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Lecture 7

Section 2.2: expected value of a discrete random variable

7.1. Distribution of discrete random variables (continued)

- Recall from the previous lecture that:
  - A **discrete random variable** is a random variable that takes only a finite number of values, or a countably infinite number of values.
  - The **probability mass function** (pmf) of a discrete random variable is $p_X(x) = \mathbb{P}(X = x)$ for $x \in \mathbb{R}$. The **support** of a discrete random variable X is the set of all points $x \in \mathbb{R}$ such that $p_X(x) \neq 0$.
  - The **cumulative distribution function** (cdf) of a random variable X is $F_X(x) = \mathbb{P}(X \leq x)$ for $x \in \mathbb{R}$.

EXAMPLE 7.1. You are given the following function

$$p(x) = \begin{cases} 0.1; & x = -1, \\ 0.3; & x = 2, \\ 0.6; & x = 5, \\ 0.0; & x = 10 \end{cases}$$

Can $p(x)$ be the pmf of a random variable? If so, find the support of the random variable.

SOLUTION. Yes, $p(x)$ can be a pmf since $p(x) \geq 0$ and $\sum_x p(x) = 1$. The support of the corresponding random variable is $\{-1, 2, 5\}$, which are the values at which $p(x) \neq 0$. Note that the probability assigned to $x = 10$ is zero. So, we should not include 10 in the support.

- **Properties of pmf:** Let X be a discrete random variable with support $\mathrm{supp}(X) = \{x_1, x_2, ...\}$ and pmf $p_X(x)$. We then have:
  (1) $p_X(x) > 0$ for all $x \in \mathrm{supp}(X)$, and $p_X(x) = 0$ for all $x \notin \mathrm{supp}(X)$.
  (2) $\sum_{x \in \mathbb{R}} p_X(x) = p_X(x_1) + p_X(x_2) + \cdots = 1$.
  (3) For any set $B \subseteq \mathbb{R}$, we must have $\mathbb{P}(X \in B) = \sum_{x \in B} p_X(x)$.

Example 7.2. You are given the following function: $F(x) = \begin{cases} 0; & x < -1, \\ 0.1; & -1 \le x < 2, \\ 0.4; & 2 \le x < 5, \\ 1; & x \ge 5. \end{cases}$

Can $F(x)$ be the cdf of a discrete random variable? If so, find the support and pmf of the random variable.

Solution. $F(x)$ is a right-continuous, non-decreasing, step function that takes value 0 for small x and takes value 1 for large x. So, it can be the cdf of a discrete random variable. The support of X is the set of all jump points, namely $\{-1, 2, 5\}$. The pmf is obtained by the size of jumps:

$$p_X(x) = \begin{cases} 0.1 - 0 = 0.1; & x = -1, \\ 0.4 - 0.1 = 0.3; & x = 2, \\ 1.0 - 0.4 = 0.6; & x = 5, \\ 0; & x \ne -1, 2, 5. \end{cases}$$

- **Properties of cdf:** In general, the cdf of a discrete random variable X has the following properties:
  (1) $F_X(x) \ge 0$ for all $x \in \mathbb{R}$.
  (2) F_X(x) is a **non-decreasing** function (it only increases or stays constant) and it is right continuous. That is $F_X(x_0) = \lim_{x \to x_0^+} F_X(x)$ for all $x_0 \in \mathbb{R}$.
  (3) $\lim\limits_{x \to -\infty} F_X(x) = 0$ and $\lim\limits_{x \to +\infty} F_X(x) = 1$.
  (4) $F_X(x)$ is a step function. It jumps at the points in supp(x) and is constant in other points. That is, if supp$(X) = \{x_1, x_2, \dots\}$,then $F_X(x)$ is constant on the interval $[x_k, x_{k+1})$, and has a jump at x_k. The size of jump is $p_X(x_k)$. At the point of jump, it is **right continuous**.
  (5) For any $a < b$, we have $\mathbb{P}(a < X \le b) = F_X(b) - F_X(a)$.
    – Proof: $\mathbb{P}(X \le b) = \mathbb{P}\left((X \le a) \cup (a < X \le b)\right) = \mathbb{P}(X \le a) + \mathbb{P}(a < X \le b)$ (why?). Therefore,
$$\mathbb{P}(a < X \le b) = \mathbb{P}(X \le b) - \mathbb{P}(X \le a) = F_X(b) - F_X(a).$$
- Any function with properties (1), (2), and (3) is the cdf of a random variable. If property (4) holds as well, then it is the cdf of a discrete random variable. Property (5), which holds for any random variable, can be used to calculate probabilities using cdf.

