$\frac{d}{dx}\sin x = \cos x$ $\quad$ $\frac{d}{dx}\cos x = -\sin x$

$\frac{d}{dx}\tan x = \sec^2 x$ $\quad$ $\frac{d}{dx}\cot x = -\csc^2 x$

$\frac{d}{dx}\sec x = \sec x \tan x$ $\quad$ $\frac{d}{dx}\csc x = -\csc x \cot x$

Pythagorean $\sin^2 x + \cos^2 x = 1$ etc...

Double angle formulas: $\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = 2\cos^2 x - 1$ ①

$\cos 2x = 1 - 2\sin^2 x$ ②

Power Reduction formulas

① solve for $\cos^2 x$ $\quad$ $\cos^2 x = \dfrac{1+\cos 2x}{2}$ ⇐

② solve for $\sin^2 x$ $\quad$ $\sin^2 x = \dfrac{1-\cos 2x}{2}$

Simple example: Evaluate

$$\int \underbrace{\sin^4 x}_{u^4} \underbrace{\cos x \, dx}_{=du}$$

$u = \sin x$ $\quad$ $du = \cos x \, dx$

$\int u^4 \, du = \frac{1}{5}u^5 + C$ $\quad$ $\int \sin^4 x \cos x \, dx = \frac{1}{5}\sin^5 x + C$

Section 6.2. Trigonometric Integrals and Substitutions..

- **Trigonometric Integrals.**

1. Integrals of the form $\boxed{\int \sin^m x \cos^n x \; dx}$ where m and n are positive integers.

   (*) see Next page

   Consider the following example: $\int \sin^6 x \cos^3 x \; dx$. We can write $\cos^3 x = \cos^2 x \quad \cos x$. We also know that $\cos^2 x = 1 - \sin^2 x$. So, the above integral can be written as

   $$\int \_\_\_ \cos x \; dx$$

   Then we can make the simple u substitution $u = \sin x$ and $du = \cos x \; dx$. The resulting integral is the integral of a polynomial. So,

   $$\int \sin^6 x \cos^3 x \; dx = \_\_\_$$

   see next page

   What conjecture can be made?

   - If the power of cosine is *odd*, isolate a factor of cosine, use $\sin^2 x + \cos^2 x = 1$, and substitute $u = \sin x$.
   - If the power of sine is *odd*, isolate a factor of sine, use $\sin^2 x + \cos^2 x = 1$, and substitute $u = \cos x$.
   - If the power of both cosine and sine is *odd*, we can isolate either a factor of cosine or sine , use $\sin^2 x + \cos^2 x = 1$, and substitute $u = \sin x$ or $u = \cos x$.
   - If the power of both cosine and sine is *even*, here we have to use the half angle identities

     $$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

     as shown in the next example.

   Example: Evaluate $\int \sin^2 x \; \cos^2 x \; dx$

$$\int \sin^6 x \cos^3 x \, dx = \int \sin^6 x \underbrace{\cos^2 x}_{= 1-\sin^2 x} \underbrace{\cos x \, dx}$$

$$= \int \underbrace{\sin^6 x \left(1-\sin^2 x\right)}_{f(\sin x)} \underbrace{\cos x \, dx}_{d(\sin x)}$$

Let $u = \sin x$ $\quad du = \cos x \, dx$

$$\int u^6 (1-u^2)\, du = \int (u^6 - u^8)\, du = \frac{1}{7} u^7 - \frac{1}{9} u^9 + c$$

$$\int \sin^6 x \cos^3 x \, dx = \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + C$$

$$\int \sin^2 x \cos^2 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{4} \int \left(1-\cos^2 2x\right) dx = \frac{1}{4} \int \sin^2 2x \, dx$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \frac{1}{4} \int \frac{1-\cos 4x}{2} \, dx$$

$$= \frac{1}{8} \int (1-\cos 4x)\, dx$$

$$= \frac{1}{8} \left[x - \frac{\sin(4x)}{4} \right] + C$$

Exercise: Evaluate [redacted]

2. Integrals of the form $\int \tan^m x \sec^n x \ \ dx$ where m and n are positive integers.
   - If the power of secant is *even*, isolate a factor of $\sec^2 x$, use $\sec^2 x = 1 + \tan^2 x$, and substitute $u = \tan x$.
   - If the power of tangent is *odd*, isolate a factor of $\tan x \sec x$, use $\tan^2 x = \sec^2 x - 1$, and substitute $u = \sec x$.

