MTH 224, Spring 2024

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Lecture 17

Section 4.4: independent continuous r.v.
Section 3.3: The normal distribution.

17.1. Exponential distribution (continued)

EXAMPLE 17.1. Customers arrive at a small store according to a Poisson process with rate 3 customers per hour.
(a) What is the probability that the shopkeeper will have to wait more than 20 minutes until the first customer arrives?
(b) If we learn that the shopkeeper has waited more than 10 minutes before the first customer arrived, what is the probability that the he waited more than 30 minutes?
(c) What is the probability that the first and second customers will enter the store at most 5 minutes apart from each other?

SOLUTION. (a) Let T be the time until the first customer enters the store. Then, $T \sim \text{Exp}(3)$. Thus

$$\mathbb{P}\left(T > \frac{1}{3}\right) = e^{-\lambda \cdot \frac{1}{3}} = e^{-1} \approx 0.37.$$

(b) By the memoryless property, $\mathbb{P}\left(X > \frac{1}{2} \,|\, X > \frac{1}{6}\right) = \mathbb{P}(X > \frac{1}{2} - \frac{1}{6} = \frac{1}{3}) = e^{-3\times\frac{1}{3}} = e^{-1} \approx 0.37$. This is the same as part (a)!

(c) Let X be the time it passes from the moment the 1^{st} customer enters the store until the 2^{nd} customer enters the store. Then, $X \sim \text{Exp}(3)$ and hence

$$\mathbb{P}\left(X \leq \frac{1}{12}\right) = 1 - e^{-3\cdot\frac{1}{12}} \approx 0.22.$$

17.2. Independent continuous random variables

- Recall our earlier definition of independent random variables. Namely, X, Y are said to be <u>independent</u> if
$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B)$$
for every sets A and B. In other words, X, Y are independent if the events $\{X \in A\}$ and $\{Y \in B\}$ are independent for every A, B.
- For continuous random variables, the following definition is equivalent to the above concept.

Definition 17.2. Continuous random variables X, Y are said to be <u>independent</u> if

$$\mathbb{P}(X \leq x \cap Y \leq y) = \mathbb{P}(X \leq x) \cdot \mathbb{P}(Y \leq y) = F_X(x) \times F_Y(y)$$

for every $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

More generally, continuous random variables $X_1, X_2, \ldots$ are independent if for any choices of $n \geq 1$, any choice of indices $i_1, \ldots, i_n \geq 1$, and any choice of real numbers $x_1, x_2, \ldots, x_n$,we have

$$\mathbb{P}(X_{i_1} \leq x_1 \cap \cdots \cap X_{i_n} \leq x_n) = \mathbb{P}(X_{i_1} \leq x_1) \ldots \mathbb{P}(X_{i_n} \leq x_n) = F_{X_{i_1}}(x_1) \times \cdots \times F_{X_{i_n}}(x_n).$$

- For independent random variables, we can define certain expectation involving both random variables, as follows.

Definition 17.3. Assume that X and Y are independent continuous random variables and $g(x)$ and $h(y)$ are two functions. We then define

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\,\mathbb{E}[h(Y)].$$

More generally, if $X_1, \ldots, X_n$ are independent and $g_1(x), \ldots, g_n(x)$ are given functions, we define

$$\mathbb{E}[g_1(X_1) \times \cdots \times g_n(X_n)] = \mathbb{E}[g_1(X_1)] \times \cdots \times \mathbb{E}[g_n(X_n)].$$

- In particular, for independent random variables X and Y, we have that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$. This is consistent with the results we shown for discrete jointly distributed random variables.

Example 17.4. Let X and Y be independent random variables. Show that $\mathrm{Var}(X+Y) = \mathrm{Var}(X)+\mathrm{Var}(Y)$.

Solution. Let us define $U = X - \mathbb{E}[X]$ and $V = Y - \mathbb{E}[Y]$ and note that U and V are also independent (why?). Furthermore, $\mathbb{E}[U] = 0$ and $\mathbb{E}[V] = 0$ (why?). We then have

$$\begin{aligned}
\mathrm{Var}(X+Y) &= \mathbb{E}\left[(X+Y-\mathbb{E}[X+Y])^2\right] = \mathbb{E}\left[(X-\mathbb{E}[X]+Y-\mathbb{E}[Y])^2\right] \\
&= \mathbb{E}\left[(U+V)^2\right] = \mathbb{E}\left[U^2+V^2+2UV\right] \\
&= \mathbb{E}[U^2]+\mathbb{E}[V^2]+2\mathbb{E}[UV] \\
&= \mathbb{E}\left[(X-\mathbb{E}[X])^2\right]+\mathbb{E}\left[(Y-\mathbb{E}[Y])^2\right]+2\mathbb{E}[U]\mathbb{E}[V] \\
&= \mathrm{Var}(X)+\mathrm{Var}(Y)+2\times 0\times 0 = \mathrm{Var}(X)+\mathrm{Var}(Y).
\end{aligned}$$