7.2. Expectation of discrete random variables

- The expectation, or mean, of a random variable provides information of the center of the values taken by the random variable. Intuitively, the expectation of a random variable is its average value if we repeat the experiment (corresponding to the underlying sample space) a large number of times.

Definition 7.3. Let X be a discrete random variable with support supp$(X) = \{x_1, x_2, \dots\}$ and pmf $p_X(x)$. For "any" function $g(x)$, we define the **expected value** of $g(X)$, denoted by $\mathbb{E}[g(X)]$, as follows

$$\mathbb{E}[g(X)] = \sum_{x \in \mathbb{R}} g(x) p_X(x) = g(x_1) p_X(x_1) + g(x_2) p_X(x_2) + \dots$$

In particular, the **expected value of** X, also called the **mean of** X, is given by

$$\mathbb{E}[X] = \sum_{x \in \mathbb{R}} x p_X(x) = x_1 p_X(x_1) + x_2 p_X(x_2) + \dots$$

It is common to use the notation μ_X for $\mathbb{E}[X]$.

- If $\text{supp}(X)$ has an infinite number of values, then the sum in the definition of $\mathbb{E}[X]$ (or $\mathbb{E}[g(X)]$) is an infinite series. Such a series may not converge. If the infinite series diverges, we say **that X does not have expectation**. More specifically, we say that $\mathbb{E}[g(X)]$ exists only if $\mathbb{E}[|g(X)|] < +\infty$. That is, when the following infinite series is convergent:

$$\sum_{x\in\mathbb{R}} |g(x)|p_X(x) = |g(x_1)|p_X(x_1) + |g(x_2)|p_X(x_2) + \cdots < +\infty.$$

- Note that $\mathbb{E}[X]$ is the **weighted average** of the values in the range of X, with weights equal to the probability of each value.

EXAMPLE 7.4. To win a state lottery, one needs to guess 6 out of 49 numbers (order irrelevant). Here's a table describing the prizes:

# correct	6	5	4	≤ 3
Prize ($)|	1.2M	800	35	0

The ticket price is \$0.14. Is this lottery profitable for the state? Would you pay \$0.14 to buy a ticket?

SOLUTION. Since there are a large number of participants, the total payout of the lottery should be close to the expected value of the prize (for one ticket) times the number of participants. Let Y be the prize, we then have

$$p_Y(1.2M) = \frac{1}{\binom{49}{6}} = \frac{1}{13,983,816}, \qquad p_Y(800) = \frac{\binom{6}{5}\binom{43}{1}}{\binom{49}{6}} = \frac{43}{2,330,636}, \qquad p_Y(35) = \frac{\binom{6}{4}\binom{43}{2}}{\binom{49}{6}} = \frac{645}{665,896}$$

$p_Y(0) = 1 - p_Y(1.2M) - p_Y(800) - p_Y(35)$.

Therefore,

$$\mathbb{E}[Y] = 1.2Mp_Y(1.2M) + 800p_Y(800) + 35p_Y(35) + 0p_Y(0) \approx 0.1345.$$

So, on average, a ticket costs \$0.1345 for the state, while it generates the revenue of \$0.14. So, it is profitable for the state.

If we pay 14 cents for a ticket, we would lose 0.5 cent **on average**, and there is more than 99% chance that we lose all the 14 cents. Most individuals, however, would take the risk of loosing a 14 cents if there is a chance (however small) of winning 1.2M.