Chapter 9. Parametric Equations and Polar Coordinates.
Sections 9.1 and 9.2. Parametric Curves

In this section, we review the properties of parametric curves, and focus our attention on applications of Calculus to sketch the graph of a parametric curve.

- **Definition:** The set of points $(x, y) = (f(t), g(t))$ where f and g are functions, and t is a real number describes a curve in parametric form.

  Examples

  1. Sketch the graph of the curve $x = 3t - 1, y = t + 2, 0 \leq t \leq 5$.

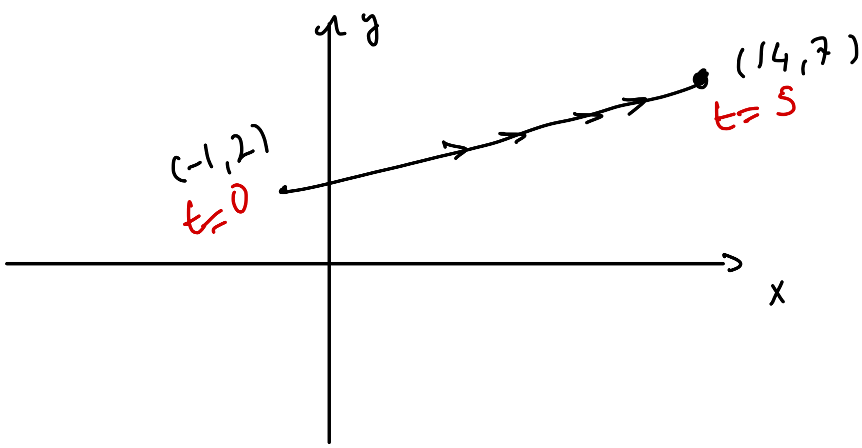
     $t = y - 2$

     $x = 3(y-2) \Rightarrow x = 3y - 6$

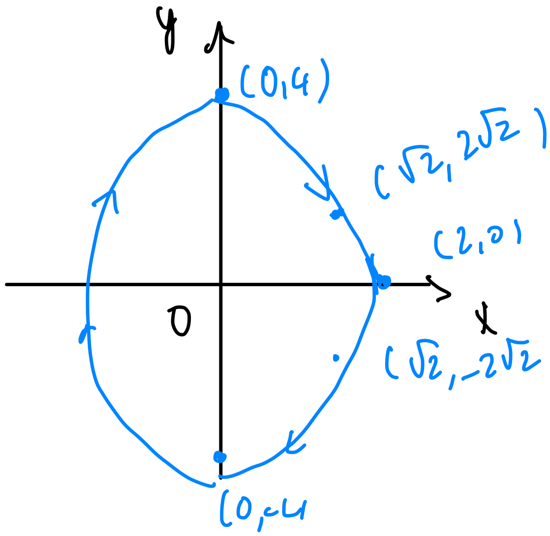
     $\boxed{x - 3y = -6}$

     $-1 \leq x \leq 14$ $\quad$ $2 \leq y \leq 7$

     Now, eliminate the parameter t to obtain an algebraic expression in x and y.



2. Sketch the graph of the curve $x = 2\sin\theta$, $y = 4\cos\theta$ for $0 \leq \theta \leq 2\pi$



θ	x	y
0	0	4
$\pi/4$	$\sqrt{2}$	$2\sqrt{2}$
$\pi/2$	2	0
$3\pi/4$	$2\sqrt{2}$	$-2\sqrt{2}$
π	0	-4

Now eliminate the parameter θ.

$x = 2\sin\theta$ $\quad$ $y = 4\cos\theta$

$\sin\theta = \frac{x}{2}$ $\quad$ $\cos\theta = \frac{y}{4}$

$\sin^2\theta + \cos^2\theta = 1$

$$\boxed{\frac{x^2}{4} + \frac{y^2}{16} = 1}$$

Equation of an ellipse

Additional Example: Eliminate the parameter to find a Cartesian equation of the curve,

