My Solutions for "Modern Quantum Mechanics (3rd Edition)" by J. J. Sakurai

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0 Introduction

This document is an archive of my solutions to Sakurai's "Modern Quantum Mechanics" textbook. None of the solutions have been verified by anyone other than myself, whom I do not consider a reliable source. Hence, it is strongly advised against to use this solution for any application where accuracy matters, especially for assignments and any academic work. Please use other readily available solutions out in the dark sides of the internet.

1 Chapter 1

1. We first recall that the average speed of a particle in a system with temperature T is given by

$$\bar{v} = \sqrt{\frac{8k_B T}{\pi m_{Ag}}}.$$

Suppose a silver atom moving at this speed passes through a region of length l_1 where it is accelerated upwards by a, followed by a region of length l_2 where it moves at constant velocity. The total upwards displacement is given by

$$\begin{aligned}\Delta z &= \frac{1}{2}a_z\left(\frac{l_1}{\bar{v}}\right)^2 + a_z\left(\frac{l_1}{\bar{v}}\right)\cdot\frac{l_2}{\bar{v}} \\ &= \frac{a_z l_1(l_1+l_2)}{2\bar{v}^2}.\end{aligned}$$

The separation between the spin-up and spin-down beams is twice this value.

The silver atom's magnetic moment has value $\mu = 9.27\times 10^{-24}J/T$ and the vertical magnetic field changes by $\frac{\partial B_z}{\partial z} = 10T/m$; hence, the upward force is given by $F_z = \mu\frac{\partial B_z}{\partial z} = 9.27\times 10^{-23}N$.

$$
\begin{aligned}
2\Delta z &= \frac{a_z l_1(l_1+2l_2)}{\bar{v}^2} \\
&= l_1(l_1+2l_2)\cdot\frac{F_z}{m_{Ag}}\cdot\frac{\pi m_{Ag}}{8k_BT} \\
&= (1m)(3m)\cdot(9.27\times10^{-23}N)\cdot\frac{\pi}{8(1.37\times10^{-23}J/K)(1273K)} \\
&\approx 6.22mm.
\end{aligned}
$$

2.

$$
\begin{aligned}
[AB,CD] &= ABCD - CDAB \\
&= -ACDB + A(CB+BC)D - CDAB \\
&= -AC(DB+BD) + A\left\{C,B\right\}D - CDAB + ACDB \\
&= -AC\left\{D,B\right\} + A\left\{C,B\right\}D - C(DA+AD)B + CADB + ACDB \\
&= -AC\left\{D,B\right\} + A\left\{C,B\right\}D - C\left\{D,A\right\}B + (CA+AC)DB \\
&= -AC\left\{D,B\right\} + A\left\{C,B\right\}D - C\left\{D,A\right\}B + \left\{C,A\right\}DB
\end{aligned}
$$

3. (i) $A = S_x,\ B = S_y$

$$
\langle S_x\rangle = \langle S_x;+|\, S_x\, |S_x;+\rangle = \left\langle S_x;+\left|\,\frac{\hbar}{2}\,\right|S_x;+\right\rangle = \frac{\hbar}{2}
$$

$$
\left\langle S_x^2\right\rangle = \left\langle S_x;+\,\middle|\, S_x^2\,\middle|\, S_x;+\right\rangle = \left\langle S_x;+\left|\left(\frac{\hbar}{2}\right)^2\right|S_x;+\right\rangle = \frac{\hbar^2}{4}
$$

$$
\Rightarrow \left\langle(\Delta S_x)^2\right\rangle = \frac{\hbar^2}{4} - \left(\frac{\hbar}{2}\right)^2 = 0 \ \Rightarrow \left\langle(\Delta S_x)^2\right\rangle\left\langle(\Delta S_y)^2\right\rangle = 0
$$

$$
[S_x,S_y] = i\hbar S_z = \frac{i\hbar^2}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)
$$

$$
\begin{aligned}
\Rightarrow \langle[S_x,S_y]\rangle &= \left\langle S_x;+\left|\,\frac{i\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)\right|S_x;-\right\rangle \\
&= \frac{i\hbar^2}{2}\langle S_x;+|\left(\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\right) \\
&= 0
\end{aligned}
$$

$$
\therefore \left\langle(\Delta S_x)^2\right\rangle\left\langle(\Delta S_y)^2\right\rangle \geq \frac{1}{4}|\langle[S_x,S_y]\rangle|^2
$$

