

Hyperfocused Derivation of Hydrogen Energy Levels: From Scratch

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1 State Representation

1.1 The Hilbert Space

The defining postulate of quantum mechanics is that the state of a system is described with an element in a Hilbert space. Say the state of a system is a ket vector $|\alpha\rangle \in \mathcal{H}$. An observable property is represented by an operator $A : \mathcal{H} \rightarrow \mathcal{H}$. We presume the operator is normal, in which case it has a spectral decomposition

$$A = \int d\mu(\lambda) \lambda |\lambda\rangle \langle \lambda|.$$

A possible observed value is an eigenvalue λ of the operator, so all eigenvalues being real is equivalent to A being Hermitian, that is $A^\dagger = A$. When the system $|\alpha\rangle$ is observed with respect to the observable A , it randomly collapses to one of its eigenkets $|\lambda\rangle$ with probability $|\langle \lambda | \alpha \rangle|^2$. The total probability adding up to 1 can be stated as

$$1 = \int d\mu(\lambda) |\langle \lambda | \alpha \rangle|^2 = \langle \alpha | \left(\int d\mu(\lambda) |\lambda\rangle \langle \lambda| \right) |\alpha\rangle = \langle \alpha | \alpha \rangle.$$

This is succinctly referred to as $|\alpha\rangle$ being normalized.

1.2 Time Evolution

Suppose a system in state $|\alpha, t_0\rangle$ at time t_0 evolves to some other state $|\alpha, t_0; t\rangle$ at time t . We make four assumptions:

1. *Time evolution is linear:* $|\alpha, t_0; t\rangle = U(t_0, t)|\alpha, t_0\rangle$ for some linear operator $U(t_0, t) : \mathcal{H} \rightarrow \mathcal{H}$.
2. *Probability is conserved:* $\langle \alpha, t_0; t | \alpha, t_0; t \rangle = 1$. This is equivalent to $U^\dagger(t_0, t) = U^{-1}(t_0, t)$, which we call unitary.
3. *Time evolution composes:* $U(t_1, t_2)U(t_0, t_1) = U(t_0, t_2)$ for $t_0 < t_1 < t_2$.
4. *Time evolution is continuous:* $\lim_{t \rightarrow t_0^+} U(t_0, t) = 1$.

If we apply these to the infinitesimal evolution $U(t_0, t_0 + dt)$, we find that there is some Hermitian operator $H(t)$ such that

$$U(t_0, t_0 + dt) = 1 - \frac{iH(t_0)dt}{\hbar}.$$

This operator H is the generator of time evolution, which is also a description of the Hamiltonian/energy of a classical system. Therefore, we extend this terminology and say $H(t)$ is the operator corresponding to the Hamiltonian of the system.

1.3 3D Space

We want to describe the Hilbert space

$$\mathcal{H} = \text{span} \{ |\mathbf{r}'\rangle \mid \mathbf{r}' \in \mathbb{R}^3 \}.$$

An element of this space would take the form

$$|\alpha\rangle = \int d^3x' |\mathbf{r}'\rangle \langle \mathbf{r}' | \alpha \rangle := \int d^3x' |\mathbf{r}'\rangle \psi_\alpha(\mathbf{r}').$$

This definition leads to the delta normalization:

$$\langle \mathbf{r}'' | \mathbf{r}' \rangle = \delta^3(\mathbf{r}'' - \mathbf{r}').$$

A translation by a vector \mathbf{a} would be represented with an operator $\mathcal{T}(\mathbf{a})$ such that $\mathcal{T}(\mathbf{a})|\mathbf{r}'\rangle = |\mathbf{r}' + \mathbf{a}\rangle$. From $\mathcal{T}(\mathbf{a}) = \int d^3x' |\mathbf{r}' + \mathbf{a}\rangle \langle \mathbf{r}'|$, we can prove compositionality $\mathcal{T}(\mathbf{b})\mathcal{T}(\mathbf{a}) = \mathcal{T}(\mathbf{a} + \mathbf{b})$ and inversion $\mathcal{T}^\dagger(\mathbf{a}) = \mathcal{T}(-\mathbf{a}) = \mathcal{T}^{-1}(\mathbf{a})$. Assuming continuity, similar arguments as the time evolution operator allows us to write

