CHAPTER 8. SEQUENCES & INFINITE SERIES
Section 8.4. Other Convergence Tests

An alternating series is any series of the form

$$\boxed{\sum_{n=1}^{\infty}(-1)^{n-1}b_n}$$

Example: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

- **The Alternating Series Test:** If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - \dots \quad \text{with} \quad b_n > 0$$

satisfies :

1. $b_{n+1} \le b_n$ for all n.
2. $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Exercises: Determine whether the series diverges or converges.

1. $\cancel{\sum_{n=1}^{\infty} \frac{(-1)^n}{n-1}}$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \underbrace{\frac{1}{n}}_{= b_n}$

Alternating series test : Let $b_n = \frac{1}{n}$

① $b_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = b_n$

so $b_{n+1} \leq b_n$

② $\lim_{n \to \infty} \frac{1}{n} = 0$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent .... (but $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent)

2. $\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n^2+1}$ (practice problem)

3. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n!} = \frac{1}{1!} + 0 - \frac{1}{3!} + 0 + \frac{1}{5!} + 0 - \frac{1}{7!} + -$

$$= \frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$$

Alternating series: $b_n = \dfrac{1}{(2n-1)!}$

- Is $b_{n+1} \leq b_n$

$$b_{n+1} = \frac{1}{(2n+1)!} \leq \frac{1}{(2n-1)!}$$

- $\displaystyle\lim_{n\to\infty} \frac{1}{(2n-1)!} = 0$

Conclusion: $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n!}$ is convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{2}\left(-\frac{1}{2}\right)^n$$ geometric series

also an alternating series

Remark: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent, but $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent,

- **Absolute Convergence:** A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

- **Theorem:** If a series is *absolutely convergent*, then it is *convergent*.

  Remark: If the a_n's are all positive, then *absolute convergence* = *convergence* .

- **Conditional Convergence:** A series that is convergent but not absolutely convergent is said to be *conditionally* convergent.

Examples: Is the series absolutely convergent, conditionally convergent or divergent. Show all work that leads to your answer.

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

Absolute convergence $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Conclusion: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is not absolutely convergent

Conditional convergence : $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

Alternating series test: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent

Conclusion: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent

2. $\sum_{n=1}^{\infty} \frac{\sin 2n}{n^2}$

Absolute convergence: $\sum_{n=1}^{\infty} \left| \frac{\sin 2n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\sin 2n|}{n^2}$

For $n \geq 1$ $\frac{|\sin 2n|}{n^2} \leq \frac{1}{n^2}$. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

By the Comparison Test $\sum_{n=1}^{\infty} \frac{|\sin 2n|}{n^2}$ is convergent

$\sum_{n=1}^{\infty} \frac{\sin(2n)}{n^2}$ is absolutely convergent.

- **The Ratio Test:**

(1) If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L < 1$ the the series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent.*

(2) If $L > 1$ or $L = \pm\infty$, then the series is *divergent.*

If $L=1$, the Ratio Test is inconclusive.

Examples: Is the series absolutely convergent, conditionally convergent or divergent. Show all work that leads to your answer.

1. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$ a_n $\lim_{n\to\infty} \frac{(-3)^n}{n!} = 0$ why? Hint: Show $\lim_{n\to\infty} \frac{3^n}{n!} = 0$ HW

Ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-3)^{\cancel{n+1}}}{\underbrace{(n+1)!}_{n!(n+1)}} \cdot \frac{n!}{(3)^{\cancel{n}}} \right| = \lim_{n\to\infty} \left| \frac{-3}{n+1} \right|$$

$$= \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$$

$\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$ is absolutely convergent

2. $\sum_{n=1}^{\infty} e^{-n} n!$ a_n

Ratio test

$$\lim_{n\to\infty} \frac{\cancel{e}^{-(n+1)} \overbrace{(n+1)!}^{\cancel{n!}(n+1)}}{\cancel{e^{-n}}\, \cancel{n!}} = \lim_{n\to\infty} e^{-1}(n+1) = \infty$$

$\sum_{n=1}^{\infty} e^{-n} n!$ is divergent

- **The Root Test:** (Rarely used)

(1) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ the the series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent.*

(2) If $L > 1$ or $L = \pm\infty$, then the series is *divergent.*

(3) $L=1$, The Root test is inconclusive.

Examples: Is the series absolutely convergent, conditionally convergent or divergent. Show all work that leads to your answer.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{(\tan^{-1} n)^n}$ Root test

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{(\tan^{-1} n)^n}\right|} = \lim_{n\to\infty} \frac{1}{\tan^{-1} n} = \frac{2}{\pi} < 1$$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{(\tan^{-1} n)^n}$ is absolutely convergent

2. $\sum_{n=1}^{\infty} \underbrace{\frac{n!}{n^n}}_{=a_n}$ Ratio test $(a_n > 0)$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\overbrace{(n+1)!}^{\cancel{n!}\,(n+1)}}{(n+1)^{n+1}} \cdot \frac{n^n}{\cancel{n!}} = \lim_{n\to\infty} \frac{\cancel{(n+1)}}{(n+1)^{\cancel{n+1}}} \cdot n^n$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\underbrace{\frac{n}{n+1}}_{\to 1^\infty}\right)^n$$

indeterminate power

$$f(x) = \left(\frac{x}{x+1}\right)^x$$

$$\lim_{x\to\infty} \ln\left(\frac{x}{x+1}\right)^x$$

$$= \lim_{x\to\infty} \underset{\to\infty}{x} \; \underset{\to 0}{\ln\left(\frac{x}{x+1}\right)}$$

$$= \lim_{x\to\infty} \frac{\ln\left(\frac{x}{x+1}\right) \to 0}{\frac{1}{x} \to 0}$$ etc...

$$= \lim_{n\to\infty} \left(\frac{1}{\frac{n+1}{n}}\right)^n = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n$$

$$= \lim_{n\to\infty} \frac{1}{\underbrace{\left(1+\frac{1}{n}\right)^n}_{e}} = \frac{1}{e} < 1$$

$$\left(\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e\right)$$

$\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is absolutely convergent.