My Solutions for "Mathematical Methods of Physics (Second Edition)" by J. Mathews, R. L. Walker

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0 Introduction

This document is an archive of my solutions to J. Mathews and R. L. Walker's "Mathematical Methods of Physics" textbook. None of the solutions have been verified by anyone other than myself, whom I do not consider a reliable source. Hence, it is strongly advised against to use this solution for any application where accuracy matters, especially for assignments and any academic work, not to mention the ethical implications of such actions where it could be considered as cheating. However, I have not yet been able to find any other solutions. Please use this manual as just a suggestion. If you spot any mistakes, please report them to the Github repository. [1]

1 Chapter 1

1. We first acknowledge that $y = 0$ is a solution, so we seek solutions that are not identically zero.

Let $y = xv$.

$$y' = v + xv' \Rightarrow x^2(v + xv') + x^2v^2 = x \cdot xv \cdot (v + xv')$$

$$\Rightarrow v + xv' = xvv' \Rightarrow \left(1 - \frac{1}{v}\right) dv = \frac{1}{x} dx \Rightarrow v - \ln v = \ln x + C$$

$$\therefore \frac{y}{x} - \ln y = C$$

2.

$$\frac{y}{\sqrt{1 + y^2}} dy = \frac{x}{\sqrt{1 + x^2}} dx$$

$$\therefore \sqrt{1 + y^2} = \sqrt{1 + x^2} + C$$

[1] https://github.com/pingpingy1/Mathews-MathPhys-Sol

3. Let $v = x + y$.

$$v' = 1 + y' \Rightarrow v' - 1 = \frac{a^2}{v^2} \Rightarrow \frac{v^2}{a^2 + v^2} dv = dx$$

$$x + C = \int dv \left(1 - \frac{a^2}{a^2 + v^2}\right) = v - a\tan^{-1}\left(\frac{v}{a}\right) = x + y - a\tan^{-1}\left(\frac{x+y}{a}\right)$$

$$\therefore y - a\tan^{-1}\left(\frac{x+y}{a}\right) = C$$

4. We first seek the complementary solutions.

$$y_c' + y_c \cos x = 0 \Rightarrow \frac{dy_c}{y_c} + \cos x dx = 0 \Rightarrow y_c = Ce^{-\sin x}$$

For the particular solution, we first observe that $\frac{1}{2}\sin 2x = \sin x \cos x$. Thus, we shall try the ansatz $y_p = \sin x + A$.

$$\cos x + (\sin x + A)\cos x = \sin x \cos x \Rightarrow A = -1$$

$$\therefore y = Ce^{-\sin x} + \cos x - 1$$

5.

$$\left(1 + x^2\right) y' = xy(y+1) \Rightarrow \frac{dy}{y(y+1)} = \frac{x}{1 - x^2} dx$$

$$\Rightarrow \ln\left(\frac{y}{y+1}\right) = -\frac{1}{2}\ln\left(1 - x^2\right) + C_1$$

$$\Rightarrow \frac{y+1}{y} = e^{-C_1}\sqrt{1 - x^2} = C\sqrt{1 - x^2}$$

$$\therefore y = \frac{1}{C\sqrt{1 - x^2} - 1}$$

6. We first take note that the equation is dimension-consistent with $[y] = [x^{-2}]$. Thus, we define $v = x^2 y$ or $y = \frac{v}{x^2}$.

$$y' = \frac{v'}{x^2} - \frac{2v}{x^3} \Rightarrow 2xv' - 4v = 1 + \sqrt{1 + 4v}$$

$$\Rightarrow \frac{v'}{1 + 4v + \sqrt{1 + 4v}} = \frac{1}{2x}$$

$$\Rightarrow \left(\frac{1}{\sqrt{1 + 4v}} - \frac{1}{\sqrt{1 + 4v} + 1}\right) dv = \frac{dx}{2x}$$

Basic calculus yields $\int \frac{dv}{\sqrt{1+4v}+1} = \frac{1}{2}\left(\sqrt{1 + 4v} - \ln\left(1 + \sqrt{1 + 4v}\right)\right) + C$. Hence,

$$\frac{1}{2}\sqrt{1 + 4v} - \frac{1}{2}\left(\sqrt{1 + 4v} - \ln\left(1 + \sqrt{1 + 4v}\right)\right) = \frac{1}{2}\ln x + C_1$$

$$\Rightarrow 1 + \sqrt{1 + 4v} = e^{2C_1} x = Cx$$

$$\therefore \sqrt{1+4x^2y} = Cx - 1$$

7. Let $v' := y'$.

$$v' + v^2 + 1 = 0 \Rightarrow \frac{dv}{v^2+1} = -1 \Rightarrow v = \tan(C_1 - x)$$

$$\therefore y = \int dx v = \ln\left(\cos(C_1 - x)\right) + C_2$$

8.

$$y'y'' = e^y y' \Rightarrow \int dy' y' = \int dy e^y \Rightarrow \frac{1}{2}y'^2 = e^y + C_1$$

$$\Rightarrow \frac{dy}{\sqrt{e^y + C_1}} = \sqrt{2}dx$$

Basic calculus yields

$$\int \frac{dy}{\sqrt{e^y + C_1}} = \frac{1}{\sqrt{C_1}} \ln\left(\frac{\sqrt{e^y + C_1} - \sqrt{A}}{\sqrt{e^y + C_1} + \sqrt{C_1}}\right) + A.$$

$$\therefore \ln\left(\frac{\sqrt{e^y + C_1} - \sqrt{C_1}}{\sqrt{e^y + C_1} + \sqrt{C_1}}\right) = \sqrt{2C_1}x + C_2$$

9. Notice how $(x(1-x))' = 1 - 2x$.

$$\begin{aligned}
0 &= x(1-x)y'' + 4y' + 2y \\
&= x(1-x)y'' + (x(1-x))'y' + (2x+3)y' + 2y \\
&= (x(1-x)y')' + ((2x+3)y)'
\end{aligned}$$

$$\Rightarrow x(1-x)y' + (2x+3)y = A$$

Let us define the integrating factor λ:

$$\begin{aligned}
\lambda &= \exp\left(\int dx \frac{2x+3}{x(1-x)}\right) \\
&= \exp\left(\int dx \left(\frac{3}{x} - \frac{5}{x-1}\right)\right) \\
&= \exp\left(3\ln x - 5\ln(x-1)\right) \\
&= \frac{x^3}{(x-1)^5}
\end{aligned}$$

$$\Rightarrow (\lambda y)' = \frac{A}{x(1-x)} \cdot \lambda = \frac{Ax^2}{(x-1)^6}$$

$$
\begin{aligned}
\Rightarrow \lambda y &= \int dx \frac{Ax^2}{(x-1)^6} \\
&= A \int dx \left(\frac{1}{(x-1)^4} + \frac{2}{(x-1)^5} + \frac{1}{(x-1)^6} \right) \\
&= -A \frac{10x^2 - 5x + 1}{30(x-1)^5} + C_2
\end{aligned}
$$

$$\therefore y = C_1 \frac{10x^2 - 5x + 1}{x^3} + C_2 \frac{(1-x)^5}{x^3}$$

10.
$$\frac{dy}{y^2} = \frac{1-x}{x^3} dx \Rightarrow -\frac{1}{y} = -\frac{1}{2x^2} + \frac{1}{x} + A$$

$$\therefore y = \frac{2x^2}{Cx^2 + 2x - 1}$$

11. Recall that the Bernoulli equation takes the form $\frac{dy}{dx} + p(x)y = q(x)y^n$. We recognize that this equation is a Bernoulli equation with $p(x) = \frac{1}{x}$, $q(x) = -x^3 e^x$, and $n = 4$. Thus, we make the substitution $v := y^{-3}$.

$$v' = -3y^{-4}y' \Rightarrow -\frac{v'}{3} + \frac{v}{x} = -x^3 e^x \Rightarrow \frac{v'}{x^3} - \frac{3}{x^4} v = 3e^x$$

$$\Rightarrow \left(\frac{v}{x^3} \right)' = 3e^x \Rightarrow v = x^3 (3e^x + C)$$

$$\therefore y = \left(x^3 (3e^x + C) \right)^{-\frac{1}{3}}$$

12. We define the integrating factor λ:

$$\lambda = \exp \left(\int \frac{dx}{1+x^2} \right) = \exp \left(\tan^{-1} x \right)$$

$$
\begin{aligned}
\Rightarrow \left(\exp \left(\tan^{-1} x \right) y \right)' &= \frac{1}{1+x^2} \cdot \tan^{-1} x \cdot \exp \left(\tan^{-1} x \right) \\
\Rightarrow \exp \left(\tan^{-1} x \right) y &= \left(\tan^{-1} x - 1 \right) \exp \left(\tan^{-1} x \right) + C \\
\therefore y &= C \exp \left(-\tan^{-1} x \right) + \tan^{-1} x - 1
\end{aligned}
$$

13. This equation is dimension-consistent with $[y] = [x^{-1}]$. Thus, let us define $v := xy$.

$$y = \frac{v}{x} \Rightarrow y' = \frac{xv' - v}{x^2}$$

$$
\begin{aligned}
\Rightarrow 0 &= (xv' - v)^2 - 2(v-4)(xv' - v) + v^2 \\
&= x^2 v'^2 - 4(v-2)xv' + 4v(v-2)
\end{aligned}
$$

$$\Rightarrow v' = \frac{2}{s} \left(v - 2 \pm \sqrt{4 - 2v} \right)$$

If $v = 2$, then we get $y = \frac{2}{x}$, which is the particular solution for this equation. To obtain the general solutions, let $u = 2 - v$.

$$
\begin{aligned}
u' &= \frac{2}{x}\left(u \pm \sqrt{2u}\right) \\
\Rightarrow \frac{du}{u \pm \sqrt{2u}} &= \frac{2}{x}dx \\
\Rightarrow 2\ln\left(\sqrt{u} \pm \sqrt{2}\right) &= 2\ln x + A \\
\Rightarrow u = \left(Cx^2 \pm \sqrt{2}\right)^2 &= 2 - xy \\
\therefore y &= \frac{2 - \left(Cx^2 \pm \sqrt{2}\right)^2}{x}
\end{aligned}
$$

14.

$$
\begin{aligned}
6x &= \frac{y''}{y} - \frac{y'}{y^2} \\
&= \left(\frac{y'}{y}\right)'
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \frac{y'}{y} &= 3x^2 + C_1 \\
\Rightarrow \ln y &= x^3 + C_1 x + A \\
\therefore y &= C_2 e^{x^3 + C_1 x}
\end{aligned}
$$

15.

$$
\begin{aligned}
\frac{1}{x} &= x^3\left(yy'' + y'^2\right) + (x^3)'yy' = \left(x^3 yy'\right)' \\
\Rightarrow x^3 yy' &= \ln x + A \Rightarrow ydy = \frac{\ln x + A}{x^3}
\end{aligned}
$$

Basic calculus yields $\int dx \frac{\ln x}{x^3} = -\frac{2\ln x + 1}{4x^2}$.

$$
\begin{aligned}
\Rightarrow \frac{y^2}{2} &= -\frac{2\ln x + 1}{4x^2} - \frac{A}{2x^2} + B \\
\therefore y &= \pm\sqrt{C_1 - \frac{\ln x + C_2}{x^2}}
\end{aligned}
$$

16. This equation is dimension-consistent with $[y] = [x]$. Thus, we define $v := \frac{y}{x}$.

$$
y = xv \Rightarrow y' = xv' + v \Rightarrow y'' = xv'' + 2v' \Rightarrow v'' + \frac{2}{x}v' - \frac{2}{x^2}v = \frac{1}{x^2}
$$

First, observe that $v = -\frac{1}{2}$ is a solution of the equation. As this equation is linear in v, we thus seek the complementary solutions v_c. We shall take the ansatz of $v_c = x^m$.

$$m(m-1)x^{m-2} + 2mx^{m-2} - 2x^{m-2} = 0 \Rightarrow m^2 + m - 2 = 0$$

$$\Rightarrow m = 1, -2 \Rightarrow v_c = C_1 x + \frac{C_2}{x^2} \Rightarrow v = C_1 x + \frac{C_2}{x^2} - \frac{1}{2}$$

$$\therefore y = C_1 x^2 + \frac{C_2}{x} - \frac{x}{2}$$

17. Notice that this equation is linear in y.

(i) Complementary solutions:
Take $y_c = e^{mx}$.

$$m^3 - 2m^2 - m + 2 = 0 \Rightarrow m = \pm 1, 2$$

$$\Rightarrow y_c = C_1 e^{2x} + C_2 e^x + C_3 e^{-x}$$

(ii) Particular solution:
Suppose $y_p = A \sin x + B \cos x$.

$$\begin{aligned}\sin x &= (-A\cos x + B\sin x) - 2(-A\sin x - B\cos x) \\ &\quad - (A\cos x - B\cos x) + 2(A\sin x + B\cos x) \\ &= (4A + 2B)\sin x + (-2A + 4B)\cos x\end{aligned}$$

$$\Rightarrow A = \frac{1}{5}, B = \frac{1}{10}$$

$$\therefore y = \frac{1}{5}\sin x + \frac{1}{10}\cos x + C_1 e^{2x} + C_2 e^x + C_3 e^{-x}$$

18. Again, this equation is linear in y.

(i) Complementary solutions:
Take $y_c = e^{mx}$.

$$m^3 + 2m^2 + 1 = 0 \Rightarrow m = m_1, m_2, m_3$$

(I refuse to write down the exact solutions of this cubic equation.)

$$\Rightarrow y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$$

(ii) Particular solution:
Suppose $y_p = A \sin x + B \cos x$.

$$\begin{aligned}\sin x &= (-A\cos x + B\sin x) + 2(-A\sin x - B\cos x) + (A\sin x + B\cos x) \\ &= (-A + B)\sin x + (-A - B)\cos x\end{aligned}$$

$$\Rightarrow A = B = -\frac{1}{2}$$

$$\therefore y = -\frac{1}{2}\sin x - \frac{1}{2}\cos x + C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$$

19. Following the same steps as before, one can easily find the complementary solutions of

$$y_c = C_1 e^{-x} + C_2 e^{-2x}.$$

Thus, we wish to find one particular solution y_p. Let us try $y_p = f(x)e^{e^x}$.

$$y_p' = \left(\frac{f'}{f} + e^x\right) y_p$$

$$\begin{aligned}
\Rightarrow y_p'' &= \left(\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 + e^x\right) y_p + \left(\frac{f'}{f} + e^x\right)^2 y_p \\
&= \left(\frac{f''}{f} + 2e^x \frac{f'}{f} + e^{2x} + e^x\right) y_p
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \left(\frac{f''}{f} + (2e^x + 3)\frac{f'}{f} + e^{2x} + 5e^x + 2\right) f e^{e^x} = e^{e^x} \\
\Rightarrow f'' + (2e^x + 3)f' + (e^{2x} + 5e^x + 2)f = 1
\end{aligned}$$

One may use an ansatz of $f = ae^{-2x} + be^{-x} + c$ to find that $f = e^{-2x}$ is a possible solution. Thus, we have found a particular solution $y_p = e^{e^x - 2x}$.

$$\therefore y = e^{e^x - 2x} + C_1 e^{-x} + C_2 e^{-2x}$$

20.

$$\frac{y''}{(1 + y'^2)^{3/2}} = \pm\frac{1}{a} \Rightarrow \frac{y'}{\sqrt{1 + y'^2}} = \pm\frac{x + C_1}{a}$$

$$\Rightarrow y'^2 = \left(\frac{x + C_1}{a}\right)^2 (1 + y'^2) \Rightarrow y' = \pm\frac{\left(\frac{x+C_1}{a}\right)^2}{\sqrt{1 - \left(\frac{x+C_1}{a}\right)^2}}$$

$$\therefore y = \pm\frac{1}{2}\left(a\sin^{-1}\left(\frac{x + C_1}{a}\right) - (x + C_1)\sqrt{1 - \left(\frac{x + C_1}{a}\right)^2}\right) + C_2$$

21. The complementary solutions are given by

$$\frac{dq_p}{dt} = -\frac{q_p}{RC} \Rightarrow q_p = Ae^{-\frac{t}{RC}}.$$

Suppose a particular solution takes the form $y_p = (at^3 + bt^2 + ct + d)e^{-\frac{t}{\tau}}$.

$$\frac{dq_p}{dt} = \left(-\frac{a}{\tau}t^3 + \left(3a - \frac{b}{\tau}\right)t^2 + \left(2b - \frac{c}{\tau}\right)t + \left(c - \frac{d}{\tau}\right)\right) e^{-\frac{t}{\tau}}$$

$$
\begin{aligned}
\Rightarrow \frac{V_0}{RC\tau^2}t^2 e^{-\frac{t}{\tau}} = \Bigg(& \left(\frac{1}{RC} - \frac{1}{\tau}\right) at^3 \\
& + \left(3a + \left(\frac{1}{RC} - \frac{1}{\tau}\right) b\right) t^2 \\
& + \left(2b + \left(\frac{1}{RC} - \frac{1}{\tau}\right) c\right) t \\
& + \left(c + \left(\frac{1}{RC} - \frac{1}{\tau}\right) d\right) \Bigg) e^{-\frac{t}{\tau}}
\end{aligned}
$$

$$
\Rightarrow q_p = \begin{cases} CV_0 \cdot \left(\frac{t^2}{\tau(\tau-RC)} - \frac{RCt}{(\tau-RC)^2} + \frac{\tau(RC)^2}{(\tau-RC)^2}\right) & (\text{if } \tau \neq RC) \\ \frac{1}{3}CV_0 \cdot \left(\frac{t}{\tau}\right)^3 e^{-\frac{t}{\tau}} & (\text{if } \tau = RC) \end{cases}
$$

We can then add an appropriate complementary solution so as to satisfy the boundary condition $q(0) = 0$.

$$
\therefore q = \begin{cases} CV_0 \cdot \frac{t}{\tau-RC} \left(\frac{t}{\tau} - \frac{RC}{\tau-RC}\right) e^{-\frac{t}{\tau}} & (\text{if } \tau \neq RC) \\ \frac{1}{3}CV_0 \cdot \left(\frac{t}{\tau}\right)^3 e^{-\frac{t}{\tau}} & (\text{if } \tau = RC) \end{cases}
$$

22.

$$
\frac{dN}{N_s - N} = \lambda dt \Rightarrow \int_0^N \frac{dN}{N_s - N} = \lambda \int_0^t dt \Rightarrow -\ln\left|\frac{N_s - N}{N_s}\right| = \lambda t
$$

$$
\therefore N = N_s\left(1 - e^{-\lambda t}\right)
$$

23. The clever idea here is that we could make a variable change from x to u, where we could choose u so as to make the equation easier to solve.

$$
\begin{aligned}
0 &= A\frac{d^2y}{dx^2} + \frac{dA}{dx}\frac{dy}{dx} + \frac{y}{A} \\
&= A\frac{du}{dx}\frac{d}{du}\left(\frac{du}{dx}\frac{dy}{dy}\right) + \frac{dA}{dx}\frac{du}{dx}\frac{dy}{du} + \frac{y}{A} \\
&= A\left(\frac{du}{dx}\right)^2\frac{d^2y}{du^2} + \frac{dy}{du}\frac{d}{dx}\left(A\frac{dy}{dx}\right)
\end{aligned}
$$

Thus, we could define $\frac{du}{dx} = \frac{1}{A(x)}$ or $u = \int \frac{dx}{A(x)}$ to obtain

$$
\frac{d^2y}{du^2} = -y \Rightarrow y(u) = C_1 \cos\left(u + C_2\right)
$$

$$
\therefore y(x) = C_1 \cos\left(\int \frac{dx}{A(x)} + C_2\right)
$$

24. Suppose we may write $y = \lambda v$ where λ and v are to be determined later.

$$
y' = \lambda v' + \lambda' v \Rightarrow y'' = \lambda v'' + 2\lambda' v' + \lambda'' v
$$

$$\Rightarrow \lambda v'' + 2\left(\lambda' + \frac{\lambda}{x}\right)v' + \left(\lambda'' + \frac{2\lambda'}{x} + n^2\lambda\right)v = \frac{\sin\omega x}{x}$$

We may choose λ so as to make the coefficient of v' vanish, i.e.,

$$\lambda' = -\frac{\lambda}{x} \Rightarrow \lambda = \frac{1}{x}.$$

$$\Rightarrow v'' + n^2 v = \sin\omega x$$

(i) $n \neq \omega$:
The complementary solutions are

$$v_c = C_1 \cos\left(nx + C_2\right).$$

Using the ansatz $v_p = A\sin\omega x$ for a particular solution, one can easily find $A = \frac{1}{n^2-\omega^2}$.

$$\Rightarrow v = \frac{\sin\omega x}{n^2-\omega^2} + C_1 \sin\left(nx + C_2\right)$$

$$\therefore y = \frac{1}{n^2-\omega^2}\frac{\sin\omega x}{x} + C_1\frac{\sin\left(nx+C_2\right)}{x}$$

(ii) $n = \omega$:
The complementary solutions are

$$v_c = C_1 \cos\left(\omega x + C_2\right).$$

Using the ansatz $v_p = Ax\sin\omega x + Bx\cos\omega x$ for a particular solution, one can easily find $A = -\frac{1}{2\omega}$ and $B = 0$.

$$\Rightarrow v = -\frac{1}{2\omega}x\cos\omega x + C_1\cos\left(\omega x + C_2\right)$$

$$\therefore y = -\frac{1}{2\omega}\cos\omega x + C_1\frac{\sin\left(\omega x + C_2\right)}{x}$$

25. From the hint, we may consider $y = xv$, where $v = 1$ is a solution if the right hand side were zero.

$$y' = xv' + v \Rightarrow y'' = xv'' + 2v'$$

$$\Rightarrow x(1-x)v'' + \left(1 + (1-x)^2\right)v' = (1-x)^2$$

$$\Rightarrow v'' + \left(\frac{2}{x} + \frac{1}{1-x} - 1\right)v' = \frac{1}{x} - 1$$

We can thus define the integration factor as

$$\lambda := \exp\left(\int dx\left(\frac{2}{x} + \frac{1}{1-x} - 1\right)\right) = \frac{x^2e^{-x}}{1-x}.$$

$$
\begin{aligned}
\Rightarrow (\lambda v')' = \lambda\left(\frac{1}{x}-1\right) = xe^{-x} \\
\Rightarrow \lambda v' = -(x+1)e^{-x} + A \\
\Rightarrow v' = 1 - \frac{1}{x^2} + A\frac{1-x}{x^2}e^x \\
\Rightarrow v = x + \frac{1}{x} - A\frac{e^x}{x} + B \\
\therefore y = x^2 + 1 + C_1 e^x + C_2 x
\end{aligned}
$$

26. While the use of Dirac delta "function" must be dealt with caution, we are physicists so we shall simply accept that they exist.

$$
\begin{aligned}
f(x) = y'' + py' + q &= \int_a^b dx' f(x')\left(\frac{\partial^2}{\partial x^2} + p(x)\frac{\partial}{\partial x} + q(x)\right)G(x,x') \\
&= \int_a^b dx' f(x')\delta(x-x')
\end{aligned}
$$

$$
\Rightarrow \left(\frac{\partial^2}{\partial x^2} + p(x)\frac{\partial}{\partial x} + q(x)\right)G(x,x') = \delta(x-x')
$$

From this, we see that $G(x,x')$ must separately equal zero for both $x < x'$ and $x > x'$. We thus write

$$
G(x,x') = \begin{cases} y_1(x)\alpha(x') + y_2(x)\beta(x') & (x < x') \\ y_1(x)\gamma(x') + y_2(x)\zeta(x') & (x > x') \end{cases}.
$$

From the definition of delta functions, we find that:

$$
\begin{aligned}
1 &= \int_{x'-\epsilon}^{x'+\epsilon} dx\delta(x-x') \\
&= \int_{x'-\epsilon}^{x'+\epsilon} dx\left(\frac{\partial^2}{\partial x^2} + p(x)\frac{\partial}{\partial x} + q(x)\right)G(x,x') \\
&\approx \left[\left(\frac{\partial}{\partial x} + p(x)\right)G(x,x')\right]_{x'-\epsilon}^{x'+\epsilon} - \int_{x'-\epsilon}^{x'+\epsilon} dxp'(x)G(x,x') \\
&\approx \left[\frac{\partial}{\partial x}G(x,x')\right]_{x'-\epsilon}^{x'+\epsilon} \\
&\approx \lim_{x\to x'+}\frac{\partial}{\partial x}G(x,x') - \lim_{x\to x'-}\frac{\partial}{\partial x}G(x,x')
\end{aligned}
$$

where we take infinitesimally small ϵ.

The boundary conditions for y imply

$$
y(a) = y(b) = 0
$$

$$\Rightarrow G(a,x') = G(b,x') = 0$$
$$\Rightarrow y_1(a)\alpha(x') + y_2(a)\beta(x') = y_1(b)\alpha(x') + y_2(b)\beta(x') = 0$$
$$\Rightarrow \beta(x') = \gamma(x') = 0.$$

The derivative condition, in turn, yields

$$y_2'(x')\zeta(x') - y_1'(x')\alpha(x') = 1.$$

As any such choice of $\zeta(x')$ and $\alpha(x')$ leads to a valid Green's function, we may arbitrarily choose

$$\alpha(x') = y_2'(x'),\ \zeta(x') = y_1'(x') + \frac{1}{y_2'(x')}$$

$$\Rightarrow G(x,x') = \begin{cases} y_1(x)y_2'(x') & (x < x') \\ y_2(x)y_1'(x') + \frac{y_2(x)}{y_2'(x')} & (x > x') \end{cases}.$$

$$\therefore y(x) = y_1(x)\int_x^b dx' f(x')y_2'(x') + y_2(x)\int_a^x f(x')\left(y_1'(x') + \frac{1}{y_2'(x')}\right)$$

For the given example $y'' + k^2y = f(x)$, we may take

$$y_1(x) = \sin k(x-a),\ y_2(x) = \sin k(b-x)$$

provided that $b-a$ is not an integer multiple of the period $\frac{2\pi}{k}$. This leads to

$$G(x,x') = \begin{cases} -k\sin k(x-a)\cos k(b-x) & (x < x') \\ \sin k(b-x)\left(k\cos k(x'-a) - \frac{1}{k\cos k(b-x')}\right) & (x > x') \end{cases}$$

$$\begin{aligned} \Rightarrow y = -k\sin k(x-a)\int_x^b dx' f(x')\cos k(b-x') \\ + \sin k(b-x)\int_a^x dx' f(x')\left(k\cos k(x'-a) - \frac{1}{k\cos k(b-x')}\right). \end{aligned}$$

One may verify that this solution satisfies both the boundary conditions and the given differential equation.

27. The complementary solutions satisfy

$$y_c'' + \frac{3}{x^2}y_c = 0.$$

Suppose $y_c = x^m$.

$$m^2 - m + 3 = 0 \Rightarrow \left(m - \frac{1}{2}\right)^2 = -\frac{11}{4} \Rightarrow m = \frac{1 \pm \sqrt{11}i}{2}$$

$$
\begin{aligned}
\Rightarrow y_c &= Ax^{\frac{1+\sqrt{11}i}{2}} + Bx^{\frac{1-\sqrt{11}i}{2}} \\
&= C_1\sqrt{x}\cos\left(\frac{\sqrt{11}}{2}\ln x\right) + C_2\sqrt{x}\sin\left(\frac{\sqrt{11}}{2}\ln x\right)
\end{aligned}
$$

We then look for two separate particular solutions:

$$y''_{p1} + \frac{3}{x^2}y_{p1} = x^2 \text{ and } y''_{p2} + \frac{3}{x^2}y_{p2} = \frac{1}{x}.$$

Using the monomial ansatz separately, that is, assuming $y_p = Ax^m$, one easily finds

$$y_{p1} = \frac{x^4}{15} \text{ and } y_{p2} = \frac{x}{3}.$$

$$\therefore y = \frac{x^4}{15} + \frac{x}{3} + C_1\sqrt{x}\cos\left(\frac{\sqrt{11}}{2}\ln x\right) + C_2\sqrt{x}\sin\left(\frac{\sqrt{11}}{2}\ln x\right)$$

28. Let us first look for asymptotic behaviors.

(i) $x \to 0+$ Suppose $y = O(x^m)$.

$$m(m-1) + 2m - l(l+1) = 0 \Rightarrow m = l \text{ or } -l-1$$

To vanish near the origin, we must have $y = O(x^l)$.

(ii) $x \to \infty$

$$\frac{d^2y}{dx^2} \approx -Ky \Rightarrow y \approx e^{\pm\sqrt{-K}x}$$

To vanish at infinity, we must have $y = x^{-\sqrt{-K}x}$ with $K < 0$.

We can thus write $y = f(x)x^l e^{-\sqrt{-K}x}$ with f regular everywhere.

$$y' = \left(\frac{f'}{f} + \frac{l}{x} - \sqrt{-K}\right)y$$

$$
\begin{aligned}
\Rightarrow y'' &= \left(\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 - \frac{l}{x^2}\right)y + \left(\frac{f'}{f} + \frac{l}{x} - \sqrt{-K}\right)^2 y \\
&= \left(\frac{f''}{f} + 2\left(\frac{l}{x} - \sqrt{-K}\right)\frac{f'}{f} + \frac{l(l-1)}{x^2} - \frac{2l\sqrt{-K}}{x} - K\right)y
\end{aligned}
$$

Tedious algebra yields

$$xf'' + 2(l+1-\sqrt{-K})f' - 2((l+1)\sqrt{-K} - 1)f = 0.$$

Let us write $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Substitution and tedious algebra yield

$$\sum_{n=0}^{\infty}((n+1)(n+2l+2)a_{n+1} - (\sqrt{-K}n + 2((l+1)\sqrt{-K} - 1))a_n)x^n = 0$$

$$\Rightarrow (\forall n)\ (n+1)(n+2l+2)a_{n+1} = (\sqrt{-K}n + 2((l+1)\sqrt{-K}-1))a_n.$$

If $\{a_n\}$ never terminated, this would mean that $a_n \approx \frac{(\sqrt{-K})^n}{n!}$ for large values of n, leading to $f(x) \approx e^{\sqrt{-K}x}$ and y diverging to infinity as x increases.

$$\Rightarrow \sqrt{-K_{nl}}n + 2((l+1)\sqrt{-K_{nl}}-1) = 0$$

$$\therefore K_{nl} = -\frac{4}{(n+2l+2)^2}\ (n = 0, 1, 2, \dots)$$

29. For very large x,

$$y'' \approx \frac{y}{4} \Rightarrow y \approx e^{\pm\frac{x}{2}}.$$

Thus, for y to vanish at infinity, we must have $y \approx e^{-x/2}$. As y must also vanish at the origin, let us write $y = f(x)xe^{-x/2}$ with $f(x)$ regular everywhere.

$$y' = \left(\frac{f'}{f} + \frac{1}{x} - \frac{1}{2}\right)y$$

$$\begin{aligned}\Rightarrow y'' &= \left(\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 - \frac{1}{x^2}\right)y + \left(\frac{f'}{f} + \frac{1}{x} - \frac{1}{2}\right)^2 y \\ &= \left(\frac{f''}{f} + \left(\frac{2}{x} - 1\right)\frac{f'}{f} - \frac{1}{x} + \frac{1}{4}\right)y \\ &\Rightarrow \frac{f''}{f} + \left(\frac{2}{x} - 1\right)\frac{f'}{f} - \frac{K+1}{x} = 0\end{aligned}$$

Let us write $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Substitution and tedious algebra yield

$$2a_1 - (K+1)a_0 + \sum_{n=1}^{\infty}((n+1)(n+2)a_{n+1} - (n+K+1)a_n)x^n = 0$$

$$\Rightarrow (\forall n)\ (n+1)(n+2)a_{n+1} = (n+K+1)a_n.$$

If $\{a_n\}$ never terminated, this would mean that $a_n \approx \frac{1}{n!}$ for large values of n, leading to $f(x) \approx e^x$ and y diverging to infinity as x increases.

$$\therefore K_n = -n\ (n = 1, 2, 3, \dots)$$

30. I assume that there must exist a "nontrivial" solution, since $y = 0$ is clearly a solution of the equation for any value of k. For large values of x,

$$y'' - 2y' - 3y \approx 0 \Rightarrow y \approx Ae^{-x} + Be^{3x}.$$

Thus, to be bounded everywhere, we must have $y \approx e^{-x}$. We then write $y = f(x)e^{-x}$ with f analytic everywhere, including the origin.

$$y' = \left(\frac{f'}{f} - 1\right)y \Rightarrow y'' = \left(\frac{f''}{f} - \frac{f'}{f} + \left(\frac{f'}{f} - 1\right)^2\right)y = \left(\frac{f''}{f} - 2\frac{f'}{f} + 1\right)y$$

Substitution yields

$$xf'' - 4xf' + kf = 0.$$

Let us write $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Substitution and tedious algebra yield

$$\sum_{n=0}^{\infty} (n(n+1)a_{n+1} - (4n-k)a_n)x^n = 0$$

$$\Rightarrow (\forall n)\ n(n+1)a_{n+1} = (4n-k)a_n.$$

If $\{a_n\}$ never terminated, this would mean that $a_n \approx \frac{4^n}{n!}$ for large values of n, leading to $f(x) \approx e^{4x}$ and y diverging to infinity as x increases.

$$\therefore k_n = 4n\ (n = 0, 1, 2, \dots)$$

31. For large values of x,

$$y'' \approx y \Rightarrow y \approx e^{\pm x}.$$

Thus, to be bounded everywhere, we must have $y \approx e^{-x}$. We then write $y = f(x)e^{-x}$ with f analytic everywhere, with $f(0) = 1$.

$$y' = \left(\frac{f'}{f} - 1\right) y \Rightarrow y'' = \left(\frac{f''}{f} - \frac{f'}{f} + \left(\frac{f'}{f} - 1\right)^2\right) y = \left(\frac{f''}{f} - 2\frac{f'}{f} + 1\right) y$$

Substitution yields

$$xf'' - 2(x-1)f' + (E-2)f = 0.$$

Let us write $f(x) = \sum_{n=0}^{\infty} a_n x^n$, with $a_0 = 1$. Substitution and tedious algebra yield

$$\sum_{n=0}^{\infty} ((n+1)(n+2)a_{n+1} - (2n+2-E)a_n)x^n = 0$$

$$\Rightarrow (\forall n)\ (n+1)(n+2)a_{n+1} = (2n+2-E)a_n.$$

If $\{a_n\}$ never terminated, this would mean that $a_n \approx \frac{2^n}{n!}$ for large values of n, leading to $f(x) \approx e^{2x}$ and y diverging to infinity as x increases.

$$\therefore E_n = 2n\ (n = 1, 2, 3, \dots)$$

32. Recall that

$$\frac{c_{r+2}}{c_r} = \frac{(r+m-n)(r+m+n+1)}{(r+1)(r+2)} \text{ and } v(x) = \sum_{r=0}^{\infty} c_r x^r$$

where the fraction is well-defined since we are assuming that $\{c_r\}$ never terminates. Notice how

$$\frac{c_{r+2}}{c_r} \approx \frac{r+2m+1}{r+3}$$

for large values of r.

On the other hand, the definition of binomial coefficients

$$\binom{-m}{r} := \frac{-m \cdot (-m-1) \cdot \dots \cdot (-m-r+1)}{r!}$$

naturally yields

$$\begin{aligned} \frac{\binom{-m}{r+2}}{\binom{-m}{r}} &= \frac{(-m-r)(-m-r-1)}{(r+1)(r+2)} \\ &= \frac{(r+m)(r+m+1)}{(r+1)(r+2)} \\ &\approx \frac{r+2m+1}{r+3} \end{aligned}$$

for large values of r.

Therefore, c_r behaves like $\binom{-m}{r}$ as r grows without bound, and consequently,

$$v(x) \approx \left(1-x^2\right)^{-m}.$$

33. Suppose $y = J_0(x)\ln x + \sum_{n=0}^{\infty} a_n x^n$ as prompted by the problem. Substitution and tedious algebra yield

$$2xJ_0'(x) + a_1 x + \sum_{n=2}^{\infty}(n^2 a_n + a_{n-2}).$$

Let us write $J_0(x) = \sum_{n=0}^{\infty} b_{2n}x^{2n}$ since we know that J_0 is even.

$$xJ_0'(x) = \sum_{n=0}^{\infty} 2nb_{2n}x^{2n}$$

$$\begin{aligned} \Rightarrow a_1 x + \sum_{n=1}^{\infty} \Big(& \left(4nb_{2n} + 4n^2 a_{2n} + a_{2n-2}\right) x^{2n} \\ & + \left((2n+1)^2 a_{2n+1} + a_{2n-1}\right) x^{2n+1} \Big) = 0 \end{aligned}$$

This immediately tells us that all odd coefficients $a_1 = a_3 = \cdots = 0$.