$$\mathcal{T}(d\mathbf{r}) = 1 - \frac{i\mathbf{p} \cdot d\mathbf{r}}{\hbar}$$

for some Hermitian operator \mathbf{p} . The two equations

$$\langle \mathbf{r}' | \mathcal{T}(d\mathbf{r}') | \alpha \rangle = \langle \mathbf{r}' - d\mathbf{r}' | \alpha \rangle = \langle \mathbf{r}' | \alpha \rangle - d\mathbf{r}' \cdot \nabla' \langle \mathbf{r}' | \alpha \rangle$$

and

$$\langle \mathbf{r}' | \mathcal{T}(d\mathbf{r}') | \alpha \rangle = \langle \mathbf{r}' | \alpha \rangle - \frac{i}{\hbar} d\mathbf{r}' \cdot \langle \mathbf{r}' | \mathbf{p} | \alpha \rangle$$

can be compared to find

$$\langle \mathbf{r}' | \mathbf{p} | \alpha \rangle = -i\hbar \nabla' \langle \mathbf{r}' | \alpha \rangle.$$

This representation allows for easy computation of the commutation relations:

$$[x_i, x_j] = [p_i, p_j] = 0, \quad [x_i, p_j] = i\hbar \delta_{ij}.$$

Here, we have defined the position operators

$$\mathbf{r} := \int d^3x' \mathbf{r}' |\mathbf{r}'\rangle \langle \mathbf{r}'|.$$

1.4 Rotation

For a 3D rotation $R \in SO(3)$, we expect there to be a corresponding rotation operator

$$\mathcal{D}(R) |\mathbf{r}'\rangle = |R\mathbf{r}'\rangle.$$

The rotation around a unit vector $\hat{\mathbf{n}}$ by an angle ϕ can be found as

$$R(\phi\hat{\mathbf{n}}) \mathbf{r}' = \cos \phi \mathbf{r}' + (1 - \cos \phi)(\mathbf{r}' \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + \sin \phi \hat{\mathbf{n}} \times \mathbf{r}',$$

which for small angles ϵ , becomes $R(\epsilon\hat{\mathbf{n}}) \mathbf{r}' = \mathbf{r}' + \epsilon\hat{\mathbf{n}} \times \mathbf{r}'$. From this, we may write

$$\mathcal{D}(R(\epsilon\hat{\mathbf{n}})) |\mathbf{r}'\rangle = |\mathbf{r}' + \epsilon\hat{\mathbf{n}} \times \mathbf{r}'\rangle = \mathcal{T}(\epsilon\hat{\mathbf{n}} \times \mathbf{r}') |\mathbf{r}'\rangle = \left(1 - \frac{i\epsilon}{\hbar} \mathbf{p} \cdot (\hat{\mathbf{n}} \times \mathbf{r}')\right) |\mathbf{r}'\rangle.$$

Hence, defining the angular momentum operator $\mathbb{L} := \mathbf{r} \times \mathbf{p}$, we obtain $\mathcal{D}(R(\epsilon\hat{\mathbf{n}})) = 1 - \frac{i\mathbb{L} \cdot \epsilon\hat{\mathbf{n}}}{\hbar}$. The definition allows the following commutation relations:

$$[L_i, x_j] = \epsilon_{ijk} x_k, \quad [L_i, r^2] = 0, \quad [L_i, p_j] = \epsilon_{ijk} p_k, \quad [L_i, p^2] = 0, \quad [L_i, L_j] = \epsilon_{ijk} L_k, \quad [L_i, \mathbb{L}^2] = 0.$$

