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Lecture 9

Section 2.6: geometric distribution, negative binomial distribution.

9.1. Binomial distribution (continued)

EXAMPLE 9.1. Let $X \sim \text{Bin}(n, p)$. Show that $\mathbb{E}[X] = np$ and $\text{Var}(X) = np(1-p)$.

SOLUTION. Let X_i be the indicator r.v. of success in the i-th trial, that is

$$X_i = \begin{cases} 1, & \text{success at trial } i \\ 0, & \text{failure at trial } i \end{cases}$$

Then, $\mathbb{E}[X_i^2] = \mathbb{E}[X_i] = p$ (show this). Furthermore, for $i \neq j$,

$$\begin{aligned} \mathbb{E}[X_i X_j] &= 1 \times \mathbb{P}(X_i X_j = 1) + 0 \times \mathbb{P}(X_i X_j = 0) \\ &= \mathbb{P}(X_i = 1 \text{ and } X_j = 1) = p^2. \end{aligned}$$

Since $X = \sum_{i=1}^n X_i$, we have

$$\mathbb{E}[X] = \sum_{i=1}^n \mathbb{E}[X_i] = np.$$

Similarly,

$$\begin{aligned} \mathbb{E}\left[X^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^n X_i\right)^2\right] = \mathbb{E}\left[\sum_{i=1}^n X_i^2 + \sum\sum_{i \neq j} X_i X_j\right] = \\ &= \sum_{i=1}^n \mathbb{E}\left[X_i^2\right] + \sum\sum_{i \neq j} \mathbb{E}\left[X_i X_j\right] \\ &= np + n(n-1)p^2 \end{aligned}$$

Finally, we obtain that

$$\text{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2 = np + n(n-1)p^2 - n^2p^2 = np(1-p).$$

9.2. The Geometric r.v.

Consider an infinite sequence of independent Bernoulli trials, with probability p of success at each trial. Let

$$X = \text{ the trial number of the first success}$$

The pmf of X is: $p_X(k) = \mathbb{P}(X = k) = (1-p)^{k-1} \cdot p$, $k = 1, 2, \ldots$, where $(1-p)^k$ is the probability that the first $k-1$ trials are failures, and p is the probability that the k^{th} trial is a success.

Definition 9.2. A r.v. X is called a geometric r.v. with parameter p if the pmf of X is

$$p_X(k) = (1-p)^{k-1} \cdot p,$$

where $k = 1, 2, \ldots$. Notation: $X \sim \mathrm{G}(p)$.

- If $X \sim \mathrm{G}(p)$, then $\mathbb{E}[X] = \frac{1}{p}$ and $\mathrm{Var}[X] = \frac{1-p}{p^2}$.

Remark. (Optional read, not covered during the lecture) To show that $\mathbb{E}[X] = \frac{1}{p}$, we can proceed as follows. By definition,

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=1}^{\infty} k(1-p)^{k-1} \cdot p = \sum_{k=1}^{\infty} (k-1+1)(1-p)^{k-1} p \\
&= \sum_{k=1}^{\infty} (k-1)(1-p)^{k-1} p + \sum_{k=1}^{\infty} (1-p)^{k-1} p \qquad \text{[change index: } \ell = k-1] \\
&= \underset{l=0}{\underset{\downarrow}{0}} + \sum_{\ell=1}^{\infty} \ell (1-p)^{\ell} p + 1 = (1-p) \sum_{l=1}^{\infty} \ell (1-p)^{\ell-1} p + 1 = (1-p)\,\mathbb{E}(X) + 1.
\end{aligned}$$

We thus have that $\mathbb{E}[X] = (1-p)\,\mathbb{E}(X) + 1$, which yields $\mathbb{E}[X] = \frac{1}{p}$. In a similar way, one can show that $\mathrm{Var}[X] = \frac{1-p}{p^2}$.

