Section 5.4. The General Exponential and Logarithmic Functions

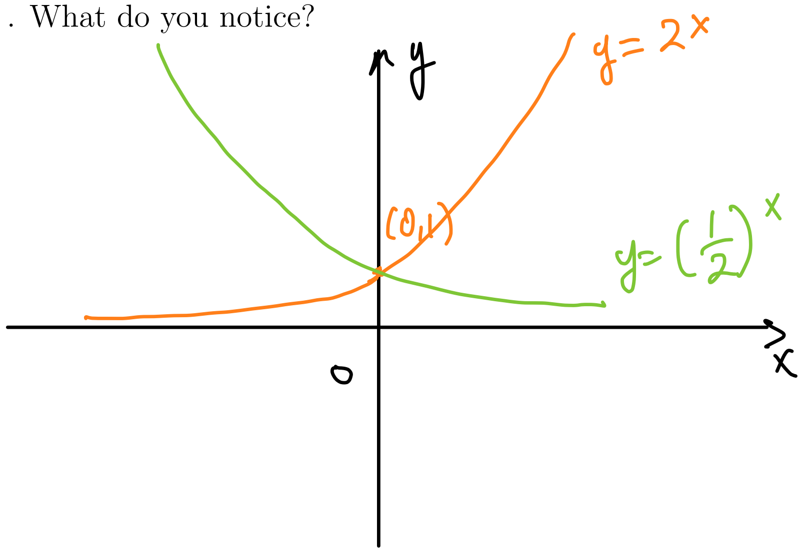
1. **The Exponential Function.**

   Let $a > 0$, we define the exponential function with base a by

$$\boxed{a^x = e^{x \ln a}}$$

- Graph of $y = a^x$. In the space provided below, sketch the graph of $y = 2^x$, and $y = \left(\frac{1}{2}\right)^x$. What do you notice?

x	2^x	$\left(\frac{1}{2}\right)^x$
-2	$1/4$	4
-1	$1/2$	2
0	1	1
1	2	$1/2$
2	4	$1/4$



- Domain and range of $y = a^x$.
  The domain of $y = a^x$ is $(-\infty, \infty)$, and its range is $(0, \infty)$.
- Properties of $y = a^x$
  - If $a > 1$, $y = a^x$ is **increasing**/**decreasing** (circle one answer), and

$$\lim_{x\to\infty} a^x = \infty, \quad \lim_{x\to-\infty} a^x = 0$$

  - If $0 < a < 1$, $y = a^x$ is **increasing**/**decreasing** (circle one answer), and

$$\lim_{x\to\infty} a^x = 0, \quad \lim_{x\to-\infty} a^x = \infty$$

  - What happens if $a = 1$?

$y = 1^x = 1$ is not an exponential function.

- Derivative.

  Use the definition of $y = a^x$ as $e^{x \ln a}$ to show that

$$\boxed{\frac{d}{dx} a^x = a^x \ln a}$$

why?:

$a^x = e^{x \ln a}$

$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \cdot \ln a$

Chain Rule

$= a^x \cdot \ln a$

- Integral. Use the information above to conclude that

$$\boxed{\int a^x \; dx = \frac{a^x}{\ln a} + C,} \text{ where } C \text{ is a constnt}$$

- Exercises:

  (a) Differentiate the function $g(x) = x^4 4^x$.