Exercises: Evaluate the integral.

(a) $\int_0^{\pi/4} \sec^4\theta \tan^4\theta \ \ d\theta = \int_0^{\pi/4} \underbrace{\sec^2\theta}_{1+\tan^2\theta} \tan^4\theta \ \underbrace{\sec^2\theta \, d\theta}_{d(\tan\theta)}$

$$\int_0^{\pi/4} \overbrace{(1+\tan^2\theta)\tan^4\theta}^{f(\tan\theta)} \ \sec^2\theta \, d\theta$$

$u = \tan\theta \qquad du = \sec^2\theta \, d\theta$

$\theta = 0 \quad u = 0$

$\theta = \pi/4 \quad u = 1$

$$\int_0^1 (1+u^2)u^4 \, du = \int_0^1 (u^4 + u^6)\, du$$

$$= \frac{u^5}{5} + \frac{u^7}{7} \Big|_0^1 = \frac{1}{5} + \frac{1}{7} = \frac{12}{35}$$

(b) $\int \cot^3\theta \; d\theta = \int \underbrace{\cot^2\theta}_{\csc^2\theta - 1} \cot\theta \, d\theta = \int (\csc^2\theta - 1)\cot\theta \, d\theta$

$= \int \csc^2\theta \underbrace{\cot\theta}_{u} \, d\theta - \int \cot\theta \, d\theta$

$u = \cot\theta \quad du = -\csc^2\theta \, d\theta$

$= -\int u \, du = -\frac{1}{2}u^2.$

$-\frac{1}{2}\cot^2\theta - \int \cot\theta \, d\theta$

$\int \frac{\cos\theta}{\sin\theta} \, du$

$u = \sin\theta$

$du = \cos\theta \, d\theta$

$\int \frac{1}{u} \, du$

$= \ln|u|$

$= \ln|\sin\theta|$

Answer: $-\frac{1}{2}\cot^2\theta - \ln|\sin\theta| + C$

$-\frac{1}{2}[\cot^2\theta + \ln|\csc\theta| + C$

$\csc^2\theta + 1$

$-\frac{1}{2}\csc^2\theta + \ln|\csc\theta| + C$

(c) $\int \sec^3 x \ dx = \int \sec x \cdot \sec^2 x \, dx$

Integration by parts:

$u = \sec x \qquad dv = \sec^2 x \, dx$

$du = \sec x \tan x \, dx \qquad v = \tan x$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \underbrace{\tan^2 x}_{= \sec^2 x - 1} dx$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$\underbrace{\int \sec^3 x \, dx} = \sec x \tan x \underbrace{- \int \sec^3 x \, dx} + \int \sec x \, dx$$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx \quad = \ln|\sec x + \tan x|$$

Table of integrals

$$\int \sec^3 x \, dx = \frac{1}{2} \left[\sec x \tan x + \ln|\sec x + \tan x| \right] + C$$

3. Integrals of the form $\int \sin Ax \cos Bx \ dx$, $\int \sin Ax \sin Bx \ dx$, and $\int \cos Ax \cos Bx \ dx$

For these type of integrals, we use product–to–sum identities.

