MTH 224, Spring 2024

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Lecture 16

Section 3.2: the exponential distribution.

16.1. Uniform distribution (continued)

EXAMPLE 16.1. Find $\mathbb{E}[X]$ and $\mathrm{Var}(X)$ if $Z \sim U(a, b)$.

SOLUTION. We can repeat the arguments in the last example of lecture 15, using the pdf $f_X(x) = 1/(b-a)$ for $x \in [a, b]$ (try this!). There is a shortcut, however. Let $Z \sim U(0, 1)$, and define $X = (b-a)Z+a$. Then, $X \sim U(a, b)$ (prove this!). Thus, using the results of the previous example and properties of expectation and variance, we obtain that

$$\mathbb{E}[X] = \mathbb{E}[(b-a)Z + a] = (b-a)\mathbb{E}[Z] + a = \frac{b-a}{2} + a = \frac{b+a}{2},$$

and

$$\mathrm{Var}(X) = \mathrm{Var}\left((b-a)Z + a\right) = (b-a)^2 \mathrm{Var}\left(Z\right) = \frac{(b-a)^2}{12}.$$

EXAMPLE 16.2. Buses arrive at the bus stop every 30 minutes, at 1:00, 1:30, 2:00, etc. Matt arrives at the bus stop at a random time, uniformly distributed between 2 and 3. What is the distribution of his waiting time? Expected wait? standard deviation?

SOLUTION. Denote Matt's arrival by $X \sim \mathrm{U}\left(2, 3\right)$, and his waiting time by W. Since X is uniform, we have that

$$f_X\left(x\right) = \begin{cases} 1, & 2 < x < 3 \\ 0, & \text{otherwise} \end{cases}.$$

In particular, note that for any $2 < a < b < 3$, we have $\mathbb{P}\left(a < X < b\right) = \int_a^b 1\, dx = b - a$.
Next, we find the cdf of Matt's waiting time W: if $t \geq \frac{1}{2}$ we clearly have that $F_W\left(t\right) = \mathbb{P}\left(W \leq t\right) = 1$ (Matt will wait at most a $\frac{1}{2}$ hour for a bus). If $t < \frac{1}{2}$ then

$$\begin{aligned} F_W\left(t\right) &= \mathbb{P}\left(W \leq t\right) = \mathbb{P}\left(X \in (2.5 - t, 2.5) \cup (3 - t, 3)\right) \\ &= \mathbb{P}\left(2.5 - t \leq X \leq 2.5\right) + \mathbb{P}\left(3 - t \leq X \leq 3\right) = \\ &= [2.5 - (2.5 - t)] + [3 - (3 - t)] = 2t. \end{aligned}$$

Therefore,

$$F_W\left(t\right) = \begin{cases} 1, & \frac{1}{2} < t \\ 2t, & 0 < t \leq \frac{1}{2} \\ 0, & t \leq 0 \end{cases}.$$

which means that $W \sim \mathrm{U}\left(0, \frac{1}{2}\right)$. Thus, $\mathbb{E}[W] = 0.5/2 = 0.25$ (or 15 minutes) and $\mathrm{Var}\,(W) = 0.2^2/12$,which yields the standard deviation of $\sqrt{\frac{0.2^2}{12}} \approx 0.12$ (or 7 minutes). So, he typically waits 15 ± 7 for the bus.

16.2. Exponential distribution

- Exponential distribution is an important continuous distribution that takes only takes positive values. It is commonly used for modeling the (continuous) time until the occurrence of an event, for instance:
  - Waiting time until getting served.
  - Lifetime of equipment, e.g. a cellphone, assuming there is no aging (can only cease to work due to an accident).
  - Time until the next email message arrives.
  - Time until next customer enters the store.
- The reason for its popularity is the following fact. If events occur according to a Poisson process, then the time between two consecutive events has exponential distribution. The following example shows this result, which we use to motivate the definition of the exponential distribution.

