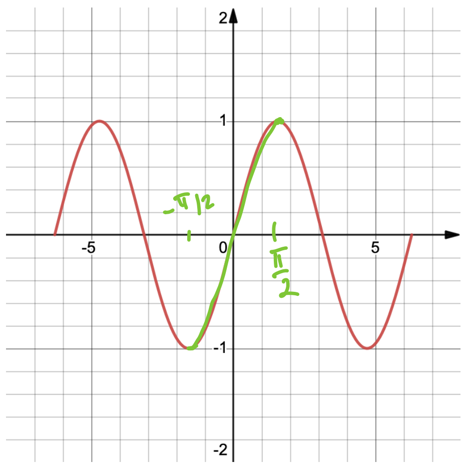
Chapter 5. Inverse Functions.
Section 5.6. Inverse Trigonometric Functions.

1. **Inverse function of $y = \sin x$.**

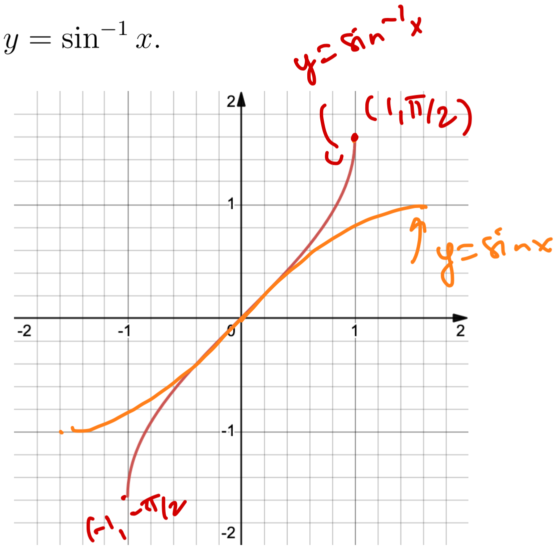
   Below is the graph of $y = \sin x$



It is clear that $y = \sin x$ is not one–to–one on its domain. But, if we restrict the domain of $y = \sin x$ to the interval $[-\pi/2, \pi/2]$, then y is increasing (hence one–to–one) with range $[-1, 1]$. Therefore, $y = \sin x$ has an inverse function denoted $\sin^{-1}$, or arcsin such that

$$\boxed{y = \sin^{-1} x \text{ if and only if } x = \sin y}$$

- Properties of $y = \sin^{-1} x$ ✓
  - The domain of $y = \sin^{-1} x$ is $[-1, 1]$ and its range is $[-\pi/2, \pi/2]$.
  - $\sin^{-1}(\sin x) = x$ if x is in $[-\pi/2, \pi/2]$
  - $\sin(\sin^{-1}(x)) = x$ if x is in $[-1, 1]$
  - Graph of $y = \sin^{-1} x$.



- Exercises: Find the exact value of

(a) $\sin^{-1}(1/2) = \theta \Rightarrow \sin\theta = \frac{1}{2}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\theta = \frac{\pi}{6}$

$\boxed{\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}}$

(b) $\sin^{-1}(\sin(3\pi/4)) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \theta \Rightarrow \sin\theta = \frac{\sqrt{2}}{2}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\sin^{-1}\left(\sin\frac{3\pi}{4}\right) = \pi/4$

$\theta = \pi/4$

(c) $\sin^{-1}(\pi/2) = \theta \Rightarrow \sin\theta = \frac{\pi}{2} > 1$

$\sin^{-1}\left(\frac{\pi}{2}\right)$ is undefined

- Derivative of $y = \sin^{-1} x$.

  Use Implicit Differentiation and trigonometric identities to show that

$y = \sin^{-1} x \iff \boxed{x = \sin y}$

$$\frac{d}{dx} \sin^{-1} x \overset{?}{=} \frac{1}{\sqrt{1-x^2}}$$

why?

