$$f(x) = \sum_{n=0}^{\infty} \overset{?}{c_n} (x-a)^n \qquad |x-a| < R$$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-4)^4 + \cdots$$

$$f(a) = c_0 = \frac{f^{(0)}(a)}{0!}$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$f'(a) = c_1 = \frac{f^{(1)}(a)}{1!}$$

$$f''(x) = 2c_2 + 2.3\,c_3(x-a) + 3.4\,c_4(x-a)^2 + \cdots$$

$$f''(a) = 2c_2 \Rightarrow c_2 = \frac{f''(a)}{2}$$

$$f^{(3)}(x) = 2.3\,c_3 + 2.3.4\,c_4(x-a) + \cdots$$

$$f^{(3)}(a) = 2.3\,c_3 \Rightarrow c_3 = \frac{f^{(3)}(a)}{1.2.3} = \frac{f^{(3)}(a)}{3!}$$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

CHAPTER 8. SEQUENCES AND SERIES.
Section 8.7. Taylor and Maclaurin Series.

Suppose that $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$ for $|x-a| < R$. We would like to find an expression for the coefficients c_n's. If we let $x = a$, then

$$c_0 = f(a)$$

Recall that we can differentiate f, and $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots$. If we substitute $x = a$,we get

$$c_1 = f'(a)$$

Now, if we differentiate f', we obtain $f''(x) = 2c_2 + 3 \cdot 2c_3(x-a) + \ldots$. Letting $x = a$ yields

$$c_2 = \frac{f''(a)}{2}$$

If we take the n–th derivative of f and substitute $x = a$, then

$$c_n = \frac{f^{(n)}(a)}{n!}$$

- **Taylor Series.**
  If f has a power series expansion at a, i.e. if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \quad |x-a| < R$$

  then its coefficients are given by

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

- f is called the *Taylor series* of the function f at a.
- If $a = 0$, then the power series expansion of f is called a *Maclaurin* series.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

Example: Find the Maclaurin series of the function and its radius of convergence.

1. $f(x) = e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = e^0 = 1, \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad I = (-\infty, \infty), \; R = \infty$

Ratio test

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{\underbrace{(n+1)!}_{n!(n+1)}} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 \text{ for all } x$$

$$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

2. $f(x) = \sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	-1
4	$\sin x$	0

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{odd exponents}$$

$\sin x \downarrow$ odd function

$I = (-\infty, \infty) \quad R = \infty$

3. McLaurin Series for $f(x) = \cos x$

$$\frac{d}{dx} \sin x = \cos x \Rightarrow \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \cos x$$

$$\frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad I = (-\infty, \infty) \quad R = \infty$$

$$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\pi}{4}\right)^{2n}$$

$\cos 0 =$

Example Find the Taylor series of the function centered at the given value.
$f(x) = \ln x$ at $a = 2$.

$$\ln x = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \quad ?$$

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\ln x$	$\ln 2$
1	$\frac{1}{x}$	$\frac{1}{2}$
2	$-\frac{1}{x^2}$	$-\frac{1}{2^2}$
3	$\frac{2}{x^3}$	$\frac{2}{2^3}$
4	$-\frac{2.3}{x^4}$	$-\frac{2.3}{2^4}$
5	$\frac{2.3.4}{x^5}$	$\frac{2.3.4}{2^5}$

$n \geq 1 \quad f^{(n)}(2) = \frac{(-1)^n (n-1)!}{2^n}$

$$f(x) = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cancel{(n-1)!}}{2^n \, n!}(x-2)^n$$

$(n-1)! \cdot n$

$$f(x) = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 2^n}(x-2)^n$$

$$f(x) = \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)2^{n+1}}(x-2)^{n+1}$$