Notice that $\mathcal{D}(R_2)\mathcal{D}(R_1) = \mathcal{D}(R_2R_1)$ and $\mathcal{D}^\dagger(R) = \mathcal{D}(R^{-1}) = \mathcal{D}^{-1}(R)$, implying $\mathbb{L}^\dagger = \mathbb{L}$. We can obtain the representation of \mathbb{L} in spherical coordinates by comparing

$$\langle \mathbf{r}' | \mathcal{D}(\epsilon\hat{\mathbf{n}}) | \alpha \rangle = \langle \mathbf{r}' - \epsilon\hat{\mathbf{n}} \times \mathbf{r}' | \alpha \rangle = \langle \mathbf{r}' | \alpha \rangle - \epsilon (\hat{\mathbf{n}} \times \mathbf{r}') \cdot \nabla' \langle \mathbf{r}' | \alpha \rangle$$

and

$$\langle \mathbf{r}' | \mathcal{D}(\epsilon\hat{\mathbf{n}}) | \alpha \rangle = \langle \mathbf{r}' | \alpha \rangle - \frac{i\epsilon\hat{\mathbf{n}}}{\hbar} \cdot \langle \mathbf{r}' | \mathbb{L} | \alpha \rangle,$$

which then yields

$$\langle \mathbf{r}' | \mathbb{L} | \alpha \rangle = -i\hbar \mathbf{r}' \times \nabla' \langle \mathbf{r}' | \alpha \rangle = \left(-i\hbar \frac{\partial}{\partial \theta'} \hat{\phi}' + i\hbar \csc \theta' \frac{\partial}{\partial \phi'} \hat{\theta}'\right) \langle \mathbf{r}' | \alpha \rangle$$

or

$$\begin{aligned}\langle \mathbf{r}' | L_x | \alpha \rangle &= \left(-i\hbar \sin \phi' \frac{\partial}{\partial \theta'} - i\hbar \cot \theta' \cos \phi' \frac{\partial}{\partial \phi'} \right) \langle \mathbf{r}' | \alpha \rangle \\ \langle \mathbf{r}' | L_y | \alpha \rangle &= \left(i\hbar \cos \phi' \frac{\partial}{\partial \theta'} - i\hbar \cot \theta' \sin \phi' \frac{\partial}{\partial \phi'} \right) \langle \mathbf{r}' | \alpha \rangle \\ \langle \mathbf{r}' | L_z | \alpha \rangle &= -i\hbar \frac{\partial}{\partial \phi'} \langle \mathbf{r}' | \alpha \rangle.\end{aligned}$$

We could square these and add them up to find the representation of \mathbb{L}^2 , but we can more easily get it by observing

$$\mathbb{L}^2 = \varepsilon_{ijk} \varepsilon_{lmk} x_i p_j x_l p_m = \dots = r^2 p^2 - (\mathbf{r} \cdot \mathbf{p})^2 + i\hbar \mathbf{r} \cdot \mathbf{p},$$

which thus leads to

$$\langle \mathbf{r}' | \mathbb{L}^2 | \alpha \rangle = -\hbar^2 r'^2 \left(\nabla'^2 - \frac{\partial^2}{\partial r'^2} - \frac{2}{r'} \frac{\partial}{\partial r'} \right) \langle \mathbf{r}' | \alpha \rangle = -\hbar^2 \left(\frac{1}{\sin \theta'} \frac{\partial}{\partial \theta'} \left(\sin \theta' \frac{\partial}{\partial \theta'} \right) + \frac{1}{\sin^2 \theta'} \frac{\partial^2}{\partial \phi'^2} \right) \langle \mathbf{r}' | \alpha \rangle.$$

