e^{-x} + C_2 e^{-2x}.$$

Thus, we wish to find one particular solution y_p. Let us try $y_p = f(x)e^{e^x}$.

$$y_p' = \left(\frac{f'}{f} + e^x\right) y_p$$

$$\Rightarrow y_p'' = \left(\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 + e^x\right) y_p + \left(\frac{f'}{f} + e^x\right)^2 y_p$$
$$= \left(\frac{f''}{f} + 2e^x \frac{f'}{f} + e^{2x} + e^x\right) y_p$$

$$\Rightarrow \left(\frac{f''}{f} + (2e^x + 3)\frac{f'}{f} + e^{2x} + 5e^x + 2\right) f e^{e^x} = e^{e^x}$$
$$\Rightarrow f'' + (2e^x + 3)f' + (e^{2x} + 5e^x + 2)f = 1$$

One may use an ansatz of $f = ae^{-2x} + be^{-x} + c$ to find that $f = e^{-2x}$ is a possible solution. Thus, we have found a particular solution $y_p = e^{e^x - 2x}$.

$$\therefore y = e^{e^x - 2x} + C_1 e^{-x} + C_2 e^{-2x}$$

20.
$$\frac{y''}{(1 + y'^2)^{3/2}} = \pm\frac{1}{a} \Rightarrow \frac{y'}{\sqrt{1 + y'^2}} = \pm\frac{x + C_1}{a}$$
$$\Rightarrow y'^2 = \left(\frac{x + C_1}{a}\right)^2 (1 + y'^2) \Rightarrow y' = \pm\frac{\left(\frac{x+C_1}{a}\right)^2}{\sqrt{1 - \left(\frac{x+C_1}{a}\right)^2}}$$
$$\therefore y = \pm\frac{1}{2}\left(a \sin^{-1}\left(\frac{x + C_1}{a}\right) - (x + C_1)\sqrt{1 - \left(\frac{x + C_1}{a}\right)^2}\right) + C_2$$

21. The complementary solutions are given by

$$\frac{dq_p}{dt} = -\frac{q_p}{RC} \Rightarrow q_p = Ae^{-\frac{t}{RC}}.$$

Suppose a particular solution takes the form $y_p = (at^3 + bt^2 + ct + d)e^{-\frac{t}{\tau}}$.

$$\frac{dq_p}{dt} = \left(-\frac{a}{\tau}t^3 + \left(3a - \frac{b}{\tau}\right)t^2 + \left(2b - \frac{c}{\tau}\right)t + \left(c - \frac{d}{\tau}\right)\right) e^{-\frac{t}{\tau}}$$

$$\begin{aligned}\Rightarrow \frac{V_0}{RC\tau^2} t^2 e^{-\frac{t}{\tau}} = \Bigg(&\left(\frac{1}{RC} - \frac{1}{\tau}\right) at^3 \\ &+ \left(3a + \left(\frac{1}{RC} - \frac{1}{\tau}\right) b\right) t^2 \\ &+ \left(2b + \left(\frac{1}{RC} - \frac{1}{\tau}\right) c\right) t \\ &+ \left(c + \left(\frac{1}{RC} - \frac{1}{\tau}\right) d\right)\Bigg) e^{-\frac{t}{\tau}}\end{aligned}$$

$$\Rightarrow q_p = \begin{cases} CV_0 \cdot \left(\frac{t^2}{\tau(\tau - RC)} - \frac{RCt}{(\tau - RC)^2} + \frac{\tau (RC)^2}{(\tau - RC)^2} \right) & (\text{if } \tau \neq RC) \\ \frac{1}{3} CV_0 \cdot \left(\frac{t}{\tau}\right)^3 e^{-\frac{t}{\tau}} & (\text{if } \tau = RC) \end{cases}$$

We can then add an appropriate complementary solution so as to satisfy the boundary condition $q(0) = 0$.

$$\therefore q = \begin{cases} CV_0 \cdot \frac{t}{\tau - RC} \left(\frac{t}{\tau} - \frac{RC}{\tau - RC} \right) e^{-\frac{t}{\tau}} & (\text{if } \tau \neq RC) \\ \frac{1}{3} CV_0 \cdot \left(\frac{t}{\tau}\right)^3 e^{-\frac{t}{\tau}} & (\text{if } \tau = RC) \end{cases}$$

22.

$$\frac{dN}{N_s - N} = \lambda dt \Rightarrow \int_0^N \frac{dN}{N_s - N} = \lambda \int_0^t dt \Rightarrow -\ln \left| \frac{N_s - N}{N_s} \right| = \lambda t$$

$$\therefore N = N_s \left(1 - e^{-\lambda t}\right)$$

23. The clever idea here is that we could make a variable change from x to u, where we could choose u so as to make the equation easier to solve.

$$\begin{aligned} 0 &= A\frac{d^2y}{dx^2} + \frac{dA}{dx}\frac{dy}{dx} + \frac{y}{A} \\ &= A\frac{du}{dx}\frac{d}{du}\left(\frac{du}{dx}\frac{dy}{dy}\right) + \frac{dA}{dx}\frac{du}{dx}\frac{dy}{du} + \frac{y}{A} \\ &= A\left(\frac{du}{dx}\right)^2\frac{d^2y}{du^2} + \frac{dy}{du}\frac{d}{dx}\left(A\frac{dy}{dx}\right) \end{aligned}$$

Thus, we could define $\frac{du}{dx} = \frac{1}{A(x)}$ or $u = \int \frac{dx}{A(x)}$ to obtain

$$\frac{d^2y}{du^2} = -y \Rightarrow y(u) = C_1 \cos\left(u + C_2\right)$$

$$\therefore y(x) = C_1 \cos\left(\int \frac{dx}{A(x)} + C_2\right)$$

24. Suppose we may write $y = \lambda v$ where λ and v are to be determined later.

$$y' = \lambda v' + \lambda' v \Rightarrow y'' = \lambda v'' + 2\lambda' v' + \lambda'' v$$

$$\Rightarrow \lambda v'' + 2\left(\lambda' + \frac{\lambda}{x}\right) v' + \left(\lambda'' + \frac{2\lambda'}{x} + n^2\lambda\right) v = \frac{\sin \omega x}{x}$$

We may choose λ so as to make the coefficient of v' vanish, i.e.,

$$\lambda' = -\frac{\lambda}{x} \Rightarrow \lambda = \frac{1}{x}.$$

$$\Rightarrow v'' + n^2 v = \sin \omega x$$

(i) $n \neq \omega$:
The complementary solutions are

$$v_c = C_1 \cos(nx + C_2).$$

Using the ansatz $v_p = A \sin \omega x$ for a particular solution, one can easily find $A = \frac{1}{n^2 - \omega^2}$.

$$\Rightarrow v = \frac{\sin \omega x}{n^2 - \omega^2} + C_1 \sin(nx + C_2)$$

$$\therefore y = \frac{1}{n^2 - \omega^2} \frac{\sin \omega x}{x} + C_1 \frac{\sin(nx + C_2)}{x}$$

(ii) $n = \omega$:
The complementary solutions are

$$v_c = C_1 \cos(\omega x + C_2).$$

Using the ansatz $v_p = Ax \sin \omega x + Bx \cos \omega x$ for a particular solution, one can easily find $A = -\frac{1}{2\omega}$ and $B = 0$.

$$\Rightarrow v = -\frac{1}{2\omega} x \cos \omega x + C_1 \cos(\omega x + C_2)$$

$$\therefore y = -\frac{1}{2\omega} \cos \omega x + C_1 \frac{\sin(\omega x + C_2)}{x}$$

25. From the hint, we may consider $y = xv$, where $v = 1$ is a solution if the right hand side were zero.

$$y' = xv' + v \Rightarrow y'' = xv'' + 2v'$$

$$\Rightarrow x(1-x)v'' + \left(1 + (1-x)^2\right) v' = (1-x)^2$$

$$\Rightarrow v'' + \left(\frac{2}{x} + \frac{1}{1-x} - 1\right) v' = \frac{1}{x} - 1$$

We can thus define the integration factor as

$$\lambda := \exp\left(\int dx \left(\frac{2}{x} + \frac{1}{1-x} - 1\right)\right) = \frac{x^2 e^{-x}}{1-x}.$$

$$\Rightarrow (\lambda v')' = \lambda \left(\frac{1}{x} - 1\right) = xe^{-x}$$

$$\Rightarrow \lambda v' = -(x+1)e^{-x} + A$$

$$\Rightarrow v' = 1 - \frac{1}{x^2} + A \frac{1-x}{x^2} e^x$$

$$\Rightarrow v = x + \frac{1}{x} - A \frac{e^x}{x} + B$$

$$\therefore y = x^2 + 1 + C_1 e^x + C_2 x$$

26. While the use of Dirac delta "function" must be dealt with caution, we are physicists so we shall simply accept that they exist.

$$
\begin{aligned}
f(x) = y'' + py' + q &= \int_a^b dx' f(x') \left(\frac{\partial^2}{\partial x^2} + p(x) \frac{\partial}{\partial x} + q(x) \right) G(x, x') \\
&= \int_a^b dx' f(x') \delta(x - x') \\
\Rightarrow \left(\frac{\partial^2}{\partial x^2} + p(x) \frac{\partial}{\partial x} + q(x) \right) G(x, x') &= \delta(x - x')
\end{aligned}
$$

From this, we see that $G(x, x')$ must separately equal zero for both $x < x'$ and $x > x'$. We thus write

$$
G(x, x') = \begin{cases} y_1(x)\alpha(x') + y_2(x)\beta(x') & (x < x') \\ y_1(x)\gamma(x') + y_2(x)\zeta(x') & (x > x') \end{cases}.
$$

From the definition of delta functions, we find that:

$$
\begin{aligned}
1 &= \int_{x'-\epsilon}^{x'+\epsilon} dx \delta(x - x') \\
&= \int_{x'-\epsilon}^{x'+\epsilon} dx \left(\frac{\partial^2}{\partial x^2} + p(x) \frac{\partial}{\partial x} + q(x) \right) G(x, x') \\
&\approx \left[\left(\frac{\partial}{\partial x} + p(x) \right) G(x, x') \right]_{x'-\epsilon}^{x'+\epsilon} - \int_{x'-\epsilon}^{x'+\epsilon} dx p'(x) G(x, x') \\
&\approx \left[\frac{\partial}{\partial x} G(x, x') \right]_{x'-\epsilon}^{x'+\epsilon} \\
&\approx \lim_{x \to x'+} \frac{\partial}{\partial x} G(x, x') - \lim_{x \to x'-} \frac{\partial}{\partial x} G(x, x')
\end{aligned}
$$

where we take infinitesimally small ϵ.

The boundary conditions for y imply

$$
\begin{gathered}
y(a) = y(b) = 0 \\
\Rightarrow G(a, x') = G(b, x') = 0 \\
\Rightarrow y_1(a)\alpha(x') + y_2(a)\beta(x') = y_1(b)\alpha(x') + y_2(b)\beta(x') = 0 \\
\Rightarrow \beta(x') = \gamma(x') = 0.
\end{gathered}
$$

The derivative condition, in turn, yields

$$
y_2'(x')\zeta(x') - y_1'(x')\alpha(x') = 1.
$$

As any such choice of $\zeta(x')$ and $\alpha(x')$ leads to a valid Green's function, we may arbitrarily choose

$$\alpha(x') = y_2'(x'),\ \zeta(x') = y_1'(x') + \frac{1}{y_2'(x')}$$

$$\Rightarrow G(x,x') = \begin{cases} y_1(x)y_2'(x') & (x < x') \\ y_2(x)y_1'(x') + \frac{y_2(x)}{y_2'(x')} & (x > x') \end{cases}.$$

$$\therefore y(x) = y_1(x)\int_x^b dx' f(x')y_2'(x') + y_2(x)\int_a^x f(x')\left(y_1'(x') + \frac{1}{y_2'(x')}\right)$$

For the given example $y'' + k^2 y = f(x)$, we may take

$$y_1(x) = \sin k(x-a),\ y_2(x) = \sin k(b-x)$$

provided that $b-a$ is not an integer multiple of the period $\frac{2\pi}{k}$. This leads to

$$G(x,x') = \begin{cases} -k\sin k(x-a)\cos k(b-x) & (x < x') \\ \sin k(b-x)\left(k\cos k(x'-a) - \frac{1}{k\cos k(b-x')}\right) & (x > x') \end{cases}$$

$$\begin{aligned}\Rightarrow y =& -k\sin k(x-a)\int_x^b dx' f(x')\cos k(b-x') \\ &+ \sin k(b-x)\int_a^x dx' f(x')\left(k\cos k(x'-a) - \frac{1}{k\cos k(b-x')}\right).\end{aligned}$$

One may verify that this solution satisfies both the boundary conditions and the given differential equation.

27. The complementary solutions satisfy

$$y_c'' + \frac{3}{x^2}y_c = 0.$$

Suppose $y_c = x^m$.

$$m^2 - m + 3 = 0 \Rightarrow \left(m - \frac{1}{2}\right)^2 = -\frac{11}{4} \Rightarrow m = \frac{1 \pm \sqrt{11}i}{2}$$

$$\begin{aligned}\Rightarrow y_c &= Ax^{\frac{1+\sqrt{11}i}{2}} + Bx^{\frac{1-\sqrt{11}i}{2}} \\ &= C_1\sqrt{x}\cos\left(\frac{\sqrt{11}}{2}\ln x\right) + C_2\sqrt{x}\sin\left(\frac{\sqrt{11}}{2}\ln x\right)\end{aligned}$$

We then look for two separate particular solutions:

$$y_{p1}'' + \frac{3}{x^2}y_{p1} = x^2 \text{ and } y_{p2}'' + \frac{3}{x^2}y_{p2} = \frac{1}{x}.$$

Using the monomial ansatz separately, that is, assuming $y_p = Ax^m$, one easily finds

$$y_{p1} = \frac{x^4}{15} \text{ and } y_{p2} = \frac{x}{3}.$$

$$\therefore y = \frac{x^4}{15} + \frac{x}{3} + C_1\sqrt{x}\cos\left(\frac{\sqrt{11}}{2}\ln x\right) + C_2\sqrt{x}\sin\left(\frac{\sqrt{11}}{2}\ln x\right)$$

28. Let us first look for asymptotic behaviors.

(i) $x \to 0+$ Suppose $y = O(x^m)$.

$$m(m-1) + 2m - l(l+1) = 0 \Rightarrow m = l \text{ or } -l-1$$

To vanish near the origin, we must have $y = O(x^l)$.

(ii) $x \to \infty$

$$\frac{d^2y}{dx^2} \approx -Ky \Rightarrow y \approx e^{\pm\sqrt{-K}x}$$

To vanish at infinity, we must have $y = x^{-\sqrt{-K}x}$ with $K < 0$.

We can thus write $y = f(x)x^l e^{-\sqrt{-K}x}$ with f regular everywhere.

$$y' = \left(\frac{f'}{f} + \frac{l}{x} - \sqrt{-K}\right) y$$

$$\Rightarrow y'' = \left(\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 - \frac{l}{x^2}\right) y + \left(\frac{f'}{f} + \frac{l}{x} - \sqrt{-K}\right)^2 y$$
$$= \left(\frac{f''}{f} + 2\left(\frac{l}{x} - \sqrt{-K}\right)\frac{f'}{f} + \frac{l(l-1)}{x^2} - \frac{2l\sqrt{-K}}{x} - K\right) y$$

Tedious algebra yields

$$xf'' + 2(l+1-\sqrt{-K})f' - 2((l+1)\sqrt{-K} - 1)f = 0.$$

Let us write $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Substitution and tedious algebra yield

$$\sum_{n=0}^{\infty}((n+1)(n+2l+2)a_{n+1} - (\sqrt{-K}n + 2((l+1)\sqrt{-K} - 1))a_n)x^n = 0$$

$$\Rightarrow (\forall n)\ (n+1)(n+2l+2)a_{n+1} = (\sqrt{-K}n + 2((l+1)\sqrt{-K} - 1))a_n.$$

If $\{a_n\}$ never terminated, this would mean that $a_n \approx \frac{(\sqrt{-K})^n}{n!}$ for large values of n, leading to $f(x) \approx e^{\sqrt{-K}x}$ and y diverging to infinity as x increases.

$$\Rightarrow \sqrt{-K_{nl}}n + 2((l+1)\sqrt{-K_{nl}} - 1) = 0$$

$$\therefore K_{nl} = -\frac{4}{(n+2l+2)^2}\ (n = 0, 1, 2, \dots)$$

29. For very large x,

$$y'' \approx \frac{y}{4} \Rightarrow y \approx e^{\pm\frac{x}{2}}.$$

Thus, for y to vanish at infinity, we must have $y \approx e^{-x/2}$. As y must also vanish at the origin, let us write $y = f(x)xe^{-x/2}$ with $f(x)$ regular everywhere.

$$y' = \left(\frac{f'}{f} + \frac{1}{x} - \frac{1}{2}\right) y$$

$$\begin{aligned}
\Rightarrow y'' &= \left(\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 - \frac{1}{x^2}\right) y + \left(\frac{f'}{f} + \frac{1}{x} - \frac{1}{2}\right)^2 y \\
&= \left(\frac{f''}{f} + \left(\frac{2}{x} - 1\right)\frac{f'}{f} - \frac{1}{x} + \frac{1}{4}\right) y \\
&\Rightarrow \frac{f''}{f} + \left(\frac{2}{x} - 1\right)\frac{f'}{f} - \frac{K+1}{x} = 0
\end{aligned}$$