Bessel's equation is linear, so addition of any multiple of J_0 to y also yields a valid solution. In other words, any variant $a_{2n}' := a_{2n} + \lambda b_{2n}$ also satisfies the above equation, for real parameter λ. Thus, we may choose $a_0 = 0$.

Using $b_0 = 1$, $b_2 = -\frac{1}{4}$, $b_4 = \frac{1}{64}$, $b_6 = -\frac{b_4}{36} = -\frac{1}{2304}$, and the recursion relation $a_2 n = -\frac{a_{2n-2}}{4n^2} - \frac{b_{2n}}{n}$ from above, we can iteratively calculate the values of a_{2n}.

$$\therefore y = J_0(x)\ln x + \frac{x^2}{4} - \frac{3}{128}x^4 + \frac{5}{4608}x^6 - \cdots$$

34.

(i) $y(0) = 1$
We may write $y = 1 + \sum_{n=1}^{\infty} a_n x^n$. Substitution and tedious algebra yields

$$2a_1 - 2 + \sum_{n=1}^{\infty} (n+2)((n+1)a_{n+1} - a_n)x^n = 0.$$

This directly leads to $a_n = \frac{1}{n!}$, and thus,

$$y = e^x.$$

(ii) $y = \frac{1}{x} + A \ln x + B$
Substitution of the above form into the differential equation and patiently performing algebraic manipulations yields

$$-\frac{1}{x} + (xA'' + (2-x)A' - 2A) \ln x + 2A' + \left(\frac{1}{x} - 1\right) A + xB''(2-x)B' - 2B = 0.$$

Read the problem carefully: we only need to "give" two solutions, not find a general form for all solutions. Thus, we could choose the oddly specific solution of $A = e^x$ to make the coefficient of the $\ln x$ term vanish. Substituting this into the above equation yields

$$xB'' + (2-x)B' - 2B + e^x + \frac{e^x - 1}{x} = 0.$$

Expanding $B = \sum_{n=0}^{\infty} b_n x^n$, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, and $\frac{e^x - 1}{x} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$, one can write

$$\sum_{n=0}^{\infty} \left((n+1)(n+2)b_{n+1} - (n+2)b_n + \frac{n+2}{(n+1)!} \right) = 0$$

$$\Rightarrow b_{n+1} = \frac{b_n}{n+1} - \frac{1}{(n+1) \cdot (n+1)!}.$$

Again, B may arbitrarily chosen, so let us choose $b_0 = 0$.

$$\therefore y = \frac{1}{x} + e^x \ln x - x - \frac{3}{4}x^2 - \frac{11}{36}x^3 - \cdots$$

35. Let $u := \frac{1}{x}$.

$$\frac{d}{dx} = \frac{du}{dx}\frac{d}{du} = -u^2 \frac{d}{du}$$

$$\Rightarrow \frac{d^2}{dx^2} = -u^2 \frac{d}{du}\left(-u^2 \frac{d}{du}\right) = u^4 \frac{d^2}{du^2} + 2u^3 \frac{d}{du}$$

$$\Rightarrow \frac{d^2 y}{du^2} + \frac{2}{u}\frac{dy}{du} = \frac{1}{(1 + u^2 y^2)^2}$$

As $u \to 0+$, $y \to 0$. Thus,

$$\frac{2}{u}\frac{dy}{du} \approx 1 \Rightarrow y \approx u^2$$

Hence, we write $y = f(u)u^2$ with f analytic everywhere.

$$\begin{aligned} u^2 f''(u) + 6uf'(u) + 6f(u) &= \frac{1}{\left(1+u^6 f(u)\right)^2} \\ &= 1 - 2u^6 f(u)^2 + 3u^{12} f(u)^4 + O\left(u^{18}\right) \end{aligned}$$

We write $f(u) = \sum_{n=0}^{\infty} a_n u^n$. The left-hand side (after tedious algebra) evaluates to

$$\sum_{n=0}^{\infty} \left(n^2 + 5n + 5\right) a_n u^n.$$

Evaluation of the right-hand side is quite more technical. We make use of the Cauchy multiplication formula:

$$f(u)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) u^n.$$

Substituting this into the equation, we get

$$\begin{aligned} &(6a_0 - 1) + \sum_{n=1}^{5} \left(n^2 + 5n + 6\right) a_n u^n \\ &+ \sum_{n=6}^{\infty} \left(\left(n^2 + 5n + 6\right) a_n + 2 \sum_{k=0}^{n-6} a_k a_{n-6-k} \right) u^n = 3u^{12} f(u)^4 + O(u^{18}). \end{aligned}$$

Let us compare each power of u separately.

$$a_0 = \frac{1}{6},\ a_1 = \cdots = a_5 = 0$$

$$\begin{gathered} O(u^6):\ 72a_6 + 2a_0^2 = 0 \Rightarrow a_6 = -\frac{1}{1296} \\ a_7 = \cdots = a_{11} = 0 \\ O(u^{12}): 210a_{12} + 2\left(a_0 a_6 + a_6 a_0\right) = 3a_0^2 \Rightarrow a_{12} = \frac{11}{816480} \\ \Rightarrow f(u) = \frac{1}{6} - \frac{u^6}{1296} + \frac{11}{816480}u^{12} - \dots \\ \therefore y = \frac{1}{6x^2} - \frac{1}{1296x^8} + \frac{11}{816480}\frac{1}{x^{14}} + \dots \end{gathered}$$

36.

(a) Assume $y = \sum_{n=0}^{\infty} c_n x^n$. Using

$$y'' = y\left(x^2 - y^2\right),\ x^2 = \sum_{n=0}^{\infty} \delta_{n2} x^n,$$

and

$$y^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} c_k c_{n-k} \right) x^n,$$

one can find (after algebraic tedium) that

$$y\left(x^2 - y^2\right) = -c_0^2 - 3c_1 c_0^2 x + \sum_{n=2}^{\infty} \left(c_{n-2} - \sum_{k=0}^{n} \sum_{l=0}^{k} c_l c_{k-l} c_{n-k} \right) x^n$$

$$\begin{aligned} &\Rightarrow \left(2c_2 + c_0^2\right) + \left(6c_3 + 3c_1 c_0^2\right) x \\ &\quad + \sum_{n=2}^{\infty} \left((n+2)(n+1) c_{n+2} - c_{n-2} + \sum_{k=0}^{n} \sum_{l=0}^{k} c_l c_{k-l} c_{n-k} \right) x^n = 0. \end{aligned}$$

We thus find two degrees of freedom, namely $c_0 = A$ and $c_1 = B$.

$$c_2 = -\frac{A^2}{2},\ c_3 = -\frac{A^2 B}{2},\ c_{n+2} = -\frac{1}{(n+2)(n+1)} \left(c_{n-2} - \sum_{k=0}^{n} \sum_{l=0}^{k} c_l c_{k-l} c_{n-k} \right)$$

We could calculate a few terms using the recurrence relation:

$$c_4 = \frac{1}{4 \cdot 3} \left(c_0 - 3c_2 c_0^2 - 3c_1^2 c_0\right) = \frac{A}{12} \left(1 + \frac{3}{2} A^3 - 3B^2\right),$$

$$\begin{aligned} c_5 &= \frac{1}{5 \cdot 4} \left(c_1 - 3c_3 c_0^2 - 6c_2 c_1 c_0 - 3c_2 c_1\right) \\ &= \frac{B}{20} \left(1 + 3A^3 \left(\frac{A}{4} + 1\right)\right) \end{aligned}$$

$$\begin{aligned} \therefore y &= A + Bx - \frac{A^2}{2} x^2 - \frac{A^2 B}{2} x^3 \\ &\quad + \frac{A}{12} \left(1 + \frac{3}{2} A^3 - 3B^2\right) x^4 + \frac{B}{20} \left(1 + 3A^3 \left(\frac{A}{4} + 1\right)\right) x^5 + \cdots \end{aligned}$$

(b) We iteratively approximate the particular nonoscillating solution, for which the second derivative would stay relatively small. That is, we use the following iterative scheme:

$$-xy^{(0)} + {y^{(0)}}^3 = 0,\ y^{(n+1)} = \sqrt[3]{xy^{(n)} - {y^{(n)}}''}$$

We thus get

$$y^{(0)} = \sqrt{x} = x^{1/2}$$

$$y^{(1)} = \sqrt[3]{x^{3/2} + \frac{1}{4x^{3/2}}} \approx x^{1/2} + \frac{1}{12}x^{-5/2}$$

$$\begin{aligned} y^{(2)} &= \sqrt[3]{x^{3/2} + \frac{1}{3}x^{-3/2} - \frac{35}{48}x^{-9/2}} \\ &\approx x^{1/2} + \frac{1}{9}x^{-5/2} - \frac{35}{144}x^{-11/2} \end{aligned}$$

$$\begin{aligned} y^{(3)} &= \sqrt[3]{x^{3/2} + \frac{13}{36}x^{-3/2} - \frac{175}{144}x^{-9/2} + \frac{5005}{576}x^{-15/2}} \\ &\approx x^{1/2} + \frac{13}{108}x^{-5/2} - \frac{175}{432}x^{-11/2} + \frac{5005}{1728}x^{-17/2} \\ \therefore y &\approx x^{1/2} + \frac{13}{108}x^{-5/2} - \frac{175}{432}x^{-11/2} + \frac{5005}{1728}x^{-17/2} \end{aligned}$$

(Note: One may find more accurate approximations by (i) iterating more times, or (ii) expanding to more terms for each binomial expansion above.)

37. We first take note that $y = x + \alpha$ is a trivial solution to the equation, as is easily verifiable. Let $z := y - x$. We directly have

$$z'' = z^2 - e^{2z}.$$

We multiply each side by z' and integrate each side to obtain

$$\frac{z'^2}{2} = \frac{1}{3}z^3 - \frac{1}{2}e^{2z} + E.$$

I find it illuminating to consider z as the position of a particle of mass 1. The above equation could then be interpreted as the energy conservation of this particle as it moves under the influence of the potential

$$V(z) = -\frac{1}{3}x^3 + \frac{1}{2}e^{2z}$$

and retains its total energy E. Thus, we could consider the infinitesimal oscillation about $z = \alpha$ as a harmonic oscillator.

$$\left.\frac{d^2V}{dz^2}\right|_{z=\alpha} = -2\alpha + 2e^{2\alpha} = 2\alpha(\alpha - 1) = m\omega^2 = \omega^2$$

$$\Rightarrow \omega = \sqrt{2\alpha(\alpha-1)}$$

$$\Rightarrow z \approx C_1 \sin\left(\sqrt{2\alpha(\alpha-1)x} + C_2\right)$$

$$\therefore y \approx x + C_1 \sin\left(\sqrt{2\alpha(\alpha-1)x} + C_2\right)$$

(Note: More accurate expressions may be obtained using the higher-order terms of the potential energy and perturbation methods.)

38.

(a) Let $y := \sum_{n=1}^{\infty} c_n (x-1)^n$.

$$\begin{aligned} e^{y/x} &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{y}{x}\right)^n = \sum_{n=0}^{\infty} \frac{y^n}{n!}(1+(x-1))^{-n} \\ &= \sum_{n=0}^{\infty} \frac{y^n}{n!}\left(\sum_{m=0}^{\infty}\binom{-n}{m}(x-1)^m\right) \\ &= 1 + \sum_{m=0}^{2}\binom{-1}{m}(x-1)^m y + \sum_{m=0}^{1}\binom{-2}{m}(x-1)^m \frac{y^2}{2} + \frac{y^3}{6} + O\left((x-1)^4\right) \\ &= 1 + c_1(x-1) + \left(c_2 - c_1 + \frac{c_1^2}{2}\right)(x-1)^2 \\ &\quad + \left(c_3 - c_2 + c_1 - c_1c_2 - c_1^2 + \frac{c_1^3}{6}\right)(x-1)^3 + O\left((x-1)^4\right) \end{aligned}$$

Comparing this with

$$y' = c_1 + 2c_2(x-1) + 3c_3(x-1)^2 + O\left((x-1)^3\right),$$

we can see that

$$c_1 = 1,\ c_2 = \frac{c_1}{2} = \frac{1}{2},\ c_3 = \frac{1}{3}\left(c_2 - c_1 + \frac{c_1^2}{2}\right) = 0$$

$$\therefore y = (x-1) + \frac{1}{2}(x-1)^2 + O\left((x-1)^4\right)$$

(b) Let $y = xv$.

$$xv' + v = e^v \Rightarrow \frac{dv}{e^v - v} = \frac{dx}{x} \Rightarrow \int_1^{y/x} \frac{dv}{e^v - v} = \ln x + C_1$$

$$\ln x_0 + C_1 = \int_1^{\infty} \frac{dv}{e^v - v} =: \alpha \Rightarrow C_1 = \alpha - \ln x_0$$

$$\int_1^{y/x} \frac{dv}{e^v - v} = \ln\frac{x}{x_0} + \alpha \Rightarrow \int_{y/x}^{\infty} \frac{dv}{e^v - v} = -\ln\frac{x}{x_0}$$

For values of x slightly less than x_0, we have $\frac{y}{x} >> 1$.

$$\Rightarrow -\ln\frac{x}{x_0} \approx \int_{y/x}^{\infty} dv e^{-v} = e^{-y/x}$$

$$\therefore y \approx -x \ln\ln\frac{x_0}{x}$$

39. From the WKB method followed the Bohr-Sommerfeld quantization condition:

$$\int_{x_1}^{x_2} dx\sqrt{E - V(x)} = \left(n + \frac{1}{2}\right)\pi.$$

($\hbar = 1$ in this unit system.)

If the particle has a total energy of $E < 0$, it then has classical turning points at $\pm a\left(1 + \frac{E}{V_0}\right)$. Hence,

$$\left(n + \frac{1}{2}\right)\pi = 2\int_0^{a\left(1+\frac{E}{V_0}\right)} dx\sqrt{V_0 + E - \frac{V_0 x}{a}} = \frac{4a}{3V_0}(V_0 + E)^{3/2}$$

$$\Rightarrow (V_0 + E_n)^{3/2} = \left(n + \frac{1}{2}\right)\frac{3\pi V_0}{4a}$$

$$\therefore E_n = -\left(V_0 - \left(\left(n + \frac{1}{2}\right)\frac{3\pi V_0}{4a}\right)^{2/3}\right)$$

Here, n takes nonnegative integers as its value such that $E_n < 0$.

40. To use the WKB method, we want to make a substitution $y = up$ for some known function p such that the first derivative term vanishes. Substituting this into the equation yields

$$pu'' + \left(2p' - \frac{3}{x}p\right)u' + \left(p'' - \frac{3}{x}p' + \left(\frac{15}{4x^2} + x^{\frac{1}{2}}\right)\right)u = 0.$$

Thus, we choose $p = x^{\frac{3}{2}}$ to obtain

$$u'' + \left(x^{\frac{1}{2}} + \frac{15}{4}x^{-2} - \frac{21}{4}x^{-2}\right)u = 0.$$

Let

$$f(x) := x^{\frac{1}{2}} + \frac{15}{4}x^{-2} - \frac{21}{4}x^{-2} = x^{\frac{1}{2}}\left(1 + \frac{15}{4}x^{-\frac{5}{2}} - \frac{21}{4}x^{-\frac{7}{2}}\right).$$

This means that, using Eq. (1–90) from the textbook, one gets

$$u \approx \frac{1}{f^{1/4}}\left(c_+\exp\left(i\int dx\sqrt{f}\right) + c_-\exp\left(-i\int dx\sqrt{f}\right)\right).$$

Evaluating these can be done using the binomial expansion for large values of x.

$$\begin{aligned}\sqrt{f} &= \sqrt{x^{\frac{1}{2}}\left(1 + \frac{15}{4}x^{-\frac{5}{2}} - \frac{21}{4}x^{-\frac{7}{2}}\right)} \\ &\approx x^{\frac{1}{4}}\left(1 + \frac{15}{8}x^{-\frac{5}{2}} - \frac{21}{8}x^{-\frac{7}{2}}\right) \\ &= x^{\frac{1}{4}} + \frac{15}{8}x^{-\frac{9}{4}} - \frac{21}{8}x^{-\frac{13}{4}}\end{aligned}$$

$$\Rightarrow \int dx\sqrt{f} = \frac{4}{5}x^{\frac{5}{4}} - \frac{3}{2}x^{-\frac{5}{4}} + \frac{7}{6}x^{-\frac{9}{4}}$$

$$\begin{aligned} f^{-\frac{1}{4}} &= \left(x^{\frac{1}{2}}\left(1 + \frac{15}{4}x^{-\frac{5}{2}} - \frac{21}{4}x^{-\frac{7}{2}}\right)\right)^{-\frac{1}{4}} \\ &\approx x^{-\frac{1}{8}}\left(1 - \frac{15}{16}x^{-\frac{5}{2}} + \frac{21}{16}x^{-\frac{7}{2}}\right) \end{aligned}$$

$$\Rightarrow u \approx x^{-\frac{1}{8}}\left(1 - \frac{15}{16}x^{-\frac{5}{2}} + \frac{21}{16}x^{-\frac{7}{2}}\right) \cdot C_1 \cos\left(\frac{4}{5}x^{\frac{5}{4}} - \frac{3}{2}x^{-\frac{5}{4}} + \frac{7}{6}x^{-\frac{9}{4}} + C_2\right)$$

$$\therefore y \approx x^{\frac{11}{8}}\left(1 - \frac{15}{16}x^{-\frac{5}{2}} + \frac{21}{16}x^{-\frac{7}{2}}\right) \cdot C_1 \cos\left(\frac{4}{5}x^{\frac{5}{4}} - \frac{3}{2}x^{-\frac{5}{4}} + \frac{7}{6}x^{-\frac{9}{4}} + C_2\right)$$

41. We again make the substitution $y = up$ to eliminate the first derivative term. Substituting this into the Bessel equation

$$x^2y'' + xy' + \left(x^2 - m^2\right)y = 0$$

yields

$$x^2pu'' + \left(2x^2p' + xp\right)u' + \left(x^2p'' + xp' + \left(x^2 - m^2\right)p\right)u = 0.$$

Thus, we choose $p = x^{-\frac{1}{2}}$ to obtain

$$u'' + \left(1 - \left(m^2 - \frac{1}{4}\right)x^{-2}\right)u = 0.$$

Let

$$f(x) := 1 - \left(m^2 - \frac{1}{4}\right)x^{-2}.$$

Now, we wish to find the asymptotic form of J_m, Bessel's function of the first kind. J_m is distinguished from that of the second kind, Y_m by its analyticity at the origin. Notice how f is a monotone increasing function with a single zero at $\sqrt{m^2 - \frac{1}{4}}$. Therefore, for $m >> \frac{1}{2}$, one can approximate the analyticity condition as the function being bounded for $x << \sqrt{m^2 - \frac{1}{4}}$. This leads us to use the connection formula Eq. (1–113).

$$\sqrt{f} = \sqrt{1 - \left(m^2 - \frac{1}{4}\right)x^{-2}} \approx 1 - \frac{1}{2}\left(m^2 - \frac{1}{4}\right)x^{-2}$$

$$\begin{aligned} \Rightarrow \int_{\sqrt{m^2-\frac{1}{4}}}^{x} dx\sqrt{f} &\approx x - \sqrt{m^2 - \frac{1}{4}} + \frac{1}{2}\left(m^2 - \frac{1}{4}\right)\left(\frac{1}{x} - \frac{1}{\sqrt{m^2 - \frac{1}{4}}}\right) \\ &= x + \frac{m^2 - \frac{1}{4}}{2x} - \frac{1}{2}\sqrt{m^2 - \frac{1}{4}} \end{aligned}$$

$$f^{-\frac{1}{4}} \approx 1 + \frac{1}{4}\left(m^2 - \frac{1}{4}\right)x^{-2}$$

$$\therefore J_m \approx 2x^{-\frac{1}{2}} f^{-\frac{1}{4}} \cos\left(\int_{\sqrt{m^2-\frac{1}{4}}}^{x} dx\sqrt{f} - \frac{\pi}{4}\right)$$
$$\approx \left(2x^{-\frac{1}{2}} + \left(\frac{m^2}{2} - \frac{1}{8}\right)x^{-\frac{5}{2}}\right)\cos\left(x + \frac{m^2 - \frac{1}{4}}{2x} - \frac{1}{2}\sqrt{m^2 - \frac{1}{4}} - \frac{\pi}{4}\right)$$

42. This problem is a simple application of the connection formulae Eq. (1–113) and (1–122) with $f(x) = x$.

(a) Here, we use Eq. (1–122) with $\phi = 0$.

$$\therefore y \sim \frac{1}{\sqrt{2}(-x)^{-\frac{1}{4}}} \exp\left(\frac{2}{3}(-x)^{\frac{3}{2}}\right)$$

(b) Here, we use Eq. (1–113).

$$\therefore y \sim \frac{2}{x^{\frac{1}{4}}} \cos\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{\pi}{4}\right)$$

(c) This problem is not defined, as explained after Eq. (1–122). In short, the large exponential alone cannot determine the phase.

43. We again use the WKB approximation with $f(x) = x^2$.

$$W_\pm := \frac{1}{f^{\frac{1}{4}}} \exp\left(\pm i \int_0^x dx\sqrt{f}\right) = x^{-\frac{1}{2}} \exp\left(\pm\frac{i}{2}x^2\right)$$

$$\Rightarrow y \approx x^{-\frac{1}{2}}\left(A\cos\frac{x^2}{2} + B\sin\frac{x^2}{2}\right)$$

Solving for A and B using the initial conditions $y(10) = 0$ and $y'(10) = 1$, we get

$$y \approx \frac{1}{\sqrt{10x}} \sin\left(\frac{x^2}{2} - 50\right).$$

(a)

$$\frac{x^2}{2} - 50 \approx \pi \Rightarrow x \approx \sqrt{100 + 2\pi} \approx 10.309$$

(b)

$$\frac{x^2}{2} - 50 \approx \frac{\pi}{2} \Rightarrow x \approx \sqrt{100 + \pi} \Rightarrow y \approx \frac{1}{\sqrt{10\sqrt{100+\pi}}} \approx 0.0992$$

44. The error estimation for this solution follows the example following Eq. (1–103). I personally am not confident that this solution is correct, but it is the best I could come up with.

The WKB approximation of y_1 is given by

$$y_1 \approx \frac{C_1}{\sqrt{5x}} \sin\left(\frac{x^2 - 25}{2}\right)$$

which can be obtained in the same way as in Problem 1–43. The 25th zero beyond $x = 5$ is then given by

$$\frac{x^2 - 25}{2} \approx 25\pi \Rightarrow x \approx 5\sqrt{2\pi + 1} \approx 13.494.$$

We now begin the error analysis. The following notation follows that of the aforementioned example. Let

$$y_1 = \alpha_+ x^{-\frac{1}{2}} \exp\left(\frac{i}{2}x^2\right) + \alpha_- x^{-\frac{1}{2}} \exp\left(-\frac{i}{2}x^2\right).$$

Near $x = 5$, we then obtain

$$\alpha_+ \approx \frac{A}{2i} e^{-\frac{25}{2}i},\ \alpha_- \approx -\frac{A}{2i} e^{\frac{25}{2}i}.$$

Let

$$g = \frac{1}{4}\frac{f''}{f} - \frac{5}{16}\left(\frac{f'}{f}\right)^2 = -\frac{3}{4x^2}$$

$$\begin{aligned}
\Rightarrow \alpha'_\pm &\approx \mp\frac{i}{2}\frac{g}{\sqrt{f}}\left(\alpha_\pm + \alpha_\mp \exp\left(\mp 2i \int dx \sqrt{f}\right)\right) \\
&= \pm\frac{3i}{8x^3}\left(\alpha_\pm + \alpha_\mp \exp\left(\mp i x^2\right)\right) \\
&\approx \frac{3iA}{32x^3} e^{\mp\frac{25}{2}i}\left(1 - e^{\pm 25i \mp ix^2}\right)
\end{aligned}$$

Now, upon integrating this from $x = 5$ to $5\sqrt{2\pi + 1}$, we may ignore the $e^{\pm 25i \mp ix^2}$ term as it rotates rapidly, thus making little contribution to the result.

$$\begin{aligned}
\Rightarrow |\Delta y_1| &\approx |\Delta \alpha_\pm| \\
&\approx \left|\int_5^{5\sqrt{2\pi+1}} dx \alpha'_\pm\right| \\
&\approx \left|\frac{3iA}{32x^3} e^{\mp\frac{25}{2}i} \int_5^{5\sqrt{2\pi+1}} \frac{dx}{x^3}\right| \\
&= \frac{3\pi A}{800(2\pi + 1)}
\end{aligned}$$

We also have

$$y_1'|_{x=5\sqrt{2\pi+1}} \approx \frac{A}{\sqrt{5}}\left(-\frac{1}{2}x^{-\frac{3}{2}}\sin\left(\frac{x^2-25}{2}\right)+x^{\frac{1}{2}}\cos\left(\frac{x^2-25}{2}\right)\right)\Big|_{x=5\sqrt{2\pi+1}}$$

$$\approx -A(2\pi+1)^{\frac{1}{4}}.$$

$$\therefore |\Delta x| \approx \left|\frac{\Delta y}{y'}\right| \approx \frac{3\pi}{800(2\pi+1)^{\frac{1}{4}}} \approx 3.488\times 10^{-4}$$

2 Chapter 2

1. A useful trick when dealing with complex sign patterns like this is to represent them using an exponential. In this case, we acknowledge that

$$\left\{\Re\left\{e^{i\left(\frac{n}{2}-\frac{1}{4}\right)\pi}\right\}\right\} = \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \cdots$$

and write the series as

$$\begin{aligned}
\text{(Given series)} &= \sum_{n=0}^{\infty}\frac{1}{4^n}\cdot\sqrt{2}\Re\left\{e^{i\left(\frac{n}{2}-\frac{1}{4}\right)\pi}\right\} \\
&= \Re\left\{\sqrt{2}e^{-i\frac{\pi}{4}}\sum_{n=0}^{\infty}\left(\frac{e^{i\frac{\pi}{4}}}{2}\right)^n\right\} \\
&= \Re\left\{(1-i)\sum_{n=0}^{\infty}\left(\frac{i}{4}\right)^n\right\} \\
&= \Re\left\{(1-i)\cdot\frac{1}{1-\frac{i}{4}}\right\} \\
&= \frac{16}{17}\Re\left\{(1-i)\left(1+\frac{i}{4}\right)\right\} \\
&= \frac{16}{17}\left(1+\frac{1}{4}\right) \\
&= \frac{20}{17}.
\end{aligned}$$

For series problems, it is always a good idea to verify the results numerically with a calculator, provided that the series converges quickly enough. In this case, evaluating up to $\frac{1}{1024}$ yields about 1.1768 while the correct answer is about 1.1765.

2.

$$
\begin{aligned}
\text{(Given series)} &= \sum_{n=0}^{\infty} \frac{1}{n(n+2)} \\
&= \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) \\
&= \frac{1}{2}\left(1 + \frac{1}{2}\right) \\
&= \frac{3}{4}
\end{aligned}
$$

3. To describe the sign pattern, we shall use

$$
\left\{\Re\left\{e^{i\left(\frac{n}{3}+\frac{1}{6}\right)\pi}\right\}\right\} = \frac{\sqrt{3}}{2},\ 0,\ -\frac{\sqrt{3}}{2},\ -\frac{\sqrt{3}}{2},\ 0,\ \frac{\sqrt{3}}{2},\ \cdots .
$$

$$
\begin{aligned}
\text{(Given series)} &= \frac{1}{1\cdot 3^0} + \frac{0}{3\cdot 3^1} + \frac{-1}{5\cdot 3^2} + \frac{-1}{7\cdot 3^3} + \frac{0}{9\cdot 3^4} + \frac{1}{11\cdot 3^5} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)3^n} \cdot \frac{2}{\sqrt{3}} \Re\left\{e^{i\left(\frac{n}{3}+\frac{1}{6}\right)\pi}\right\} \\
&= \Re\left\{\frac{2}{\sqrt{3}} e^{i\frac{\pi}{6}} \sum_{n=0}^{\infty} \frac{1}{2n+1}\left(\frac{1}{3}e^{i\frac{\pi}{3}}\right)^n\right\} \\
&= \Re\left\{2\sum_{n=0}^{\infty} \frac{1}{2n+1}\left(\frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}\right)^{2n+1}\right\} \\
&= \Re\left\{2\tanh^{-1}\left(\frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}\right)\right\} \\
&= \Re\left\{\ln\left(\frac{1+\frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}}{1-\frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}}\right)\right\} \\
&= \ln\left|\frac{1+\frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}}{1-\frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}}\right| \\
&= \ln\left|\frac{5-\sqrt{3}i}{2}\right| \\
&= \frac{1}{2}\ln 7
\end{aligned}
$$

4.

$$
\begin{aligned}
(\text{Given series}) &= \sum_{n=0}^{\infty} \frac{n+1}{n!} \\
&= \left. \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!} \right|_{x=1} \\
&= \left. \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} \right|_{x=1} \\
&= \left. \frac{d}{dx} \left(xe^x\right) \right|_{x=1} \\
&= \left. \left((x+1)e^x\right) \right|_{x=1} \\
&= 2e
\end{aligned}
$$

5.

$$
\begin{aligned}
(\text{Given series}) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\
&= \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= \frac{3}{4} \cdot \frac{\pi^2}{6} \\
&= \frac{\pi^2}{8}
\end{aligned}
$$

6.

$$
\begin{aligned}
(\text{Given series}) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^4} - \sum_{n=1}^{\infty} \frac{1}{(2n)^4} \\
&= \frac{15}{16} \sum_{n=1}^{\infty} \frac{1}{n^4} \\
&= \frac{15}{16} \cdot \frac{\pi^4}{90} \\
&= \frac{\pi^4}{96}
\end{aligned}
$$

7.

$$\begin{aligned}
(\text{Given series}) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} - 2\sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n^2} \\
&= \frac{1}{2} \cdot \frac{\pi^2}{6} \\
&= \frac{\pi^2}{12}
\end{aligned}$$

8.

$$\begin{aligned}
f(\theta) &= \sum_{n=0}^{\infty} \frac{\sin{(n+1)\,\theta}}{2n+1} \\
&= \Im\left\{ \sum_{n=0}^{\infty} \frac{1}{2n+1} e^{i(n+1)\theta} \right\} \\
&= \Im\left\{ e^{i\frac{\theta}{2}} \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(e^{i\frac{\theta}{2}}\right)^{2n+1} \right\} \\
&= \Im\left\{ e^{i\frac{\theta}{2}} \tanh^{-1}\left(e^{i\frac{\theta}{2}}\right) \right\} \\
&= \frac{1}{2}\Im\left\{ e^{i\frac{\theta}{2}} \ln\left(\frac{1+e^{i\frac{\theta}{2}}}{1-e^{i\frac{\theta}{2}}}\right) \right\} \\
&= \frac{1}{2}\Im\left\{ e^{i\frac{\theta}{2}} \ln\left(i\cot\frac{\theta}{4}\right) \right\} \\
&= \frac{1}{2}\Im\left\{ \left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \left(\ln\left(\cot\frac{\theta}{4}\right) + i\frac{\pi}{2}\right) \right\} \\
&= \frac{\pi}{4}\cos\frac{\theta}{2} + \frac{1}{2}\ln\left(\cot\frac{\theta}{4}\right)\sin\frac{\theta}{2}
\end{aligned}$$

9. Half of the challenge of this problem is expressing the nth term in closed

form. Let us denote the first term as a_1.

$$
\begin{aligned}
a_n &= \frac{\left(\prod_{j=1}^{n-1} \frac{2j-1}{2}\right) \cdot \left(\frac{2n-1}{2}\right)^2}{(2n+7)(2n+5)(2n+3)^2 n!} \\
&= \frac{(2n-1)!!(2n-1)}{2^{n+1}(2n+7)(2n+5)(2n+3)^2 n!} \\
&= \frac{\dfrac{(2n)!}{2^n n!} \cdot (2n-1)}{2^{n+1}(2n+7)(2n+5)(2n+3)^2 n!} \\
&= \frac{(2n-1) \cdot (2n)!}{2^{2n+1}(2n+7)(2n+5)(2n+3)(n!)^2} \\
&= \frac{2n-1}{2^{2n+1}(2n+7)(2n+5)(2n+3)^2} \binom{2n}{n}
\end{aligned}
$$

Unfortunately, the ratio test of this series collapses. We thus try to find an upper/lower bound for this sequence.

$$
\begin{aligned}
a_n &= \frac{2n-1}{2^{2n+1}(2n+7)(2n+5)(2n+3)^2} \binom{2n}{n} \\
&< \frac{2n-1}{2^{2n+1}(2n+7)(2n+5)(2n+3)^2} \cdot 2^{2n} \\
&= \frac{2n-1}{2(2n+7)(2n+5)(2n+3)^2} \\
&< \frac{2n}{2(2n)(2n)(2n)^2} \\
&= \frac{1}{16n^4}
\end{aligned}
$$

The series of the last sequence converges by the p-test. Therefore, by comparison test, the given series converges.

10. Again, we shall denote the first term as a_1.

$$
\begin{aligned}
a_n &= \frac{((2n+1)!!)^2}{4^{n-1} \cdot n \cdot (n!)^2} \\
&= \frac{\left(\dfrac{(2n+1)!}{2^n n!}\right)^2}{4^{n-1} \cdot n \cdot (n!)^2} \\
&= \frac{((2n+1)!)^2}{4^{2n+1} \cdot n \cdot (n!)^4}
\end{aligned}
$$

To deduce the limit of this sequence, we employ Stirling's approximation:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \text{ (for large } n\text{).}$$

$$
\begin{aligned}
\Rightarrow a_n &\approx \frac{2\pi(2n+1)\left(\dfrac{2n+1}{e}\right)^{2n+1}}{4^{2n+1}\cdot n\cdot 4\pi^2 n^2\left(\dfrac{n}{e}\right)^{n}} \\
&= \frac{(2n+1)^{4n+3}}{\pi e^2 2^{8n+3} n^{4n+3}} \\
&= \frac{4}{\pi e}\left(1+\frac{1}{2n}\right)^{4n+3} \\
&= \frac{4}{\pi e}\left(\left(1+\frac{1}{2n}\right)^{2n}\right)^{\frac{4n+3}{2n}} \\
&\xrightarrow{n\to\infty} \frac{4}{\pi} \neq 0
\end{aligned}
$$

Therefore, the given series diverges.

11. We want to evaluate

$$
f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)^2 x^{2n+1}}{(2n+1)!}.
$$

To this end, we want to express $(n+1)^2 = n^2+2n+1$ in terms of $2n+2$ and $(2n+1)(2n+3)$ as they can easily be obtained by differentiation. We employ Horner's schema to do so.

$$
\begin{array}{r|ccc}
 & 1 & 2 & 1 \\
-1 & & -1 & -1 \\
\hline
 & 1 & 1 & \multicolumn{1}{|c}{0} \\
-\frac{3}{2} & & -\frac{3}{2} & \\
\hline
 & \multicolumn{1}{c|}{1} & -\frac{1}{2} &
\end{array}
$$

$$
\begin{aligned}
\Rightarrow n^2+2n+1 &= \left(\left(n+\frac{3}{2}\right)-\frac{1}{2}\right)(n+1) \\
&= \frac{1}{4}(2n+3)(2n+2) - \frac{1}{4}(2n+2)
\end{aligned}
$$

$$
\begin{aligned}
\therefore f(x) &= \frac{1}{4}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2n+3)(2n+2)x^{2n+1} - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2n+2)x^{2n+1}\right) \\
&= \frac{1}{4}\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+3}\right) - \frac{1}{4}\frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+2}\right) \\
&= \frac{1}{4}\frac{d^2}{dx^2}\left(x^2\sin x\right) - \frac{1}{4}\frac{d}{dx}\left(x\sin x\right) \\
&= \frac{1}{4}\left(2\sin x + 4x\cos x - x^2\sin x\right) - \frac{1}{4}\left(\sin x + x\cos x\right) \\
&= \frac{1-x^2}{4}\sin x + \frac{3}{4}x\cos x
\end{aligned}
$$

12.

