Solution of practice final exam MTH 224-O, Spring 2024

Question 1. (a) Let the Ukrainian diplomat be the point of reference. The Russian diplomat is equally likely seated in any of the remaining 19 chairs. Only 2 chairs are next to the Ukrainian. So, the probability is $\frac{2}{19}$.

(b) $\frac{\binom{5}{2}}{\binom{6}{2}} = \frac{\frac{5\times4}{2}}{\frac{6\cdot5}{2}} = \frac{2}{3}$.

(c) $\frac{\binom{9}{2}}{\binom{15}{2}} \cdot \frac{\binom{6}{3}}{\binom{13}{3}} = \frac{9\times8\times6\times5\times4}{15\times14\times13\times12\times11}$.

(d) By Bayes' formula,

$$\mathbb{P}\left(\text{first three days are } SRR | 6 \text{ sunny days total}\right)$$

$$= \frac{\mathbb{P}\left(6 \text{ sunny days total}|\text{first three days are} SRR\right)\,\mathbb{P}\left(\text{first three days are} SRR\right)}{\mathbb{P}\left(6 \text{ sunny days total}\right)}$$

$$= \frac{\mathbb{P}\left(5 \text{ sunny days among the last 7 days}\right)\mathbb{P}\left(\text{first three days are} SRR\right)}{\mathbb{P}\left(6 \text{ sunny days total}\right)}$$

$$= \frac{\left[\binom{7}{5}p^5\left(1-p\right)^2\right]\cdot\left[p\left(1-p\right)^2\right]}{\binom{10}{6}p^6\left(1-p\right)^4} = \frac{1}{10}.$$

Question 2.
(a) First, we find:

$$\mathbb{P}\left(M \geq 6.5\right) = \mathbb{P}\left(5+T \geq 6.5\right) = \mathbb{P}\left(T \geq 1.5\right) = e^{-\ln 10\cdot 1.5} = 10^{-1.5} \approx 0.032.$$

Now, let X_i be the indicator that the i^{th} earthquake in 2023 is of magnitude 6.5 or more. Then

$$\mathbb{E}\left(X_i\right) = 1\cdot\mathbb{P}\left(M \geq 6.5\right) + 0\cdot\mathbb{P}\left(M < 6.5\right) = 0.032.$$

Let $U = X_1 + \cdots + X_N$ be the number of earthquakes in 2023 of magnitude 6.5 or more. Then, by the tower property of the conditional expectation,

$$\begin{aligned}\mathbb{E}\left(U\right) &= \mathbb{E}\left(\mathbb{E}\left(X_1+\cdots+X_N \mid N\right)\right) = \mathbb{E}\left[N\cdot 0.032\right] = 0.032\cdot\mathbb{E}\left[N\right] = 0.032\cdot\left(30+100\cdot\frac{1}{2}\right)\\ &= 0.032\cdot 80 \approx 2.56.\end{aligned}$$

(b) Since Z and M are independent, we have that

$$\mathbb{E}[D] = \mathbb{E}\left[e^{M-5}Z\right] = \mathbb{E}\left[Z\right]\mathbb{E}\left[e^{M-5}\right] = 0.1\mathbb{E}\left[e^{T}\right]$$

where $T = M-5 \sim \text{Exp}(\lambda = \ln(10))$. Note that:

$$\mathbb{E}[\mathrm{e}^T] = \int_0^{+\infty} \mathrm{e}^t\lambda\mathrm{e}^{-\lambda t}\mathrm{d}t = \int_0^{+\infty}\lambda\mathrm{e}^{-(\lambda-1)t}\mathrm{d}t = \frac{\lambda}{\lambda-1}\int_0^{+\infty}(\lambda-1)\mathrm{e}^{-(\lambda-1)t}\mathrm{d}t = \frac{\lambda}{\lambda-1}.$$

Therefore, $\mathbb{E}[D] = 0.1\mathbb{E}\left[e^T\right] = \frac{0.1\ln(10)}{\ln(10)-1} \approx 0.17677$.