Let us write $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Substitution and tedious algebra yield

$$2a_1 - (K+1)a_0 + \sum_{n=1}^{\infty}((n+1)(n+2)a_{n+1} - (n+K+1)a_n)x^n = 0$$

$$\Rightarrow (\forall n)\ (n+1)(n+2)a_{n+1} = (n+K+1)a_n.$$

If $\{a_n\}$ never terminated, this would mean that $a_n \approx \frac{1}{n!}$ for large values of n, leading to $f(x) \approx e^x$ and y diverging to infinity as x increases.

$$\therefore K_n = -n\ (n = 1, 2, 3, \dots)$$

30. I assume that there must exist a "nontrivial" solution, since $y = 0$ is clearly a solution of the equation for any value of k. For large values of x,

$$y'' - 2y' - 3y \approx 0 \Rightarrow y \approx Ae^{-x} + Be^{3x}.$$

Thus, to be bounded everywhere, we must have $y \approx e^{-x}$. We then write $y = f(x)e^{-x}$ with f analytic everywhere, including the origin.

$$y' = \left(\frac{f'}{f} - 1\right) y \Rightarrow y'' = \left(\frac{f''}{f} - \frac{f'}{f} + \left(\frac{f'}{f} - 1\right)^2\right) y = \left(\frac{f''}{f} - 2\frac{f'}{f} + 1\right) y$$

Substitution yields

$$xf'' - 4xf' + kf = 0.$$

Let us write $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Substitution and tedious algebra yield

$$\sum_{n=0}^{\infty}(n(n+1)a_{n+1} - (4n-k)a_n)x^n = 0$$

$$\Rightarrow (\forall n)\ n(n+1)a_{n+1} = (4n-k)a_n.$$

If $\{a_n\}$ never terminated, this would mean that $a_n \approx \frac{4^n}{n!}$ for large values of n, leading to $f(x) \approx e^{4x}$ and y diverging to infinity as x increases.

$$\therefore k_n = 4n\ (n = 0, 1, 2, \dots)$$

31. For large values of x,

$$y'' \approx y \Rightarrow y \approx e^{\pm x}.$$

Thus, to be bounded everywhere, we must have $y \approx e^{-x}$. We then write $y = f(x)e^{-x}$ with f analytic everywhere, with $f(0) = 1$.

$$y' = \left(\frac{f'}{f} - 1\right) y \Rightarrow y'' = \left(\frac{f''}{f} - \frac{f'}{f} + \left(\frac{f'}{f} - 1\right)^2\right) y = \left(\frac{f''}{f} - 2\frac{f'}{f} + 1\right) y$$

Substitution yields

$$xf'' - 2(x-1)f' + (E-2)f = 0.$$

Let us write $f(x) = \sum_{n=0}^{\infty} a_n x^n$, with $a_0 = 1$. Substitution and tedious algebra yield

$$\sum_{n=0}^{\infty}((n+1)(n+2)a_{n+1} - (2n+2-E)a_n)x^n = 0$$

$$\Rightarrow (\forall n)\ (n+1)(n+2)a_{n+1} = (2n+2-E)a_n.$$

If $\{a_n\}$ never terminated, this would mean that $a_n \approx \frac{2^n}{n!}$ for large values of n, leading to $f(x) \approx e^{2x}$ and y diverging to infinity as x increases.

$$\therefore E_n = 2n\ (n = 1, 2, 3, \dots)$$

32. Recall that

$$\frac{c_{r+2}}{c_r} = \frac{(r+m-n)(r+m+n+1)}{(r+1)(r+2)} \text{ and } v(x) = \sum_{r=0}^{\infty} c_r x^r$$

where the fraction is well-defined since we are assuming that $\{c_r\}$ never terminates. Notice how

$$\frac{c_{r+2}}{c_r} \approx \frac{r+2m+1}{r+3}$$

for large values of r.

On the other hand, the definition of binomial coefficients

$$\binom{-m}{r} := \frac{-m \cdot (-m-1) \cdot \cdots \cdot (-m-r+1)}{r!}$$

naturally yields

$$\begin{aligned}
\frac{\binom{-m}{r+2}}{\binom{-m}{r}} &= \frac{(-m-r)(-m-r-1)}{(r+1)(r+2)} \\
&= \frac{(r+m)(r+m+1)}{(r+1)(r+2)} \\
&\approx \frac{r+2m+1}{r+3}
\end{aligned}$$

for large values of r.

Therefore, c_r behaves like $\binom{-m}{r}$ as r grows without bound, and consequently,

$$v(x) \approx \left(1-x^2\right)^{-m}.$$

33. Suppose $y = J_0(x)\ln x + \sum_{n=0}^{\infty} a_n x^n$ as prompted by the problem. Substitution and tedious algebra yield

$$2xJ_0'(x) + a_1 x + \sum_{n=2}^{\infty}(n^2 a_n + a_{n-2}).$$

Let us write $J_0(x) = \sum_{n=0}^{\infty} b_{2n}x^{2n}$ since we know that J_0 is even.

$$xJ_0'(x) = \sum_{n=0}^{\infty} 2nb_{2n}x^{2n}$$

$$\begin{aligned}
\Rightarrow a_1 x + \sum_{n=1}^{\infty}\Bigg(&\left(4nb_{2n} + 4n^2 a_{2n} + a_{2n-2}\right)x^{2n} \\
&+ \left((2n+1)^2 a_{2n+1} + a_{2n-1}\right)x^{2n+1}\Bigg) = 0
\end{aligned}$$

This immediately tells us that all odd coefficients $a_1 = a_3 = \cdots = 0$.

Bessel's equation is linear, so addition of any multiple of J_0 to y also yields a valid solution. In other words, any variant $a_{2n}' := a_{2n} + \lambda b_{2n}$ also satisfies the above equation, for real parameter λ. Thus, we may choose $a_0 = 0$.

Using $b_0 = 1$, $b_2 = -\frac{1}{4}$, $b_4 = \frac{1}{64}$, $b_6 = -\frac{b_4}{36} = -\frac{1}{2304}$, and the recursion relation $a_2 n = -\frac{a_{2n-2}}{4n^2} - \frac{b_{2n}}{n}$ from above, we can iteratively calculate the values of a_{2n}.

$$\therefore y = J_0(x)\ln x + \frac{x^2}{4} - \frac{3}{128}x^4 + \frac{5}{4608}x^6 - \cdots$$

34. (i) $y(0) = 1$
We may write $y = 1 + \sum_{n=1}^{\infty} a_n x^n$. Substitution and tedious algebra yields

$$2a_1 - 2 + \sum_{n=1}^{\infty}(n+2)((n+1)a_{n+1} - a_n)x^n = 0.$$

This directly leads to $a_n = \frac{1}{n!}$, and thus,

$$y = e^x.$$

(ii) $y = \frac{1}{x} + A \ln x + B$
Substitution of the above form into the differential equation and patiently performing algebraic manipulations yields

$$-\frac{1}{x} + (xA'' + (2-x)A' - 2A)\ln x + 2A' + \left(\frac{1}{x} - 1\right) A + xB''(2-x)B' - 2B = 0.$$

Read the problem carefully: we only need to "give" two solutions, not find a general form for all solutions. Thus, we could choose the oddly specific solution of $A = e^x$ to make the coefficient of the $\ln x$ term vanish. Substituting this into the above equation yields

$$xB'' + (2-x)B' - 2B + e^x + \frac{e^x - 1}{x} = 0.$$

Expanding $B = \sum_{n=0}^{\infty} b_n x^n$, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, and $\frac{e^x-1}{x} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$, one can write

$$\sum_{n=0}^{\infty} \left((n+1)(n+2)b_{n+1} - (n+2)b_n + \frac{n+2}{(n+1)!} \right) = 0$$

$$\Rightarrow b_{n+1} = \frac{b_n}{n+1} - \frac{1}{(n+1)\cdot(n+1)!}.$$

Again, B may arbitrarily chosen, so let us choose $b_0 = 0$.

$$\therefore y = \frac{1}{x} + e^x \ln x - x - \frac{3}{4}x^2 - \frac{11}{36}x^3 - \cdots$$

35. Let $u := \frac{1}{x}$.

$$\frac{d}{dx} = \frac{du}{dx}\frac{d}{du} = -u^2 \frac{d}{du}$$

$$\Rightarrow \frac{d^2}{dx^2} = -u^2 \frac{d}{du}\left(-u^2 \frac{d}{du}\right) = u^4 \frac{d^2}{du^2} + 2u^3 \frac{d}{du}$$

$$\Rightarrow \frac{d^2 y}{du^2} + \frac{2}{u}\frac{dy}{du} = \frac{1}{\left(1 + u^2 y^2\right)^2}$$

As $u \to 0+$, $y \to 0$. Thus,

$$\frac{2}{u}\frac{dy}{du} \approx 1 \Rightarrow y \approx u^2$$

Hence, we write $y = f(u)u^2$ with f analytic everywhere.

$$\begin{aligned} u^2 f''(u) + 6uf'(u) + 6f(u) &= \frac{1}{\left(1 + u^6 f(u)\right)^2} \\ &= 1 - 2u^6 f(u)^2 + 3u^{12} f(u)^4 + O\left(u^{18}\right) \end{aligned}$$

We write $f(u)=\sum_{n=0}^{\infty} a_n u^n$. The left-hand side (after tedious algebra) evaluates to

$$\sum_{n=0}^{\infty}\left(n^2+5n+5\right)a_n u^n.$$

Evaluation of the right-hand side is quite more technical. We make use of the Cauchy multiplication formula:

$$f(u)^2=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k a_{n-k}\right)u^n.$$

Substituting this into the equation, we get

$$\begin{aligned}&(6a_0-1)+\sum_{n=1}^{5}\left(n^2+5n+6\right)a_n u^n\\ &+\sum_{n=6}^{\infty}\left(\left(n^2+5n+6\right)a_n+2\sum_{k=0}^{n-6}a_k a_{n-6-k}\right)u^n=3u^{12}f(u)^4+O(u^{18}).\end{aligned}$$

Let us compare each power of u separately.

$$a_0=\frac{1}{6},\ a_1=\cdots=a_5=0$$

$$O(u^6):\ 72a_6+2a_0^2=0\Rightarrow a_6=-\frac{1}{1296}$$

$$a_7=\cdots=a_{11}=0$$

$$O(u^{12}):210a_{12}+2\left(a_0a_6+a_6a_0\right)=3a_0^2\Rightarrow a_{12}=\frac{11}{816480}$$

$$\Rightarrow f(u)=\frac{1}{6}-\frac{u^6}{1296}+\frac{11}{816480}u^{12}-\dots$$

$$\therefore y=\frac{1}{6x^2}-\frac{1}{1296x^8}+\frac{11}{816480}\frac{1}{x^{14}}+\dots$$

36. (a) Assume $y=\sum_{n=0}^{\infty}c_n x^n$. Using

$$y''=y\left(x^2-y^2\right),\ x^2=\sum_{n=0}^{\infty}\delta_{n2}x^n,$$

and

$$y^2=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}c_k c_{n-k}\right)x^n,$$

one can find (after algebraic tedium) that

$$y\left(x^2-y^2\right)=-c_0^2-3c_1c_0^2x+\sum_{n=2}^{\infty}\left(c_{n-2}-\sum_{k=0}^{n}\sum_{l=0}^{k}c_l c_{k-l}c_{n-k}\right)x^n$$

$$\begin{aligned}\Rightarrow &\left(2c_2 + c_0^2\right) + \left(6c_3 + 3c_1c_0^2\right)x \\ &+ \sum_{n=2}^{\infty}\left((n+2)(n+1)c_{n+2} - c_{n-2} + \sum_{k=0}^{n}\sum_{l=0}^{k} c_l c_{k-l} c_{n-k}\right)x^n = 0.\end{aligned}$$

We thus find two degrees of freedom, namely $c_0 = A$ and $c_1 = B$.

$$c_2 = -\frac{A^2}{2},\ c_3 = -\frac{A^2B}{2},\ c_{n+2} = -\frac{1}{(n+2)(n+1)}\left(c_{n-2} - \sum_{k=0}^{n}\sum_{l=0}^{k} c_l c_{k-l} c_{n-k}\right)$$

We could calculate a few terms using the recurrence relation:

$$c_4 = \frac{1}{4\cdot 3}\left(c_0 - 3c_2c_0^2 - 3c_1^2c_0\right) = \frac{A}{12}\left(1 + \frac{3}{2}A^3 - 3B^2\right),$$

$$\begin{aligned}c_5 &= \frac{1}{5\cdot 4}\left(c_1 - 3c_3c_0^2 - 6c_2c_1c_0 - 3c_2c_1\right) \\ &= \frac{B}{20}\left(1 + 3A^3\left(\frac{A}{4} + 1\right)\right)\end{aligned}$$

$$\begin{aligned}\therefore y =& A + Bx - \frac{A^2}{2}x^2 - \frac{A^2B}{2}x^3 \\ &+ \frac{A}{12}\left(1 + \frac{3}{2}A^3 - 3B^2\right)x^4 + \frac{B}{20}\left(1 + 3A^3\left(\frac{A}{4} + 1\right)\right)x^5 + \cdots\end{aligned}$$

(b) We iteratively approximate the particular nonoscillating solution, for which the second derivative would stay relatively small. That is, we use the following iterative scheme:

$$-xy^{(0)} + {y^{(0)}}^3 = 0,\ y^{(n+1)} = \sqrt[3]{xy^{(n)} - {y^{(n)}}''}$$

We thus get

$$y^{(0)} = \sqrt{x} = x^{1/2}$$

$$y^{(1)} = \sqrt[3]{x^{3/2} + \frac{1}{4x^{3/2}}} \approx x^{1/2} + \frac{1}{12}x^{-5/2}$$

$$\begin{aligned}y^{(2)} &= \sqrt[3]{x^{3/2} + \frac{1}{3}x^{-3/2} - \frac{35}{48}x^{-9/2}} \\ &\approx x^{1/2} + \frac{1}{9}x^{-5/2} - \frac{35}{144}x^{-11/2}\end{aligned}$$

$$\begin{aligned}y^{(3)} &= \sqrt[3]{x^{3/2} + \frac{13}{36}x^{-3/2} - \frac{175}{144}x^{-9/2} + \frac{5005}{576}x^{-15/2}} \\ &\approx x^{1/2} + \frac{13}{108}x^{-5/2} - \frac{175}{432}x^{-11/2} + \frac{5005}{1728}x^{-17/2}\end{aligned}$$

$$\therefore y \approx x^{1/2} + \frac{13}{108}x^{-5/2} - \frac{175}{432}x^{-11/2} + \frac{5005}{1728}x^{-17/2}$$

(Note: One may find more accurate approximations by (i) iterating more times, or (ii) expanding to more terms for each binomial expansion above.)

37. We first take note that $y = x + \alpha$ is a trivial solution to the equation, as is easily verifiable. Let $z := y - x$. We directly have

$$z'' = z^2 - e^{2z}.$$

We multiply each side by z' and integrate each side to obtain

$$\frac{z'^2}{2} = \frac{1}{3}z^3 - \frac{1}{2}e^{2z} + E.$$

I find it illuminating to consider z as the position of a particle of mass 1. The above equation could then be interpreted as the energy conservation of this particle as it moves under the influence of the potential

$$V(z) = -\frac{1}{3}x^3 + \frac{1}{2}e^{2z}$$

and retains its total energy E. Thus, we could consider the infinitesimal oscillation about $z = \alpha$ as a harmonic oscillator.

$$\left.\frac{d^2V}{dz^2}\right|_{z=\alpha} = -2\alpha + 2e^{2\alpha} = 2\alpha(\alpha - 1) = m\omega^2 = \omega^2$$

$$\Rightarrow \omega = \sqrt{2\alpha(\alpha-1)}$$

$$\Rightarrow z \approx C_1 \sin\left(\sqrt{2\alpha(\alpha-1)x} + C_2\right)$$

$$\therefore y \approx x + C_1 \sin\left(\sqrt{2\alpha(\alpha-1)x} + C_2\right)$$

(Note: More accurate expressions may be obtained using the higher-order terms of the potential energy and perturbation methods.)

