MATH 210 FINAL 2024

These problems all have to be done "by hand". Any arithmetic involved should be fairly simple. You may use your notes and the textbook, but once you access the exam you are on your own- you may not discuss the exam with anyone else or look at other sources for help.

1. In this problem we have the matrix

$$A = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}.$$

Find the eigenvalues and corresponding eigenvectors for A. Verify the orthogonality relations. Find an explicit formula for A^n (for n any integer, positive or negative).

2. Consider the system of three equations in three unknowns:

$$\begin{aligned} x + 2y + 3z &= 14 \\ x - 2y + 6z &= 15 \\ 2x \quad + 10z &= 32. \end{aligned}$$

You need to solve this system two different ways. One way is to find the coefficient matrix A, so the system becomes A X $= \begin{pmatrix} 14 \\ 15 \\ 32 \end{pmatrix}$. Find det A; find the inverse of A and use that to solve AX=B. Also, solve the system by using elementary row operations.

3. We have five "data points": (1,3), (3,7), (3,11), (4,11) and (5,18). Find the best (least squares) linear fit and the correlation coefficient. Find a linear relation y = mx + b which gives a smaller maximum error than the least squares fit. Find a linear relation which gives a smaller total error than the least squares fit. (You don't have to find the best fit in these two examples, just something better.)

4. (i) Solve the recursion relation f(n+2)- 7 f(n+1) + 12 f(n) = 0, with f(0)= f(1) = 1. Show that f(n) is eventually negative.

(ii) Solve the recursion relation f(n+2)- 7 f(n+1) + 12 f(n) = 1, with f(0) = f(1) = 1. Show that f(n) is eventually negative.

5. Suppose the matrix A is given by

$$A = \begin{pmatrix} .1 & .3 & .3 & .3 \\ .2 & .1 & .3 & .4 \\ .3 & .4 & .2 & .1 \\ .4 & .2 & .2 & .2 \end{pmatrix}.$$

Compute det(A). Find the largest eigenvalue of A and a corresponding eigenvector. Show that the rows of A^n all sum to 1, for every positive integer n. Find the limit as n tends to infin-

ity of A^n. In particular, if X = $\begin{pmatrix} 10 \\ 20 \\ 30 \\ 40 \end{pmatrix}$, find the limit as n tends to infinity of A^n X .

6. In R^3 we can define a kind of multiplication by (a,b,c) x (d,e,f) = (bf-ce, cd-af, ae-bd). (I am writing vectors as rows rather than columns.) Verify that if X and Y are vectors then X x Y = - Y x X, that X x X = 0, and that X x Y is orthogonal to X (with the usual dot product.) If |X| denotes the length of a vector X, show that |X x Y| = |X| |Y| |sin (α)|, where α is the angle between X and Y. Show by example that this multiplication is not always associative: X x (Y x Z) is not always the same as (X x Y) x Z.

7. Suppose P is the vector space of all polynomials and we define an inner product < , > on P by

$$<f, g> = 2 \int_0^1 x\, f(x)\, g(x)\, dx.$$

Verify that this satisfies the conditions for an inner product. Find <1,1>, <1,x> , <x,x>. Find two orthonormal vectors that span the space of all linear polynomials. (In other words, convert {1, x} to an ON basis for the linear polynomials.)