Next, we calculate $\mathrm{Var}(D) = \mathbb{E}[D^2] - \Big(\mathbb{E}[D]\Big)^2$. We have,

$$\mathbb{E}[D^2] = \mathbb{E}[Z^2 \mathrm{e}^{2T}] = \mathbb{E}[Z^2]\mathbb{E}[\mathrm{e}^{2T}] = 0.1 \int_0^{+\infty} \mathrm{e}^{2t} \lambda \mathrm{e}^{-\lambda t} \mathrm{d}t$$
$$= 0.1 \int_0^{+\infty} \lambda \mathrm{e}^{-(\lambda-2)t} \mathrm{d}t = \frac{0.1\lambda}{\lambda - 2} \int_0^{+\infty} (\lambda-2)\mathrm{e}^{-(\lambda-2)t}\mathrm{d}t = \frac{0.1\lambda}{\lambda-2}.$$

Therefore,

$$\mathrm{Var}(D) = \mathbb{E}[D^2] - \Big(\mathbb{E}[D]\Big)^2 = \frac{0.1\ln(10)}{\ln(10)-2} - (0.17677)^2 \approx 0.729723.$$

(c) Let $D_i, i = 1, \ldots, 80$,be the damage caused by the i-th earthquake. From part (b), we know that $\mathbb{E}[D_i] \approx 0.17677$ and $\mathrm{Var}(D_i) \approx 0.729723$. Let $S = D_1 + \cdots + D_{80}$ be the total damage, such that $\mathbb{E}[S] \approx 80 \times 0.17677 = 14.14$ and $\mathrm{Var}(S) \approx 80 \times 0.729723 = 58.38$. Finally, CLT yields that

$$\mathbb{P}\left(S \geq 20\right) = \mathbb{P}\left(\frac{S - 14.14}{\sqrt{58.38}} \geq \frac{20-14.14}{\sqrt{58.38}}\right) \approx \mathbb{P}\left(Z \geq 0.767\right) = 1 - \phi\left(0.767\right) \approx 0.22.$$

Question 3. (a) We have:

$$\overline{X} = \frac{\sum_{i=1}^N X_i}{N} = 0.52$$
$$S_X^2 = \frac{\sum_i X_i^2 - N\overline{X}^2}{N-1} = 0.261$$
$$\overline{Y} = \frac{\sum_{i=1}^N Y_i}{N} = 3.022$$
$$S_Y^2 = \frac{\sum_i Y_i^2 - N\overline{Y}^2}{N-1} = 1.101$$

(b) We have $S_{XY} = \frac{1}{N-1}\left(\sum_{i=1}^N X_i Y_i - N\overline{X}\,\overline{Y}\right) = -0.534$. The regression line is $Y = \beta_0 + \beta_1 X$, in which

$$\beta_1 = \frac{S_{XY}}{S_X^2} = \frac{-0.534}{0.261} \approx -2.047$$
$$\beta_0 = \overline{Y} - \beta_1 \overline{X} \approx 4.085.$$

Question 4. (a) The log-likelihood function is:

$$\mathscr{L}(\mu) = \sum_{n=1}^N \log f_{X_n}(X_n) = \sum_{n=1}^N \log\left(\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(X_n-\mu)^2}{2}}\right)$$
$$= \sum_{n=1}^N \log\left((2\pi)^{-\frac{1}{2}}\right) + \sum_{n=1}^N \log\left(\mathrm{e}^{-\frac{(X_n-\mu)^2}{2}}\right) = -\frac{1}{2}\sum_{n=1}^N \log\left(2\pi\right) - \frac{1}{2}\sum_{n=1}^N (X_n-\mu)^2.$$

By differentiating with respect to μ, we obtain that

$$\mathscr{L}'(\mu) = \sum_{n=1}^N (X_n - \mu) = \sum_{n=1}^N X_n - N\mu,$$

and $\mathscr{L}''(\mu) = -N$. Since $\mathscr{L}''(\mu) < 0$, the unique maximizer of the log-likelihood function satisfies:

$$\mathscr{L}'(\hat{\mu}) = \sum_{n=1}^{N} X_n - N\hat{\mu} = 0 \quad \Longrightarrow \quad \hat{\mu} = \frac{1}{N}\sum_{n=1}^{N} X_n,$$

In other words, the sample mean $\hat{\mu} = \frac{1}{N}\sum_{n=1}^{N} X_n$ is the MLE for μ.

(b) Note that

$$\mathbb{E}[\hat{\mu}] = \mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N} X_n\right] = \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left[X_n\right] = \frac{1}{N}\sum_{n=1}^{N}\mu = \mu,$$

Thus, $\text{Bias}_{\hat{\mu}} = 0$. Furthermore,

$$\text{Var}\left(\hat{\mu}\right) = \text{Var}\left(\frac{1}{N}\sum_{n=1}^{N} X_n\right) = \frac{1}{N^2}\sum_{n=1}^{N}\text{Var}(X_n) = \frac{1}{N^2}\sum_{n=1}^{N} 1 = \frac{N}{N^2} = \frac{1}{N}.$$

Finally, we obtain that $\text{MSE}_{\hat{\mu}} = (\text{Bias}_{\hat{\mu}})^2 + \text{Var}(\hat{\mu}) = 1/N$.