$$① \quad x = \sqrt{t+1}$$

$$② \quad y = \sqrt{t-1} \quad \Rightarrow \quad t \geq 1$$

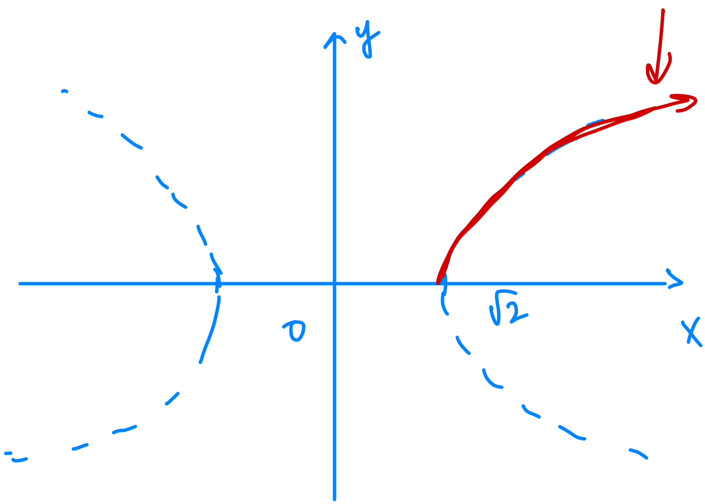
$x \geq \sqrt{2}$ and $y \geq 0$

$① \Rightarrow \quad t = x^2 - 1$
$② \Rightarrow \quad t = y^2 + 1$

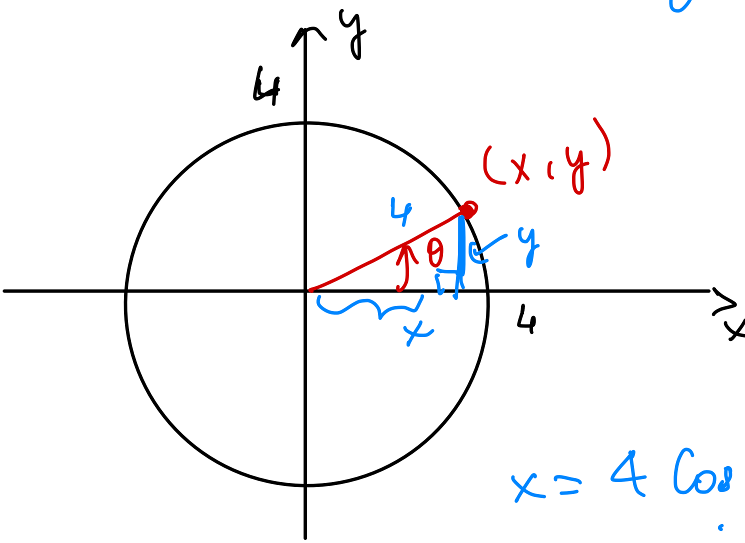
$x^2 - 1 = y^2 + 1$

$$\boxed{x^2 - y^2 = 2}$$

Hyperbola
$x \geq \sqrt{2}$
$y \geq 0$



Write the equation of a circle centered at (0,0) with radius r, as a set of parametric equation with parameter θ



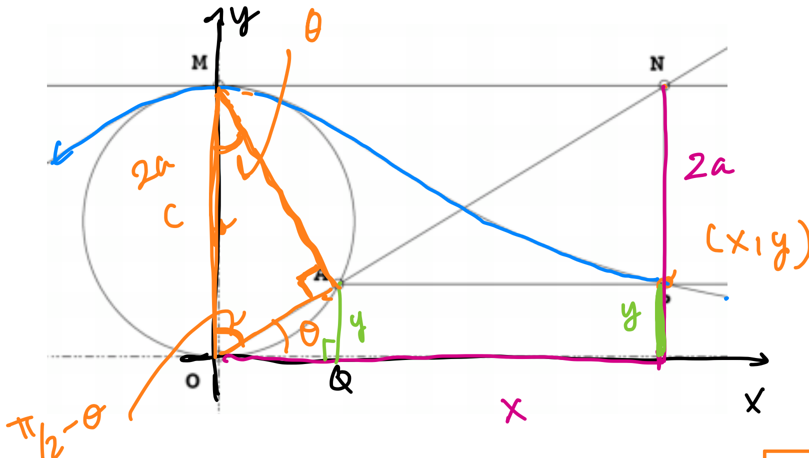
$x = 4\cos\theta$

$y = 4\sin\theta$

- A curve, called a **witch of Maria Agnesi**, consists of all possible positions of the point P in the figure. Show that the parametric equations for this curve are

$$x = 2a\cot\theta, \quad y = 2a\sin^2\theta$$

where θ is the angle between the positive x–axis and OA.



