Chapter 5 INVERSE FUNCTIONS
Section 5.2. The Natural Logarithmic Function

The often forgotten Fundamental Theorem of Calculus.

Theorem: Let f be a continuous function on $[a,b]$. Let $a < x < b$. If

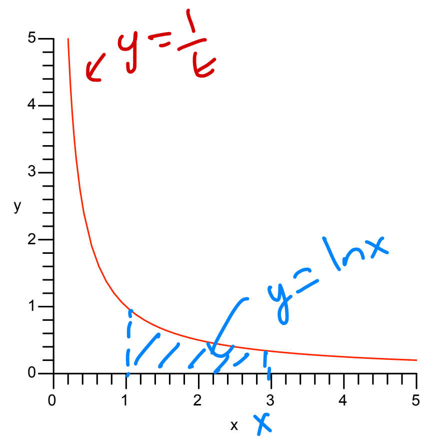
$$g(x) = \int_a^x f(t)\ dt, \text{ then } g'(x) = f(x).$$

$$\frac{d}{dx}\int_a^{u(x)} f(t)dt = f(u(x)).u'(x)$$

For example, $\dfrac{d}{dx}\displaystyle\int_2^{x^3} \sin(t^2)\ dt =$ $\sin(x^3)^2 \cdot 3x^2$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \qquad \int \frac{1}{x} dx = ?$$

The definition of the natural logarithmic function is given as the area under the curve $y = \dfrac{1}{t}$ for $1 \leq t \leq x$.



- **Definition.**

$$\boxed{\ln x = \int_1^x \frac{1}{t}\ dt, \quad x > 0}$$

If $x > 1$ $\ln x > 0$

If $0 < x < 1$, then $\ln x = \displaystyle\int_x^1 \frac{1}{t}\ dt = -\int_1^x \frac{1}{t}\ dt \leq 0$. Also, $\ln 1 = 0$.

if $0 < x < 1$ $\ln x < 0$

$$\ln 1 = \int_1^1 \frac{1}{t}\ dt = 0$$

- **Properties of the natural logarithmic function.**
  - **Derivative.**

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

Proof: $\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$

  - **Chain rule.**

$(f(g(x)))' = f'(g(x)) \cdot g'(x)$

$$\boxed{\frac{d}{dx}\ln u(x) = \frac{1}{u(x)}\frac{du}{dx}}$$

Example: Differentiate the function. $h(x) = \ln(\underbrace{x + \sqrt{x^2 - 1}}_{u})$. Simplify

$$h'(x) = \frac{1}{x+\sqrt{x^2-1}} \cdot \left(1 + \frac{1}{\cancel{2}}(x^2-1)^{-1/2}(\cancel{2}x)\right) = \frac{1}{x+\sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right)$$

$$= \frac{1}{\cancel{x+\sqrt{x^2-1}}} \cdot \left(\frac{\cancel{\sqrt{x^2-1}+x}}{\sqrt{x^2-1}}\right)$$

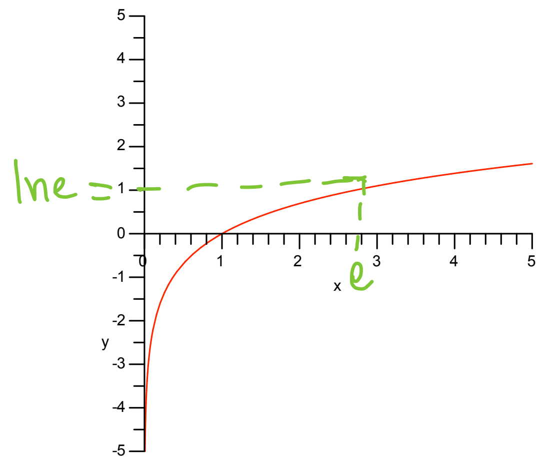
$$= \frac{1}{\sqrt{x^2-1}}$$

– **Graph of $y = \ln x$.**

We know that if $x > 1$, $\ln x > 0$, and if $0 < x < 1$, $\ln x < 0$.

Since $\dfrac{d}{dx} \ln x = \dfrac{1}{x} > 0$, $y = \ln x$ is increasing on $(0, \infty)$.

$\dfrac{d^2}{dx^2} \ln x = -\dfrac{1}{x^2} < 0$ for $x > 0$, the graph of $y = \ln x$ is concave (or concave "down".)



$y = \ln x$ is an increasing function with domain $(0, \infty)$ and range $(-\infty, \infty)$.

Important limits.

$$\lim_{x\to\infty} \ln x = \infty, \quad \lim_{x\to 0^+} \ln x = -\infty$$

$x = 0$ is a vertical asymptote

Definition: e is the number such that $\ln e = 1$. (We know that $\ln 1 = 0$, and $y = \ln x$ is a continuous increasing function. Therefore, by the Intermediate Value Theorem there is a number such that $\ln x = 1$.) e is an irrational number between 2 and 3. We will come back to the properties of e later in the chapter.

