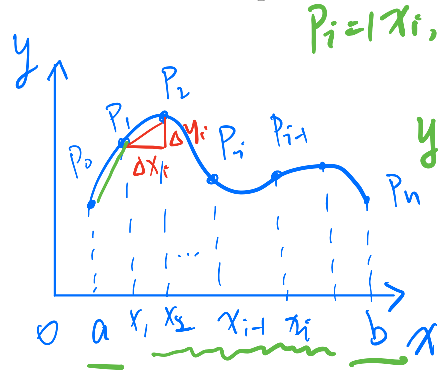
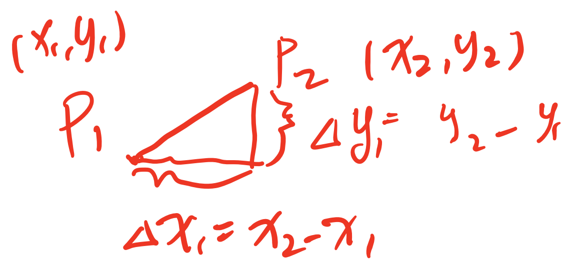
Chapter 7. Applications of Integrations.
Section 7.4. Arc Length.

In this section we will learn how to find the length of a curve described by a continuous function $y = f(x)$ for $a \le x \le b$. The method is to approximate the length of the curve by adding the length of line segments $|P_{i-1}P_i|$ for $0 \le i \le n$. Let us draw a picture that describe the above concept.

$P_i = (x_i, y_i)$



$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| \cdots\cdots ①$$



Write an expression for $|P_{i-1}P_i|$. (Hint: Use the Pythagorean Theorem.)

$$|P_{i-1}P_i| = \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

for f on the terval $[x_{i-1}, x_i]$, can always find $x_i^* \in [x_{i-1}, x_i]$, s.t. $\boxed{\begin{aligned} f(x_i) - f(x_{i-1}) &= f'(x^*)(x_i - x_{i-1}) \\ \Delta y_i &= f'(x^*)\,\Delta x_i \end{aligned}}$

Use the Mean Value Theorem (what is this again?) and the limit of a Riemann sum to show that

- **Definition:** If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \le x \le b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx$$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

MVT

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2} \cdot \Delta x_i$$

limit of Riemann sum

$$= \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

An alternate notation is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

- **Definition.** If g' is continuous on $[c, d]$, then the length of the curve $x = g(y)$, $c \le y \le d$, is

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\ dy$$

An alternate notation is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

Example: Find the length of the curve $y = \frac{x^2}{2} - \frac{\ln x}{4}$, $2 \le x \le 4$.

$$L = \int_2^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \int_2^4 \sqrt{1 + \left(x - \frac{1}{4x}\right)^2}\ dx$$

$$= \int_2^4 \sqrt{\frac{16x^2 + (4x^2-1)^2}{16x^2}}\ dx$$

$16x^2 + 16x^4 - 8x^2 + 1$

$16x^4 + 8x^2 + 1$

$(4x^2+1)^2$

$$= \int_2^4 \sqrt{\frac{(4x^2+1)^2}{16x^2}}\ dx$$

$$= \int_2^4 \frac{4x^2+1}{4x}\ dx$$

$$= \int_2^4 x + \frac{1}{4x}\ dx = \int_2^4 x\,dx + \frac{1}{4}\int_2^4 \frac{1}{x}\,dx$$

$$= \frac{1}{2}x^2\Big|_2^4 + \frac{1}{4}\ln x\Big|_2^4 = 6 + \frac{1}{4}\ln 2$$

- The Arc Length Function.

  Rather than calculating a different integral each time we want to find the length of the curve C defined by $y = f(x)$ for different end points, we can define the arc length function $s(x)$ to be the distance along C from an initial point $P(a, f(a))$ to the point $Q(x, f(x))$ by

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\ dt$$

Example: Find the arc length function for the curve $y = 2x^{3/2}$ with starting point $P(1, 2)$.

$x = 1$

$y' = 3x^{1/2}$

$$S(x) = \int_1^x \sqrt{1+(3t^{1/2})^2}\ dt$$

y

$$= \int_1^x \sqrt{1+9t}\ dt$$

$$= \frac{1}{9} \cdot \frac{2}{3} (1+9t)^{3/2} \Big|_1^x$$

$$= \frac{2}{27}(1+9x)^{3/2} - \frac{1}{9} \cdot \frac{2}{3} \cdot (10)^{\frac{3}{2}}$$

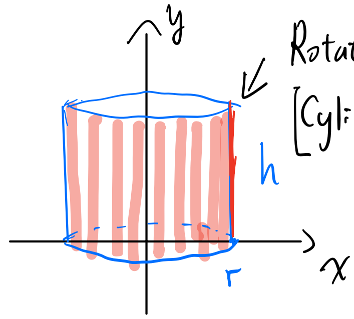
$$= \frac{2}{27}(1+9x)^{3/2} - \frac{20}{27}\sqrt{10}$$

Section 7.5. Area of a surface of revolution

A surface of revolution can be obtained by rotating a curve about a line.

