MTH 224, Spring 2024

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Lecture 3

Section 1.2: basic multiplication principle, permutations, general permutations

3.1. Counting Methods

EXAMPLE 3.1. We roll a fair die 3 times. What is the probability that it lands on 4 exactly once?

SOLUTION. Let an outcome be the ordered list of the outcomes of each roll. For example, $(2, 5, 1)$ is the outcome that first we roll a 2, then a 5, and finally a 1.

- How many outcomes are there? What is the probability of each outcome?
  - There are $6 \times 6 \times 6 = 216$ possible outcomes. Each outcome is equally likely to occur (convince yourself). So, the probability of each outcome is $1/216 \approx 0.00463$.
- In how many outcomes, will we get exactly one 4?
  - $3 \times 5 \times 5 = 75$. So, the probability of getting exactly one 4 is $75/216 \approx 0.3472$.

- **Basic multiplication principle:** Consider an operations with K consecutive steps. There are n_1 possible ways to perform step 1. For each choice in step 1, there are n_2 possible ways to choose from in step 2. Regardless of what we choose in steps 1 and 2, there are n_3 possible ways to perform step 3. And so forth. Then, there are a total of $n_1 \times n_2 \times \cdots \times n_K$ different ways to perform the operation.

Examples:

- When ordering a certain type of computer, there are 3 choices of hard drive, 4 choices RAM, 2 choices of graphics card, and 3 choices of CPU. In how many ways can a computer be ordered?
  - $3 \times 4 \times 2 \times 3 = 72$
- How many words are there with 5 letters (meaningless words count as well)?
  - 26^5
- We roll a die three times. How many outcomes are possible if order is important?
  - $6^3 = 216$
- We roll three dice once. How many outcomes are possible if we do **not** distinguish between the dice?
  - A bit more difficult. The answer is $6 + 6 \times 5 + \dfrac{6 \times 5 \times 4}{6} = 56$.
  - In the last term on the left side, you should divide by 6 because, in outcomes with 3 different numbers, order is not important. That is, (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), and (3,2,1) all count as one outcome.

3.2. Permutations

- In how many ways can one arrange 8 people in a line?
  - $8! = 8 \times 7 \times 6 \times 5 \times 3 \times 2 \times 1 = 40320$

- A **permutation** is an ordering of a collection of objects. The number of permutation of N objects is $N!$ (read "N factorial"). Here:

$$N! = N \times (N-1) \times (N-2) \times \cdots \times 1.$$

- We take the convention that $0! = 1$.

EXAMPLE 3.2. In how many ways can we arrange 8 couples in a line such that each couple is standing next to each other?

SOLUTION. There are $8! = 40320$ possible ways to order the different couples (as if each couple is one object). There are $2^8 = 256$ ways to order each pair. So, in total, there are $8! \times 2^8 = 10,321,920$ ways to arrange the couples.

EXAMPLE 3.3. In how many ways can we arrange 8 people in a round table with 8 seats? Here, only relative positions of the individuals with respect to each other matter.

SOLUTION. Since only relative positions matter, we can assume that the first individual sits at the top of the table (essentially, the top of the table is where the first individual sits). The remaining 7 individuals can then be seated in $7!$ ways. So, the answer is $7! = 5040$.

3.3. General Permutations

EXAMPLE 3.4. **(General Permutation)** In how many ways can we arrange 5 identical red balloons, 3 identical blue balloons, and 2 identical green balloons in a row?

SOLUTION. First, suppose that we distinguish the balloons by labeling them individually. For example:

$$R_1, R_2, R_3, R_4, R_5, B_1, B_2, B_3, G_1, G_2$$

There are $10!$ arrangement of the balloons if they were individually distinguishable. Now, we can think of arranging the individually distinguishable balloons as a 2-step operation: 1) arrange the unlabeled balloons in a line; and 2) label the balloons. We don't know in how many ways we can perform step 1. This is actually what the question asks. Let us call it x.

We know how many ways step 2 is possible: $5! \times 3! \times 2!$. So, by the multiplication principle, we must have

$$10! = x \times 5! \times 3! \times 2! \qquad \Longrightarrow \qquad x = \frac{10!}{5! \times 3! \times 2!} = 2520.$$

- **In general:** the number of possible arrangements of n objects in a row; k_1 of type 1; k_2 of type 2, ..., k_r of type r (where objects of each type are identical, and $k_1 + \ldots + k_r = n$) is

$$\binom{n}{k_1, k_2, \ldots, k_r} := \frac{n!}{k_1! \times k_2! \times \cdots \times k_r!}.$$

EXAMPLE 3.5. We have 10 white doors. In how many ways can we color 5 of them red, 3 of them blue, and 2 of them green?

SOLUTION. Note that each such coloring is essentially a general permutation of 5 red door, 3 blue door, and 2 green door. So, this is exactly the same problem as in the previous example! The answer is the same: $\binom{10}{5,3,2} = \frac{10!}{5!3!2!} = 2520$.

- **In general:** the number of possible ways to divide a group of n objects into r sub-groups, such that the i-th sub-group has k_i objects, ... , is

$$\binom{n}{k_1, k_2, \ldots, k_r} := \frac{n!}{k_1! \times k_2! \times \cdots \times k_r!}.$$

REMARK. $\binom{n}{k_1,k_2,\ldots,k_r}$ is called a **multinomial coefficient**, because they appear in the multinomial theorem:

- **The multinomial theorem:** For any $n, r \in \{1, 2, \ldots\}$, we have

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{k_1, \ldots, k_r \\ k_1 + \cdots + k_r = n}} \cdots \sum \binom{n}{k_1, \ldots, k_r} x_1^{k_1} \ldots x_r^{k_r}.$$