Solution of Exam 2 (make-up) MTH 224-O, Spring 2024

Question 1. (a) (3 points) Flight A: 10 tickets sold and the plane has 9 seats. Flight B: 20 tickets sold and the plane has 18 seats. Passengers show up with probability 0.9 each, and independently of each other. Which flight is more likely to get overbooked? Show your calculations. A flight is overbooked if more passengers show up than available seats.

Solution: Let X_A = number of passengers who show up for flight A, and X_B = the number of passengers who show up for flight B. Note that $X_A \sim \text{bin}(10, 0.9)$, and $X_B \sim \text{Bin}(20, 0.9)$. Hence,

$$\mathbb{P}(\text{A overbooked}) = \mathbb{P}(X_A = 10) = \binom{10}{10} \cdot 0.9^{10} = 0.35.$$

$$\mathbb{P}(\text{B overbooked}) = \mathbb{P}(X_B = 19) + \mathbb{P}(X_B = 20) = \binom{20}{19} 0.9^{19} 0.1^1 + \binom{20}{20} \cdot 0.9^{20} = 0.39.$$

Therefore, B is more likely to get overbooked.

(b) (3 points) You have a box containing 25 LEGO bricks, 7 of which are broken. You randomly take 3 bricks out of the box. Let X be the number of broken LEGO out of the 3 bricks. Find $\mathbb{E}(X)$, $\text{Var}(X)$, and $\text{Var}(20 - X)$.

Solution: We have $X \sim HG(N = 25, D = 7, n = 3)$. So,$\mathbb{E}[X] = \frac{nD}{N} = \frac{3\times 7}{25} = 0.84$ and

$$\text{Var}(X) = \frac{nD}{N}\left(1 - \frac{D}{N}\right)\left(\frac{N-n}{N-1}\right) = \frac{3 \times 7}{25} \times \left(1 - \frac{7}{25}\right) \times \left(\frac{25-3}{25-1}\right) = \frac{693}{1250} \approx 0.554.$$

Finally, $\text{Var}(20 - X) = |-1|^2 \, \text{Var}(X) = \text{Var}(X) \approx 0.554$.

(c) (2 points) Cars enter a crossroads according to a Poisson process. On average, 10 cars enter the crossroads every hour. Given that between 4pm to 5pm, exactly 10 cars have entered the crossroads, what is the probability that between 4pm to 6pm exactly 19 cars have entered the crossroads?.

Solution: We have that

$$\mathbb{P}(\text{19 cars from 4 - 6} | \text{10 cars from 4 - 5}) = \mathbb{P}(\text{9 cars from 5 - 6} | \text{10 cars from 4 - 5}).$$

By independence of the latter events, this is equal to

$$\mathbb{P}(\text{9 cars from 5pm - 6pm}) = \frac{10^9}{9!} \cdot e^{-10} \approx 0.125.$$

Note that if X = number of cars between 5pm and 6pm, then $X \sim \text{Pois}(10)$.

Question 2. (6 points) We roll a fair die N times, where N is a Poisson random variable with parameter 4. Let X be the number of times we get an outcome of six. What is the distribution of X? Show your calculations.

Solution: It is given that $N \sim \text{Pois}(4)$, thus $\mathbb{P}(N=n) = e^{-4}\frac{4^n}{n!}$, $n = 0, 1, \dots$.
Let X be the total number of sixes we get. We can use the law of total probability to find the probability that $X = m$, by conditioning on the events $\{N = n\}$. Notice that if $N = n$ is fixed, then $X \sim \text{Bin}(n, \frac{1}{6}))$. We must flip at least k dice in order to get k sixes. Therefore,

$$\begin{aligned}
\mathbb{P}(X=k) &= \sum_{n=k}^{\infty} \mathbb{P}(X=k \mid N=n)\,\mathbb{P}(N=n) = \sum_{n=k}^{\infty} \binom{n}{k}\left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{n-k} \times e^{-4} \times \frac{4^n}{n!} \\
&= e^{-4} \times \sum_{n=k}^{\infty} \frac{n!}{(n-k)!k!} \times \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{n-k} \times \frac{4^k \times 4^{n-k}}{n!} = \frac{e^{-4}}{k!}\left(\frac{4}{6}\right)^k \sum_{n=k}^{\infty} \frac{\left(\frac{4\times 5}{6}\right)^{n-k}}{(n-k)!} \\
&= \frac{e^{-4}}{k!}\left(\frac{4}{6}\right)^k \sum_{m=0}^{\infty} \frac{\left(\frac{10}{3}\right)^m}{m!} = \frac{e^{-4}}{k!}\left(\frac{4}{6}\right)^k e^{\frac{10}{3}},
\end{aligned}$$

in which we have used the Taylor's series $e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$. Therefore,

$$\mathbb{P}(X=k) = \frac{e^{-4+\frac{10}{3}}}{k!}\left(\frac{2}{3}\right)^k = \frac{(2/3)^k}{k!}e^{-2/3}.$$

This is the pmf of a Poisson random variable with parameter $\lambda = 2/3$.