2 Hydrogen Atom

2.1 Compatible Observables

Suppose two observables A and B commute, i.e., $[A, B] = 0$. We prove that the two have a common eigenbasis of \mathcal{H} . Let $|a, \lambda\rangle$ denote an eigenbasis of A with eigenvalue a . Notice that

$$A(B|a, \lambda\rangle) = BA|a, \lambda\rangle = a(B|a, \lambda\rangle),$$

so $B|a, \lambda\rangle$ is also an eigenket of A with eigenvalue a . This allows us to write

$$B|a, \lambda\rangle = \int d\mu(\lambda') K(\lambda', \lambda) |a, \lambda'\rangle$$

where the kernel K is defined to be

$$K(\lambda', \lambda) := \langle a, \lambda' | B | a, \lambda \rangle.$$

Assuming compactness of this kernel, therefore, we can find another basis $\{|a, \rho\rangle\}$ for this eigenspace, where

$$|a, \rho\rangle = \int d\mu(\lambda) L(\lambda, \rho) |a, \lambda\rangle$$

and

$$A|a, \rho\rangle = a|a, \rho\rangle, \quad B|a, \rho\rangle = b(\rho)|a, \rho\rangle.$$

Therefore, A and B are simultaneously diagonalizable.

2.2 Angular Momentum Basis

The Hamiltonian of a hydrogen atom is given by

$$H = \frac{p^2}{2\mu} - \frac{k}{r} \quad \left(k = \frac{e^2}{4\pi\epsilon_0} \right),$$

which incidentally commutes with \mathbb{L}^2 and L_z because the Coulomb potential shows spherical symmetry. This allows us to declare a simultaneous eigenbasis $\{|Eab\rangle\}$ with eigenvalues E, a, b for H, \mathbb{L}^2, L_z , respectively.

Let $L_{\pm} := L_x \pm iL_y$.

$$[L_z, L_{\pm}] = \pm\hbar L_{\pm}, \quad [L_+, L_-] = 2\hbar L_z, \quad L_+ L_- = L_x^2 + L_y^2 + \hbar L_z \Rightarrow \mathbb{L}^2 = L_+ L_- + L_z^2 - \hbar L_z = L_- L_+ + L_z^2 + \hbar L_z$$

We now try to find what eigenvalues a, b are possible.

$$a = \langle Eab | \mathbb{L}^2 | Eab \rangle = \|L_-|Eab\rangle\|^2 + b^2 - \hbar b = \|L_+|Eab\rangle\|^2 + b^2 + \hbar b$$

$$\Rightarrow a \geq b^2 + \hbar |b|$$

Hence, the range of b is bounded for a given a , say between b_{min} and b_{max} . But notice that $L_z(L_{\pm}|Eab\rangle) = (b \pm \hbar)(L_{\pm}|Eab\rangle)$. Thus, we must have $L_+|Eab_{max}\rangle = L_-|Eab_{min}\rangle = 0$, and also $b_{max} - b_{min} \in \hbar\mathbb{N}$. This also implies

$$a = b_{max}^2 - \hbar b_{max} = b_{min}^2 + \hbar b_{min}$$

$$\Rightarrow b_{max} = -b_{min} = \hbar j \left(j \in \frac{1}{2}\mathbb{N} \right), \quad a = j(j+1)\hbar^2, \quad b = -j, -j+1, \dots, j-1, j.$$

Now, let us employ the position-basis representation of \mathbb{L} to get further restrictions.

$$\langle \mathbf{r}' | L_z | Ejm \rangle = -i\hbar \frac{\partial}{\partial \phi'} \langle \mathbf{r}' | Ejm \rangle = m\hbar \langle \mathbf{r}' | Ejm \rangle \Rightarrow \langle \mathbf{r}' | Ejm \rangle \propto e^{im\phi'}$$

However, the wave function must be single-valued for ϕ' and $\phi' + 2\pi$; hence, m must be an integer and thus $j = l \in \mathbb{N}$.

$$\langle \mathbf{r}' | \mathbb{L}^2 | Elm \rangle = -\hbar^2 \left(\frac{1}{\sin \theta'} \frac{\partial}{\partial \theta'} \left(\sin \theta' \frac{\partial}{\partial \theta'} \right) + \frac{1}{\sin^2 \theta'} \frac{\partial^2}{\partial \phi'^2} \right) \langle \mathbf{r}' | Elm \rangle = \hbar^2 l(l+1) \langle \mathbf{r}' | Elm \rangle$$

$$\Rightarrow \left(\frac{1}{\sin \theta'} \frac{\partial}{\partial \theta'} \left(\sin \theta' \frac{\partial}{\partial \theta'} \right) - \frac{m^2}{\sin^2 \theta'} - l(l+1) \right) \langle \mathbf{r}' | Elm \rangle = 0$$