Example 16.3. Consider a Poisson process with rate λ (events per unit time) (see Lectures 10 and 11). For example, suppose $N_t \sim \mathrm{Pois}\,(\lambda t)$ is the number of emails received t minutes after midnight of 01/01/2024. Let $T =$ the time, in minutes after midnight 01/01/2024, of the first email. Find the pdf of T.

Solution. Let us first find the cdf of T. Note that $T > t$ if and only if $N_t = 0$ (the first email arrives after t minutes if and only if there is no email during time period $[0, t]$). Therefore,

$$\mathbb{P}\,(T > t) = \mathbb{P}\,(N_t = 0) = \frac{(\lambda t)^0\, e^{-\lambda t}}{0!} = e^{-\lambda t}.$$

It then follows that $F_T(t) = \mathbb{P}(T \le t) = 1 - \mathrm{e}^{-\lambda t}$, for $t > 0$ (and $F_T(t) = 0$ for $t \le 0$). Thus, the pdf of T is $f_T(t) = F_T'(t) = \lambda \mathrm{e}^{-\lambda t}$ for $t > 0$ (and $f_T(t) = 0$ for $t \le 0$). This is the exponential pdf.

Definition 16.4. A r.v. T has the <u>exponential distribution</u> with <u>rate</u> $\lambda > 0$ if :

$$f_T\,(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0 \\ 0, & t \le 0 \end{cases}.$$

Notation: $T \sim \mathrm{Exp}\,(\lambda)$.

- cdf of T: $F_T\,(t) = \int_{-\infty}^{t} f_T(t)\mathrm{d}t = \begin{cases} \int_0^t \lambda e^{-\lambda x} dx = -\, e^{-\lambda x}\big|_t^{\infty} = 1 - e^{-\lambda t}, & t \ge 0 \\ 0, & t < 0 \end{cases}.$
- Hence, the so-called "survival function", defined as $\mathbb{P}(T > t) = 1 - F_T(t)$, has exponential decay:

$$\mathbb{P}\,(T > t) = 1 - F_T\,(t) = e^{-\lambda t},\ \forall t > 0.$$

- Expected value: $\mathbb{E}\,[T] = \frac{1}{\lambda}$. Indeed, integration by parts yields

$$\mathbb{E}\,[T] = \int_0^{\infty} t\lambda \mathrm{e}^{-\lambda t} dt = t\left(-\mathrm{e}^{-\lambda t}\right)\Big|_{t=0}^{t=\infty} - \int_0^{\infty}\left(-\mathrm{e}^{-\lambda t}\right)\mathrm{d}t = 0 - 0 + \int_0^{\infty} \mathrm{e}^{-\lambda t}\mathrm{d}t = \frac{1}{\lambda}.$$

Remark 16.5. Recall the integration-by-parts formula. For any two functions $u(t)$ and $v(t)$, we have

$$\int_a^b u(t)v'(t)\mathrm{d}t = \underbrace{u(b)v(b) - u(a)v(a)}_{=u(t)v(t)|_{t=a}^{t=b}} - \int_a^b u'(t)v(t)\mathrm{d}t.$$

- $\mathrm{Var}\,(X) = \frac{1}{\lambda^2}$ ($\sigma_X = \frac{1}{\lambda}$). Try to calculate using integration by parts, twice.
- A very important property of the exponential distribution is that it is *memoryless.*

THEOREM 16.6. *(Memoryless property of the exponential distribution) Let* $T \sim \mathrm{Exp}\,(\lambda)$. *Then,* $\mathbb{P}\,(T > t + s \,|\, T > t) = \mathbb{P}\,(T > s)$ *for any* $s, t > 0$.

PROOF. Using the definition of conditional probability, we have:

$$\mathbb{P}(T > t + s | T > t) = \frac{\mathbb{P}(T > t + s \cap T > t)}{\mathbb{P}(T > t)} = \frac{\mathbb{P}(T > t + s)}{\mathbb{P}(T > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = \mathbb{P}(T > s).$$

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