Chapter 5 INVERSE FUNCTIONS

Section 5.1. Inverse Functions

example of inverse functions: Solve $x^2 = 5$ for $x \geq 0$

$x = \sqrt{x^2} = \sqrt{5}$

Idea of an inverse function

$A \xrightarrow{f} B$

$x \longrightarrow y = f(x)$

$? \longleftarrow y$

f^{-1} inverse function

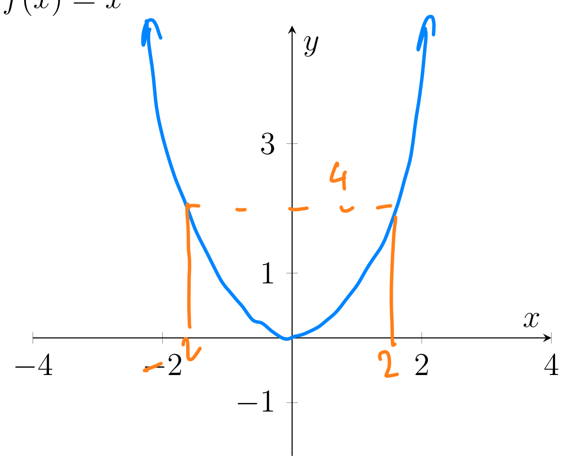
In this section, we study the properties of inverse functions.

- **One–to–one function:** A function f is one–to–one if

$$\text{if } x_1 \neq x_2, \text{ then } f(x_1) \neq f(x_2)$$

How does the graph of a one–to–one function look like? Let's graph the following function and decide whether they are one–to–one or not.

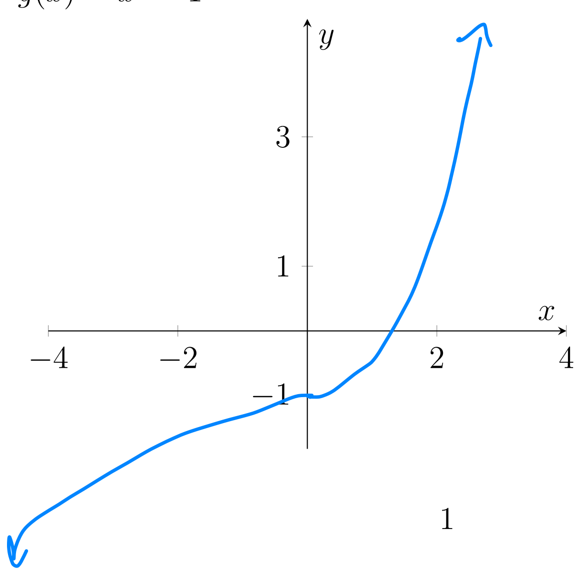
1. $f(x) = x^2$



not one-to-one

$(-2)^2 = 2^2 = 4$

2. $g(x) = x^3 - 1$



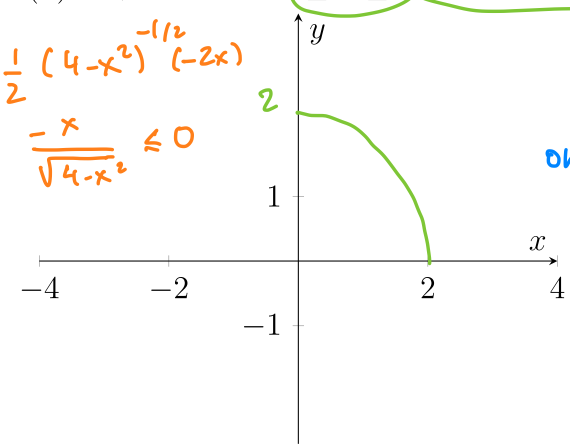
one-to-one

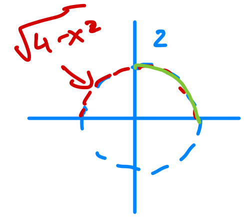
$y=\sqrt{4-x^2} \Rightarrow x^2+y^2=4$

3. $h(x)=\sqrt{4-x^2}$ for $0\le x\le 2$.

$h'(x)=\frac{1}{2}(4-x^2)^{-1/2}(-2x)$

$h'(x)=\frac{-x}{\sqrt{4-x^2}}\le 0$





one-to-one

There are different ways to show that a function is one–to–one.

- From the graph of the function: A function is one–to–one if and only if no horizontal line intersects the graph more than once.
- Notice that a one–to–one function is either always decreasing or always increasing on its domain, which means f' does not change sign on its domain, then f is one–to–one.

- **Inverse Function:** Let f be a one–to–one function with domain A and range B. Then f has an inverse function f^{-1} such that

$$\boxed{y=f^{-1}(x) \text{ is equivalent to } f(y)=x}$$

as for example, $f^{-1}(3)=5$ is equivalent to $f(5)=3$

Properties of Inverse Functions.

Domain of f^{-1} = Range of f.

Range of f^{-1} = Domain of f.

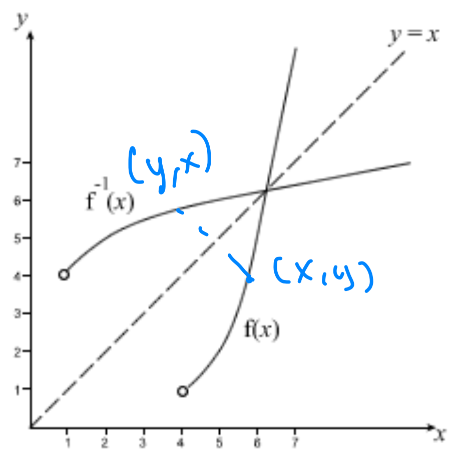
$f^{-1}(f(x))=x$ for every x in A.

