Section 5.8. Indeterminate forms

1. **Indeterminate Quotients.**

Theorem: Suppose that f and g are differentiable and $g'(x) \neq 0$. Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$. Then

$\lim_{x\to a} f(x) = 0 = f(a)$

$\lim_{x\to a} g(x) = 0 = g(a)$

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

Exercise. Evaluate the limit. $\lim_{x\to 1} \dfrac{\ln x}{\sin \pi x}$

$$\lim_{x\to 1} \frac{\ln x \to 0}{\sin \pi x \to 0} = \lim_{x\to 1} \frac{\frac{1}{x}}{\pi \cos \pi x} = -\frac{1}{\pi}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Additional examples

$$\lim_{x\to 0} \frac{e^x - e^{-x} - 2x \to 0}{x - \sin x \to 0} = \lim_{x\to 0} \frac{e^x + e^{-x} - 2 \to 0}{1 - \cos x \to 0} = \lim_{x\to 0} \frac{e^x - e^{-x} \to 0}{\sin x \to 0}$$

$$= \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x}$$

$$= 2$$

$$\lim_{x\to\infty} \frac{x \to \infty}{\sqrt{x^2+5} \to \infty} = \lim_{x\to\infty} \frac{1}{\frac{x}{\sqrt{x^2+5}}} = \lim_{x\to\infty} \frac{\sqrt{x^2+5} \to \infty}{x \to \infty}$$

$$= \lim_{x\to\infty} \frac{\frac{x}{\sqrt{x^2+5}}}{1} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+5}} ??$$

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2\left(1+\frac{5}{x}\right)}} = \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}\sqrt{1+\frac{5}{x} \to 0}}$$

$$= 1$$

2. **Indeterminate Products.**

When $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$. What is $\lim_{x\to a} f(x)g(x)$? Well, it depends! It is not always zero. How can we use the above theorem to eliminate the indetermination in the case of the products of the form "$0 \cdot \pm\infty$?

$$\lim_{x\to a} f(x)\cdot g(x) = \lim_{x\to a} \frac{f(x) \to 0}{\frac{1}{g(x)} \to 0} \quad \text{use L'Hospital's Rule}$$

$$\text{or } \lim_{x\to a} f(x)\cdot g(x) = \lim_{x\to a} \frac{g(x) \to \pm\infty}{\frac{1}{f(x)} \to \pm\infty}$$

Exercise: Evaluate the limit. $\lim_{x\to\infty} x\tan(1/x)$

$$\lim_{x\to\infty} x \tan\left(\tfrac{1}{x}\right) = \lim_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x} \to 0} = \lim_{x\to\infty} \frac{\sec^2\left(\frac{1}{x}\right)\left(\cancel{-\frac{1}{x^2}}\right)}{\cancel{-\frac{1}{x^2}}}$$

$$= \lim_{x\to\infty} \sec^2\left(\tfrac{1}{x}\right) = \sec^2(0) = 1$$

$$\lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1$$

$$\lim_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{u\to 0} \frac{\tan u}{u} = 1$$

Let $u = \frac{1}{x}$. As $x \to \infty$, $u \to 0$

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta\cos\theta} = 1$$

3. **Indeterminate Differences.**

When evaluating the limit $\lim_{x\to a}(f(x) - g(x))$, where $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$, we have an indeterminate case of the form "$\infty - \infty$".

Exercise. Evaluate the limit. $\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

"$\infty - \infty$"

$$\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1}\frac{\overbrace{x-1-\ln x}^{\to 0}}{\underbrace{(x-1)\ln x}_{0}} = \lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x + (x-1)\frac{1}{x}}$$

$$= \lim_{x\to 1}\frac{\overbrace{1-\frac{1}{x}}^{\to 0}}{\underbrace{\ln x + 1}_{0} - \frac{1}{x}} = \lim_{x\to 1}\frac{\frac{1}{x^2}}{\frac{1}{x}+\frac{1}{x^2}} = \frac{1}{2}$$

4. **Indeterminate Powers.**

When we evaluate a limit of the form $\lim_{x\to a}[f(x)]^{g(x)}$, we will have an indeterminate form in the following cases.

(a) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ type 0^0.

(b) $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ type ∞^0.

(c) $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ type 1^∞.

We first find $\lim_{x\to a} \ln[f(x)]^{g(x)}$ and then use the properties of the logarithmic functions to find $\lim_{x\to a}[f(x)]^{g(x)}$

Exercises. Evaluate the limit.

(a) $\lim_{x\to\infty} (1 + \underbrace{\frac{1}{x}}_{\to 0})^{\overbrace{x}^{\to\infty}}$ "1^∞"

Step1: $\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} \overbrace{x}^{\infty} \cdot \overbrace{\ln\left(1+\frac{1}{x}\right)}^{\to 0} = \lim_{x\to\infty} \frac{\overbrace{\ln\left(1+\frac{1}{x}\right)}^{\to 0}}{\underbrace{\frac{1}{x}}_{\to 0}}$

Use L'Hospital's Rule

$$\lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{-\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{1+\underbrace{\frac{1}{x}}_{\to 0}} = 1 = \lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x$$

So, $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e^1$
$= e$

(b) $\lim_{x\to 0^+} (\tan 2x)^x$ $\xrightarrow{} 0$ "0^0"

$$\lim_{x\to 0^+} \ln(\tan 2x)^x = \lim_{x\to 0^+} x \cdot \ln(\tan 2x) = \lim_{x\to 0^+} \frac{\ln(\tan 2x)}{\frac{1}{x}}$$

$\to -\infty$, $\to \infty$

use L'Hospital's Rule.

$$\lim_{x\to 0^+} \frac{\frac{1}{\tan 2x}\, 2\sec^2(2x)}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{2\sec^2(2x)}{\tan(2x)}(-x^2)$$

$$= \lim_{x\to 0^+} \frac{2\,\cancel{\cos(2x)}}{\cos^{\cancel{2}}(2x)\,\sin(2x)}(-x^2)$$

$$= \lim_{x\to 0} \frac{-2x^2}{\cos(2x)\sin(2x)}$$

$$= \lim_{x\to 0} -\frac{x}{\cos(2x)}\cdot\frac{2x}{\sin(2x)} = 0$$

$\frac{x}{\cos(2x)} \to 0$, $\frac{2x}{\sin(2x)} \to 1$

$\lim_{x\to 0^+} (\tan 2x)^x = e^0 = 1$

(c) $\lim_{x\to 0} \frac{1}{x^3} \int_0^x \sin(t^2)\ dt$

$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

$$= \lim_{x\to 0} \frac{\int_0^x \sin(t^2)\,dt}{x^3} = \lim_{x\to 0} \frac{\sin x^2}{3x^2} = \frac{1}{3}\lim_{x\to 0} \frac{\sin x^2}{x^2} = \frac{1}{3}$$

or L'Hospital's Rule

$$= \lim_{x\to 0} \frac{2x\cos(x^2)}{6x} = \lim_{x\to 0} \frac{\cos(x^2)}{3} = \frac{1}{3}$$

Important antiderivative

$$\int \frac{1}{x}\,dx = \ln|x| + C, \quad \int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C, \quad \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$