Then eliminate the parameter θ.

$\cot\theta = \frac{x}{2a} \Rightarrow \boxed{x = 2a\cot\theta}$ ✓

?

Right triangle AOQ : $\sin\theta = \frac{y}{OA} \Rightarrow y = OA\sin\theta$

$\sin\theta = \frac{OA}{2a} \Rightarrow OA = 2a\sin\theta$

$\boxed{y = 2a\sin^2\theta}$

- **Calculus with parametric curves.**

  We use tools such as the first derivative to find horizontal/vertical tangents and sketch the graph of parametric curves.

  First, let us derive an expression for the first derivative of a curve given by $x = f(t)$, $y = g(t)$.

why?

$x = f(t)$

$y = F(x(t)) \Rightarrow \frac{dy}{dt} = F'(x(t))\frac{dx}{dt}$

or $\underbrace{F'(x(t))}_{\frac{dy}{dx}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

Now, we can use the above expression to find the horizontal and vertical tangent(s) to a curve in parametric form.

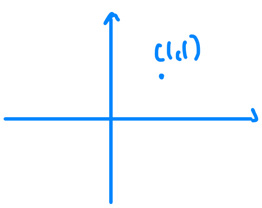
Exampless:

$y - 1 = ?\,(x-1)$

1. Find an equation of the tangent line to the curve $x = 2\cos 2t$, $y = 2\sin t$ at the point $(1, 1)$.

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t}{-4\sin(2t)}$

$t = ?$



$1 = 2\cos 2t$ (If $t = \pi/6$ $1 \overset{?}{=} 2\underbrace{\cos(\pi/3)}_{1/2}$ ✓

$1 = 2\sin t \to \sin t = \frac{1}{2}$ $\boxed{t = \pi/6}$

At $t = \pi/6$ $\frac{dy}{dx} = \frac{2(\sqrt{3}/2)}{-4(\sqrt{3}/2)} = -1/2$

Equation of the tangent at $(1,1)$ $y - 1 = -\frac{1}{2}(x-1)$

or $y = -\frac{1}{2}x + \frac{3}{2}$

$x = 2\cos(2t)$, $y = 2\sin t$, eliminate t.

$(\cos(2t) = \cos^2 t - \sin^2 t)$

$\cos(2t) = 1 - 2\sin^2 t$

$x = 2 - 4\sin^2 t$

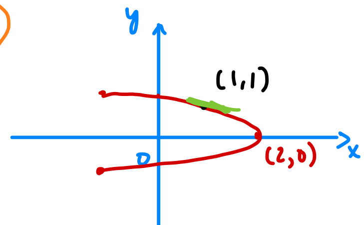
$x = 2 - (2\sin t)^2$

$x = 2 - y^2$

parabola

$-2 \le x \le 2$

$-2 \le y \le 2$



Curve C $x = f(t)$

$y = g(t)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

C has a horizontal tangent when $\frac{dy}{dt} = 0$

C has a vertical tangent when $\frac{dx}{dt} = 0$

An expression for $\dfrac{d^2y}{dx^2}$ is given by $\quad \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$

Proof:?

$$\boxed{\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}}$$

Examples

1. Find $\dfrac{d^2y}{dx^2}$ for $x = 1 + t^2$, $y = \ln t$. $(t>0)$ Curve C

a) Find the intervals of increase/decrease of C.

$\dfrac{dy}{dx} = \dfrac{\frac{1}{t}}{2t} = \dfrac{1}{2t^2} > 0$ for $t>0$

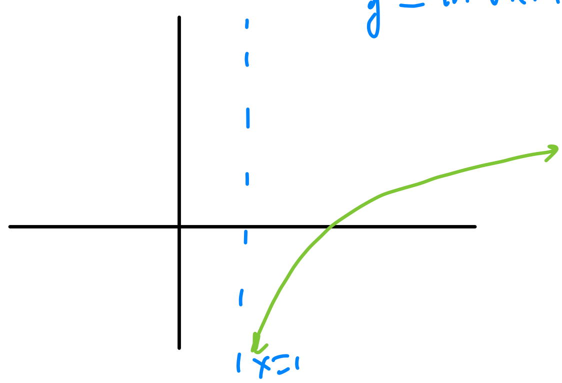
C is increasing

b) concavity and points of inflection. $\quad \dfrac{d^2y}{dx^2} = \dfrac{-\frac{1}{t^3}}{2t} = \dfrac{-1}{2t^4} < 0$