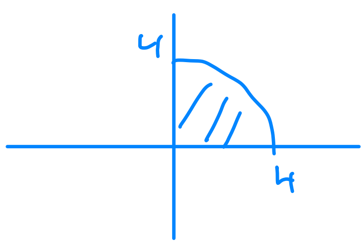
$$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)].$$
$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)].$$
$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)].$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$
$$+ \quad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$
$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$$

Exercise: Evaluate the integral.

$$\int \sin \underset{}{\overset{Ax}{3x}} \cos \overset{Bx}{x} \, dx = \frac{1}{2}\int \left(\sin(2x) + \sin(4x)\right) dx$$
$$= \frac{1}{2}\left[-\frac{\cos(2x)}{2} - \frac{\cos(4x)}{4}\right] + C$$

Evaluate $\int_0^4 \sqrt{16-x^2}\, dx = \frac{1}{4}\pi(4)^2 = 4\pi$



$$\int \sqrt{16-x^2}\, dx = ??$$

- **Trigonometric Subsitutions.**

A trigonometric substitution may be useful when evaluating integrals whose integrand contains a radical of the form $\sqrt{x^2 - a^2}$, $\sqrt{x^2 + a^2}$, and $\sqrt{a^2 - x^2}$. We can eliminate the radical and then evaluate a trigonometric integral as seen in section 6.2.

Table of Trigonometric Substitutions.

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\pi/2 \le \theta \le \pi/2$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\pi/2 < \theta < \pi/2$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \le \theta < \pi/2$ or $\pi \le \theta < 3\pi/2$	$\sec^2\theta - 1 = \tan^2\theta$

Examples: Evaluate the integral.

1. $\displaystyle\int \frac{\sqrt{x^2 - a^2}}{x^4}\,dx$ where $a \neq 0$ $\quad \boxed{x = a\sec\theta} \quad dx = a\sec\theta\tan\theta\,d\theta$

$\sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = a\tan\theta$

$$\int \frac{\sqrt{a^2\sec^2\theta - a^2}}{a^4\sec^4\theta}\, a\sec\theta\tan\theta\, d\theta = \int \frac{\cancel{a}\tan\theta}{a^{\cancel{4}}\sec^{\cancel{4}3}\theta}\,\cancel{a\sec\theta}\tan\theta\, d\theta$$

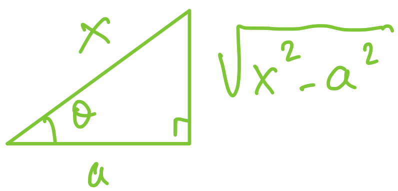
$$= \frac{1}{a^2}\int \frac{\tan^2\theta}{\sec^3\theta}\, d\theta = \frac{1}{a^2}\int \tan^2\theta\cos^3\theta\, d\theta = \frac{1}{a^2}\int \frac{\sin^2\theta}{\cos^2\theta}\cos^3\theta\, d\theta$$

$$= \frac{1}{a^2}\int \sin^2\theta\cos\theta\, d\theta \quad , \quad u = \sin\theta \quad du = \cos\theta\, d\theta$$

$$\left(\frac{1}{a^2}\int u^2\, du = \frac{1}{3a^2}u^3 + C\right)$$

$$= \frac{1}{3a^2}\sin^3\theta + C = \frac{1}{3a^2}\left(\frac{\sqrt{x^2 - a^2}}{x}\right)^3 + C$$

$$\sec\theta = \frac{x}{a}$$



Trigonometric substitutions : $(a>0)$

- $\sqrt{a^2-x^2}$, $x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \pi/2$

$$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)}$$
$$= a\sqrt{1-\sin^2\theta}$$
$$= a\sqrt{\cos^2\theta}$$
$$= a\cos\theta$$

- $\sqrt{a^2+x^2}$, $x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \pi/2$

$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2 x} = \sqrt{a^2(1+\tan^2 x)}$$
$$= \sqrt{a^2\sec^2\theta}$$
$$= a\sec\theta$$

Evaluate the integral:

$$\int \frac{1}{x\sqrt{9-x^2}}\,dx \qquad x = 3\sin\theta \qquad dx = 3\cos\theta\, d\theta$$

$$\int \frac{1}{3\sin\theta\,\underbrace{\sqrt{9-9\sin^2\theta}}}\,3\cos\theta\, d\theta = \int \frac{1}{3\sin\theta\,\cancel{3\cos\theta}}\,\cancel{3\cos\theta}\, d\theta$$