Example: Find the derivative of $y = \ln|x|$.

$$\ln|x| = \begin{cases} \ln x, \text{ if } x > 0 \\ \ln(-x), \text{ if } x < 0 \end{cases} \qquad \frac{d}{dx}\ln|x| = \begin{cases} \frac{1}{x} \text{ if } x > 0 \\ \frac{1}{-x}\cdot(-1) = \frac{1}{x}, \text{ if } x < 0 \end{cases}$$

$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}}$$

Integral:

$$\boxed{\int \frac{1}{x}\, dx = \ln|x| + C}$$

$\boxed{\begin{array}{l}\textbf{Algebraic properties of the natural logarithmic function.}\\[1ex] \text{Let } x > 0 \text{ and } y > 0. \text{ Then}\\ \quad * \ \ln(xy) = \ln x + \ln y \ \ (a)\\ \quad * \ \ln\left(\dfrac{x}{y}\right) = \ln x - \ln y\\ \quad * \ \ln(x^r) = r\ln x,\ r \text{ is a rational number.}\end{array}}$

Proof of (a):

Assume a is a constant $(a>0,\ x>0)$

$$\frac{d}{dx}\ln(ax) = \frac{1}{ax}\cdot a = \frac{1}{x}$$

$$\frac{d}{dx}\ln(ax) = \frac{1}{x}$$

Integrate: $\int \frac{d}{dx}\ln(ax)\,dx = \int \frac{1}{x}\,dx \Rightarrow \ln(ax) = \ln x + C$?

if $x=1$ $\quad \ln a = 0 + C$

$$\boxed{\ln(ax) = \ln x + \ln a}$$

Exercises:

1. Find an equation of the tangent line to the $y = \ln(x^3 - 7)$ at $(2, 0)$.

$$\frac{dy}{dx} = \frac{3x^2}{x^3-7} \qquad \text{at } x=2 \quad \frac{dy}{dx} = \underbrace{12}_{\text{slope}}$$

Equation of the tangent line at $(2,0)$

$$\boxed{y = 12(x-2)}$$

2. Evaluate the integral $\int \frac{\cos x}{2+\sin x}\,dx$ $\quad u = 2+\sin x \quad du = \cos x\,dx$

$$\int \frac{\overbrace{\cos x \, dx}^{du}}{\underbrace{2+\sin x}_{u}}$$

$$\int \frac{1}{u}\,du = \ln|u| + C$$

$$\int \frac{\cos x}{2+\sin x}\,dx = \ln|2+\sin x| + C$$

or $\ln(2+\sin x) + C$

3. Evaluate the integral $\int_e^6 \frac{1}{x \ln x}\,dx$ $\quad u = \ln x \quad , du = \frac{1}{x}\,dx$

$$\int_e^6 \frac{\overbrace{\frac{1}{x}\,dx}^{=du}}{\underbrace{\ln x}_{=u}}$$

$x = e \quad , u = 1$

$x = 6 \quad u = \ln 6$

$$\int_1^{\ln 6} \frac{1}{u}\,du = \ln|u| \Big|_1^{\ln 6} = \ln(\ln 6) - \cancel{\ln 1}$$

$$= \ln(\ln 6)$$

4. Use the properties of logarithmic functions to express the given quantity as a single logarithm.

$$\frac{1}{2}\ln x - 5\ln(x^2+1) = \ln x^{1/2} - \ln(x^2+1)^5$$

$$= \ln \frac{x^{1/2}}{(x^2+1)^5}$$

5. Evaluate $\int \tan x \; dx$ $= \int \frac{\sin x \; dx}{\cos x}$ $= -du$ $u = \cos x$ $du = -\sin x dx$

u

$-\int \frac{1}{u} \, du = -\ln|u| + C$

$\int \tan x \, dx = -\ln|\cos x| + C$ $^{-1}$

or $\boxed{\ln|\sec x| + C}$

6. Differentiate the function $y = \ln \frac{1}{x}$.

$y = \ln 1 - \ln x = -\ln x \Rightarrow \frac{dy}{dx} = -\frac{1}{x}$

long way $y = \ln\left(\frac{1}{x}\right)$

$\frac{dy}{dx} = \frac{1}{\frac{1}{x}} \cdot -\frac{1}{x^2} = -\frac{1}{x}$

7. Find y' if $\ln(xy) = y \sin x$.

$\left(\frac{1}{xy}\left(y + x\frac{dy}{dx}\right) = \frac{dy}{dx}\sin x + y\cos x\right) xy$

$y + x\frac{dy}{dx} = xy\sin x \frac{dy}{dx} + xy^2\cos x$

$\frac{dy}{dx}(x - xy\sin x) = xy^2\cos x - y$

$\frac{dy}{dx} = \frac{xy^2\cos x - y}{x - xy\sin x}$

- **Logarithmic Differentiation.**

  The logarithmic differentiation provide a simpler alternative to the differentiation of functions like $y = \dfrac{x^2+1}{\sqrt[4]{x^2-1}}$.

  We first take the natural logarithm of each side, then we use properties of the logarithmic functions to simplify the expression, and then, we differentiate implicitly with respect to x.

① $\ln y = \ln\left(\dfrac{x^2+1}{\sqrt[4]{x^2-1}}\right)$

$\ln y = \ln(x^2+1) - \frac{1}{4}\ln(x^2-1)$

② Use implicit differentiation

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+1} - \frac{1}{4}\frac{2x}{x^2-1} = \frac{2x}{x^2+1} - \frac{x}{2(x^2-1)}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{4x(x^2-1) - x(x^2+1)}{2(x^2-1)(x^2+1)} = \frac{3x^3-5x}{2(x^2-1)(x^2+1)}$$

$$\frac{dy}{dx} = \frac{3x^3-5x}{2(x^2-1)\cancel{(x^2+1)}} \cdot \frac{\cancel{x^2+1}}{\sqrt[4]{x^2-1}}$$