C is concave

Eliminate t: $\quad x = 1+t^2 \implies t = \sqrt{x-1} \quad (t>0)$

$y = \ln\sqrt{x-1} = \frac{1}{2}\ln(x-1)$



$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

2. Find dy/dx, and d^2y/dx^2 for the curve given by the parametric equations $x = e^t$, $y = te^{-t}$. For which value(s) of t is the curve concave up?

$$\frac{dy}{dx} = \frac{e^{-t} - te^{-t}}{e^t} = \frac{e^{-t}(1-t)}{e^t} = e^{-2t}(1-t)$$

$$\frac{d^2y}{dx^2} = \frac{-2e^{-2t}(1-t) - e^{-2t}}{e^t} = \frac{e^{-2t}(2t-3)}{e^t} = \frac{2t-3}{e^t}$$

$\frac{d^2y}{dx^2} > 0$ when $2t - 3 > 0$ $t > 3/2$

3. Let $x = 2\cos\theta$ and $y = \sin(2\theta)$ for $0 \le \theta < 2\pi$ be the parametric equations for a curve C. Find the points on the curve where the tangent is horizontal or vertical.

Horizontal tangent $\frac{dy}{dt} = 2\cos(2\theta)$, $\frac{dy}{dt} = 0$ when $\cos 2\theta = 0$ or $2\cos^2\theta - 1 = 0$

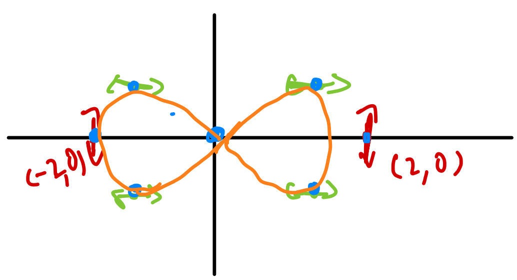
$\cos^2\theta = \frac{1}{2} \Rightarrow \cos\theta = \pm\frac{1}{\sqrt{2}}$

or $\cos\theta = \pm\frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$

Points $(\sqrt{2}, 1), (\sqrt{2}, -1), (-\sqrt{2}, 1), (-\sqrt{2}, -1)$

Vertical tangents: $\frac{dx}{dt} = -2\sin\theta$ $\quad \frac{dx}{dt} = 0$, $\theta = 0$, $\theta = \pi$

Points: $(2, 0)$; $(-2, 0)$



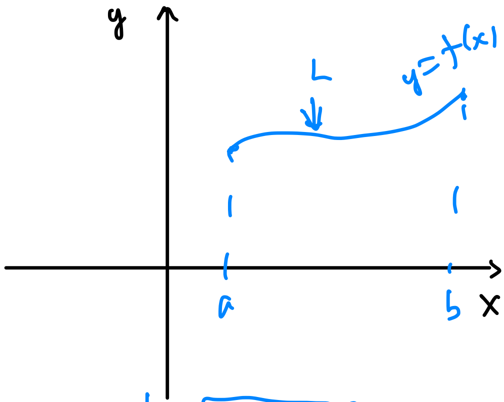
Eliminate θ:

$\cos\theta = \frac{x}{2}$ $\qquad y = 2\sin\theta\cos\theta$

$\cos^2\theta = \frac{x^2}{4}$ $\qquad y^2 = 4\underbrace{\sin^2\theta}_{(1-\cos^2\theta)}\cos^2\theta$

$$\boxed{y^2 = 4\left(1 - \frac{x^2}{4}\right)\frac{x^2}{4}}$$

Is $(0,0)$ on the graph of C? yes.