9.3. The negative binomial r.v.

Consider, again, an infinite sequence of independent Bernoulli trials, with probability p of success at each trial. Let

$$X = \text{ the trial number of the } r\text{-th success}$$

The pmf is now

$$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \ldots$$

Explanation: $\binom{k-1}{r-1}$ is the number of possibilities for $r-1$ successes among the $k-1$ first trials (the k^{th} trail is the r^{th} success) and $p^r(1-p)^{k-r}$ is the probability of each such possibility.

Definition 9.3. A r.v. X has negative binomial distribution, denoted by $X \sim \mathrm{NB}(r,p)$ r.v. with parameter p if the pmf of X is

$$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \ldots$$

Remark 9.4. If $X \sim \mathrm{NB}(r,p)$, then we have $X = X_1 + \cdots + X_r$ where $X_i \sim \mathrm{G}(p)$ (why?). We can use this relationship to show that $\mathbb{E}[X] = \frac{r}{p}$, $\mathrm{Var}(X) = r \cdot \frac{1-p}{p^2}$. We will do this later after we learn about covariance and independent random variables.

9.4. Examples on binomial, geometric, and negative binomial random variables

Example 9.5. A miner is trapped in a mine with three doors. One door leads outside through a tunnel, in 2 hours. The other two doors are connected by another tunnel, which one can walk through in 3 hours. The miner is equally likely to choose any of the three doors. If he finds himself back in the mine, he again chooses one of the 3 doors at random (the miner is tired and disoriented, which makes him forget which doors he had chosen before). What is the expected time it takes the miner to escape the mine?

Solution. The number of times it takes until the door leading outside is chosen is $X \sim \mathrm{G}\left(\frac{1}{3}\right)$. Therefore $\mathbb{E}(X) = 3$. This means that the average time it would take to reach safety is $3+3+2=8$ hours (three hours for choosing the first two connected doors, and another two hours in the tunnel leading outside). More formally, the r.v. representing the time passing until reaching safety is: $T = 3(X-1)+2$, where $X-1$ is # of times choosing one of the wrong doors. Thus,

$$\mathbb{E}(T) = 3 \cdot \mathbb{E}(X-1) + 2 = 3 \cdot 2 + 2 = 8.$$

Example 9.6. In a class of 40 students, on average, 2 students have been sick in the past. What is the probability that in the next lecture, 4 students will be sick?

Solution. $X \sim \mathrm{Bin}(40, p)$, and $\mathbb{E}[X] = 40p = 2 \implies p = 1/20$. Therefore

$$p(4) = \binom{40}{4}\left(\frac{1}{20}\right)^4\left(1-\frac{1}{20}\right)^{36} = 0.09012.$$

Example 9.7. Consider a roulette wheel consisting of 38 numbers 1 through 36, 0, and 00. Jeff always bets that the outcome will be any one of the numbers 1 through 12.
(a) What is the probability that his first win occurs on the fourth round?
(b) If we learn that he has lost the first four rounds, what is the probability that his first win occurs on his seventh round?
(c) What is the probability that his third win occurs on his ninth round?

Solution. (a) The round number of Jeff's first win is a geometric r.v. $X \sim \mathrm{G}(p)$ with $p = \frac{12}{38}$. Thus, $\mathbb{P}(X=4) = \left(1-\frac{12}{38}\right)^3 \cdot \frac{12}{38} \approx 0.1$.
(b) Let $A = \{\text{Jeff losses his first four rounds}\}$. Let Y be the round number of his first win, starting counting from the 5^{th} round. Then, given that A has occurred, $Y \sim \mathrm{G}\left(\frac{12}{38}\right)$. By the independence of the results in each round,

$$\mathbb{P}(X=7 \mid A) = \mathbb{P}(Y=3) = \left(1-\frac{12}{38}\right)^2 \cdot \frac{12}{38} \approx 0.15.$$

(c) Let Z denote the round number of the third win. Then $Z \sim \mathrm{NB}\left(3, \frac{12}{38}\right)$ and hence

$$\mathbb{P}(Z=9) = \binom{8}{2}\left(\frac{12}{38}\right)^3\left(1-\frac{12}{38}\right)^6 \approx 0.09.$$