38. (a) Let $y := \sum_{n=1}^{\infty} c_n (x-1)^n$.

$$\begin{aligned}
e^{y/x} &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{y}{x}\right)^n = \sum_{n=0}^{\infty} \frac{y^n}{n!}(1 + (x-1))^{-n} \\
&= \sum_{n=0}^{\infty} \frac{y^n}{n!}\left(\sum_{m=0}^{\infty} \binom{-n}{m}(x-1)^m\right) \\
&= 1 + \sum_{m=0}^{2} \binom{-1}{m}(x-1)^m y + \sum_{m=0}^{1} \binom{-2}{m}(x-1)^m \frac{y^2}{2} + \frac{y^3}{6} + O\left((x-1)^4\right) \\
&= 1 + c_1(x-1) + \left(c_2 - c_1 + \frac{c_1^2}{2}\right)(x-1)^2 \\
&\quad + \left(c_3 - c_2 + c_1 - c_1c_2 - c_1^2 + \frac{c_1^3}{6}\right)(x-1)^3 + O\left((x-1)^4\right)
\end{aligned}$$

Comparing this with

$$y' = c_1 + 2c_2(x-1) + 3c_3(x-1)^2 + O\left((x-1)^3\right),$$

we can see that

$$c_1 = 1,\ c_2 = \frac{c_1}{2} = \frac{1}{2},\ c_3 = \frac{1}{3}\left(c_2 - c_1 + \frac{c_1^2}{2}\right) = 0$$

$$\therefore y = (x-1) + \frac{1}{2}(x-1)^2 + O\left((x-1)^4\right)$$

(b) Let $y = xv$.

$$xv' + v = e^v \Rightarrow \frac{dv}{e^v - v} = \frac{dx}{x} \Rightarrow \int_1^{y/x} \frac{dv}{e^v - v} = \ln x + C_1$$

$$\ln x_0 + C_1 = \int_1^{\infty} \frac{dv}{e^v - v} =: \alpha \Rightarrow C_1 = \alpha - \ln x_0$$

$$\int_1^{y/x} \frac{dv}{e^v - v} = \ln \frac{x}{x_0} + \alpha \Rightarrow \int_{y/x}^{\infty} \frac{dv}{e^v - v} = -\ln \frac{x}{x_0}$$

For values of x slightly less than x_0, we have $\frac{y}{x} >> 1$.

$$\Rightarrow -\ln \frac{x}{x_0} \approx \int_{y/x}^{\infty} dv e^{-v} = e^{-y/x}$$

$$\therefore y \approx -x \ln \ln \frac{x_0}{x}$$

39. From the WKB method followed the Bohr-Sommerfeld quantization condition:

$$\int_{x_1}^{x_2} dx \sqrt{E - V(x)} = \left(n + \frac{1}{2}\right)\pi.$$

($\hbar = 1$ in this unit system.)

If the particle has a total energy of $E < 0$, it then has classical turning points at $\pm a\left(1 + \frac{E}{V_0}\right)$. Hence,

$$\left(n + \frac{1}{2}\right)\pi = 2\int_0^{a\left(1+\frac{E}{V_0}\right)} dx \sqrt{V_0 + E - \frac{V_0 x}{a}} = \frac{4a}{3V_0}(V_0 + E)^{3/2}$$

$$\Rightarrow (V_0 + E_n)^{3/2} = \left(n + \frac{1}{2}\right)\frac{3\pi V_0}{4a}$$

$$\therefore E_n = -\left(V_0 - \left(\left(n + \frac{1}{2}\right)\frac{3\pi V_0}{4a}\right)^{2/3}\right)$$

Here, n takes nonnegative integers as its value such that $E_n < 0$.

40. To use the WKB method, we want to make a substitution $y = up$ for some known function p such that the first derivative term vanishes. Substituting this into the equation yields

$$pu'' + \left(2p' - \frac{3}{x}p\right)u' + \left(p'' - \frac{3}{x}p' + \left(\frac{15}{4x^2} + x^{\frac{1}{2}}\right)\right)u = 0.$$

Thus, we choose $p = x^{\frac{3}{2}}$ to obtain

$$u'' + \left(x^{\frac{1}{2}} + \frac{15}{4}x^{-2} - \frac{21}{4}x^{-2}\right)u = 0.$$

Let

$$f(x) := x^{\frac{1}{2}} + \frac{15}{4}x^{-2} - \frac{21}{4}x^{-2} = x^{\frac{1}{2}}\left(1 + \frac{15}{4}x^{-\frac{5}{2}} - \frac{21}{4}x^{-\frac{7}{2}}\right).$$

This means that, using Eq. (1–90) from the textbook, one gets

$$u \approx \frac{1}{f^{1/4}}\left(c_+ \exp\left(i\int dx\sqrt{f}\right) + c_- \exp\left(-i\int dx\sqrt{f}\right)\right).$$

Evaluating these can be done using the binomial expansion for large values of x.

$$\begin{aligned}
\sqrt{f} &= \sqrt{x^{\frac{1}{2}}\left(1 + \frac{15}{4}x^{-\frac{5}{2}} - \frac{21}{4}x^{-\frac{7}{2}}\right)} \\
&\approx x^{\frac{1}{4}}\left(1 + \frac{15}{8}x^{-\frac{5}{2}} - \frac{21}{8}x^{-\frac{7}{2}}\right) \\
&= x^{\frac{1}{4}} + \frac{15}{8}x^{-\frac{9}{4}} - \frac{21}{8}x^{-\frac{13}{4}} \\
\Rightarrow \int dx\sqrt{f} &= \frac{4}{5}x^{\frac{5}{4}} - \frac{3}{2}x^{-\frac{5}{4}} + \frac{7}{6}x^{-\frac{9}{4}}
\end{aligned}$$

$$\begin{aligned}
f^{-\frac{1}{4}} &= \left(x^{\frac{1}{2}}\left(1 + \frac{15}{4}x^{-\frac{5}{2}} - \frac{21}{4}x^{-\frac{7}{2}}\right)\right)^{-\frac{1}{4}} \\
&\approx x^{-\frac{1}{8}}\left(1 - \frac{15}{16}x^{-\frac{5}{2}} + \frac{21}{16}x^{-\frac{7}{2}}\right)
\end{aligned}$$

$$\Rightarrow u \approx x^{-\frac{1}{8}}\left(1 - \frac{15}{16}x^{-\frac{5}{2}} + \frac{21}{16}x^{-\frac{7}{2}}\right) \cdot C_1 \cos\left(\frac{4}{5}x^{\frac{5}{4}} - \frac{3}{2}x^{-\frac{5}{4}} + \frac{7}{6}x^{-\frac{9}{4}} + C_2\right)$$

$$\therefore y \approx x^{\frac{11}{8}}\left(1 - \frac{15}{16}x^{-\frac{5}{2}} + \frac{21}{16}x^{-\frac{7}{2}}\right) \cdot C_1 \cos\left(\frac{4}{5}x^{\frac{5}{4}} - \frac{3}{2}x^{-\frac{5}{4}} + \frac{7}{6}x^{-\frac{9}{4}} + C_2\right)$$

41. We again make the substitution $y = up$ to eliminate the first derivative term. Substituting this into the Bessel equation

$$x^2y'' + xy' + \left(x^2 - m^2\right)y = 0$$

yields

$$x^2pu'' + \left(2x^2p' + xp\right)u' + \left(x^2p'' + xp' + \left(x^2 - m^2\right)p\right)u = 0.$$

Thus, we choose $p = x^{-\frac{1}{2}}$ to obtain

$$u'' + \left(1 - \left(m^2 - \frac{1}{4}\right)x^{-2}\right)u = 0.$$

Let

$$f(x) := 1 - \left(m^2 - \frac{1}{4}\right)x^{-2}.$$

Now, we wish to find the asymptotic form of J_m, Bessel's function of the first kind. J_m is distinguished from that of the second kind, Y_m by its analyticity at the origin. Notice how f is a monotone increasing function with a single zero at $\sqrt{m^2 - \frac{1}{4}}$. Therefore, for $m >> \frac{1}{2}$, one can approximate the analyticity condition as the function being bounded for $x << \sqrt{m^2 - \frac{1}{4}}$. This leads us to use the connection formula Eq. (1–113).

$$\sqrt{f} = \sqrt{1 - \left(m^2 - \frac{1}{4}\right)x^{-2}} \approx 1 - \frac{1}{2}\left(m^2 - \frac{1}{4}\right)x^{-2}$$

$$\begin{aligned}
\Rightarrow \int_{\sqrt{m^2-\frac{1}{4}}}^{x} dx\sqrt{f} &\approx x - \sqrt{m^2 - \frac{1}{4}} + \frac{1}{2}\left(m^2 - \frac{1}{4}\right)\left(\frac{1}{x} - \frac{1}{\sqrt{m^2 - \frac{1}{4}}}\right) \\
&= x + \frac{m^2 - \frac{1}{4}}{2x} - \frac{1}{2}\sqrt{m^2 - \frac{1}{4}} \\
f^{-\frac{1}{4}} &\approx 1 + \frac{1}{4}\left(m^2 - \frac{1}{4}\right)x^{-2}
\end{aligned}$$

$$\begin{aligned}
\therefore J_m &\approx 2x^{-\frac{1}{2}}f^{-\frac{1}{4}}\cos\left(\int_{\sqrt{m^2-\frac{1}{4}}}^{x} dx\sqrt{f} - \frac{\pi}{4}\right) \\
&\approx \left(2x^{-\frac{1}{2}} + \left(\frac{m^2}{2} - \frac{1}{8}\right)x^{-\frac{5}{2}}\right)\cos\left(x + \frac{m^2 - \frac{1}{4}}{2x} - \frac{1}{2}\sqrt{m^2 - \frac{1}{4}} - \frac{\pi}{4}\right)
\end{aligned}$$

42. This problem is a simple application of the connection formulae Eq. (1–113) and (1–122) with $f(x) = x$.

(a) Here, we use Eq. (1–122) with $\phi = 0$.

$$\therefore y \sim \frac{1}{\sqrt{2}(-x)^{-\frac{1}{4}}}\exp\left(\frac{2}{3}(-x)^{\frac{3}{2}}\right)$$

(b) Here, we use Eq. (1–113).

$$\therefore y \sim \frac{2}{x^{\frac{1}{4}}} \cos\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{\pi}{4}\right)$$

(c) This problem is not defined, as explained after Eq. (1–122). In short, the large exponential alone cannot determine the phase.

43. We again use the WKB approximation with $f(x) = x^2$.

$$W_{\pm} := \frac{1}{f^{\frac{1}{4}}} \exp\left(\pm i \int_0^x dx \sqrt{f}\right) = x^{-\frac{1}{2}} \exp\left(\pm \frac{i}{2} x^2\right)$$

$$\Rightarrow y \approx x^{-\frac{1}{2}} \left(A \cos \frac{x^2}{2} + B \sin \frac{x^2}{2}\right)$$

Solving for A and B using the initial conditions $y(10) = 0$ and $y'(10) = 1$, we get

$$y \approx \frac{1}{\sqrt{10x}} \sin\left(\frac{x^2}{2} - 50\right).$$

(a)

$$\frac{x^2}{2} - 50 \approx \pi \Rightarrow x \approx \sqrt{100 + 2\pi} \approx 10.309$$

(b)

$$\frac{x^2}{2} - 50 \approx \frac{\pi}{2} \Rightarrow x \approx \sqrt{100 + \pi} \Rightarrow y \approx \frac{1}{\sqrt{10\sqrt{100 + \pi}}} \approx 0.0992$$

44. The error estimation for this solution follows the example following Eq. (1–103). I personally am not confident that this solution is correct, but it is the best I could come up with.

The WKB approximation of y_1 is given by

$$y_1 \approx \frac{C_1}{\sqrt{5x}} \sin\left(\frac{x^2 - 25}{2}\right)$$

which can be obtained in the same way as in Problem 1–43. The 25th zero beyond $x = 5$ is then given by

$$\frac{x^2 - 25}{2} \approx 25\pi \Rightarrow x \approx 5\sqrt{2\pi + 1} \approx 13.494.$$

We now begin the error analysis. The following notation follows that of the aforementioned example. Let

$$y_1 = \alpha_+ x^{-\frac{1}{2}} \exp\left(\frac{i}{2}x^2\right) + \alpha_- x^{-\frac{1}{2}} \exp\left(-\frac{i}{2}x^2\right).$$

Near $x = 5$, we then obtain

$$\alpha_+ \approx \frac{A}{2i} e^{-\frac{25}{2}i}, \ \alpha_- \approx -\frac{A}{2i} e^{\frac{25}{2}i}.$$

Let

$$g = \frac{1}{4}\frac{f''}{f} - \frac{5}{16}\left(\frac{f'}{f}\right)^2 = -\frac{3}{4x^2}$$

$$\begin{aligned}
\Rightarrow \alpha'_\pm &\approx \mp \frac{i}{2}\frac{g}{\sqrt{f}} \left(\alpha_\pm + \alpha_\mp \exp\left(\mp 2i \int dx \sqrt{f}\right)\right) \\
&= \pm \frac{3i}{8x^3} \left(\alpha_\pm + \alpha_\mp \exp\left(\mp i x^2\right)\right) \\
&\approx \frac{3iA}{32x^3} e^{\mp \frac{25}{2}i} \left(1 - e^{\pm 25i \mp ix^2}\right)
\end{aligned}$$

Now, upon integrating this from $x = 5$ to $5\sqrt{2\pi + 1}$, we may ignore the $e^{\pm 25i \mp ix^2}$ term as it rotates rapidly, thus making little contribution to the result.

$$\begin{aligned}
\Rightarrow |\Delta y_1| &\approx |\Delta \alpha_\pm| \\
&\approx \left| \int_5^{5\sqrt{2\pi+1}} dx \alpha'_\pm \right| \\
&\approx \left| \frac{3iA}{32x^3} e^{\mp \frac{25}{2}i} \int_5^{5\sqrt{2\pi+1}} \frac{dx}{x^3} \right| \\
&= \frac{3\pi A}{800(2\pi + 1)}
\end{aligned}$$

We also have

$$\begin{aligned}
y'_1|_{x=5\sqrt{2\pi+1}} &\approx \frac{A}{\sqrt{5}} \left(-\frac{1}{2} x^{-\frac{3}{2}} \sin\left(\frac{x^2 - 25}{2}\right) + x^{\frac{1}{2}} \cos\left(\frac{x^2 - 25}{2}\right)\right)\Bigg|_{x=5\sqrt{2\pi+1}} \\
&\approx -A(2\pi + 1)^{\frac{1}{4}}.
\end{aligned}$$

$$\therefore |\Delta x| \approx \left|\frac{\Delta y}{y'}\right| \approx \frac{3\pi}{800(2\pi + 1)^{\frac{1}{4}}} \approx 3.488 \times 10^{-4}$$

2 Chapter 2

1. A useful trick when dealing with complex sign patterns like this is to represent them using an exponential. In this case, we acknowledge that

$$\left\{\Re\left\{e^{i\left(\frac{n}{2} - \frac{1}{4}\right)\pi}\right\}\right\} = \frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}}, \ -\frac{1}{\sqrt{2}}, \ -\frac{1}{\sqrt{2}}, \cdots$$

and write the series as

$$
\begin{aligned}
\text{(Given series)} &= \sum_{n=0}^{\infty} \frac{1}{4^n} \cdot \sqrt{2}\Re\left\{e^{i\left(\frac{n}{2}-\frac{1}{4}\right)\pi}\right\} \\
&= \Re\left\{\sqrt{2}e^{-i\frac{\pi}{4}} \sum_{n=0}^{\infty} \left(\frac{e^{i\frac{\pi}{4}}}{2}\right)^n\right\} \\
&= \Re\left\{(1-i)\sum_{n=0}^{\infty} \left(\frac{i}{4}\right)^n\right\} \\
&= \Re\left\{(1-i)\cdot\frac{1}{1-\frac{i}{4}}\right\} \\
&= \frac{16}{17}\Re\left\{(1-i)\left(1+\frac{i}{4}\right)\right\} \\
&= \frac{16}{17}\left(1+\frac{1}{4}\right) \\
&= \frac{20}{17}.
\end{aligned}
$$

For series problems, it is always a good idea to verify the results numerically with a calculator, provided that the series converges quickly enough. In this case, evaluating up to $\frac{1}{1024}$ yields about 1.1768 while the correct answer is about 1.1765.

