CHAPTER 8. SEQUENCES AND SERIES.
SECTION 8.6. Power Series Representation.

Recall that $\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$ for $|x| < 1$. (Interval of convergence : $(-1,1)$)

Question: Use the above series to express $f(x) = \frac{2}{1+x}$ as a power series.

$$\frac{2}{1+x} = 2\,\frac{1}{1+x} = 2\,\frac{1}{1-(-x)} = 2\sum_{n=0}^{\infty}(-x)^n = 2\sum_{n=0}^{\infty}(-1)^n x^n$$

sum of a geometric with $r=-x$

$$\frac{2}{1+x} = 2\left(1 - x + x^2 - x^3 + \dots\right)$$

Interval of convergence $|-x| = |x| < 1$

$-1 < x < 1$

- **Differentiation and integration of a power series.**

  If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$\boxed{f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots = \sum_{n=0}^{\infty} c_n(x-a)^n}$$

  is differentiable on the interval $(a-R, a+R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\int f(x)\ dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + \dots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x|<1$$

Examples: Find a power series representation for the function and determine the radius of convergence.

1. $f(x) = \dfrac{x}{2-x}$ $= x \dfrac{1}{2-x} = x \dfrac{1}{2(1-\frac{x}{2})} = \dfrac{x}{2} \cdot \dfrac{1}{1-\frac{x}{2}}$

$$= \frac{x}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \frac{x}{2} \sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{2^{n+1}} = \frac{x}{2-x}$$

Interval of convergence:

$r = \frac{x}{2}$, $\left|\frac{x}{2}\right| < 1$

$|x| < 2$ $\quad \boxed{(-2, 2)}$

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{x^n}{2^n}$$

2. $f(x) = \dfrac{x^2}{x-5}$ $= x^2 \dfrac{1}{x-5} = -x^2 \dfrac{1}{5-x} = -x^2 \dfrac{1}{5(1-\frac{x}{5})}$

$$= -\frac{x^2}{5} \cdot \frac{1}{1-\frac{x}{5}} = -\frac{x^2}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n$$

$$= -\frac{x^2}{5} \sum_{n=0}^{\infty} \frac{x^n}{5^n}$$

$$\frac{x^2}{x-5} = \sum_{n=0}^{\infty} -\frac{x^{n+2}}{5^{n+1}}$$

Interval of convergenc : $r = \frac{x}{5}$

$\left|\frac{x}{5}\right| < 1 \Rightarrow -5 < x < 5$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x|<1$$

3. $f(x) = \dfrac{1}{(1-x)^2}$.

$$y = \frac{1}{1-x} \qquad \frac{dy}{dx} = \frac{1}{(1-x)^2}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\frac{d}{dx}\left(1+x+x^2+x^3+\cdots\right) = 1+2x+3x^2+\cdots = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\boxed{\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}} = \sum_{n=0}^{\infty} (n+1) x^n$$

Interval of convergence $(-1,1)$

4. $f(x) = \dfrac{x^2}{(1+x)^2} = x^2 \cdot \dfrac{1}{(1+x)^2}$

$$\text{Let } y = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{-1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$$

$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} n x^{n-1}$$

$$f(x) = x^2 \sum_{n=1}^{\infty} (-1)^{n-1} n x^{n-1}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} n x^{n+1}} \quad \text{or} \quad \sum_{n=0}^{\infty} (-1)^n (n+1) x^{n+2}$$

Interval of convergence: $(-1,1)$

$$\rightarrow \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad , \; |x|<1$$

Given $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$

$$f'(x) = \sum_{n=1}^{\infty} c_n \, n \, (x-a)^{n-1}$$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

Example: Find the power series representation for $f(x) = \frac{x}{1+2x^2}$. State the radius and interval of convergence

$$f(x) = x \, \frac{1}{1+2x^2} = x \cdot \frac{1}{1-(-2x^2)} = x \sum_{n=0}^{\infty} (-2x^2)^n = x \sum_{n=0}^{\infty} (-1)^n 2^n x^{2n}$$

$$= \sum_{n=0}^{\infty} (-1)^n 2^n x^{2n+1}$$

$r = -2x^2$

$$|-2x^2| = 2x^2 < 1 \implies -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \qquad R = \frac{1}{\sqrt{2}}$$

$$I = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

If $|x|<1$, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x_n$

$$\frac{d}{dx}\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=1}^{\infty} c_n\, n (x-a)^{n-1}, \quad \int \sum_{n=0}^{\infty} c_n (x-a)^n dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

5. $f(x) = \ln(1+x)$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx \qquad ① \ \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$② \ \int \frac{1}{1+x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^n\, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$\ln(1+x) = \overset{?}{C} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}, \quad -1<x<1$$

$$x=0 \quad \ln 1 = 0 = C + \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{0^{n+1}}{n+1}}_{0} \Rightarrow C = 0, \quad \ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$\left(\text{If } x = -\tfrac{1}{2}, \ \ln(\tfrac{1}{2}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(-\tfrac{1}{2}\right)^{n+1}\right)$$

6. $\tan^{-1} x$

$$\tan^{-1}x = \int \frac{1}{1+x^2}\,dx$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x|<1$$

$$\int \frac{1}{1+x^2}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

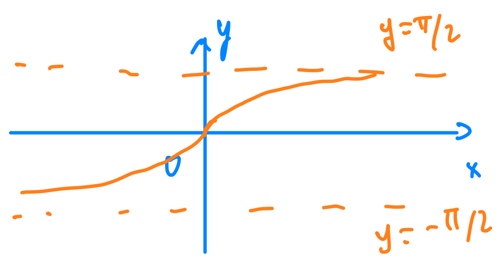
$$\tan^{-1}x = \overset{?}{C} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad -1<x<1$$

If $x=0$

$$0 = C + 0 \Rightarrow C = 0$$

$$\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

odd exponents.



$y = \tan^{-1}x$ is an odd function

$f(-x) = -f(x)$

$$\tan^{-1}x = \sum_{n=0}^{\infty} \overbrace{\frac{(-1)^n}{2n+1}}^{c_n} x^{2n+1}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} 1\, x^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)}{n+1} x^{n+1}$$

Power series representation for $f(x) = e^x$

$g(x) = \sin x$??

?

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

what are c_n's?

All of the above examples use the fact that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$, and derivatives/integrals of this function.

Now, let us say that we would like to find the power series expansion of $f(x) = e^x$. Well, this time, I will not be able to use the above series. It may be a good idea to see if we can obtain a general form for the coefficients c_n's in $\sum_{n=0}^{\infty} c_n(x-a)^n$.