Thus, we may decompose the wavefunction into the radial part and the angular part:

$$\langle \mathbf{r}' | Elm \rangle = R(r') \Theta_{lm}(\theta') e^{im\phi'}.$$

2.3 Energy Levels

We find the bound states of the hydrogen atom, i.e., $E < 0$.

$$\langle \mathbf{r}' | H | Elm \rangle = \left(-\frac{\hbar^2}{2\mu} \nabla'^2 - \frac{k}{r'} \right) \langle \mathbf{r}' | Elm \rangle = \left(-\frac{\hbar^2}{2\mu} \left(\frac{\partial^2}{\partial r'^2} + \frac{2}{r'} \frac{\partial}{\partial r'} - \frac{l(l+1)}{r'^2} \right) - \frac{k}{r'} \right) \langle \mathbf{r}' | Elm \rangle$$

$$\Rightarrow \frac{d^2 R}{dr'^2} + \frac{2}{r'} \frac{dR}{dr'} + \left(\frac{2\mu E}{\hbar^2} - \frac{l(l+1)}{r'^2} + \frac{2\mu k}{\hbar^2 r'} \right) R = 0$$

Let us define $\kappa := \frac{\sqrt{2\mu|E|}}{\hbar}$ and $\lambda := \frac{\mu k}{\hbar}$. For $r' \rightarrow \infty$, $\frac{d^2 R}{dr'^2} \approx \kappa^2 R \Rightarrow R \sim e^{\pm \kappa r'}$. For $r' \rightarrow 0+$, similarly, we have

$$\frac{d^2 R}{dr'^2} + \frac{2}{r'} \frac{dR}{dr'} - \frac{l(l+1)}{r'^2} \approx 0 \Rightarrow R \sim r'^l \text{ or } r'^{-l-1}.$$

For normalizability, hence, we require that $R(r') = f(r') r'^l e^{-\kappa r'}$ for some “well-behaved” function $f(r')$. Substitution yields the new differential equation

$$r' \frac{d^2 f}{dr'^2} + 2(l+1-\kappa r') \frac{df}{dr'} + 2(\lambda - (l+1)\kappa) f = 0.$$

Suppose $f(r') = \sum_{k=0}^{\infty} c_k r'^k$ for unknown coefficients c_k . The differential equation then becomes

$$\sum_{k=0}^{\infty} ((k+1)(k+2l+2)c_{k+1} - 2(\kappa(k+l+1) - \lambda)c_k) r'^k = 0$$

$$\Rightarrow c_{k+1} = \frac{2(\kappa(k+l+1) - \lambda)}{(k+1)(k+2l+2)} c_k.$$

Now, if every coefficient were nonzero, then for large k ,

$$c_k \sim \frac{(2\kappa)^k}{k!} \Rightarrow f(r') \sim e^{2\kappa r'},$$

which would render $R(r')$ non-normalizable. Therefore, the sequence $\{c_k\}$ must terminate, i.e., there must exist some $k_0 \in \mathbb{N}$ such that $\kappa(k_0 + l + 1) - \lambda = 0$. Relabelling as $n := k_0 + l + 1 \in \mathbb{Z}_>$ (rendering $l = 0, 1, \dots, n - 1$), we finally obtain

$$E_n = -\frac{\mu k^2}{2\hbar^2 n^2} = -\frac{Z^2 e^4 \mu}{32\pi^2 \epsilon_0^2 \hbar^2 n^2},$$

or using the fine-structure constant $\alpha := \frac{e^2}{4\pi\epsilon_0\hbar c}$,

$$E_n = -\frac{\mu c^2 Z^2 \alpha^2}{2n^2}$$

thus concluding the derivation. \square