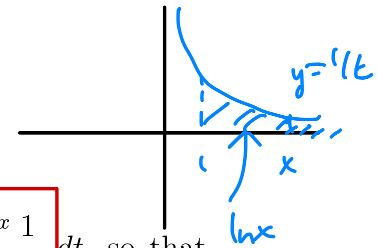
Section 6.6. Improper Integrals.



- **Infinite Integrals.**

We defined the Natural Logarithmic function by $= \boxed{\ln x = \int_1^x \frac{1}{t}}\, dt$, so that $y = \ln x$ represent the area under $y = \frac{1}{t}$ for $1 \le t \le x$. What happens to this area when x becomes "very large"?

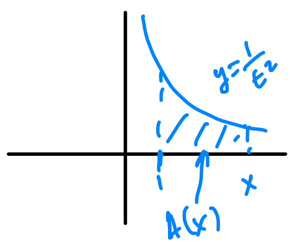
$\int_1^\infty \frac{1}{t}\,dt = \infty = \lim_{x\to\infty} \int_1^x \frac{1}{t}\,dt$ divergent

When $x \to \infty$ we know that $\ln x \to \infty$. Therefore, the area under $y = \frac{1}{t}$ is infinite. Consider now the following situation.

Let $A(x) = \displaystyle\int_1^x \frac{1}{t^2}\, dt$.

1. Evaluate $A(x)$

$A(x) = -\frac{1}{t}\Big|_1^x = -\frac{1}{x} + 1$



$\lim_{x\to\infty} A(x) = \lim_{x\to\infty} \left(-\frac{1}{x} + 1\right) = 1$

$\int_1^\infty \frac{1}{t^2}\,dt = 1$ convergent

2. What happens to $A(x)$ as $x \to \infty$?

- **Definition of an Infinite Integral.**

(a) If $\int_a^t f(x)\ dx$ exists for every $t \geq a$, then

$$\boxed{\int_a^\infty f(x)\ dx = \lim_{t \to \infty} \int_a^t f(x)\ dx}$$

provided this limit exists as a finite number.

(b) If $\int_t^b f(x)\ dx$ exists for every $t \leq b$, then

$$\boxed{\int_{-\infty}^b f(x)\ dx = \lim_{t \to -\infty} \int_t^b f(x)\ dx}$$

provided this limit exists as a finite number.

$\int_a^\infty f(x)\ dx$ and $\int_{-\infty}^b f(x)\ dx$ are called *convergent* if the corresponding limit exists and *divergent* if the limit does not exist.

(c) If both $\int_a^\infty f(x)\ dx$ and $\int_{-\infty}^a f(x)\ dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\ dx = \int_{-\infty}^a f(x)\ dx + \int_a^\infty f(x)\ dx$$

$\int_3^\infty f(x)dx$ is divergent where f is increasing on $(3,\infty)$

Examples

1. Determine whether the integral is convergent or divergent. Evaluate those that are convergent.

$$\int_{-\infty}^{6} xe^{x/3}\,dx = \lim_{t\to-\infty}\int_t^6 xe^{x/3}\,dx$$

① Evaluate $\int_t^6 xe^{x/3}\,dx$

IBP: $u = x \quad dv = e^{x/3}\,dx$

$du = dx \quad v = 3e^{x/3}$

$$\int_t^6 xe^{x/3}\,dx = 3xe^{x/3}\Big|_t^6 - 3\underbrace{\int_t^6 e^{x/3}\,dx}_{3e^{x/3}\big|_t^6}$$

$$\int_t^6 xe^{x/3}\,dx = 18e^2 - 3te^{t/3} - 9e^2 + 9e^{t/3} = 9e^2 - 3te^{t/3} + 9e^{t/3}$$

② $$\lim_{t\to-\infty}\left(9e^2 - 3\underbrace{\overset{-\infty}{t}\,\overset{\to 0}{e^{t/3}}}_{\frac{t\to-\infty}{e^{-t/3}\to\infty}} + \underbrace{9e^{t/3}}_{0}\right) = 9e^2$$

$$\lim_{t\to-\infty}\frac{t}{e^{-t/3}} = \lim_{t\to-\infty}\frac{1}{-\frac{1}{3}e^{-t/3}} = \lim_{t\to-\infty} -3e^{t/3} = 0$$