$\sqrt{9(1-\sin^2\theta)}$

$= \sqrt{9\cos^2\theta}$

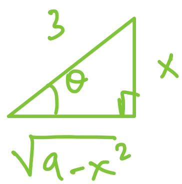
$= 3\cos\theta$

$$= \frac{1}{3}\int \frac{1}{\sin\theta}\, d\theta$$

$$= \frac{1}{3}\int \csc\theta\, d\theta$$

Integral #15 $\quad = \frac{1}{3}\ln\left|\csc\theta - \cot\theta\right| + C$

$\sin\theta = \frac{x}{3}$



$$\int \frac{1}{x\sqrt{9-x^2}}\,dx = \frac{1}{3}\ln\left|\frac{3}{x} - \frac{\sqrt{9-x^2}}{x}\right| + C$$

2. $\int_0^1 x\sqrt{x^2+4}\ dx$. Let $u = x^2+4$ $du = 2x\,dx$

$x = 0$ $u = 4$

$x = 1$ $u = 5$

$\frac{1}{2}\int_4^5 u^{1/2}\,du = \frac{1}{2}\,\frac{2}{3}\,u^{3/2}\Big|_4^5 = \frac{1}{3}\left(5\sqrt{5} - 8\right)$

Completing the square:

Recall that

- $(x + a)^2 = x^2 + 2ax + a^2$
- $(x - a)^2 = x^2 - 2ax + a^2$

  In completing the square, we are given $x^2 \pm 2ax$. What must be added so that we can obtain a perfect square, i.e. $(x \pm a)^2$?

  Consider the following example: Complete the square for $x^2 - 6x$. We have a number "?" such that $(x \pm a)^2 = x^2 \pm 2ax + a^2$

$$x^2 - 6x + 3^2 = (x - 3)^2$$

$\frac{6}{2} = 3$

  The middle term is $6x$ which can be written as $2(x)(3)$ (as in $2ax$), therefore we need to add 3^2 to get a perfect square.

  Examples: Complete the square.

  (a) $x^2 + 5x + \left(\frac{5}{2}\right)^2 = (x + \frac{5}{2})^2$

  $\frac{5}{2}$

  (b) $3x^2 - 12x + \_\_\_ = 3(x - \_\_\_)^2$

  $3(x^2 - 4x + 2^2) = 3(x-2)^2$

  $\frac{4}{2} = 2$

  (c) $x^2 + 10x - 3 = (x + \_\_\_)^2 + \_\_\_$

  $x^2 + 10x + 5^2 - 5^2 - 3 = (x+5)^2 - 28$

  $\frac{10}{2} = 5$

Radical	Substitution
$\sqrt{a^2-x^2}$	$x = a\sin\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$

3. $\displaystyle\int \frac{dt}{\sqrt{t^2-6t+13}}$

Complete the square: $\underbrace{t^2-6t}+13 = \underbrace{t^2-6t+9}+4$

$= (t-3)^2+4$

$= (t-3)^2+2^2$

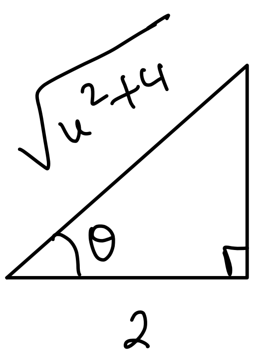
$$\int \frac{1}{\sqrt{(t-3)^2+2^2}}\,dt \qquad u = t-3 \qquad du = dt$$

$$\int \frac{1}{\sqrt{u^2+2^2}}\,du \qquad u = 2\tan\theta \qquad du = 2\sec^2\theta\,d\theta$$

$$\int \frac{1}{\sqrt{4\tan^2\theta+4}}\,2\sec^2\theta\,d\theta = \int \frac{2\sec^2\theta}{2\underbrace{\sqrt{\tan^2\theta+1}}_{\sec^2\theta}}\,d\theta$$