(ii) $A = S_z,\ B = S_y$

$$
\begin{aligned}
\langle S_z \rangle &= \left\langle S_x; + \left| \frac{\hbar}{2} \left(|+\rangle \langle +| - |-\rangle \langle -| \right) \right| S_x; + \right\rangle \\
&= \frac{\hbar}{2} \langle S_x; +| \left(\frac{1}{\sqrt{2}} (|+\rangle - |-\rangle) \right) \\
&= 0
\end{aligned}
$$

$$
\left\langle S_z^2 \right\rangle = \left\langle S_x; + \left| \frac{\hbar^2}{4} \left(|+\rangle \langle +| + |-\rangle \langle -| \right) \right| S_x; + \right\rangle = \frac{\hbar^2}{4}
$$

$$
\Rightarrow \left\langle (\Delta S_x)^2 \right\rangle = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4}
$$

One can similarly find

$$
\left\langle (\Delta S_y)^2 \right\rangle = \frac{\hbar^2}{4} \Rightarrow \left\langle (\Delta S_z)^2 \right\rangle \left\langle (\Delta S_y)^2 \right\rangle = \frac{\hbar^4}{16}.
$$

$$
[S_z, S_y] = -i\hbar S_x
$$

$$
\Rightarrow \langle [S_z, S_y] \rangle = \langle S_x; + \,|\, (-i\hbar S_x) \,|\, S_x; - \rangle = -\frac{i\hbar^2}{2}
$$

$$
\Rightarrow \frac{1}{4} |\langle [S_x, S_y] \rangle|^2 = \frac{\hbar^4}{16}
$$

$$
\therefore \left\langle (\Delta S_x)^2 \right\rangle \left\langle (\Delta S_y)^2 \right\rangle \geq \frac{1}{4} |\langle [S_x, S_y] \rangle|^2
$$

4. We first observe that

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

$$
\Rightarrow X = \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix}
$$

(a)

$$
\sigma_1 X = \begin{pmatrix} a_1 + ia_2 & a_0 - a_3 \\ a_0 + a_3 & a_1 - ia_2 \end{pmatrix}
$$

$$
\sigma_2 X = \begin{pmatrix} -ia_1 + a_2 & -ia_0 + ia_3 \\ ia_0 + ia_3 & ia_1 + a_2 \end{pmatrix}
$$

$$
\sigma_3 X = \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ -a_1 - ia_2 & -a_0 + a_3 \end{pmatrix}
$$

$$
\therefore \mathrm{tr}(X) = 2a_0,\ \mathrm{tr}(\sigma_k X) = 2a_k
$$

(b)

$$\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & -i & 0\\ 0 & 1 & i & 0\\ 1 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ a_2\\ a_3\end{pmatrix}=\begin{pmatrix}X_{11}\\ X_{12}\\ X_{21}\\ X_{22}\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}a_0\\ a_1\\ a_2\\ a_3\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & 0 & 0 & \frac{1}{2}\\ 0 & \frac{1}{2} & \frac{1}{2} & 0\\ 0 & \frac{i}{2} & -\frac{i}{2} & 0\\ \frac{1}{2} & 0 & 0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}X_{11}\\ X_{12}\\ X_{21}\\ X_{22}\end{pmatrix}$$

$$\therefore a_0=\frac{1}{2}(X_{11}+X_{22}),\ a_1=\frac{1}{2}(X_{12}+X_{21}),\ a_2=\frac{i}{2}(X_{12}-X_{21}),\ a_3=\frac{1}{2}(X_{11}-X_{22})$$