If a is a constant

- $\int \frac{1}{x+a}\,dx = \ln|x+a| + c$

if $n \neq 1$ • $\int \frac{1}{(x+a)^n}\,dx = \frac{(x+a)^{-n+1}}{-n+1} + c = \frac{1}{(1-n)(x+a)^{1-n}} + c$

- $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

- $\int \frac{x}{\underbrace{x^2+a}}\,dx = \frac{1}{2}\ln|x^2+a| + c$

$u = x^2+a$

$du = 2x\,dx$

Section 6.3. Integration of Rational Functions by Partial Fractions.

In this section, we will develop a way to evaluate integrals of the form $\int \frac{P(x)}{Q(x)}\ dx$, where $P(x)$ and $Q(x)$ are polynomials. The idea is to write a rational function as a sum of partial fractions whose denominators are simple linear factors, linear factors of multiplicity m, irreducible factors of degree 2.

Consider the following example.

$$\int \frac{x^3 - 4x - 10}{x^2 - x - 6}\ dx$$

It is clear that a direct substitution will not work. To evaluate a rational integral of the form $\int \frac{P(x)}{Q(x)}\ dx$ we perform the following tasks:

1. If the degree of $P(x)$ is greater $\geq$ than or equal to the degree of $Q(x)$ then divide $P(x)$ by $Q(x)$ so that $\frac{P(x)}{Q(x)} = \underbrace{S(x)}_{\text{polynomial}} + \frac{R(x)}{Q(x)}$.

2. Factor the denominator of $\frac{R(x)}{Q(x)}$ as far as possible. Then, the following situations can occur:

   - **The denominator $Q(x)$ is a product of linear factors.**
     $Q(x) = (a_1x + b_1)(a_2x + b_2)\dots(a_kx + b_k)$
     then there exists real numbers $A_1, A_2, \dots A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots \frac{A_k}{a_kx + b_k}$$

Example: Write
$\dfrac{x^3 - 4x - 10}{x^2 - x - 6}$ as a sum of partial fractions.

Divide:

quotient

$$
\begin{array}{r|l}
 & x + 1 \\
x^2 - x - 6 & x^3 + 0x^2 - 4x - 10 \\
 & -(x^3 - x^2 - 6x) \\
 & \quad x^2 + 2x - 10 \\
 & \quad -(x^2 - x - 6) \\
 & \qquad 3x - 4
\end{array}
$$

remainder

$$\frac{x^3 - 4x - 10}{x^2 - x - 6} = x + 1 + \frac{3x-4}{x^2 - x - 6}$$

?

$\left(\frac{1}{12} = \frac{1}{3} - \frac{1}{4}\right.$

Partial fractions:

$$\frac{3x-4}{x^2-x-6} = \frac{3x-4}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$ where A & B are constants

Find A & B

$$\left(\frac{3x-4}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}\right)(x-3)(x+2)$$

$$3x - 4 = A(x+2) + B(x-3)$$

Elimination method:

$x = -2 \qquad -10 = -5B$

$\boxed{B = 2}$

$x = 3 \qquad 5 = 5A$

$\boxed{A = 1}$

Identifying coefficient method

$3x - 4 = (A+B)x + 2A - 3B$

$\begin{cases} A + B = 3 \\ 2A - 3B = -4 \end{cases}$ solve for A & B

$$\int \frac{x^3 - 4x - 10}{x^2 - x - 6}\,dx = \int \left(x + 1 + \frac{1}{x-3} + \frac{2}{x+2}\right) dx$$

$$= \frac{x^2}{2} + x + \ln|x-3| + 2\ln|x+2| + C$$

- **The denominator $Q(x)$ is a product of linear factors, some of which are repeated.**

  If $Q(x)$ contains a factor (a_1x+b_1) that is repeated m times, then the sum of partial fractions will contain the following

$$\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\ldots+\frac{A_m}{(a_1x+b_1)^m}$$

Example: Write $\dfrac{x-5}{x^2(x-1)^3(x+3)}$ as a sum of partial fractions.