$\frac{d}{dx} y = \frac{d}{dx} \sin^{-1} x$ is equivalent to $\frac{d}{dx} x = \frac{d}{dx} \sin y$

$1 = \cos y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$

$\cos^2 y + \sin^2 y = 1$ (x^2) $\Rightarrow \cos^2 y = 1 - x^2$

$\cos y = \pm\sqrt{1-x^2}$

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ $\quad \cos y = \sqrt{1-x^2}$

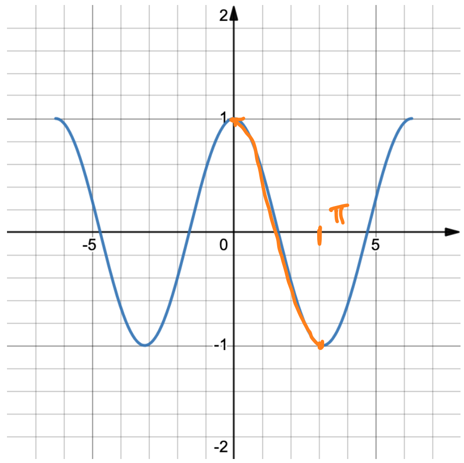
The Chain rule follows: $\boxed{\frac{d}{dx} \sin^{-1} u(x) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}}$

- Integral.

$$\boxed{\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C, \quad \text{where } C \text{ is a constant}}$$

2. **Inverse function of $y = \cos x$.**

Show that $y = \cos x$ is one–to–one on $[0, \pi]$ with range $[-1, 1]$.

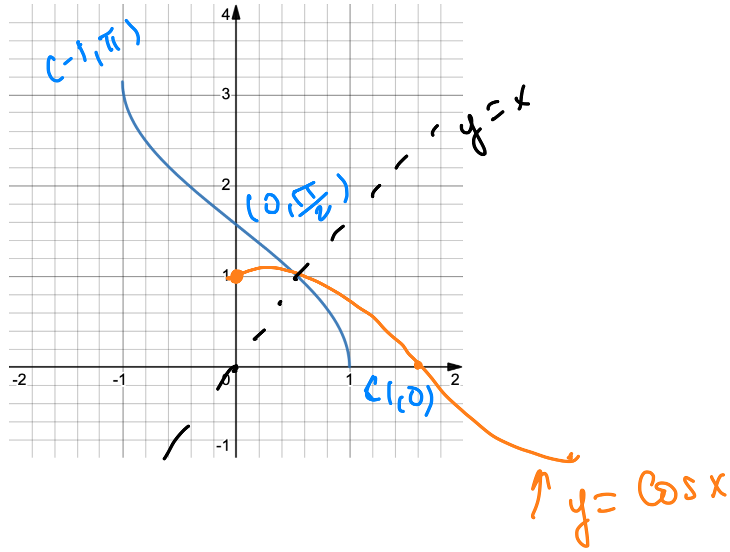


$y = \cos x$ has an inverse function denoted $\cos^{-1}$, or arccos such that

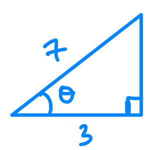
$$\boxed{y = \cos^{-1} x \ \text{ if and only if } \ x = \cos y}$$

- Properties of $y = \cos^{-1} x$
  - The domain of $y = \cos^{-1} x$ is $[-1, 1]$ and its range is $[0, \pi]$.
  - $\cos^{-1}(\cos x) = x$ if x is in $[0, \pi]$.
  - $\cos(\cos^{-1} x) = x$ if x is in $[-1, 1]$

– Graph of $y = \cos^{-1} x$.



- Exercise: Find the exact value of $\csc\left(\underbrace{\arccos}_{\theta} \frac{3}{7}\right) = \csc\theta$ where $\cos\theta = \frac{3}{7}$



$\sqrt{40} = 2\sqrt{10}$

$\csc\theta = \frac{7}{2\sqrt{10}}$

$\csc\left(\arccos\frac{3}{7}\right) = \frac{7}{2\sqrt{10}} = \frac{7\sqrt{10}}{20}$

- Derivative of $y = \cos^{-1} x$.

Use Implicit Differentiation and trigonometric identities to show that

$$\boxed{\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}}$$

$y = \cos^{-1} x \iff x = \cos y$

$\frac{dy}{dx} = \frac{d}{dx}\cos^{-1} x \iff \frac{d}{dx} x = \frac{d}{dx}\cos y$

$1 = -\sin y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-1}{\sin y}$ etc. . . . . .