(a) We should first formally state what the problem is asking of us.

$$
\begin{aligned}
(E+1)^k + (E-1)^k &= \sum_{i=0}^{k}\binom{k}{i}E_i + \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}E_i \\
&= \sum_{i=0}^{k}\left(1+(-1)^i\right)\binom{k}{i}E_i \\
&= 2\sum_{i=0}^{k/2}\binom{k}{2i}E_{2i}
\end{aligned}
$$

Therefore, we need to show that

$$
\sum_{i=0}^{l}\binom{2l}{2i}E_{2i} = 0
$$

for all integers l.

$$
\begin{aligned}
1 &= \sec z\cdot\cos z \\
&= \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}E_{2n}z^{2n}\right)\cdot\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}\right) \\
&= \sum_{l=0}^{\infty}\left(\sum_{i=0}^{l}\frac{(-1)^i}{(2i)!}E_{2i}\cdot\frac{(-1)^{2l-i}}{(k-2i)!}\right)z^{2l} \\
&= \sum_{l=0}^{\infty}\left(\frac{1}{(2l)!}\sum_{i=0}^{l}\binom{2l}{2i}E_{2i}\right)z^{2l}
\end{aligned}
$$

$$
\therefore (\forall l\geq 1)\sum_{i=0}^{l}\binom{2l}{2i}E_{2i} = 0
$$

We could use this formula to determine successive terms of this sequence.

$$l = 1: \; E_2 + E_0 = 0 \Rightarrow E_2 = -1$$

$$l = 2: \; E_4 + 6E_2 + E_0 = 0 \Rightarrow E_4 = 5$$

$$l = 3: \; E_6 + 15E_4 + 15E_2 + E_0 = 0 \Rightarrow E_6 = -61$$

$$l = 4: \; E_8 + 28E_6 + 70E_4 + 28E_2 + E_0 = 0 \Rightarrow E_9 = 1385$$

(b) It is sometimes a good idea to start with the more general formula then apply it to the more concrete problem, as this could lead to less work.

$$\begin{aligned}
\sec \pi x &= \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i (2i+1)}{(2i+1)^2 - 4x^2} \\
&= \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \left(1 - \left(\frac{2x}{2i+1} \right)^2 \right)^{-1} \\
&= \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \sum_{n=0}^{\infty} \left(\frac{2x}{2i+1} \right)^{2n} \\
&= \frac{4}{\pi} \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} (-1)^i \frac{4^n}{2i+1^{2n+1}} x^{2n} \\
&= \sum_{n=0}^{\infty} \left(\frac{4^{n+1}}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^{2n+1}} \right) x^{2n} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} (\pi x)^{2n}
\end{aligned}$$

We thus compare the last two series term-by-term to obtain

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^{2n+1}} = \frac{\pi^{2n+1} (-1)^n E_{2n}}{4^{n+1} (2n)!}$$

as the answer to part (b).

Substituting $n = 1$ for part (a), we get

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^3} = \frac{\pi^3 (-1)^1 E_2}{4^2 \cdot 2!} = \frac{\pi^3}{32}.$$

3 Chapter 3

1. Let us define

$$I(a, b) = \int_0^{\infty} dy \frac{e^{-ay} - e^{-by}}{y}.$$

$$\frac{\partial I}{\partial a} = -\int_0^\infty dy e^{-ay} = -\frac{1}{a}, \; \frac{\partial I}{\partial b} = \int_0^\infty dy e^{-by} = \frac{1}{b}$$

$$\Rightarrow I(a,b) = \ln\left(\frac{b}{a}\right) + C$$

Now, notice that

$$C = I(a,a) = \int_0^\infty dy 0 = 0.$$

$$\therefore I(a,b) = \ln\left(\frac{b}{a}\right)$$

2. (If your knee-jerk reaction to this problem is any form of skepticism, then you are probably correct; if not, you are a perfect fit for a physicist in my opinion.)

$$\begin{aligned}
\int_0^\infty dx \sin bx &= \lim_{a\to 0+} \int_0^\infty dx e^{-ax} \sin bx \\
&= \lim_{a\to 0+} \Im\left\{\int_0^\infty dx e^{-ax} e^{ibx}\right\} \\
&= \lim_{a\to 0+} \Im\left\{\frac{1}{a-ib}\right\} \\
&= \lim_{a\to 0+} \frac{b}{a^2+b^2} \\
&= \frac{1}{b}
\end{aligned}$$

3.

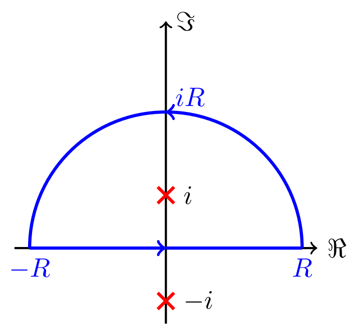


Figure 1: Contour for Problem 3–3, 3–4, and 3–9.

We shall perform a contour integral of $f(z) := \frac{e^{iaz}}{1+z^2}$ along Figure 1.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=i} f(z) = 2i\pi \cdot \left.\frac{e^{iaz}}{z+i}\right|_{z=i} = \pi e^{-a}$$

We also have

$$\oint_C dx f(z) = \int_{-R}^{R} dx \frac{e^{iax}}{1+x^2} + \int_0^{\pi} d\theta i R e^{i\theta} \frac{e^{iaRe^{i\theta}}}{1+R^2 e^{2i\theta}}$$
$$\xrightarrow{R\to\infty} \int_{-\infty}^{\infty} dx \frac{e^{iax}}{1+x^2}.$$

$$\begin{aligned}
\therefore \int_0^{\infty} dx \frac{\cos ax}{1+x^2} &= \frac{1}{2}\int_{-\infty}^{\infty} dx \frac{\cos ax}{1+x^2} \\
&= \frac{1}{2}\cdot \Re\left\{\int_{-\infty}^{\infty} dx \frac{e^{iax}}{1+x^2}\right\} \\
&= \frac{\pi}{2} e^{-a}
\end{aligned}$$

4. Let us integrate $f(z) := \frac{e^{iaz}}{(1+z^2)^2}$, again along Figure 1.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=i} f(z) = 2i\pi \cdot \left.\frac{d}{dz}\frac{e^{iaz}}{(z+i)^2}\right|_{z=i} = \frac{\pi}{2}(a+1)e^{-a}$$

and

$$\oint_C dx f(z) = \int_{-R}^{R} dx \frac{e^{iax}}{(1+x^2)^2} + \int_0^{\pi} d\theta i R e^{i\theta} \frac{e^{iaRe^{i\theta}}}{(1+R^2 e^{2i\theta})^2}$$
$$\xrightarrow{R\to\infty} \int_{-\infty}^{\infty} dx \frac{e^{iax}}{(1+x^2)^2}.$$

$$\begin{aligned}
\therefore \int_0^{\infty} dx \frac{\cos ax}{(1+x^2)^2} &= \frac{1}{2}\int_{-\infty}^{\infty} dx \frac{\cos ax}{(1+x^2)^2} \\
&= \frac{1}{2}\cdot \Re\left\{\int_{-\infty}^{\infty} dx \frac{e^{iax}}{(1+x^2)^2}\right\} \\
&= \frac{\pi}{4}(a+1)e^{-a}
\end{aligned}$$

5. Let $I(\mathbf{a}, b) = \int d^3x e^{i\mathbf{a}\cdot\mathbf{x}} e^{-br^2}$. Notice that this integral depends on a vector and results in a scalar. Thus, we must conclude that only the magnitude of $\mathbf{a}$ matters and not its direction.

Let $a := ||\mathbb{a}||$.

$$\begin{aligned}
I(\mathbb{a}, b) &= I(a\hat{\mathbb{z}}, b) \\
&= \int d^3x e^{iaz} e^{-b\left(x^2+y^2+z^2\right)} \\
&= \int_{-\infty}^{\infty} dx e^{-bx^2} \int_{-\infty}^{\infty} dy e^{-by^2} \int_{-\infty}^{\infty} dz e^{iaz-bz^2} \\
&= \sqrt{\frac{\pi}{b}} \cdot \sqrt{\frac{\pi}{b}} \cdot \int_{-\infty}^{\infty} dz e^{-b\left(z-\frac{ia}{2b}\right)^2-\frac{a^2}{4b}} \\
&= \left(\frac{\pi}{b}\right)^{3/2} e^{-\frac{a^2}{4b}}
\end{aligned}$$

$$\therefore I(\mathbb{a}, b) = \left(\frac{\pi}{b}\right)^{3/2} e^{-\frac{\mathbb{a}^2}{4b}}$$

6. Let $\mathbb{I}(\mathbb{a}, b) = \int d^3x \mathbb{x} e^{i\mathbb{a}\cdot\mathbb{x}} e^{-br^2}$. Notice that this vector-valued integral depends on only one vector, namely, $\mathbb{a}$. Therefore, this integral must be parallel to this vector.

Let $a = ||\mathbb{a}||$ and $\mathbb{I}(\mathbb{a}, b) = F(a, b)\hat{\mathbb{a}}$.

$$\mathbb{I}(a\hat{\mathbb{z}}, b) = F(a, b)\hat{\mathbb{z}}$$

$$\begin{aligned}
\Rightarrow F(a, b) &= \mathbb{I}(a\hat{\mathbb{z}}, b) \cdot \hat{\mathbb{z}} \\
&= \int d^3x z e^{iaz} e^{-b\left(x^2+y^2+z^2\right)} \\
&= \int_{-\infty}^{\infty} dx e^{-bx^2} \int_{-\infty}^{\infty} dy e^{-by^2} \int_{-\infty}^{\infty} dz z e^{iaz-bz^2} \\
&= \sqrt{\frac{\pi}{b}} \cdot \sqrt{\frac{\pi}{b}} \cdot \frac{\partial}{\partial\alpha} \int_{-\infty}^{\infty} dz e^{-b\left(z-\frac{\alpha}{2b}\right)^2+\frac{\alpha^2}{4b}} \Bigg|_{\alpha=ia} \\
&= \frac{\pi}{b} \cdot \frac{\partial}{\partial\alpha} \left(\sqrt{\frac{\pi}{b}} e^{\frac{\alpha^2}{4b}}\right)\Bigg|_{\alpha=ia} \\
&= \left(\frac{\pi}{b}\right)^{3/2} \cdot \frac{\alpha}{2b} e^{\frac{\alpha^2}{4b}}\Bigg|_{\alpha=ia} \\
&= \left(\frac{\pi}{b}\right)^{3/2} \frac{ia}{2b} e^{-\frac{a^2}{4b}}
\end{aligned}$$

$$\therefore I(\mathbb{a}, b) = F(||\mathbb{a}||, b)\hat{\mathbb{a}} = \left(\frac{\pi}{b}\right)^{3/2} \frac{i\mathbb{a}}{2b} e^{-\frac{\mathbb{a}^2}{4b}}$$

7.

$$\begin{aligned}
\ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (-x)^n \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(1 + (-1)^{n-1}\right) x^n \\
&= \sum_{n=0}^{\infty} \frac{2}{2n+1} x^{2n+1}
\end{aligned}$$

$$\begin{aligned}
\therefore \int_0^1 \frac{dx}{x} \ln\left(\frac{1+x}{1-x}\right) &= \int_0^1 dx \sum_{n=0}^{\infty} \frac{2}{2n+1} x^{2n} \\
&= \sum_{n=0}^{\infty} \frac{2}{(2n+1)^2} \\
&= 2\left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2}\right) \\
&= \frac{3}{2} \cdot \frac{\pi^2}{6} \\
&= \frac{\pi^2}{4}
\end{aligned}$$

8.

$$\begin{aligned}
\int_0^{\infty} \frac{dx}{\cosh x} &= \int_0^{\infty} dx \frac{2}{e^x + e^{-x}} \\
&= \int_1^{\infty} du \frac{2}{u^2+1} \qquad\qquad (u := e^x) \\
&= \left[2\tan^{-1} u\right]_1^{\infty} \\
&= \frac{\pi}{2}
\end{aligned}$$

9. Let us integrate $f(z) := \frac{1}{(1+z^2)^2}$, again along Figure 1.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=i} f(z) = 2i\pi \cdot \left.\frac{d}{dz}\frac{1}{(z+i)^2}\right|_{z=i} = \frac{\pi}{2}$$

and

$$\oint_C dx f(z) = \int_{-R}^{R} dx \frac{1}{(1+x^2)^2} + \int_0^{\pi} d\theta \frac{iRe^{i\theta}}{(1+R^2e^{2i\theta})^2} \xrightarrow{R\to\infty} \int_{-\infty}^{\infty} dx \frac{e^{iax}}{(1+x^2)^2}.$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{\left(1+x^2\right)^2} = \frac{\pi}{2}$$

10.

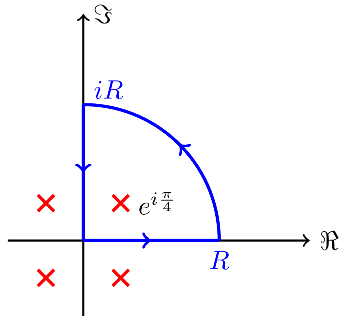


Figure 2: Contour for Problem 3–10.

We shall perform a contour integral of $f(z) := \frac{1}{1+z^4}$ along Figure 2.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=e^{i\frac{\pi}{4}}} f(z) = 2i\pi \cdot \left.\frac{1}{4x^3}\right|_{z=e^{i\frac{\pi}{4}}} = \frac{\pi}{2}e^{-i\frac{\pi}{4}}$$

We also have

$$\begin{aligned}
\oint_C dx f(z) &= \int_0^R \frac{dx}{1+x^4} + \int_0^{\frac{\pi}{2}} d\theta \frac{iRe^{i\theta}}{1+R^4e^{4i\theta}} + \int_R^0 dy \frac{i}{1+(iy)^4} \\
&= (1-i)\int_0^R \frac{dx}{1+x^4} + O\left(R^{-3}\right) \\
&\xrightarrow{R\to\infty} (1-i)\int_0^\infty \frac{dx}{1+x^4}.
\end{aligned}$$

$$\therefore \int_0^\infty \frac{dx}{1+x^4} = \frac{1}{1-i}\cdot\frac{\pi}{2}e^{-i\frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}}$$

11.
We shall perform a contour integral of $f(\omega) := \frac{e^{i\omega t}}{\omega^2-\omega_0^2}$ along Figure 3.

$$\oint_C d\omega f(\omega) = 2i\pi\left(\operatorname*{Res}_{\omega=-\omega_0} f(\omega) + \operatorname*{Res}_{\omega=\omega_0} f(\omega)\right) = -\frac{2\pi}{\omega_0}\sin\omega_0 t$$

We also have

$$\begin{aligned}
\oint_C d\omega f(\omega) &= \int_{-R}^R d\omega \frac{e^{i\omega t}}{\omega^2-\omega_0^2} + \int_0^\pi d\theta iRe^{i\theta}\frac{e^{iRte^{i\theta}}}{R^2e^{2i\theta}-\omega_0^2} \\
&\xrightarrow{R\to\infty} \int_{-\infty}^\infty d\omega \frac{e^{i\omega t}}{\omega^2-\omega_0^2}
\end{aligned}$$

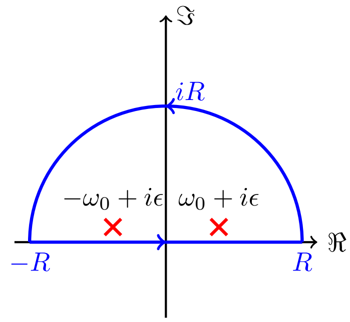


Figure 3: Contour for Problem 3–11.

$$\therefore \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega t}}{\omega^2 - \omega_0^2} = -\frac{2\pi}{\omega_0} \sin \omega_0 t$$

(Note: As there are poles on the real axis, this integral technically diverges without the "slightly above the axis" condition. If this integral were to have physical meaning, then this assumption must be physically explained, for example using causality arguments. Another way around this is to use the "Cauchy principal value" of the integral, whose formula is shown in Eq. (A–17).)

12.

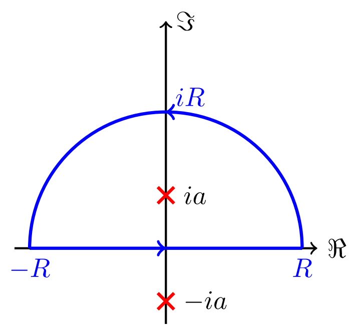


Figure 4: Contour for Problem 3–12, 3–13, 3–21, and 3–22.

We shall perform a contour integral of $f(z) := \frac{z^2}{(a^2+z^2)^2}$ along Figure 4.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=ia} f(z) = 2i\pi \cdot \frac{d}{dz} \frac{z^2}{(z+ia)^2}\bigg|_{z=ia} = \frac{\pi}{2a}$$

We also have

$$\oint_C dx f(z) = \int_{-R}^{R} dx \frac{x^2}{(a^2+x^2)^2} + \int_0^{\pi} d\theta i R e^{i\theta} \frac{R^2 e^{2i\theta}}{(a^2+R^2e^{2i\theta})^2}$$
$$\xrightarrow{R\to\infty} \int_{-\infty}^{\infty} dx \frac{x^2}{(a^2+x^2)^2}.$$
$$\therefore \int_{-\infty}^{\infty} dx \frac{x^2}{a^2+x^2{}^2} = \frac{\pi}{2a}$$

13.

$$\int \frac{d^3x}{(a^2+r^2)^3} = \int_0^{\infty} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \frac{r^2 \sin\theta}{(a^2+r^2)^3}$$
$$= 4\pi \int_0^{\infty} dr \frac{r^2}{(a^2+r^2)^3}$$

We shall perform a contour integral of $f(z) := \frac{z^2}{(a^2+z^2)^3}$, again along Figure 4.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=ia} f(z) = \frac{2i\pi}{2!} \cdot \frac{d^2}{dz^2} \frac{z^2}{(z+ia)^3}\bigg|_{z=ia} = \frac{\pi}{8a^3}$$

We also have

$$\oint_C dx f(z) = \int_{-R}^{R} dr \frac{r^2}{(a^2+r^2)^3} + \int_0^{\pi} d\theta i R e^{i\theta} \frac{R^2 e^{2i\theta}}{(a^2+R^2e^{2i\theta})^3}$$
$$\xrightarrow{R\to\infty} \int_{-\infty}^{\infty} dx \frac{r^2}{(a^2+r^2)^3}.$$
$$\Rightarrow \int_0^{\infty} dr \frac{r^2}{(a^2+r^2)^3} = \frac{1}{2}\int_{-\infty}^{\infty} dx \frac{r^2}{(a^2+r^2)^3} = \frac{\pi}{16a^3}$$
$$\therefore \int \frac{d^3x}{(a^2+r^2)^3} = \frac{\pi^2}{4a^3}$$

14.

$$I := \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}(a+bx)}$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \frac{\cos\theta}{\cos\theta\,(a+b\sin\theta)} \qquad (x = \sin\theta)$$
$$= \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$$
$$= \oint_{|z|=1} \frac{dz/iz}{a+\frac{b}{2i}\left(z-\frac{1}{z}\right)} \qquad (z = e^{i\theta})$$
$$= \oint_{|z|=1} \frac{dz}{bz^2+2iaz-b}$$

Notice that the last integrand has simple poles at $z_\pm^* := \frac{-a\pm\sqrt{a^2-b^2}}{b} i$, of which only z_+^* lies inside the unit circle.

$$\therefore I = 2i\pi \operatorname*{Res}_{z=z_+^*} \frac{1}{bz^2 + 2iaz - b} = \frac{\pi}{\sqrt{a^2-b^2}}$$

15.

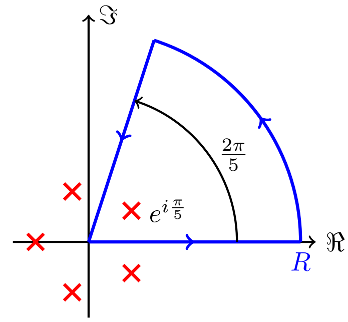


Figure 5: Contour for Problem 3–15.

We shall perform a contour integral of $f(z) := \frac{z}{1+z^5}$ along Figure 5.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=e^{i\frac{\pi}{5}}} f(z) = 2i\pi \cdot \left.\frac{z}{5z^4}\right|_{z=e^{i\frac{\pi}{5}}} = \frac{2\pi}{5} e^{-i\frac{\pi}{10}}$$

We also have

$$\begin{aligned}\oint_C dx f(z) &= \int_0^R dx \frac{x}{1+x^5} + \int_0^{\frac{2\pi}{5}} d\theta i R e^{i\theta} \frac{Re^{i\theta}}{1+R^5 e^{5i\theta}} + \int_R^0 dx e^{i\frac{2\pi}{5}} \frac{xe^{i\frac{2\pi}{5}}}{1+x^5} \\ &\xrightarrow{R\to\infty} \left(1+e^{-i\frac{\pi}{5}}\right) \int_0^\infty dx \frac{x}{1+x^5}.\end{aligned}$$

$$\therefore \int_0^\infty dx \frac{x}{1+x^5} = \frac{\pi}{5\cos\frac{\pi}{10}}$$

16. Let $z := e^{i\theta}$.

$$dz = ie^{i\theta} d\theta,\ \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right),\ \sin^2\theta = \frac{1-\cos 2\theta}{2} = \frac{1}{4}\left(2 - z^2 - \frac{1}{z^2}\right)$$

$$\begin{aligned} I &:= \int_0^{2\pi} d\theta \frac{\sin^2\theta}{a + b\cos\theta} \\ &= \frac{1}{2}\oint_{|z|=1} \frac{-\left(z^2-1\right)^2}{z\left(bz^2 + 2az + b\right)}\end{aligned}$$

Let $f(z)$ be the integrand here. $f(z)$ has simple poles at

$$0,\ z_\pm^* := \frac{-a \pm \sqrt{a^2-b^2}}{b},$$

among which 0 and z_+^* lie within the unit circle (which is our contour).

$$\begin{aligned}
\therefore I &= \frac{1}{2}\cdot 2i\pi \left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=z_+^*} f(z) \right) \\
&= i\pi \left(\left. \frac{-\left(z^2-1\right)^2}{bz^2+2az+b} \right|_{z=0} + \left. \frac{-\left(z^2-1\right)^2}{z\left(bz+a+\sqrt{a^2-b^2}\right)} \right|_{z=z_+^*} \right) \\
&= \frac{i\pi}{b} \left(-1 + 4\frac{a^4+b^4-2ab\left((a-b)\sqrt{a^2-b^2}+b^2\right)}{-a+\sqrt{a^2-b^2}} \right)
\end{aligned}$$

17.

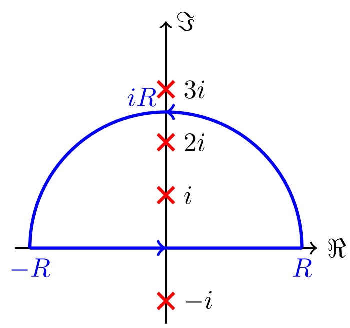


Figure 6: Contour for Problem 3–17.

We must first determine whether, or more precisely, for which values of a, this integral converges. As $|x| \to \infty$, $\sinh ax \sim e^{|ax|}$. Thus, this integral converges whenever $|a| < \pi$.

Let $f(z) := \frac{\sinh az}{\sinh \pi z}$ The singularity at the origin is removable, as

$$\lim_{z\to 0} \frac{\sinh az}{\sinh \pi z} = \lim_{z\to 0} \frac{a\cosh az}{\pi \cosh \pi z} = \frac{a}{\pi}$$

via L'Hôpital's rule. (Note the sloppy math here: the first equality hold only because the second expression does indeed converge to a finite number. If this were a calculus class, this equation is prone to points lost; since this is physics, we can proceed as is.) Thus, $f(z)$ has poles at $z = im,\ m \in \mathbb{Z}\backslash\{0\}$. We ignore cases where a is a rational multiple of π for convinience; these cases are left as exercise for the reader.[2]

[2]Just kidding! I will come back for this later, but if you'd like to help, please consider discussing on `https://github.com/pingpingy1/Mathews-MathPhys-Sol/issues/1`.

Consider the contour shown in Figure 6.

$$\begin{aligned}
\oint_C dz f(z) &\xrightarrow{R\to\infty} 2i\pi \sum_{m=1}^{\infty} \operatorname*{Res}_{z=im} f(z) \\
&= 2i \sum_{m=1}^{\infty} \frac{\sinh iam}{\cosh i\pi m} \\
&= 2 \sum_{m=1}^{\infty} (-1)^{m-1} \sin am \\
&= 2\Im \left\{ \sum_{m=1}^{\infty} e^{i(m-1)\pi} e^{iam} \right\} \\
&= \tan \frac{a}{2}
\end{aligned}$$

We also have

$$\begin{aligned}
\oint_C dz f(z) &= \int_{-R}^{R} dx \frac{\sinh ax}{\sinh \pi x} + \int_0^{\pi} d\theta i R e^{i\theta} \frac{\sinh\left(aRe^{i\theta}\right)}{\sinh\left(\pi R e^{i\theta}\right)} \\
&= \int_{-R}^{R} dx \frac{\sinh ax}{\sinh \pi x} + \mathrm{O}\left(\frac{R}{e^{(\pi-|a|)}R}\right) \\
&\xrightarrow{R\to\infty} \int_{-\infty}^{\infty} dx \frac{\sinh ax}{\sinh \pi x}
\end{aligned}$$

$$\therefore \int_{-\infty}^{\infty} dx \frac{\sinh ax}{\sinh \pi x} = \tan \frac{a}{2}$$

18.

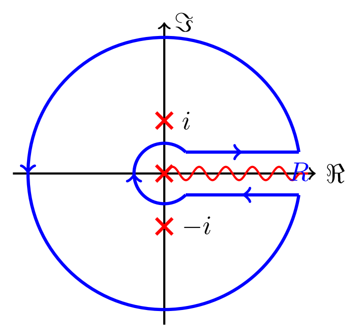


Figure 7: Contour for Problem 3–18.

Recalling the exercise in the main text regarding Eq. (3–37), we could infer that a direct contour integration of the integrand along some contour analogous to Figure 3–3 in the textbook would lead to a loss in the exponent of the logarithm (or one could just do it and observe).

Thus, let $f(z) := \frac{(\ln z)^3}{1+z^2}$, where we place the branch cut on the real axis and $\arg z = 0$ just above the real axis. As such, the poles of $f(z)$ have arguments $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Consider the contour shown in Figure 7.

$$\begin{aligned}\oint_C dz f(z) &= 2i\pi \left(\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-i} f(z) \right) \\ &= 2i\pi \left(\left. \frac{(\ln z)^3}{2z} \right|_{z=i} + \left. \frac{(\ln z)^3}{2z} \right|_{z=-i} \right) \\ &= \frac{13}{4} i\pi\end{aligned}$$

We also have

$$\begin{aligned}\oint_C dz f(z) = &\int_\epsilon^R dx \frac{(\ln x)^3}{1+x^2} + \int_0^{2\pi} d\theta i R e^{i\theta} \frac{(\ln R + i\theta)^3}{R^2 e^{2i\theta}+1} \\ &+ \int_R^\epsilon dx \frac{(\ln x + 2i\pi)^3}{1+x^2} + \int_{2\pi}^0 d\theta i \epsilon e^{i\theta} \frac{(\ln \epsilon + i\theta)^3}{\epsilon^2 e^{2i\theta}+1} \\ &\xrightarrow[\epsilon\to 0]{R\to\infty} -6i\pi \int_0^\infty dx \frac{(\ln x)^2}{1+x^2} + 12\pi^2 \int_0^\infty dx \frac{\ln x}{1+x^2} + 4i\pi^4.\end{aligned}$$

This same procedure can be used to show that

$$\int_0^\infty dx \frac{\ln x}{1+x^2} = 0,$$

which is left as an exercise to the reader[3].

$$\therefore \int_0^\infty dx \frac{\ln x^2}{1+x^2} = \frac{\pi^3}{8}$$

19.

Let $f(z) := \frac{1}{1+z^2+z^4}$. Notice that $z^6 - 1 = \left(z^2-1\right)\left(z^4+z^2+1\right)$, so the poles of $f(z)$ lie at $e^{i\frac{\pi}{3}}$, $e^{i\frac{2\pi}{3}}$, $e^{i\frac{4\pi}{3}}$, and $e^{i\frac{5\pi}{3}}$.

Consider the contour shown in Figure 8.

$$\begin{aligned}\oint_C dz f(z) &= 2i\pi \left(\operatorname*{Res}_{z=e^{i\frac{\pi}{3}}} f(z) + \operatorname*{Res}_{z=e^{i\frac{2\pi}{3}}} f(z) \right) \\ &= 2i\pi \left(\left. \frac{1}{4z^3+2z} \right|_{z=e^{i\frac{\pi}{3}}} + \left. \frac{1}{4z^3+2z} \right|_{z=e^{i\frac{2\pi}{3}}} \right) \\ &= \frac{\pi}{\sqrt{3}}\end{aligned}$$

[3] Just kidding! I will get back to this, but if you could, please consider discussing on `https://github.com/pingpingy1/Mathews-MathPhys-Sol/issues/2`.

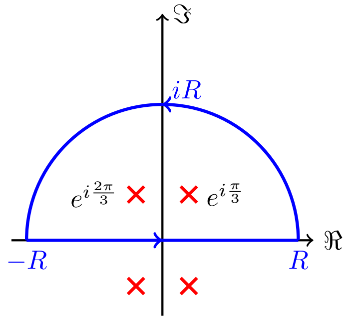


Figure 8: Contour for Problem 3–19.

We also have

$$\oint_C dz f(z) = \int_{-R}^{R} \frac{dx}{1+x^2+x^4} + \int_0^{\pi} d\theta \frac{iRe^{i\theta}}{1+R^2e^{2i\theta}+R^4e^{4i\theta}} \xrightarrow{R\to\infty} 2\int_0^{\infty} \frac{dx}{1+x^2+x^4}.$$

$$\therefore \int_0^{\infty} \frac{dx}{1+x^2+x^4} = \frac{\pi}{2\sqrt{3}}$$

20.

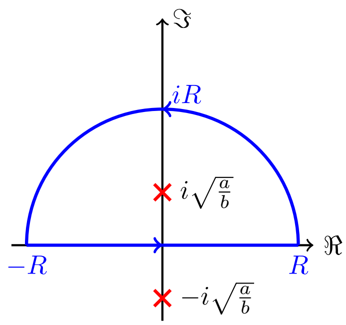


Figure 9: Contour for Problem 3–20.

We shall perform a contour integral of $f(z) := \frac{1}{(a+bz^2)^3}$ along Figure 9.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=i\sqrt{\frac{a}{b}}} f(z) = \frac{2i\pi}{2!} \cdot \frac{d^2}{dz^2} \frac{1}{b^{3/2}\left(\sqrt{b}z+i\sqrt{a}\right)^3}\Bigg|_{z=i\sqrt{\frac{a}{b}}} = \frac{3\pi}{8a^{5/2}b^{1/2}}$$

We also have

$$\oint_C dx f(z) = \int_{-R}^{R} \frac{dx}{(a+bx^2)^3} + \int_0^{\pi} d\theta \frac{iRe^{i\theta}}{(a+bR^2e^{2i\theta})^3} \xrightarrow{R\to\infty} 2\int_0^{\infty} \frac{dx}{(a+bx^3)^3}.$$

$$\therefore \int_0^\infty \frac{dx}{\left(a+bx^3\right)^3} = \frac{3\pi}{16a^{5/2}b^{1/2}}$$

21. We shall perform a contour integral of $f(z) := \frac{z^2}{(a^2+z^2)^3}$ along Figure 4.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=ia} f(z) = \frac{2i\pi}{2!} \cdot \frac{d^2}{dz^2} \frac{z^2}{\left(z+ia\right)^3}\bigg|_{z=ia} = \frac{\pi}{8a^3}$$

We also have

$$\begin{aligned}\oint_C dz f(z) &= \int_{-R}^{R} dx \frac{x^2}{\left(a^2+x^2\right)^3} + \int_0^\pi d\theta i R e^{i\theta} \frac{R^2 e^{2i\theta}}{\left(a^2+R^2e^{2i\theta}\right)^3} \\ &\xrightarrow{R\to\infty} 2\int_0^\infty dx \frac{x^2}{\left(a^2+x^2\right)^3}.\end{aligned}$$

$$\therefore \int_0^\infty dx \frac{x^2}{a^2+x^{2^3}} = \frac{\pi}{16a^3}$$

22. We shall perform a contour integral of $f(z) := \frac{\sin z}{z(a^2+z^2)}$ along Figure 4. Note that the singularity at the origin is removable, as we may define $f(0) := \lim_{z\to 0} f(z) = \frac{1}{a^2}$.

$$\oint_C dz f(z) = 2i\pi \operatorname*{Res}_{z=ia} f(z) = 2i\pi \cdot \frac{\sin z}{z\left(z+ia\right)}\bigg|_{z=ia} = \frac{\pi \sin a}{a^2}$$

We also have

$$\begin{aligned}\oint_C dz f(z) &= \int_{-R}^{R} dx \frac{\sin x}{x\left(a^2+x^2\right)} + \int_0^\pi d\theta i R e^{i\theta} \frac{\sin\left(Re^{i\theta}\right)}{Re^{i\theta}\left(a^2+R^2e^{2i\theta}\right)} \\ &\xrightarrow{R\to\infty} 2\int_0^\infty dx \frac{\sin x}{x\left(a^2+x^2\right)}.\end{aligned}$$

(Note: One may object that it is nontrivial that the second integral vanishes in the limit, as $\sin z$ grows without bound along the imaginary axis. A physicist like myself would sweep this fact under the rug, but if any mathematician would like to give a more rigorous proof of this fact, I'm all ears!)

$$\therefore \int_0^\infty dx \frac{x^2}{a^2+x^{2^3}} = \frac{\pi \sin a}{2a^2}$$

23. Let us perform the change of variable $z := e^{i\theta}$, such that the contour of integration is around the unit circle.

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{\left(a+b\cos\theta\right)^2} &= \oint_{|z|=1} \frac{dz/iz}{\left(a+\frac{b}{2}\left(z+\frac{1}{z}\right)\right)^2} \\ &= -4i \oint_{|z|=1} dz \frac{z}{\left(bz^2+2az+b\right)^2}\end{aligned}$$

The integrand has poles of second order at

$$z_{\pm}^{*} := \frac{-a \pm \sqrt{a^2 - b^2}}{b},$$

but only z_{+}^{*} lies inside the unit circle.

$$\begin{aligned}
\therefore \int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2} &= -4i \cdot 2i\pi \operatorname*{Res}_{z=z_{+}^{*}} f(z) \\
&= 8\pi \cdot \frac{d}{dz} \frac{z}{\left(bz + a + \sqrt{a^2 - b^2}\right)^2} \Bigg|_{z=z_{+}^{*}} \\
&= \frac{2\pi a}{(a^2 - b^2)^{3/2}}
\end{aligned}$$

24.

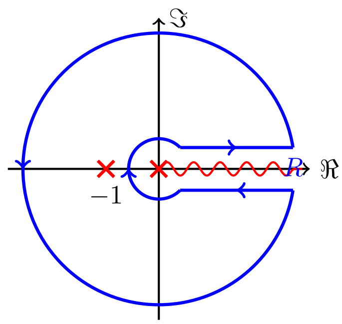


Figure 10: Contour for Problem 3–24.

