The determinant is defined for any square matrix. An nxn determinant 3628000 terms which involve multiplication and addition.

Permutations

{1,2,3} 1 2 3 1 3 2 2 3 1 2 1 3 3 1 2 3 2 1

A permutation is just an arrangement of the integers 1,...,n in some order. There are six permutations of 1,2,3.

{1,2} 1 2 2 1

{1,2,3,4} 24 permutations

1 2 3 4 1 3 2 4 1 4 2 3 1 3 42 1 4 3 2 1 2 4 3

2 1 3 4 2 3 1 4 ....

In general there are n! permutations of {1,2,..,n}. There are n choices for the first integer, n-1 for the second, n-2 for the third, etc. N! grows very rapidly. It grows faster than exponential speed.

Every permutation can be transformed into the identity permutation on {1,...,n}.

$1\,3\,5\,4\,2 \longrightarrow 1\,3\,4\,5\,2 \longrightarrow 1\,3\,4\,2\,5 \longrightarrow 1\,3\,2\,4\,5 \longrightarrow 1\,2\,3\,4\,5$

4 steps

Either the number of transpositions required to transform the permutation into the identity is an even integer, or it is an odd integer. 1 3 5 4 2 is an even permutation .

$2\,3\,4\,5\,1 \longrightarrow 1\,3\,4\,5\,2 \longrightarrow 1\,2\,4\,5\,3 \longrightarrow 1\,2\,3\,5\,4 \longrightarrow 1\,2\,3\,4\,5$

4 steps even permutation

transpositions- interchanges of 2 elements

If π is a permutation the sign of π (sgn(π)) is +1 if π is even and -1 if π is odd.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Here is how we compute the determinant. I want pick one element from each row and each column and multiply them together. I want to do this in every possible way and then certain expressions that result are added.

1 2 3 1 3 2 2 1 3 2 3 1 3 1 2 3 2 1

$$\det(A) = aei - afh - bdi + bfg + cdh - iec$$

In general, for an n x n matrix for every permutation of {1,2,...,n} we have a choice of one element from each row and each column of the matrix corresponding to that permutation. We multiply the corresponding elements of the matrix together, multiply by the sign of the permutation, and then add. That is the determinant of A.

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a d - bc. \quad .$$

For a 10 x 10 matrix we would have more than 3 million terms. (Without the + and - signs what you get is called the permanent of the matrix.) It turns out that the determinant can be computed another way, so that computing a determinant is feasible. How is it possible to compute a determinant? What is the relevance of the determinant?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \det = a d - b c.$$

$$\begin{pmatrix} c & d \\ a & b \end{pmatrix} \quad \det = c b - a d \quad \text{the negative}$$

THEOREM If B is obtained from A by interchanging two rows, then det(B) =- det (A).

THEOREM If you multiply a row of a matrix by a number a the determinant gets multiplied by a.

THEOREM If you add a multiple of one row to another you do not change the determinant.

If you go through a sequence of elementary row operations on a square matrix you can keep track easily of the effect on the determinant. If a matrix is triangular then the determinant is just the product of the diagonal entries.

$$\det \begin{pmatrix} 1 & 3 & 4 & 5 \\ 0 & 3 & 5 & 3 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 13 \end{pmatrix} = 1 \times 3 \times 4 \times 13 = 156 .$$

The idea in evaluating the determinant is to transform the matrix by elementary row operations into a triangular matrix- keep track of the row interchanges and the row multiplications.

THEOREM If A is a square matrix then A is invertible if and only if its determinant is NOT 0.

If A can be transformed to B by elementary row operations, then A is invertible if and only if B is invertible.

THEOREM If A and B are two n x n matrices then A can be transformed to B by elementary row operations if A and B are both invertible. Any invertible n x n matrix can be transformed to any other invertible n x n matrix by elementary row operations. If B is the identity matrix then we can read off the solution to the system of simultaneous equations.

If A and B are not invertible you may or may not be able to transform one to the other by elementary row operations.

We have been talking about row operations. You could also do column operations. We concentrated on row operations because they have obvious effects on the simultaneous equations. If you did column operations you would be changing the original system.

In older linear algebra books instead of writing A X = B the book may have written X A = B, where X is now a row vector and the coefficient matrix A has columns as the coefficients.

When interchange two rows or multiply a row by a number you don't change the span of the rows. You also don't change the span of the rows when you add a multiple of one row to another.

Is the vector (1,2,4) a linear combination of these 3 vectors?

(1,2,4) = x(1,2,3)+y (4,5,6) + z(7,8,9) =(x+4y+7z,2z+5y+8z,3x+6y+9z) = (1,2,4)??

$x + 4 y + 7 z = 1$

$2 x+ 5 y+8 z = 2$

$3 x+6 y+9 z =4$

A X = B

The span of (1,2,3),(4,5,6),(7,8,9) is the set of vectors which are linear combinations of these three. Is the span all of 3 space? In this case the answer is no- the span is not all of 3-space. It is 2-dimensional. The three vectors are not linearly independent. The determinant of the corresponding matrix is 0.

If A is an n x n matrix the row span (or column span) of A is all of R^n if and only if the determinant of the matrix is not 0, i.e., if and only if the matrix A is invertible.

dimension of a vector space

Given a 4 x 4 matrix evaluate its determinant.