$\int_{-\infty}^{6} xe^{x/3}\,dx$ is convergent and $\int_{-\infty}^{6} xe^{x/3}\,dx = 9e^2$

2. For what values of n is the integral $\int_1^{\infty} \frac{1}{x^n}\, dx$ convergent?

* $n=1$ $\int_1^{\infty} \frac{1}{x}\,dx$ is divergent

$n=2$ $\int_1^{\infty} \frac{1}{x^2}\,dx$ is convergent

$$\int_1^{\infty} \frac{1}{x^n}\,dx = \lim_{t\to\infty} \int_1^{t} x^{-n}\,dx = \lim_{t\to\infty} \left. \frac{x^{-n+1}}{-n+1} \right|_1^t \qquad n \neq 1$$

$$= \lim_{t\to\infty} \left(\frac{1}{1-n} \left(t^{-n+1} - 1 \right) \right)$$

$\lim_{t\to\infty} t^{-n+1} = 0$ if $-n+1 < 0$ or $(n>1)$

$\lim_{t\to\infty} t^{-n+1} = \infty$ if $-n+1 \geq 0$ (or $n \leq 1$)

"Test" integrals

$a>0$

$$\boxed{\int_{a}^{\infty} \frac{1}{x^p}\,dx \text{ is } \begin{cases} \text{convergent if } p>1 \\ \text{divergent if } p \leq 1 \end{cases}}$$

Warmup examples: Is the integral convergent or divergent ?

$\int_6^{\infty} \frac{5}{x^4}\,dx$ $\quad 4>1$, convergent

$\int_{-\infty}^{5} x^3\,dx$, divergent

$\int_2^{\infty} \frac{1}{\sqrt{x}}\,dx$ $\quad x^{1/2}$ $\quad 1/2<1$, divergent

- **Discontinuous Integrands.**

  When evaluating $\int_0^3 \frac{1}{x\sqrt{x}}\ dx$, the Fundamental Theorem of Calculus will not work because $y = \frac{1}{x\sqrt{x}}$ is not continuous at $x = 0$.

- **Definition of an Improper Integral with discontinuous Integrand.**

  (a) If f is continuous on $(a, b]$, then

  $$\boxed{\int_a^b f(x)\ dx = \lim_{t\to a^+} \int_t^b f(x)\ dx}$$

  provided this limit exists as a finite number.

  (b) If f is continuous on $[a, b)$, then

  $$\boxed{\int_a^b f(x)\ dx = \lim_{t\to b^-} \int_a^t f(x)\ dx}$$

  provided this limit exists as a finite number.

  $\int_a^b f(x)\ dx$ is called *convergent* if the corresponding limit exists, and *divergent* if the limit does not exist.

  $a < c < b$

  (c) If f has a discontinuity at c, where $a < \cancel{b < c}$, and both $\int_a^c f(x)\ dx$ and $\int_c^b f(x)\ dx$ are convergent, then we define $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$.

Example Determine whether the integral is convergent or divergent. Evaluate the integral if it is convergent.

$$\int_0^3 \frac{1}{x\sqrt{x}}\ dx = \lim_{t\to 0^+} \int_t^3 x^{-3/2}\ dx = \lim_{t\to 0^+} \left[-2x^{-1/2}\right]_t^3$$

$$= \lim_{t\to 0^+} -2 \left.\frac{1}{\sqrt{x}}\right|_t^3 = \lim_{t\to 0^+} -2\left(\frac{1}{\sqrt{3}} - \overset{\to\infty}{\frac{1}{\sqrt{t}}}\right) = \infty$$

$\int_0^3 \frac{1}{x\sqrt{x}}\ dx$ is divergent

If $a>0$ (*) $\int_a^\infty \frac{1}{x^p}\,dx$ is convergent if $p>1$, divergent if $p\le 1$

If $a>0$ (*) $\int_0^a \frac{1}{x^p}\,dx$ is convergent if $p<1$, divergent if $p\geq 1$

$= \lim_{t\to 0^+} \int_t^a x^{-p}\,dx$ etc. . .

- A Comparison Test for Improper Integrals.