Let $f(z) := \frac{(\ln z)^2}{(z+1)^2}$, where we place the branch cut on the real axis and $\arg z = 0$ just above the real axis. (Recall the exercise in the main text regarding Eq. (3–37); we need an extra $\ln z$ for this procedure!) As such, the pole of $f(z)$ has argument π. Consider the contour shown in Figure 10.

$$\begin{aligned}
\oint_C dz f(z) &= 2i\pi \operatorname*{Res}_{z=-1} f(z) \\
&= 2i\pi \cdot \frac{d^2}{dz^2} (\ln z)^2 \Bigg|_{z=e^{i\pi}} \\
&= 4\pi^2
\end{aligned}$$

We also have

$$\oint_C dz f(z) = \int_\epsilon^R dx \frac{(\ln x)^2}{(x+1)^2} + \int_0^{2\pi} d\theta i Re^{i\theta} \frac{(\ln R + i\theta)^2}{(Re^{i\theta}+1)^2}$$
$$+ \int_R^\epsilon dx \frac{(\ln x + 2i\pi)^2}{(x+1)^2} + \int_{2\pi}^0 d\theta i\epsilon e^{i\theta} \frac{(\ln \epsilon + i\theta)^2}{(\epsilon e^{i\theta}+1)^2}$$
$$\xrightarrow[\epsilon\to 0]{R\to\infty} \int_0^\infty dx \frac{(\ln x)^2}{(x+1)^2} + 0$$
$$+ \left(-\int_0^\infty dx \frac{(\ln x)^2}{(x+1)^2} - 4i\pi \int_0^\infty dx \frac{\ln x}{(x+1)^2} + 4\pi^2 \int_0^\infty \frac{dx}{(x+1)^2} \right)$$
$$+ 0$$
$$= -4i\pi \int_0^\infty dx \frac{\ln x}{(x+1)^2} + 4\pi^2.$$

$$\therefore \int_0^\infty dx \frac{\ln x}{(x+1)^2} = 0$$

25. Let us define

$$I(a) := \int_0^\infty dx e^{-x^2} \operatorname{Ci}(ax).$$

$$\begin{aligned} I'(a) &= \int_0^\infty dx \frac{\partial}{\partial a} \left(e^{-x^2} \operatorname{Ci}(ax) \right) \\ &= \frac{1}{a} \int_0^\infty dx e^{-x^2} \cos ax \\ &= \frac{1}{2a} \int_{-\infty}^\infty dx e^{-x^2} \cos ax \\ &= \frac{1}{2a} \Re \left\{ \int_{-\infty}^\infty dx e^{-x^2+iax} \right\} \\ &= \frac{e^{-a^2/4}}{2a} \Re \left\{ \int_{-\infty}^\infty dx e^{-\left(x-\frac{ia}{2}\right)^2} \right\} \\ &= \frac{\sqrt{\pi} e^{-a^2/4}}{2a} \end{aligned}$$

Also, $\lim_{x\to\infty} \operatorname{Ci}(x) = 0$, so $\lim_{a\to\infty} I(a) = 0$.

$$\begin{aligned} \therefore I(a) &= \frac{\sqrt{\pi}}{2} \int_\infty^a dt \frac{e^{-t^2/4}}{t} \\ &= -\frac{\sqrt{\pi}}{4} \int_{-\infty}^{-a^2/4} dv \frac{e^v}{v} \qquad\qquad (v := -\frac{t^2}{4}) \\ &= -\frac{\sqrt{\pi}}{4} \operatorname{Ei}\left(-\frac{a^2}{4} \right) \end{aligned}$$

(Note: What if we used Si instead of Ci inside the integrand? This is a harder problem since the integral

$$\int_0^\infty dx e^{-x^2} \sin ax$$

is rather tricky. The reader is encouraged to explore this, possibly with the help of the Dawson function[4].)

26.

$$\begin{aligned}
\int_0^\infty dx e^{-ax} \operatorname{erf} x &= \left[-\frac{1}{a} e^{-ax} \operatorname{erf} x\right]_0^\infty + \frac{1}{a} \int_0^\infty dx e^{-ax} \cdot \frac{2}{\sqrt{\pi}} e^{-x^2} \\
&= 0 + \frac{2}{\sqrt{\pi} a} \int_0^\infty dx e^{-x^2 - ax} \\
&= \frac{2}{\sqrt{\pi} a} \int_0^\infty dx e^{-\left(x + \frac{a}{2}\right)^2 + \frac{a^2}{4}} \\
&= \frac{2 e^{a^2/4}}{\sqrt{\pi} a} \int_{a/2}^\infty dx e^{-x^2} \\
&= \frac{1}{a} e^{a^2/4} \left(1 - \operatorname{erf} \frac{a}{2}\right)
\end{aligned}$$

27.

(a)

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\Gamma\left(\frac{1}{2}\right)^2} = \sqrt{\frac{\pi}{\sin \frac{\pi}{2}}} = \sqrt{\pi}$$

(b)

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \Gamma\left(\frac{3}{2}\right) = \frac{3}{4} \Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{\pi}}{4}$$

(c)

$$\mathrm{B}(1,3) = \frac{\Gamma(1)\Gamma(3)}{\Gamma(4)} = \frac{0!2!}{3!} = \frac{1}{3}$$

(d)

$$\mathrm{B}\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma(0)} = \frac{\sqrt{\pi} \cdot \frac{\sqrt{\pi}}{-1/2}}{\infty} = 0$$

(e)

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(-\frac{1}{3}\right) = -3\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = -3 \cdot \frac{\pi}{\sin \frac{\pi}{3}} = -2\sqrt{3}\pi$$

[4] `https://en.wikipedia.org/wiki/Dawson_function`

28.

$$-\operatorname{Ei}(-x) = \int_{-\infty}^{-x} dy \frac{e^y}{y} = \int_{\infty}^{x} dt \frac{e^{-t}}{t}$$

$$\begin{aligned}
\therefore \int_0^\infty dx e^{-ax}\left(-\operatorname{Ei}(-x)\right) &= \int_0^\infty dx \int_\infty^x dt \frac{e^{-ax-t}}{t} \\
&= -\int_0^\infty dx \int_{-(a+1)x}^\infty du \frac{e^{-u}}{u-ax} \qquad\qquad (u := ax+t) \\
&= -\int_0^\infty du \int_0^{u/(a+1)} dx \frac{e^{-u}}{u-ax} \\
&= -\int_0^\infty du \int_0^{1/a(a+1)} dv \frac{u}{a}\frac{e^{-u}}{u(1-v)} \quad (v := \frac{ax}{u}) \\
&= -\frac{1}{a}\int_0^\infty du e^{-u} \cdot \int_0^{1/a(a+1)} \frac{dv}{1-v} \\
&= \frac{1}{a}\ln\left(1 - \frac{1}{a(a+1)}\right)
\end{aligned}$$

29. We suspect that changing the range of integration to $[0,1]$ might be beneficial in expressing this integral as a Beta function. Thus, let $u := \frac{x+2}{4}$.

$$\begin{aligned}
\int_{-2}^{2} dx \big(4-x^2\big)^{1/6} &= \int_0^1 4du \cdot \left(4 - (4u-2)^2\right)^{1/6} \\
&= 2^{8/3}\int_0^1 du u^{1/6}(1-u)^{1/6} \\
&= 2^{8/3}\,\mathrm{B}\left(\frac{1}{6},\frac{1}{6}\right) = 2^{8/3}\frac{\Gamma\left(\frac{1}{6}\right)^2}{\Gamma\left(\frac{1}{3}\right)}
\end{aligned}$$

30.

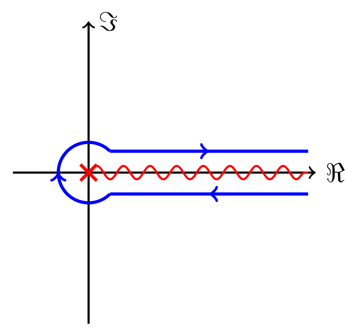


Figure 11: Contour for Problem 3–30.

The contour described in the problem is shown in Figure 11. The choice of branch is such that $\Im\{\ln(-t)\}$ equals π just below the branch cut and $-\pi$ just above it.

$$
\begin{aligned}
F(z) &= \int_C dt(-t)^{z-1}e^{-t} \\
&= \int_\infty^\epsilon dx e^{(z-1)(\ln x + i\pi) - x} \\
&+ \int_0^{2\pi} d\theta i\epsilon e^{i\theta} e^{(z-1)(\ln\epsilon + i(\pi-\theta)) - \epsilon e^{i\theta}} \\
&+ \int_\epsilon^\infty dx e^{(z-1)(\ln x - i\pi) - x} \\
&\xrightarrow{\epsilon\to 0+} -\int_0^\infty dx e^{(z-1)(\ln x + i\pi) - x} + \int_0^\infty dx e^{(z-1)(\ln x - i\pi) - x} \\
&= 2i \sin\pi z \Gamma(z)
\end{aligned}
$$

$$
\therefore (\text{something}) = 2i\sin\pi z
$$

31.

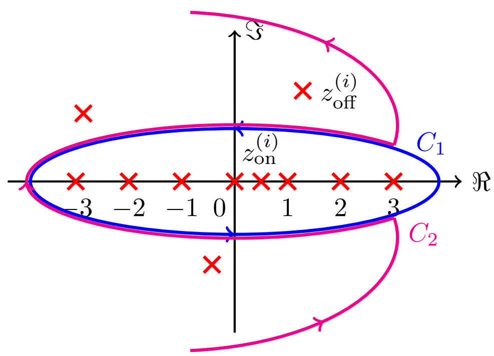


Figure 12: Contour for Problem 3–31.

(a) For simplicity, we shall assume that $f(z)$ has no poles exactly on any of the real integers. (In fact, I am not sure if this procedure could even be done with them!) Let us denote the poles of $f(z)$ off of the real axis as $z_{\text{off}}^{(i)}$ and those on the real axis as $z_{\text{on}}^{(i)}$. (Of course, we are also assuming that $f(z)$ only has isolated singularities on the entire plane. Note that this is a tiny bit more general than meromorphic functions[5], since we allow the singularities to be essential, not poles. We just need them to be discrete, i.e., no cluster points.)

[5] `https://en.wikipedia.org/wiki/Meromorphic_function`

We first consider the contour C_1 in Figure 12 to obtain

$$\begin{aligned}\oint_{C_1} dz f(z)\cot\pi z &= 2i\pi\sum_n \operatorname*{Res}_{z=n} f(z)\cot\pi z + 2i\pi\sum_i \operatorname*{Res}_{z=z_{\text{on}}^{(i)}} f(z)\cot\pi z \\ &= 2i\pi\sum_n \left.\frac{f(z)\cos\pi z}{\pi\cos\pi z}\right|_{z=n} + 2i\pi\sum_i \operatorname*{Res}_{z=z_{\text{on}}^{(i)}} f(z)\cot\pi z \\ &=: 2i\sum_n f(n) + 2i\pi\sum_i R_{\text{on}}^{(i)}\end{aligned}$$

where $R_{\text{on}}^{(i)} := \operatorname{Res}_{z=z_{\text{on}}^{(i)}} f(z)\cot\pi z$. Similarly for C_2,

$$\oint_{C_2} dz f(z)\cot\pi z = 2i\pi\sum_i \operatorname*{Res}_{z=z_{\text{off}}^{(i)}} f(z)\cot\pi z =: 2i\sum_n f(n) + 2i\pi\sum_i R_{\text{off}}^{(i)}$$

where $R_{\text{off}}^{(i)} := \operatorname{Res}_{z=z_{\text{off}}^{(i)}} f(z)\cot\pi z$.

Now, following the exercise regarding Eq. (3–43), we note that if $|f(z)|$ vanishes sufficiently quickly[6] as z approaches infinity, then

$$\oint_{C_1+C_2} dz f(z)\cot\pi z \xrightarrow{R\to\infty} 0.$$

$$\therefore \sum_n f(n) = -\pi\sum_i R_{\text{on}}^{(i)} - \pi\sum_i R_{\text{off}}^{(i)} = -\pi\sum_i R^{(i)}$$

where we drop the subscript as the distinction between the two types of residues have disappeared.

(Note: One may be tempted to write

$$R^{(i)} = \cot\pi z^{(i)}\cdot\operatorname*{Res}_{z=z^{(i)}} f(z)$$

which may seem so since $\cot\pi z$ is analytic on a neighborhood of $z^{(i)}$; goodness knows I fell for this. However, if one considers the entire Laurent series, they can show that this is not the case. In particular, if $z^{(i)}$ is a pole of nth order, then the residue can be caluclated using the first n Taylor coefficients of $\cot\pi z$ around $z^{(i)}$. This means that this naive equation holds for simple poles, specifically! See also: Convolutions[7])

(b) Let $f(z) := \frac{1}{z^2+a^2}$. This function has simple poles at $\pm ia$. We require that ia not be an integer, as per the above discussion.

$$\operatorname*{Res}_{z=\pm ia} f(z) = \left.\frac{\cot\pi z}{z\pm ia}\right|_{z=\pm ia} = \frac{\cot\pm i\pi a}{\pm 2ia} = -\frac{1}{2a}\coth\pi a$$

$$\therefore g(a) = -\pi\left(R_+ + R_-\right) = \frac{\pi}{a}\coth\pi a$$

[6]But what does this mean??!

[7]`https://en.wikipedia.org/wiki/Convolution`

32. Let θ denote the angle away from the dotted axis in the figure. We need not consider the azimuthal angle around it as the system is cylindrically symmetric. The intensity of neutrons as emitted by the source into a range $[\theta, \theta + d\theta]$ is given by

$$dI_{\text{source}} = I_0 \frac{d\Omega}{4\pi} = \frac{I_0}{2} \sin\theta d\theta$$

where I_0 is the total intensity of neutrons. Such neutrons travel a distance of $T\sec\theta$ through the absorber, so the intensity of neutrons that arrive at the film is given by

$$dI_{\text{film}} = \frac{I_0}{2} e^{-\frac{T}{\lambda}\sec\theta} \sin\theta d\theta.$$

These neutrons travel through the film, again, a distance proportional to $\sec\theta$. Thus, one can introduce a proportionality constant C to obtain

$$dA = Ce^{-\frac{T}{\lambda}\sec\theta} \sin\theta d\theta \cdot \sec\theta = Ce^{-\frac{T}{\lambda}\sec\theta} \tan\theta d\theta.$$

Therefore, integrating over the film, we obtain the following formula for the total activity:

$$\begin{aligned}
A(\lambda) &= \int_0^{\tan^{-1}\left(\frac{b}{a}\right)} d\theta C e^{-\frac{T}{\lambda}\sec\theta} \tan\theta \\
&= C \int_1^{\frac{\sqrt{a^2+b^2}}{a}} du \frac{\exp\left(-\frac{T}{\lambda}u\right)}{u} && (u := \sec\theta) \\
&= C \int_{\frac{T}{\lambda}}^{\frac{T}{\lambda}\frac{\sqrt{a^2+b^2}}{a}} dv \frac{\exp(-v)}{v} && (v := \frac{T}{\lambda}u) \\
&= C\left(\left(-\operatorname{Ei}\left(-\frac{T}{\lambda}\frac{\sqrt{a^2+b^2}}{a}\right)\right) - \left(-\operatorname{Ei}\left(-\frac{T}{\lambda}\right)\right)\right).
\end{aligned}$$

The activity without the absorber is given by

$$A_0 = \lim_{T\to 0} A(\lambda) = \int_0^{\tan^{-1}\left(\frac{b}{a}\right)} d\theta C \tan\theta = C \ln\left(\frac{\sqrt{a^2+b^2}}{a}\right).$$

Hence, we obtain

$$\frac{A}{A_0} = \frac{\left(-\operatorname{Ei}\left(-\frac{T}{\lambda}\frac{\sqrt{a^2+b^2}}{a}\right)\right) - \left(-\operatorname{Ei}\left(-\frac{T}{\lambda}\right)\right)}{\ln\left(\frac{\sqrt{a^2+b^2}}{a}\right)}$$

which, given the numerical constants, is an equation that the almighty WolframAlpha could solve! Plugging in the given constants and assuming λ is in units of centimeters, we get

$$0.25 = \frac{\left(-\operatorname{Ei}\left(-\frac{\sqrt{2}}{\lambda}\right)\right) - \left(-\operatorname{Ei}\left(-\frac{1}{\lambda}\right)\right)}{\ln\left(\sqrt{2}\right)}.$$

$$\therefore \lambda \approx 0.856\,\mathrm{cm}$$

33. As $e^{\alpha z}$ is analytic everywhere, $\Gamma(z)e^{\alpha z}$ has poles precisely at the non-positive integers.

$$I := \oint_{|z|=\frac{5}{2}} dz\Gamma(z)e^{\alpha z} = 2i\pi\left(\operatorname*{Res}_{z=0}\Gamma(z)e^{\alpha z} + \operatorname*{Res}_{z=-1}\Gamma(z)e^{\alpha z} + \operatorname*{Res}_{z=-2}\Gamma(z)e^{\alpha z}\right)$$

To calculate these residues, we wish to transform the equation

$$\operatorname*{Res}_{z=-n}\Gamma(z)e^{\alpha z} = \lim_{z\to -n}(z+n)\Gamma(z)e^{\alpha z}$$

in such a way to make the argument of the gamma function positive, where n is a nonnegative integer.

$$\begin{aligned}
\operatorname*{Res}_{z=-n}\Gamma(z)e^{\alpha z} &= \lim_{z\to -n}(z+n)\Gamma(z)e^{\alpha z} \\
&= \lim_{z\to -n}(z+n)\cdot\frac{z(z+1)\cdots(z+n)}{z(z+1)\cdots(z+n)}\Gamma(z)e^{\alpha z} \\
&= \lim_{z\to -n}\frac{\Gamma(z+n+1)e^{\alpha z}}{z(z+1)\cdots(z+n-1)} \\
&= \frac{\Gamma(0)e^{-n\alpha}}{(-n)(-n+1)\cdots(-1)} \\
&= \frac{(-e^{\alpha})^n}{n!}
\end{aligned}$$

$$\therefore I = 2i\pi\left(1 - e^{\alpha} + \frac{e^{2\alpha}}{2}\right)$$

34.

(i) $|x| << 1$: We use the familiar Taylor expansion of $\sin t$.

$$\begin{aligned}
\int_0^x dt\frac{\sin t}{t} &= \int_0^x dt\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}t^{2k} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)\cdot(2k+1)!}x^{2k+1} \\
&= x - \frac{x^3}{18} + \frac{x^5}{600} - \cdots
\end{aligned}$$

Note that this series absolutely converges for all values of x.

(ii) $|x| >> 1$: Here, following the exercise deriving Eq. (3–72), we perform repeated integration by parts. To deal with the trigonometric functions more easily, we express them as complex exponentials.

$$\begin{aligned}
\int_0^x dt \frac{\sin t}{t} &= \frac{\pi}{2} - \Im\left\{\int_x^\infty dt \frac{e^{it}}{t}\right\} \\
&= \frac{\pi}{2} + \Im\left\{\int_x^\infty dt \frac{e^{i(t-\pi)}}{t}\right\} \\
&= \frac{\pi}{2} + \Im\left\{\left[\frac{-ie^{i(t-\pi)}}{t}\right]_x^\infty - \int_x^\infty dt \frac{-ie^{i(t-\pi)}}{-t^2}\right\} \\
&= \frac{\pi}{2} + \Im\left\{\frac{e^{i\left(x-\frac{\pi}{2}\right)}}{x}\right\} + \Im\left\{\int_x^\infty dt \frac{e^{i\left(t-\frac{3\pi}{2}\right)}}{t^2}\right\} \\
&= \frac{\pi}{2} + \Im\left\{\frac{e^{i\left(x-\frac{\pi}{2}\right)}}{x}\right\} + \Im\left\{\left[\frac{-ie^{i\left(t-\frac{3\pi}{2}\right)}}{t}\right]_x^\infty - 2\int_x^\infty dt \frac{-ie^{i\left(t-\frac{3\pi}{2}\right)}}{-t^3}\right\} \\
&= \frac{\pi}{2} + \Im\left\{\frac{e^{i\left(x-\frac{\pi}{2}\right)}}{x}\right\} + \Im\left\{\frac{e^{i(x-\pi)}}{x^2}\right\} + 2\Im\left\{\int_x^\infty dt \frac{e^{i(t-2\pi)}}{t^3}\right\} \\
&= \cdots \\
&= \frac{\pi}{2} + \sum_{k=0}^{n} k!\Im\left\{\frac{e^{i\left(x-\frac{k+1}{2}\pi\right)}}{x^{k+1}}\right\} + (n+1)!\Im\left\{\int_x^\infty dt \frac{e^{i\left(t-\frac{n+3}{2}\pi\right)}}{t^{n+2}}\right\}
\end{aligned}$$

Note that the sum in this expression diverges for all finite values of x, since the summand grows like $k!$.

35. We make use of the saddle-point approximation for this integral. Let $f(t) := xt - e^t$ such that $I(x) = \int_0^\infty dt \exp(f(t))$. $f(t)$ has a single stationary point, which is a global maximum at $(\ln x, x\ln x - x)$. The second derivative of $f(t)$ is evaluated to $-x$.

$$\begin{aligned}
\therefore I &\approx \int_{-\infty}^{\infty} dt e^{-\frac{x}{2}(t-\ln x)^2 + x\ln x - x} \\
&= e^{x\ln x - x}\int_{-\infty}^{\infty} dt e^{-\frac{x}{2}t^2} \\
&= \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x
\end{aligned}$$

What might be more important than this calculation is defending that the saddle-point approximation holds in this case. To this end, one should argue that the integral around the "hump" of $f(x)$ contributes the most to the integral. For this, one notices that the height of the hump grows like $x\ln x$, while the curvature at that point grows like x. Thus, the width of the hump, which can be approximated by the full width at half maximum of the parabola it is approximated as, grows like $\sqrt{\ln x}$. While the exact value of the width tends to grow with x, it is positioned at $t = \ln x$, which moves away from the origin quadratically faster than the growth of the width. Thus, as x becomes larger, the integral is better approximated via the parabola at the hump, i.e., the saddle-point approximation.

36. The average number of reactions per time is given by the integral

$$I := \int_0^\infty dENEe^{-E/kT} \cdot Me^{-\alpha/\sqrt{E}} = NM \int_0^\infty dE \exp\left(-\frac{E}{kT} - \frac{\alpha}{\sqrt{E}} + \ln E\right).$$

To employ the saddle-point approximation, let

$$f(E) := -\frac{E}{kT} - \frac{\alpha}{\sqrt{E}} + \ln E.$$

$$\Rightarrow f'(E) = -\frac{1}{kT} + \frac{\alpha}{2E^{3/2}} + \frac{1}{E}$$

Suppose $f(E)$ has a stationary point at $E_0 = \lambda_0 kT$.

$$f'(E_0) = -\frac{1}{kT} + \frac{\alpha}{2(kT)^{3/2}}\lambda_0^{-3/2} + \frac{1}{\lambda_0 kT} = 0 \Rightarrow \lambda_0 \approx 1$$

$$\Rightarrow f(kT) = -1 - \frac{\alpha}{\sqrt{kT}} + \ln(kT) \approx \ln(kT) - 1$$

$$f''(kT) = -\frac{3}{4}\frac{\alpha}{(kT)^{5/2}} - \frac{1}{(kT)^2} \approx -\frac{1}{(kT)^2}$$

$$\begin{aligned}\therefore I &\approx NM \int_{-\infty}^{\infty} dv \exp\left(\ln(kT) - 1 - \frac{v^2}{2(kT)^2}\right) \\ &= NM\frac{kT}{e} \cdot \sqrt{2\pi(kT)^2} \\ &= \frac{\sqrt{2\pi}}{e} NM(kT)^2\end{aligned}$$

37. We must evaluate the integral

$$\psi(\mathbb{k}, \mathbb{l}) := \int \frac{d\Omega}{(1 + \mathbb{k} \cdot \hat{\mathbb{r}})(1 + \mathbb{l} \cdot \hat{\mathbb{r}})}$$

using Feynman's identity (Eq. (3–28)).

$$\begin{aligned}\Rightarrow \psi(\mathbb{k}, \mathbb{l}) &= \int d\Omega \int_0^1 du \frac{1}{((1 + \mathbb{k} \cdot \hat{\mathbb{r}})u + (1 + \mathbb{l} \cdot \hat{\mathbb{r}})(1 - u))^2} \\ &= \int_0^1 du \int \frac{d\Omega}{(1 + (u\mathbb{k} + (1 - u)\mathbb{l}) \cdot \hat{\mathbb{r}})^2}\end{aligned}$$

Let

$$F(\mathbb{v}) := \int \frac{d\Omega}{(1 + \mathbb{v} \cdot \hat{\mathbb{r}})^2}.$$

This scalar-valued integral is a function of a vector; thus, it may only depend on the vector's magnitude. Let $v := ||\mathbb{v}||$.

$$
\begin{aligned}
\Rightarrow F(\mathbb{v}) &= F\left(v\hat{\mathbb{z}}\right) \\
&= \int \frac{d\Omega}{\left(1+vz\right)^2} \\
&= \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin\theta \cdot \frac{1}{\left(1+v\cos\theta\right)^2} \\
&= 2\pi \left[\frac{1}{v(1+v\cos\theta)}\right]_0^\pi \\
&= \frac{4\pi}{1-v^2}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \psi\left(\mathbb{k},\mathbb{l}\right) &= \int_0^1 duF\left(u\mathbb{k}+(1-u)\mathbb{l}\right) \\
&= 4\pi \int_0^1 \frac{du}{1-\left||u\mathbb{k}+(1-u)\mathbb{l}|\right|^2} \\
&= 4\pi \int_0^1 \frac{du}{1-l^2-2\left(\mathbb{k}\cdot\mathbb{l}-l^2\right)u+\left(k^2+l^2-2\mathbb{k}\cdot\mathbb{l}\right)u^2}
\end{aligned}
$$

where $k := ||\mathbb{k}||$ and $l := ||\mathbb{l}||$. Now, notice that

$$\frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2}.$$

$$
\begin{aligned}
\Rightarrow \frac{1}{c-2bx-ax^2} &= \frac{1}{\sqrt{ac+b^2}} \cdot \frac{\frac{a}{\sqrt{ac+b^2}}}{1-\left(\frac{a}{\sqrt{ac+b^2}}\left(x+\frac{b}{a}\right)\right)^2} \\
&= \frac{d}{dx}\left(\frac{1}{\sqrt{ac+b^2}}\tanh^{-1}\left(\frac{ax+b}{\sqrt{ac+b^2}}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \psi\left(\mathbb{k},\mathbb{l}\right) = &\frac{4\pi}{\sqrt{\left(1-\mathbb{k}\cdot\mathbb{l}\right)^2-\left(1-k^2\right)\left(1-l^2\right)}} \\
&\times \left[\tanh^{-1}\left(\frac{\left(k^2+l^2-2\mathbb{k}\cdot\mathbb{l}\right)u+\mathbb{k}\cdot\mathbb{l}-l^2}{\sqrt{\left(1-\mathbb{k}\cdot\mathbb{l}\right)^2-\left(1-k^2\right)\left(1-l^2\right)}}\right)\right]_0^1 \\
= &\frac{4\pi}{\sqrt{A^2-B^2}}\left(\tanh^{-1}\left(\frac{A-1+k^2}{\sqrt{A^2-B^2}}\right)+\tanh^{-1}\left(\frac{A-1+l^2}{\sqrt{A^2-B^2}}\right)\right)
\end{aligned}
$$

where $A := 1-\mathbb{k}\cdot\mathbb{l}$ and $B := \sqrt{\left(1-k^2\right)\left(1-l^2\right)}$.

Now, note that

$$\begin{aligned}
\cosh\left(\tanh^{-1}x+\tanh^{-1}y\right) &= \cosh\left(\tanh^{-1}x\right)\cosh\left(\tanh^{-1}y\right) \\
&\quad+\sin\left(\tanh^{-1}x\right)\sinh\left(\tanh^{-1}y\right) \\
&= \frac{1}{(1-x^2)}\cdot\frac{1}{(1-x^2)}+\frac{x}{(1-x^2)}\cdot\frac{y}{(1-y^2)} \\
&= \frac{1+xy}{\sqrt{(1-x^2)(1-y^2)}}.
\end{aligned}$$

In our case, we have

$$x=\frac{A-1+k^2}{\sqrt{A^2-B^2}} \text{ and } y=\frac{A-1+l^2}{\sqrt{A^2-B^2}}$$

which yields

$$1+xy=\frac{A}{A^2-B^2}\left(2A+k^2+l^2-2\right)$$

and

$$\sqrt{(1-x^2)(1-y^2)}=\frac{B}{A^2-B^2}\left(2A+k^2+l^2-2\right).$$

$$\Rightarrow \tanh^{-1}x+\tanh^{-1}y=\cosh^{-1}\left(\frac{1+xy}{\sqrt{(1-x^2)(1-y^2)}}\right)=\cosh^{-1}\frac{A}{B}$$

$$\therefore \psi\left(\mathbb{k},\mathbb{l}\right)=\frac{4\pi}{\sqrt{A^2-B^2}}\cosh^{-1}\frac{A}{B}$$

38. If you have read the textbook thoroughly (unlike myself), then you may recall the "method of stationary phase" explained around Eq. (3–88).

$$f_n(x)=\int_C dt\exp\left(ix\left(\frac{n}{x}t-\sin t\right)\right)$$

Let $g(t) := \frac{n}{x}t-\sin t$. For large positive values of x, nearly all contribution to the integral comes near

$$g'(t_0)=\frac{n}{x}-\cos t_0=0\Rightarrow t_0=-\cos^{-1}\frac{n}{x}.$$

$$g(t_0)=-\frac{n}{x}\cos^{-1}\frac{n}{x}+\sqrt{1-\left(\frac{n}{x}\right)^2}$$

$$g''(t_0)=-\sqrt{1-\left(\frac{n}{x}\right)^2}$$

$$\begin{aligned}
\therefore f_n(x) &\approx \int_{-\infty}^{\infty} dt \exp\left(ix\left(g(t_0)+\frac{g''(t_0)}{2}(t-t_0)^2\right)\right)\\
&= e^{ixg(t_0)}\sqrt{\frac{2\pi}{ixg''(t_0)}}\\
&= e^{i\left(\sqrt{x^2-n^2}-n\cos^{-1}\frac{n}{x}-\frac{\pi}{4}\right)}\sqrt{\frac{2\pi}{x\left(1-\frac{1}{2}\left(\frac{n}{x}\right)^2\right)}}
\end{aligned}$$

For large values of x, we can simplify this expression to only consider leading powers of x.

$$\begin{aligned}
\therefore f_n(x) &\approx e^{i\left(x\left(1-\frac{1}{2}\left(\frac{n}{x}\right)^2\right)-n\left(\frac{\pi}{2}-\frac{n}{x}\right)-\frac{\pi}{4}\right)}\sqrt{\frac{2\pi}{x}}\left(1+\frac{1}{4}\left(\frac{n}{x}\right)^2\right)\\
&\approx \sqrt{\frac{2\pi}{x}}e^{i\left(x-\frac{2n+1}{4}\pi\right)}\left(1+\frac{in^2}{2x}\right)
\end{aligned}$$

4 Chapter 4

1. As $f(x)$ is an even function, we immediately see that no sine terms will be present in the series. Let

$$f(x) = \frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\cos\frac{2n\pi}{L}x.$$

$$A_0 = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx f(x) = \frac{4}{L}\int_0^{\frac{L}{2}} dx\left(1-\frac{2}{L}x\right) = 1$$

$$\begin{aligned}
A_n &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx f(x)\cos\frac{2n\pi}{L}x\\
&= \frac{4}{L}\int_0^{\frac{L}{2}} dx\left(1-\frac{2}{L}x\right)\cos\frac{2n\pi}{L}x\\
&= \frac{2\left(1-(-1)^n\right)}{\pi^2 n^2}\\
&= \begin{cases}\frac{4}{\pi^2 n^2} & (2\nmid n)\\ 0 & (2\mid n)\end{cases}\quad (n\neq 0)
\end{aligned}$$

$$\begin{aligned}
\therefore f(x) &= \frac{1}{2}+\frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{2(2n+1)\pi}{L}x\\
&= \frac{1}{2}+\frac{4}{\pi^2}\left(\cos\frac{2\pi}{L}x+\frac{1}{9}\cos\frac{6\pi}{L}x+\frac{1}{25}\cos\frac{10\pi}{L}x+\cdots\right)
\end{aligned}$$

2.

(a) The Fourier series is given by Eq. (4–4):

$$f(\theta) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\theta}{2n+1}.$$

Thus, let us denote the partial sums as

$$f_n(\theta) := \frac{4}{\pi}\sum_{k=0}^{n}\frac{\sin(2k+1)\theta}{2k+1}.$$

$$\begin{aligned}
\Rightarrow f_n'(\theta) &= \frac{4}{\pi}\sum_{k=0}^{n}\cos(2k+1)\theta \\
&= \frac{4}{\pi}\Re\left\{\sum_{k=0}^{n} e^{i(2k+1)\theta}\right\} \\
&= \frac{4}{\pi}\Re\left\{\frac{e^{i\theta}\left(1-e^{2in\theta}\right)}{1-e^{2i\theta}}\right\} \\
&= \frac{2}{\pi}\frac{\sin 2n\theta}{\sin\theta}
\end{aligned}$$

Hence, the first overshoot, i.e., local maximum occurs at $\frac{\pi}{2n}$.

$$\therefore \delta_n = f_n\left(\frac{\pi}{2n}\right) - 1 = \frac{4}{\pi}\sum_{k=0}^{n}\frac{1}{2k+1}\sin\left(\frac{2k+1}{2n}\right)\pi - 1$$

(b)

$$\begin{aligned}
\lim_{n\to\infty}\delta_n &= \frac{4}{\pi}\lim_{n\to\infty}\sum_{k=0}^{n}\frac{\sin\dfrac{2k+1}{2n}\pi}{\dfrac{2k+1}{2n}}\cdot\frac{1}{2n} - 1 \\
&= \frac{2}{\pi}\int_0^1 dx\frac{\sin\pi x}{x} - 1 \\
&= \frac{2}{\pi}\int_0^{\pi} du\frac{\sin u}{u} - 1 && (u := \pi x) \\
&= \frac{2}{\pi}\,\mathrm{Si}(\pi) - 1
\end{aligned}$$

3.

(a) We basically express the piecewise function

$$f(x) = \begin{cases} -e^{x} & (-1 < x < 0) \\ e^{-x} & (0 < x < 1) \end{cases}$$

as a Fourier series. Since this function is defined to be odd, we will only obtain the sine terms:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin n\pi x.$$

$$\Rightarrow B_n = 2\int_0^1 dx e^{-x} \sin n\pi x = \frac{2n\pi}{n^2\pi^2+1}\left(1 - \frac{(-1)^n}{e}\right)$$

$$\begin{aligned}
\therefore f(x) &= \sum_{n=1}^{\infty} \frac{2\pi n}{n^2\pi^2+1}\left(1 - \frac{(-1)^n}{e}\right)\sin n\pi x \\
&= \frac{\pi}{\pi^2+1}\left(1+\frac{1}{e}\right)\sin\pi x + \frac{2\pi}{4\pi^2+1}\left(1-\frac{1}{e}\right)\sin 2\pi x \\
&\quad + \frac{3\pi}{9\pi^2+1}\left(1-\frac{1}{e}\right)\sin 3\pi x + \cdots
\end{aligned}$$

(b) Let

$$f(x) = e^{-x} = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos 2n\pi x + B_n \sin 2n\pi x) \quad (0 < x < 1).$$

$$\begin{aligned}
\Rightarrow A_n &= 2\int_0^1 dx e^{-x}\cos 2n\pi x = \frac{2}{4n^2\pi^2+1}\left(1-\frac{1}{e}\right) \\
B_n &= 2\int_0^1 dx e^{-x}\sin 2n\pi x = \frac{4n\pi}{4n^2\pi^2+1}\left(1-\frac{1}{e}\right)
\end{aligned}$$

$$\begin{aligned}
\therefore f(x) &= \left(1-\frac{1}{e}\right)\left(1+\sum_{n=1}^{\infty}\frac{2\cos 2n\pi x + 4n\pi \sin 2n\pi}{4n^2\pi^2+1}\right) \\
&= \left(1-\frac{1}{e}\right)\left(1+\frac{2\cos 2\pi x + 4\pi\sin 2\pi x}{4\pi^2+1}\right. \\
&\quad \left. + \frac{2\cos 4\pi x + 8\pi \sin 4\pi x}{16\pi^2+1} + \frac{2\cos 6\pi x + 12\pi\sin 6\pi x}{36\pi^2+1} + \cdots\right)
\end{aligned}$$

4. For sinusoidal input functions, we can rewrite the linear system's outputs as

$$\begin{aligned}
\frac{1}{T}\int_0^T dt' G(t-t')\sin\frac{2n\pi}{T}t' &= \left(\frac{\omega_0 T}{2n\pi}\right)^2 \sin\frac{2n\pi}{T}t, \\
\frac{1}{T}\int_0^T dt' G(t-t')\cos\frac{2n\pi}{T}t' &= \left(\frac{\omega_0 T}{2n\pi}\right)^2 \cos\frac{2n\pi}{T}t,
\end{aligned}$$

and

$$\frac{1}{T}\int_0^T dt' G(t-t') = 0.$$

For "sufficiently well-behaved" inputs[8] f, these conditions are sufficient to ensure the correct behavior of the linear system, as these inputs may be expanded into Fourier series, which the system acts on termwise.

Now, let us consider the Fourier expansion of $G(t)$ itself as

$$G(-(t-t_0)) := \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos \frac{2n\pi}{T} t + B_n \sin \frac{2n\pi}{T} t \right)$$

where the coefficients A_n and B_n will, in general, depend on t_0.

$$A_0 = \frac{2}{T} \int_0^T dt' G(t_0 - t') = 0$$

$$\Rightarrow A_n = \frac{2}{T} \int_0^T dt' G(t_0 - t') \cos \frac{2n\pi}{T} t' = 2 \left(\frac{\omega_0 T}{2n\pi} \right)^2 \cos \frac{2n\pi}{T} t_0$$

$$B_n = \frac{2}{T} \int_0^T dt' G(t_0 - t') \sin \frac{2n\pi}{T} t' = 2 \left(\frac{\omega_0 T}{2n\pi} \right)^2 \sin \frac{2n\pi}{T} t_0$$

$$\begin{aligned} \Rightarrow G(t_0 - t) &= \frac{\omega_0^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\cos \frac{2n\pi}{T} t_0 \cos \frac{2n\pi}{T} t_0 + \cos \frac{2n\pi}{T} t_0 \cos \frac{2n\pi}{T} t_0 \right) \\ &= \frac{\omega_0^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{2n\pi}{T} (t_0 - t) \end{aligned}$$

$$\therefore G(t) = \frac{\omega_0^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{2n\pi}{T} t$$

(Note: This series might look like it could be further simplified; however, to do so requires defining something like the polylogarithm function[9], which is a step I will not take.)