The Chain rule follows: $\boxed{\dfrac{d}{dx}\cos^{-1}u(x) = \dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx}}$

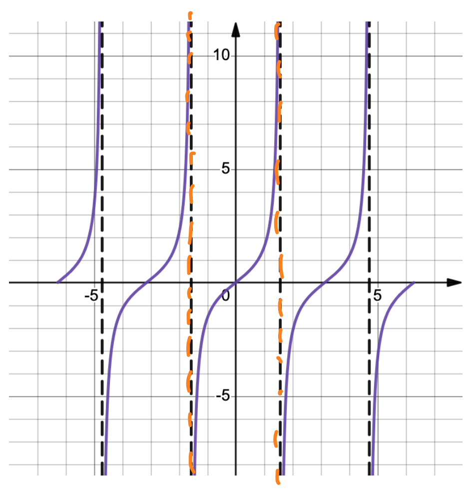
- **Example**

  Find the limit. $\displaystyle\lim_{x\to\infty}\cos^{-1}\left(\frac{1+x^2}{1+2x^2}\right)$

$$\lim_{x\to\infty}\frac{1+x^2}{1+2x^2} = \lim_{x\to\infty}\frac{x^2}{2x^2} = \frac{1}{2} \Rightarrow \lim_{x\to\infty}\cos^{-1}\left(\frac{1+x^2}{1+2x^2}\right) = \cos^{-1}\left(\frac{1}{2}\right)$$
$$= \frac{\pi}{3}$$

3. **Inverse function of $y = \tan x$.**

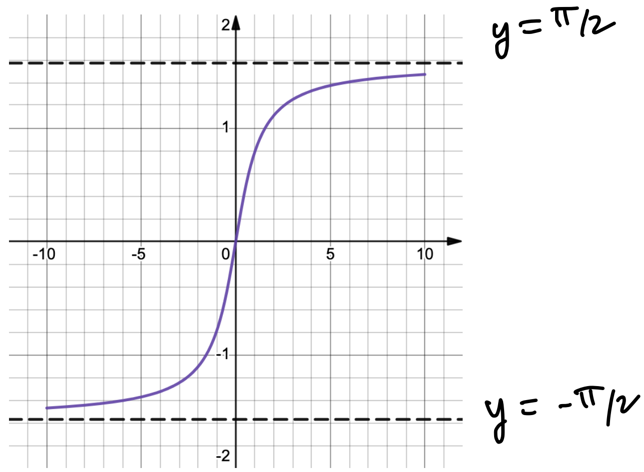
   Show that $y = \tan x$ is one–to–one on $(-\pi/2, \pi/2)$ with range $(-\infty, \infty)$.



$y = \tan x$ has an inverse function denoted $\tan^{-1}$, or arctan such that

$$\boxed{y = \tan^{-1} x \ \text{ if and only if } \ x = \tan y}$$

- Properties of $y = \tan^{-1} x$
  - The domain of $y = \tan^{-1} x$ is $(-\infty, \infty)$ and its range is $(-\pi/2, \pi/2)$.
  - $\tan^{-1}(\tan x) = x$ if x is in $(-\pi/2, \pi/2)$.
  - $\tan(\tan^{-1} x) = x$ if x is in $(-\infty, \infty)$
  - Graph of $y = \tan^{-1} x$.

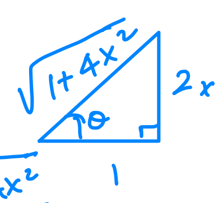


  – Limits.
  $\lim_{x\to\infty} \tan^{-1} x =$ $\pi/2$, $\lim_{x\to-\infty} \tan^{-1} x =$ $-\pi/2$

- Exercise: Find the exact value of $\csc(\underbrace{\arctan 2x}_{\theta})$

write $\csc(\arctan 2x)$ as an algebraic equation (with no trigonometric or inverse trigonometric functions

$\csc\theta = ?$ if $\tan\theta = 2x = \frac{2x}{1}$



$$\csc\theta = \frac{\sqrt{1+4x^2}}{2x}$$

- Derivative of $y = \tan^{-1} x$.
  Use Implicit Differentiation and trigonometric identities to show that

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$ why?

