MTH 224, Spring 2024

Instructor: Bahman Angoshtari

Lecture 4

Section 1.2: combinations**;**
Section 1.3: conditional probability

4.1. Combinations

- In some cases, when choosing a set of objects from a larger set, we don't care about the ordering of the chosen objects; we care only which objects are chosen. Each distinct group of objects that can be selected, without regard to order, is called a **combination**.

EXAMPLE 4.1. In how many ways can we choose 6 different courses out of 10 possible course offerings (order is not important)? In how may ways can we choose 4 different courses out of the 10 courses?

SOLUTION. We can solve this problem using general permutations. Specifically, we have 10 courses and we want to divide them into two groups, 6 "chosen" courses, and 4 "unchosen" courses. The number of such general permutations are

$$\binom{10}{6, 4} = \frac{10!}{6!4!} = 210.$$

If we were to choose 4 courses, then we would divide the 10 courses to two groups of 4 "chosen" courses and 10 "unchosen" courses. Therefore, the answer is the same, there are 210 different ways.

- **Binomial coefficients:** The number of combinations of k objects chosen from a group of n objects is

$$\binom{n}{k} := \binom{n}{k, (n-k)} = \frac{n!}{k!(n-k)!}.$$

  $\binom{n}{k}$ is read as "n choose k."
- Basic properties:
  - $\binom{n}{k} = \binom{n}{n-k} = \binom{n}{k, n-k}$
  - $\binom{n}{0} = \binom{n}{n} = 1$. Recall that we have defined $0! = 1$.
  - It is common to define $\binom{n}{r} = 0$ if either $r < 0$ or $r > n$.
- $\binom{n}{k}$ is called a binomial coefficient because of the following result:
  The binomial theorem: For any $n \in \{1, 2, \dots\}$, we have $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k}$.
- Let $x = y = 1$ in the binomial theorem. Then, $2^n = \sum_{k=0}^{n} \binom{n}{k}$. This equation answers the question "how many subsets does a set with n elements have?" in two ways. Can you guess how?

Example 4.2. How many non-negative integer solutions does the following equation have?

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10.$$

Solution. The answer can be found easily once we notice that each solution of the equation can be represented as a combination. Assume that we have 10 blue balloons and 5 red ribbons. Then the solution

$$x_1 = 1, x_2 = 0, x_3 = 2, x_4 = 3, x_5 = 2, x_6 = 2$$

is reperesented by the following ordering of the 10 balloons and 5 ribbons:

$$\bigcirc\wr\wr\bigcirc\bigcirc\wr\bigcirc\bigcirc\bigcirc\wr\bigcirc\bigcirc\wr\bigcirc\bigcirc$$

Can you see the patern? x_1 is the number of balloons till the first ribbon, x_2 is the number of ballons between the first and second ribbons, ..., and x_6 is the number of balloons to the right of the fifth ribbon. So, the number of solutions is the same as the number of ways to order 10 balloons and 5 ribbons in a line:

$$\binom{15}{5} = \binom{15}{10, 5} = \frac{15!}{10!5!} = 3003.$$

There are 3003 such solutions.

Example 4.3. Suppose that to win the lottery we have to pick 6 correct (and different) numbers out of 49 possible choices. What is the probability that we pick 5 correct numbers and one incorrect one?

Solution. The sample space S consists of all possible subsets of 6 numbers, from the possible 49 choices. Therefore, $|S| = \binom{49}{6}$. All outcomes are equally likely.

Let A be the event that we choose 5 correct numbers and 1 incorrect one. There are $\binom{6}{5}$ ways of choosing 5 correct numbers. There are $\binom{49-6}{1}$ ways of choosing an incorrect number. So, the probability of the A is

$$\mathbb{P}(A) = \frac{|A|}{|S|} = \frac{\binom{6}{5}\binom{43}{1}}{\binom{49}{6}} = \frac{6 \times 43}{13,983,816} \approx 0.00001845$$

.

4.2. Conditional probability

- Let A and B be two events. Intuitively speaking, the **conditional probability** of A given B is the probability of A **if we know that B has occurred**.

Example 4.4. 32 teams including Argentina and England are participating in the World Cup. Let A be the event that England wins the tournament. Let B be the event that Argentina and England make it to the final game. Then, assuming all teams have the same skill level (obviously, not a correct assumption), the unconditional probability of A is $\frac{1}{32}$. The conditional probability of A given B is $\frac{1}{2}$.

Definition 4.5. Let A and B be two events such that $\mathbb{P}(B) \neq 0$. We then define the conditional probability of A given B, denoted by $\mathbb{P}(A|B)$, as follows:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Example 4.6. A family has two children. Each child is equally likely to be a girl or a boy. Do you think that the following conditional probabilities are the same or not?
(a) Given that the older child is a girl, the probability that the younger child is a girl.
(b) Given that at least one child is a girl, the probability that the other child is a girl.

Solution. $S = \{GG, GB, BG, BB\}$ in which, say, GB denotes the outcome that the older child is a girl and the younger is a boy. Since that are equally likely, we have that $\mathbb{P}(GG) = \mathbb{P}(GB) = \mathbb{P}(BG) = \mathbb{P}(BB) = \frac{1}{4}$.

(a) older is G$= \{GG, GB\}$, younger is G $= \{GG, BG\}$

$$\mathbb{P}(\text{younger is G}|\text{older is G}) = \frac{\mathbb{P}(\{GG, BG\} \cap \{GG, GB\})}{\mathbb{P}(\{GG, GB\})} = \frac{\mathbb{P}(\{GG\})}{\mathbb{P}(\{GG, GB\})} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$$

(b) at least one G$= \{GG, GB, BG\}$, at least one G $\cap$ the other is G $= \{GG\}$

$$\mathbb{P}(\text{the other is G}|\text{at least one G}) = \frac{\mathbb{P}(\{GG\})}{\mathbb{P}(\{GG, GB, BG\})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$