2.

$$
\begin{aligned}
\text{(Given series)} &= \sum_{n=0}^{\infty} \frac{1}{n(n+2)} \\
&= \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) \\
&= \frac{1}{2}\left(1+\frac{1}{2}\right) \\
&= \frac{3}{4}
\end{aligned}
$$

3. To describe the sign pattern, we shall use

$$
\left\{\Re\left\{e^{i\left(\frac{n}{3}+\frac{1}{6}\right)\pi}\right\}\right\} = \frac{\sqrt{3}}{2},\ 0,\ -\frac{\sqrt{3}}{2},\ -\frac{\sqrt{3}}{2},\ 0,\ \frac{\sqrt{3}}{2},\ \cdots.
$$

$$
\begin{aligned}
\text{(Given series)} &= \frac{1}{1\cdot 3^0} + \frac{0}{3\cdot 3^1} + \frac{-1}{5\cdot 3^2} + \frac{-1}{7\cdot 3^3} + \frac{0}{9\cdot 3^4} + \frac{1}{11\cdot 3^5} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)3^n} \cdot \frac{2}{\sqrt{3}} \Re\left\{ e^{i\left(\frac{n}{3}+\frac{1}{6}\right)\pi} \right\} \\
&= \Re\left\{ \frac{2}{\sqrt{3}} e^{i\frac{\pi}{6}} \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{1}{3} e^{i\frac{\pi}{3}} \right)^n \right\} \\
&= \Re\left\{ 2 \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{1}{\sqrt{3}} e^{i\frac{\pi}{6}} \right)^{2n+1} \right\} \\
&= \Re\left\{ 2\tanh^{-1}\left(\frac{1}{\sqrt{3}} e^{i\frac{\pi}{6}} \right) \right\} \\
&= \Re\left\{ \ln\left(\frac{1+\frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}}{1-\frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}} \right) \right\} \\
&= \ln\left| \frac{1+\frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}}{1-\frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}} \right| \\
&= \ln\left| \frac{5-\sqrt{3}i}{2} \right| \\
&= \frac{1}{2}\ln 7
\end{aligned}
$$

4.

$$
\begin{aligned}
\text{(Given series)} &= \sum_{n=0}^{\infty} \frac{n+1}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!} \Bigg|_{x=1} \\
&= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} \Bigg|_{x=1} \\
&= \frac{d}{dx}\left(xe^x\right) \Bigg|_{x=1} \\
&= \left((x+1)e^x\right)\Big|_{x=1} \\
&= 2e
\end{aligned}
$$

5.

$$\begin{aligned}
\text{(Given series)} &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\
&= \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= \frac{3}{4} \cdot \frac{\pi^2}{6} \\
&= \frac{\pi^2}{8}
\end{aligned}$$

6.

$$\begin{aligned}
\text{(Given series)} &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^4} - \sum_{n=1}^{\infty} \frac{1}{(2n)^4} \\
&= \frac{15}{16} \sum_{n=1}^{\infty} \frac{1}{n^4} \\
&= \frac{15}{16} \cdot \frac{\pi^4}{90} \\
&= \frac{\pi^4}{96}
\end{aligned}$$

7.

$$\begin{aligned}
\text{(Given series)} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} - 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n^2} \\
&= \frac{1}{2} \cdot \frac{\pi^2}{6} \\
&= \frac{\pi^2}{12}
\end{aligned}$$

8.

$$
\begin{aligned}
f(\theta) &= \sum_{n=0}^{\infty} \frac{\sin(n+1)\theta}{2n+1} \\
&= \Im\left\{\sum_{n=0}^{\infty} \frac{1}{2n+1} e^{i(n+1)\theta}\right\} \\
&= \Im\left\{e^{i\frac{\theta}{2}} \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(e^{i\frac{\theta}{2}}\right)^{2n+1}\right\} \\
&= \Im\left\{e^{i\frac{\theta}{2}} \tanh^{-1}\left(e^{i\frac{\theta}{2}}\right)\right\} \\
&= \frac{1}{2}\Im\left\{e^{i\frac{\theta}{2}} \ln\left(\frac{1+e^{i\frac{\theta}{2}}}{1-e^{i\frac{\theta}{2}}}\right)\right\} \\
&= \frac{1}{2}\Im\left\{e^{i\frac{\theta}{2}} \ln\left(i\cot\frac{\theta}{4}\right)\right\} \\
&= \frac{1}{2}\Im\left\{\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\left(\ln\left(\cot\frac{\theta}{4}\right) + i\frac{\pi}{2}\right)\right\} \\
&= \frac{\pi}{4}\cos\frac{\theta}{2} + \frac{1}{2}\ln\left(\cot\frac{\theta}{4}\right)\sin\frac{\theta}{2}
\end{aligned}
$$

9. Half of the challenge of this problem is expressing the nth term in closed form. Let us denote the first term as a_1.

$$
\begin{aligned}
a_n &= \frac{\left(\prod_{j=1}^{n-1} \frac{2j-1}{2}\right)\cdot\left(\frac{2n-1}{2}\right)^2}{(2n+7)(2n+5)(2n+3)^2 n!} \\
&= \frac{(2n-1)!!(2n-1)}{2^{n+1}(2n+7)(2n+5)(2n+3)^2 n!} \\
&= \frac{\frac{(2n)!}{2^n n!}\cdot(2n-1)}{2^{n+1}(2n+7)(2n+5)(2n+3)^2 n!} \\
&= \frac{(2n-1)\cdot(2n)!}{2^{2n+1}(2n+7)(2n+5)(2n+3)(n!)^2}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \frac{a_{n+1}}{a_n} &= \frac{(2n+1)^2(2n+2)(2n+3)^2}{(2n+9)(2n+5)(n+1)^2(2n-1)} \\
&\xrightarrow{n\to\infty} 4 > 1
\end{aligned}
$$

Therefore, the given series diverges.

10. Again, we shall denote the first term as a_1.

$$\begin{aligned}
a_n &= \frac{\left((2n+1)!!\right)^2}{4^{n-1} \cdot n \cdot (n!)^2} \\
&= \frac{\left(\dfrac{(2n+1)!}{2^n n!}\right)^2}{4^{n-1} \cdot n \cdot (n!)^2} \\
&= \frac{\left((2n+1)!\right)^2}{4^{2n+1} \cdot n \cdot (n!)^4}
\end{aligned}$$

To deduce the limit of this sequence, we employ Stirling's approximation:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \text{ (for large } n\text{).}$$

$$\begin{aligned}
\Rightarrow a_n &\approx \frac{2\pi(2n+1)\left(\dfrac{2n+1}{e}\right)^{2n+1}}{4^{2n+1} \cdot n \cdot 4\pi^2 n^2 \left(\dfrac{n}{e}\right)^n} \\
&= \frac{(2n+1)^{4n+3}}{\pi e^2 2^{8n+3} n^{4n+3}} \\
&= \frac{4}{\pi e}\left(1+\frac{1}{2n}\right)^{4n+3} \\
&= \frac{4}{\pi e}\left(\left(1+\frac{1}{2n}\right)^{2n}\right)^{\frac{4n+3}{2n}} \\
&\xrightarrow{n\to\infty} \frac{4}{\pi} \neq 0
\end{aligned}$$

Therefore, the given series diverges.

11. We want to evaluate

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)^2 x^{2n+1}}{(2n+1)!}.$$

To this end, we want to express $(n+1)^2 = n^2 + 2n + 1$ in terms of $2n+2$ and $(2n+1)(2n+3)$ as they can easily be obtained by differentiation. We employ Horner's schema to do so.

$$\begin{array}{c|ccc}
 & 1 & 2 & 1 \\
-1 & & -1 & -1 \\
\hline
 & 1 & 1 & \vert\ 0 \\
-\frac{3}{2} & & -\frac{3}{2} & \\
\hline
 & 1 & \vert\ -\frac{1}{2} &
\end{array}$$

$$
\begin{aligned}
\Rightarrow n^2+2n+1 &= \left(\left(n+\frac{3}{2}\right)-\frac{1}{2}\right)(n+1)\\
&= \frac{1}{4}(2n+3)(2n+2)-\frac{1}{4}(2n+2)
\end{aligned}
$$

$$
\begin{aligned}
\therefore f(x) &= \frac{1}{4}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2n+3)(2n+2)x^{2n+1}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2n+2)x^{2n+1}\right)\\
&= \frac{1}{4}\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+3}\right)-\frac{1}{4}\frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+2}\right)\\
&= \frac{1}{4}\frac{d^2}{dx^2}\left(x^2\sin x\right)-\frac{1}{4}\frac{d}{dx}\left(x\sin x\right)\\
&= \frac{1}{4}\left(2\sin x+4x\cos x-x^2\sin x\right)-\frac{1}{4}\left(\sin x+x\cos x\right)\\
&= \frac{1-x^2}{4}\sin x+\frac{3}{4}x\cos x
\end{aligned}
$$

12. (a) We should first formally state what the problem is asking of us.

$$
\begin{aligned}
(E+1)^k+(E-1)^k &= \sum_{i=0}^{k}\binom{k}{i}E_i+\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}E_i\\
&= \sum_{i=0}^{k}\left(1+(-1)^i\right)\binom{k}{i}E_i\\
&= 2\sum_{i=0}^{k/2}\binom{k}{2i}E_{2i}
\end{aligned}
$$

Therefore, we need to show that

$$
\sum_{i=0}^{l}\binom{2l}{2i}E_{2i}=0
$$

for all integers l.

$$
\begin{aligned}
1 &= \sec z\cdot\cos z\\
&= \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}E_{2n}z^{2n}\right)\cdot\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}\right)\\
&= \sum_{l=0}^{\infty}\left(\sum_{i=0}^{l}\frac{(-1)^i}{(2i)!}E_{2i}\cdot\frac{(-1)^{2l-i}}{(k-2i)!}\right)z^{2l}\\
&= \sum_{l=0}^{\infty}\left(\frac{1}{(2l)!}\sum_{i=0}^{l}\binom{2l}{2i}E_{2i}\right)z^{2l}
\end{aligned}
$$

$$\therefore (\forall l \geq 1) \sum_{i=0}^{l} \binom{2l}{2i} E_{2i} = 0$$

We could use this formula to determine successive terms of this sequence.

$$l = 1: \;\; E_2 + E_0 = 0 \Rightarrow E_2 = -1$$

$$l = 2: \;\; E_4 + 6E_2 + E_0 = 0 \Rightarrow E_4 = 5$$

$$l = 3: \;\; E_6 + 15E_4 + 15E_2 + E_0 = 0 \Rightarrow E_6 = -61$$

$$l = 4: \;\; E_8 + 28E_6 + 70E_4 + 28E_2 + E_0 = 0 \Rightarrow E_9 = 1385$$

(b) It is sometimes a good idea to start with the more general formula then apply it to the more concrete problem, as this could lead to less work.

$$\begin{aligned}
\sec \pi x &= \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i (2i+1)}{(2i+1)^2 - 4x^2} \\
&= \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \left(1 - \left(\frac{2x}{2i+1} \right)^2 \right)^{-1} \\
&= \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \sum_{n=0}^{\infty} \left(\frac{2x}{2i+1} \right)^{2n} \\
&= \frac{4}{\pi} \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} (-1)^i \frac{4^n}{2i+1^{2n+1}} x^{2n} \\
&= \sum_{n=0}^{\infty} \left(\frac{4^{n+1}}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^{2n+1}} \right) x^{2n} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} (\pi x)^{2n}
\end{aligned}$$

We thus compare the last two series term-by-term to obtain

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^{2n+1}} = \frac{\pi^{2n+1} (-1)^n E_{2n}}{4^{n+1} (2n)!}$$

as the answer to part (b).

Substituting $n = 1$ for part (a), we get

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^3} = \frac{\pi^3 (-1)^1 E_2}{4^2 \cdot 2!} = \frac{\pi^3}{32}.$$

3 Chapter 3

1. Let us define

$$I(a,b) = \int_0^\infty dy \frac{e^{-ay} - e^{-by}}{y}.$$

$$\frac{\partial I}{\partial a} = -\int_0^\infty dy e^{-ay} = -\frac{1}{a}, \ \frac{\partial I}{\partial b} = \int_0^\infty dy e^{-by} = \frac{1}{b}$$

$$\Rightarrow I(a,b) = \ln\left(\frac{b}{a}\right) + C$$

Now, notice that

$$C = I(a,a) = \int_0^\infty dy 0 = 0.$$

$$\therefore I(a,b) = \ln\left(\frac{b}{a}\right)$$

2. (If your knee-jerk reaction to this problem is any form of skepticism, then you are probably correct; if not, you are a perfect fit for a physicist in my opinion.)

$$\begin{aligned}
\int_0^\infty dx \sin bx &= \lim_{a \to 0+} \int_0^\infty dx e^{-ax} \sin bx \\
&= \lim_{a \to 0+} \Im \left\{ \int_0^\infty dx e^{-ax} e^{ibx} \right\} \\
&= \lim_{a \to 0+} \Im \left\{ \frac{1}{a - ib} \right\} \\
&= \lim_{a \to 0+} \frac{b}{a^2 + b^2} \\
&= \frac{1}{b}
\end{aligned}$$

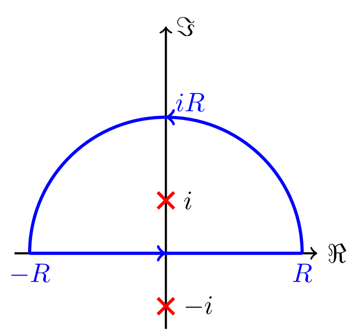

Figure 1: Contour for Problem 3–3, 3–4, and 3–9.

3. We shall perform a contour integral of $f(z) := \frac{e^{iaz}}{1+z^2}$ along Figure 1.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=i} f(z) = 2i\pi \cdot \left.\frac{e^{iaz}}{z+i}\right|_{z=i} = \pi e^{-a}$$

We also have

$$\begin{aligned}\oint_C dx f(z) &= \int_{-R}^{R} dx \frac{e^{iax}}{1+x^2} + \int_0^{\pi} d\theta i Re^{i\theta} \frac{e^{iaRe^{i\theta}}}{1+R^2e^{2i\theta}} \\ &\xrightarrow{R\to\infty} \int_{-\infty}^{\infty} dx \frac{e^{iax}}{1+x^2}.\end{aligned}$$

$$\begin{aligned}\therefore \int_0^{\infty} dx \frac{\cos ax}{1+x^2} &= \frac{1}{2}\int_{-\infty}^{\infty} dx \frac{\cos ax}{1+x^2} \\ &= \frac{1}{2}\cdot \Re\left\{\int_{-\infty}^{\infty} dx \frac{e^{iax}}{1+x^2}\right\} \\ &= \frac{\pi}{2} e^{-a}\end{aligned}$$

4. Let us integrate $f(z) := \frac{e^{iaz}}{(1+z^2)^2}$, again along Figure 1.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=i} f(z) = 2i\pi \cdot \left.\frac{d}{dz}\frac{e^{iaz}}{(z+i)^2}\right|_{z=i} = \frac{\pi}{2}(a+1)e^{-a}$$

and

$$\begin{aligned}\oint_C dx f(z) &= \int_{-R}^{R} dx \frac{e^{iax}}{(1+x^2)^2} + \int_0^{\pi} d\theta i Re^{i\theta} \frac{e^{iaRe^{i\theta}}}{(1+R^2e^{2i\theta})^2} \\ &\xrightarrow{R\to\infty} \int_{-\infty}^{\infty} dx \frac{e^{iax}}{(1+x^2)^2}.\end{aligned}$$

$$\begin{aligned}\therefore \int_0^{\infty} dx \frac{\cos ax}{(1+x^2)^2} &= \frac{1}{2}\int_{-\infty}^{\infty} dx \frac{\cos ax}{(1+x^2)^2} \\ &= \frac{1}{2}\cdot \Re\left\{\int_{-\infty}^{\infty} dx \frac{e^{iax}}{(1+x^2)^2}\right\} \\ &= \frac{\pi}{4}(a+1) e^{-a}\end{aligned}$$

5. Let $I(\mathbb{a}, b) = \int d^3x e^{i\mathbb{a}\cdot\mathbb{x}} e^{-br^2}$. Notice that this integral depends on a vector and results in a scalar. Thus, we must conclude that only the magnitude of $\mathbb{a}$ matters and not its direction.

Let $a := ||\mathbb{a}||$.

$$\begin{aligned} I(\mathbb{a}, b) &= I(a\hat{\mathbb{z}}, b) \\ &= \int d^3x e^{iaz} e^{-b\left(x^2+y^2+z^2\right)} \\ &= \int_{-\infty}^{\infty} dx e^{-bx^2} \int_{-\infty}^{\infty} dy e^{-by^2} \int_{-\infty}^{\infty} dz e^{iaz-bz^2} \\ &= \sqrt{\frac{\pi}{b}} \cdot \sqrt{\frac{\pi}{b}} \cdot \int_{-\infty}^{\infty} dz e^{-b\left(z-\frac{ia}{2b}\right)^2 - \frac{a^2}{4b}} \\ &= \left(\frac{\pi}{b}\right)^{3/2} e^{-\frac{a^2}{4b}} \end{aligned}$$

$$\therefore I(\mathbb{a}, b) = \left(\frac{\pi}{b}\right)^{3/2} e^{-\frac{\mathbb{a}^2}{4b}}$$

6. Let $\mathbb{I}(\mathbb{a}, b) = \int d^3x \mathbb{x} e^{i\mathbb{a}\cdot\mathbb{x}} e^{-br^2}$. Notice that this vector-valued integral depends on only one vector, namely, $\mathbb{a}$. Therefore, this integral must be parallel to this vector.

Let $a = ||\mathbb{a}||$ and $\mathbb{I}(\mathbb{a}, b) = F(a, b)\hat{\mathbb{a}}$.

$$\mathbb{I}(a\hat{\mathbb{z}}, b) = F(a, b)\hat{\mathbb{z}}$$

$$\begin{aligned} \Rightarrow F(a, b) &= \mathbb{I}(a\hat{\mathbb{z}}, b) \cdot \hat{\mathbb{z}} \\ &= \int d^3x z e^{iaz} e^{-b\left(x^2+y^2+z^2\right)} \\ &= \int_{-\infty}^{\infty} dx e^{-bx^2} \int_{-\infty}^{\infty} dy e^{-by^2} \int_{-\infty}^{\infty} dz z e^{iaz-bz^2} \\ &= \sqrt{\frac{\pi}{b}} \cdot \sqrt{\frac{\pi}{b}} \cdot \frac{\partial}{\partial \alpha} \int_{-\infty}^{\infty} dz e^{-b\left(z-\frac{\alpha}{2b}\right)^2 + \frac{\alpha^2}{4b}} \Big|_{\alpha=ia} \\ &= \frac{\pi}{b} \cdot \frac{\partial}{\partial \alpha} \left(\sqrt{\frac{\pi}{b}} e^{\frac{\alpha^2}{4b}}\right)\Big|_{\alpha=ia} \\ &= \left(\frac{\pi}{b}\right)^{3/2} \cdot \frac{\alpha}{2b} e^{\frac{\alpha^2}{4b}} \Big|_{\alpha=ia} \\ &= \left(\frac{\pi}{b}\right)^{3/2} \frac{ia}{2b} e^{-\frac{a^2}{4b}} \end{aligned}$$

$$\therefore I(\mathbb{a}, b) = F(||a||, b)\hat{\mathbb{a}} = \left(\frac{\pi}{b}\right)^{3/2} \frac{i\mathbb{a}}{2b} e^{-\frac{\mathbb{a}^2}{4b}}$$

7.

$$\begin{aligned}\ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (-x)^n \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(1 + (-1)^{n-1}\right) x^n \\ &= \sum_{n=0}^{\infty} \frac{2}{2n+1} x^{2n+1}\end{aligned}$$

$$\begin{aligned}\therefore \int_0^1 \frac{dx}{x} \ln\left(\frac{1+x}{1-x}\right) &= \int_0^1 dx \sum_{n=0}^{\infty} \frac{2}{2n+1} x^{2n} \\ &= \sum_{n=0}^{\infty} \frac{2}{(2n+1)^2} \\ &= 2\left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2}\right) \\ &= \frac{3}{2} \cdot \frac{\pi^2}{6} \\ &= \frac{\pi^2}{4}\end{aligned}$$

8.

$$\begin{aligned}\int_0^{\infty} \frac{dx}{\cosh x} &= \int_0^{\infty} dx \frac{2}{e^x + e^{-x}} \\ &= \int_1^{\infty} du \frac{2}{u^2+1} && (u := e^x) \\ &= \left[2 \tan^{-1} u\right]_1^{\infty} \\ &= \frac{\pi}{2}\end{aligned}$$

9. Let us integrate $f(z) := \frac{1}{(1+z^2)^2}$, again along Figure 1.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=i} f(z) = 2i\pi \cdot \left. \frac{d}{dz} \frac{1}{(z+i)^2} \right|_{z=i} = \frac{\pi}{2}$$

and

$$\oint_C dx f(z) = \int_{-R}^{R} dx \frac{1}{(1+x^2)^2} + \int_0^{\pi} d\theta \frac{iRe^{i\theta}}{(1+R^2 e^{2i\theta})^2} \xrightarrow{R\to\infty} \int_{-\infty}^{\infty} dx \frac{e^{iax}}{(1+x^2)^2}.$$

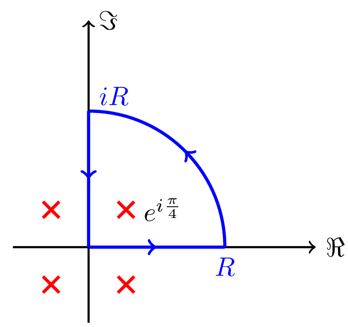

Figure 2: Contour for Problem 3–10.