5. Let

$$f(\theta) := \sum_{n=0}^{\infty} \frac{\cos(2n+1)\theta}{(2n+1)^2}.$$

$$\Rightarrow f''(\theta) = -\sum_{n=0}^{\infty} \cos(2n+1)\theta = -\Re \left\{ \sum_{n=0}^{\infty} e^{i(2n+1)\theta} \right\} = 0$$

$$\Rightarrow f'(\theta) = f'\left(\frac{\pi}{2}\right) = -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = -\frac{\pi}{4}$$

$$\Rightarrow f(\theta) = f(0) - \frac{\pi}{4}\theta = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} - \frac{\pi}{4}\theta = \frac{\pi^2}{8} \left(1 - \frac{2}{\pi}\theta \right)$$

[8] As in, inputs in the domain densly spanned by the sinusoidal functions.
[9] https://en.wikipedia.org/wiki/Polylogarithm

Now note that f must be periodic and even, which this linear function definitely is not. Yet, nowhere in our derivation did we specify the range of θ! What went wrong?

The answer is that our $f''(\theta)$ is not strictly true; it diverges at integral multiples of π. Hence, we cannot invoke the fundamental theorem of calculus to obtain $f'(\theta)$ in one go. However, the above derivation works on the interval $(0, \pi)$, which we periodically and evenly extend to the entire real line.

$$\therefore f(\theta) = \begin{cases} \frac{\pi^2}{8}\left(1 - \frac{2}{\pi}(\theta - 2k\pi)\right) & (2k\pi < \theta < (2k+1)\pi) \\ \frac{\pi^2}{8}\left(1 + \frac{2}{\pi}(\theta - 2k\pi)\right) & ((2k-1)\pi < \theta < 2k\pi) \end{cases}$$

(Note: What about the part where we evaluate $f(0)$? Is that also erroneous? Technically yes, but the diligent mathematicians have proven Abel's theorem[10], with which we may correctly write

$$\lim_{\theta \to 0+} f(\theta) = \frac{\pi^2}{8}.$$

Of course, we are physicists, so *obviously* it works!)

6. Equalities between delta "function"-like things should be dealt with care. A rigorous-enough definition of equality for physicists is given by

$$A(x) = B(x) \Leftrightarrow (\forall f(x)) \int dx f(x) A(x) = \int dx f(x) B(x).$$

Hence, this problem could be viewed as proving the equality

$$\delta\left[g(x)\right] = \frac{\delta(x - x_0)}{|g'(x_0)|}.$$

$$\begin{aligned} \int_a^b dx f(x) \delta\left[g(x)\right] &= \int_{g(a)}^{g(b)} \frac{du}{g'(x(u))} f(x(u)) \delta(u) && (u := g(x)) \\ &= \operatorname{sign}\left(g(b) - g(a)\right) \left.\frac{f(x(u))}{g'(x(u))}\right|_{u=0} \\ &= \operatorname{sign}\left(g'(x_0)\right) \frac{f(x_0)}{g'(x_0)} \\ &= \frac{f(x_0)}{|g'(x_0)|} \end{aligned}$$

7. It is 0. It is just 0. No doubt.

You need an explanation? The integrand is identically zero on the integration range $[0, \pi] \times [1, 2]$. Just look! Never are both of the arguments to the delta functions zero.

[10] `https://en.wikipedia.org/wiki/Abel%27s_theorem`

Fine, I'll give a formal proof. It's really no need, you see.

$$\begin{aligned}
I &:= \int_0^\pi dx \int_1^2 dy \delta\left(\sin x\right) \delta\left(x^2 - y^2\right) \\
&= \int_0^\pi dx \delta\left(\sin x\right) \left(\mathbb{1}\left\{x \in [1,2]\right\} \cdot \left.\frac{1}{|-2y|}\right|_{y=x} + \mathbb{1}\left\{x \in [1,2]\right\} \cdot \left.\frac{1}{|-2y|}\right|_{y=-x} \right) \\
&= 0
\end{aligned}$$

Here, $\mathbb{1}\left\{p\right\}$ equals 1 if p is true and 0 if false.

8.

$$\begin{aligned}
\tilde{\psi}(\mathbb{k}) &= \int \frac{d^3x}{(2\pi)^{3/2}} \psi(\mathbb{x}) e^{-i\mathbb{k}\cdot\mathbb{x}} \\
&= \frac{1}{16\pi^2 a_0^{5/2}} \int d^3x z e^{-r/2a_0 - i\mathbb{k}\cdot\mathbb{x}} \\
&= \frac{i}{16\pi^2 a_0^{5/2}} \frac{\partial}{\partial k_z} \int d^3x e^{-r/2a_0 - i\mathbb{k}\cdot\mathbb{x}} \\
&= \frac{i}{16\pi^2 a_0^{5/2}} \frac{\partial}{\partial k_z} \int_0^\infty dr r^2 \int_{-1}^1 d(\cos\theta) \int_0^{2\pi} d\phi e^{-r/2a_0 - ikr\cos\theta} \\
&= \frac{i}{8\pi a_0^{5/2}} \frac{\partial}{\partial k_z} \int_0^\infty dr r^2 e^{-r/2a_0} \left[-\frac{i}{kr} e^{-ikr\cos\theta} \right]_{\theta=0}^{\theta=\pi} \\
&= \frac{1}{8\pi a_0^{5/2}} \frac{\partial}{\partial k_z} \left(\frac{1}{k} \int_0^\infty dr r e^{-r/2a_0} \left(e^{ikr} - e^{-ikr}\right) \right) \\
&= \frac{i}{8\pi a_0^{5/2}} \frac{\partial k}{\partial k_z} \frac{\partial}{\partial k} \left(k^2 + \frac{1}{4a_0^2} \right)^{-2} \\
&= \frac{1}{2i\pi a_0^{3/2}} \frac{k_z}{\left(k^2 + 1/4a_0^2\right)^3}
\end{aligned}$$

(Note: The original integral is not spherically symmetric, since the integrand is not a coordinate-independent scalar (it depends on z). Consequently, the resulting expression depends on more than just the magnitude of $\mathbb{k}$. However, on the fourth equality sign, we arbitrarily chose the z axis to be along $\mathbb{k}$. This is because upon applying Feynman's trick, the integrand becomes spherically symmetric again. This exercise is a good training for applying symmetry arguments only when applicable.)

9.

(a) The Fourier transform of the input function is given by

$$\begin{aligned}\tilde{f}(\omega) &= \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} f(t)e^{-i\omega t} \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} dte^{-(\lambda+i\omega)t} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\lambda + i\omega}\end{aligned}$$

while that of the response function is given by

$$\begin{aligned}\tilde{F}(\omega) &= \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} F(t)e^{-i\omega t} \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} dt \left(e^{-(\lambda+i\omega)t} - e^{-(\lambda+\alpha+i\omega)t}\right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\lambda + i\omega} - \frac{1}{\lambda+\alpha+i\omega}\right).\end{aligned}$$

Meanwhile, since the system is linear, we know that

$$\tilde{F}(\omega) = G(-\omega)\tilde{f}(\omega)$$

since each oscillatory mode $e^{i\omega t}$ transforms to $G(-\omega)e^{-i\omega t}$.

$$\therefore G(\omega) = 1 + \frac{\lambda - i\omega}{\lambda + \alpha - i\omega}$$

(b)

$$\begin{aligned}\tilde{f}(\omega) &= \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} A\delta(x)e^{-i\omega t} \\ &= \frac{A}{\sqrt{2\pi}}\end{aligned}$$

$$\begin{aligned}\Rightarrow \tilde{F}(\omega) &= G(-\omega)\tilde{f}(\omega) \\ &= \frac{A}{\sqrt{2\pi}} \left(1 + \frac{\lambda - i\omega}{\lambda + \alpha - i\omega}\right) \\ &= \frac{A}{\sqrt{2\pi}} \left(2 - \frac{\alpha}{\lambda + \alpha - i\omega}\right)\end{aligned}$$

$$\therefore F(t) = A\left(2\delta(t) - \alpha e^{-(\lambda+\alpha+i\omega)t}\right)$$

10.

$$\begin{aligned}
\mathcal{L}[f] &= \int_0^\infty dx f(x) e^{-sx} \\
&= \sum_{n=0}^\infty \int_{2n}^{2n+1} dx e^{-sx} \\
&= \frac{1}{s} \sum_{n=0}^\infty \left(e^{-2ns} - e^{-(2n+1)s} \right) \\
&= \frac{1}{s} \left(\frac{1}{1-e^{-2s}} - \frac{e^{-s}}{1-e^{-2s}} \right) \\
&= \frac{e^{s/2}}{s} \frac{\sinh s/2}{\sinh s}
\end{aligned}$$

11. The crucial insight to solve this problem (at least in my case) was the realization that the transformation $f(x) \mapsto g(y)$ is linear in its arguments. Hence, describing the transformation for only a set of basis functions is enough!

$$f(x) := e^{i\omega x} = \sum_{n=0}^\infty \frac{(i\omega)^n}{n!} x^n$$

$$\mapsto g(y) = \sum_{n=0}^\infty (i\omega y)^n = \frac{1}{1-i\omega y}$$

$$\begin{aligned}
\therefore g(y) &= \int \frac{d\omega}{\sqrt{2\pi}} \tilde{f}(\omega) \cdot \frac{1}{1-i\omega y} \\
&= \int \frac{d\omega}{\sqrt{2\pi}} \frac{1}{1-i\omega y} \cdot \int \frac{dx}{\sqrt{2\pi}} f(x) e^{-i\omega x} \\
&= \frac{1}{2\pi} \int d\omega \int dx f(x) \frac{e^{-i\omega x}}{1-i\omega y}
\end{aligned}$$

Here, the ranges of the integrals are assumed to be over the entire real axis.

12.

$$\begin{aligned}
J_0(x) &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{e^{i\cos\theta x} + e^{-i\cos\theta x}}{2} \\
&= \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{i\cos\theta x} \quad \left(\because \cos\theta \text{ symmetric about } \left(0, \frac{\pi}{2}\right) \right) \\
&= \frac{1}{2\pi} \int_{-1}^{1} \frac{dk}{\sqrt{1-k^2}} e^{ikx} \quad (k := \cos\theta) \\
&= \frac{1}{2i\pi} \int_{-i}^{i} \frac{ds}{\sqrt{1+s^2}} e^{sx} \quad (s := ik) \\
&= \frac{1}{2i\pi} \int_{-i\infty}^{i\infty} ds \frac{\mathbb{1}\{-1 < \Im s < 1\}}{\sqrt{1+s^2}} e^{sx}
\end{aligned}$$

$$\therefore \mathcal{L}\left[J_0\right](s) = \frac{\mathbb{1}\left\{-1 < \Im s < 1\right\}}{\sqrt{1+s^2}}$$

Here, $\mathbb{1}\left\{p\right\}$ evaluates to 1 if p is true and 0 if p is false. This function is used to constrain s to be inside a horizontal strip, where the Laplace transform stays analytic.

13.

(Solution 1) We perform a partial fraction decomposition to obtain formulae whose inverse Laplace transforms are trivial (or at least well documented).

$$\frac{1}{\left(s^2+1\right)(s-1)} = \frac{-s/2-1/2}{s^2+1} + \frac{1/2}{s-1} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\right)$$

$$\therefore \mathcal{L}^{-1}\left[\frac{1}{\left(s^2+1\right)(s-1)}\right](x) = \frac{1}{2}\left(e^x - \cos x - \sin x\right)$$

(Solution 2) We apply properties of Laplace transforms to gradually reduce the problem.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{\left(s^2+1\right)(s-1)}\right](x) &= e^x \mathcal{L}^{-1}\left[\frac{1}{\left(\left(s+1\right)^2+1\right)s}\right](x) \\
&= e^x \int_0^x dt \mathcal{L}^{-1}\left[\frac{1}{\left(s+1\right)^2+1}\right](t) \\
&= e^x \int_0^x dt e^{-t} \mathcal{L}^{-1}\left[\frac{1}{s^2+1}(t)\right] \\
&= e^x \int_0^x dt e^{-t} \sin t \\
&= \frac{1}{2}\left(e^x - \cos x - \sin x\right)
\end{aligned}$$

14. In matrix form, the coupled differential equations can be written as

$$\frac{d}{dt}\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} -\lambda_1 & & \\ \lambda_1 & -\lambda_2 & \\ & \lambda_2 & -\lambda_3 \end{pmatrix}\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}.$$

Let

$$F_i := \mathcal{L}\left[N_i\right](s).$$

$$\Rightarrow \mathcal{L}\left[\dot{N}_i\right](s) = sF_i - N_i(0)$$

$$\Rightarrow \begin{pmatrix} sF_1 - N \\ sF_2 \\ sF_3 - n \end{pmatrix} = \begin{pmatrix} -\lambda_1 & & \\ \lambda_1 & -\lambda_2 & \\ & \lambda_2 & -\lambda_3 \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} s+\lambda_1 & & \\ -\lambda_1 & s+\lambda_2 & \\ & -\lambda_2 & s+\lambda_3 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} N \\ \\ m \end{pmatrix}$$

$$\begin{aligned} \Rightarrow F_3 = \frac{n}{s+\lambda_3} - &\frac{N\lambda_1\lambda_2}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)} \\ &\cdot \left(\frac{\lambda_2-\lambda_3}{s-\lambda_1} + \frac{\lambda_3-\lambda_1}{s-\lambda_2} + \frac{\lambda_1-\lambda_2}{s-\lambda_3} \right) \end{aligned}$$

$$\begin{aligned} \therefore N_3(t) = ne^{-\lambda_3 t} - &\frac{N\lambda_1\lambda_2}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)} \\ &\cdot \left((\lambda_2-\lambda_3)e^{-\lambda_1} + (\lambda_3-\lambda_1)e^{-\lambda_2} + (\lambda_1-\lambda_2)e^{-\lambda_3} \right) \end{aligned}$$

(Note 1: What if two of the λ_i's are the same? In this case, the inverse Laplace transform becomes more complicated, and the solution would take the form of either te^{-t} (multiplicity 2) or t^2e^{-t} (multiplicity 3).)

(Note 2: Solving the linear system of equations, in this case is straightforward since the matrix is already lower triangular. However, since only F_3 is of interest in our case, the Cramer's rule[11] may be more efficient, especially because the determinant calculations are also relatively simple.)

5 Chapter 5

Note of Warning To be perfectly candid, I kind of hate solutions that use a wholly unmotivated conformal mapping, since they feel more like things to memorize than a skill I can train and apply to other problems. I will try to provide as much motivation as possible, but please keep this mindset of mine in mind.

1. The problem describes a linear array of charged stripts, as shown in Figure 13a. We wish to find a conformal map that transforms this into a problem with a known solution. It is helpful to approach this step by step.

The first problem we see is that there are infinitely many charged sheets; we wish to collapse them into a single strip. To do so, we should employ a map that is periodic with period $2(a+b)$. Hence, we employ the map

$$\zeta := \exp\left(i\frac{\pi z}{a+b} \right) \Leftrightarrow z = -i(a+b)\ln\zeta.$$

This yields Figure 13b, where the charged strips collapse into a single one shaped as a circular arc (or a cylindrical arc in 3D).

We now want to map this circular arc into a well-known shape. One way to do so is given by the Möbius transformation[12], which is often introduced

[11] https://en.wikipedia.org/wiki/Cramer%27s_rule

[12] https://en.wikipedia.org/wiki/M%C3%B6bius_transformation

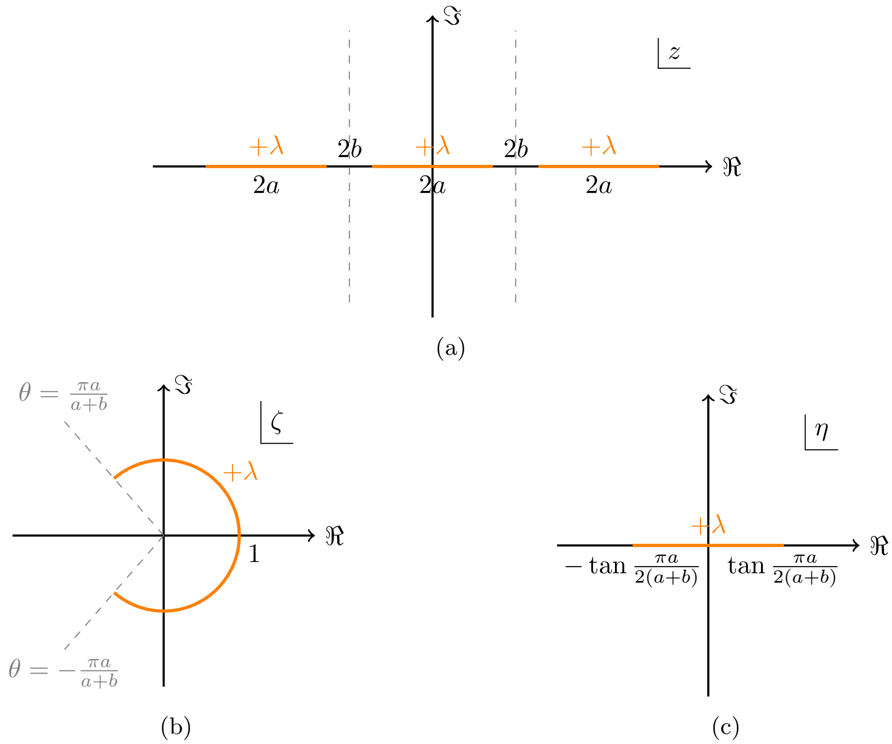


Figure 13: Results of conformal mappings used in Problem 5–1.

as an important class of conformal mappings in other textbooks. With that motivation-via-divine-intervention aside, let us consider the map

$$\eta = \frac{a\zeta + b}{c\zeta + d}.$$

We will attempt to transform the unit circle in the ζ plane into the real axis in the η plane. Also, it would be nice to have 1 map to the origin for more symmetry and -1 to infinity because it is not part of the charged strip. These conditions constrain the transformation to

$$\eta = a\frac{1-\zeta}{1+\zeta}.$$

Finally, requiring that i maps to a point on the real axis, we get

$$\eta := i\frac{1-\zeta}{1+\zeta} \Leftrightarrow \zeta = \frac{i-\eta}{i+\eta}.$$

This transformation actually maps the point $e^{i\alpha}$ to $\tan\frac{\alpha}{2}$, so it places the charged strip on the real interval from $-\tan\frac{\pi a}{2(a+b)}$ to $\tan\frac{\pi a}{2(a+b)}$. This is shown in Figure 13c.

This final situation in the η was dealt with in the main text and is solved by

$$f(\eta) = -\frac{\lambda}{2\pi\epsilon_0}\sin^{-1}\left(\frac{\eta}{\tan(\pi a/2(a+b))}\right).$$

Composing the two conformal maps yields

$$\eta(z) = \tan\left(\frac{\pi z}{2(a+b)}\right).$$

$$\begin{aligned}\therefore F(z) &= f(\eta(z)) \\ &= -\frac{\lambda}{2\pi\epsilon_0}\sin^{-1}\left(\frac{\tan\left(\frac{\pi z}{2(a+b)}\right)}{\tan\left(\frac{\pi a}{2(a+b)}\right)}\right)\end{aligned}$$

The potential is given by the imaginary part of this function.

(Note: It may be possible to directly motivate the conformal map $\eta(z)$ as follows: The map should be periodic as discussed earlier, but we also require that real values are mapped to real values. This would ensure that the charged strips stay straight, which may be easier to calculate. The tangent function achieves this as shown in this solution, but one may be tempted to use one of the other trigonometric functions. The reader is encouraged to try out other possible mappings.)

2. First, consider a single line charge λ at the origin. We want to express the potential $V(x, y)$ as the imaginary component of some complex function

$$F(z) = U(x, y) + iV(x, y).$$

Rotational symmetry implies that V is solely a function of $r := |z|$; consequently, the streamlines formed by $U =$ const. must be radial. Moreover, U must be multivalued since the integral

$$\oint_{r=\text{const.}} dz(\nabla V)_n = \Delta U$$

is nonzero (unless V itself is zero). In other words, following a circular contour around the origin increments U by a fixed amount. "Therefore," the only analytic function that satisfies these conditions is

$$F(z) := iC\ln z = C\left(-\theta + i\ln r\right) \quad (z = re^{i\theta}).$$

(Note: I really hoped that I could come up with an air-tight argument for the logarithm to appear here; if you have any ideas other than direct calculation of the potential, please let me know!)

Integration around the origin then should represent the total linear charge density enclosed:

$$\frac{\lambda}{\epsilon_0} = \oint_{r=\text{const.}} dz(\nabla V)_n = \Delta U = -2\pi C.$$

$$\Rightarrow C = -\frac{\lambda}{2\pi\epsilon_0} \Rightarrow F(z) = -\frac{i\lambda}{2\pi\epsilon_0}\ln z$$

Confusingly enough, this problem in particular requires that the *real* part be the potential. Thus, we divide by i and use

$$F(z) = -\frac{\lambda}{2\pi\epsilon_0}\ln z.$$

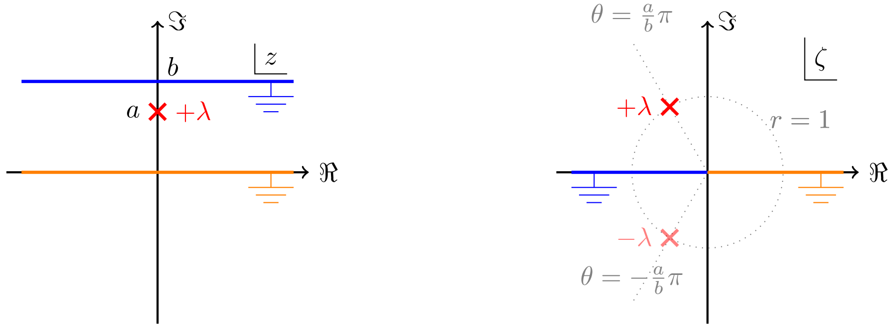
For the 2D-equivalent of the electric dipole as described in the problem, we simply superpose the functions for each of the line charge.

$$\begin{aligned}
\therefore W(z) &= \lim_{d\to 0^+}\left(-\frac{p/2d}{2\pi\epsilon_0}\ln(z-id) + \left(-\frac{-p/2d}{2\pi\epsilon_0}\ln(z+id)\right)\right)\\
&= \frac{ip}{2\pi\epsilon_0}\lim_{d\to 0^+}\frac{\ln(z+id)-\ln(z-id)}{2id}\\
&= \frac{ip}{2\pi\epsilon_0 z}
\end{aligned}$$

Notice that the electric potential is given by

$$\Re\{W(z)\} = \frac{p}{2\pi\epsilon_0}\frac{y}{x^2+y^2}.$$

3.



(a) (b)

Figure 14: Results of conformal mappings used in Problem 5–3. Note that the grounded symbols have not been mapped conformally.

The effects of the given transformation is shown in Figure 14. The region between the plates in the z plane is mapped to the upper half plane in the ζ plane. Moreover, the two charged plates are mapped onto the positive and negative halves of the real axis. Therefore, we should find a complex function $F(\zeta)$ such that its imaginary part vanishes on the real axis while showing a log-like behavior near $e^{i\frac{a}{b}\pi}$. Those with some electrostatic problem-solving experience may recall the image charge method[13]. Those without could tinker around, perhaps placing some negative charges trying to cancel out the potential on the real axis. This may lead you to placing a charged line of charge density $-\lambda$ at $\zeta = e^{-i\frac{a}{b}\pi}$.

$$\Rightarrow F(\zeta) = \frac{i\lambda}{2\pi\epsilon_0}\left(\ln\left(\zeta - e^{-i\frac{a}{b}\pi}\right) - \ln\left(\zeta - e^{i\frac{a}{b}\pi}\right)\right)$$

$$\begin{aligned}\Rightarrow f(z) &= F(\zeta(z)) \\ &= \frac{i\lambda}{2\pi\epsilon_0}\left(\ln\left(e^{i\frac{z}{b}} - e^{-i\frac{a}{b}\pi}\right) - \ln\left(e^{i\frac{z}{b}} - e^{i\frac{a}{b}\pi}\right)\right)\end{aligned}$$

$$\begin{aligned}\therefore V(x,y) &= \Im\left\{f(z)\right\} \\ &= \frac{\lambda}{2\pi\epsilon_0}\Re\left\{\left(\ln\left(e^{i\frac{x+iy}{b}} - e^{-i\frac{a}{b}\pi}\right) - \ln\left(e^{i\frac{x+iy}{b}} - e^{i\frac{a}{b}\pi}\right)\right)\right\}\end{aligned}$$

4.

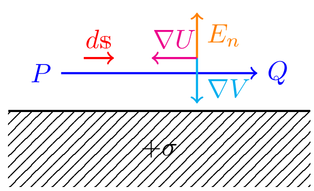


Figure 15: The surface of a charged conductor, used in Problem 5–4.

Suppose we traverse near the surface of a sheet of conductor from P to Q, such that the conductor stays on the right of the direction of traversal. Such a contour is shown in Figure 15. The electric field and the gradients of the potential and the streaming function are also shown, where ∇V must be counterclockwise to ∇U.

For computational ease, let us equate the 2D vector (a, b) with the complex number $a + ib$. This leads to some rather uncomfortable, yet wieldy expressions such as

$$\nabla V = -E_n = i\nabla U.$$

$$\begin{aligned}\Rightarrow \frac{\sigma}{\epsilon_0} &= E_n \\ &= -\nabla V \cdot \hat{\mathbb{n}} \\ &= -\left(i\nabla U\right) \cdot \left(i\hat{\mathbb{s}}\right) \\ &= \nabla U \cdot \hat{\mathbb{s}}\end{aligned}$$

[13] https://en.wikipedia.org/wiki/Method_of_image_charges

$$\begin{aligned}\therefore C(P,Q) &= \int_P^Q ds\sigma \\ &= \epsilon_0 \int_P^Q ds \nabla U \cdot \hat{\mathbb{s}} \\ &= \epsilon_0 \int_P^Q d\mathbb{s} \cdot \nabla U \\ &= \epsilon_0 \left(U(Q) - U(P)\right)\end{aligned}$$

5.

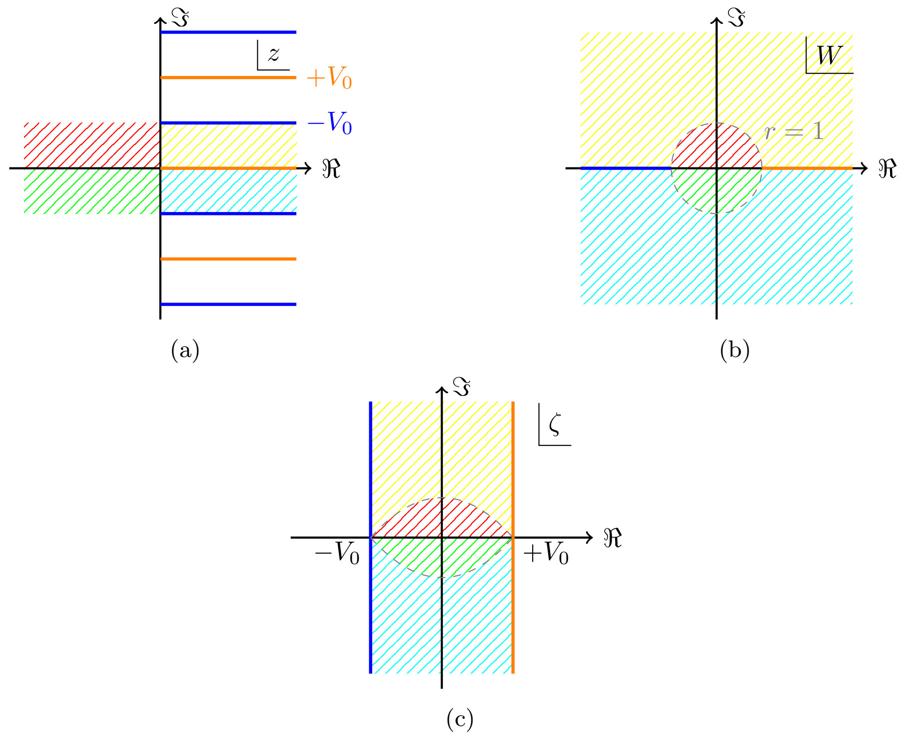


Figure 16: Results of conformal mappings used in Problem 5–5.

(a) Let's think step by step.[14] That is, let us consider the intermediary tranformation

$$W := e^{\frac{\pi z}{d}},$$

which is plausible since it is periodic with period id. Its effect is shown in Figure 16b, with matching regions correspondingly shaded.

Of course, a manual that claims to be useful should explain how this can be drawn by hand. The orange conductor(s) is (are) mapped to the real interval

[14] `https://arxiv.org/pdf/2205.11916`

$[1, \infty)$, while the blue conductors to the interval $(-\infty, -1]$, which is straightforward to see. (Note that for the blue conductor,

$$e^{\frac{\pi}{d}(x+(2n+1)id)} = -e^{\frac{\pi x}{d}}$$

would account for the negative sign.) The gray boundary is the image of the imaginary interval $[-id, id]$ (if such a notation makes sense), which is given by

$$\left\{e^{\frac{i\pi t}{d}} : t \in [-d, d]\right\}$$

which traces out the unit circle.

We finally apply the arcsine transformation, which we have encountered in both the example regarding Eq. (5–9) in the main text and Problem 5–1. The result of this transformation is given in Figure 16c, where the two conductors are now parallel plates, and we only consider the region between them. At this point, how the regions are transformed doesn't really matter, but they are pretty so they get to stay.[15]

The problem states that now the potential-finding problem is trivial. I don't like the word 'trivial,' but it seems safe to say that this problem is way easier now. We need to find an analytic function $F(\zeta)$ such that

$$\Im F(\zeta)|_{\Re\zeta=\pm V_0} = \pm V_0.$$

This can be achieved by the function $F(\zeta) := i\zeta$ (quite simple indeed!).

$$\Rightarrow F(\zeta(z)) = \frac{2iV_0}{\pi} \sin^{-1} e^{\frac{\pi z}{d}}$$

$$\begin{aligned} \therefore V(x, y) &= \Im F(\zeta(z)) \\ &= \frac{2V_0}{\pi} \Re\left\{\sin^{-1} e^{\frac{\pi z}{d}}\right\} \end{aligned}$$

(b) We use the stream function to calculate the charge accumulated on each plate. By symmetry, we only need to calculate the charge density on $x + i0^+$.

$$\begin{aligned} \Re F(x + i0^+) &= \frac{2V_0}{\pi} \Im\left\{\sin^{-1} e^{\frac{\pi x}{d} + i0^+}\right\} \\ &= \frac{2V_0}{\pi} \Im\left\{\frac{\pi}{2} + i\cosh^{-1}\left(e^{\frac{\pi x}{d}}\right)\right\} \\ &= \frac{2V_0}{\pi} \ln\left(e^{\frac{\pi x}{d}} + \sqrt{e^{\frac{2\pi x}{d}} - 1}\right) \end{aligned}$$

[15]Note that the boundary is only approximately drawn; the actual boundary is given by (ignoring scales) $y = \pm\cosh^{-1}\left(\sqrt{1 + \cos^2 x}\right)$, while I have drawn two parabolas with the same zeros and peaks. They do look quite similar according to Desmos, but I felt like I should mention it.

Thus, the argument of Problem 5–4 then tells us that

$$\lambda(0+i0^+, l+i0^+) = \frac{2\epsilon_0 V_0}{\pi} \ln\left(e^{\frac{\pi l}{d}} + \sqrt{e^{\frac{2\pi l}{d}} - 1}\right)$$

where λ denotes the charge per unit length perpendicular to the z plane. We should note that this is the total charge accumulated, which is equally divided between the top and bottom of each conductor. Thus, the capacitance per unit length is given by

$$\begin{aligned} C &= \frac{\lambda/2}{V_0} \\ &= \frac{\epsilon_0}{\pi} \ln\left(e^{\frac{\pi l}{d}} + \sqrt{e^{\frac{2\pi l}{d}} - 1}\right) \\ &\approx \frac{\epsilon_0}{\pi} \ln\left(2e^{\frac{\pi l}{d}}\right) \\ &= \frac{\epsilon_0}{d}\left(l + \frac{d}{\pi}\ln 2\right) \end{aligned}$$

where the approximation holds for $l >> d$ (i.e., semi-infinite plates). Identifying $\frac{\epsilon_0 l}{d}$ as the capacitance of infinite-plate capacitors, we conclude that the additional term $\frac{d}{\pi}\ln 2$ must come from edge effects.

6.

(a) I'm not quite sure how much rigor one can show this correspondence with. My argument is something that goes like this: Suppose the potential of this situation is the stream function of another. This means that on the real axis, the field lines should be perfectly horizontal, with discontinuities at precisely $x = \pm a$. This exact situation is described by two line charges at $x = \pm a$. Therefore, by the "boundary conditions pin down the potential" lemma[16], these two situations (OP and the two line charges) precisely correspond by exchanging the potential and the stream functions.

The above argument allows us to write

$$F(z) := C_1 \ln(z-a) + C_2 \ln(z+a)$$

where C_1 and C_2 are real constants to be determined and

$$\begin{aligned} \phi(x,y) &= \Im F(z) \\ &= C_1 \Im \ln(x-a+iy) + C_2 \Im \ln(x+a+iy) \\ &= C_1 \tan^{-1}\frac{y}{x-a} + C_2 \tan^{-1}\frac{y}{x+a}. \end{aligned}$$

Therefore, the boundary conditions imply $C_1 = -C_2 = \frac{V_0}{\pi}$, giving

$$\phi(x,y) = \frac{V_0}{\pi}\left(\tan^{-1}\frac{y}{x-a} - \tan^{-1}\frac{y}{x+a}\right).$$

[16] https://en.wikipedia.org/wiki/Laplace%27s_equation#Electrostatics_2

(b) Place the origin at the lower left corner of the metal plate and promptly disregard the 2 in the denominator[17], that is, let

$$\zeta := \cosh \frac{\pi z}{d} = \cosh \frac{\pi x}{d} \cos \frac{\pi y}{d} + i \sinh \frac{\pi x}{d} \sin \frac{\pi y}{d}.$$

The given conformal transformation will then map the plate to the upper half plane, with boundary conditions

$$T(x,0) = \begin{cases} 0 & (x < -1) \\ T_0 & (-1 < x < 1) \\ 0 & (x > 1) \end{cases}.$$

To convince ourselves of this fact, consider tracing out the boundary of the plate.

- The point $(x,0)$ is mapped to $(\cosh \frac{\pi x}{d}, 0)$, which traces out the real interval $(1,\infty)$.
- The point $(0,y)$ is mapped to $(\cos \frac{\pi y}{d}, 0)$, which traces out the real interval $(-1,1)$.
- The point (x,d) is mapped to $(-\cosh \frac{\pi x}{d}, 0)$, which traces out the real interval $(-\infty,-1)$.

$$\begin{aligned} \therefore T(x,y) &= \frac{T_0}{\pi}\left(\frac{\Im\zeta}{\Re\zeta - 1} - \tan^{-1}\frac{\Im\zeta}{\Re\zeta+1}\right) \\ &= \frac{T_0}{\pi}\left(\frac{\sinh\frac{\pi x}{d}\sin\frac{\pi y}{d}}{\cosh\frac{\pi x}{d}\cos\frac{\pi y}{d} - 1} - \frac{\sinh\frac{\pi x}{d}\sin\frac{\pi y}{d}}{\cosh\frac{\pi x}{d}\cos\frac{\pi y}{d} + 1}\right) \end{aligned}$$

7.

(a) The proof has basically been outlined in Figure 17b. This transformation, which can be searched for asthe Schwarz-Christoffel mapping[18][19], uses the key insight that the phase of z^s takes a jump discontinuity on the real axis at the origin. Quantitatively, it is πs for $x < 0$ and 0 for $x > 0$ (slightly above the real axis). Therefore, the slope of the mapping, which is basically the phase of $\frac{d\zeta}{dz}$, changes discontinuously at $z = a_1, a_2$. How discontinuous? Well, it decreases by πs_1 at $z = a_1$ and πs_2 at $z = a_2$, just as the problem claimed!

