CHAPTER 8. SEQUENCES & SERIES
Section 8.1. Sequences.(Part II)

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- Convergence (divergence) of a sequence $\{a_n\}$ defined recursively.

  Definition: A sequence $\{a_n\}$ is called *increasing* if $a_n \leq a_{n+1}$ for all $n \geq 1$. It is called *decreasing* if $a_n \geq a_{n+1}$ for all $n \geq 1$. It is called *monotonic* if it is either increasing or decreasing.

- Proof by Induction: The following reasoning is a powerful tool to prove statements in the context of sequences defined recursively. (It is used for all kinds of proofs.)

  Suppose we wish to show that a property $P(n)$ is true for every $n \geq 1$ where n is a natural number.

Step 1 Show that $P(1)$ is true.

Step 2 Assume $P(k)$ is true for some $k \leq n$. Use this assumption to show that $P(n+1)$ is true. The conclusion then, is that $P(n)$ is true for all n.

Example: Show that the sequence $\boxed{a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2 + a_n} \text{ is increasing.}}$

$a_1 = \sqrt{2}$, $a_2 = \sqrt{2+\sqrt{2}}$, $a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$ . . .

$P(n)$: " $a_{n+1} \geq a_n$ "

- Show $P(1)$ is true: $a_2 \overset{?}{\geq} a_1$

  $a_2 = \sqrt{2+\sqrt{2}} \geq \sqrt{2}$ ✓

- Assume $P(k)$ is true for $k \leq n$, i.e. $a_{k+1} \geq a_k$ Assumption

  Show $P(k+1)$ is true ($a_{k+2} \geq a_{k+1}$)

  or ($\sqrt{2 + a_{k+1}} \geq \sqrt{2 + a_k}$)

$2 + a_{k+1} \geq 2 + a_k$

$\sqrt{2 + a_{k+1}} \geq \sqrt{2 + a_k}$

$\boxed{a_{k+2} \geq a_{k+1}}$

Conclusion:
By induction $a_{n+1} \geq a_n$

a_n is a increasing sequence

$a_1 = \sqrt{2}$

$a_{n+1} = \sqrt{2 + a_n}$

Definition: A sequence $\{a_n\}$ is *bounded above* if there is a number M such that $a_n \le M$ for all $n \ge 1$.

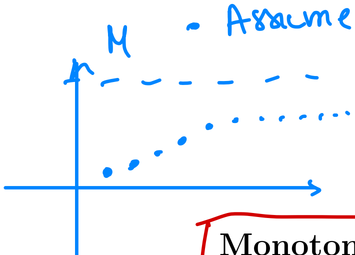
It is $\{a_n\}$ is *bounded below* if there is a number m such that $a_n \ge m$ for all $n \ge 1$.

If it bounded above and below, then $\{a_n\}$ is a *bounded* sequence.

Example: Show that the sequence in the previous example is bounded above by 3.

$P(n)$: $a_n \le 3$. Proof by induction:

- $P(1)$ $a_1 = \sqrt{2} < 3$ ✓
- Assume $P(k)$ is true: "$a_k \le 3$". Show that $a_{k+1} \le 3$ ($\sqrt{2 + a_k} \le 3$)



$2 + a_k \le 5$

$\sqrt{2 + a_k} \le \sqrt{5} \le 3$

therefore $a_{k+1} \le 3$. Conclusion $a_n \le 3$ for $n \ge 1$

Monotonic Sequence Theorem: Every bounded monotonic sequence is convergent.

Remark: This theorem means that any increasing bounded above sequence is convergent, any decreasing bounded below sequence is convergent.

Theorem: If $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$$

$a_n = \frac{1}{n}$ $\lim_{n\to\infty} \frac{1}{n} = 0$

$a_{n+1} = \frac{1}{n+1}$, $\lim_{n\to\infty} \frac{1}{n+1} = 0$

Example: Use the above theorem to find where the sequence in the previous example converges to.

$a_1 = \sqrt{2}$

$a_{n+1} = \sqrt{2 + a_n}$

Since a_n is increasing and bounded ($\sqrt{2} \le a_n \le 3$), therefore a_n is convergent: $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} a_{n+1}$

$L = \sqrt{2 + L}$

Solve for L $L^2 = 2 + L$

or $L^2 - L - 2 = 0$

$(L+1)(L-2) = 0$

$L = -1$ or $\boxed{L = 2}$