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$$

10. We shall perform a contour integral of $f(z) := \frac{1}{1+z^4}$ along Figure 2.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=e^{i\frac{\pi}{4}}} f(z) = 2i\pi \cdot \left.\frac{1}{4x^3}\right|_{z=e^{i\frac{\pi}{4}}} = \frac{\pi}{2} e^{-i\frac{\pi}{4}}$$

We also have

$$\begin{aligned}
\oint_C dx f(z) &= \int_0^R \frac{dx}{1+x^4} + \int_0^{\frac{\pi}{2}} d\theta \frac{iRe^{i\theta}}{1+R^4e^{4i\theta}} + \int_R^0 dy \frac{i}{1+(iy)^4} \\
&= (1-i) \int_0^R \frac{dx}{1+x^4} + O\left(R^{-3}\right) \\
&\xrightarrow{R\to\infty} (1-i) \int_0^{\infty} \frac{dx}{1+x^4}.
\end{aligned}$$

$$\therefore \int_0^{\infty} \frac{dx}{1+x^4} = \frac{1}{1-i} \cdot \frac{\pi}{2} e^{-i\frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}}$$

11. We shall perform a contour integral of $f(\omega) := \frac{e^{i\omega t}}{\omega^2-\omega_0^2}$ along Figure 3.

$$\oint_C d\omega f(\omega) = 2i\pi \left(\operatorname*{Res}_{\omega=-\omega_0} f(\omega) + \operatorname*{Res}_{\omega=\omega_0} f(\omega) \right) = -\frac{2\pi}{\omega_0} \sin \omega_0 t$$

We also have

$$\begin{aligned}
\oint_C d\omega f(\omega) &= \int_{-R}^{R} d\omega \frac{e^{i\omega t}}{\omega^2-\omega_0^2} + \int_0^{\pi} d\theta iRe^{i\theta} \frac{e^{iRte^{i\theta}}}{R^2e^{2i\theta}-\omega_0^2} \\
&\xrightarrow{R\to\infty} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega t}}{\omega^2-\omega_0^2}
\end{aligned}$$

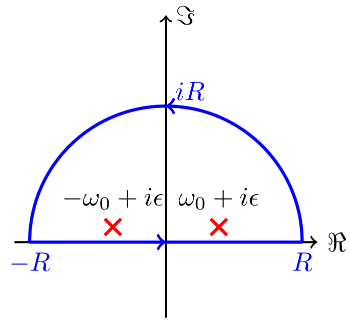

Figure 3: Contour for Problem 3–11.

$$\therefore \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega t}}{\omega^2 - \omega_0^2} = -\frac{2\pi}{\omega_0} \sin \omega_0 t$$

(Note: As there are poles on the real axis, this integral technically diverges without the "slightly above the axis" condition. If this integral were to have physical meaning, then this assumption must be physically explained, for example using causality arguments. Another way around this is to use the "Cauchy principal value" of the integral, whose formula is shown in Eq. (A–17).)

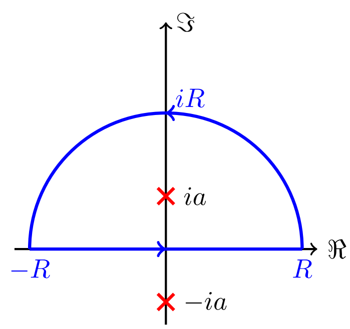

Figure 4: Contour for Problem 3–12, 3–13, 3–21, and 3–22.

12. We shall perform a contour integral of $f(z) := \frac{z^2}{(a^2+z^2)^2}$ along Figure 4.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=ia} f(z) = 2i\pi \cdot \frac{d}{dz} \frac{z^2}{(z+ia)^2}\bigg|_{z=ia} = \frac{\pi}{2a}$$

We also have

$$
\begin{aligned}
\oint_C dx f(z) &= \int_{-R}^{R} dx \frac{x^2}{\left(a^2+x^2\right)^2} + \int_0^{\pi} d\theta i R e^{i\theta} \frac{R^2 e^{2i\theta}}{\left(a^2+R^2 e^{2i\theta}\right)^2} \\
&\xrightarrow{R\to\infty} \int_{-\infty}^{\infty} dx \frac{x^2}{\left(a^2+x^2\right)^2}. \\
\therefore \int_{-\infty}^{\infty} dx \frac{x^2}{a^2+x^{2^2}} &= \frac{\pi}{2a}
\end{aligned}
$$

13.

$$
\begin{aligned}
\int \frac{d^3x}{\left(a^2+r^2\right)^3} &= \int_0^{\infty} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \frac{r^2 \sin\theta}{\left(a^2+r^2\right)^3} \\
&= 4\pi \int_0^{\infty} dr \frac{r^2}{\left(a^2+r^2\right)^3}
\end{aligned}
$$

We shall perform a contour integral of $f(z) := \frac{z^2}{\left(a^2+z^2\right)^3}$, again along Figure 4.

$$
\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=ia} f(z) = \frac{2i\pi}{2!} \cdot \frac{d^2}{dz^2} \frac{z^2}{\left(z+ia\right)^3}\bigg|_{z=ia} = \frac{\pi}{8a^3}
$$

We also have

$$
\begin{aligned}
\oint_C dx f(z) &= \int_{-R}^{R} dr \frac{r^2}{\left(a^2+r^2\right)^3} + \int_0^{\pi} d\theta i R e^{i\theta} \frac{R^2 e^{2i\theta}}{\left(a^2+R^2 e^{2i\theta}\right)^3} \\
&\xrightarrow{R\to\infty} \int_{-\infty}^{\infty} dx \frac{r^2}{\left(a^2+r^2\right)^3}. \\
\Rightarrow \int_0^{\infty} dr \frac{r^2}{\left(a^2+r^2\right)^3} &= \frac{1}{2} \int_{-\infty}^{\infty} dx \frac{r^2}{\left(a^2+r^2\right)^3} = \frac{\pi}{16a^3} \\
\therefore \int \frac{d^3x}{\left(a^2+r^2\right)^3} &= \frac{\pi^2}{4a^3}
\end{aligned}
$$

14.

$$
\begin{aligned}
I &:= \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}(a+bx)} \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \frac{\cos\theta}{\cos\theta\,(a+b\sin\theta)} && (x=\sin\theta) \\
&= \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} \\
&= \oint_{|z|=1} \frac{dz/iz}{a+\frac{b}{2i}\left(z-\frac{1}{z}\right)} && (z=e^{i\theta}) \\
&= \oint_{|z|=1} \frac{dz}{bz^2+2iaz-b}
\end{aligned}
$$

Notice that the last integrand has simple poles at $z_\pm^* := \frac{-a\pm\sqrt{a^2-b^2}}{b}i$, of which only the positive one lies inside the unit circle.

$$\therefore I = 2i\pi \operatorname*{Res}_{z=z_+^*} \frac{1}{bz^2+2iaz-b} = \frac{\pi}{\sqrt{a^2-b^2}}$$

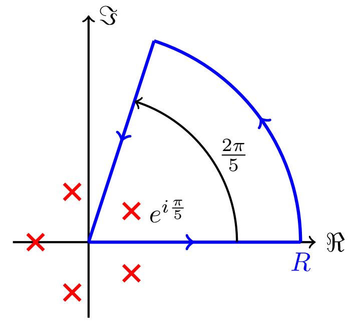

Figure 5: Contour for Problem 3–15.

15. We shall perform a contour integral of $f(z) := \frac{z}{1+z^5}$ along Figure 5.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=e^{i\frac{\pi}{5}}} f(z) = 2i\pi \cdot \left.\frac{z}{5z^4}\right|_{z=e^{i\frac{\pi}{5}}} = \frac{2\pi}{5}e^{-i\frac{\pi}{10}}$$

We also have

$$\begin{aligned}\oint_C dx f(z) &= \int_0^R dx \frac{x}{1+x^5} + \int_0^{\frac{2\pi}{5}} d\theta iRe^{i\theta}\frac{Re^{i\theta}}{1+R^5e^{5i\theta}} + \int_R^0 dxe^{i\frac{2\pi}{5}}\frac{xe^{i\frac{2\pi}{5}}}{1+x^5} \\ &\xrightarrow{R\to\infty} \left(1+e^{-i\frac{\pi}{5}}\right)\int_0^\infty dx\frac{x}{1+x^5}.\end{aligned}$$

$$\therefore \int_0^\infty dx \frac{x}{1+x^5} = \frac{\pi}{5\cos\frac{\pi}{10}}$$

16. Let $z := e^{i\theta}$.

$$dz = ie^{i\theta}d\theta,\ \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right),\ \sin^2\theta = \frac{1-\cos 2\theta}{2} = \frac{1}{4}\left(2-z^2-\frac{1}{z^2}\right)$$

$$\begin{aligned}I &:= \int_0^{2\pi} d\theta \frac{\sin^2\theta}{a+b\cos\theta} \\ &= \frac{1}{2}\oint_{|z|=1}\frac{-\left(z^2-1\right)^2}{z\left(bz^2+2az+b\right)}\end{aligned}$$

Let $f(z)$ be the integrand here. $f(z)$ has simple poles at

$$0,\ z_\pm^* := \frac{-a \pm \sqrt{a^2 - b^2}}{b},$$

among which 0 and z_+^* lie within the unit circle (which is our contour).

$$\begin{aligned}
\therefore I &= \frac{1}{2} \cdot 2i\pi \left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=z_+^*} f(z) \right) \\
&= i\pi \left(\left. \frac{-\left(z^2-1\right)^2}{bz^2 + 2az + b} \right|_{z=0} + \left. \frac{-\left(z^2-1\right)^2}{z\left(bz + a + \sqrt{a^2-b^2}\right)} \right|_{z=z_+^*} \right) \\
&= \frac{i\pi}{b} \left(-1 + 4 \frac{a^4 + b^4 - 2ab\left((a-b)\sqrt{a^2-b^2} + b^2\right)}{-a + \sqrt{a^2-b^2}} \right)
\end{aligned}$$

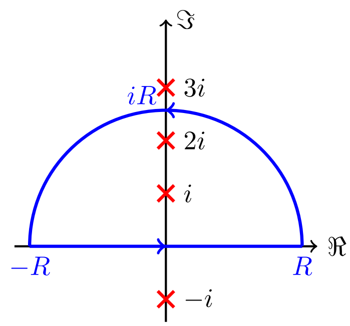

Figure 6: Contour for Problem 3–17.

17. We must first determine whether, or more precisely, for which values of a, this integral converges. As $|x| \to \infty$, $\sinh ax \sim e^{|ax|}$. Thus, this integral converges whenever $|a| < \pi$.

Let $f(z) := \frac{\sinh az}{\sinh \pi z}$ The singularity at the origin is removable, as

$$\lim z \to 0 \frac{\sinh az}{\sinh \pi z} = \lim_{z \to 0} \frac{a \cosh az}{\pi \cosh \pi z} = \frac{a}{\pi}$$

via L'Hôpital's rule. (Note the sloppy math here: the first equality hold only because the second expression does indeed converge to a finite number. If this were a calculus class, this equation is prone to points lost; since this is physics, we can proceed as is.) Thus, $f(z)$ has poles at $z = im,\ m \in \mathbb{Z}\backslash\{0\}$. We ignore cases where a is a rational multiple of π for convinience; these cases are left as exercise for the reader.[2]

[2] Just kidding! I will come back for this later, but if you'd like to help, please consider discussing on `https://github.com/pingpingy1/Mathews-MathPhys-Sol/issues/1`.

Consider the contour shown in Figure 6.

$$\begin{aligned}
\oint_C dz f(z) &\xrightarrow{R\to\infty} 2i\pi \sum_{m=1}^{\infty} \operatorname*{Res}_{z=im} f(z) \\
&= 2i \sum_{m=1}^{\infty} \frac{\sinh iam}{\cosh i\pi m} \\
&= 2 \sum_{m=1}^{\infty} (-1)^{m-1} \sin am \\
&= 2\Im\left\{ \sum_{m=1}^{\infty} e^{i(m-1)\pi} e^{iam} \right\} \\
&= \tan\frac{a}{2}
\end{aligned}$$

We also have

$$\begin{aligned}
\oint_C dz f(z) &= \int_{-R}^{R} dx \frac{\sinh ax}{\sinh \pi x} + \int_0^{\pi} d\theta i R e^{i\theta} \frac{\sinh\left(aRe^{i\theta}\right)}{\sinh\left(\pi R e^{i\theta}\right)} \\
&= \int_{-R}^{R} dx \frac{\sinh ax}{\sinh \pi x} + O\left(\frac{R}{e^{(\pi-|a|)}R}\right) \\
&\xrightarrow{R\to\infty} \int_{-\infty}^{\infty} dx \frac{\sinh ax}{\sinh \pi x}
\end{aligned}$$

$$\therefore \int_{-\infty}^{\infty} dx \frac{\sinh ax}{\sinh \pi x} = \tan\frac{a}{2}$$

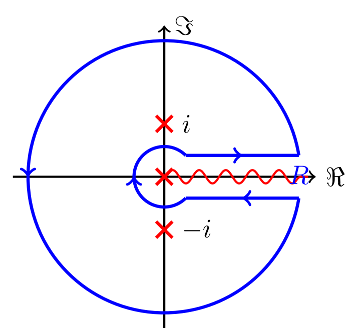

Figure 7: Contour for Problem 3–18.

18. Recalling the exercise in the main text regarding Eq. (3–37), we could infer that a direct contour integration of the integrand along some contour analogous to Figure 3–3 in the textbook would lead to a loss in the exponent of the logarithm (or one could just do it and observe).

Thus, let $f(z) := \frac{(\ln z)^3}{1+z^2}$, where we place the branch cut on the real axis and $\arg z = 0$ just above the real axis. As such, the poles of $f(z)$ have arguments $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Consider the contour shown in Figure 7.

$$\begin{aligned}
\oint_C dz f(z) &= 2i\pi \left(\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-i} f(z) \right) \\
&= 2i\pi \left(\left. \frac{(\ln z)^3}{2z} \right|_{z=i} + \left. \frac{(\ln z)^3}{2z} \right|_{z=-i} \right) \\
&= \frac{13}{4} i\pi
\end{aligned}$$

We also have

$$\begin{aligned}
\oint_C dz f(z) = &\int_\epsilon^R dx \frac{(\ln x)^3}{1+x^2} + \int_0^{2\pi} d\theta i R e^{i\theta} \frac{(\ln R + i\theta)^3}{R^2 e^{2i\theta} + 1} \\
&+ \int_R^\epsilon dx \frac{(\ln x + 2i\pi)^3}{1+x^2} + \int_{2\pi}^0 d\theta i \epsilon e^{i\theta} \frac{(\ln \epsilon + i\theta)^3}{\epsilon^2 e^{2i\theta} + 1} \\
&\xrightarrow[\epsilon \to 0]{R \to \infty} -6i\pi \int_0^\infty dx \frac{(\ln x)^2}{1+x^2} + 12\pi^2 \int_0^\infty dx \frac{\ln x}{1+x^2} + 4i\pi^4.
\end{aligned}$$

This same procedure can be used to show that

$$\int_0^\infty dx \frac{\ln x}{1+x^2} = 0,$$

which is left as an exercise to the reader[3].

$$\therefore \int_0^\infty dx \frac{\ln x^2}{1+x^2} = \frac{\pi^3}{8}$$

19. Let $f(z) := \frac{1}{1+z^2+z^4}$. Notice that $z^6 - 1 = \left(z^2 - 1\right)\left(z^4 + z^2 + 1\right)$, so the poles of $f(z)$ lie at $e^{i\frac{\pi}{3}}$, $e^{i\frac{2\pi}{3}}$, $e^{i\frac{4\pi}{3}}$, and $e^{i\frac{5\pi}{3}}$.

