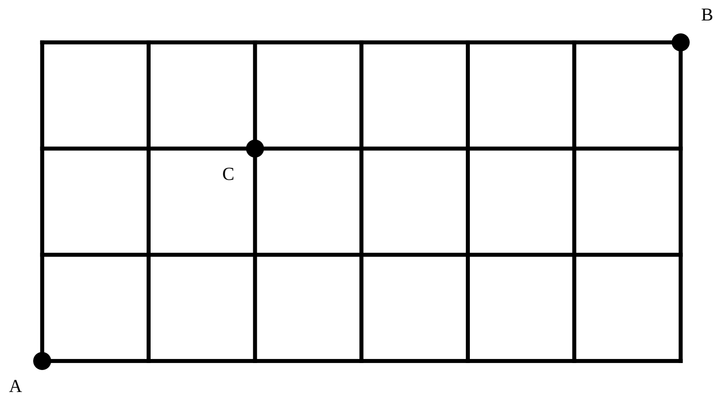
Solution of Exam 1 MTH 224-O, Spring 2024

Question 1. (a) (2 points) You are given the grid below. You can only move along the lines. At each step, you can only move up or move to the right.



How many paths are there that start at A and end at B?

Solution: There are a total of 9 steps; 6 steps to the right, and 3 steps up. We have $\binom{9}{3} = 84$ possibilities to choose when we take a step up in a path, which determines the entire path.

(b) (3 points) In part (a), how many paths pass through C? How many paths avoid C?

Solution: We compute the number of possibilities to get from A to C: $\binom{4}{2} = 6$ (same argument as in (a)), and the number of possibilities to get from C to B: $\binom{5}{1} = 5$ (again, same argument). In total, there are $\binom{4}{2}\binom{5}{1} = 30$ such paths.

To determine how many paths are there in which C is avoided, we simply subtract the number of paths going through C from the number of paths from A to B: $\binom{9}{3} - \binom{4}{2}\binom{5}{1} = 84 - 30 = 54$.

(c) (2 points) Considered an ordered list $(x_1, x_2, ..., x_{10})$ of ten binary digits $x_i \in \{0, 1\}$. In how many of such lists is the sum $x_1 + x_2 + \cdots + x_{100}$ an odd number? That is, the sume is either 1, or 3, or 5,...

Solution: Given the sum of the first $n-1$ numbers in the vectors, adding the n^{th} number, which can be either 0 or 1, results in one even sum, and one odd sum. Since for $n = 1$, there is one even sum $(x_1 = 0)$ and one odd sum $(x_1 = 1)$, we see that for any n, half of the possible sums are even, and half of them are odd.

Therefore, it remains to compute the total number of vectors of length $n = 100$. Since each number x_i can be one of two possibilities (0 or 1), we have 2^{100} possible sums. Thus, there are $2^{100}/2 = 2^{99}$ such lists with odd sums.

Question 2. (a) (2 points) There are two identical cabinets, each with three drawers. One cabinet contains a silver coin in one drawer and a gold coin in each of the remaining drawers (so, one silver coin and 2 gold coins). The other cabinet contains a gold coin in one drawer and a silver coin in each of the remaining drawers (so, 2 silver coins and one gold coin). We randomly select a cabinet, and then open one drawer at random.

If a silver coin is found, what is the probability that both other drawers of the chosen cabinet contain gold coins?

Solution: Let the first cabinet be the one with 2 gold coins and the second cabinet be the one with 2 silver coins. Define $C_i = \{\text{cabinet i was chosen}\}$, $S = \{\text{Silver coin is found}\}$. Then:
$\mathbb{P}(C_1|S) = \frac{\mathbb{P}(S|C_1)\mathbb{P}(C_1)}{\mathbb{P}(S)} = \frac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{1}{3}\cdot\frac{1}{2}+\frac{2}{3}\cdot\frac{1}{2}} = \frac{1}{3}$.

Alternative solution: There are 3 silver coins and only one of them is in the first cabinet. Since silver coins are chosen with equal probability, there is a $\frac{1}{3}$ chance that the chosen silver coin is in the first cabinet. So, $\mathbb{P}(C_1|S) = \frac{1}{3}$.

(b) (3 points) In part (a), if a silver coin is found, what is the probability that a drawer randomly chosen among the two remaining drawers in the same cabinet contains a gold coin?