$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Example: Find the length of the curve $y = 1 + x^{3/2}$
for $0 \le x \le 1$

$$\frac{dy}{dx} = \frac{3}{2} x^{1/2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{9x}{4}$$

$$L = \int_0^1 \sqrt{1+\frac{9x}{4}}\, dx = \frac{1}{2}\int_0^1 \sqrt{4+9x}\, dx$$

$u = 4+9x$

$du = 9dx$

$x=0 \quad u=4$

$x=1 \quad u=13$

$$= \frac{1}{18}\int_4^{13} u^{1/2}\, du = \frac{1}{18}\cdot\frac{2}{3} u^{3/2}\Big|_4^{13}$$

$$= \frac{1}{27}\left[13\sqrt{13} - 8\right]$$

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

A Curve C has parametric equations $(x(t), y(t))$

Find the length of C for $t_1 \le t \le t_2$

$$L = \int_{t_1}^{t_2} \sqrt{1+\left(\frac{dy}{dt}\Big/\frac{dx}{dt}\right)^2}\, dx = \int_{t_1}^{t_2} \sqrt{1+\frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\, dx$$

$$= \int_{t_1}^{t_2} \sqrt{\frac{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\, dx = \int_{t_1}^{t_2} \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{\frac{d\cancel{x}}{dt}} \cdot d\cancel{x}$$

$$\boxed{L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

- Arc Length.

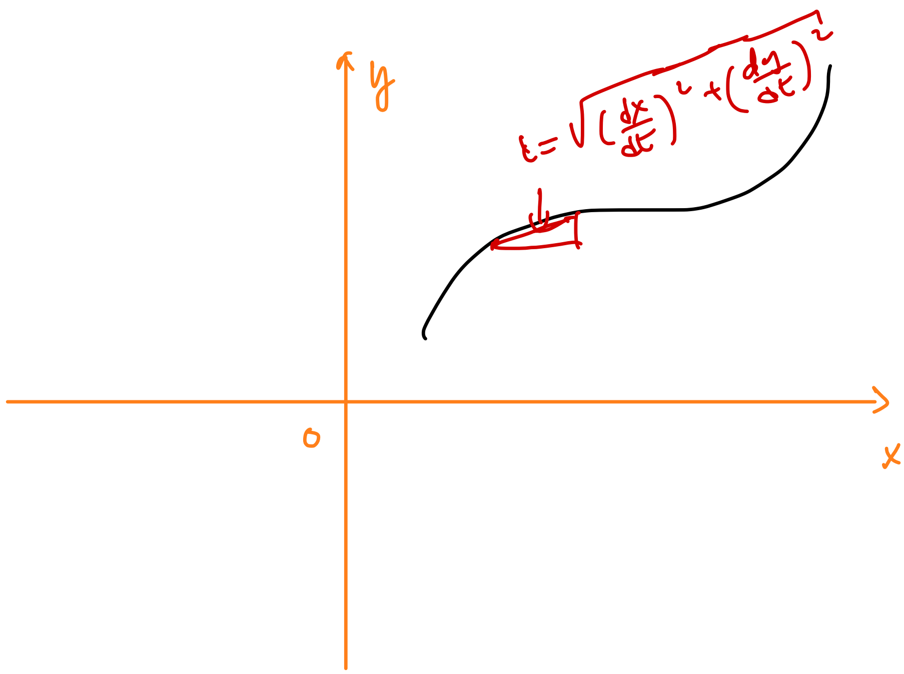
  Recall that the length L of a curve C given by $y = F(x)$ $a \le x \le b$ is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

assuming that F' is continuous. Suppose now that C is defined by parametric equations $x = f(t)$, $y = g(t)$ for $\alpha \le t \le \beta$. Since $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

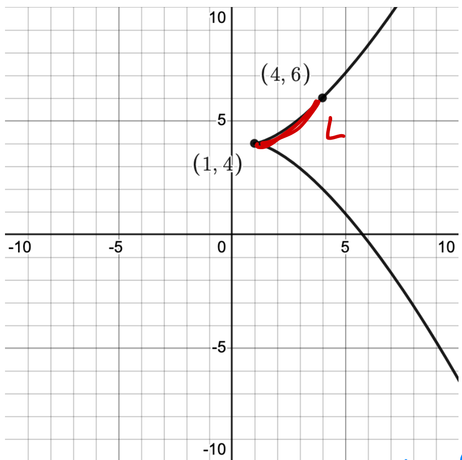
provided the curve is traversed once when t increases from α to β.