- More generally, we can show that if $X_1, \ldots, X_n$ are independent, then

$$\mathrm{Var}(X_1+\cdots+X_n) = \mathrm{Var}(X_1)+\cdots+\mathrm{Var}(X_n).$$

17.3. The normal distribution

Definition 17.5. A r.v. has the <u>standard normal distribution</u> if the pdf of X is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty.$$

Notation: $X \sim \mathrm{N}(0,1)$.

Remark 17.6. Gauss used normal distributions to model his observations in astronomy. Therefore, normal distributions are often referred to as Gaussian distributions.

THEOREM 17.7. *$f(x)$ above is a pdf, that is* $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} dx = 1$.

PROOF. Outside the scope of the class. □

DEFINITION 17.8. The cdf of a standard normal r.v. is denoted by $\phi(a)$. We have

$$\phi(a) = \mathbb{P}(X \leq a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx, \quad a \in \mathbb{R}.$$

REMARK 17.9. The function $\phi(x)$ cannot be expressed in terms of elementary functions, such as x^n, $\sin(x)$, $\cos(x)$, e^x, $\ln(x)$, etc. It is a "special function", tabulated on p.818 of the textbook. Many software calculate the normal cdf, for example, "pnorm(x)" returns $\phi(x)$ in R. Similarly, "NORM.DIST(x,0,1,TRUE)" is the value of $\phi(x)$ in Excel.

REMARK 17.10. By the symmetry of the density of $X \sim N(0,1)$, one has $\phi(-x) = 1 - \phi(x)$.

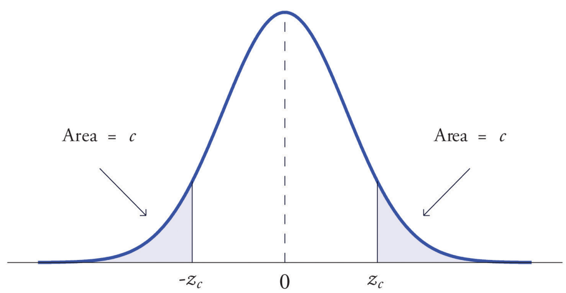

FIGURE 1. By symmetry of the density of $N(0,1)$,$\phi(-z) = 1 - \phi(z)$

EXAMPLE 17.11. Let $X \sim N(0,1)$ .Find $\mathbb{P}(|X| \leq 2)$.

SOLUTION. We have

$$\mathbb{P}(-2 \leq X \leq 2) = \phi(2) - \phi(-2) = 2\phi(2) - 1 = 0.954.$$

In the this example, you see how high the probability of a small interval around the origin is. This is a demonstration of the "fast tail decay of $f(x)$".

EXAMPLE 17.12. Let $Z \sim \mathrm{N}(0,1)$. Show that $\mathbb{E}[Z] = 0$.

SOLUTION. By definition $\mathbb{E}[Z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underset{\substack{\downarrow \\ \text{odd function}}}{ze^{-z^2/2}} \, dz = 0$. Recall that $g(x)$ is an odd function if $g(-x) = -g(x)$. If $g(x)$ is an odd function, the $\int_{-a}^{a} g(x) \mathrm{d}x = 0$.

EXAMPLE 17.13. Let $X \sim \mathrm{N}(0,1)$. Show that $\mathrm{Var}(X) = 1$.

SOLUTION. To find the variance, we use the fact that $\mathbb{E}[X] = 0$:

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2 = \mathbb{E}\left[X^2\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx \\
&= \frac{1}{\sqrt{2\pi}} \left(- xe^{-x^2/2} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2} dx \right) \qquad \begin{bmatrix} u = x, & v' = xe^{-x^2/2} \\ u' = 1, & v = \int xe^{-x^2/2} = -e^{-x^2/2} \end{bmatrix} \\
&= \frac{1}{\sqrt{2\pi}} \left(0 + \sqrt{2\pi} \right) = 1.
\end{aligned}$$