Consider the contour shown in Figure 8.

$$\begin{aligned}
\oint_C dz f(z) &= 2i\pi \left(\operatorname*{Res}_{z=e^{i\frac{\pi}{3}}} f(z) + \operatorname*{Res}_{z=e^{i\frac{2\pi}{3}}} f(z) \right) \\
&= 2i\pi \left(\left. \frac{1}{4z^3+2z} \right|_{z=e^{i\frac{\pi}{3}}} + \left. \frac{1}{4z^3+2z} \right|_{z=e^{i\frac{2\pi}{3}}} \right) \\
&= \frac{\pi}{\sqrt{3}}
\end{aligned}$$

[3] Just kidding! I will get back to this, but if you could, please consider discussing on `https://github.com/pingpingy1/Mathews-MathPhys-Sol/issues/2`.

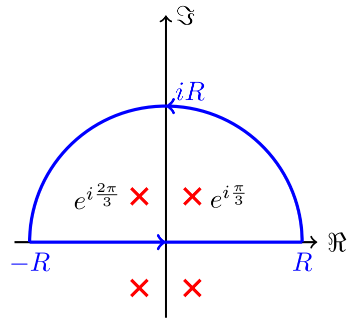

Figure 8: Contour for Problem 3–19.

We also have

$$\oint_C dz f(z) = \int_{-R}^{R} \frac{dx}{1+x^2+x^4} + \int_0^{\pi} d\theta \frac{iRe^{i\theta}}{1+R^2e^{2i\theta}+R^4e^{4i\theta}} \xrightarrow{R\to\infty} 2\int_0^{\infty} \frac{dx}{1+x^2+x^4}.$$

$$\therefore \int_0^{\infty} \frac{dx}{1+x^2+x^4} = \frac{\pi}{2\sqrt{3}}$$

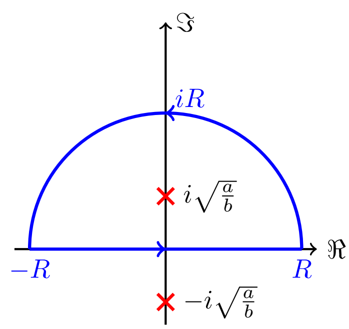

Figure 9: Contour for Problem 3–20.

20. We shall perform a contour integral of $f(z) := \frac{1}{(a+bz^2)^3}$ along Figure 9.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=i\sqrt{\frac{a}{b}}} f(z) = \frac{2i\pi}{2!} \cdot \frac{d^2}{dz^2} \frac{1}{b^{3/2}\left(\sqrt{b}z+i\sqrt{a}\right)^3}\Bigg|_{z=i\sqrt{\frac{a}{b}}} = \frac{3\pi}{8a^{5/2}b^{1/2}}$$

We also have

$$\oint_C dx f(z) = \int_{-R}^{R} \frac{dx}{(a+bx^2)^3} + \int_0^{\pi} d\theta \frac{iRe^{i\theta}}{(a+bR^2e^{2i\theta})^3} \xrightarrow{R\to\infty} 2\int_0^{\infty} \frac{dx}{(a+bx^3)^3}.$$

$$\therefore \int_0^\infty \frac{dx}{\left(a+bx^3\right)^3} = \frac{3\pi}{16a^{5/2}b^{1/2}}$$

21. We shall perform a contour integral of $f(z) := \frac{z^2}{(a^2+z^2)^3}$ along Figure 4.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=ia} f(z) = \frac{2i\pi}{2!} \cdot \frac{d^2}{dz^2} \frac{z^2}{\left(z+ia\right)^3}\bigg|_{z=ia} = \frac{\pi}{8a^3}$$

We also have

$$\begin{aligned}\oint_C dx f(z) &= \int_{-R}^R dx \frac{x^2}{\left(a^2+x^2\right)^3} + \int_0^\pi d\theta i R e^{i\theta} \frac{R^2 e^{2i\theta}}{\left(a^2+R^2e^{2i\theta}\right)^3} \\ &\xrightarrow{R\to\infty} 2\int_0^\infty dx \frac{x^2}{\left(a^2+x^2\right)^3}.\end{aligned}$$

$$\therefore \int_0^\infty dx \frac{x^2}{a^2+x^{2^3}} = \frac{\pi}{16a^3}$$

22. We shall perform a contour integral of $f(z) := \frac{\sin z}{z(a^2+z^2)}$ along Figure 4. Note that the singularity at the origin is removable, as we may define $f(0) := \lim_{z\to 0} f(z) = \frac{1}{a^2}$.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=ia} f(z) = 2i\pi \cdot \frac{\sin z}{z\left(z+ia\right)}\bigg|_{z=ia} = \frac{\pi \sin a}{a^2}$$

We also have

$$\begin{aligned}\oint_C dx f(z) &= \int_{-R}^R dx \frac{\sin x}{x\left(a^2+x^2\right)} + \int_0^\pi d\theta i R e^{i\theta} \frac{\sin\left(Re^{i\theta}\right)}{Re^{i\theta}\left(a^2+R^2e^{2i\theta}\right)} \\ &\xrightarrow{R\to\infty} 2\int_0^\infty dx \frac{\sin x}{x\left(a^2+x^2\right)}.\end{aligned}$$

(Note: One may object that it is nontrivial that the second integral vanishes in the limit, as $\sin z$ grows without bound along the imaginary axis. A physicist like myself would sweep this fact under the rug, but if any mathematician would like to give a more rigorous proof of this fact, I'm all ears!)

$$\therefore \int_0^\infty dx \frac{x^2}{a^2+x^{2^3}} = \frac{\pi \sin a}{2a^2}$$

23. Let us perform the change of variable $z := e^{i\theta}$, such that the contour of integration is around the unit circle.

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{\left(a+b\cos\theta\right)^2} &= \oint_{|z|=1} \frac{dz/iz}{\left(a+\frac{b}{2}\left(z+\frac{1}{z}\right)\right)^2} \\ &= -4i \oint_{|z|=1} dz \frac{z}{\left(bz^2+2az+b\right)^2}\end{aligned}$$

The integrand has poles of second order at

$$z_\pm^* := \frac{-a \pm \sqrt{a^2 - b^2}}{b},$$

but only z_+^* lies inside the unit circle.

$$\begin{aligned}
\therefore \int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2} &= -4i \cdot 2i\pi \operatorname*{Res}_{z=z_+^*} f(z) \\
&= 8\pi \cdot \frac{d}{dz} \frac{z}{\left(bz + a + \sqrt{a^2 - b^2}\right)^2}\Bigg|_{z=z_+^*} \\
&= \frac{2\pi a}{(a^2 - b^2)^{3/2}}
\end{aligned}$$

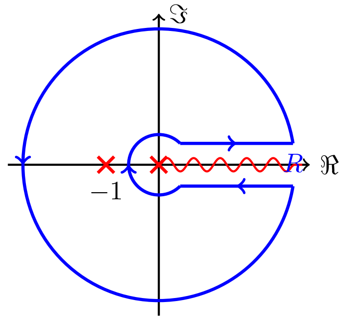

Figure 10: Contour for Problem 3–24.

24. Let $f(z) := \frac{(\ln z)^2}{(z+1)^2}$, where we place the branch cut on the real axis and $\arg z = 0$ just above the real axis. (Recall the exercise in the main text regarding Eq. (3–37); we need an extra $\ln z$ for this procedure!) As such, the pole of $f(z)$ has argument π. Consider the contour shown in Figure 10.

$$\begin{aligned}
\oint_C dz f(z) &= 2i\pi \operatorname*{Res}_{z=-1} f(z) \\
&= 2i\pi \cdot \frac{d^2}{dz^2}(\ln z)^2\Bigg|_{z=e^{i\pi}} \\
&= 4\pi^2
\end{aligned}$$

We also have

$$
\begin{aligned}
\oint_C dz f(z) &= \int_\epsilon^R dx \frac{(\ln x)^2}{(x+1)^2} + \int_0^{2\pi} d\theta i R e^{i\theta} \frac{(\ln R + i\theta)^2}{(Re^{i\theta}+1)^2} \\
&+ \int_R^\epsilon dx \frac{(\ln x + 2i\pi)^2}{(x+1)^2} + \int_{2\pi}^0 d\theta i \epsilon e^{i\theta} \frac{(\ln \epsilon + i\theta)^2}{(\epsilon e^{i\theta}+1)^2} \\
&\xrightarrow[\epsilon\to 0]{R\to\infty} \\
&\int_0^\infty dx \frac{(\ln x)^2}{(x+1)^2} + 0 \\
&+ \left(-\int_0^\infty dx \frac{(\ln x)^2}{(x+1)^2} - 4i\pi \int_0^\infty dx \frac{\ln x}{(x+1)^2} + 4\pi^2 \int_0^\infty \frac{dx}{(x+1)^2} \right) \\
&+ 0 \\
&= -4i\pi \int_0^\infty dx \frac{\ln x}{(x+1)^2} + 4\pi^2.
\end{aligned}
$$

$$
\therefore \int_0^\infty dx \frac{\ln x}{(x+1)^2} = 0
$$

25. Let us define

$$
I(a) := \int_0^\infty dx e^{-x^2} \operatorname{Ci}(ax).
$$

$$
\begin{aligned}
I'(a) &= \int_0^\infty dx \frac{\partial}{\partial a} \left(e^{-x^2} \operatorname{Ci}(ax) \right) \\
&= \frac{1}{a} \int_0^\infty dx e^{-x^2} \cos ax \\
&= \frac{1}{2a} \int_{-\infty}^\infty dx e^{-x^2} \cos ax \\
&= \frac{1}{2a} \Re \left\{ \int_{-\infty}^\infty dx e^{-x^2 + iax} \right\} \\
&= \frac{e^{-a^2/4}}{2a} \Re \left\{ \int_{-\infty}^\infty dx e^{-\left(x - \frac{ia}{2}\right)^2} \right\} \\
&= \frac{\sqrt{\pi} e^{-a^2/4}}{2a}
\end{aligned}
$$

Also, $\lim_{x\to\infty} \operatorname{Ci}(x) = 0$, so $\lim_{a\to\infty} I(a) = 0$.

$$\begin{aligned}
\therefore I(a) &= \frac{\sqrt{\pi}}{2}\int_{\infty}^{a} dt\frac{e^{-t^2/4}}{t} \\
&= -\frac{\sqrt{\pi}}{4}\int_{-\infty}^{-a^2/4} dv\frac{e^{v}}{v} && (v := -\frac{t^2}{4}) \\
&= -\frac{\sqrt{\pi}}{4}\operatorname{Ei}\left(-\frac{a^2}{4}\right)
\end{aligned}$$

(Note: What if we used Si instead of Ci inside the integrand? This is a harder problem since the integral

$$\int_0^{\infty} dxe^{-x^2}\sin ax$$

is rather tricky. The reader is encouraged to explore this, possibly with the help of the Dawson function[4].)

26.

$$\begin{aligned}
\int_0^{\infty} dxe^{-ax}\operatorname{erf} x &= \left[-\frac{1}{a}e^{-ax}\operatorname{erf} x\right]_0^{\infty} + \frac{1}{a}\int_0^{\infty} dxe^{-ax}\cdot\frac{2}{\sqrt{\pi}}e^{-x^2} \\
&= 0 + \frac{2}{\sqrt{\pi}a}\int_0^{\infty} dxe^{-x^2-ax} \\
&= \frac{2}{\sqrt{\pi}a}\int_0^{\infty} dxe^{-\left(x+\frac{a}{2}\right)^2+\frac{a^2}{4}} \\
&= \frac{2e^{a^2/4}}{\sqrt{\pi}a}\int_{a/2}^{\infty} dxe^{-x^2} \\
&= \frac{1}{a}e^{a^2/4}\left(1-\operatorname{erf}\frac{a}{2}\right)
\end{aligned}$$

27. (a)

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\Gamma\left(\frac{1}{2}\right)^2} = \sqrt{\frac{\pi}{\sin\frac{\pi}{2}}} = \sqrt{\pi}$$

(b)

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{4}\Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{\pi}}{4}$$

(c)

$$\mathrm{B}(1,3) = \frac{\Gamma(1)\Gamma(3)}{\Gamma(4)} = \frac{0!2!}{3!} = \frac{1}{3}$$

[4] https://en.wikipedia.org/wiki/Dawson_function

(d)

$$\mathrm{B}\left(\frac{1}{2},-\frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(0\right)}=\frac{\sqrt{\pi}\cdot\frac{\sqrt{\pi}}{-1/2}}{\infty}=0$$

(e)

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(-\frac{1}{3}\right)=-3\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)=-3\cdot\frac{\pi}{\sin\frac{\pi}{3}}=-2\sqrt{3}\pi$$

28.

$$-\operatorname{Ei}(-x)=\int_{-\infty}^{-x}dy\frac{e^{y}}{y}=\int_{\infty}^{x}dt\frac{e^{-t}}{t}$$

$$\begin{aligned}
\therefore\int_{0}^{\infty}dxe^{-ax}\left(-\operatorname{Ei}(-x)\right)&=\int_{0}^{\infty}dx\int_{\infty}^{x}dt\frac{e^{-ax-t}}{t}\\
&=-\int_{0}^{\infty}dx\int_{-(a+1)x}^{\infty}du\frac{e^{-u}}{u-ax} && (u:=ax+t)\\
&=-\int_{0}^{\infty}du\int_{0}^{u/(a+1)}dx\frac{e^{-u}}{u-ax}\\
&=-\int_{0}^{\infty}du\int_{0}^{1/a(a+1)}dv\frac{u}{a}\frac{e^{-u}}{u(1-v)} && (v:=\frac{ax}{u})\\
&=-\frac{1}{a}\left(\int_{0}^{\infty}due^{-u}\right)\cdot\left(\int_{0}^{1/a(a+1)}\frac{dv}{1-v}\right)\\
&=\frac{1}{a}\ln\left(1-\frac{1}{a(a+1)}\right)
\end{aligned}$$

29. We suspect that changing the range of integration to $[0,1]$ might be beneficial in expressing this integral as a Beta function. Thus, let $u:=\frac{x+2}{4}$.

$$\begin{aligned}
\int_{-2}^{2}dx\left(4-x^{2}\right)^{1/6}&=\int_{0}^{1}4du\cdot\left(4-(4u-2)^{2}\right)^{1/6}\\
&=2^{8/3}\int_{0}^{1}duu^{1/6}(1-u)^{1/6}\\
&=2^{8/3}\,\mathrm{B}\left(\frac{1}{6},\frac{1}{6}\right)=2^{8/3}\frac{\Gamma\left(\frac{1}{6}\right)^{2}}{\Gamma\left(\frac{1}{3}\right)}
\end{aligned}$$

30. The contour described in the problem is shown in Figure 11. The choice of branch is such that $\Im\left\{\ln(-t)\right\}$ equals π just below the branch cut and $-\pi$ just above it.

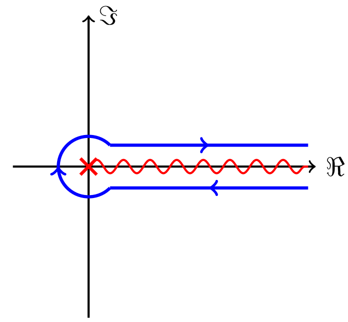

Figure 11: Contour for Problem 3–30.

$$
\begin{aligned}
F(z) &= \int_C dt(-t)^{z-1}e^{-t} \\
&= \int_\infty^\epsilon dx e^{(z-1)(\ln x+i\pi)-x} \\
&+ \int_0^{2\pi} d\theta i\epsilon e^{i\theta} e^{(z-1)(\ln\epsilon+i(\pi-\theta))-\epsilon e^{i\theta}} \\
&+ \int_\epsilon^\infty dx e^{(z-1)(\ln x-i\pi)-x} \\
&\xrightarrow{\epsilon\to 0+} -\int_0^\infty dx e^{(z-1)(\ln x+i\pi)-x} + \int_0^\infty dx e^{(z-1)(\ln x-i\pi)-x} \\
&= 2i\sin\pi z\Gamma(z)
\end{aligned}
$$

$$\therefore \text{(something)} = 2i\sin\pi z$$

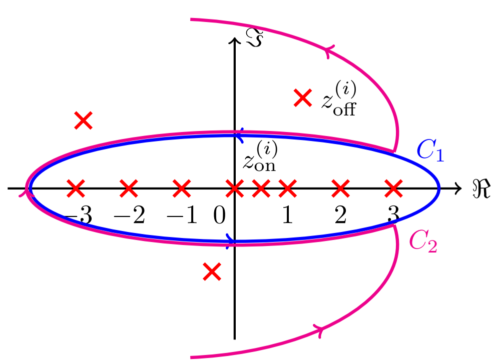

Figure 12: Contour for Problem 3–31.

31. (a) For simplicity, we shall assume that $f(z)$ has no poles exactly on any of the real integers. (In fact, I am not sure if this procedure could even be done with them!) Let us denote the poles of $f(z)$ off of the real axis as $z_{\text{off}}^{(i)}$ and those on the real axis as $z_{\text{on}}^{(i)}$. (Of course, we are also assuming that $f(z)$ only has isolated singularities on the entire plane. Note that this is a tiny bit more general than meromorphic functions[5], since we allow the singularities to be essential, not poles. We just need them to be discrete, i.e., no cluster points.)

