Recap:

- Is $\sum_{n\geq 0}^{\infty} a_n$ convergent or divergent?

  * Geometric series: $\sum_{n\geq 0}^{\infty} a(r)^{\text{power of } n}$

    - If $|r|<1$, $\sum_{n\geq 0}^{\infty} a(r)^{\text{power of } n} = \dfrac{\text{First term}}{1-r}$

    - If $|r|\geq 1$, divergent

  * $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

  * Telescoping & sequence of partial sums.

- Divergence test:

  If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n\geq 0}^{\infty} a_n$ is divergent

  If $\lim_{n\to\infty} a_n = 0$ then THERE IS WORK TO DO

If $a>0$ $\int_a^{\infty} \frac{1}{x^p}\,dx$ is $\to$ CV when $p>1$, $\to$ DV when $p\leq 1$

question: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\to$ is CV if $p>1$, $\to$ is DV if $p\leq 1$

p-series.

CHAPTER 8. SEQUENCES & INFINITE SERIES
Section 8.3. The Integral and Comparison Tests.

- **Theorem.** Suppose that f is a continuous, positive function and decreasing on $[1, \infty)$. Let $a_n = f(n)$. Then the series

$$\sum_{n=1}^{\infty} a_n \text{ is convergent if and only if } \int_1^{\infty} f(x)\, dx \text{ is convergent.}$$

This means:

1. If $\displaystyle\int_1^{\infty} f(x)\ dx$ is convergent then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.
2. If $\displaystyle\int_1^{\infty} f(x)\ dx$ is divergent then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

- **Consequence.**

Recall from section 6.6 that $\displaystyle\int_1^{\infty} \frac{1}{x^p}\ dx$ is convergent if $p > 1$ and divergent if $p \leq 1$. Therefore, by the Integral Test, we can conclude that

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is convergent if } p > 1 \text{ and divergent if } p \leq 1} \quad *$$

Series of the form $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ are called p–series.

- **Exercise:** Determine whether the series is convergent or divergent.

$\sum_{n=1}^{\infty} \frac{n}{e^n}$

① Divergence test: $\lim_{n\to\infty} \frac{n}{e^n} = 0$ ☹

② Integral test: let $f(x) = \frac{x}{e^x} = xe^{-x}$ for $x \geq 1$

f is continuous, $f(x) > 0$ on $[1, \infty)$

$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) \leq 0$ for $x \geq 1$

f is decreasing on $[1, \infty)$

$$\int_1^{\infty} xe^{-x}\,dx = \lim_{t\to\infty} \int_1^t xe^{-x}\,dx$$

$$\int_1^t xe^{-x}\,dx = -xe^{-x}\Big|_1^t + \underbrace{\int_1^t e^{-x}\,dx}_{-e^{-x}\big|_1^t} = -te^{-t} + e^{-1} - e^{-t} + e^{-1}$$

$$= \underbrace{2e^{-1} - te^{-t} - e^{-t}}$$

IBP $u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

$$\lim_{t\to\infty} (2e^{-1} - \underbrace{te^{-t}}_{\to 0} - \underbrace{e^{-t}}_{\to 0}) = 2e^{-1}$$

Conclusion: $\int_1^{\infty} xe^{-x}\,dx$ is convergent

$\sum_{n=1}^{\infty} ne^{-n}$ is convergent

very complicated

"Test" series:
- p series
- geometric series

?

- **The Comparison Test.** Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with *positive* terms.

  1. (a) If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

     (b) If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

- Exercises: Determine whether the series diverges or converges.

  1. $\sum_{n=1}^{\infty} \frac{2}{n^3+4}$ $= a_n$ ("guess": As $n \to \infty$ $\frac{2}{n^3+4} \sim \frac{2}{n^3}$. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent)

     For $n \geq 1$ $\underbrace{\frac{2}{n^3+4}}_{a_n} \leq \underbrace{\frac{2}{n^3}}_{b_n}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent, by the Comparison test $\sum_{n=1}^{\infty} \frac{2}{n^3+4}$ is convergent.