  Consider the improper integral $\displaystyle\int_1^\infty \frac{1}{\sqrt{1+x^3}}\ dx$. If we use material from the first paragraph the idea is to find

  ____

$$\int_1^\infty \frac{1}{\sqrt{1+x^3}}\,dx = \lim_{t\to\infty} \int_1^t \underbrace{\frac{1}{\sqrt{1+x^3}}}_{\text{does not have an antiderivative.}}\,dx$$

$f(x) = \frac{1}{\sqrt{1+x^3}}$

as $x\to\infty$ $\frac{1}{\sqrt{1+x^3}} \sim \frac{1}{x^{3/2}}$

$\int_1^\infty \frac{1}{x^{3/2}}\,dx$ is cv

Comparison Theorem. Rewrite

Suppose that f and g are continuous functions with $0 \leq g(x) \leq f(x)$ for $x \geq a$.

(a) If $\displaystyle\int_a^\infty f(x)\ dx$ is convergent, $\displaystyle\int_a^\infty g(x)\ dx$ is convergent.

(b) If $\displaystyle\int_a^\infty g(x)\ dx$ is divergent, $\displaystyle\int_a^\infty f(x)\ dx$ is divergent.

Remark: This theorem remains valid for all types of improper integrals.

Test "integral

Important: Assume $a > 0$

- $\displaystyle\int_a^\infty \frac{1}{x^p}\ dx$ is covergent if $p > 1$, divergent if $p \leq 1$
- $\displaystyle\int_0^a \frac{1}{x^p}\ dx$ is covergent if $p < 1$, divergent if $p \geq 1$

$$\int_1^{\infty} \frac{1}{\sqrt{x^3+1}}\,dx = ?$$

As $x \to \infty$ $\quad \frac{1}{\sqrt{x^3+1}} \sim \frac{1}{x^{3/2}}$ $\quad \int_1^{\infty} \frac{1}{x^{3/2}}\,dx$ is convergent $\quad 3/2 > 1$

"Educated guess" $\int_1^{\infty} \frac{1}{\sqrt{x^3+1}}\,dx$ is cv

The comparison Theorem: $a>0$

Start with $\int_a^{\infty} g(x)\,dx = ?$ and $\int_a^{\infty} f(x)\,dx$ (test integral)

Case 1: If $g(x) \leq f(x)$ for $x>a$ and $\int_a^{\infty} f(x)\,dx$ is convergent,

then $\int_a^{\infty} g(x)\,dx$ is also convergent.

Case 2: If $g(x) \geq f(x)$ for $x>a$ and $\int_a^{\infty} f(x)\,dx$ is divergent

then $\int_a^{\infty} g(x)\,dx$ is also divergent

Back to $\int_1^{\infty} \underbrace{\frac{1}{\sqrt{x^3+1}}}_{g(x)}\,dx$. Compare $g(x) = \frac{1}{\sqrt{x^3+1}}$

to $\frac{1}{x^{3/2}} = f(x)$

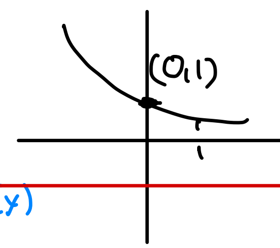
$$\boxed{\text{If } x \geq 1 \quad \frac{1}{\sqrt{x^3+1}} \leq \frac{1}{x^{3/2}}. \text{ Since } \int_1^{\infty} \frac{1}{x^{3/2}}\,dx \text{ is convergent, } \int_1^{\infty} \frac{1}{\sqrt{x^3+1}}\,dx \text{ is convergent}}$$

Exercises. Use the Comparison Theorem to determine whether the integral is convergent or divergent.

1. $\displaystyle\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$ (on Wednesday)

$$\int_0^1 \frac{1}{x^p}\,dx \text{ is } \begin{cases} \text{convergent if } p<1 \\ \text{divergent if } p \geq 1 \end{cases}$$

2. $\displaystyle\int_0^1 \underbrace{\frac{e^{-x}}{\sqrt{x}}}_{g(x)}\,dx$



$g(x) = \dfrac{e^{-x}}{\sqrt{x}}$ $\qquad$ $f(x) = \dfrac{1}{\sqrt{x}}$

if $0 < x \leq 1$ $\quad e^{-x} \leq 1 \Rightarrow \underbrace{\dfrac{e^{-x}}{\sqrt{x}}}_{g(x)} \leq \underbrace{\dfrac{1}{\sqrt{x}}}_{f(x)}$. Since $\displaystyle\int_0^1 \frac{1}{x^{1/2}}\,dx$ is convergent, therefore $\displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$ is also convergent