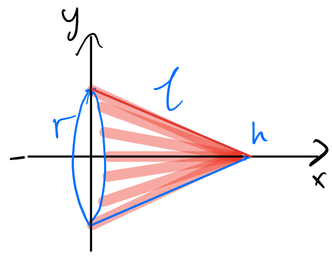
A solid of revolution can be obtained by rotating an area about a line. For example, a cylinder can be formed by rotating a rectangular surface about the y–axis.

Ex.



Rotate $x = g(y) = r$, $y \in [0, h]$, about the y-axis

[Cylinder]



[Cone]

Rotate $y = -\frac{r}{h}x + r$, $x \in [0, h]$ about the x-axis.

We know the volume of a cylinder with a circular base of radius r, and height h is $V = \pi r^2 h$. The lateral surface area A is the area of a rectangle with length h, and width $2\pi r$. Therefore,

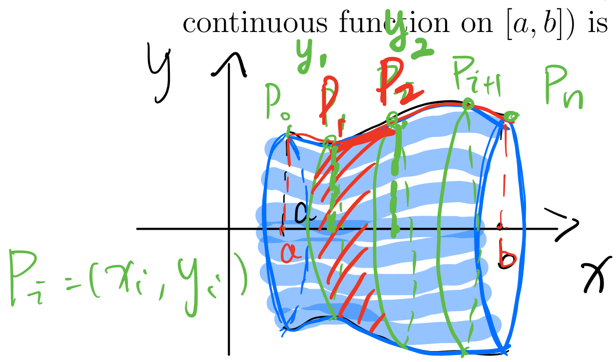
$$A = 2\pi r h$$

Using a similar argument, it can be shown that the surface area of a cone with baas radius r, and slant height l is

$$A = \pi r l$$

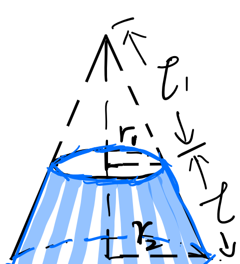
What happens to other solids of revotion?

Assume the area of the region enclose by $y = f(x)$, $a \le x \le b$ (where f is a continuous function on $[a, b]$) is rotated about the x-axis.



$S = \lim_{n\to\infty} \sum_{i=1}^{n} A_i$, A_i is the area of the band with slant height $|P_{i-1}P_i|$ and lower radius y_{i-1}, upper radius y_i.

- To find the formula for lateral area of band:



$A = \pi r_2 (l_1 + l) - \pi r_1 l_1$

$= \pi[(r_2 - r_1) l_1 + r_2 l]$

$\frac{r_1}{l_1} = \frac{r_2}{l_1 + l} \Rightarrow (r_2 - r_1) l_1 = r_1 l$

So $A = \pi(r_1 + r_2) l$, or $A = 2\pi r l$, $r = \frac{1}{2}(r_1 + r_2)$.

Then $S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \frac{(y_i + y_{i-1})}{2} \sqrt{1+f'(x_i^*)^2}\, \Delta x_i$.

Δx small, $y_i \approx y_{i-1} \approx f(x_i^*)$

The surface area S of a slice is $A = 2\pi r l$. To obtain the total surface area, we add all the areas of all the "slices" of the solid. Therefore,

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\ dx$$

or, equivalently

$$S = \int_a^b 2\pi f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

If the region enclosed by $x = g(y)$, $c \le y \le d$ is rotated about the x–axis

$$S = \int_c^d 2\pi g(y)\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

- Find the exact area of the surface obtained by rotating the curve about the x–axis.

  $y = x^3$, $0 \le x \le 2$.

$$S = \int_0^2 2\pi \overset{y}{x^3} \sqrt{1 + (\underbrace{3x^2}_{y'})^2}\ dx$$

$$= \int_0^2 2\pi x^3 \sqrt{1+9x^4}\ dx$$

$u = x^4 \qquad du = 4x^3 dx$

$$= \int_0^{16} \frac{\pi}{2}\sqrt{1+9u}\ du$$

$$= \frac{\pi}{2}\cdot\frac{1}{9}\cdot\frac{2}{3}(1+9u)^{3/2}\Big|_0^{16}$$

$$= \frac{\pi}{27}(145\sqrt{145} - 1)$$

- Find the exact area of the surface obtained by rotating the curve about the y–axis.

  $y = 1 - x^2$, $0 \le x \le 1$.

$x = g(y)$

$$\begin{cases} x = g(y) = \sqrt{1-y} \\ 0 \le y \le 1 \end{cases}$$

$$S = \int_a^b 2\pi g(y) \sqrt{1 + [g'(y)]^2}\, dy$$

$$S = \int_0^1 2\pi \sqrt{1-y} \sqrt{1 + \left[\tfrac{1}{2}(1-y)^{-\frac{1}{2}}\right]^2}\, dy$$

$$= \int_0^1 2\pi \sqrt{1-y} \sqrt{1 + \frac{1}{4(1-y)}}\, dy$$

$$= \int_0^1 2\pi \sqrt{1-y+\tfrac{1}{4}}\, dy$$

$$= 2\pi \cdot \frac{2}{3} \left(\frac{5}{4} - y\right)^{\frac{3}{2}} \Big|_1^0$$

$$= \frac{4}{3}\pi \cdot \left(\frac{5}{8}\sqrt{5} - \frac{1}{8}\right)$$

$$= \frac{\pi}{6}(5\sqrt{5} - 1)$$