Question 3. Let X be a continuous r.v. with pdf $f_X(x) = c\,\mathrm{e}^{2x}$ for $x \in (-\infty, 2]$ and $f_X(x) = 0$ for $x > 2$, in which c is a constant.

(a) (2 points) Find the value of c.

Solution: We have $1 = \int_{-\infty}^{2} c\mathrm{e}^{2x}\mathrm{d}x = \frac{c}{2}\mathrm{e}^{2x}\big|_{-\infty}^{2} = \frac{\mathrm{e}^4}{2}\,c \quad \Longrightarrow \quad c = 2\,\mathrm{e}^{-4} \approx 0.03663$

(b) (3 points) Find $\mathrm{Var}(X)$. **Hint:** Recall the integration-by-parts formula:

$$\int_a^b u(x)v'(x)\mathrm{d}x = \big(u(b)v(b) - u(a)v(a)\big) - \int_a^b u'(x)v(x)\mathrm{d}x.$$

Solution: We have,

$\mathbb{E}[X] = \int_{-\infty}^{2} cx\mathrm{e}^{2x}\mathrm{d}x = \frac{c}{2}x\mathrm{e}^{2x}\big|_{-\infty}^{2} - \int_{-\infty}^{2}\frac{c}{2}\mathrm{e}^{2x}\mathrm{d}x = \frac{c}{2}\left(2\mathrm{e}^4 - 0 - \frac{1}{2}\mathrm{e}^{2x}\big|_{-\infty}^{2}\right) = \mathrm{e}^{-4}\left(2\mathrm{e}^4 - \frac{1}{2}\mathrm{e}^4\right) = \frac{3}{2} = 1.5.$

Furthermore,

$\mathbb{E}[X^2] = \int_{-\infty}^{2} cx^2\mathrm{e}^{2x}\mathrm{d}x = \frac{c}{2}x^2\mathrm{e}^{2x}\big|_{-\infty}^{2} - \underbrace{\int_{-\infty}^{2} cx\mathrm{e}^{2x}\mathrm{d}x}_{=\mathbb{E}[X]=1.5} = \frac{c}{2}\left(4\mathrm{e}^4 - 0\right) - 1.5 = \mathrm{e}^{-4}\times 4\mathrm{e}^4 - 1.5 = 2.5.$

Finally, we obtain that $\mathrm{Var}(X) = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2 = 2.5 - 1.5^2 = 0.25.$

(c) (3 points) Find the pdf of the random variable $Y = \ln(2 - X)$.

Solution: Since $-\infty < X \leq 2 \Leftrightarrow 0 \leq 2 - X < +\infty \Leftrightarrow -\infty < \ln(2 - X) < +\infty$, the support of Y is $(-\infty, +\infty)$. To find the pdf of Y, we First find its cdf. For any $y \in \mathbb{R}$,

$$F_Y(y) = \mathbb{P}\big(\ln(2 - X) \leq y\big) = \mathbb{P}\left(2 - X \leq \mathrm{e}^y\right) = \mathbb{P}\left(X \geq 2 - \mathrm{e}^y\right) = 1 - F_X\left(2 - \mathrm{e}^y\right).$$

Finally, we obtain the pdf of Y by differentiating $F_Y(y)$:

$$f_Y(y) = F_Y'(y) = \frac{\mathrm{d}}{\mathrm{d}y}\left(1 - F_X\left(2 - \mathrm{e}^y\right)\right) = -f_X\left(2 - \mathrm{e}^y\right)\left(-\mathrm{e}^y\right) = 2\mathrm{e}^{2(2-\mathrm{e}^y-2)}\mathrm{e}^y = 2\mathrm{e}^{y-2\mathrm{e}^y},$$

for $y \in \mathbb{R}$. Note that from part (a), $F_X'(x) = f_X(x) = 2\mathrm{e}^{2(x-2)}$ for $x \leq 2$.