MTH 224O - Spring 2024

Practice Problems for Exam 2

Problem 1:

(a) $1 = \displaystyle\int_{10}^{+\infty} \frac{c}{x^2} dx = \frac{c}{10} \quad \implies \quad c = 10.$

(b) $\mathbb{E}(X)$ and $\mathrm{Var}(X)$ are not defined because $\displaystyle\int_{10}^{+\infty} x\frac{10}{x^2} dx = +\infty.$

(c) We assume that light bulbs stop working independently of each other, so their lifetimes are independent. Under this assumption, the number of working light bulbs after 15 years, denoted by N, is a binomial random variable with parameters $n = 6$ and $p = \int_{15}^{+\infty} \frac{10}{x^2} dx = \frac{2}{3}$. Thus:

$$\mathbb{P}(N \geq 3) = 1 - (1/3)^6 - 6 \times (1/3)^5 \times (2/3) - 15 \times (1/3)^4 \times (2/3)^2 = \frac{656}{729} \approx 0.89986.$$

Problem 2:

(a) The probability that Jane stops after each roll is $\frac{1}{6}$, therefore $N \sim G\left(\frac{1}{6}\right)$. If $N = n$, then the number of 1s in n rolls (where the n-th one is known to be 6), is binomial. Thus $X|_{N=n} \sim \mathrm{Bin}\left(n-1, \frac{1}{5}\right)$ (notice that the parameter for the binomial is $\frac{1}{5}$, since we know that the number 6 did not appear in the first n-1 rolls). In addition, if $N = 1$, then $X = 0$. Thus, the joint pmf is given by

$$p_{N,X}(n,k) = p_{X|N}(k \mid n)\, p_N(n) = \begin{cases} \binom{n-1}{k} \left(\frac{1}{5}\right)^k \left(\frac{4}{5}\right)^{n-1-k} \cdot \left(\frac{5}{6}\right)^{n-1} \frac{1}{6} & k \in \{0, \ldots, n-1\}; \\ 0 & \text{otherwise.} \end{cases}$$

Remark: We assume here that $\binom{0}{0} = 1$, so that $p_{N,X}(1,0) = \frac{1}{6}$.

(b) We can use the law of total probability:

$$\begin{aligned} P(X = N-1) &= \sum_{n=1}^{\infty} P(X = N-1 = n-1) = \sum_{n=1}^{\infty} P(X = n-1,\, N = n) \\ &= \sum_{n=1}^{\infty} \binom{n-1}{n-1} \left(\frac{1}{5}\right)^{n-1} \left(\frac{4}{5}\right)^{n-1-(n-1)} \left(\frac{5}{6}\right)^{n-1} \frac{1}{6} \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1} \cdot \frac{1}{6} = \frac{1}{6} \cdot \sum_{m=0}^{\infty} \left(\frac{1}{6}\right)^m = \frac{1}{6} \cdot \frac{1}{1-\frac{1}{6}} = \frac{1}{5}. \end{aligned}$$

Problem 3:

(a) Let X be the number of typos in page 3. Assuming that there are many possibilities for typos ($n \to \infty$), and probability of a typo in a word is small ($p \to 0$), we can approximate X by a Poisson random variable with parameter $\lambda = 0.1$ typos per page. Therefore, $\mathbb{P}(X \geq 1) = 1 - e^{-0.1} \approx 0.095$.

(b) Let X = number mistakes on page 5. Then, $X \sim \text{Pois}(0.1)$. Therefore:

$$\mathbb{P}(X = 3 | X \geq 1) = \frac{\mathbb{P}(X = 3)}{\mathbb{P}(X \geq 1)} = \frac{\frac{0.1^3}{3!} e^{-0.1}}{1 - e^{-0.1}} \approx 0.00158472.$$

(c) Let X = be the number of misprints on page 8. Note that $X|_A \sim \text{Pois}(3)$ and $X|_B \sim \text{Pois}(4.2)$. Therefore, LTP yields that

$$\mathbb{P}(X = 0) = \mathbb{P}(A) \times \mathbb{P}(X = 0|A) + \mathbb{P}(B) \times \mathbb{P}(X = 0|B) = \frac{1}{2} e^{-3} + \frac{1}{2} e^{-4.2} = 0.03239.$$