$y = \tan^{-1} x \iff x = \tan y$

$\frac{dy}{dx} = \frac{d}{dx} \tan^{-1} x \iff \frac{d}{dx} x = \frac{d}{dx} \tan y$

$1 = \sec^2 y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$

$\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$

$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

The Chain rule follows: $\boxed{\frac{d}{dx} \tan^{-1} u(x) = \frac{1}{1+u^2} \frac{du}{dx}}$

$\int \frac{1}{x} dx = \ln|x| + c$

$\int e^x dx = e^x + c$

$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$

- Integral.

$$\boxed{\int \frac{1}{1+x^2}\, dx = \arctan x + C}$$

- **Example.**
  Evaluate the integral.(Here, $a \neq 0$ and $a \neq 1$.)

$\int \frac{1}{9+x^2} dx = \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + C$

$$\int \frac{1}{a^2+x^2}\, dx$$

$= \int \frac{1}{a^2(1+\frac{x^2}{a^2})} dx = \frac{1}{a^2} \int \frac{1}{1+(\frac{x}{a})^2} dx$ $dx = a\,du$ $u = \frac{x}{a}$ $du = \frac{1}{a} dx$

$= \frac{1}{a^2} \int \frac{a\,du}{1+u^2} = \frac{1}{a} \int \frac{1}{1+u^2} du$

$= \frac{1}{a} \tan^{-1}(u) + C$

$$\boxed{\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C}$$

Hw: $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

$\rightarrow \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$, $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C$

Exercises: Evaluate the integral.

(a) $\int \frac{1}{\sqrt{1-4t^2}}\,dt = \int \frac{1}{\sqrt{1-(\underbrace{2t}_{u})^2}}\,dt$ $= \frac{1}{2}du$ $\quad u = 2t \quad du = 2\,dt$

$\frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{2}\sin^{-1}(u)+C$

$\int \frac{1}{\sqrt{1-4t^2}}\,dt = \frac{1}{2}\sin^{-1}(2t)+C$

(b) $\int \frac{\sin x}{1+\cos^2 x}\,dx$ $= -du$ $\quad u^2$

$u = \cos x, \; du = -\sin x\,dx$

$\int -\frac{1}{1+u^2}\,du = -\tan^{-1}(u)+C$, $\int \frac{\sin x}{1+\cos^2 x}\,dx$

$= -\tan^{-1}(\cos x)+C$

(c) $\int \frac{x+9}{x^2+9}\,dx$ $= \frac{1}{2}du$

$\int \frac{x+9}{x^2+9}\,dx = \int \frac{x}{\underbrace{x^2+9}_{u}}\,dx + \int \frac{9}{x^2+9}\,dx$

$u = x^2+9, \; du = 2x\,dx$

$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u|+C$

$\frac{1}{2}\ln(x^2+9)$

$9\int \frac{1}{9+x^2}\,dx = 9\left(\frac{1}{3}\right)\tan^{-1}\left(\frac{x}{3}\right)$

$= 3\tan^{-1}\left(\frac{x}{3}\right)$

$\int \frac{x+9}{x^2+9}\,dx = \frac{1}{2}\ln(x^2+9)+3\tan^{-1}\left(\frac{x}{3}\right)+C$

- Other Inverse Functions.

  1. $y = \csc^{-1} x$.
     Domain: $(|x| \geq 1)$
     Range: $(0, \pi/2] \cup (\pi, 3\pi/2]$.
  2. $y = \sec^{-1} x$.
     Domain: $(|x| \geq 1)$
     Range: $[0, \pi/2) \cup [\pi, 3\pi/2)$.
  3. $y = \cot^{-1} x$.
     Domain: $(-\infty, \infty)$
     Range: $(0, \pi)$.

  Derivatives.

  $$\frac{d}{dx} \csc^{-1} x = -\frac{1}{x\sqrt{x^2 - 1}} \qquad \frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{1 - x^2}} \qquad \frac{d}{dx} \cot^{-1} x = -\frac{1}{x^2 + 1}$$