CHAPTER 8. SEQUENCES & INFINITE SERIES
Section 8.2. Series.

Basic definitions and results about infinite series.

- Definition: A series is the sum of all the terms of sequence $\{a_n\}_{n=1}^{\infty}$.
  Notation: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots$. (n≥0) Here again, note that a series does not always have to start at $n = 1$.

- A series is an infinite sum. The question we will ask ourselves is: When is an infinite sum finite, when is it infinite? In other words, we will explore the convergence or divergence of an infinite series. This can be summarized by finding $\lim_{n\to\infty} \sum_{i=1}^{n} a_i$.

- Examples: Consider the following series.

  1. $\sum_{n=1}^{\infty} n = 1 + 2 + \ldots$ is divergent . Why?
  2. $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \ldots = 1$. This series is convergent.

- In order to show the convergence (or divergence) of a series, we will be using ideas developped in the previous section. We can view a series as being a sequence of partial sums. A partial sum is simply the sum of a finite number of terms of a series.

  $S_1 = a_1$
  $S_2 = a_1 + a_2$
  $S_3 = a_1 + a_2 + a_3$

  $S_n = a_1 + a_2 + a_3 + \ldots + a_n$

  The convergence of $\sum_{n=1}^{\infty} a_n$ is the same as the convergenc e of the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$.

- Definition: A series is convergent if

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots = s$$

where s is a finite number. Otherwise, the series is said to be divergent.

An application of partial sums: Geometric series.

- A **geometric series** is a series of the form

example:

$$\sum_{n=0}^{\infty} 4\, 3^{n+1} 5^{-n} = \sum_{n=0}^{\infty} 4(3)\frac{3^n}{5^n} = \sum_{n=0}^{\infty} 12\left(\frac{3}{5}\right)^n \quad \boxed{\sum_{n=1}^{\infty} ar^{n-1}} \quad \left(\sum_{n\geq 0}^{\infty} \text{constant}\,(\underbrace{r}_{\text{constant}})^{\text{power of } n}\right)$$

where a and r are constants (r is called the common ratio). When is $\sum_{n=1}^{\infty} ar^{n-1}$ convergent? Let us look at the n–th partial sum.

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}.$$

Then multiply S_n by r. $rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$.
Now subtract: $\underbrace{S_n - rS_n}_{S_n(1-r)} = a - ar^n = a(1 - r^n)$. So

$$\boxed{S_n = \frac{a(1 - r^n)}{1 - r}}.$$

What is $\lim_{n\to\infty} S_n$? From Section ~~12~~ 8.1, we know that

$$\lim_{n\to\infty} r^n = 0 \text{ if } -1 < r < 1$$

and $\{r^n\}$ is divergent if $|r| > 1$. Here

First term of the series.

$$\boxed{\lim_{n\to\infty} S_n = \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{divergent} & \text{if } |r| > 1 \end{cases}}$$

Hence $\sum_{n=1}^{\infty} ar^{n-1} = \dfrac{a}{1-r}$ if $|r| < 1$, and $\sum_{n=1}^{\infty} ar^{n-1}$ is divergent if $|r| > 1$.

- **Important remark.** a is the first term of the series.
- Examples: Test the convergence of each series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}}{5^n} = \sum_{n=1}^{\infty} 4\left(\frac{4}{5}\right)^n \quad r = \frac{4}{5}$

$|r|<1$, $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}}{5^n}$ is convergent

$$\sum_{n=1}^{\infty} \frac{4^{n+1}}{5^n} = \frac{16/5}{1-\frac{4}{5}} = 16$$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}} = \sum_{n=1}^{\infty} \left(-\frac{6}{5}\right)^{n-1}$

$r = -\frac{6}{5}$, $|r|>1$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}}$

is divergent

③ $\displaystyle\sum_{n=0}^{\infty} 2\,\pi^n\, 4^{-n+1}$

$\displaystyle = \sum_{n=0}^{\infty} 2\left(\frac{\pi}{4}\right)^n 4 = \sum_{n=0}^{\infty} 8\left(\frac{\pi}{4}\right)^n$

$r = \frac{\pi}{4}$, $|r|<1$, $\displaystyle\sum_{n=0}^{\infty} 8\left(\frac{\pi}{4}\right)^n$ is convergent

$$\sum_{n=0}^{\infty} 8\left(\frac{\pi}{4}\right)^n = \frac{8}{1-\frac{\pi}{4}} = \frac{32}{4-\pi}$$

④ $\displaystyle\sum_{n=1}^{\infty} 2\left(\frac{n}{5}\right)^n$

Is not a geometric series

3. For what values of x does the series $\displaystyle\sum_{n=0}^{\infty}(x-4)^n$ converge?

Power series

$$\sum_{n=0}^{\infty}(x-4)^n = 1 + (x-4) + (x-4)^2 + (x-4)^3 + \cdots$$

"infinite polynomial"

$\displaystyle\sum_{n=0}^{\infty}(x-4)^n$ is a geometric series with $r = x-4$

$\displaystyle\sum_{n=0}^{\infty}(x-4)^n$ is convergent when $|x-4|<1 \Rightarrow -1<x-4<1$

$3<x<5$

If $3<x<5$, $\displaystyle\sum_{n=0}^{\infty}(x-4)^n = \frac{1}{1-(x-4)} = \frac{1}{5-x}$

Rational function

If $-1<x<1$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

To evaluate $\int \frac{1}{x(x+1)}\,dx$ use partial fraction

Here is another example of partial sums: The telescoping sum.