[17] If anyone knows why the 2 appears here, please let me know! I honestly have no clue.

[18] `https://en.wikipedia.org/wiki/Schwarz%E2%80%93Christoffel_mapping`

[19] The lack of 't' in 'Schwarz' here hurts my brain. Which is right — the textbook or Wikipedia? Probably Wikipedia; it claims that this is Schwarz in the Cauchy-Schwarz inequality.

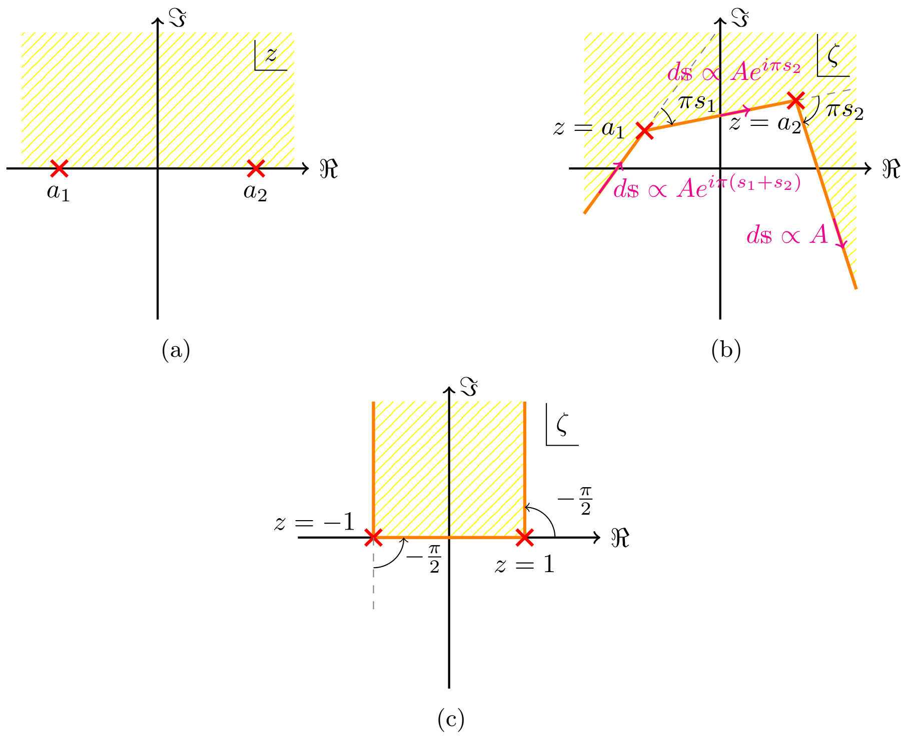


Figure 17: Results of conformal mappings used in Problem 5–7.

(b) For simplicity, suppose the vertices correspond to $z = \pm 1$. As shown in Figure 17c, this leads to phase shifts of $-\frac{\pi}{2}$.

$$\begin{aligned}
\Rightarrow \zeta(z) &= A \int \frac{dz}{(z+1)^{1/2}(z-1)^{1/2}} \\
&= A \int \frac{dz}{(z^2-1)^{1/2}} \\
&= A \cosh^{-1} z + B
\end{aligned}$$

The constants A, B are fixed by the conditions

$$\zeta(z=-1) = i\pi A + B = -a,\ \zeta(z=1) = B = a.$$

$$\Rightarrow A = \frac{2ia}{\pi},\ B = a$$

$$\begin{aligned}
\therefore \zeta(z) &= \frac{2ia}{\pi}\cosh^{-1} z + a \\
&= \frac{2a}{\pi}\cos^{-1} z + a \\
&= \frac{2a}{\pi}\left(\cos^{-1} z + \frac{\pi}{2}\right) \\
&= \frac{2a}{\pi}\sin^{-1} z
\end{aligned}$$

Thus, we have recovered the arcsine transformation as shown in Figure 5–3 from the main text.

8. Recall that the complex potential is given by

$$f(\zeta) = -\frac{\lambda}{2\pi\epsilon_0}\sin^{-1}\frac{\zeta}{a}.$$

Thus, our problem reduces to expressing approximate forms of the arcsine for large arguments. To this end, let $w := \sin^{-1} z$.

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$$

$$\begin{aligned}
&\Rightarrow e^{2iw} - 2ize^{iw} - 1 = 0 \\
&\Rightarrow e^{iw} = iz \pm \sqrt{1 - z^2}
\end{aligned}$$

The arcsine is conventionally defined using the plus sign here (since we want $w \in \left[0, \frac{\pi}{2}\right]$ when $z \in [0, 1]$), which yields

$$w = \sin^{-1} z = -i\ln\left(iz + \sqrt{1 - z^2}\right).$$

$$\begin{aligned}
\Rightarrow f(\zeta) &= \frac{i\lambda}{2\pi\epsilon_0}\ln\left(i\frac{\zeta}{a} + \sqrt{1 - \left(\frac{\zeta}{a}\right)^2}\right) \\
&\approx \frac{i\lambda}{2\pi\epsilon_0}\ln\left(2i\frac{\zeta}{a}\right)
\end{aligned}$$

$$\begin{aligned}
\therefore V(x, y) &= \Im\left\{f(\zeta)\right\} \\
&\approx \frac{\lambda}{2\pi\epsilon_0}\Re\left\{\ln(\zeta)\right\}
\end{aligned}$$

This is exactly the potential created by a line charge of λ at the origin.

9.

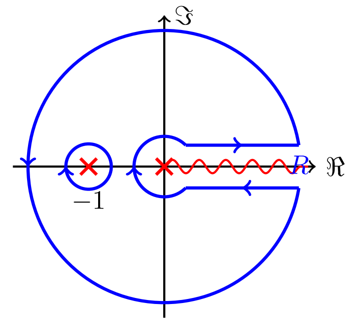


Figure 18: Contour for Problem 5–9.

We will integrate $\frac{f(\zeta)}{\zeta - z}$ along the contour shown in Figure 18, akin to the procedure described in the main text. The integral along the outer circle vanishes by condition (2), while we may use the Schwarz reflection principle[20][21] by condition (3).

$$
\begin{aligned}
\Rightarrow f(z) &= \frac{1}{2i\pi} \oint_C d\zeta \frac{f(\zeta)}{\zeta - z} \\
&= \frac{2}{z+2} + \frac{1}{2i\pi} \lim_{\epsilon \to 0^+} \left(\int_0^\infty dx' \frac{f(x'+i\epsilon)}{x'+i\epsilon - z} - \int_0^\infty dx' \frac{f(x'-i\epsilon)}{x'-i\epsilon - z} \right) \\
&= \frac{2}{z+2} + \frac{1}{\pi} \lim_{\epsilon \to 0^+} \left(\int_0^\infty dx' \frac{\Im\left\{f(x'+i\epsilon)\right\}}{x'-z} \right) \\
&= \frac{2}{z+2} + \frac{1}{\pi} \int_0^\infty dx' \frac{x'}{\left(x'^2+1\right)(x'-z)} \\
&= \frac{2}{z+2} + \frac{\pi - 2z\ln(-z)}{2\pi\left(z^2+1\right)}
\end{aligned}
$$

$$
\begin{aligned}
\therefore F(x) &= \Re\left\{f(x+i\epsilon)\right\} \\
&= \begin{cases} \frac{2}{x+2} + \frac{\pi - 2x\ln(-x)}{2\pi(x^2+1)} & (x<0) \\ \frac{2}{x+2} + \frac{\pi - 2x\ln x + 2i\pi x}{2\pi(x^2+1)} & (x>0) \end{cases}
\end{aligned}
$$

10.

(a)

$$
\Re F(x) = \Re F(x_0) + \frac{x - x_0}{\pi} \mathrm{P} \int_{-\infty}^{\infty} dx' \frac{\Im F(x')}{(x'-x_0)(x'-x)}
$$

$$
\Rightarrow \frac{\Re F(x) - \Re F(x_0)}{x - x_0} = \frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} dx' \frac{\Im F(x')}{(x'-x_0)(x'-x)}
$$

[20] `https://en.wikipedia.org/wiki/Schwarz_reflection_principle`
[21] Why the heck is the "Method of image charges" linked to and from this document?

$$\xRightarrow{x_0 \to x} \Re F'(x) = \frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} dx' \frac{\Im F(x')}{(x'-x)^2}$$
$$= \frac{1}{\pi} \left(\left[-\frac{\Im F(x')}{x'-x} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} dx' \frac{F'(x')}{x'-x} \right)$$
$$= \frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} dx' \frac{F'(x')}{x'-x}$$

Therefore, replacing F' with F and substituting $x = 0$ yields the desired sum rule.

(b) You substitute $x = 0$. I really don't understand what the problem is here.

(c) 😫 Whoops! This problem is under construction ($\because$ author's skill issue). Please wait or consider contributing![22]

6 Chapter 6

Note of Warning Simply stating the answers to problems such as these have zero value, as they can be found with one query to the almighty WolframAlpha immediately. Hence, this manual is dedicated to a more proof-of-effort style of solution, where every nitty gritty detail is to be written. Proceed with caution.

1.

(a)

$$\left(\widetilde{(AB)} \right)_{ij} = (BA)_{ji}$$
$$= \sum_k B_{jk} A_{ki}$$
$$= \sum_k \left(\tilde{A} \right)_{ik} \left(\tilde{B} \right)_{kj}$$
$$= \left(\tilde{A} \tilde{B} \right)_{ij}$$
$$\therefore \widetilde{(AB)} = \tilde{B} \tilde{A}$$

[22] But my beef with this problem lies beyond the simple "What should I do?" confusion. I feel like whatever we derived in part (a) is a large nothing-burger; it is simply the unsubtracted relation evaluated at 0, as stated in part (b). What am I proving? What is the goal? If we want to express some rule such as

$$\Re F(x) = \mathrm{P} \int dx' \Im F(x) \cdot (\text{something}),$$

when what kind of properties are we trying to get for (something)? How close of an analogy are we talking about? Could you tell I'm losing my mind here?

(b)

$$\begin{aligned}
\left((AB)^\dagger\right)_{ij} &= \left((AB)_{ji}\right)^* \\
&= \left(\sum_k A_{jk} B_{ki}\right)^* \\
&= \sum_k {B_{ki}}^* {A_{jk}}^* \\
&= \sum_k \left(B^\dagger\right)_{ik} \left(A^\dagger\right)_{kj} \\
&= \left(B^\dagger A^\dagger\right)_{ij}
\end{aligned}$$

$$\therefore (AB)^\dagger = B^\dagger A^\dagger$$

(c)

$$\begin{aligned}
(A(BC))_{ij} &= \sum_k A_{ik} (BC)_{kj} \\
&= \sum_k \sum_l A_{ik} B_{kl} C_{lj} \\
&= \sum_l (AB)_{il} C_{lj} \\
&= ((AB)C)_{ij}
\end{aligned}$$

$$\therefore A(BC) = (AB)C$$

(d)

$$\begin{aligned}
\operatorname{Tr} ABC &= \sum_i (ABC)_{ii} \\
&= \sum_i \sum_j A_{ij} (BC)_{ji} \\
&= \sum_i \sum_j \sum_k A_{ij} B_{jk} C_{ki} \\
&= \sum_j \sum_k B_{jk} (CA)_{kj} \\
&= \sum_j (BCA)_{jj} \\
&= \operatorname{Tr} BCA
\end{aligned}$$

2.

(a)

$$\begin{aligned} x_1^\dagger x_1 &= x_1^\dagger 1 x_1 \\ &= x_1^\dagger U^\dagger U x_1 \\ &= (U x_1)^\dagger (U x_1) \\ &= (\lambda_1 x_1)^\dagger (\lambda_1 x_1) \\ &= |\lambda_1|^2 x_1^\dagger x_1 \end{aligned}$$

Since x_1 is nonzero, we therefore obtain

$$|\lambda_1|^2 = 1 \Leftrightarrow |\lambda_1| = 1.$$

(b)

$$\begin{aligned} x_1^\dagger x_2 &= x_1^\dagger 1 x_2 \\ &= x_1^\dagger U^\dagger U x_2 \\ &= (U x_1)^\dagger (U x_2) \\ &= (\lambda_1 x_1)^\dagger (\lambda_2 x_2) \\ &= \lambda_1^* \lambda_2 x_1^\dagger x_2 \end{aligned}$$

By part (a), $\lambda_1 \neq \lambda_2$ implies $\lambda_1^* \lambda_2 \neq 1$.

$$\therefore x_1^\dagger x_2 = 0$$

3.

(a)

$$\begin{aligned} \left(C^{-1} A C\right)^\dagger &= C^\dagger A^\dagger \left(C^{-1}\right)^\dagger \\ &= C^{-1} A \left(C^{-1}\right)^\dagger \end{aligned}$$

Now, notice that for a general invertible matrix M,

$$\begin{aligned} \left(M^{-1}\right)^\dagger M^\dagger &= \left(M M^{-1}\right)^\dagger \\ &= 1^\dagger \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} M^\dagger \left(M^{-1}\right)^\dagger &= \left(M^{-1} M\right)^\dagger \\ &= 1^\dagger \\ &= 1, \end{aligned}$$

hence implying

$$\left(M^{-1}\right)^{\dagger} = \left(M^{\dagger}\right)^{-1}.$$

$$\begin{aligned}\therefore \left(C^{-1}AC\right)^{\dagger} &= C^{-1}A\left(C^{-1}\right)^{\dagger} \\ &= C^{-1}A\left(C^{\dagger}\right)^{-1} \\ &= C^{-1}AC^{-1^{-1}} \\ &= C^{-1}AC\end{aligned}$$

(b)

$$\begin{aligned}\left(C^{-1}DC\right)^{\dagger} &= C^{\dagger}D^{\dagger}\left(C^{-1}\right)^{\dagger} \\ &= C^{-1}D^{-1}\left(C^{\dagger}\right)^{-1} \\ &= C^{-1}D^{-1}\left(C^{-1}\right)^{-1} \\ &= \left(C^{-1}DC\right)^{-1}\end{aligned}$$

(c)

$$\begin{aligned}(i(AB - BA))^{\dagger} &= i^{*}\left(B^{\dagger}A^{\dagger} - A^{\dagger}B^{\dagger}\right) \\ &= -i(BA - AB) \\ &= i(AB - BA)\end{aligned}$$

4. The characteristic polynomial for this matrix is given by

$$\begin{aligned}p(\lambda) &= \det\left(A - \lambda I_3\right) \\ &= \begin{vmatrix} 1-\lambda & 2 & 3 \\ 4 & 5-\lambda & 6 \\ 7 & 8 & 9-\lambda \end{vmatrix} \\ &= (1-\lambda)\left((5-\lambda)(9-\lambda) - 48\right) - 2\left(4(9-\lambda) - 42\right) + 3\left(32 - 7(5-\lambda)\right) \\ &= (1-\lambda)\left(-3 - 14\lambda + \lambda^2\right) - 2(-6 - 4\lambda) + 3(-3 + 7\lambda) \\ &= \left(-3 - 11\lambda + 15\lambda^2 - \lambda^3\right) + (12 + 8\lambda) + (-9 + 21\lambda) \\ &= 18\lambda + 15\lambda^2 - \lambda^3 \\ &= -\lambda\left(\lambda^2 - 15\lambda - 18\right).\end{aligned}$$

The eigenvalues are the zeros of this cubic polynomial, so

$$\lambda_0 = 0,\ \lambda_{\pm} = \frac{15 \pm 3\sqrt{33}}{2} \approx 16.117, -1.117.$$

The eigenvectors for each eigenvalue can be found via Gauss-Jordan elimination.

(i) $\lambda_0 = 0$

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix} \xrightarrow[R_3\leftarrow\frac{1}{7}R_3]{R_2\leftarrow\frac{1}{4}R_2} \begin{pmatrix}1&2&3\\1&\frac{5}{4}&\frac{3}{2}\\1&\frac{8}{7}&\frac{9}{7}\end{pmatrix} \xrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-R_1} \begin{pmatrix}1&2&3\\0&-\frac{3}{4}&-\frac{3}{2}\\0&-\frac{6}{7}&-\frac{12}{7}\end{pmatrix}$$

$$\xrightarrow[R_3\leftarrow-\frac{7}{6}R_3]{R_2\leftarrow\frac{4}{3}R_2} \begin{pmatrix}1&2&3\\0&1&2\\0&1&2\end{pmatrix} \xrightarrow[R_3\leftarrow R_3-R_2]{R_1\leftarrow R_1-2R_2} \begin{pmatrix}1&0&-1\\0&1&2\\0&0&0\end{pmatrix}$$

$$\Rightarrow \mathbb{u}_0 = \begin{pmatrix}s\\-2s\\s\end{pmatrix}$$

$$\therefore \mathbb{u}_0 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix} \approx \begin{pmatrix}0.408\\-0.816\\0.408\end{pmatrix}$$

(Note: This eigenvector reflects the fact that each row of this matrix is an arithmetic progression!)

(ii) $\lambda_\pm = \frac{15\pm3\sqrt{33}}{2}$

$$\begin{pmatrix}\frac{-13\mp3\sqrt{33}}{2}&2&3\\4&\frac{-5\mp3\sqrt{33}}{2}&6\\7&8&\frac{3\mp3\sqrt{33}}{2}\end{pmatrix} \xrightarrow{R_i\leftarrow\frac{1}{a_{i1}}R_i} \begin{pmatrix}1&\frac{13\mp3\sqrt{33}}{32}&\frac{39\mp9\sqrt{33}}{64}\\1&\frac{-5\mp3\sqrt{33}}{8}&\frac{3}{2}\\1&\frac{8}{7}&\frac{3\mp3\sqrt{33}}{14}\end{pmatrix}$$

$$\xrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-R_1} \begin{pmatrix}1&\frac{13\mp3\sqrt{33}}{32}&\frac{39\mp9\sqrt{33}}{64}\\0&\frac{-33\mp9\sqrt{33}}{32}&\frac{57\pm9\sqrt{33}}{64}\\0&\frac{165\pm21\sqrt{33}}{224}&\frac{-177\mp33\sqrt{33}}{448}\end{pmatrix}$$

$$\xrightarrow[R_3\leftarrow\frac{1}{a_{31}}R_3]{R_2\leftarrow\frac{1}{a_{21}}R_2} \begin{pmatrix}1&\frac{13\mp3\sqrt{33}}{32}&\frac{39\mp9\sqrt{33}}{64}\\0&1&\frac{-11\mp3\sqrt{33}}{44}\\0&1&\frac{-11\mp3\sqrt{33}}{44}\end{pmatrix}$$

$$\xrightarrow[R_3\leftarrow R_3-R_2]{R_1\leftarrow R_1-a_{12}R_2} \begin{pmatrix}1&0&\frac{11\mp3\sqrt{33}}{22}\\0&1&\frac{-11\mp3\sqrt{33}}{44}\\0&0&0\end{pmatrix}$$

$$\Rightarrow \mathbb{u}_\pm = \begin{pmatrix}\frac{-11\pm3\sqrt{33}}{22}s\\\frac{11\pm3\sqrt{33}}{44}s\\s\end{pmatrix}$$

$$||\mathbb{u}_\pm|| = s^2\sqrt{\frac{11}{30}}$$

$$\therefore \mathbb{u}_\pm = \begin{pmatrix}-\frac{-\sqrt{330}\pm9\sqrt{10}}{60}\\-\frac{\sqrt{330}\pm9\sqrt{10}}{60}\\-\frac{-\sqrt{330}\pm9\sqrt{10}}{60}\end{pmatrix} \approx \begin{pmatrix}0.172\\0.777\\0.606\end{pmatrix}, \begin{pmatrix}-0.777\\-0.172\\0.606\end{pmatrix}$$

5. Let the linear transformation and the vector be denoted as $\mathcal{A}$ and $\mathbb{x}$, respectively. Let us also denote the three given basis vector as $\mathbb{e}_i$ $(i = 1, 2, 3)$.

$$\mathcal{A}\mathbb{e}_1 = 3\mathbb{e}_1, ; \mathcal{A}\mathbb{e}_2 = \mathbb{e}_2, ; \mathcal{A}\mathbb{e}_3 = 5\mathbb{e}_3$$

Therefore, the matrix representation of $\mathcal{A}$ in this basis is given by

$$A' = \begin{pmatrix} 3 & & \\ & 1 & \\ & & 5 \end{pmatrix}.$$

$$\mathbb{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{\mathbb{e}_1 + \mathbb{e}_2}{2} + 2\frac{\mathbb{e}_1 - \mathbb{e}_2}{2} + 3\mathbb{e}_3 = \frac{3}{2}\mathbb{e}_1 - \frac{1}{2}\mathbb{e}_2 + 3\mathbb{e}_3$$

$$\therefore x' = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 3 \end{pmatrix}$$

6. The matrix A is thankfully block-diagonal as

$$A = \left(\begin{array}{cc|c|cc} 0 & -i & & & \\ i & 0 & & & \\ \hline & & 3 & & \\ \hline & & & 1 & -i \\ & & & i & 1 \end{array}\right).$$

Thus, the matrix is diagonal in the basis

$$e_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}, \; e_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

$$\therefore A' = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 3 & & \\ & & & 2 & \\ & & & & 0 \end{pmatrix}$$

Noting that these eigenvectors are orthonormal (since A is Hermitian), the components of x in the new coordinate system can be found as inner products with the basis vectors.

$$\therefore x' = \begin{pmatrix} -\frac{a}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \\ i \\ \frac{b-i}{\sqrt{2}} \\ \frac{1+ib}{\sqrt{2}} \end{pmatrix}$$

7.

$$\begin{aligned}
p(\lambda) &= \begin{vmatrix} \frac{5}{2}-\lambda & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{7}{3}-\lambda & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6}-\lambda \end{vmatrix} \\
&= \left(\frac{5}{2}-\lambda\right)\left(\frac{7}{3}-\lambda\right)\left(\frac{13}{6}-\lambda\right) \\
&\quad + \sqrt{\frac{3}{2}}\sqrt{\frac{1}{18}}\sqrt{\frac{3}{4}} \\
&\quad + \sqrt{\frac{3}{4}}\sqrt{\frac{3}{2}}\sqrt{\frac{1}{18}} \\
&\quad - \frac{1}{18}\left(\frac{5}{2}-\lambda\right) \\
&\quad - \frac{3}{2}\left(\frac{13}{6}-\lambda\right) \\
&\quad - \frac{3}{4}\left(\frac{7}{3}-\lambda\right) \\
&= 8 - 14\lambda + 7\lambda^2 - \lambda^3 \\
&= (1-\lambda)(2-\lambda)(4-\lambda)
\end{aligned}$$

(Imagine messing up this polynomial expansion, getting an irreducible cubic polynomial, finding its roots using Newton's method, trying and failing to find the eigenvectors because the eigenvalues are wrong but instead blaming the numerical instability of Gauss-Jordan elimination, typing up the power method and the inverse power method in Python, running the code on this matrix, only to find the eigenvalues to be about a millionth away from clean integers! How big a fool must you be to fall for that?)

(i) $\lambda = 1$

$$\begin{pmatrix} \frac{3}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{4}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{7}{6} \end{pmatrix} \xrightarrow{R_i \leftarrow \frac{1}{a_{i1}} R_i} \begin{pmatrix} 1 & \sqrt{\frac{2}{3}} & \frac{\sqrt{3}}{2} \\ 1 & \frac{4\sqrt{2}}{3\sqrt{3}} & \frac{1}{3\sqrt{3}} \\ 1 & \frac{\sqrt{2}}{3\sqrt{3}} & \frac{7}{3\sqrt{3}} \end{pmatrix}$$

$$\xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & \sqrt{\frac{2}{3}} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{2}}{3\sqrt{3}} & -\frac{2}{3\sqrt{3}} \\ 0 & -\frac{2\sqrt{2}}{3\sqrt{3}} & \frac{4}{3\sqrt{3}} \end{pmatrix}$$

$$\xrightarrow[R_3 \leftarrow -\frac{3\sqrt{3}}{2\sqrt{2}} R_3]{R_2 \leftarrow \frac{3\sqrt{3}}{2} R_2} \begin{pmatrix} 1 & \sqrt{\frac{2}{3}} & \frac{\sqrt{3}}{2} \\ 0 & 1 & -\sqrt{2} \\ 0 & 1 & -\sqrt{2} \end{pmatrix}$$

$$\xrightarrow[R_3 \leftarrow R_3 - R_2]{R_1 \leftarrow R_1 - \sqrt{\frac{2}{3}} R_2} \begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \mathbb{u} = \begin{pmatrix} -\sqrt{3}s \\ \sqrt{2}s \\ s \end{pmatrix}$$

$$\therefore \mathbb{u} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

(ii) $\lambda = 2$

$$\begin{pmatrix} \frac{1}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{1}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{1}{6} \end{pmatrix} \xrightarrow{R_i \leftarrow \frac{1}{a_{i1}} R_i} \begin{pmatrix} 1 & \sqrt{6} & \sqrt{3} \\ 1 & \frac{\sqrt{2}}{3\sqrt{3}} & \frac{1}{3\sqrt{2}} \\ 1 & \frac{\sqrt{2}}{3\sqrt{3}} & \frac{7}{3\sqrt{3}} \end{pmatrix}$$

$$\xrightarrow[R_3 \leftarrow R_3 - R_2]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & \sqrt{6} & \sqrt{3} \\ 0 & -\frac{8\sqrt{2}}{3\sqrt{3}} & -\frac{8}{3\sqrt{3}} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow -\frac{3\sqrt{3}}{8\sqrt{2}} R_2} \begin{pmatrix} 1 & \sqrt{6} & \sqrt{3} \\ 0 & 1 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 - \sqrt{6} R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \mathbb{u} = \begin{pmatrix} 0 \\ s \\ \sqrt{2}s \end{pmatrix}$$

$$\therefore \mathbb{u} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\frac{2}{3}} \end{pmatrix}$$

(iii) $\lambda = 4$

$$\begin{pmatrix} -\frac{3}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & -\frac{5}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & -\frac{11}{6} \end{pmatrix} \xrightarrow{R_i \leftarrow \frac{1}{a_{i1}} R_i} \begin{pmatrix} 1 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} \\ 1 & -\frac{5\sqrt{2}}{3\sqrt{3}} & \frac{1}{3\sqrt{3}} \\ 1 & \frac{\sqrt{2}}{3\sqrt{3}} & -\frac{11}{3\sqrt{3}} \end{pmatrix}$$

$$\xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} \\ 0 & -\frac{2\sqrt{2}}{3\sqrt{3}} & \frac{4}{3\sqrt{3}} \\ 0 & \frac{4\sqrt{2}}{3\sqrt{3}} & -\frac{8}{3\sqrt{3}} \end{pmatrix}$$

$$\xrightarrow[R_3 \leftarrow -\frac{3\sqrt{3}}{4\sqrt{2}} R_3]{R_2 \leftarrow -\frac{3\sqrt{3}}{2\sqrt{2}} R_2} \begin{pmatrix} 1 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} \\ 0 & 1 & -\sqrt{2} \\ 0 & 1 & -\sqrt{2} \end{pmatrix}$$

$$\xrightarrow[R_3 \leftarrow R_3 - R_2]{R_1 \leftarrow R_1 + \sqrt{\frac{2}{3}} R_2} \begin{pmatrix} 1 & 0 & -\sqrt{3} \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \mathbb{u} = \begin{pmatrix} \sqrt{3}s \\ \sqrt{2}s \\ s \end{pmatrix}$$

$$\therefore \mathbb{u} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

8. Let m_o and m_c denote the masses of the oxygen and carbon atoms, respectively. The three equations of motion are given by

$$\begin{cases} m_o \ddot{x}_1 = k_1 (x_2 - x_1) + k_2 (x_3 - x_1) \\ m_c \ddot{x}_2 = k_1 (x_1 - x_2) + k_1 (x_3 - x_2) \\ m_o \ddot{x}_3 = k_2 (x_1 - x_3) + k_1 (x_2 - x_3) \end{cases} .$$

Denoting the Fourier transform of these as $X_i(\omega)$ $(i = 1, 2, 3)$, we obtain

$$\begin{pmatrix} -m_c \omega^2 X_1 \\ -m_o \omega^2 X_2 \\ -m_c \omega^2 X_3 \end{pmatrix} = \begin{pmatrix} -k_1 - k_2 & k_1 & k_2 \\ k_1 & -2k_1 & k_1 \\ k_2 & k_1 & -k_1 - k_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} .$$

$$\Rightarrow -\omega^2 \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} -\frac{k_1+k_2}{m_c} & \frac{k_1}{m_c} & \frac{k_2}{m_c} \\ \frac{k_1}{m_o} & -\frac{2k_1}{m_o} & \frac{k_1}{m_o} \\ \frac{k_2}{m_c} & \frac{k_1}{m_c} & -\frac{k_1+k_2}{m_c} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

Let

$$\alpha := \frac{k_1}{m_o},\ \beta := \frac{k_2}{m_o},\ \gamma := \frac{m_o}{m_c}.$$

We can then see that $-\omega^2$ must be the eigenvalues of the coefficient matrix

$$\begin{pmatrix} -(\alpha+\beta) & \alpha & \beta \\ \alpha\gamma & -2\alpha\gamma & \alpha\gamma \\ \beta & \alpha & -(\alpha+\beta) \end{pmatrix}.$$

$$\begin{aligned} p(\lambda) &= \begin{vmatrix} -\alpha-\beta-\lambda & \alpha & \beta \\ \alpha\gamma & -2\alpha\gamma-\lambda & \alpha\gamma \\ \beta & \alpha & -\alpha-\beta-\lambda \end{vmatrix} \\ &= -\left(4\alpha\beta\gamma + 2\alpha^2\gamma + \alpha^2 + 2\alpha\beta\right)\lambda - 2(\alpha+\beta+\alpha\gamma)\lambda^2 - \lambda^3 \\ &= -\lambda(\lambda+\alpha+2\beta)(\lambda+\alpha+2\alpha\gamma) \end{aligned}$$

Therefore, the allowed modes of vibrations are given by

$$\omega = 0,\ \pm\sqrt{\frac{k_1+2k_2}{m_o}},\ \pm\sqrt{\left(\frac{1}{m_o}+\frac{2}{m_c}\right)k_1}.$$

The interpretation of these frequencies is rather physically intuitive, as the terms appearing in them indicate which components of the system play an active role in the vibrations. The frequency-0 mode must be translation, as we only consider 1D motion. The second frequency does not involve the motion of the carbon atom but does involve all three "springs." Hence, the two carbon atoms must oscillate symmetrically about the oxygen atom. The third frequency does not involve the k_2 "spring," so it must have the carbon atom oscillating totally out of phase with the two carbon atoms such that the total length of the molecule stays fixed.[23]

9. Orthogonality in modes a, b is given by

$$\sum_i m_i \mathbf{q_i}^{(a)} \cdot \mathbf{q_i}^{(b)} = 0$$

where m_i is the mass of the i^{th} particle and $\mathbf{q_i}^{(a)}$ is the displacement vector of the i^{th} particle in mode a.
In translation mode, all particles move in the same direction with same magnitude.

$$\Longrightarrow \mathbf{q_i}^{(t)} = \mathbf{c} \quad \forall i$$

[23] I better draw some diagrams for these, huh?

where $\mathbf{c}$ is a constant vector.
If a mode a is orthogonal to translation mode, then

$$\sum_i m_i \mathbf{q_i}^{(a)} \cdot \mathbf{c} = 0$$

as this is true for any $\mathbf{c}$,

$$\sum_i m_i \mathbf{q_i}^{(a)} = 0$$

Now, center of mass is given by

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r_i}}{\sum_i m_i}$$

where $\mathbf{r_i}$ is the position vector of the i^{th} particle.
For the displacement from equilibrium position, we have

$$\mathbf{r_i} = \mathbf{r_i^0} + \mathbf{q_i}$$

where $\mathbf{r_i^0}$ is the equilibrium position of the i^{th} particle.
So,

$$\mathbf{R} = \frac{\sum_i m_i (\mathbf{r_i^0} + \mathbf{q_i})}{\sum_i m_i}$$

$$= \frac{\sum_i m_i \mathbf{r_i^0}}{\sum_i m_i} + \frac{\sum_i m_i \mathbf{q_i}}{\sum_i m_i}$$

For mode a orthogonal to translation mode,

$$\implies \sum_i m_i \mathbf{q_i}^{(a)} = 0$$

$$\therefore \quad \mathbf{R} = \frac{\sum_i m_i \mathbf{r_i^0}}{\sum_i m_i}$$

$$\implies \mathbf{R} = \text{constant}$$

For a rotation mode, all particles move in a direction perpendicular to the line joining the particle and the axis of rotation.

$$\implies \mathbf{q_i}^{(r)} = \boldsymbol{\omega} \times \mathbf{r_i}$$

where $\boldsymbol{\omega}$ is the angular velocity vector and $\mathbf{r_i}$ is the position vector of the i^{th} particle.
If a mode a is orthogonal to rotation mode, then

$$\sum_i m_i (\boldsymbol{\omega} \times \mathbf{r_i}) \cdot \mathbf{q_i}^{(a)} = 0$$

as this is true for any $\boldsymbol{\omega}$,

$$\sum_i m_i(\mathbf{r_i} \times \mathbf{q_i}^{(a)}) = 0$$

Now, angular momentum is given by

$$\mathbf{L} = \sum_i \mathbf{r_i} \times m_i \mathbf{v_i}$$

where $\mathbf{v_i}$ is the velocity of the i^{th} particle.
For the displacement from equilibrium position, we have

$$\mathbf{v_i} = \dot{\mathbf{q}}_\mathbf{i}$$

So,

$$\mathbf{L} = \sum_i \mathbf{r_i} \times m_i \dot{\mathbf{q}}_\mathbf{i}$$

For mode a orthogonal to rotation mode,

$$\implies \sum_i m_i(\mathbf{r_i} \times \mathbf{q_i}^{(a)}) = 0$$

$$\therefore \quad \mathbf{L} = \sum_i \mathbf{r_i} \times m_i \dot{\mathbf{q}}_\mathbf{i} = 0 \quad (\text{as } \dot{\mathbf{r}}_\mathbf{i} = 0)$$

$$\implies \mathbf{L} = 0$$

10.

(a) The relation holds when $\mathbb{b} =$, as both sides are zero. When $\mathbb{b} \neq$, define $\mathbb{c} := \mathbb{a} + \lambda \mathbb{b}$.

$$\begin{aligned} 0 &\leq \mathbb{c} \cdot \mathbb{c} \\ &= (\mathbb{a} + \lambda \mathbb{b}) \cdot (\mathbb{a} + \lambda \mathbb{b}) \\ &= \mathbb{a} \cdot \mathbb{a} + 2\Re\{\lambda \mathbb{a} \cdot \mathbb{b}\} + \mathbb{b} \cdot \mathbb{b} \lambda^2 \end{aligned}$$

As this holds for any λ, let

$$\lambda = -\frac{\mathbb{b} \cdot \mathbb{a}}{\mathbb{b} \cdot \mathbb{b}}.$$

$$\Rightarrow 0 \leq \mathbb{a} \cdot \mathbb{a} - \frac{|\mathbb{a} \cdot \mathbb{b}|^2}{\mathbb{b} \cdot \mathbb{b}}$$

$$\therefore (\mathbb{a} \cdot \mathbb{a})(\mathbb{b} \cdot \mathbb{b}) \geq |\mathbb{a} \cdot \mathbb{b}|^2$$

(b)

$$\mathbb{x} \cdot \mathbb{x}^{(n)} = \sum_{i=1}^{n} x_i(\mathbb{x} \cdot \mathbb{e}_i) = \sum_{i=1}^{n} |x_i|^2$$

$$\mathbb{x}^{(n)} \cdot \mathbb{x}^{(n)} = \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i|^2 \delta_{ij} = \sum_{i=1}^{n} |x_i|^2$$

$$\Rightarrow \mathbb{x} \cdot \mathbb{x} \left(\sum_{i=1}^{n} |x_i|^2 \right) = \left(\sum_{i=1}^{n} |x_i|^2 \right)^2$$

$$\therefore \mathbb{x} \cdot \mathbb{x} \geq \sum_{i=1}^{n} |x_i|^2$$

(When $\sum_{i=1}^{n} |x_i|^2 = 0$, then the norm must be no less than 0.)

11. The problem, as stated, is wrong; its proposed solution is for when the cyclic boundary condition has been lifted. Let us solve the problem backwards, incrementing the imposed conditions.

When there are no constraints, the system is N independent atoms, each with 3 allowed states. Thus, $W = 3^N$ and $S = k \ln 3$.