  2. $\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$ a_n ("Guess": as $n \to \infty$ $\frac{1}{n-\sqrt{n}} \sim \frac{1}{n}$, $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent)

For $n \geq 2$, $n - \sqrt{n} \leq n \Rightarrow \underbrace{\frac{1}{n-\sqrt{n}}}_{a_n} \geq \underbrace{\frac{1}{n}}_{b_n}$. Since $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent, by the comparison test, $\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$ is divergent.

3. $\sum_{n=1}^{\infty} \frac{1}{n!}$ a_n $b_n = ?$

$$n! = 1.2.3.4 \cdots n \geq 1.\underbrace{2.2.2 \cdots 2}_{2^{n-1}}$$

For $n \geq 1$ $n! \geq 2^{n-1} \Rightarrow \underset{a_n}{\frac{1}{n!}} \leq \underset{b_n}{\frac{1}{2^{n-1}}}$. Since $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$ $r = \frac{1}{2}$

is convergent, $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent.

4. $\sum_{n=1}^{\infty} \frac{1}{n+1}$ a_n ("Guess" As $n \to \infty$ $\frac{1}{n+1} \sim \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent)

For $n \geq 1$ $\underset{a_n}{\frac{1}{n+1}} \cancel{\leq} \underset{b_n}{\frac{1}{n}}$. The Comparison test is inconclusive.

? test series.

- **The Limit Comparison Test.** Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with *positive* terms. If

  1. (a) $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then either both series diverge or both series converge.

     (b) $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

     (c) $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

  We can now do Exercise (3).

- Exercises. Determine whether the series diverges or cenverges .

  1. $\sum_{n=3}^{\infty} \underbrace{\frac{1}{n^2-4}}_{a_n}$ (Guess: as $n \to \infty$ $\frac{1}{n^2-4} \sim \frac{1}{n^2}$, $\sum_{n=3}^{\infty} \frac{1}{n^2}$ is convergent)

     Limit comparison test with $b_n = \frac{1}{n^2}$

     $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2-4}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-4} = 1 > 0$$

     $\sum_{n=3}^{\infty} \frac{1}{n^2-4}$ is convergent.

  2. $\sum_{n=1}^{\infty} \underbrace{\frac{2n^2+7n}{3^n(n^2+5n-1)}}_{b_n}$ (Guess: As $n \to \infty$ $\frac{2n^2+7n \to 2}{3^n(n^2+5n-1)} \sim \frac{2}{3^n} = 2\left(\frac{1}{3}\right)^n$

     $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$, convergent geometric series($|r| = \frac{1}{3} < 1$)) $b_n = \left(\frac{1}{3}\right)^n = \frac{1}{3^n}$

     $$\lim_{n\to\infty} \frac{\frac{2n^2+7n}{3^n(n^2+5n-1)}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{\cancel{3^n}(2n^2+7n)}{\cancel{3^n}(n^2+5n-1)} = 2 > 0$$

     $\sum_{n=1}^{\infty} \frac{2n^2+7n}{3^n(n^2+5n-1)}$ is convergent.

Additional Examples: Test the series for convergence or divergence. Show all steps that lead to your answer.

1. $\sum_{n=0}^{\infty} \overbrace{\frac{1+\sin n}{10^n}}^{a_n}$ (Guess: $n\to\infty$ $\frac{1+\sin n}{10^n} \approx \frac{\text{number}}{10^n}$, $\sum_{n=0}^{\infty} \frac{1}{10^n} = \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n$ convergent)

Comparison test:

For $n \geq 0$ $\quad \frac{1+\sin n}{10^n} \leq \frac{2}{10^n} = 2\left(\frac{1}{10}\right)^n$. Since $\sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n$ is convergent

$\sum_{n=0}^{\infty} \frac{1+\sin n}{10^n}$ is convergent

2. $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ Divergence test: $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ no conclusion

For $n \geq 3$ $\quad \underbrace{\frac{\ln n}{n}}_{a_n} \geq \underbrace{\frac{1}{n}}_{b_n}$. $\sum_{n=3}^{\infty} \frac{1}{n}$ is divergent. Therefore by the comparison test $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ is divergent.

Limit comparison test: $a_n = \frac{\ln n}{n}$, $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{\frac{\ln n}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \ln n = \infty$. Since $\sum_{n=3}^{\infty} \frac{1}{n}$ is divergent,

$\sum_{n=3}^{\infty} \frac{\ln n}{n}$ is divergent