$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Examples:

1. Find the length of the curve $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \leq t \leq 1$.



$$\frac{dx}{dt} = 6t$$

$$\frac{dy}{dt} = 6t^2$$

$$L = \int_0^1 \sqrt{36t^2 + 36t^4}\, dt$$

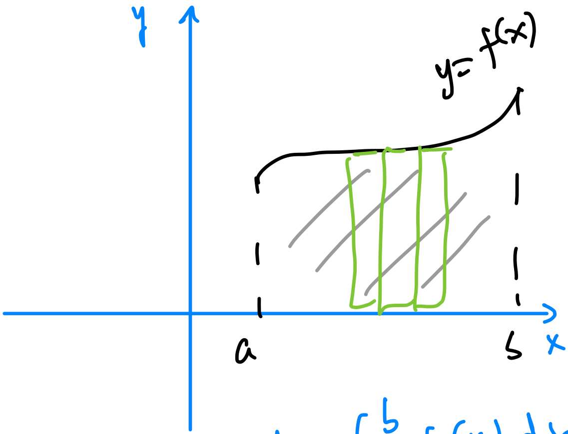
$$L = \int_0^1 6t\sqrt{1 + t^2}\, dt$$

$u = 1 + t^2 \quad du = 2t\, dt$

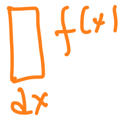
$t = 0 \quad u = 1$

$t = 1 \quad u = 2$

$$L = 3\int_1^2 u^{1/2}\, du = \cancel{3}\frac{2}{\cancel{3}} u^{3/2} \Big|_1^2 = 2\left(2\sqrt{2} - 1\right)$$



$$A = \int_a^b \underbrace{f(x)\,dx} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$



$$A = \int_a^b y \, dx$$

Curve c given by $x = f(t)$

$y = g(t)$

- **Areas:**

  Recall that the area under the graph of a function y from a to b where $y \geq 0$ is given by

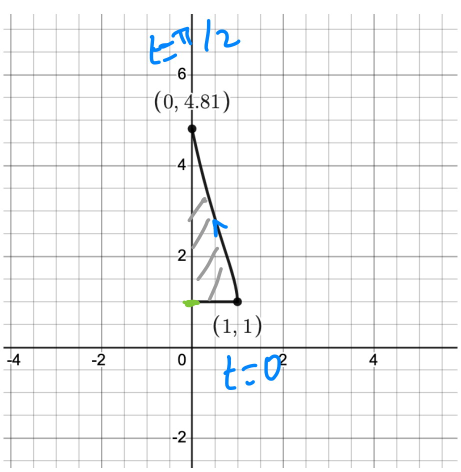
$$A = \int_a^b y \; dx$$

When a curve is defined with parametric equations $x = f(t)$, $y = g(t)$ and is traversed once when $\alpha \leq t \leq \beta$, we can write,

$$\frac{dx}{dt} = f'(t) \Rightarrow dx = f'(t)\,dt$$

$$A = \int_\alpha^\beta \underbrace{g(t)}_{y}\underbrace{f'(t)dt}_{dx}$$

Example: Find the area bounded by the curve $x = \cos t$, $y = e^t$, $0 \leq t \leq \pi/2$ and the lines $y = 1$, and $x = 0$.



$$A = -\int_0^{\pi/2} e^t (-\sin t)\,dt$$

$$A = \int_0^{\pi/2} e^t \sin t\,dt$$

$u = \sin t \qquad dv = e^t\,dt$

$du = \cos t\,dt \qquad v = e^t$

$$\int_0^{\pi/2} e^t \sin t\,dt = e^t \sin t \Big|_0^{\pi/2} - \int_0^{\pi/2} e^t \cos t\,dt$$

$$\int_0^{\pi/2} e^t \sin t\,dt = e^{\pi/2} - \int_0^{\pi/2} e^t \cos t\,dt$$

$u = \cos t \qquad dv = e^t\,dt$

$du = -\sin t\,dt \qquad v = e^t$

$$\int_0^{\pi/2} e^t \sin t\,dt = e^{\pi/2} - \left(e^t \cos t \Big|_0^{\pi/2} + \int_0^{\pi/2} e^t \sin t\,dt \right)$$

$$2\int_0^{\pi/2} e^t \sin t\,dt = e^{\pi/2} - (-1) = e^{\pi/2} + 1$$

$$\boxed{A = \frac{e^{\pi/2}+1}{2}}$$