When the AC condition is imposed, let $v_A^{(j)}, v_B^{(j)},$ and $v_C^{(j)}$ denote the possible number of states with j atoms such that the last atom is in state $A, B,$ and C, respectively. Finding a recurrence relation for these is then a classic combinatorics problem. Every $(j+1)$-atom state ending in an A tom can be bijectively mapped to a j-atom state ending in a non-C atom. That is, $v_A^{(j+1)} = v_A^{(j)} + v_B^{(j)}$. Similar arguments yield the transfer matrix given in the problem. Therefore, we have

$$\begin{aligned}
W &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} M^N \begin{pmatrix} v_A^{(1)} \\ v_B^{(1)} \\ v_C^{(1)} \end{pmatrix} \\
&\approx \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \cdot \left(1+\sqrt{2}\right)^N \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \left(1 + \frac{1}{\sqrt{2}}\right)^2 \left(1+\sqrt{2}\right)^N
\end{aligned}$$

where the vector $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$ is the unit eigenvector corresponding to the largest eigenvalue of $1 + \sqrt{2}$.

$$\therefore S = \lim_{N \to \infty} \frac{k \ln W}{N} = k \ln \left(1 + \sqrt{2}\right)$$

When the circular boundary condition is imposed, we must define separate variables for the starting atom's state as well. Hence, we define the 7-

dimensional vector

$$\mathbb{v}^{(j)} := \begin{pmatrix} v_{AA}^{(j)} \\ v_{AB}^{(j)} \\ v_{BA}^{(j)} \\ v_{BB}^{(j)} \\ v_{BC}^{(j)} \\ v_{CB}^{(j)} \\ v_{CC}^{(j)} \end{pmatrix}.$$

The transfer matrix is then defined by

$$M = \left(\begin{array}{cc|ccc|cc} 1 & 1 & & & & & \\ 1 & 1 & & & & & \\ \hline & & 1 & 1 & & & \\ & & 1 & 1 & 1 & & \\ & & & 1 & 1 & & \\ \hline & & & & & 1 & 1 \\ & & & & & 1 & 1 \end{array}\right).$$

The dominating eigenvalue and the corresponding eigenvector are equal to the previous case.

$$\Rightarrow W = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} M^N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\approx \left(1+\sqrt{2}\right)^N \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$$

$$\times \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \left(1 + \frac{1}{\sqrt{2}}\right) \left(1+\sqrt{2}\right)^N$$

The entropy is therefore the same as before.

7 Chapter 7

1. Let us denote the Legendre coefficients as $\{c_n\}$.

$$\begin{aligned}
c_n &= \frac{2n+1}{2}\int_{-1}^{1} dx P_n(x) f(x) \\
&= \frac{2n+1}{2}\left(-\int_{-1}^{0} dx P_n(x) + \int_{0}^{1} dx P_n(x)\right) \\
&= \frac{2n+1}{2}\left(-(-1)^n + 1\right)\int_{0}^{1} dx P_n(x) \\
&= \begin{cases} 0 & (2 \mid n) \\ (2n+1)\int_0^1 dx P_n(x) & (2 \nmid n) \end{cases}
\end{aligned}$$

To evaluate the integral, we use the generating function.

$$\begin{aligned}
\sum_{n=0}^{\infty} h^n \int_0^1 dx P_n(x) &= \int_0^1 dx \left(\sum_{n=0}^{\infty} h^n P_n(x)\right) \\
&= \int_0^1 \frac{dx}{\sqrt{1-2hx+h^2}} \\
&= \left[-\frac{\sqrt{1-2hx+h^2}}{h}\right]_0^1 \\
&= \frac{\sqrt{1+h^2}-(1-h)}{h} \\
&= \frac{1}{h}\left(\sum_{l=0}^{\infty}\binom{\frac{1}{2}}{l} h^{2l} - 1 + h\right) \\
&= 1 + \sum_{l=1}^{\infty}\binom{\frac{1}{2}}{l} h^{2l-1}
\end{aligned}$$

Here, we used the fact that $|h| < 1$.

$$\begin{aligned}
\Rightarrow c_{2l+1} &= (4l+3)\binom{\frac{1}{2}}{l+1} \\
&= (4l+3)\frac{\prod_{j=0}^{l}\left(\frac{1}{2}-j\right)}{(l+1)!} \\
&= (4l+3)\frac{(-1)^{l+1}\prod_{j=0}^{l}(2j-1)}{2^{l+1}(l+1)!} \\
&= (-1)^l (4l+3)\frac{(2l-1)!!}{(2l+2)!!}
\end{aligned}$$

$$\begin{aligned}
\therefore f(x) &= \sum_{n=0}^{\infty} (-1)^n (4n+3) \frac{(2n-1)!!}{(2n+2)!!} P_n(x) \\
&= \frac{3}{2} P_1(x) - \frac{7}{8} P_3(x) + \frac{11}{16} P_5(x) - \frac{75}{128} P_7(x) + \cdots
\end{aligned}$$

2.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} P_n(x) &= \int_0^x dh \left(\sum_{n=0}^{\infty} h^n P_n(x) \right) \\
&= \int_0^x \frac{dh}{\sqrt{1 - 2hx + h^2}} \\
&= \left[-\ln \left(\sqrt{1 - 2hx + h^2} + x - h \right) \right]_{h=0}^{h=x} \\
&= \ln \left(\frac{1+x}{\sqrt{1-x^2}} \right)
\end{aligned}$$

Of course, the integral formulation restricts x such that $|x| < 1$. It seems likely that the domain of convergence is this disc (plus some boundary points). As evidence, I point out that $x = 1$ yields the harmonic series. If anyone could find the domain of convergence, it would be greatly appreciated!

3. (1) We will make the change of variables

$$x = \cos\alpha \text{ and } y = \cos\beta.$$

Without loss of generality, suppose we have defined the coordinate system such that the direction for α is parallel to the z axis and that for β lies in the zx plane. The coordinates (x, y) maps to exactly two points on the unit sphere, where the two points have the same θ but ϕ's that add to 2π. (In a coordinate-independent formulation, (x, y) represents a unique point on one of the two half-spheres cut by the plane containing the two fixed directions.) The integrals evaluated on each half-sphere are equal, so we may perform the change of variables for one of the half-spheres and multiply the result by 2.

Let γ denote the angle between the two fixed directions.

$$x = \cos\theta, \ y = \cos\gamma\cos\theta + \sin\gamma\sin\theta\cos\phi.$$

$$\Rightarrow \begin{cases} \cos\theta = x & \sin\theta = \sqrt{1-x^2} \\ \cos\phi = \frac{y - x\cos\gamma}{\sin\gamma\sqrt{1-x^2}} & \end{cases}$$

$$
\begin{aligned}
\Rightarrow \frac{\partial(x,y)}{\partial(\theta,\phi)} &= \begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{vmatrix} \\
&= \sin\gamma \sin^2\theta \sin\phi \\
&= \sin\gamma \left(1 - x^2\right) \sqrt{1 - \left(\frac{y - x\cos\gamma}{\sin\gamma\sqrt{1-x^2}}\right)^2} \\
&= \sqrt{\left(1-x^2\right)\left(\sin^2\gamma - x^2 - y^2 + 2xy\cos\gamma\right)}
\end{aligned}
$$

$$
\therefore I = \int_{-1}^{1} dx \int_{-1}^{1} dy f(x) g(y) \cdot \frac{2}{\sqrt{\left(1-x^2\right)\left(\sin^2\gamma - x^2 - y^2 + 2xy\cos\gamma\right)}}
$$

(2) ☹ Whoops! This problem is under construction ($\because$ author's skill issue). Please wait or consider contributing!

4. (a)

$$
\begin{aligned}
P_n'(1) &= \frac{1}{2^n n!} \frac{d^{n+1}}{dx^{n+1}} \left(x^2 - 1\right)^n \\
&= \frac{1}{2^n n!} \frac{d^{n+1}}{dx^{n+1}} (x-1)^n (x+1)^n
\end{aligned}
$$

Imagining the $n+1$ derivatives as repeated applications of the product rule, we see that all n factors of $x-1$ must be differentiated, leaving a single derivative of $x+1$. There are $(n+1)!$ such orders to perform such derivatives.

$$
\therefore P_n'(1) = \frac{1}{2^n n!} \cdot (n+1)! n 2^{n-1} = \frac{n(n+1)}{2}
$$

(b)

$$\begin{aligned}
\sum_{n=0}^{\infty} h^n P_n'(1) &= \frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} h^n P_n(x)\right)\bigg|_{x=1} \\
&= \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{1-2hx+x^2}}\right)\bigg|_{x=1} \\
&= \frac{h}{(1-2hx+x^2)^{\frac{3}{2}}}\bigg|_{x=1} \\
&= h(1-h)^{-3} \\
&= h\sum_{n=0}^{\infty} \frac{3(3+1)\cdots(3+n-1)}{n!} h^n \\
&= \sum_{n=0}^{\infty} \frac{(n+2)!}{2n!} h^{n+1} \\
&= \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} h^{n+1} \\
&= \sum_{n=0}^{\infty} \frac{n(n+1)}{2} h^n
\end{aligned}$$

$$\therefore P_n'(1) = \frac{n(n+1)}{2}$$

5. Recall that

$$Y_{lm}(\Omega) = \left(\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\right)^{\frac{1}{2}} P_l^{|m|}(\cos\theta)e^{im\phi} \times \begin{cases} (-1)^m & (m \geq 0) \\ 1 & (m < 0) \end{cases}.$$

(i) $m \geq 0$:

$$\begin{aligned}
(-1)^m P_l^m(\cos\theta) &= \frac{(-1)^m}{2^l l!}\left(1-\cos^2\theta\right)^{\frac{m}{2}} \frac{d^{l+m}}{d(\cos\theta)^{l+m}}\left(\cos^2\theta - 1\right)^l \\
&= \frac{1}{2^l l!}(-\sin\theta)^m \frac{d^{l+m}}{d(\cos\theta)^{l+m}}\left(\cos^2\theta - 1\right)^l
\end{aligned}$$

(ii) $m < 0$: ☹ Whoops! This problem is under construction (∵ author's skill issue). Please wait or consider contributing!

☹ Whoops! This problem is under construction (∵ author's skill issue). Please wait or consider contributing!

6. 7.

$$\sum_{n=0}^{\infty} c_n P_n(z) = \frac{1}{\pi}\int_0^{2\pi} d\phi \sum_{n=0}^{\infty} c_n \left(z + \sqrt{z^2-1}\cos\phi\right)^n$$

For the left-hand side to converge, the integrand of the right-hand side must also converge.

$$\begin{aligned}
\therefore \sup_{\phi\in[0,2\pi]} \left|\left(z+\sqrt{z^2-1}\right)\right| &< R \\
\Leftrightarrow |z| + \sqrt{|z^2-1|} &< R \\
\Leftrightarrow |z| < R \wedge \left|z^2-1\right| &< (R-|z|)^2
\end{aligned}$$

(Note: This is most definitely <u>not</u> correct! Simple graphing shows that the shapes are not ellipses; but rather ovals with bulges on each end. The shape does seem to converge to ellipses for large values of R, which might be sufficient for a physicist to say "It is an ellipse." That said, I wouldn't be surprised when somebody finds a flaw in this solution.

It also seems that the first condition is implied by the second. This is plausible at first glance, since as both R and $|z|$ grow without bound, the second condition reduces to $|z| \lesssim \frac{R}{2}$. I however failed to prove this rigorously; it is left as an exercise to the reader.)

8. (a)

$$\begin{aligned}
\sum_{n=0}^{\infty} H_{n+1}(x)\frac{h^n}{n!} &= \frac{\partial}{\partial h}\left(\sum_{n=0}^{\infty} H_n(x)\frac{h^n}{n!}\right) \\
&= \frac{\partial}{\partial h} e^{2hx-h^2} \\
&= 2(x-h)e^{2hx-h^2} \\
&= 2(x-h)\sum_{n=0}^{\infty} H_n(x)\frac{h^n}{n!} \\
&= \sum_{n=0}^{\infty} 2\left(xH_n(x) - nH_{n-1}(x)\right)\frac{h^n}{n!} \\
\therefore H_{n+1} &= 2xH_n(x) - 2nH_{n-1}(x)
\end{aligned}$$

(b)

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{h^n}{n!}\int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2}} H_n(x) &= \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2}}\left(\sum_{n=0}^{\infty} H_n(x)\frac{h^n}{n!}\right) \\
&= \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2}+2hx-h^2} \\
&= \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2}} e^{-\frac{1}{2}(x-2h)^2+h^2} \\
&= \sqrt{2\pi}e^{h^2} \\
&= \sum_{n=0}^{\infty}\frac{\sqrt{2\pi}}{n!}h^{2n}
\end{aligned}$$

$$\therefore \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2}} H_n(x) = \begin{cases} 0 & (2 \nmid n) \\ \frac{\sqrt{2\pi} n!}{\left(\frac{n}{2}\right)!} & (2 \mid n) \end{cases}$$

9.

$$\begin{aligned} \frac{\partial G}{\partial x}(x,t) &= \sum_n f_n'(x) t^n \\ &= \sum_n f_{n-1}(x) t^n \\ &= \sum_n f_n(x) t^{n+1} \\ &= tG(x,t) \\ \Rightarrow G(x,t) &= A(t) e^{xt} \end{aligned}$$

$$\begin{aligned} \frac{\partial G}{\partial t}(x,t) &= \sum_n n f_n(x) t^{n-1} \\ &= \sum_n (n+1) f_{n+1}(x) t^n \\ &= \sum_n \left(x f_n(x) - f_{n+2}(x) \right) t^n \\ &= \left(x - \frac{1}{t^2} \right) G(x,t) \\ \Rightarrow G(x,t) &= B(x) e^{xt + \frac{1}{t}} \end{aligned}$$

For these two conditions to simultaneously hold, we must have

$$G(x,t) = C e^{xt + \frac{1}{t}}.$$

We will find C using condition (a):

$$\begin{aligned} G(x,t) &= C \sum_{k=0}^{\infty} \frac{\left(xt + \frac{1}{t}\right)^k}{k!} \\ &= C \sum_{k=0}^{\infty} \sum_{l=0}^{k} \binom{k!}{l!} \frac{(xt)^l t^{-n+l}}{k!} \\ &= C \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{x^l t^{2l-k}}{l!(k-l)!} \end{aligned}$$

$$\Rightarrow f_0(x) = C \sum_{k,l:2l-k=0} \frac{x^l}{l!(k-l)!} = C \sum_{l=0}^{\infty} \frac{x^l}{(l!)^2} \Rightarrow C = 1$$

$$\therefore G(x,t) = e^{xt + \frac{1}{t}}$$

10. Let us approach the problem using a general second-order linear ODE. That is, suppose f and g are solutions to the differential equation

$$p(x)\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x) = 0.$$

Define the Wronskian as

$$W(x) := f(x)g'(x) - f'(x)g(x).$$

We then have

$$\begin{aligned} p(x)W'(x) &= p(x)\left(f'(x)g'(x) + f(x)g''(x) - f''(x)g(x) - f'(x)g'(x)\right) \\ &= f(x)\left(p(x)g''(x)\right) - \left(p(x)f''(x)\right)g(x) \\ &= f(x)\left(-q(x)g'(x) - r(x)g(x)\right) - \left(-q(x)f'(x) - r(x)f(x)\right)g(x) \\ &= -q(x)W(x). \end{aligned}$$

Therefore, the Wronskian follows the first-order linear ODE

$$p(x)\frac{dW}{dx}(x) + q(x)W(x) = 0.$$

In the case of Bessel's equation, $p(x) = x^2$ and $q(x) = x$.

$$\Rightarrow W'(x) = -\frac{1}{x}W(x)$$

$$\therefore W(x) = \frac{C}{x}$$

11. As shown in the previous exercise, we now know that

$$W(x) := J_m(x)J'_{-m}(x) - J'_m(x)J_{-m}(x) = \frac{C_m}{x}$$

for some constant C_m. As this holds for all x, we may write

$$\begin{aligned} C_m &= \lim_{x\to 0} x\left(J_m(x)J'_{-m}(x) - J'_m(x)J_{-m}(x)\right) \\ &= \lim_{x\to 0} \frac{x}{2}\left(J_m\left(J_{-m-1} - J_{-m+1}\right) - \left(J_{m-1} - J_{m+1}\right)J_{-m}\right). \end{aligned}$$

Notice that as $x \to 0$, each term behaves like $J_m \sim x^m$. This argument shows that the terms like xJ_mJ_{-m-1} vanish at the origin.

$$\begin{aligned}
\Rightarrow C_m &= \lim_{x\to 0}\frac{x}{2}\left(J_m(x)J_{-m-1}(x) - J_{m-1}(x)J_{-m}(x)\right)\\
&= \lim_{x\to 0}\frac{x}{2}\left(\frac{x^m}{2^m m!}\frac{x^{-m-1}}{2^{-m-1}(-m-1)!} - \frac{x^{m-1}}{2^{m-1}(m-1)!}\frac{x^{-m}}{2^{-m}(-m)!}\right)\\
&= \frac{1}{\Gamma(m+1)\Gamma(-m)} - \frac{1}{\Gamma(m)\Gamma(-m+1)}\\
&= \frac{1}{\Gamma(m)\Gamma(-m)}\left(\frac{1}{-m} - \frac{1}{-m}\right)\\
&= -\frac{2}{\Gamma(m)\Gamma(1-m)}\\
&= -\frac{2}{\pi}\sin m\pi
\end{aligned}$$

$$\therefore W(x) = -\frac{2\sin m\pi}{\pi x}$$

We can verify that $W(x) = 0$ for integral values of m.

12.

$$\begin{aligned}
W(x) &:= J_m(x)Y_m'(x) - J_m'(x)Y_m(x)\\
&= \frac{J_m\left(\cos m\pi J_m'(x) - J_{-m}'(x)\right) - J_m'(x)\left(\cos m\pi J_m(x) - J_{-m}(x)\right)}{\sin m\pi}\\
&= -\frac{J_m(x)J_{-m}'(x) - J_m'(x)J_{-m}(x)}{\sin m\pi}\\
&= \frac{2}{\pi x}
\end{aligned}$$

13. Following the argument of Problem 7–10, we know that the Wronskian satisfies

$$W'(x) = \frac{2x}{1-x^2}W(x).$$

$$\Rightarrow W(x) = P_l(x)Q_l'(x) - P_l'(x)Q_l(x) = \frac{C_l}{1-x^2}$$

$$\Rightarrow C_l = \left(1-x^2\right)\left(P_l(x)Q_l'(x) - P_l'(x)Q_l(x)\right).$$

This time, let us find C_l using the asymptotic behavior at infinity.

$$\begin{aligned}
P_l(x) &= \frac{1}{2^l l!}\frac{d^l}{dx^l}\left(x^2-1\right)^l\\
&\sim \frac{1}{2^l l!}\frac{d^l}{dx^l}x^{2l}\\
&= \frac{(2l)!}{2^l (l!)^2}x^l
\end{aligned}$$

$$
\begin{aligned}
Q_l(x) &= -P_l(x)\int^x \frac{dx}{(x^2-1)\,P_l(x)^2} \\
&\sim -\frac{(2l)!}{2^l (l!)^2} x^l \cdot \int^x \frac{dx}{x^2\left(\frac{(2l)!}{2^l (l!)^2} x^l\right)^2} \\
&= -\frac{2^l (l!)^2}{(2l)!} x^l \int^x \frac{dx}{x^{2l+2}} \\
&= \frac{2^l (l!)^2}{(2l+1)(2l)!} x^{-l-1}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow C_l &= \lim_{x\to\infty} \left(1-x^2\right)\left(P_l(x)Q_l'(x) - P_l'(x)Q_l(x)\right) \\
&= -\lim_{x\to\infty} x^2 \left(-\frac{(2l)!}{2^l (l!)^2} x^l \cdot \frac{(l+1)2^l (l!)^2}{(2l+1)(2l)!} x^{-l-2} \right. \\
&\quad \left. - \frac{(2l)!}{2^l (l!)^2} l x^{l-1} \cdot \frac{2^l (l!)^2}{(2l+1)(2l)!} x^{-l-1} \right) \\
&= 1
\end{aligned}
$$

$$
\therefore W(x) = \frac{1}{1-x^2}
$$

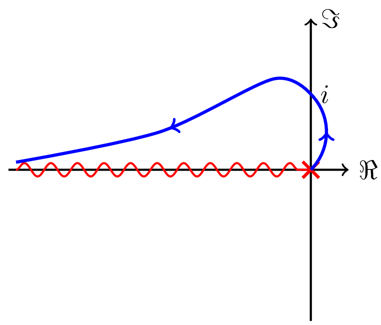


Figure 19: Contour for Problem 7–14.

14. The integral formulation for the Hankel function is given by

$$
H_n^{(1)}(z) = \frac{1}{i\pi}\int_C dt \frac{\exp\left(\frac{z}{2}\left(t-\frac{1}{t}\right)\right)}{t^{n+1}}
$$

along the contour shown in Figure 19 The numerator has a saddle point at $t = i$ along the contour. Around this point,

$$
t + \frac{1}{t} \approx 2i - e^{i\frac{\pi}{2}} (t-i)^2.
$$

$$\Rightarrow H_n^{(1)}(z) \approx \frac{1}{i\pi}\int dt \frac{iz - \frac{z}{2}e^{i\pi}(t-i)^2}{t^{n+1}}$$

By what angle should we cross the $t = i$ point to perform the saddle-point method? Since the coefficient of the quadratic term has argument π, the contour must pass the point with angle $\frac{3\pi}{4}$ or $\frac{5\pi}{4}$. The shape of the contour demands $\phi = \frac{3\pi}{4}$.

$$\begin{aligned}
\therefore H_n^{(1)}(z) &\approx \frac{e^{i\left(\frac{3\pi}{4}+z\right)}}{i^{n+2}\pi} \cdot \int_{-\infty}^{\infty} dr \exp\left(-\frac{z}{2}r^2\right) \\
&= \sqrt{\frac{2}{\pi z}} \exp\left(iz - i\frac{2n+1}{4}\pi\right)
\end{aligned}$$

15. Without loss of generality, we may only prove the statement for $m \geq 0$.

$$\begin{aligned}
J_{-m}(x) &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(r-m+1)}\left(\frac{x}{2}\right)^{-m+2r} \\
&= \sum_{r=m}^{\infty} \frac{(-1)^r}{r!(r-m)!}\left(\frac{x}{2}\right)^{-m+2r} \\
&= \sum_{r=0}^{\infty} \frac{(-1)^{r+m}}{(r+m)!r!}\left(\frac{x}{2}\right)^{-m+2r+2m} \\
&= (-1)^m \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(r+m+1)}\left(\frac{x}{2}\right)^{m+2r} \\
&= (-1)^m J_m(x)
\end{aligned}$$

16. (a)

$$\begin{aligned}
J_{\frac{1}{2}}(x) &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma\left(r+\frac{3}{2}\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}+2r} \\
&= \frac{1}{\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\prod_{j=0}^{r}\left(\frac{1}{2}+j\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}+2r} \\
&= \frac{1}{\frac{2}{\pi}} \sum_{r=0}^{\infty} \frac{(-1)^r}{2^r r!(2r+1)!!} x^{\frac{1}{2}+2r} \\
&= \sqrt{\frac{2}{\pi x}} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)!} x^{2r+1} \\
&= \sqrt{\frac{2}{\pi x}} \sin x
\end{aligned}$$

(b)

$$\begin{aligned}
J_{-\frac{1}{2}} &= \frac{\frac{1}{2}}{x} J_{\frac{1}{2}}(x) + J'_{\frac{1}{2}}(x) \\
&= \frac{1}{\sqrt{2\pi x^3}} \sin x + \sqrt{\frac{2}{\pi}} \left(-\frac{1}{2} x^{-\frac{3}{2}} \sin x + x^{-\frac{1}{2}} \cos x \right) \\
&= \sqrt{\frac{2}{\pi x}} \cos x
\end{aligned}$$

17. Recall that

$$f(x) = \sum_n c_n J_m(k_n x), \; c_n = 2(J_{m+1}(k_n a))^{-2} \int_0^a dx x J_m(k_n x) f(x)$$

where $k_1 < k_2 < \cdots$ satisfy $J_m(k_i a) = 0$. The asymptotic forms of Bessel functions are given by

$$J_m(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4} \right) \text{ as } x \to \infty.$$

$$\Rightarrow (a J_{m+1}(k_n a))^2 \sim \frac{2a}{\pi k_n} \cos^2 (k_n a - \cdots) \sim \frac{2a}{\pi k_n}$$

$$\Rightarrow c_n \sim \frac{\pi k_n}{a} \int_0^a dx x J_m(k_n x) f(x)$$

$$\begin{aligned}
\Rightarrow f(x) &= \sum_{n=1}^{\infty} \left(\frac{\pi k_n}{a} \int_0^a dx' x' J_m(k_n x') f(x') \right) J_m(k_n x) \\
&\approx \sum_{n=1}^{\infty} \left(\Delta k_n k_n \int_0^a dx' x' J_m(k_n x') f(x') \right) J_m(k_n x) \qquad \left(\Delta k_n \sim \frac{\pi}{a} \right) \\
&\xrightarrow{a \to \infty} \int_0^{\infty} dy y \left(\int_0^{\infty} dx' x' J_m(x' y) f(x') \right) J_m(xy)
\end{aligned}$$

$$\therefore f(x) = \int_0^{\infty} dy y J_m(xy) g(y), \; g(y) = \int_0^{\infty} dx x J_m(xy) f(x)$$

18.

$$\begin{aligned}
Y_n(x) &= \lim_{m \to n} \frac{\cos m\pi J_m(x) - J_{-m}(x)}{\sin m\pi} \\
&= \lim_{m \to n} \frac{-\pi \sin m\pi J_m(x) + \cos m\pi \frac{\partial J_m}{\partial m} - \frac{\partial J_{-m}}{\partial m}}{\pi \cos m\pi} \qquad \text{(L'Hôspital's rule)} \\
&= \frac{1}{\pi} \left(\frac{\partial J_n}{\partial n} - (-1)^n \frac{\partial J_{-n}}{\partial n} \right)
\end{aligned}$$

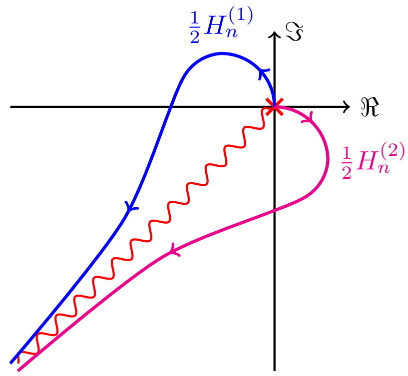


Figure 20: Contour for Problem 7–19.

19. Notice that the region of validity for a specific contour for the Hankel functions is the half-plane in $\mathbb{C}$ perpendicular to the branch cut. Thus, we make the cut along $\arg z = -\frac{3\pi}{4}$, as shown in Figure 20.

20. The differential equation is given by

$$\begin{aligned}0 &= (\beta x^\gamma)^2 \frac{d^2}{d(\beta x^\gamma)^2}\left(x^{-\alpha}y\right) - 2\left(\beta x^\gamma\right)\frac{d}{d\left(\beta x^\gamma\right)}\left(x^{-\alpha}y\right)\\ &\quad + \left((\beta x^\gamma)^2 - m^2\right)\left(x^{-\alpha}y\right).\end{aligned}$$

$$\begin{aligned}\frac{d}{d\left(\beta x^\gamma\right)}\left(x^{-\alpha}y\right) &= \frac{1}{\beta\gamma x^{\gamma-1}}\frac{d}{dx}\left(x^{-\alpha}y\right)\\ &= \frac{1}{\beta\gamma x^{\gamma-1}}\left(x^{-\alpha}y' - \alpha x^{-\alpha-1}y\right)\\ &= \frac{1}{\beta\gamma}\left(x^{-\alpha-\gamma+1}y' - \alpha x^{-\alpha-\gamma}y\right)\end{aligned}$$

$$\begin{aligned}&\Rightarrow \frac{d^2}{d(\beta x^\gamma)^2}\left(x^{-\alpha}y\right)\\ &= \frac{1}{\beta\gamma x^{\gamma-1}}\frac{d}{dx}\left(\frac{1}{\beta\gamma}\left(x^{-\alpha-\gamma+1}y' - \alpha x^{-\alpha-\gamma}y\right)\right)\\ &= \frac{1}{\beta^2\gamma^2}\left(x^{-\alpha-2\gamma+2}y'' - (2\alpha+\gamma-1)x^{-\alpha-2\gamma+1}y' + \alpha(\alpha+\gamma)x^{-\alpha-2\gamma}y\right)\end{aligned}$$

$$
\begin{aligned}
\Rightarrow 0 &= \frac{1}{\gamma^2 x^\alpha}\left(x^2 y'' - (2\alpha+\gamma-1)xy' + \alpha(\alpha+\gamma)y\right) \\
&\quad - \frac{2}{\gamma x^\alpha}\left(xy' - \alpha y\right) \\
&\quad + \frac{\left(\beta x^\gamma\right)^2 - m^2}{x^\alpha} y \\
&= \frac{1}{\gamma^2 x^{-\alpha}}\left(x^2 y'' - (2\alpha-1)y' + \left(\beta^2\gamma^2 x^{2\gamma} + \alpha^2 - m^2\gamma^2\right)\right) \\
&\therefore x^2 y'' - (2\alpha-1)y' + \left(\beta^2\gamma^2 x^{2\gamma} + \alpha^2 - m^2\gamma^2\right) = 0
\end{aligned}
$$

As a sanity check, substituting $\alpha = 0,\ \beta = \gamma = 1$ yields the original Bessel equation.

The equation

$$y'' + x^2 y = 0$$

can be obtained via the substitution $\alpha = \beta = \frac{1}{2},\ \gamma = 2, m = \frac{1}{4}$.

$$\therefore y = C_1\sqrt{x} J_{\frac{1}{4}}\left(\frac{1}{2}x^2\right) + C_2\sqrt{x} J_{-\frac{1}{4}}\left(\frac{1}{2}x^2\right)$$

21. The given arguments seem flawed, as the differential equation has regular singularities at 0, ± 1, and ∞. Hence, we must remove one of the singularities before we may employ hypergeometric functions. Let $u := z^2$.

$$\frac{d}{dz} = 2\sqrt{u}\frac{d}{du}$$

$$\Rightarrow \frac{d^2 y}{du^2} + \frac{1}{u}\frac{dy}{du} + \frac{1}{64u^2(u-1)} y = 0$$

Suppose $y = u^\alpha v$ with v analytic near the origin.

$$\Rightarrow \frac{d^2 v}{du^2} + \frac{5}{4u}\frac{dv}{du} + \frac{1}{u^2}\left(\alpha^2 + \frac{1}{64(u-1)}\right) v = 0$$

Thus, the analyticity of v implies

$$\left.\left(\alpha^2 + \frac{1}{64(u-1)}\right)\right|_{u=0} \Rightarrow \alpha = \pm\frac{1}{8}.$$

$$\therefore y = u^{\frac{1}{8}} v \Rightarrow u(1-u)\frac{d^2 v}{du^2} + \left(\frac{5}{4} - \frac{5}{4}u\right)\frac{dv}{du} - \frac{1}{64} v = 0$$

This, finally, is the hypergeometric equation with $a = b = \frac{1}{8},\ c = \frac{5}{4}$.

$$\Rightarrow v = C_1\, {}_2F_1\left(\frac{1}{8}, \frac{1}{8}; \frac{5}{4}; u\right) + C_2 u^{-\frac{1}{4}}\, {}_2F_1\left(-\frac{1}{8}, -\frac{1}{8}; \frac{3}{4}; u\right)$$

$$\therefore y = C_1 x^{\frac{1}{4}}\, {}_2F_1\left(\frac{1}{8}, \frac{1}{8}; \frac{5}{4}; x^2\right) + C_2 x^{-\frac{1}{4}}\, {}_2F_1\left(-\frac{1}{8}, -\frac{1}{8}; \frac{3}{4}; x^2\right)$$

22. (a)

$$
\begin{aligned}
{}_2F_1(1,\alpha;2;z) &= \frac{\Gamma(2)}{\Gamma(1)\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\Gamma(1+n)\Gamma(\alpha+n)}{\Gamma(2+n)}\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!(n+1)}z^n \\
&= \sum_{n=0}^{\infty}(-1)^n\binom{\alpha+n}{n}\frac{z^n}{n+1} \\
&= \frac{1}{z}\int_0^z dt\sum_{n=0}^{\infty}\binom{\alpha+n}{n}(-t)^n \\
&= \frac{1}{z}\int_0^z \frac{dt}{(1-t)^{\alpha}} \\
&= \frac{1-(1-z)^{1-\alpha}}{(1-\alpha)z}
\end{aligned}
$$

(b)

$$
\begin{aligned}
{}_2F_1(1,1;2;z) &= \lim_{\alpha\to 1}\frac{1-(1-z)^{1-\alpha}}{(1-\alpha)z} \\
&= \lim_{\beta\to 0}\frac{1-(1-z)^{\beta}}{\beta z} && (\beta := 1-\alpha) \\
&= \lim_{\beta\to 0}\frac{(1-z)^{\beta}\ln(1-z)}{z} && (\text{L'Hôsptial's rule}) \\
&= \frac{\ln(1-z)}{z}
\end{aligned}
$$

23. We transpose the two regular singularities 0 and 1 with the transformation $u = 1-z$. This transforms the hypergeometric equation as

$$
u(1-u)\frac{d^2y}{du^2} + (-c-(a+b+1)u)\frac{dy}{du} - aby = 0.
$$

$$
\begin{aligned}
\Rightarrow y &= C_1\,{}_2F_1(a,b;-c;u) \\
&\quad + C_2u^{1+c}\,{}_2F_1(1+a+c,1+b+c;2+c;u) \\
&= C_1\,{}_2F_1(a,b;-c;1-z) \\
&\quad + C_2(1-z)^{1+c}\,{}_2F_1(1+a+c,1+b+c;2+c;1-z)
\end{aligned}
$$

24. Parallelling the textbook, suppose the differential equation is

$$
y'' + \frac{\mathrm{poly}_n(z)}{(z-\xi)(x-\eta)}y' + \frac{\mathrm{poly}_m(z)}{(z-\xi)^2(z-\eta)^2}y = 0
$$

where n and m are the degrees of the respective polynomials. The variable change $u = \frac{1}{z}$ yields

$$\frac{d^2y}{du^2} + \left(\frac{2}{u} - \frac{\mathrm{poly}_n\left(u^{-1}\right)}{(1-\xi u)(1-\eta u)}\right)\frac{dy}{du} + \frac{\mathrm{poly}_m\left(u^{-1}\right)}{(1-\xi u)^2(1-\eta u)^2}y = 0.$$

For $z = \infty \Leftrightarrow u = 0$ to be ordinary, we must have $n = 1$ and $m = 0$.

$$\Rightarrow y'' + \left(\frac{A}{z-\xi} + \frac{B}{z-\eta}\right)y' + \frac{C}{(z-\xi)^2(z-\eta)^2}y = 0$$

$$\frac{d^2y}{du^2} + \frac{1}{u}\left(2 - \frac{A}{1-\xi u} - \frac{B}{1-\eta u}\right)\frac{dy}{du} + \frac{C}{(1-\xi u)^2(1-\eta u)^2}y = 0$$

$u = 0$ being an ordinary point further demands that $A + B = 2$.