- Example: Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

Sequence of partial sums:

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{6}$

$S_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12}$

$\vdots$

$\rightarrow S_n = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{n(n+1)}$

Partial fractions

$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \Rightarrow \quad 1 = A(n+1) + Bn$

$n=0, \quad A = 1$

$n=-1, \quad B=-1$

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$S_1 = 1 - \frac{1}{2}$

$S_2 = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)$

$S_3 = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$

$\vdots$

$S_n = \left(1-\cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\frac{1}{n-1} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$

$S_n = 1 - \frac{1}{n+1}$ $\qquad \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$

Conclusion: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is DV

Is $\int_1^{\infty} \frac{1}{x}\,dx$ convergent, divergent?

- The harmonic series.

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent.

Proof: Let us look at the S_{2^n} partial sums of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

$n = 0,\ S_1 = 1$

$n = 1,\ S_2 = 1 + \dfrac{1}{2}$

$n = 2,\ S_4 = 1 + \dfrac{1}{2} + \overset{>\frac{1}{4}}{\dfrac{1}{3}} + \dfrac{1}{4} > 1 + \dfrac{1}{2} + \overbrace{\left(\dfrac{1}{4} + \dfrac{1}{4}\right)}^{\frac{1}{2}} = 1 + \dfrac{2}{2}$

$n = 3,\ S_8 = 1 + \dfrac{1}{2} + \left(\dfrac{1}{3} + \dfrac{1}{4}\right) + \left(\dfrac{1}{5} + \dfrac{1}{6} + \dfrac{1}{7} + \dfrac{1}{8}\right) > 1 + \dfrac{1}{2} + \dfrac{1}{2} + \dfrac{1}{2} = 1 + \dfrac{3}{2}$

$S_{2^n} = 1 + \dfrac{1}{2} + \ldots + \dfrac{1}{n} > 1 + \dfrac{n}{2}$.

Then $\displaystyle\lim_{n\to\infty} 1 + \frac{n}{2} = \infty$ and $S_{2^n} > 1 + \dfrac{n}{2}$. Therefore $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent.}}$

Infinite series: $\sum_{n\geq 0}^{\infty} a_n$. $\lim_{n\to\infty} \sum_{n\geq 0}^{\infty} a_n = ?$

- Sequence of partial sums: S_n

$S_1 = a_1$, $S_2 = a_1 + a_2$, ... $S_n = a_1 + a_2 + \cdots a_n$

If $\lim_{n\to\infty} S_n = L$ $\sum_{n\geq 0}^{\infty} a_n$ is convergent

- Geometric series $\sum_{n\geq 0}^{\infty} a\,(r)^{\text{power of } n}$

a ↑ constant, r ↑ common ratio)

A geometric series is convergent

when $|r| < 1$ $(-1 < r < 1)$

If $|r| < 1$, sum of the geometric series $= \dfrac{\text{First term}}{1-r}$

- THEOREM: If $\sum_{n\geq 0}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.
This theorem is not very useful in this form. We usually use its contrapositive form to show the divergence of $\sum_{n=1}^{\infty} a_n$.

- TEST for DIVERGENCE. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

contrapositive of

- THEOREM. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, then so are $\sum_{n=1}^{\infty}(a_n + b_n)$ and $\sum_{n=1}^{\infty}(a_n - b_n)$. We also have the following:

1. $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$, where c is a constant
2. $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
3. $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Practice problem: Determine whether the series is convergent or divergent. If it is convergent, find its sum.

① $\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = \sum_{n=1}^{\infty} 3\left(\frac{e}{3}\right)^n$ [Geometric series with $r = \frac{e}{3}$

$|r| < 1$, $\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$ is convergent, $\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = \frac{e}{1-\frac{e}{3}} = \frac{3e}{3-e}$]

② $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$

Divergence test $\lim_{n\to\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right) = \ln\frac{1}{2} \neq 0$ $\quad \frac{1}{2}$

$\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$ is divergent

③ $\sum_{n=1}^{\infty} \frac{4}{n}$ is divergent $\sum_{n=1}^{\infty} \frac{4}{n} = 4\sum_{n=1}^{\infty} \frac{1}{n}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, therefore

$\sum_{n=1}^{\infty} \frac{4}{n}$ is divergent

$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$? $\quad \lim_{n\to\infty} \frac{\cos n\pi}{n^2} = 0$

$\cos(n\pi) = (-1)^n$

The divergence test is inconclusive.

if n is even $\frac{\cos n\pi}{n^2} = \frac{1}{n^2}$

If n is odd $\frac{\cos n\pi}{n^2} = -\frac{1}{n^2}$

$\lim_{n\to\infty} -\frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n^2} = 0$