The Fibonacci Sequence

Suppose rabbits live forever and that every month each pair produces a new pair which becomes productive at age 2 months.
If we start with one newborn pair, how many pairs of rabbits will we have in the n–th month? The answer is

$$f_n = f_{n-1} + f_{n-2}$$

for $n \geq 3$

Let $a_n = \dfrac{f_{n+1}}{f_n}$, and show that $a_{n-1} = 1 + \dfrac{1}{a_{n-2}}$. Assume that $\{a_n\}$ is convergent, find its limit.

$a_{n-1} = \dfrac{f_n}{f_{n-1}}$

$a_{n-1} = \dfrac{f_n}{f_{n-1}} = \dfrac{f_{n-1}}{f_{n-1}} + \dfrac{f_{n-2}}{f_{n-1}} = 1 + \dfrac{1}{\frac{f_{n-1}}{f_{n-2}}} = 1 + \dfrac{1}{a_{n-2}}$

So $\boxed{a_{n-1} = 1 + \dfrac{1}{a_{n-2}}}$

a_n is convergent : $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} a_{n-1} = L$ and $\lim_{n\to\infty} a_{n-2} = L$

$L = 1 + \dfrac{1}{L} \Rightarrow L^2 - L - 1 = 0$

$L = \dfrac{1 \pm \sqrt{5}}{2} \Rightarrow \dfrac{1+\sqrt{5}}{2}$, $\dfrac{1-\sqrt{5}}{2}$

$\boxed{L = \dfrac{1+\sqrt{5}}{2}}$ Golden Ratio