CHAPTER 8. SEQUENCES & SERIES
Section 8.1. SEQUENCES

- **Definition of a sequence.**

  A sequence is an infinite set of numbers $a_1, a_2, \ldots, a_n, \ldots$. The terms a_i are indexed by natural numbers.

  Notation: $\{a_n\}_{n=1}^{\infty}$.

  A sequence can be defined by its nth term.

  Example: List the first 3 terms of the sequence.

  1. $a_n = \dfrac{1}{n+1}$ for $n \geq 1$. $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{3}$, $a_3 = \frac{1}{4}$

  $a_{99} = \frac{1}{100}$

  2. $b_n = \dfrac{(-1)^n}{2^n}$ for $n \geq 0$. $b_0 = 1$, $b_1 = -\frac{1}{2}$, $b_2 = \frac{1}{4}$, $b_3 = -\frac{1}{8}$

  ...

  A sequence can also be defined is a *recursive* fashion: The first term(s) of the sequences is (are) known, as well as a relationship between consecutive terms.

  Example: Write the first 3 terms of the sequence defined by

$$a_1 = 2, \;\; a_{n+1} = \frac{1}{3 - a_n}$$

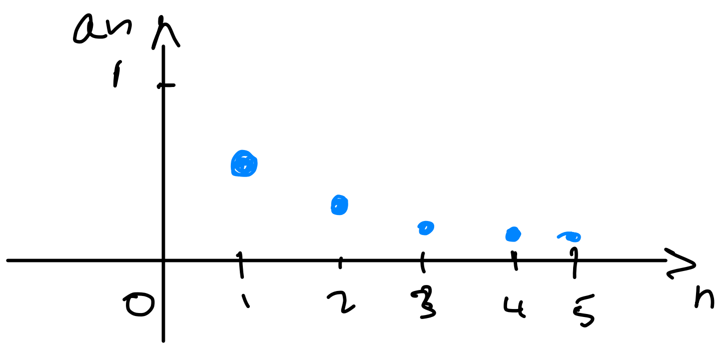
$a_1 = 2$ $a_2 = \frac{1}{3-a_1} = 1$, $a_3 = \frac{1}{3-a_2} = \frac{1}{2}$, $a_4 = \frac{1}{3-\frac{1}{2}}$

$= 5$

$a_{99} = ?$

- **Graph of a sequence.**

  Example: Sketch the graph of the sequence in example 1. $a_n = \frac{1}{n+1}$, $n \geq 1$



- **Convergence of a sequence.** $\lim_{n\to\infty} a_n = ?$

  Given a sequence, what happen the the values of the terms of a sequence a_n as n approaches ∞? From the graph of the sequence in the previous example, it appears that a_n approaches ____0____ as n approaches ∞.

  Definition: A sequence $\{a_n\}$ has the limit L, or

  $$\lim_{n\to\infty} a_n = L$$

  if we can make the terms a_n as close to L as we like by taking n sufficiently large. If the limit exists and is a finite number, we say that the sequence is *convergent*, otherwise the sequence is *divergent*.

  Here is a list of theorems that are very useful to find the limit of a sequence.

  L'Hospital's Rule

  Theorem: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$

  Theorem: If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

  (geometric sequence)

  Theorem: The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

  $$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if} \quad -1 < r < 1 \\ 1 & \text{if} \quad r = 1 \end{cases}$$

$$\lim_{n\to\infty} b_n = ?$$

The Squeeze Theorem: If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

- **The limit laws for sequences.**

  If $\{a_n\}$ and $\{b_n\}$ are *convergent* sequences and c is a constant, then

  1. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
  2. $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
  3. $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$
  4. $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
  5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$
  6. $\lim_{n\to\infty}[a_n]^p = [\lim_{n\to\infty} a_n]^p$ if $p > 0$ and $a_n > 0$.

  Exercises: Determine whether the sequence converges or diverges. If it converges, find its limit.

  1. $a_n = \frac{\sqrt{n}}{1+\sqrt{n}}$.

     $$\lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}} = 1$$

     a_n converges to 1

  2. $a_n = \frac{\ln n}{\ln 2n}$. $(n \geqslant 2)$

     $$\lim_{n\to\infty} \frac{\ln n}{\ln(2n)} = \lim_{n\to\infty} \frac{\ln n}{\ln 2 + \ln n} = \lim_{n\to\infty} \frac{\ln n}{\ln n} = 1$$

     a_n converges to 1

Assume $n \geq 1$

alternating sequen

3. $a_n = \dfrac{(-1)^{n-1}n^2}{n^3 + 2n^2 + 1}$.

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^2}{n^3+2n^2+1} = \lim_{n\to\infty} \frac{n^2}{n^3} = \lim_{n\to\infty} \frac{1}{n} = 0$$

a_n converges to 0.

4. $a_n = \dfrac{\sin 2n}{1 + \sqrt{n}}$.

$-1 \leq \sin 2n \leq 1 \Rightarrow \dfrac{-1}{1+\sqrt{n}} \leq \dfrac{\sin 2n}{1+\sqrt{n}} \leq \dfrac{1}{1+\sqrt{n}}$

$\lim_{n\to\infty} \dfrac{-1}{1+\sqrt{n}} = 0$, $\lim_{n\to\infty} \dfrac{1}{1+\sqrt{n}} = 0$, therefore $\lim_{n\to\infty} \dfrac{\sin 2n}{1+\sqrt{n}} = 0$

5. $a_n = ne^{-n}$.

$$\lim_{n\to\infty} ne^{-n} = \lim_{n\to\infty} \frac{n}{e^n}$$

Let $f(x) = \dfrac{x}{e^x}$ $\quad \lim_{x\to\infty} \dfrac{x}{e^x} = \lim_{x\to\infty} \dfrac{1}{e^x} = 0$

therefore $\lim_{n\to\infty} \dfrac{n}{e^n} = 0$