$g'(x) = 4x^3 4^x + x^4 4^x \ln 4$

$= x^3 4^x (4 + x \ln 4)$

  (b) $\displaystyle\int \frac{2^x}{2^x + 1} dx$

$\frac{1}{\ln 2} du$

$= u$

$u = 2^x + 1 \qquad du = 2^x \ln 2 \, dx$

$\frac{1}{\ln 2} \int \frac{1}{u} du = \frac{1}{\ln 2} \ln|u| + C$

$\int \frac{2^x}{2^x+1} dx = \frac{1}{\ln 2} \ln(2^x + 1) + C$

or $\frac{1}{\ln 2} \ln|2^x + 1| + C$

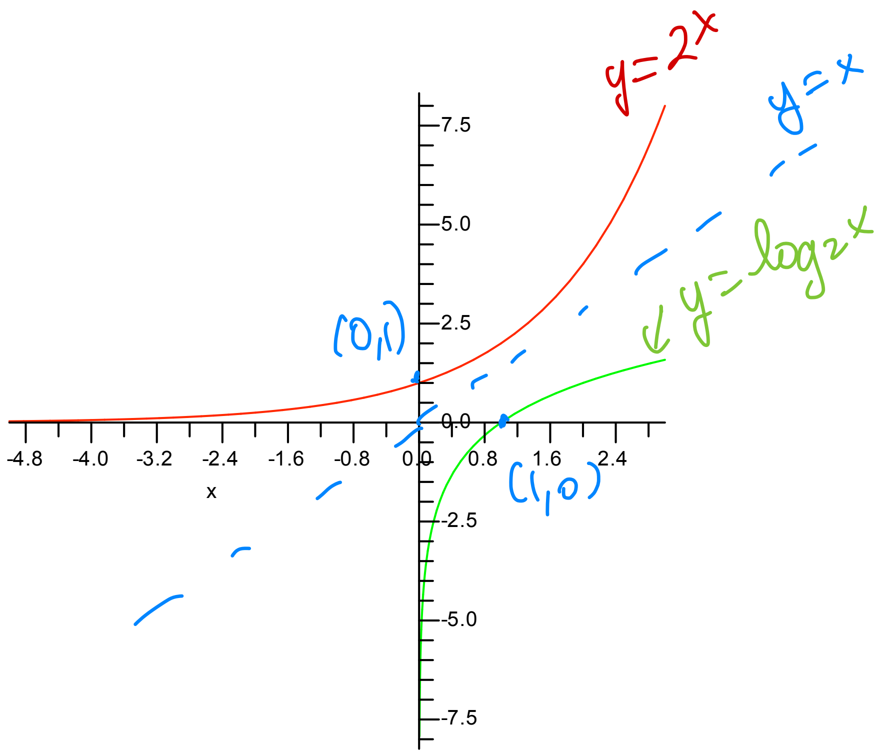
2. **The General Logarithmic Function.**

   Although it is possible to define a logarithmic function with base $0 < a < 1$, we only study the case where $a > 1$.

   Definition: Since $y = a^x$ is increasing for $a > 1$, it is one–to–one. Therefore, it has an inverse function denoted $\log_a$ such that

$$\boxed{y = \log_a x \text{ if and only if } x = a^y} \qquad (\ln x = \log_e x)$$

   - Graph of $y = \log_a x$.
     In the space provided below, sketch the graph of $y = 2^x$, and $y = \log_2 x$.



What do you notice?

- Domain and range of $y = \log_a x$.
  The domain of $y = \log_a x$ is $(0, \infty)$, and its range is $(-\infty, \infty)$.
- Properties of $y = \log_a x$
  - $\log_a(a^x) = x$ if x is in $(-\infty, \infty)$
  - $a^{\log_a x} = x$ if x is in $(0, \infty)$.
  - $\lim_{x\to\infty} \log_a x = \infty$, $\lim_{x\to 0^+} \log_a x = -\infty$
- Derivative.
  Use the definition of $y = \log_a x$ as the inverse function of $y = a^x$ to show that

$$\boxed{\frac{d}{dx}\log_a x = \frac{1}{x\ln a}}$$

why?

$y = \log_a x$ is equivalent to $x = a^y$

$\frac{dy}{dx} = \frac{d}{dx}\log_a x$

$\frac{d}{dx}x = \frac{d}{dx}a^y$

$1 = a^y \ln a \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{a^y \ln a}$

x

- Another definition of e.

$$\boxed{\lim_{x\to 0}(1+x)^{1/x} = e}$$

$x \to 0$

"1^∞"

indeterminate power

- Exercises:
  - Evaluate the limit. $\lim_{x\to 3^+} \log_{10}(x^2 - 5x + 6)$

$y = \log_{10} x = \log x$

"common log"

$= \lim_{x\to 3^+} \log_{10}[(x-3)(x-2)] = -\infty$

As $x \to 3^+$ $(x-3)(x-2) \to 0^+$

$\log_{10}[(x-3)(x-2)] \to -\infty$

  - Differentiate the function $y = x^{\ln x}$ $= (f(x))^{g(x)}$

logarithmic differentiation

$\ln y = \ln x^{\ln x}$

$\ln y = \ln x \cdot \ln x = (\ln x)^2$

$\frac{1}{y}\frac{dy}{dx} = 2\ln x \cdot \frac{1}{x}$

$$\boxed{\frac{dy}{dx} = 2\frac{\ln x}{x}\, x^{\ln x}}$$

(or $y = x^{\ln x} = e^{\ln x \cdot \ln x} = e^{(\ln x)^2}$)