$a>0$ $\int_a^{\infty} \frac{1}{x^p}$ is $\begin{cases} \to \text{convergent if } p>1 \\ \to \text{divergent if } p \le 1 \end{cases}$

$a>0$ $\int_0^{a} \frac{1}{x^p}dx$ is $\begin{cases} \to \text{convergent if } p<1 \\ \to \text{divergent if } p \ge 1 \end{cases}$

Section 6.6 Improper Integrals worksheet

- Use the comparison theorem to decide whether the integral convergent, or divergent. Here, you need to say more than just yes, or no. Your answer has to contain an inequality "algebraically sound" as well as an appropriate conclusion.

1. $\int_1^{\infty} \frac{\sin x + 2}{x^2}\ dx$

If $x \ge 1$ $\sin x \le 1 \Rightarrow \sin x + 2 \le 3 \Rightarrow \frac{\sin x+2}{x^2} \le \frac{3}{x^2}$. Since $\overbrace{\int_1^{\infty} \frac{1}{x^2}dx}^{CV}$ is convergent, $\int_1^{\infty} \frac{\sin x+2}{x^2}dx$ is convergent.

2. $\int_1^{\infty} \frac{\sin x + 2}{x}\ dx$ (guess work: as $x \to \infty$ $\frac{\sin x+2}{x} \sim \frac{\text{number}}{x}$, $\int_1^{\infty}\frac{1}{x}dx$ is divergent)

For x $x \ge 1$ $\sin x \ge -1 \Rightarrow \sin x + 2 \ge 1$ and $\frac{\sin x+2}{x} \ge \frac{1}{x}$. Since $\int_1^{\infty} \frac{1}{x}dx$ is divergent, therefore $\int_1^{\infty} \frac{\sin x+2}{x}dx$ is divergent.

3. $\int_1^{\infty} \frac{1}{x^2 + 5x + 1}\ dx$

For $x \ge 1$ $\frac{1}{x^2+5x+1} \le \frac{1}{x^2}$ Since $\int_1^{\infty} \frac{1}{x^2}dx$ is convergent, $\int_1^{\infty} \frac{1}{x^2+5x+1}dx$ is convergent

Another "test" integral

For any real number a $\int_a^\infty e^{-x}\,dx$ is convergent

4. $\displaystyle\int_3^\infty \frac{1}{x+e^x}\,dx$ (Guess: As $x \to \infty$ $\frac{1}{x+e^x} \sim \frac{1}{e^x} = e^{-x}$

$\int_3^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_3^t e^{-x}\,dx = \lim_{t\to\infty} -e^{-x}\Big|_3^t$ is cv

$= \lim_{t\to\infty} (\underbrace{-e^{-t}}_{\to 0} + e^{3}) = e^{3}$

For $x \geq 3$, $x+e^x \geq e^x \Rightarrow \frac{1}{x+e^x} \leq \frac{1}{e^x}$. Since $\int_3^\infty e^{-x}\,dx$ is convergent, $\int_3^\infty \frac{1}{x+e^x}\,dx$ is convergent

5. $\displaystyle\int_0^\infty \frac{\tan^{-1} x}{2+e^x}\,dx$

If $x \geq 0$ $\tan^{-1}x \leq \frac{\pi}{2} \Rightarrow \frac{\tan^{-1}x}{2+e^x} \leq \frac{\pi}{2}\frac{1}{2+e^x} \leq \frac{\pi}{2e^x}$

Since $\int_0^\infty \frac{1}{e^x}\,dx$ is convergent, $\int_0^\infty \frac{\tan^{-1}x}{2+e^x}\,dx$ is convergent.

6. $\displaystyle\int_1^\infty \frac{1+3\sin^4(2x)}{\sqrt{x}}\,dx$

For $x \geq 1$ $\sin^4(2x) \geq 0$ $1+3\sin^4(2x) \geq 1 \Rightarrow \frac{1+3\sin^4(2x)}{\sqrt{x}} \geq \frac{1}{\sqrt{x}}$

Since $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is divergent, $\int_1^\infty \frac{1+3\sin^4(2x)}{\sqrt{x}}\,dx$ is divergent.

7. $\int_1^\infty e^{-x^2}\ dx$

For $x \geq 1$ $\quad x^2 \geq x \implies -x^2 \leq -x$ and $e^{-x^2} \leq e^{-x}$.

Since $\int_1^\infty e^{-x} dx$ is convergent, $\int_1^\infty e^{-x^2} dx$ is convergent

8. $\int_0^1 \frac{1}{\sqrt{2x - x^2}}\ dx$

For $0 < x < 1$,

$x^2 \leq x$, $\quad -x^2 \geq -x$ and $2x - x^2 \geq x$.

Since the square root function is increasing

$\sqrt{2x - x^2} \geq x$ and $\frac{1}{\sqrt{2x-x^2}} \leq \frac{1}{\sqrt{x}}$.

$\int_0^1 \frac{1}{\sqrt{x}} dx$ is convergent. Therefore $\int_0^1 \frac{1}{\sqrt{2x-x^2}} dx$ is convergent.

(or $\int_0^1 \frac{1}{\sqrt{2x-x^2}} dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{\sqrt{2x-x^2}} dx$

$\longrightarrow$ Next page

$$\int_t^1 \frac{1}{\sqrt{2x-x^2}}\,dx = \int_t^1 \frac{1}{\sqrt{-(x^2-2x)}}\,dx$$

$$= \int_t^1 \frac{1}{\sqrt{1-(x-1)^2}}\,dx = \sin^{-1}(x-1)\Big|_t^1$$

$$= \sin^{-1}(0) - \sin^{-1}(t-1)$$

$$= 0 - \sin^{-1}(t-1)$$

$$\lim_{t\to 0^+}\left(-\sin^{-1}(t-1)\right) = -\sin^{-1}(-1) = -\left(-\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2}$$

$\int_0^1 \frac{1}{\sqrt{2x-x^2}}\,dx$ is convergent