$$\frac{x-5}{x^2(x-1)^3(x+3)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1}{x-1} + \frac{B_2}{(x-1)^2} + \frac{B_3}{(x-1)^3} + \frac{C}{(x+3)}$$

$\ln|x| \qquad -\frac{1}{x} \qquad \ln|x-1| \qquad -\frac{1}{x-1} \qquad \int u^{-3}\,du.$

"Warm-up" example: Evaluate the integral.

$$\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\,dx$$

Partial Fractions

$$\left(\frac{x^2-5x+16}{(2x+1)(x-2)^2} = \frac{A}{2x+1} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}\right)(2x+1)(x-2)^2$$

$$x^2-5x+16 = A(x-2)^2 + B(x-2)(2x+1) + C(2x+1)$$

- $x=-\frac{1}{2}$, $\frac{1}{4}+\frac{5}{2}+16 = \frac{75}{4} = A\left(-\frac{5}{2}\right)^2 = \frac{25}{4}A \Rightarrow \boxed{A=3}$

- $x=2$, $10 = 5C \Rightarrow \boxed{C=2}$

$B=?$, identify coefficients: (x^2) $1 = \underbrace{A}_{3} + 2B$ $\quad 2B=-2 \quad \boxed{B=-1}$

$$\int \left(\frac{3}{\underbrace{2x+1}_{u=2x+1}} - \frac{1}{x-2} + \frac{2}{(x-2)^2}\right) dx$$

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x}$$

$$= \frac{3}{2}\ln|2x+1| - \ln|x-2| - \frac{2}{(x-2)} + C$$

- **The denominator $Q(x)$ contains irreducible factors, none of which is repeated.**

  When is a polynomial of the form $ax^2 + bx + c$ irreducible?

  $ax^2+bx+c=0$

  $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ discriminant

  ax^2+bx+c is irreducible when $b^2-4ac<0$

  $y = x^2+1 > 0$

  The decomposition of $\dfrac{R(x)}{Q(x)}$ will contain terms of the form

  $$\boxed{\frac{Ax + B}{ax^2 + bx + c}}$$

  where A and B are real numbers.

  Example: Write $\dfrac{x^2 - 2x - 1}{(x-1)^2(x^2+1)}$ as a sum of partial fractions, then integrate the function.

  $$\frac{x^2-2x-1}{(x-1)^2(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

  $(x-1)^2$: linear factor repeated twice

  (x^2+1): irreducible factor

  $\dfrac{Cx+D}{x^2+1} \to \dfrac{Cx}{x^2+1} + \dfrac{D}{x^2+1}$

  $\dfrac{Cx}{x^2+1} \to \frac{1}{2}\ln(x^2+1)$, $\quad \dfrac{D}{x^2+1} \to \tan^{-1}x$

$x^2-2x-1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$

$x=1$, $-2 = 2B \Rightarrow \boxed{B=-1}$

Identify coefficients:

x^3, $\boxed{0 = A+C}$

$x=0$ $\quad -1 = -A + \overset{-1}{B} + D \Rightarrow \boxed{-A+D=0}$

$x=2$ $\quad -1 = 5A + \underbrace{5B}_{=-5} + 2C + D$

$4 = 5A + 2C + D$, $D = A$

$4 = 6A + 2C$ $\quad C = -A$

$4 = 6A - 2A = 4A \Rightarrow \boxed{A=1}$

$\boxed{C=-1}$ $\quad \boxed{D=1}$

$$\int \left(\frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{-x+1}{x^2+1} \right) dx$$

$$= \ln|x-1| + \frac{1}{x-1} + \int \left(-\frac{x}{x^2+1} + \frac{1}{x^2+1} \right) dx$$

$u = x^2+1$

$$= \ln|x-1| + \frac{1}{x-1} - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + C$$

- **The denominator $Q(x)$ contains a repeated irreducible factor.**
  If the factor $ax^2 + bx + c$ is an irreducible factor repeated m times, then we will the sum of partial fractions.