We first consider the contour C_1 in Figure 12 to obtain

$$\begin{aligned}\oint_{C_1} dz f(z)\cot\pi z &= 2i\pi\sum_n \operatorname*{Res}_{z=n} f(z)\cot\pi z + 2i\pi\sum_i \operatorname*{Res}_{z=z_{\text{on}}^{(i)}} f(z)\cot\pi z\\ &= 2i\pi\sum_n \left.\frac{f(z)\cos\pi z}{\pi\cos\pi z}\right|_{z=n} + 2i\pi\sum_i \operatorname*{Res}_{z=z_{\text{on}}^{(i)}} f(z)\cot\pi z\\ &=: 2i\sum_n f(n) + 2i\pi\sum_i R_{\text{on}}^{(i)}\end{aligned}$$

where $R_{\text{on}}^{(i)} := \operatorname{Res}_{z=z_{\text{on}}^{(i)}} f(z)\cot\pi z$. Similarly for C_2,

$$\oint_{C_2} dz f(z)\cot\pi z = 2i\pi\sum_i \operatorname*{Res}_{z=z_{\text{off}}^{(i)}} f(z)\cot\pi z =: 2i\sum_n f(n) + 2i\pi\sum_i R_{\text{off}}^{(i)}$$

where $R_{\text{off}}^{(i)} := \operatorname{Res}_{z=z_{\text{off}}^{(i)}} f(z)\cot\pi z$.

Now, following the exercise regarding Eq. (3–43), we note that if $|f(z)|$ vanishes sufficiently quickly[6] as z approaches infinity, then

$$\oint_{C_1+C_2} dz f(z)\cot\pi z \xrightarrow{R\to\infty} 0.$$

$$\therefore \sum_n f(n) = -\pi\sum_i R_{\text{on}}^{(i)} - \pi\sum_i R_{\text{off}}^{(i)} = -\pi\sum_i R^{(i)}$$

where we drop the subscript as the distinction between the two types of residues have disappeared.

(Note: One may be tempted to write

$$R^{(i)} = \cot\pi z^{(i)}\cdot \operatorname*{Res}_{z=z^{(i)}} f(z)$$

which may seem so since $\cot\pi z$ is analytic on a neighborhood of $z^{(i)}$; goodness knows I fell for this. However, if one considers the entire Laurent series, they can show that this is not the case. In particular, if $z^{(i)}$ is a pole of nth order, then the residue can be caluclated using the first n Taylor coefficients of $\cot\pi z$ around $z^{(i)}$. This means that this naive equation holds for simple poles, specifically! See also: Convolutions[7])

[5] `https://en.wikipedia.org/wiki/Meromorphic_function`
[6] But what does this mean??!
[7] `https://en.wikipedia.org/wiki/Convolution`

(b) Let $f(z) := \frac{1}{z^2+a^2}$. This function has simple poles at $\pm ia$. We require that ia not be an integer, as per the above discussion.

$$\operatorname*{Res}_{z=\pm ia} f(z) = \left.\frac{\cot \pi z}{z \pm ia}\right|_{z=\pm ia} = \frac{\cot \pm i\pi a}{\pm 2ia} = -\frac{1}{2a} \coth \pi a$$

$$\therefore g(a) = -\pi \left(R_+ + R_-\right) = \frac{\pi}{a} \coth \pi a$$

32. Let θ denote the angle away from the dotted axis in the figure. We need not consider the azimuthal angle around it as the system is cylindrically symmetric. The intensity of neutrons as emitted by the source into a range $[\theta, \theta + d\theta]$ is given by

$$dI_{\text{source}} = I_0 \frac{d\Omega}{4\pi} = \frac{I_0}{2} \sin\theta d\theta$$

where I_0 is the total intensity of neutrons. Such neutrons travel a distance of $T \sec\theta$ through the absorber, so the intensity of neutrons that arrive at the film is given by

$$dI_{\text{film}} = \frac{I_0}{2} e^{-\frac{T}{\lambda}\sec\theta} \sin\theta d\theta.$$

These neutrons travel through the film, again, a distance proportional to $\sec\theta$. Thus, one can introduce a proportionality constant C to obtain

$$dA = Ce^{-\frac{T}{\lambda}\sec\theta} \sin\theta d\theta \cdot \sec\theta = Ce^{-\frac{T}{\lambda}\sec\theta} \tan\theta d\theta.$$

Therefore, integrating over the film, we obtain the following formula for the total activity:

$$\begin{aligned}
A(\lambda) &= \int_0^{\tan^{-1}\left(\frac{b}{a}\right)} d\theta C e^{-\frac{T}{\lambda}\sec\theta} \tan\theta \\
&= C \int_1^{\frac{\sqrt{a^2+b^2}}{a}} du \frac{\exp\left(-\frac{T}{\lambda}u\right)}{u} && (u := \sec\theta) \\
&= C \int_{\frac{T}{\lambda}}^{\frac{T}{\lambda}\frac{\sqrt{a^2+b^2}}{a}} dv \frac{\exp(-v)}{v} && (v := \frac{T}{\lambda}u) \\
&= C \left(\left(-\operatorname{Ei}\left(-\frac{T}{\lambda} \frac{\sqrt{a^2+b^2}}{a} \right) \right) - \left(-\operatorname{Ei}\left(-\frac{T}{\lambda} \right) \right) \right).
\end{aligned}$$

The activity without the absorber is given by

$$A_0 = \lim_{T\to 0} A(\lambda) = \int_0^{\tan^{-1}\left(\frac{b}{a}\right)} d\theta C \tan\theta = C \ln\left(\frac{\sqrt{a^2+b^2}}{a} \right).$$

Hence, we obtain

$$\frac{A}{A_0} = \frac{\left(-\operatorname{Ei}\left(-\frac{T}{\lambda} \frac{\sqrt{a^2+b^2}}{a} \right) \right) - \left(-\operatorname{Ei}\left(-\frac{T}{\lambda} \right) \right)}{\ln\left(\frac{\sqrt{a^2+b^2}}{a} \right)}$$

which, given the numerical constants, is an equation that the almighty WolframAlpha could solve! Plugging in the given constants and assuming λ is in units of centimeters, we get

$$0.25 = \frac{\left(-\operatorname{Ei}\left(-\frac{\sqrt{2}}{\lambda}\right)\right) - \left(-\operatorname{Ei}\left(-\frac{1}{\lambda}\right)\right)}{\ln\left(\sqrt{2}\right)}.$$

$$\therefore \lambda \approx 0.856\,\mathrm{cm}$$

33. As $e^{\alpha z}$ is analytic everywhere, $\Gamma(z)e^{\alpha z}$ has poles precisely at the nonpositive integers.

$$I := \oint_{|z|=\frac{5}{2}} dz \Gamma(z)e^{\alpha z} = 2i\pi \left(\operatorname*{Res}_{z=0} \Gamma(z)e^{\alpha z} + \operatorname*{Res}_{z=-1} \Gamma(z)e^{\alpha z} + \operatorname*{Res}_{z=-2} \Gamma(z)e^{\alpha z} \right)$$

To calculate these residues, we wish to transform the equation

$$\operatorname*{Res}_{z=-n} \Gamma(z)e^{\alpha z} = \lim_{z\to -n} (z+n)\Gamma(z)e^{\alpha z}$$

in such a way to make the argument of the gamma function positive, where n is a nonnegative integer.

$$\begin{aligned}
\operatorname*{Res}_{z=-n} \Gamma(z)e^{\alpha z} &= \lim_{z\to -n} (z+n)\Gamma(z)e^{\alpha z} \\
&= \lim_{z\to -n} (z+n) \cdot \frac{z(z+1)\cdots(z+n)}{z(z+1)\cdots(z+n)} \Gamma(z)e^{\alpha z} \\
&= \lim_{z\to -n} \frac{\Gamma(z+n+1)e^{\alpha z}}{z(z+1)\cdots(z+n-1)} \\
&= \frac{\Gamma(0)e^{-n\alpha}}{(-n)(-n+1)\cdots(-1)} \\
&= \frac{(-e^{\alpha})^n}{n!}
\end{aligned}$$

$$\therefore I = 2i\pi \left(1 - e^{\alpha} + \frac{e^{2\alpha}}{2} \right)$$

34. (i) $|x| << 1$: We use the familiar Taylor expansion of $\sin t$.

$$\begin{aligned}
\int_0^x dt \frac{\sin t}{t} &= \int_0^x dt \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\cdot(2k+1)!} x^{2k+1} \\
&= x - \frac{x^3}{18} + \frac{x^5}{600} - \cdots
\end{aligned}$$

Note that this series absolutely converges for all values of x.

(ii) $|x| >> 1$: Here, following the exercise deriving Eq. (3–72), we perform repeated integration by parts. To deal with the trigonometric functions more easily, we express them as complex exponentials.

$$\begin{aligned}
\int_0^x dt \frac{\sin t}{t} &= \frac{\pi}{2} - \Im\left\{\int_x^\infty dt \frac{e^{it}}{t}\right\} \\
&= \frac{\pi}{2} + \Im\left\{\int_x^\infty dt \frac{e^{i(t-\pi)}}{t}\right\} \\
&= \frac{\pi}{2} + \Im\left\{\left[\frac{-ie^{i(t-\pi)}}{t}\right]_x^\infty - \int_x^\infty dt \frac{-ie^{i(t-\pi)}}{-t^2}\right\} \\
&= \frac{\pi}{2} + \Im\left\{\frac{e^{i\left(x-\frac{\pi}{2}\right)}}{x}\right\} + \Im\left\{\int_x^\infty dt \frac{e^{i\left(t-\frac{3\pi}{2}\right)}}{t^2}\right\} \\
&= \frac{\pi}{2} + \Im\left\{\frac{e^{i\left(x-\frac{\pi}{2}\right)}}{x}\right\} + \Im\left\{\left[\frac{-ie^{i\left(t-\frac{3\pi}{2}\right)}}{t}\right]_x^\infty - 2\int_x^\infty dt \frac{-ie^{i\left(t-\frac{3\pi}{2}\right)}}{-t^3}\right\} \\
&= \frac{\pi}{2} + \Im\left\{\frac{e^{i\left(x-\frac{\pi}{2}\right)}}{x}\right\} + \Im\left\{\frac{e^{i(x-\pi)}}{x^2}\right\} + 2\Im\left\{\int_x^\infty dt \frac{e^{i(t-2\pi)}}{t^3}\right\} \\
&= \cdots \\
&= \frac{\pi}{2} + \sum_{k=0}^{n} k!\Im\left\{\frac{e^{i\left(x-\frac{k+1}{2}\pi\right)}}{x^{k+1}}\right\} + (n+1)!\Im\left\{\int_x^\infty dt \frac{e^{i\left(t-\frac{n+3}{2}\pi\right)}}{t^{n+2}}\right\}
\end{aligned}$$

Note that the sum in this expression diverges for all finite values of x, since the summand grows like $k!$.

35. We make use of the saddle-point approximation for this integral. Let $f(t) := xt - e^t$ such that $I(x) = \int_0^\infty dt \exp(f(t))$. $f(t)$ has a single stationary point, which is a global maximum at $(\ln x, x\ln x - x)$. The second derivative of $f(t)$ is evaluated to $-x$.

$$\begin{aligned}
\therefore I &\approx \int_{-\infty}^{\infty} dt e^{-\frac{x}{2}(t-\ln x)^2 + x\ln x - x} \\
&= e^{x\ln x - x}\int_{-\infty}^{\infty} dt e^{-\frac{x}{2}t^2} \\
&= \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x
\end{aligned}$$

What might be more important than this calculation is defending that the saddle-point approximation holds in this case. To this end, one should argue that the integral around the "hump" of $f(x)$ contributes the most to the integral. For this, one notices that the height of the hump grows like $x\ln x$, while the curvature at that point grows like x. Thus, the width of the hump, which can be approximated by the full width at half maximum of the parabola it is approximated as, grows like $\sqrt{\ln x}$. While the exact value of the width tends to grow with x, it is positioned at $t = \ln x$, which moves away from the origin quadratically faster than the growth of the width. Thus, as x becomes larger, the integral is better approximated via the parabola at the hump, i.e., the saddle-point approximation.

36. The average number of reactions per time is given by the integral

$$I := \int_0^\infty dENEe^{-E/kT} \cdot Me^{-\alpha/\sqrt{E}} = NM \int_0^\infty dE \exp\left(-\frac{E}{kT} - \frac{\alpha}{\sqrt{E}} + \ln E\right).$$

To employ the saddle-point approximation, let

$$f(E) := -\frac{E}{kT} - \frac{\alpha}{\sqrt{E}} + \ln E.$$

$$\Rightarrow f'(E) = -\frac{1}{kT} + \frac{\alpha}{2E^{3/2}} + \frac{1}{E}$$

Suppose $f(E)$ has a stationary point at $E_0 = \lambda_0 kT$.

$$f'(E_0) = -\frac{1}{kT} + \frac{\alpha}{2(kT)^{3/2}}\lambda_0^{-3/2} + \frac{1}{\lambda_0 kT} = 0 \Rightarrow \lambda_0 \approx 1$$

$$\Rightarrow f(kT) = -1 - \frac{\alpha}{\sqrt{kT}} + \ln(kT) \approx \ln(kT) - 1$$

$$f''(kT) = -\frac{3}{4}\frac{\alpha}{(kT)^{5/2}} - \frac{1}{(kT)^2} \approx -\frac{1}{(kT)^2}$$

$$\begin{aligned}\therefore I &\approx NM \int_{-\infty}^{\infty} dv \exp\left(\ln(kT) - 1 - \frac{v^2}{2(kT)^2}\right)\\ &= NM\frac{kT}{e} \cdot \sqrt{2\pi(kT)^2}\\ &= \frac{\sqrt{2\pi}}{e} NM(kT)^2\end{aligned}$$

37. We must evaluate the integral

$$\psi(\mathbb{k}, \mathbb{l}) := \int \frac{d\Omega}{(1 + \mathbb{k} \cdot \hat{\mathfrak{r}})(1 + \mathbb{l} \cdot \hat{\mathfrak{r}})}$$

using Feynman's identity (Eq. (3–28)).

$$\begin{aligned}\Rightarrow \psi(\mathbb{k}, \mathbb{l}) &= \int d\Omega \int_0^1 du \frac{1}{((1 + \mathbb{k} \cdot \hat{\mathfrak{r}})u + (1 + \mathbb{l} \cdot \hat{\mathfrak{r}})(1-u))^2}\\ &= \int_0^1 du \int \frac{d\Omega}{(1 + (u\mathbb{k} + (1-u)\mathbb{l}) \cdot \hat{\mathfrak{r}})^2}\end{aligned}$$

Let

$$F(\mathbb{v}) := \int \frac{d\Omega}{\left(1 + \mathbb{v} \cdot \hat{\mathbb{r}}\right)^2}.$$

This scalar-valued integral is a function of a vector; thus, it may only depend on the vector's magnitude. Let $v := ||\mathbb{v}||$.

$$\begin{aligned}
\Rightarrow F(\mathbb{v}) &= F\left(v\hat{\mathbb{z}}\right) \\
&= \int \frac{d\Omega}{\left(1 + vz\right)^2} \\
&= \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin\theta \cdot \frac{1}{\left(1 + v\cos\theta\right)^2} \\
&= 2\pi \left[\frac{1}{v(1 + v\cos\theta)}\right]_0^\pi \\
&= \frac{4\pi}{1 - v^2}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \psi\left(\mathbb{k}, \mathbb{l}\right) &= \int_0^1 du F\left(u\mathbb{k} + (1-u)\mathbb{l}\right) \\
&= 4\pi \int_0^1 \frac{du}{1 - \left|\left|u\mathbb{k} + (1-u)\mathbb{l}\right|\right|^2} \\
&= 4\pi \int_0^1 \frac{du}{1 - l^2 - 2\left(\mathbb{k} \cdot \mathbb{l} - l^2\right)u + \left(k^2 + l^2 - 2\mathbb{k} \cdot \mathbb{l}\right)u^2}
\end{aligned}$$

where $k := ||\mathbb{k}||$ and $l := ||\mathbb{l}||$. Now, notice that

$$\frac{d}{dx} \tanh^{-1} x = \frac{1}{1 - x^2}.$$

$$\begin{aligned}
\Rightarrow \frac{1}{c - 2bx - ax^2} &= \frac{1}{\sqrt{ac + b^2}} \cdot \frac{\frac{a}{\sqrt{ac+b^2}}}{1 - \left(\frac{a}{\sqrt{ac+b^2}}\left(x + \frac{b}{a}\right)\right)^2} \\
&= \frac{d}{dx}\left(\frac{1}{\sqrt{ac + b^2}} \tanh^{-1}\left(\frac{ax + b}{\sqrt{ac + b^2}}\right)\right)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \psi\left(\mathbb{k}, \mathbb{l}\right) &= \frac{4\pi}{\sqrt{\left(1 - \mathbb{k} \cdot \mathbb{l}\right)^2 - \left(1 - k^2\right)\left(1 - l^2\right)}} \\
&\quad \times \left[\tanh^{-1}\left(\frac{\left(k^2 + l^2 - 2\mathbb{k} \cdot \mathbb{l}\right)u + \mathbb{k} \cdot \mathbb{l} - l^2}{\sqrt{\left(1 - \mathbb{k} \cdot \mathbb{l}\right)^2 - \left(1 - k^2\right)\left(1 - l^2\right)}}\right)\right]_0^1 \\
&= \frac{4\pi}{\sqrt{A^2 - B^2}}\left(\tanh^{-1}\left(\frac{A - 1 + k^2}{\sqrt{A^2 - B^2}}\right) + \tanh^{-1}\left(\frac{A - 1 + l^2}{\sqrt{A^2 - B^2}}\right)\right)
\end{aligned}$$

where $A := 1 - \Bbbk \cdot \mathbb{l}$ and $B := \sqrt{(1-k^2)(1-l^2)}$.

