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Lecture 5

Independent events (Section 1.4), multiplication rule (Section 1.3)
Law of total probability (LTP) (Section 1.5)

5.1. Independent events

- Intuitively, two events are **independent** if knowing that one has occurred (or has not occurred) gives us no information about the other event. Similarly, $A_1, A_2, ..., A_n$ are independent if probability of each remains the same no matter **which of the others** occur.

Example 5.1. (a) We draw a card from a standard deck of 52 playing cards. Let A= getting a king, B=getting a heart. These two events are independent.
(b) We roll a dice three times. Let A= getting a 6 on the first roll, B=getting an even number in the second roll, C=getting an odd number in the third roll. These events are independent.

- Don't confuse independent events with disjoint events. In part (a) of the previous example, let C=getting a Queen. Then, A and C are disjoint (they cannot happen at the same time). But, they are not independent! Indeed, if A occurs, then we know that C cannot occur. So, occurrence of A gives us information about (non-)occurrence of C.
- Listing all the possible conditional probabilities is cumbersome. The following definition is a better way of defining independence using unconditional probabilities.

Definition 5.2. We define independence for any number of events as follows.

- Two events A_1 and A_2 are independent if $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$.
- Three events A_1, A_2, and A_3 are independent if:
  - Every two of them are independent, and
  - $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$.
- Four events A_1, A_2, A_3 and A_4 are independent if:
  - Every three of them are independent, and
  - $\mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)\mathbb{P}(A_4)$.

$\vdots$

- Events $A_1, A_2, ..., A_n$ are independent if:
  - Every $(n-1)$ of them are independent, and
  - $\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2)\dots\mathbb{P}(A_n)$.

- We can check that if A and B are independent, then

$$\mathbb{P}(A|B)=\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}=\frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)}=\mathbb{P}(A).$$

  and

$$\mathbb{P}(B|A)=\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)}=\frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(A)}=\mathbb{P}(B).$$

- Similarly, assume that $A_1, A_2, ..., A_n$ are independent. Then, for any event A_i and any choice of m other events $A_{j_1}, A_{j_2}, \ldots, A_{j_m}$, we have

$$\mathbb{P}(A_i|A_{j_1}\cap A_{j_2}\cap ... \cap A_{j_m})=\mathbb{P}(A_i).$$

EXAMPLE 5.3. We roll 2 fair dice. $A_1 =$ first die is 4, $A_2 =$ second die is 4, $A_3 =$ the sum is 7. Are A_1, A_2, and A_3 independent?

SOLUTION. The three events are clearly not independent, can you see why?
But, to practice with the definition, let us show this. We need to check the following four relationships

$$\mathbb{P}(A_1\cap A_2)\stackrel{?}{=}\mathbb{P}(A_1)\mathbb{P}(A_2), \qquad \mathbb{P}(A_1\cap A_3)\stackrel{?}{=}\mathbb{P}(A_1)\mathbb{P}(A_3),$$
$$\mathbb{P}(A_2\cap A_3)\stackrel{?}{=}\mathbb{P}(A_2)\mathbb{P}(A_3), \qquad \mathbb{P}(A_1\cap A_2\cap A_3)\stackrel{?}{=}\mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3).$$

Let $S=\{11,12,\ldots,66\}$ which has 36 equally likely outcomes. Note that

$$A_1=\{41,42,43,44,45,46\}$$
$$A_2=\{14,24,34,44,54,64\}$$
$$A_3=\{16,25,34,43,52,61\}$$

We then have

$$\mathbb{P}(A_1)=\mathbb{P}(A_2)=\mathbb{P}(A_3)=\frac{6}{36}=\frac{1}{6}.$$

We can checked that any two of the three events are independent:

$$\mathbb{P}(A_1\cap A_2)=\mathbb{P}(\{44\})=\frac{1}{36}=\mathbb{P}(A_1)\mathbb{P}(A_2)\Longrightarrow A_1 \text{ and } A_2 \text{ are independent}$$
$$\mathbb{P}(A_1\cap A_3)=\mathbb{P}(\{43\})=\frac{1}{36}=\mathbb{P}(A_1)\mathbb{P}(A_3)\Longrightarrow A_1 \text{ and } A_3 \text{ are independent}$$
$$\mathbb{P}(A_2\cap A_3)=\mathbb{P}(\{34\})=\frac{1}{36}=\mathbb{P}(A_2)\mathbb{P}(A_3)\Longrightarrow A_2 \text{ and } A_3 \text{ are independent}$$

However, we have that

$$\mathbb{P}(A_1\cap A_2\cap A_3)=\mathbb{P}(\emptyset)=0\neq\frac{1}{216}=\mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3).$$

So, the three events A_1, A_2, and A_3 are dependent (that is, they are NOT independent).

5.2. The multiplication rule

- Sometimes, calculating conditional probabilities is easier than unconditional probabilities. In such cases, conditional probabilities can be very helpful in computing the unconditional probabilities.
- The **multiplication rule** (also known as the **chain rule for probability**) lets you calculate the probability of intersection of events using conditional probabilities.

EXAMPLE 5.4. The numbers 1, 2, 3, 4, 5, and 6 were randomly written on the sides of a blank six-sided die. What is the probability that the numbers on opposite sides sum to 7?

SOLUTION. A_{16}= 1 and 6 are on opposite sides; A_{25}= 2 and 5 are on opposite sides; A_{34}= 3 and 4 are on opposite sides. We are interested in probability of $A_{16} \cap A_{25} \cap A_{34}$. Note, however, that $A_{16} \cap A_{25} \cap A_{34} = A_{16} \cap A_{25}$ (why?)

