Chapter 8: Sections 8.2 to 8.7

1 a) $\sum_{n=0}^{\infty} \frac{1+4^n}{1+3^n}$ is divergent. $\lim_{n\to\infty} \frac{1+4^n}{1+3^n} = \lim_{n\to\infty} \frac{4^n}{3^n} = \lim \left(\frac{4}{3}\right)^n = \infty \neq 0$

b) $\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1} \left(\frac{2}{3}\right)^n$. Geometric series with $r = \frac{2}{3}$. Since $|r|<1$, $\sum_{n=1} \frac{2^n}{3^n}$ is convergent

$\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \frac{2/3}{1-2/3} = 2$

c) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ $\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos 0 = 1 \neq 0$, $\sum_{n=1} \cos\left(\frac{1}{n}\right)$ is divergent

c) $\sum_{n=1}^{\infty} \frac{2^{2n}+(-\pi)^n}{5^{n-1}} = \sum_{n=1}^{\infty} 5\left(\frac{4}{5}\right)^n + \sum_{n=1}^{\infty} 5\left(\frac{-\pi}{5}\right)^n$ Sum of 2 convergent geometric series with $r=\frac{4}{5}$ and $r=-\pi/5$ respectively.

d) $\sum_{n=1}^{\infty} n^{-1/3} = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ p-series with $p=1/3<1$ divergent.

e) $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ Let $b_n = \frac{1}{n^2}$. For $n\geq 1$ $\frac{1}{n^2+n} \leq \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent,

$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ is convergent.

2 $\sum_{n=1}^{\infty} 2^n 3^{1-n} = \sum_{n=1}^{\infty} 3\left(\frac{2}{3}\right)^n$. Convergent geometric series with $r=\frac{2}{3}$

$\sum_{n=1}^{\infty} 2^n 3^{1-n} = \frac{2}{1-\frac{2}{3}} = 6$

3 a) $\sum_{n=1}^{\infty} (-1)^n \frac{\arctan n}{n^2}$

Absolute convergence $\sum_{n=1}^{\infty} \left|\frac{(-1)^n \arctan n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{\arctan n}{n^2}$

For $n\geq 1$ $\frac{\arctan n}{n^2} \leq \frac{\pi}{2n^2}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2}$ is convergent

Therefore $\sum_{n=1}^{\infty} (-1)^n \frac{\arctan n}{n^2}$ is Absolutely convergent

b) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$

Absolute convergence $\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\ln n}\right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ For $n\geq 2$ $\ln n \leq n \Rightarrow \frac{1}{\ln n} \geq \frac{1}{n}$.

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent, $\sum_{n=1}^{\infty} \frac{1}{\ln n}$ is divergent. Therefore, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is not absolutely cv

Conditional convergence (see Next page)

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$. Alternating Series Test with $b_n = \frac{1}{\ln n}$. $\lim_{n\to\infty} \frac{1}{\ln n} = 0$, $\frac{1}{\ln(n+1)} \leq \frac{1}{\ln n}$

Conclusion: $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is <u>conditionally convergent</u>

c) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$ Ratio test. $\lim_{n\to\infty} \left| \frac{(-2)^{n+1}}{(n+1)!} \frac{n!}{(-2)^n} \right| = \lim_{n\to\infty} \frac{2\,n!}{(n+1)!} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$

$\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$ is <u>absolutely convergent</u>.

4) $\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n+1}$. Ratio test. $\lim_{n\to\infty} \left| \frac{3^{n+1}(x-2)^{n+1}}{n+2} \cdot \frac{n+1}{3^n(x-2)^n} \right| = 3|x+2|$.

If $3|x+2| < 1$, $\sum_{n=1}^{\infty} \frac{3(x-2)^n}{n+1}$ is convergent with $R = \frac{1}{3}$

(or $|x+2| < \frac{1}{3}$). If $|x+2| = \frac{1}{3}$, $x = \frac{5}{3}$ or $x = 7/3$.

If $x = \frac{5}{3}$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$ is convergent (AST), if $x = \frac{7}{3}$ $\sum_{n=1}^{\infty} \frac{1}{n+1}$ is divergent. (Limit comp with $b_n = \frac{1}{n}$)

So $I = [5/3, 7/3)$

5) $f(x) = \frac{x^2}{(1+3x)^2}$. Let $g(x) = \frac{1}{1+3x} = \sum_{n=0}^{\infty} (-1)^n 3^n x^n$

$$g'(x) = \frac{-3}{(1+3x)^2} = \sum_{n=1}^{\infty} (-1)^n 3^n n x^{n-1}, \quad \frac{1}{(3x+1)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} 3^{n-1} n x^{n-1}$$

$$f(x) = \frac{x^2}{(1+3x)^2} = \boxed{\sum_{n=1}^{\infty} (-1)^{n-1} 3^{n-1} n x^{n+1}}$$

6) $$f(x) = \frac{1}{3-2x} = \frac{1}{3} \frac{1}{1-\frac{2x}{3}} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{2^n}{3^n} x^n = \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} x^n$$

$$f'(x) = \frac{2}{(3-2x)^2} = \sum_{n=1}^{\infty} \frac{2^n}{3^{n+1}} n x^{n-1} = \frac{2}{9} + \frac{8}{27} x + \frac{8}{27} x^2 + \frac{16}{243} x^3 + \cdots$$

7) $\ln(3+x) = \int \frac{1}{3+x}\,dx$. $\frac{1}{3+x} = \frac{1}{3} \frac{1}{1-\left(-\frac{x}{3}\right)} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}}$

$\ln(3+x) = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{3^{n+1}(n+1)}$. If $x=0$ $C = \ln 3$

$$\ln(3+x) = \ln 3 + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{3^{n+1}(n+1)} = \ln 3 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{3^n n}$$

(8) $\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{2})}{n!}(x-\frac{\pi}{2})^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x-\frac{\pi}{2})^{2n}$

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1

(9) $f(x) = x^2 \ln(1+x^3)$

$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$ (From table). So $\ln(1+x^3) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{3n}}{n}$

and $x^2 \ln(1+x^3) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{3n+2}}{n}$.

(10) $\sum_{n=2}^{\infty} \frac{3^n}{5^n n!}$ $\quad \sum_{n=0}^{\infty} \frac{3^n}{5^n n!} = e^{3/5} = 1 + \frac{3}{5} + \sum_{n=2}^{\infty} \frac{3^n}{5^n n!}$. So $\sum_{n=2}^{\infty} \frac{3^n}{5^n n!} = e^{3/5} - \frac{8}{5}$