Solution: Define $G = \left\{\text{gold coin is found in the } 2^{nd}\text{drawer}\right\}$. For an event A, define $\mathbb{P}'(A) = \mathbb{P}(A|S)$. Note that:
$\mathbb{P}'(A|B) = \frac{\mathbb{P}'(A\cap B)}{\mathbb{P}'(B)} = \frac{\mathbb{P}(A\cap B|S)}{\mathbb{P}(B|S)} = \frac{\frac{\mathbb{P}(A\cap B\cap S)}{\mathbb{P}(S)}}{\frac{\mathbb{P}(B\cap S)}{\mathbb{P}(S)}} = \frac{\mathbb{P}(A\cap B\cap S)}{\mathbb{P}(B\cap S)} = \mathbb{P}(A|B\cap S)$,
for an event B.

By LTP, we have that

$$\begin{aligned}
\mathbb{P}(G|S) = \mathbb{P}'(G) &= \mathbb{P}'(G|C_1)\cdot\mathbb{P}'(C_1) + \mathbb{P}'(G|C_2)\cdot\mathbb{P}'(C_2) \\
&= \mathbb{P}(G|S\cap C_1)\mathbb{P}(C_1|S) + \mathbb{P}(G|S\cap C_2)\mathbb{P}(C_2|S=1) \\
&= \frac{1}{3} + \frac{1}{2}\times\frac{2}{3} = \frac{2}{3}.
\end{aligned}$$

(c) (3 points) Batman and Robin go target practice. Batman hits the target with probability p_1, while Robin hits it with probability p_2. Assume that they hit the target independently of each other. They simultaneously shoot at the same target and knock it over. Find the probability that they both had hit the target. You should express the probability in terms of p_1 and p_2.

Solution: Let R = Robin hits, and B = Batman hits. Since the two hit the target independently of each other, we have: $\mathbb{P}(B\cap R) = p_1p_2$. Then:

$\mathbb{P}(\text{both hit}|\text{there was a hit}) = \mathbb{P}(B\cap R|B\cup R) = \frac{\mathbb{P}\left((B\cap R)\cap(B\cup R)\right)}{\mathbb{P}(B\cup R)} = \frac{\mathbb{P}\left(B\cap R\right)}{\mathbb{P}(B)+\mathbb{P}(R)-\mathbb{P}(B\cap R)} = \frac{p_1p_2}{p_1+p_2-p_1p_2}$.

Question 3. (7 points) You have a box containing 25 LEGO bricks, 7 of which are broken. You randomly take 3 bricks out of the box. Let X be the number of broken LEGO out of the 3 bricks. Find:

(a) $\mathbb{E}[X]$.

(b) $\mathrm{Var}(X)$.

(c) $\mathrm{Var}(20 - X)$.

Solution: We have:

$$\begin{aligned}
\mathbb{E}[X] &= 0 \times \mathbb{P}(X = 0) + 1 \times \mathbb{P}(X = 1) + 2 \times \mathbb{P}(X = 2) + 3 \times \mathbb{P}(X = 3) \\
&= 0 + \frac{\binom{7}{1}\binom{18}{2}}{\binom{25}{3}} + 2\frac{\binom{7}{2}\binom{18}{1}}{\binom{25}{3}} + 3\frac{\binom{7}{3}}{\binom{25}{3}} \\
&= \frac{21}{25} = 0.84.
\end{aligned}$$

Similarly, we compute:

$$\begin{aligned}
\mathbb{E}[X^2] &= 0^2 \times \mathbb{P}(X = 0) + 1^2 \times \mathbb{P}(X = 1)\, 2^2 \times \mathbb{P}(X = 2) + 3^2 \cdot \mathbb{P}(X = 3) \\
&= 0 + 1 \cdot \frac{\binom{7}{1}\binom{18}{2}}{\binom{25}{3}} + 4 \cdot \frac{\binom{7}{2}\binom{18}{1}}{\binom{25}{3}} + 9 \cdot \frac{\binom{7}{3}}{\binom{25}{3}} \\
&= \frac{63}{50} \approx 1.26.
\end{aligned}$$

Hence, $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{21}{50} = 0.42$.

Finally, $\mathrm{Var}(20 - X) = |-1|^2\, \mathrm{Var}(X) = \mathrm{Var}(X) = 0.42$.