$$= \int \frac{\cancel{2}\sec^2\theta}{\cancel{2}\sec\theta}\,d\theta$$

$\tan\theta = \dfrac{u}{2}$



$u = t-3$

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta+\tan\theta|+C$$

$$= \ln\left|\frac{\sqrt{(t-3)^2+4}}{2}+\frac{t-3}{2}\right|+C$$

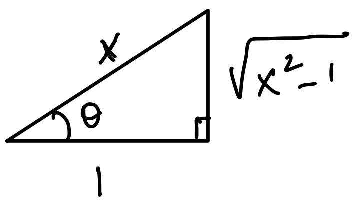
4. $\int \frac{\sqrt{x^2-1}}{x}\,dx$

$$\frac{x}{1} = \sec\theta \qquad dx = \sec\theta\tan\theta\,d\theta$$

$$\sqrt{\tan^2\theta} = \tan\theta$$

$$\int \frac{\sqrt{\sec^2\theta-1}}{\sec\theta}\sec\theta\tan\theta\,d\theta = \int \frac{\tan\theta}{\cancel{\sec\theta}}\cancel{\sec\theta}\tan\theta\,d\theta$$

$$= \int \tan^2\theta\,d\theta = \int (\sec^2\theta - 1)\,d\theta = \tan\theta - \theta + C$$



$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \sqrt{x^2-1} - \sec^{-1}x + C$$

5. $\int \frac{x^2}{\sqrt{9-25x^2}}dx = \int \frac{x^2}{\sqrt{3^2-(5x)^2}}\,dx$

Trigonometric substitution $5x = 3\sin\theta$

$(\text{or } x = \frac{3}{5}\sin\theta) \quad dx = \frac{3}{5}\cos\theta\, d\theta$

$$\int \frac{\frac{3^2}{5^2}\sin^2\theta}{\underbrace{\sqrt{9-9\sin^2\theta}}_{3\sqrt{1-\sin^2\theta} = 3\sqrt{\cos^2\theta} = 3\cos\theta}} \frac{3}{5}\cos\theta\, d\theta = \int \frac{\frac{3^2}{5^2}\sin^2\theta}{\cancel{3}\cancel{\cos\theta}} \cancel{\frac{3}{5}}\, \cancel{\cos}\theta\, d\theta$$

$$= \frac{9}{125}\int \sin^2\theta\, d\theta = \frac{9}{125}\int \frac{(1-\cos 2\theta)}{2}d\theta = \frac{9}{250}\int (1-\cos 2\theta)\, d\theta$$

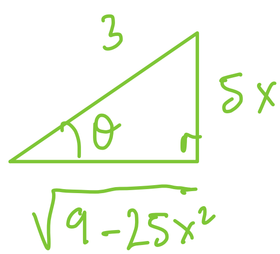
$$= \frac{9}{250}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{9}{250}\left(\theta - \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right) + C$$

$$= \frac{9}{250}\left(\sin^{-1}\left(\frac{5x}{3}\right) - \frac{5x}{3}\cdot\frac{\sqrt{9-25x^2}}{3}\right)$$

$+C$

$\sin\theta = \frac{5x}{3}$

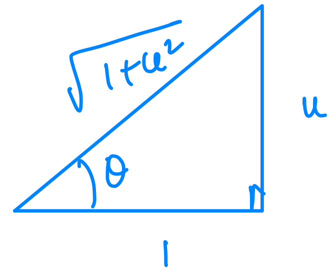


6. $\displaystyle\int \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt$ Let $u = \sin t$ $\quad du = \cos t\,dt$

$\displaystyle\int \frac{1}{\sqrt{1+u^2}}\,du$ $\quad u = \tan\theta \quad du = \sec^2\theta\,d\theta$

$$\int \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\sqrt{1+u^2} + u\right| + C$$



$$\int \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt = \ln\left|\sqrt{1+\sin^2 t} + \sin t\right| + C$$