CHAPTER 8. SEQUENCES AND SERIES.
SECTION 8.5. Power Series

Example: For what values of x does the series $\sum_{n=0}^{\infty} x^n$ converge? What is its sum?

Geometric with $r = x$. $\sum_{n=0}^{\infty} x^n$ is convergent when $|x|<1$ or $-1<x<1$

when $\underbrace{-1<x<1}_{\text{Interval of convergence.}}$, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = 1+x+x^2+x^3+\ldots$

- <u>Definitions.</u>

1. A *power series* is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

where the c_n's are called the coefficients of the series. When the series converges, the sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

2. A *power series centered at* a is a power series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$$

Example: For what values of x does the series $\sum_{n=1}^{\infty} \underbrace{\frac{(-2)^n (x+3)^n}{\sqrt{n}}}_{= a_n}$ converge?

Ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\cancel{(-2)^{n+1}} \cancel{(x+3)^{n+1}}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{\cancel{(-2)^n} \cancel{(x+3)^n}} \right|$$

$$= \lim_{n\to\infty} \underbrace{\frac{2\sqrt{n}}{\sqrt{n+1}}}_{\to 1} |x+3| = 2|x+3| = L$$

Radius of convergence

- If $2|x+3| < 1$ or $|x+3| < \frac{1}{2}$, $\sum_{n=1}^{\infty} \frac{(-2)^n (x+3)^n}{\sqrt{n}}$ is convergent.

$\left(-\frac{1}{2} < x+3 < \frac{1}{2} \implies -\frac{7}{2} < x < -\frac{5}{2} \right)$

- If $2|x+3| = 1 \implies x = -\frac{7}{2}$ or $x = -\frac{5}{2}$

- If $x = -\frac{7}{2}$, $\sum_{n=1}^{\infty} \frac{(-2)^n \left(-\frac{7}{2}+3\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-2)^n \left(-\frac{1}{2}\right)^n}{\sqrt{n}}$

$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent

- If $x = -\frac{5}{2}$, $\sum_{n=1}^{\infty} \frac{(-2)^n \left(-\frac{5}{2}+3\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-2)^n \left(\frac{1}{2}\right)^n}{\sqrt{n}}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. Alternating series

$b_n = \frac{1}{\sqrt{n}}$ $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

$\frac{1}{\sqrt{n+1}} \le \frac{1}{\sqrt{n}}$

convergent.

Conclusion:

$\sum_{n=1}^{\infty} \frac{(-2)^n (x+3)^n}{\sqrt{n}}$ is convergent when $-\frac{7}{2} < x \le -\frac{5}{2}$

$\left(-\frac{7}{2}, -\frac{5}{2}\right]$ is the interval of convergence

- **Theorem.** For a given power series $\boxed{\sum_{n=0}^{\infty} c_n(x-a)^n}$ there are only three possibilities:

  - The series converges only when $x = a$.
  - The series converges for all x.
  - There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x - a| > R$.

Remark: The theorem does not say anything about what happens when $|x-a| = R$.

The number R is called the *radius of convergence* of the series. In the first two cases, we say that $R = 0$ and $R = \infty$ respectively. The *interval of convergence* of a power series is the interval that consists of all values of x for which the series converges.

Examples: Find the radius of convergence and interval of convergence of the series.

1. $\sum_{n=1}^{\infty} \underbrace{n!(2x-1)^n}_{a_n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(n+1)!}^{\,\cancel{n!}(n+1)}\,(2x-1)^{\cancel{n+1}}}{\cancel{n!}\,\cancel{(2x-1)^n}}\right|$$

$$= \lim_{n\to\infty} \underbrace{(n+1)}_{\downarrow\ \infty}|2x-1| = \begin{cases} \infty & \text{if } x \neq \frac{1}{2} \\ 0 & \text{if } x = \frac{1}{2} \end{cases}$$

$\sum_{n=1}^{\infty} n!(2x-1)^n$ is convergent when $x = \frac{1}{2}$

$$\boxed{\begin{array}{c}\text{Interval of convergence } I = \{\tfrac{1}{2}\} \\ \text{Radius of convergence} \\ R = 0\end{array}}$$

Qu

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$ Root test

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-2)^n}{n^n}\right|} = \lim_{n\to\infty} \frac{|x-2|}{n} = 0$$

$\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$ is convergent for all real numbers

$I = (-\infty, \infty)$

$R = \infty$

$|x-a|<R$

• $\sum_{n=2}^{\infty} \frac{(2x+3)^n}{n\ln n} = a_n$ Ratio test $\lim_{n\to\infty}\left|\frac{(2x+\cancel{3})^{n+1}}{(n+1)\ln(n+1)} \cdot \frac{n\ln n}{\cancel{(2x+3)^n}}\right|$

$= \lim_{n\to\infty} \underbrace{\frac{n}{n+1}}_{\to 1} \underbrace{\frac{\ln n}{\ln(n+1)}}_{\to 1} |2x+3| = |2x+3|$

. If $|2x+3|<1$, $\sum_{n=2}^{\infty} \frac{(2x+3)^n}{n\ln n}$ is convergent with $R=\frac{1}{2}$

$|2(x+\frac{3}{2})|<1$

or $|x+\frac{3}{2}|<\frac{1}{2}$

• If $|2x+3|=1 \Rightarrow$ $2x+3=1$ or $2x+3=-1$

$x=-1$ or $x=-2$

. If $x=-2$, $\sum_{n=2}^{\infty} \frac{(-4+3)^n}{n\ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$ is a convergent alternating series

. If $x=-1$, $\sum_{n=2}^{\infty} \frac{(-2+3)^n}{n\ln n} = \sum_{n=2}^{\infty} \frac{1}{n\ln n}$ is divergent by the integral test.

Conclusion: $I=[-2,-1)$

Let $f(x)=\frac{1}{x\ln x}$ f is continuous, positive and decreasing for $x\geq 2$

$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x\underbrace{\ln x}_{u}}\,dx = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u}\,du$

$= \lim_{t\to\infty} \ln|u| \Big|_{\ln 2}^{\ln t}$

$\int_2^{\infty} \frac{1}{x\ln x}\,dx$ is divergent

$\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ is divergent $= \lim_{t\to\infty}\left(\ln(\ln t) - \ln(\ln 2)\right)$

$=\infty$