MTH 224, Spring 2024

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Lecture 1

Section 1.1: sample space and events, set operations

1.1. Sample space and events

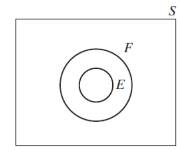
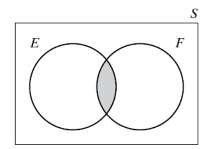
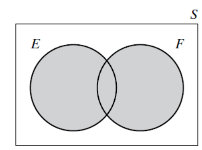
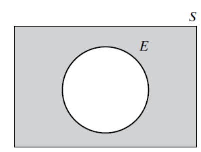
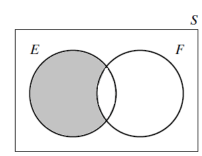
- An **experiment** is a process which results in an outcome that cannot be predicted in advance with certainty. The set of all possible outcomes of an experiment is called the **sample space**.
- Examples:
  - rolling a six-sided die: the sample space is $S = \{1, 2, 3, 4, 5, 6\}$.
  - tossing a coin: the sample space is $S = \{H, T\}$.
  - the result of this year's Super Bowl: What is the sample space?
  - weighing the contents of a box of cereal: The sample space is $S = (0, +\infty)$. More reasonably, $S = (12, 20)$ for a 16oz box.
  - the temperature tomorrow at noon in this classroom in degree Fahrenheit: The sample space is $S = (-459.67, +\infty)$.
- Any subset E of the sample space S is called an **event**. We say that **an event has occurred** if the outcome of the experiment is one of the outcomes in that event.
- Examples:
  - When rolling a six-sided die, getting an odd outcome is the event $A = \{1, 3, 5\}$.
  - Another event is getting an even outcome $B = \{1, 3, 5\}$.
  - Yet another event is getting 1, $C = \{1\}$.
  - Usually, we don't distinguish between one element events such as $\{1\}$ and the outcome 1.

1.2. Operations on events

- Events can be combined in various ways to yield new events. Such combination can easily be expressed using the basic set operations **union**, **intersection**, **taking complement**, and **set difference**.
- Let A and B be two sets. Recall from the set theory that:

$$\begin{aligned}
\text{Union: } & A \cup B = \{x : x \in A \text{ or } x \in B\} \\
\text{Intersection: } & A \cap B = \{x : x \in A \text{ and } x \in B\} \\
\text{Complement: } & A^c = \{x : x \notin A\} \\
\text{Set difference: } & A \backslash B = \{x : x \in A \text{ and } x \notin B\} = A \cap B^c
\end{aligned}$$

- Let E, F be events in a sample space S. Then:

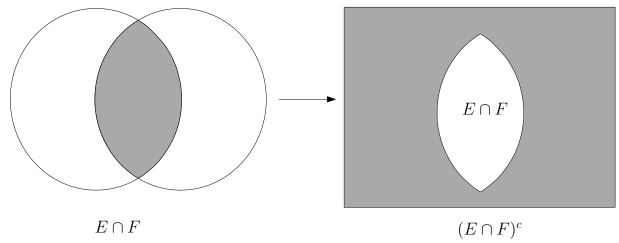
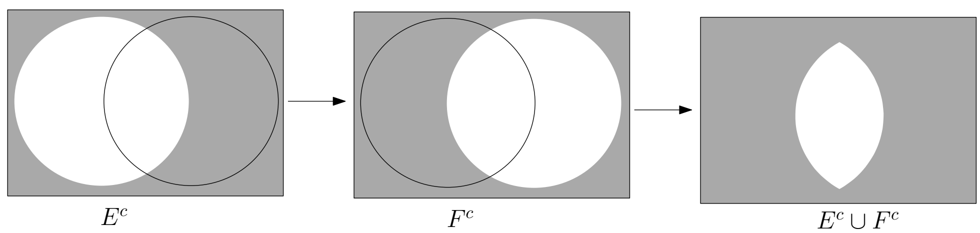
Operation\Relation	Probabilistic Meaning	Example $S = \{1, \ldots, 6\}$	Venn diagram
E is a subset of F $E \subset F$	If E occurs, then F occurs	$E = \{1\}$ $F = \{\text{an odd number}\}$	
E equals F $E = F$	E, F both occur, or both don't occur	$E = \{1, 3, 5\}$ $F = \{\text{an odd number}\}$	
intersection of E and F $E \cap F$	all outcomes that are both in E and in F, $E \cap F$ occurs if both E and F occur at the same time	$E = \{1, 4, 6\}$ $F = \{\text{an odd number}\}$ $E \cap F = \{1\}$	
union of E and F $E \cup F$	outcomes that either in E or in F (or both), $E \cup F$ occurs if E, or F, or both occur.	$E = \{1, 4, 6\}$ $F = \{\text{an odd number}\}$ $E \cup F = \{1, 3, 4, 5, 6\}$	
E complement E^c	all outcomes not in E, E^c occurs if E doesn't	$E = \{\text{an even number}\}$ $F = \{\text{an odd number}\}$ $E = F^c$, $E^c = F$	
E minus F $E \backslash F = E \cap F^c$	all outcomes in E and not F. $E \backslash F$ occurs if E occurs and F doesn't	$E = \{1, 3, 4, 6\}$ $F = \{\text{an even number}\}$ $E \backslash F = \{1, 3\}$	

- Basic set rules:
  (1) Associative laws: $(E \cap F) \cap G = E \cap (F \cap G)$ and $(E \cup F) \cup G = E \cup (F \cup G)$
  In particular, this means that $E \cap F \cap G$ (respectively $E \cup F \cup G$) is well defined since the order in which we take the intersection operations does not matter.

  (2) Distributive laws: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$ and $(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$

(3) De Morgan's Laws: $(E \cap F)^c = E^c \cup F^c$ and $(E \cup F)^c = E^c \cap F^c$

- Showing the first De Morgan's Law by Venn diagram:

Left hand side:	
Right hand side:	

- To save space, we use the following notations for operations on multiple (possibly infinitely many) events:
  - Intersection: $\bigcap_{i=1}^{n} E_i = E_1 \cap E_2 \cap \cdots \cap E_n$.
    Meaning: $\bigcap_{i=1}^{n} E_i$ occurs if and only if all the events E_i occur.

  - Union: $\bigcup_{i=1}^{n} E_i = E_1 \cup E_2 \cup \cdots \cup E_n$.
    Meaning: $\bigcup_{i=1}^{n} E_i$ occurs if and only if at least one of the events E_i occurs.

- We have the following generalizations of the De Morgan's laws:
  - $\left(\bigcap_{i=1}^{n} E_i \right)^c = \bigcup_{i=1}^{n} E_i^c$
  - $\left(\bigcup_{i=1}^{n} E_i \right)^c = \bigcap_{i=1}^{n} E_i^c$
- The events A and B are said to be **mutually exclusive** or **disjoint** if they have no outcomes in common. That is, if $A \cap B = \emptyset$.
  - Example: In rolling a 6-sided die, the event of getting an even outcome and the event of getting an odd outcome are disjoint.
- More generally, a collection of events $A_1, A_2, \ldots, A_k$ are mutually exclusive (or disjoint) if no two of them have any outcomes in common, that is, if $A_i \cap A_j = \emptyset$ for all $i, j \in \{1, \ldots, k\}$ such that $i \neq j$.