Calculating $\mathbb{P}(A_{16} \cap A_{25})$ by counting can be a bit complicated (give it a try). Let us take a shortcut. First, we recall the definition of conditional probability

$$\mathbb{P}(A_{25}|A_{16}) = \frac{\mathbb{P}(A_{25} \cap A_{16})}{\mathbb{P}(A_{16})} \Longrightarrow \mathbb{P}(A_{25} \cap A_{16}) = \mathbb{P}(A_{16})\mathbb{P}(A_{25}|A_{16}).$$

We have $\mathbb{P}(A_{16}) = \frac{1}{5}$ and $\mathbb{P}(A_{25}|A_{16}) = \frac{1}{3}$ (why are these correct?). So, we obtain that

$$\mathbb{P}(A_{16} \cap A_{25} \cap A_{34}) = \mathbb{P}(A_{25} \cap A_{16}) = \mathbb{P}(A_{16})\mathbb{P}(A_{25}|A_{16}) = \frac{1}{5} \times \frac{1}{3} = \frac{1}{15}.$$

THEOREM 5.5. *(Multiplication rule)*

- *For any two events A and B, we have that $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A)$.*
- *More generally, for events $A_1, A_2, \dots, A_N$, we have that $\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_N) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 \cap A_2) \dots \mathbb{P}(A_N|A_1 \cap \cdots \cap A_{N-1})$.*

- To prove multiplication rules, use the definition of conditional probabilities (try this!).
- Also, write down the multiplication rule for independent events and compare it with the definition of independence.

5.3. Law of total probability

- The **law of total probability (LTP)** is another very useful result. It lets you calculate the **unconditional probability** of an event using the conditional probability of that event given some other events.

EXAMPLE 5.6. We need to know the probability that a United flight will arrive on time. We have the following information from past trips of that flight.

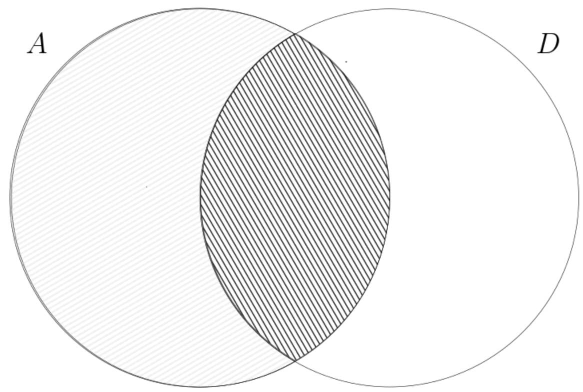
- 70% of past flights depart on time.
- From the flights that depart on time, 80% arrive on time.
- From the flights that depart late, 90% arrive late.

What is your estimate of the probability that the flight tomorrow will be on time?

SOLUTION. Let A = arrive on time; D = depart on time. We are interested in $\mathbb{P}(A)$, the **unconditional probability** of A. From the problem statement, we know that:

$$\mathbb{P}(D) = 0.7, \quad \mathbb{P}(A|D) = 0.8, \quad \mathbb{P}(A^c|D^c) = 0.9.$$

Let us express $\mathbb{P}(A)$ in terms of the above probabilities. Note that $A = \big(D \cap A\big) \cup \big(D^c \cap A\big)$



By using (A3) (third axiom of probability) and the multiplication rule, we obtain that:

$$\mathbb{P}(A) = \mathbb{P}(D \cap A) + \mathbb{P}(D^c \cap A) = \mathbb{P}(D)\mathbb{P}(A|D) + \mathbb{P}(D^c)\mathbb{P}(A|D^c).$$

From the problem statement, we get

$$\mathbb{P}(D) = 0.7, \mathbb{P}(D^c) = 0.3, \mathbb{P}(A|D) = 0.8, \mathbb{P}(A|D^c) = 1 - \mathbb{P}(A^c|D^c) = 0.1.$$

Finally, we obtain that the probability of on-time arrival is 59%:

$$\mathbb{P}(A) = \mathbb{P}(D)\mathbb{P}(A|D) + \mathbb{P}(D^c)\mathbb{P}(A|D^c) = 0.7 \times 0.8 + 0.3 \times 0.1 = 0.59.$$

- In the above example, we used the law of total probability (LTP) to calculate the probability of an event (on-time arrival) by conditioning on another event (on-time departure).

Law of total probability for two events: Let A and B be two events such that $\mathbb{P}(A) \neq 0$ and $\mathbb{P}(A) \neq 1$. Then,

$$\text{LTP:} \quad \mathbb{P}(B) = \mathbb{P}(A)\mathbb{P}(B|A) + \mathbb{P}(A^c)\mathbb{P}(B|A^c).$$

- **Proof:** Start from $B = \big(A \cap B\big) \cup \big(A^c \cap B\big)$, then apply (A3) (third axiom of probability) and the multiplication rule.

LTP for multiple events: Let $A_1, A_2, ..., A_N$ be N events such that

(1) They are disjoint, that is, $A_i \cap A_j = \emptyset$ for all $i \neq j$.
(2) $A_1 \cup A_2 \cup ... \cup A_N =$ sample space.

We usually summarize (1.) and (2.) by saying that $A_1, A_2, ..., A_N$ is a **partition** of the sample space. Then, for any event B, we have

$$\text{LTP:} \quad \mathbb{P}(B) = \mathbb{P}(A_1)\mathbb{P}(B|A_1) + \cdots + \mathbb{P}(A_N)\mathbb{P}(B|A_N)$$

- **Proof:** Start from $B = \big(A_1 \cap B\big) \cup ... \cup \big(A_N \cap B\big)$, then apply (A3) (third axiom of probability) and the multiplication rule.