Now, note that

$$\begin{aligned}\cosh\left(\tanh^{-1} x + \tanh^{-1} y\right) &= \cosh\left(\tanh^{-1} x\right)\cosh\left(\tanh^{-1} y\right) \\ &+ \sin\left(\tanh^{-1} x\right)\sinh\left(\tanh^{-1} y\right) \\ &= \frac{1}{(1-x^2)} \cdot \frac{1}{(1-x^2)} + \frac{x}{(1-x^2)} \cdot \frac{y}{(1-y^2)} \\ &= \frac{1+xy}{\sqrt{(1-x^2)(1-y^2)}}.\end{aligned}$$

In our case, we have

$$x = \frac{A-1+k^2}{\sqrt{A^2-B^2}} \text{ and } y = \frac{A-1+l^2}{\sqrt{A^2-B^2}}$$

which yields

$$1 + xy = \frac{A}{A^2-B^2}\left(2A+k^2+l^2-2\right)$$

and

$$\sqrt{(1-x^2)(1-y^2)} = \frac{B}{A^2-B^2}\left(2A+k^2+l^2-2\right).$$

$$\Rightarrow \tanh^{-1} x + \tanh^{-1} y = \cosh^{-1}\left(\frac{1+xy}{\sqrt{(1-x^2)(1-y^2)}}\right) = \cosh^{-1}\frac{A}{B}$$

$$\therefore \psi\left(\Bbbk, \mathbb{l}\right) = \frac{4\pi}{\sqrt{A^2-B^2}}\cosh^{-1}\frac{A}{B}$$

38. If you have read the textbook thoroughly (unlike myself), then you may recall the "method of stationary phase" explained around Eq. (3–88).

$$f_n(x) = \int_C dt \exp\left(ix\left(\frac{n}{x}t - \sin t\right)\right)$$

Let $g(t) := \frac{n}{x}t - \sin t$. For large positive values of x, nearly all contribution to the integral comes near

$$g'(t_0) = \frac{n}{x} - \cos t_0 = 0 \Rightarrow t_0 = -\cos^{-1}\frac{n}{x}.$$

$$g(t_0) = -\frac{n}{x}\cos^{-1}\frac{n}{x} + \sqrt{1-\left(\frac{n}{x}\right)^2}$$

$$g''(t_0) = -\sqrt{1-\left(\frac{n}{x}\right)^2}$$

$$
\begin{aligned}
\therefore f_n(x) &\approx \int_{-\infty}^{\infty} dt \exp\left(ix\left(g(t_0)+\frac{g''(t_0)}{2}(t-t_0)^2\right)\right)\\
&= e^{ixg(t_0)}\sqrt{\frac{2\pi}{ixg''(t_0)}}\\
&= e^{i\left(\sqrt{x^2-n^2}-n\cos^{-1}\frac{n}{x}-\frac{\pi}{4}\right)}\sqrt{\frac{2\pi}{x\left(1-\frac{1}{2}\left(\frac{n}{x}\right)^2\right)}}
\end{aligned}
$$

For large values of x, we can simplify this expression to only consider leading powers of x.

$$
\begin{aligned}
\therefore f_n(x) &\approx e^{i\left(x\left(1-\frac{1}{2}\left(\frac{n}{x}\right)^2\right)-n\left(\frac{\pi}{2}-\frac{n}{x}\right)-\frac{\pi}{4}\right)}\sqrt{\frac{2\pi}{x}}\left(1+\frac{1}{4}\left(\frac{n}{x}\right)^2\right)\\
&\approx \sqrt{\frac{2\pi}{x}}e^{i\left(x-\frac{2n+1}{4}\pi\right)}\left(1+\frac{in^2}{2x}\right)
\end{aligned}
$$

A Appendix

6. (This single problem had cost me months of my life, and here's the unsatisfying conclusion of that journey. I am fairly certain that there are a myriad of different possible answers for this problem with varying levels of sophistication, and possibly even more ways to derive said solutions. For my purposes, I have settled on a solution that is semi-straightforward with an answer that I can plug into almighty WolframAlpha.)

We will derive a procedure to (painstakingly) calculate the Laurent series of the given function. First, let us define some notations. Recall that if a function $f(z)$ is analytic on an annulus around a point z_0, then it can be expanded into a unique Laurent series

$$
\begin{aligned}
f(z) &= \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k\\
&= \cdots + \frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-1}}{z-z_0}+a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots.
\end{aligned}
$$

For Laurent series specifically centered around the origin, let us denote the Laurent coefficients as

$$\left[z^k\right](f(z)) := a_k.$$

Then, multiplying a monomial corresponds to

$$\left[z^k\right]\left(z^l f(z)\right) = \left[z^{k-l}\right](f(z)).$$

Also, the residue at the origin can be compactly expressed as

$$\operatorname*{Res}_{z=0} f(z) = \left[z^{-1}\right](f(z)).$$

The residue asked of us is given by

$$\begin{aligned}
\operatorname*{Res}_{z=\pi} z^2 e^{\frac{1}{\sin z}} &= \operatorname*{Res}_{z=0} (z+\pi)^2 e^{\frac{1}{\sin(z+\pi)}} \\
&= \left[z^{-1}\right]\left(\left(z^2+2\pi z+\pi^2\right)e^{-\frac{1}{\sin z}}\right) \\
&= \left[z^{-3}\right]\left(e^{-\frac{1}{\sin z}}\right) + 2\pi\left[z^{-2}\right]\left(e^{-\frac{1}{\sin z}}\right) + \pi^2\left[z^{-1}\right]\left(e^{-\frac{1}{\sin z}}\right).
\end{aligned}$$

Thus, our job is reduced to calculating the Laurent coefficients of $e^{-\frac{1}{\sin z}}$. We expand the exponential into its Taylor series and obtain

$$\left[z^{-n}\right]\left(e^{-\frac{1}{\sin z}}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\left[z^{-n}\right]\left((\sin z)^{-k}\right).$$

Squaring, cubing, and so forth a power series (or even a Laurent series) is something achievable, but inverting a power series is a much more complicated task. We thus investigate a general method of calculating the inverse powers of a general series. Consider the following manipulation:

$$\begin{aligned}
\left[z^{-n}\right]\left((f(z))^{-k}\right) &= \left[z^{-1}\right]\left(z^{n-1}(f(z))^{-k}\right) \\
&= \left[z^{-1}\right]\left(\left(\frac{z^n}{n}\right)'(f(z))^{-k}\right) \\
&= \left[z^{-1}\right]\left(\left(\frac{z^n}{n}(f(z))^{-k}\right)'\right) - \left[z^{-1}\right]\left(\frac{z^n}{n}\left((f(z))^{-k}\right)'\right)
\end{aligned}$$

which comes mostly from playing around, but perhaps reasonable since the -1th[8] coefficient seems to be special in certain ways. In fact, it is so special that we can immediately see that the first term must be zero, regardless of whether $f(z)$ is analytic at the origin! This is because no derivative of a monomial yields the -1th power term.[9] Thus,

$$\begin{aligned}
\left[z^{-n}\right]\left((f(z))^{-k}\right) &= -\left[z^{-1}\right]\left(\frac{z^n}{n}\left((f(z))^{-k}\right)'\right) \\
&= \frac{k}{n}\left[z^{-1}\right]\left(z^n(f(z))^{-k-1}f'(z)\right).
\end{aligned}$$

Let $g(z) := z^n(f(z))^{-k-1}f'(z)$ and $g_{-1} := \left[z^{-1}\right](g(z))$. The Laurent series of $g(z) - g_{-1}z^{-1}$ contains no z^{-1} term, and thus can be integrated to an analytic function; let us call it $G(z)$. Hence, $g(z) = G'(z) + g_{-1}z^{-1}$.

$$g(f^{-1}(z))\left(f^{-1}(z)\right)' = \left(G(f^{-1}(z))\right)' + g_{-1}\frac{\left(f^{-1}(z)\right)'}{f^{-1}(z)}$$

[8]I had to search for whether it should be 'st' or 'th'; see `https://english.stackexchange.com/questions/326604/is-it-correct-to-say-1th-or-1st`.

[9]This can also be seen via $\operatorname{Res}_{z=0} f(z) = \frac{1}{2i\pi}\oint dz f'(z) = 0$ around the origin.

Again, $\left(G(f^{-1}(z))\right)'$ has zero residue. If the residue of $\frac{\left(f^{-1}(z)\right)'}{f^{-1}(z)}$ could be easily calculated and is nonzero, then we get an elegant formula for g_{-1} as

$$
\begin{aligned}
g_{-1} &= \frac{\left[z^{-1}\right]\left(g(f^{-1}(z))\left(f^{-1}(z)\right)'\right)}{\left[z^{-1}\right]\left(\frac{(f^{-1}(z))'}{f^{-1}(z)}\right)} \\
&= \frac{\left[z^{-1}\right]\left(\left(f^{-1}(z)\right)^{n} z^{-k-1} f'\left(f^{-1}(z)\right)\left(f^{-1}(z)\right)'\right)}{\left[z^{-1}\right]\left(\frac{(f^{-1}(z))'}{f^{-1}(z)}\right)} \\
&= \frac{\left[z^{k}\right]\left(\left(f^{-1}(z)\right)^{n}\right)}{\left[z^{-1}\right]\left(\frac{(f^{-1}(z))'}{f^{-1}(z)}\right)}
\end{aligned}
$$

and consequently

$$
\left[z^{-n}\right]\left((f(z))^{-k}\right) = \frac{k}{n}\frac{\left[z^{k}\right]\left(\left(f^{-1}(z)\right)^{n}\right)}{\left[z^{-1}\right]\left(\frac{(f^{-1}(z))'}{f^{-1}(z)}\right)}.
$$

Now, *if* $f(z)$ is analytic at the origin *and* $f(0) = 0$ *and* $f'(0) \neq 0$ (which just so happens to hold for $f(z) = \sin z$), then one could show that these same conditions hold for $f^{-1}(z)$ and consequently

$$
\left[z^{-1}\right]\left(\frac{\left(f^{-1}(z)\right)'}{f^{-1}(z)}\right) = 1.
$$

(This could be shown by substituting $n = -1$ for Problem A–4, since we did not require that n be positive anyway!) Therefore, we arrive at the pivotal formula

$$
\left[z^{-n}\right]\left((f(z))^{-k}\right) = \frac{k}{n}\left[z^{k}\right]\left(\left(f^{-1}(z)\right)^{n}\right).
$$

This is known in the maths world as the Lagrange inversion formula,[10] and it is what we desperately needed to calculate the residue in question!

Having sufficiently patted ourselves on our respective backs after that long derivation, let us go back to the original question. We have essentially derived the formula

$$
\left[z^{-n}\right]\left((\sin z)^{-k}\right) = \frac{k}{n}\left[z^{k}\right]\left((\arcsin z)^{n}\right).
$$

(Note that I use arcsin in lieu of $\sin^{-1}$ as all these exponents are getting con-

[10] `https://en.wikipedia.org/wiki/Formal_power_series#The_Lagrange_inversion_formula`

fusing.) Let us substitute this formula into the previous equation:

$$\begin{aligned}
\left[z^{-n}\right]\left(e^{-\frac{1}{\sin z}}\right) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[z^{-n}\right]\left((\sin z)^{-k}\right) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \cdot \frac{k}{n} \left[z^k\right]\left((\arcsin z)^n\right) \\
&= \frac{1}{n} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} \left[z^k\right]\left((\arcsin z)^n\right).
\end{aligned}$$

As for the powers of $\arcsin z$, let us simply expand out the multiplications using the Taylor series of $\arcsin z$ only.

$$\begin{aligned}
\arcsin z &= \int_0^z \frac{du}{\sqrt{1-u^2}} \\
&= \int_0^z dt \sum_{t=0}^{\infty} \binom{-\frac{1}{2}}{t} u^{2t} \\
&= \sum_{t=0}^{\infty} \frac{(-1/2)(-3/2)\cdots(1/2-t)}{2^t t!} \cdot \frac{u^{2t+1}}{2t+1} \\
&= \sum_{t=0}^{\infty} \frac{(-1)^t (2t)!}{4^t (2t+1)(t!)^2} u^{2t+1}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \left[z^k\right]\left((\arcsin z)^n\right) &= \sum_{t_1+\cdots+t_k=n} \prod_{j=1}^{n} \left[z^{t_j}\right](\arcsin z) \\
&= \sum_{(2l_1+1)+\cdots+(2l_k+1)=n} \prod_j \left[z^{2l_j+1}\right](\arcsin z) \\
&= \sum_{2\sum_j l_j + k = n} \left(-\frac{1}{4}\right)^{\sum_j l_j} \prod_j \left(\frac{(2l_j)!}{(2l_j+1)(l_j!)^2}\right)
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow \left[z^{-n}\right]\left(e^{-\frac{1}{\sin z}}\right) &= \frac{1}{n}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k-1)!}\left[z^k\right]\left((\arcsin z)^n\right) \\
&= \frac{1}{n}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k-1)!} \\
&\times \sum_{2\sum_j l_j + k = n}\left(-\frac{1}{4}\right)^{\sum_j l_j}\prod_j\left(\frac{(2l_j)!}{(2l_j+1)(l_j!)^2}\right) \\
&= \frac{1}{n}\sum_{l_1=0}^{\infty}\cdots\sum_{l_n=0}^{\infty}\frac{(-1)^{n-2\sum_j l_j}}{\left(n-1+2\sum_j l_j\right)!} \\
&\times \left(-\frac{1}{4}\right)^{\sum_j l_j}\prod_j\left(\frac{(2l_j)!}{(2l_j+1)(l_j!)^2}\right) \\
&= \frac{1}{n}\sum_{l_1=0}^{\infty}\cdots\sum_{l_n=0}^{\infty}\frac{(-1)^{\sum_j l_j + n}}{4^{\sum_j l_j}\left(n-1+2\sum_j l_j\right)!} \\
&\times \prod_j\left(\frac{(2l_j)!}{(2l_j+1)(l_j!)^2}\right)
\end{aligned}
$$

We evaluate this formula for $n = 1, 2, 3$.

$$
\left[z^{-1}\right]\left(e^{-\frac{1}{\sin z}}\right) = \sum_{l_1=0}^{\infty}\frac{(-1)^{l_1+1}}{4^{l_1}(2l_1+1)(l_1!)^2}
$$

$$
\left[z^{-2}\right]\left(e^{-\frac{1}{\sin z}}\right) = \frac{1}{2}\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\frac{(-1)^{l_1+l_2}}{4^{l_1+l_2}(2l_1+2l_2+1)!}\frac{(2l_1)!}{(2l_1+1)(l_1!)^2}\frac{(2l_2)!}{(2l_2+1)(l_2!)^2}
$$

$$
\begin{aligned}
\left[z^{-3}\right]\left(e^{-\frac{1}{\sin z}}\right) &= \frac{1}{3}\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{l_3=0}^{\infty}\frac{(-1)^{l_1+l_2+l_3+1}}{4^{l_1+l_2+l_3}(2l_1+2l_2+2l_3+2)!} \\
&\times \frac{(2l_1)!}{(2l_1+1)(l_1!)^2}\frac{(2l_2)!}{(2l_2+1)(l_2!)^2}\frac{(2l_3)!}{(2l_3+1)(l_3!)^2}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \operatorname*{Res}_{z=\pi} z^2 e^{\frac{1}{\sin z}} &= \sum_{l_1=0}^{\infty}\frac{(-1)^{l_1+1}(2l_1)!}{4^{l_1}(2l_1+1)(l_1!)^2}\left(\frac{\pi^2}{(2l_1)!}\right. \\
&- \sum_{l_2=0}^{\infty}\frac{(-1)^{l_2}(2l_2)!}{4^{l_2}(2l_2+1)(l_2!)^2}\left(\frac{\pi}{(2l_1+2l_2+1)!}\right. \\
&- \left.\left.\sum_{l_3=0}^{\infty}\frac{(-1)^{l_3}(2l_3)!}{4^{l_3}(2l_3+1)(l_3!)^2}\cdot\frac{1}{3(2l_1+2l_2+2l_3+2)!}\right)\right)
\end{aligned}
$$

Concluding remarks:

(i) This result, while hard achieved on my part, is still pretty shitty; for one, it cannot be calculated via WolframAlpha alone.[11] Some points for improvement include:

- More compact expressions for the Taylor coefficients of $(\arcsin z)^k$
- A numerical evaluation of the residue
- Even a closed-form solution? (This sounds crazy, but $\left[z^{-1}\right]\left(e^{-\frac{1}{\sin z}}\right)$ *does* have a closed-form expression! It uses a hypergeometric function, somehow.)

And we all know that "We leave so-and-so to future work." translates to either "I give up." or "Give me more money."

(ii) I have been hinted at that this problem could also be solved via a contour integral method, where we try to evaluate

$$\oint_{|z|=1} dz z^2 e^{\frac{1}{\sin z}}$$

using some tabulated-function trickery. I'm too tired to pursue this path, but it is still another viable method.

[11] If any Mathematica wizards could numerically calculate this, I'm all ears.