Now, suppose $y \sim (z-\xi)^\alpha$ near $z = \xi$.

$$0 = \alpha(\alpha-1) + A\alpha + \frac{C}{(\xi-\eta)^2} = \alpha^2 - (A-1)\alpha + \frac{C}{(\xi-\eta)^2}$$

$$\Rightarrow \alpha_1 + \alpha_2 = A - 1,\ \alpha_1\alpha_2 = \frac{C}{(\xi-\eta)^2}$$

Similar analysis for $y \sim (z-\eta)^\beta$ near $z = \eta$ yields

$$\beta_1 + \beta_2 = 1 - B,\ \beta_1\beta_2 = \frac{C}{(\xi-\eta)^2}.$$

Therefore, eliminating A, B, and C, we get the most general form

$$y'' + \left(\frac{1-\alpha_1-\alpha_2}{z-\xi} + \frac{1-\beta_1-\beta_2}{z-\eta}\right)y' + \frac{(\xi-\eta)^2\alpha_1\alpha_2}{(z-\xi)^2(z-\eta)^2}y = 0$$

with conditions

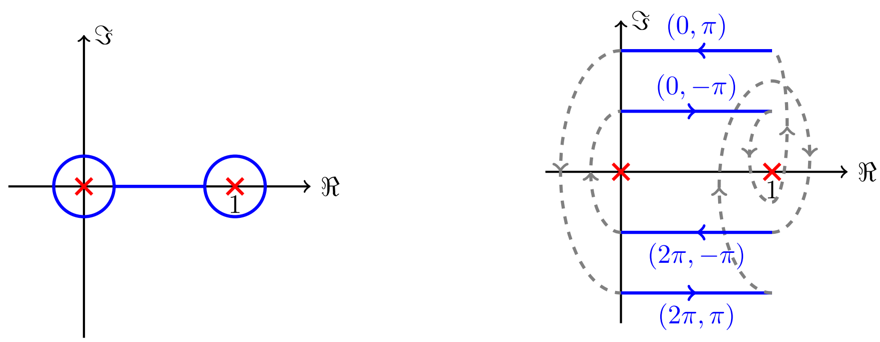
$$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = 0,\ \alpha_1\alpha_2 = \beta_1\beta_2.$$

The given transformation of exponentials leaves the sum of them fixed. As for the product,

$$\begin{aligned}(\alpha_1+\lambda)(\alpha_2+\lambda) &= \alpha_1\alpha_2 + \lambda(\alpha_1+\alpha_2) + \lambda^2\\ &= \beta_1\beta_2 - \lambda(\beta_1+\beta_2) + \lambda^2\\ &= (\beta_1-\lambda)(\beta_2-\lambda).\end{aligned}$$

Therefore, the given transformation leaves the condition satisfied.

25. It might seem at first glance that even properly defining this contour is challenging. However, two observations make this evaluation possible:



(a) (b)

Figure 21: Contour for Problem 7–25.

(a) The contour can be constrained to the "dumbbell" shape, that is, two circles around the two poles along with the segment of the real axis connecting them. (See Figure 21a.)

(b) As we make the circles as smaller and smaller, the contributions of such segments to the integral vanish.

Therefore, we can effectively only consider the integral on the segment $[0, 1]$; we only need to take note of the phases of z and $z - 1$ while doing so. Spreading out the overlapped contours, we obtain Figure 21b. Please do verify that this is homotopic to the Pochhammer contour in the problem. Also recorded in the figure are the phases $(\arg z, \arg(z-1))$ along the straight sections. Again, the gray curving sections will be rendered negligible and we imagine the straight contours exactly on the real axis.

$$
\begin{aligned}
\therefore I &= \left(-e^{i\pi b} + e^{i\pi(2a+b)} - e^{i\pi(2a-b)} + e^{-i\pi b}\right) \int_0^1 dx x^a (1-x)^b \\
&= -4e^{i\pi a} \sin \pi a \sin \pi b \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}
\end{aligned}
$$

26.

$$
y'' + \left(\frac{1}{z} + \frac{1}{z-1}\right) y' + \frac{1}{z(z-1)(z-2)} \left(\frac{-\frac{2}{9}}{z}\right) y = 0
$$

This equation is the hypergeometric equation[24][25] with the regular singularities at $z = 0, 1, 2$ and corresponding powers

$$
\alpha_1 = -\frac{1}{3},\ \alpha_2 = \frac{1}{3},\ \beta_1 = \beta_2 = \gamma_1 = 0,\ \gamma_2 = 1.
$$

[24] https://en.wikipedia.org/wiki/Riemann%27s_differential_equation

[25] Also known as Eq. (7–80) of the textbook, Riemann's equation, or Papperitz equation.

$$\Rightarrow y = P\begin{Bmatrix} 0 & 1 & 2 & \\ -\frac{1}{3} & 0 & 0 & z \\ \frac{1}{3} & 0 & 1 & \end{Bmatrix} = \left(\frac{z-2}{z}\right)^{\frac{1}{3}} P\begin{Bmatrix} 0 & 1 & 2 & \\ 0 & 0 & -\frac{1}{3} & z \\ \frac{2}{3} & 0 & \frac{2}{3} & \end{Bmatrix}$$

We wish to transform z such that 0 and 1 remain fixed, while 2 transforms to infinity. This is achieved by the homographic transformation $u = \frac{z}{2-z}$.

$$\Rightarrow y = (-u)^{-\frac{1}{3}} P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & -\frac{1}{3} & u \\ \frac{2}{3} & 0 & \frac{2}{3} & \end{Bmatrix}$$

This is the P symbol corresponding to the hypergeometric equation with $a = -\frac{1}{3}$, $b = \frac{2}{3}$, and $c = \frac{1}{3}$. Therefore, substituting u back in, we obtain the final solution

$$\begin{aligned} y = {} & C_1\left(\frac{z}{2-z}\right)^{\frac{1}{3}} {}_2F_1\left(-\frac{1}{3}, \frac{2}{3}; \frac{1}{3}; \frac{z}{2-z}\right) \\ & + C_2\left(\frac{2-z}{z}\right)^{\frac{1}{3}} {}_2F_1\left(\frac{1}{3}, \frac{4}{3}; \frac{5}{3}; \frac{z}{2-z}\right). \end{aligned}$$

27. Dividing by $z^2\left(z^2-1\right)^2$ and taking partial fractions, we obtain

$$\begin{aligned} 0 = {} & y'' + \left(\frac{1}{z} + \frac{\frac{1}{2}}{z+1} + \frac{\frac{1}{2}}{z-1}\right) y' \\ & + \frac{1}{z(z+1)(z-1)}\left(\frac{\alpha^2}{z} + \frac{\frac{1}{8} - 2\alpha^2}{z+1} + \frac{\frac{1}{8} - 2\alpha^2}{z-1}\right) y. \end{aligned}$$

This is the hypergeometric equation with $-\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \gamma_1 = \frac{1}{4} + \alpha$, and $\beta_2 = \gamma_2 = \frac{1}{4} - \alpha$.

$$\begin{aligned} \therefore y &= P\begin{Bmatrix} 0 & 1 & -1 & \\ -\alpha & \frac{1}{4}+\alpha & \frac{1}{4}+\alpha & z \\ \alpha & \frac{1}{4}-\alpha & \frac{1}{4}-\alpha & \end{Bmatrix} \\ &= \left(\frac{z}{z-1}\right)^{\alpha} P\begin{Bmatrix} 0 & 1 & -1 & \\ 0 & \frac{1}{4} & \frac{1}{4}+\alpha & z \\ 2\alpha & \frac{1}{4}-2\alpha & \frac{1}{4}-\alpha & \end{Bmatrix} \\ &= \frac{z^\alpha}{(z-1)^{\alpha-\frac{1}{4}}(z+1)^{\frac{1}{4}}} P\begin{Bmatrix} 0 & 1 & -1 & \\ 0 & 0 & \frac{1}{2}+\alpha & z \\ 2\alpha & -2\alpha & \frac{1}{2}-\alpha & \end{Bmatrix} \\ &= \frac{z^\alpha}{(z-1)^{\alpha-\frac{1}{4}}(z+1)^{\frac{1}{4}}} P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \frac{1}{2}+\alpha & \frac{2z}{z+1} \\ 2\alpha & -2\alpha & \frac{1}{2}-\alpha & \end{Bmatrix} \\ &= \frac{z^\alpha}{(z-1)^{\alpha-\frac{1}{4}}(z+1)^{\frac{1}{4}}}\left(C_1\, {}_2F_1\left(\alpha - \frac{1}{4}, \alpha + \frac{1}{4}; 1-2\alpha; \frac{2z}{z+1}\right)\right. \\ &\qquad \left. + C_2\left(\frac{2z}{z+1}\right)^{2\alpha} {}_2F_1\left(\frac{1}{2}+3\alpha, \frac{1}{2}+\alpha; 1+2\alpha; \left(\frac{2z}{z+1}\right)\right)\right) \end{aligned}$$

28. Following the discussion in Problem 7–10, the Wronskian must satisfy

$$W'(z) = -\frac{c-z}{z}W(z) = \left(-\frac{c}{z}+1\right)W(z).$$

$$\Rightarrow W(z) = Cz^{-c}e^{-z}$$

$$\begin{aligned}\Rightarrow C = \lim_{z\to 0} z^c \Big(&{}_1F_1(a;c;z)\left(z^{1-c}\,{}_1F_1(1+a-c;2-c;z)\right)' \\ &- {}_1F_1'(a;c;z)\cdot z^{1-c}\,{}_1F_1(1+a-c;2-c;z)\Big) \\ = 1-c&\end{aligned}$$

$$\therefore W(z) = (1-c)z^{-c}e^{z}$$

29.

$$\begin{aligned}\int_0^\infty dxe^{-sx}\,{}_1F_1(a;c;x) &= \frac{\Gamma(c)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(c+n)}\frac{1}{n!}\int_0^\infty dxx^n e^{-sx} \\ &= \frac{\Gamma(c)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(c+n)}s^{-(n+1)} \\ &= \frac{1}{s}\,{}_1F_1\left(a;c;s^{-1}\right)\end{aligned}$$

30.

$$\begin{aligned}0 &= x\left(e^{-\lambda x}y\right)'' + (c-x)\left(e^{-\lambda x}y\right)' - a\left(e^{-\lambda x}y\right) \\ &= xe^{-\lambda x}\left(\left(y''-2\lambda y'+\lambda^2 y\right) + \left(\frac{c}{x}-1\right)(y'-\lambda y) - \frac{a}{x}y\right)\end{aligned}$$

$$\therefore y'' + \left(\frac{c}{x}-2\lambda-1\right)y' + \left(-\frac{\lambda c+a}{x}+\lambda(\lambda+1)\right)y = 0$$

The given differential equation is a special case of this one with $c=\frac{3}{2}$, $\lambda=-\frac{1}{3}$, and $a=\frac{1}{2}-E$.

$$\Rightarrow y = e^{-\frac{x}{3}}\left(C_1\,{}_1F_1\left(\frac{1}{2}-E;\frac{3}{2};x\right) + C_2x^{-\frac{1}{2}}\,{}_1F_1\left(\frac{1}{2}-E;\frac{1}{2};x\right)\right)$$

The second solution diverges as $x\to 0+$; the first solution converges as $x\to\infty$ iff the hypergeometric series terminates.

$$\therefore E_n = n+\frac{1}{2}\ (n=0,1,2,\dots)$$

(Does this not remind you of the energies of a harmonic oscillator?)

31. We use the recursion formulae derived from Eq. (7–129):

$$B_3 = \left(\frac{2}{\beta}(\alpha - 1) + 1\right) B_1,\ B_{n+2} = \frac{2}{\beta}\left(n^2 - \alpha\right) B_n - B_{n-2}$$

as the $se_{2n+1}(x)$ solutions contain B_{2n+1} terms.

$$\begin{aligned}
\frac{B_n}{B_{n-2}} &= \cfrac{1}{\frac{2}{\beta}(n^2 - \alpha^2) - \frac{B_{n+2}}{B_n}} \\
&= \cfrac{-\beta}{2(\alpha^2 - n^2) + \beta\frac{B_{n+2}}{B_n}} \\
&= \cfrac{-\beta}{2(\alpha^2 - n^2) - \cfrac{\beta^2}{2\left(\alpha^2 - (n+1)^2\right) - \cfrac{\beta^2}{\ddots}}}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{B_3}{B_1} &= \frac{2}{\beta}(1 - \alpha) + 1 \\
&= \cfrac{-\beta}{2(\alpha^2 - 9) - \cfrac{\beta^2}{2(\alpha^2 - 16) - \cfrac{\beta^2}{\ddots}}}
\end{aligned}$$

$$\therefore \alpha(\beta) = 1 + \frac{\beta}{2} + \cfrac{\beta^2/2}{2(\alpha^2 - 9) - \cfrac{\beta^2}{2(\alpha^2 - 25) - \cfrac{\beta^2}{\ddots}}}$$

$se_1(x)$ has $\alpha = 1$ for $\beta = 0$.

$$\Rightarrow \alpha \approx 1 + \frac{\beta}{2}$$

$$\Rightarrow \alpha \approx 1 + \frac{\beta}{2} + \frac{\beta^2/2}{2(1-9)} = 1 + \frac{\beta}{2} - \frac{\beta^2}{16}$$

$$\Rightarrow \alpha \approx 1 + \frac{\beta}{2} + \cfrac{\beta^2/2}{2\left(\left(1 + \frac{\beta}{2}\right)^2 - 9\right) - \cfrac{\beta^2}{2(1-25)}} \approx 1 + \frac{\beta}{2} - \frac{\beta^2}{32} - \frac{\beta^3}{256}$$

32. Shifting the time by $\frac{\pi}{2\omega}$, we may replace sin with cos without loss of generality. Defining $\alpha := \frac{k_0}{m}$ yields the differential equation

$$\ddot{x} + \alpha \cos \omega t x = 0.$$

As before, we assume that $x(t)$ is periodic at the threshold frequency and consider its Fourier transform

$$x(t) = \frac{A_0}{2} + \sum_n \left(A_n \cos \omega_0 t + B_n \sin \omega_0 t \right).$$

This yields the recursion formulae

$$A_1 = 0,\ -n^2\omega_0^2 A_n + \frac{\alpha}{2}\left(A_{n+1} + A_{n-1}\right) = 0,$$

$$-n^2\omega_0^2 B_n + \frac{\alpha}{2}\left(B_{n+1} + B_{n-1}\right) = 0.$$

We only focus on the A_n formula.

$$\frac{A_{n-1}}{A_n} + \frac{A_{n+1}}{A_n} = \frac{2n^2\omega_0^2}{\alpha}$$

$$\Rightarrow \frac{A_{n-1}}{A_n} = \frac{2n^2\omega_0^2}{\alpha} - \cfrac{1}{\cfrac{2(n+1)^2\omega_0^2}{\alpha} - \cfrac{1}{\cfrac{2(n+2)^2\omega_0^2}{\alpha} - \cfrac{1}{\ddots}}}$$

$$\therefore 2 \cdot 2^2 \cdot \frac{\omega_0^2}{\alpha} - \cfrac{1}{2 \cdot 3^2 \cdot \cfrac{\omega_0^2}{\alpha} - \cfrac{1}{\ddots}} = 0$$

A short Python code to calculate $\frac{\omega_0^2}{\alpha}$ is provided in Listing 1. It repeatedly bisects the interval between 10^{-9} and 1. A tolerable cut-off point for the continued fraction is found at 31738, and we obtain $\frac{\omega_0^2}{\alpha} \approx 0.0939027\ldots$.

$$\therefore \omega_0 \sqrt{\frac{m}{k_0}} \approx 0.306\ldots$$

33. The simple nature of $f(x)$ prompts us not to blindly make a Fourier expansion. Indeed, y is linear when $f(x) = 0$ and sinusoidal when $f(x) = C$ with angular frequency $\sqrt{C}$. One possible method is to define the general solution

$$y(x) = \begin{cases} A_n x + B_n & (2n\pi < x < (2n+1)\pi) \\ D_n \cos \sqrt{C}x + E_n \sin \sqrt{C}x & ((2n+1)\pi < x < 2(n+1)\pi) \end{cases}$$

```python
EPS: float = 1e-9


def cont_frac(x: float, n: int) -> float:
    res: float = 2 * n * n * x
    for i in range(n - 1, 1, -1):
        res = 2 * i * i * x - 1 / res
    return res


def find_tolerable_n() -> int:
    n: int = 2
    prev: float = cont_frac(EPS, n)
    while True:
        n += 1
        curr: float = cont_frac(EPS, n)
        ##        if n % 1000 == 0:
        ##            print(f"{n} {abs(curr - prev)}")
        if abs(curr - prev) < EPS:
            break
        prev = curr
    return n


def find_root(n: int, l: float, r: float) -> float:
    l_val: float = cont_frac(l, n)
    r_val: float = cont_frac(r, n)
    assert l_val * r_val < 0

    mid: float = (l + r) / 2
    mid_val: float = cont_frac(mid, n)
    if abs(mid_val) < 0:
        return mid
    if l_val * mid_val < 0:
        return find_root(n, l, mid)
    return find_root(n, mid, r)


if __name__ == "__main__":
    n: int = find_tolerable_n()
    print(n)
    x = find_root(n, EPS, 1)
    print(f"{x} {cont_frac(x, n)}")
```

Listing 1: Python script for finding the value of $\frac{\omega_0^2}{\alpha}$.

and find recursion formulae for the coefficients. However, after days of making algebraic mistakes and starting over and over again, we may take a simpler route.

Suppose $y_1(x)$ and $y_2(x)$ are two independent solutions. As noted in the textbook, we may write

$$\begin{pmatrix} y_1(x+2\pi) \\ y_2(x+2\pi) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

for some coefficients $A_{ij}(i, j = 1, 2)$. Then, $e^{2\pi\mu}$ is an eigenvalue of the coefficient matrix. That is, we only need to find the coefficient matrix even if y_1 and y_2 are not exactly found.

As such, we arbitrarily assume $y_1(x) = 1$ and $y_2(x) = x$ for $0 < x < \pi$. Using the continuity of y and y', we extend these solutions up to $2\pi < x < 3\pi$ and extract the coefficient matrix.

$$y_1(x) = \begin{cases} 1 & (0 < x < \pi) \\ \cos\left(\sqrt{C}(x-\pi)\right) & (\pi < x < 2\pi) \\ -\sqrt{C}\sin\left(\pi\sqrt{C}\right)(x-2\pi) + \cos\left(\pi\sqrt{C}\right) & (2\pi < x < 3\pi) \end{cases}$$

$$y_2(x) = \begin{cases} x-\pi & (0 < x < \pi) \\ \frac{1}{\sqrt{C}}\sin\left(\sqrt{C}(x-\pi)\right) & (\pi < x < 2\pi) \\ \cos\left(\pi\sqrt{C}\right)(x-2\pi) + \frac{1}{\sqrt{C}}\sin\left(\pi\sqrt{C}\right) & (2\pi < x < 3\pi) \end{cases}$$

$$\Rightarrow (A_{ij}) = \begin{pmatrix} -\sqrt{C}\sin\left(\pi\sqrt{C}\right) + \cos\left(\pi\sqrt{C}\right) & -\sqrt{C}\sin\left(\pi\sqrt{C}\right) \\ \pi\cos\left(\pi\sqrt{C}\right) + \frac{1}{\sqrt{C}}\sin\left(\pi\sqrt{C}\right) & \cos\left(\pi\sqrt{C}\right) \end{pmatrix}$$

$$\begin{aligned} \therefore \mu = \frac{1}{2\pi}\ln\Bigg(&\cos\left(\pi\sqrt{C}\right) - \frac{\pi\sqrt{C}}{2}\sin\left(\pi\sqrt{C}\right) \\ &\pm\sqrt{-\pi^2\sqrt{C}\cos^2\left(\pi\sqrt{C}\right) - \pi\sin\left(\pi\sqrt{C}\right)\cos\left(\pi\sqrt{C}\right) + \frac{\pi^2 C}{4}\sin^2\left(\pi\sqrt{C}\right)}\Bigg) \end{aligned}$$

A solution with period 2π is possible when either one of μ equals 0; a periodic solution exists when either one is a rational multiple of $2i\pi$.

34. 😫 Whoops! This problem is under construction ($\because$ author's skill issue). Please wait or consider contributing!

35. (a) We compare the integrals obtained by Figure 22a and 22b.

$$x_1 = r + 2,\ x_2 = r - s = 2r - x_1$$

$$r = \int_0^1 \frac{dy}{\sqrt{1-y^2}} = \frac{\pi}{2}$$

$$\therefore \sin(x) = \sin(\pi - x)$$

(b)

$$x_1 = r + s + t,\ x_2 = r - s + t = x_1 - 2s$$

$$s = \int_{-1}^1 \frac{dy}{\sqrt{1-y^2}} = \pi$$

$$\therefore \sin(x) = \sin(x - 2\pi)$$

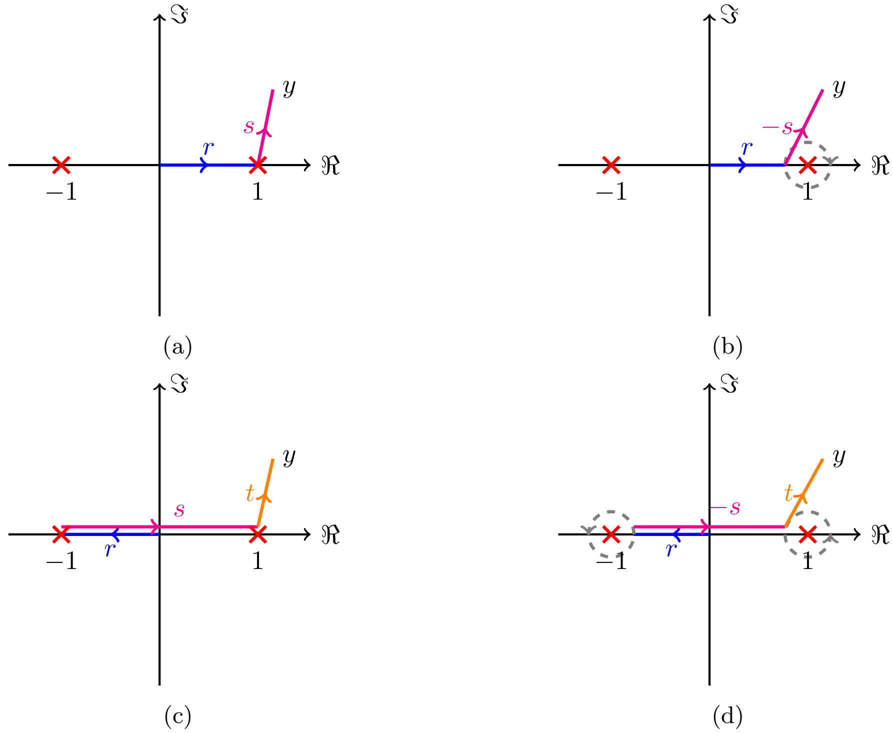


Figure 22: Results of conformal mappings used in Problem 5–7.

36. 😫 Whoops! This problem is under construction (∵ author's skill issue). Please wait or consider contributing![26]

37. (a) We define $\arg t = -\pi$ just under the branch cut and $\arg t = \pi$ just

[26]For the record, the given Airy equation can be obtained by

$$\alpha = \frac{1}{2},\ \beta = \frac{2}{3},\ \gamma = \frac{3}{2},\ m = \frac{1}{3}.$$

above.

$$
\begin{aligned}
\Rightarrow 2i\pi I_n &= \int_{-\infty}^{0} d\left(-r\right) e^{-r} \left(re^{-i\pi}\right)^{n+1} + \int_{0}^{-\infty} d\left(-r\right) e^{-r} \left(re^{i\pi}\right)^{n+1} \\
&= \left(e^{i(n+1)\pi} - e^{-i(n+1)\pi}\right) \int_0^\infty dr e^{-r} r^{-n-1} \\
&= 2i \sin((n+1)\pi)\Gamma(-n) \\
&= \frac{2i\pi}{\Gamma(n+2)\Gamma(-n-1)} \cdot \Gamma(-n) \\
&= \frac{2i\pi}{\Gamma(n+1)}
\end{aligned}
$$

$$
\therefore I_n = \frac{1}{\Gamma(n+1)}
$$

(b) 😫 Whoops! This problem is under construction ($\because$ author's skill issue). Please wait or consider contributing!

A Appendix

6. (This single problem had cost me months of my life, and here's the unsatisfying conclusion of that journey. I am fairly certain that there are a myriad of different possible answers for this problem with varying levels of sophistication, and possibly even more ways to derive said solutions. For my purposes, I have settled on a solution that is semi-straightforward with an answer that I can plug into almighty WolframAlpha.)

We will derive a procedure to (painstakingly) calculate the Laurent series of the given function. First, let us define some notations. Recall that if a function $f(z)$ is analytic on an annulus around a point z_0, then it can be expanded into a unique Laurent series

$$
\begin{aligned}
f(z) &= \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k \\
&= \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots .
\end{aligned}
$$

For Laurent series specifically centered around the origin, let us denote the Laurent coefficients as

$$
\left[z^k\right](f(z)) := a_k.
$$

Then, multiplying a monomial corresponds to

$$
\left[z^k\right]\left(z^l f(z)\right) = \left[z^{k-l}\right](f(z)).
$$

Also, the residue at the origin can be compactly expressed as

$$
\operatorname*{Res}_{z=0} f(z) = \left[z^{-1}\right](f(z)).
$$

The residue asked of us is given by

$$\begin{aligned}
\operatorname*{Res}_{z=\pi} z^2 e^{\frac{1}{\sin z}} &= \operatorname*{Res}_{z=0} (z+\pi)^2 e^{\frac{1}{\sin(z+\pi)}} \\
&= \left[z^{-1}\right] \left(\left(z^2 + 2\pi z + \pi^2\right) e^{-\frac{1}{\sin z}} \right) \\
&= \left[z^{-3}\right] \left(e^{-\frac{1}{\sin z}} \right) + 2\pi \left[z^{-2}\right] \left(e^{-\frac{1}{\sin z}} \right) + \pi^2 \left[z^{-1}\right] \left(e^{-\frac{1}{\sin z}} \right).
\end{aligned}$$

Thus, our job is reduced to calculating the Laurent coefficients of $e^{-\frac{1}{\sin z}}$. We expand the exponential into its Taylor series and obtain

$$\left[z^{-n}\right] \left(e^{-\frac{1}{\sin z}} \right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[z^{-n}\right] \left((\sin z)^{-k} \right).$$

Squaring, cubing, and so forth a power series (or even a Laurent series) is something achievable, but inverting a power series is a much more complicated task. We thus investigate a general method of calculating the inverse powers of a general series. Consider the following manipulation:

$$\begin{aligned}
\left[z^{-n}\right] \left((f(z))^{-k} \right) &= \left[z^{-1}\right] \left(z^{n-1} (f(z))^{-k} \right) \\
&= \left[z^{-1}\right] \left(\left(\frac{z^n}{n} \right)' (f(z))^{-k} \right) \\
&= \left[z^{-1}\right] \left(\left(\frac{z^n}{n} (f(z))^{-k} \right)' \right) - \left[z^{-1}\right] \left(\frac{z^n}{n} \left((f(z))^{-k} \right)' \right)
\end{aligned}$$

which comes mostly from playing around, but perhaps reasonable since the -1th[27] coefficient seems to be special in certain ways. In fact, it is so special that we can immediately see that the first term must be zero, regardless of whether $f(z)$ is analytic at the origin! This is because no derivative of a monomial yields the -1th power term.[28] Thus,

$$\begin{aligned}
\left[z^{-n}\right] \left((f(z))^{-k} \right) &= -\left[z^{-1}\right] \left(\frac{z^n}{n} \left((f(z))^{-k} \right)' \right) \\
&= \frac{k}{n} \left[z^{-1}\right] \left(z^n (f(z))^{-k-1} f'(z) \right).
\end{aligned}$$

Let $g(z) := z^n (f(z))^{-k-1} f'(z)$ and $g_{-1} := \left[z^{-1}\right] (g(z))$. The Laurent series of $g(z) - g_{-1} z^{-1}$ contains no z^{-1} term, and thus can be integrated to an analytic function; let us call it $G(z)$. Hence, $g(z) = G'(z) + g_{-1} z^{-1}$.

$$g(f^{-1}(z)) \left(f^{-1}(z)\right)' = \left(G(f^{-1}(z))\right)' + g_{-1} \frac{\left(f^{-1}(z)\right)'}{f^{-1}(z)}$$

[27]I had to search for whether it should be 'st' or 'th'; see `https://english.stackexchange.com/questions/326604/is-it-correct-to-say-1th-or-1st`.

[28]This can also be seen via $\operatorname{Res}_{z=0} f(z) = \frac{1}{2i\pi} \oint dz f'(z) = 0$ around the origin.

Again, $\big(G(f^{-1}(z))\big)'$ has zero residue. If the residue of $\frac{(f^{-1}(z))'}{f^{-1}(z)}$ could be easily calculated and is nonzero, then we get an elegant formula for g_{-1} as

$$\begin{aligned} g_{-1} &= \frac{\left[z^{-1}\right]\left(g(f^{-1}(z))\big(f^{-1}(z)\big)'\right)}{\left[z^{-1}\right]\left(\frac{(f^{-1}(z))'}{f^{-1}(z)}\right)} \\ &= \frac{\left[z^{-1}\right]\left(\big(f^{-1}(z)\big)^{n} z^{-k-1} f'\left(f^{-1}(z)\right)\left(f^{-1}(z)\right)'\right)}{\left[z^{-1}\right]\left(\frac{(f^{-1}(z))'}{f^{-1}(z)}\right)} \\ &= \frac{\left[z^{k}\right]\left(\big(f^{-1}(z)\big)^{n}\right)}{\left[z^{-1}\right]\left(\frac{(f^{-1}(z))'}{f^{-1}(z)}\right)} \end{aligned}$$

and consequently

$$\left[z^{-n}\right]\left((f(z))^{-k}\right) = \frac{k}{n}\frac{\left[z^{k}\right]\left(\big(f^{-1}(z)\big)^{n}\right)}{\left[z^{-1}\right]\left(\frac{(f^{-1}(z))'}{f^{-1}(z)}\right)}.$$

Now, *if* $f(z)$ is analytic at the origin *and* $f(0)=0$ *and* $f'(0)\neq 0$ (which just so happens to hold for $f(z)=\sin z$), then one could show that these same conditions hold for $f^{-1}(z)$ and consequently

$$\left[z^{-1}\right]\left(\frac{\big(f^{-1}(z)\big)'}{f^{-1}(z)}\right) = 1.$$

(This could be shown by substituting $n=-1$ for Problem A–4, since we did not require that n be positive anyway!) Therefore, we arrive at the pivotal formula

$$\left[z^{-n}\right]\left((f(z))^{-k}\right) = \frac{k}{n}\left[z^{k}\right]\left(\big(f^{-1}(z)\big)^{n}\right).$$

This is known in the maths world as the Lagrange inversion formula[29], and it is what we desperately needed to calculate the residue in question!

Having sufficiently patted ourselves on our respective backs after that long derivation, let us go back to the original question. We have essentially derived the formula

$$\left[z^{-n}\right]\left((\sin z)^{-k}\right) = \frac{k}{n}\left[z^{k}\right]\left((\arcsin z)^{n}\right).$$

(Note that I use arcsin in lieu of $\sin^{-1}$ as all these exponents are getting con-

[29] `https://en.wikipedia.org/wiki/Formal_power_series#The_Lagrange_inversion_formula`

fusing.) Let us substitute this formula into the previous equation:

$$\begin{aligned}
\left[z^{-n}\right]\left(e^{-\frac{1}{\sin z}}\right) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[z^{-n}\right]\left((\sin z)^{-k}\right) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \cdot \frac{k}{n} \left[z^k\right]\left((\arcsin z)^n\right) \\
&= \frac{1}{n} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} \left[z^k\right]\left((\arcsin z)^n\right).
\end{aligned}$$

As for the powers of $\arcsin z$, let us simply expand out the multiplications using the Taylor series of $\arcsin z$ only.

$$\begin{aligned}
\arcsin z &= \int_0^z \frac{du}{\sqrt{1-u^2}} \\
&= \int_0^z dt \sum_{t=0}^{\infty} \binom{-\frac{1}{2}}{t} u^{2t} \\
&= \sum_{t=0}^{\infty} \frac{(-1/2)(-3/2)\cdots(1/2-t)}{2^t t!} \cdot \frac{u^{2t+1}}{2t+1} \\
&= \sum_{t=0}^{\infty} \frac{(-1)^t (2t)!}{4^t (2t+1)(t!)^2} u^{2t+1}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \left[z^k\right]\left((\arcsin z)^n\right) &= \sum_{t_1+\cdots+t_k=n} \prod_{j=1}^{n} \left[z^{t_j}\right](\arcsin z) \\
&= \sum_{(2l_1+1)+\cdots+(2l_k+1)=n} \prod_j \left[z^{2l_j+1}\right](\arcsin z) \\
&= \sum_{2\sum_j l_j+k=n} \left(-\frac{1}{4}\right)^{\sum_j l_j} \prod_j \left(\frac{(2l_j)!}{(2l_j+1)(l_j!)^2}\right)
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow \left[z^{-n}\right]\left(e^{-\frac{1}{\sin z}}\right) &= \frac{1}{n}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k-1)!}\left[z^k\right]\left((\arcsin z)^n\right) \\
&= \frac{1}{n}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k-1)!} \\
&\quad \times \sum_{2\sum_j l_j + k = n}\left(-\frac{1}{4}\right)^{\sum_j l_j}\prod_j\left(\frac{(2l_j)!}{(2l_j+1)(l_j!)^2}\right) \\
&= \frac{1}{n}\sum_{l_1=0}^{\infty}\cdots\sum_{l_n=0}^{\infty}\frac{(-1)^{n-2\sum_j l_j}}{\left(n-1+2\sum_j l_j\right)!} \\
&\quad \times \left(-\frac{1}{4}\right)^{\sum_j l_j}\prod_j\left(\frac{(2l_j)!}{(2l_j+1)(l_j!)^2}\right) \\
&= \frac{1}{n}\sum_{l_1=0}^{\infty}\cdots\sum_{l_n=0}^{\infty}\frac{(-1)^{\sum_j l_j + n}}{4^{\sum_j l_j}\left(n-1+2\sum_j l_j\right)!} \\
&\quad \times \prod_j\left(\frac{(2l_j)!}{(2l_j+1)(l_j!)^2}\right)
\end{aligned}
$$

We evaluate this formula for $n = 1, 2, 3$.

$$
\left[z^{-1}\right]\left(e^{-\frac{1}{\sin z}}\right) = \sum_{l_1=0}^{\infty}\frac{(-1)^{l_1+1}}{4^{l_1}(2l_1+1)(l_1!)^2}
$$

$$
\left[z^{-2}\right]\left(e^{-\frac{1}{\sin z}}\right) = \frac{1}{2}\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\frac{(-1)^{l_1+l_2}}{4^{l_1+l_2}(2l_1+2l_2+1)!}\frac{(2l_1)!}{(2l_1+1)(l_1!)^2}\frac{(2l_2)!}{(2l_2+1)(l_2!)^2}
$$

$$
\begin{aligned}
\left[z^{-3}\right]\left(e^{-\frac{1}{\sin z}}\right) &= \frac{1}{3}\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{l_3=0}^{\infty}\frac{(-1)^{l_1+l_2+l_3+1}}{4^{l_1+l_2+l_3}(2l_1+2l_2+2l_3+2)!} \\
&\quad \times \frac{(2l_1)!}{(2l_1+1)(l_1!)^2}\frac{(2l_2)!}{(2l_2+1)(l_2!)^2}\frac{(2l_3)!}{(2l_3+1)(l_3!)^2}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \operatorname*{Res}_{z=\pi} z^2 e^{\frac{1}{\sin z}} &= \sum_{l_1=0}^{\infty}\frac{(-1)^{l_1+1}(2l_1)!}{4^{l_1}(2l_1+1)(l_1!)^2}\left(\frac{\pi^2}{(2l_1)!}\right. \\
&\quad - \sum_{l_2=0}^{\infty}\frac{(-1)^{l_2}(2l_2)!}{4^{l_2}(2l_2+1)(l_2!)^2}\left(\frac{\pi}{(2l_1+2l_2+1)!}\right. \\
&\quad\quad \left.\left. - \sum_{l_3=0}^{\infty}\frac{(-1)^{l_3}(2l_3)!}{4^{l_3}(2l_3+1)(l_3!)^2}\cdot\frac{1}{3(2l_1+2l_2+2l_3+2)!}\right)\right)
\end{aligned}
$$

Concluding remarks:

(i) This result, while hard achieved on my part, is still pretty shitty; for one, it cannot be calculated via WolframAlpha alone.[30] Some points for improvement include:

- More compact expressions for the Taylor coefficients of $(\arcsin z)^k$
- A numerical evaluation of the residue
- Even a closed-form solution? (This sounds crazy, but $\left[z^{-1}\right]\left(e^{-\frac{1}{\sin z}}\right)$ *does* have a closed-form expression! It uses a hypergeometric function, somehow.)

And we all know that "We leave so-and-so to future work." translates to either "I give up." or "Give me more money."

(ii) I have been hinted at that this problem could also be solved via a contour integral method, where we try to evaluate

$$\oint_{|z|=1} dz z^2 e^{\frac{1}{\sin z}}$$

using some tabulated-function trickery. I'm too tired to pursue this path, but it is still another viable method.

(iii) The threefold condition given in the middle of this proof (f analytic and vanishing at the origin, while $f'(0) \neq 0$) may seem arbitrary; however, it would appear that these are the exact conditions such that $e^{\frac{1}{f(z)}}$ has a nonzero (or more precisely, not necessarily zero) residue. Coincidence?

[30] If any Mathematica wizards could numerically calculate this, I'm all ears.