$f(f^{-1}(x))=x$ for every x in B.

} cancellation equations.

The graphs of f and f^{-1} are symmetric with respect to the line $y=x$.

$\underset{x}{A} \xrightarrow{f} \underset{y=f(x)}{B} \xrightarrow{f^{-1}} \underset{f^{-1}(f(x))=x}{A}$, $x\geq 0$ $\sqrt{x^2}=x$



Examples:

1. If $f(2) = 3$, then $f^{-1}(3) =$ 2

2. If $f^{-1}(5) = 0$, then $f(0) =$ 5

3. Let y $f(x)$ $= x^2 + 5$ for $x \leq 0$

a) State the range of y : $[5, \infty)$

b) Is y one–to–one ? Why or why not?

$f'(x) = 2x$, $f'(x) \leq 0$ when $x \leq 0$.

f is one-to-one

c) Write an expression for f^{-1} the inverse function of y.

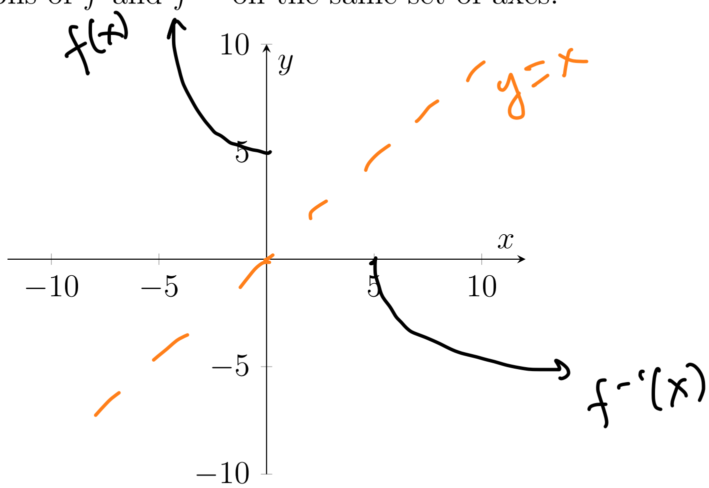
Step1: solve $y = x^2 + 5$ for x.

$x^2 = y - 5 \Rightarrow x = \pm\sqrt{y-5}$

Since $x \leq 0$ $x = -\sqrt{y-5}$

Step2: interchange x & y $f^{-1}(x) = -\sqrt{x-5}$

d) Sketch the graphs of f and f^{-1} on the same set of axes.



- **Calculus of Inverse Functions.**

1. **Theorem:** If f is a one–to–one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

2. Derivative of an inverse function at a point a. Let's look at the following example:
$(f^{-1}(a))' = ?$
Let $f(x) = x^5 - x^3 + 2x$. Find $(f^{-1})'(2)$.
How do we solve $y = x^5 - x^3 + 2x$ for x to find $f^{-1}(x)$? The following theorem will help.

3. **Theorem:** If f is a one–to–one function with inverse function f^{-1}, and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1}(a))' = \frac{1}{f'(f^{-1}(a))}$$

A simple proof will be given using Implicit differentiation.

Definition of an inverse function. If f is one-to-one

$y = f^{-1}(x)$ is equivalent to $x = f(y)$

$$\frac{d}{dx} y = \frac{d}{dx} f^{-1}(x) \longleftrightarrow \frac{d}{dx} x = \frac{d}{dx} f(y)$$

$$1 = f'(y) \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

Example

$$f(x) = x^5 - x^3 + 2x$$

$$(f^{-1}(2))' = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{4}$$

$f^{-1}(2) = b$ or $f(b) = 2$

$b^5 - b^3 + 2b = 2$

Solve by inspection

$b = 1 = f^{-1}(2)$

$f'(x) = 5x^4 - 3x^2 + 2$

$f'(1) = 4$

$$(f^{-1}(a))' = \frac{1}{f'(f^{-1}(a))}$$

Examples:

1. Let $f(x) = 5x + \cos 2x$. Find $(f^{-1}(1))'$.

$$(f^{-1}(1))' = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{5}$$

$f^{-1}(1) = b$ or $f(b) = 1 \Rightarrow 5b + \cos 2b = 1$

$b = 0$

$f^{-1}(1) = 0$

$f'(x) = 5 - 2\sin 2x$

$f'(0) = 5$

2. Let $f(x) = \sqrt{x-4}$. Find $(f^{-1}(3))'$ in 2 different ways.

(a) By first finding $f^{-1}(x)$ explicitly.

$y = \sqrt{x-4}$. Solve for x $\quad y^2 = x - 4$

$x = y^2 + 4$

interchange x & y : $f^{-1}(x) = x^2 + 4$

$(f^{-1}(x))' = 2x$

$\boxed{(f^{-1}(3))' = 6}$

(b) By using the theorem. $(f^{-1}(3))' = \dfrac{1}{f'(f^{-1}(3))}$

$f^{-1}(3) = b \Rightarrow f(b) = 3$

$\sqrt{b-4} = 3 \Rightarrow b = 13$

$(f^{-1}(3))' = \dfrac{1}{f'(13)}$

$f'(x) = \dfrac{1}{2\sqrt{x-4}} \Rightarrow f'(13) = \dfrac{1}{6}$

$(f^{-1}(3))' = \dfrac{1}{1/6} = 6$