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \ldots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}$$

Example: Write $\dfrac{x-5}{x^2(x^2+1)^3}$ as a sum of partial fractions.

linear repeated twice

irreducible repeated 3 times:

$$\frac{x-5}{x^2(x^2+1)^3} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1x+C_1}{(x^2+1)} + \frac{B_2x+C_2}{(x^2+1)^2} + \frac{B_3x+C_3}{(x^2+1)^3}$$

$$\int \sec x \, dx \overset{?}{=} \ln|\sec x + \tan x| + C$$

$$\int \frac{1}{\cos x} \, dx = \int \frac{\cos x}{\cos^2 x} \, dx = \int \frac{\cos x}{1 - \sin^2 x} \, dx$$

Let $u = \sin x \qquad du = \cos x \, dx$

$$\int \frac{1}{1-u^2} \, du = \int \frac{1}{(1+u)(1-u)} \, du \quad (*)$$

Partial fractions:

$$\frac{1}{(1+u)(1-u)} = \frac{A}{1+u} + \frac{B}{1-u}$$

$$1 = A(1-u) + B(1+u)$$

$u = 1 \qquad 1 = 2B \implies B = \frac{1}{2}$

$u = -1 \qquad 1 = 2A \implies A = \frac{1}{2}$

$$(*) = \frac{1}{2} \int \left(\frac{1}{1+u} + \frac{1}{1-u} \right) du = \frac{1}{2} \Big[\ln|1+u| - \ln|1-u| \Big] + C$$

$$= \frac{1}{2} \Big[\ln|1+\sin x| - \ln|1 - \sin x| \Big] + C$$

$$= \frac{1}{2} \ln\left| \frac{1+\sin x}{1-\sin x} \right| + C = \frac{1}{2} \ln\left| \frac{(1+\sin x)(1+\sin x)}{(1-\sin x)(1+\sin x)} \right| + C$$

$$= \frac{1}{2} \ln\left| \frac{(1+\sin x)^2}{1-\sin^2 x} \right| = \frac{1}{2} \ln\left| \frac{(1+\sin x)^2}{\cos^2 x} \right| + C$$

$$= \ln|\sec x + \tan x| + C = \ln\left| \frac{1+\sin x}{\cos x} \right| + C$$

Rationalizing Substitutions.

Some integrals involving radicals can be treated like integrals of rational functions by letting $u = \sqrt[n]{g(x)}$.

Example: Evaluate the integral.

$$\int \frac{\sqrt{x}}{x+1}\ dx$$

Rationalizing substitution : $u = \sqrt{x} \Rightarrow x = u^2$

$dx = 2u\,du$

$$\int \frac{u}{u^2+1}(2u\,du) = 2\int \frac{u^2}{u^2+1}\,du$$

Divide: $\dfrac{u^2}{u^2+1} = \dfrac{u^2+1-1}{u^2+1} = 1 - \dfrac{1}{u^2+1}$

$$2\int \frac{u^2}{u^2+1}\,du = 2\int \left(1 - \frac{1}{u^2+1}\right)du = 2\left[u - \tan^{-1}u\right] + C$$

$$\int \frac{\sqrt{x}}{x+1}\,dx = 2\left(\sqrt{x} - \tan^{-1}(\sqrt{x})\right) + C$$

Additional example:

$$\int \frac{x^5+x-1}{x^3+1}\,dx$$

$$\begin{array}{r|l} & x^2 \\ \hline x^3+1 & x^5+0x^4+0x^3+0x^2+x-1 \\ & -(x^5 \qquad\qquad\qquad +x^2) \\ \hline & \qquad\qquad\qquad -x^2+x-1 \end{array}$$

$$\int \left(x^2+\frac{-x^2+x-1}{x^3+1}\right)dx = \frac{x^3}{3}+\int \frac{-x^2+x-1}{x^3+1}\,dx$$

$$\frac{-x^2+x-1}{x^3+1} = \frac{\overset{-(x^2\cancel{-x+1})}{-x^2+x-1}}{(x+1)(x^2\cancel{-x+1})} = \frac{-1}{x+1}$$

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

$$\frac{x^3}{3}+\int \frac{-1}{x+1}\,dx = \frac{x^3}{3}-\ln|x+1|+C$$