Radius of convergence: Ratio test

$$\lim_{n\to\infty}\left|\frac{(-1)^n (x-2)^{n+1}}{(n+1)\,2^{n+1}} \cdot \frac{n 2^n}{(-1)^{n-1}(x-2)^n}\right| = \lim_{x\to\infty} \frac{n}{2(n+1)}|x-2|$$

$$= \frac{|x-2|}{2} < 1$$

$$|x-2| < 2$$

$$R = 2$$

Important Mclaurin series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},\ R = \infty,\ I = (-\infty, \infty)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},\ R = \infty,\ I = (-\infty, \infty)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!},\ R = \infty,\ I = (-\infty, \infty)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,\ R = 1,\ I = (-1, 1)$$

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$$

Additional Problems

1. $f(x) = e^{x^2}$ at $a = 0$ (a McLaurin Series)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}, \quad e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad , e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

2. $f(x) = \sinh x = \dfrac{e^x - e^{-x}}{2}$ at $a = 0$

$$\frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[\left(\cancel{1} + \frac{x}{1!} + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \cancel{\frac{x^4}{4!}} \cdots\right) - \left(\cancel{1} - \frac{x}{1!} + \cancel{\frac{x^2}{2!}} - \frac{x^3}{3!} + \cancel{\frac{x^4}{4!}} + \cdots\right)\right.$$

$$= \frac{1}{2}\left(2x + 2\frac{x^3}{3!} + 2\frac{x^5}{5!} + \cdots\right) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

- Evaluate the indefinite integral as an infinite series.

$$\int \frac{e^x - 1}{x}\, dx$$

Find the McLaurin series of $f(x) = \dfrac{e^x - 1}{x}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad , \quad \frac{e^x - 1}{x} = \frac{\left(\cancel{1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \cancel{-1}}{x}$$

$$\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$$

$$\int \frac{e^x - 1}{x}\, dx = C + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \frac{x^{n+1}}{n+1}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

- Use series to evaluate the limit.

$$\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \lim_{x\to 0} \frac{\left(\cancel{x} - \cancel{\frac{x^3}{3!}} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right) - \cancel{x} + \cancel{\frac{x^3}{6}}}{x^5}$$

$$= \lim_{x\to 0} \left(\frac{1}{5!} - \frac{x^2}{7!} + \frac{x^4}{9!} + \cdots\right) = \frac{1}{5!} = \frac{1}{120}$$

or L'Hospital's Rule

$$\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{5x^4} = \lim_{x\to 0} \frac{-\sin x + x}{20x^3} = \lim_{x\to 0} \frac{-\cos x + 1}{60x^2}$$

$$= \lim_{x\to 0} \frac{\sin x}{120x} = \frac{1}{120}$$

- Find the sum of the series.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{4n}}{n!} = \sum_{n=0}^{\infty} \frac{(x^4)^n}{n!} = e^{x^4}$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n}(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{4}\right)^{2n}}{(2n)!} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

3. $\displaystyle\sum_{n=3}^{\infty} \frac{(3x)^n}{n!}$

$$\sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = e^{3x} = \underbrace{1}_{n=0} + \underbrace{3x}_{n=1} + \underbrace{\frac{9x^2}{2}}_{n=2} + \sum_{n=3}^{\infty} \frac{(3x)^n}{n!}$$

$$\sum_{n=3}^{\infty} \frac{(3x)^n}{n!} = e^{3x} - 1 - 3x - \frac{9x^2}{2}$$

Taylor Polynomial : The nth degree Taylor Polynomial of $f(x)$, $T_n(x)$ is the partial sum of f as sin

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Examples: ① $f(x) = \ln(3-x)$. Find the 3rd degree Taylor polynomial of f at $a=2$

$$T_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f^{(3)}(2)}{3!}(x-2)^3$$

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\ln(3-x)$	0
1	$\frac{-1}{3-x}$	-1
2	$-\frac{1}{(3-x)^2}$	-1
3	$\frac{-2}{(3-x)^3}$	-2

$$T_3(x) = -(x-2) - \frac{1}{2}(x-2)^2 - \frac{1}{3}(x-2)^3$$

② $P(x) = 3x^2 - 5x^3 + 7x^4 + 3x^5$ is the 5th degree Taylor Polynomial for a function f about $a=0$. What is the value of $f'''(0)$?

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

$$\frac{f^{(3)}(0)}{3!} = -5$$

$$f^{(3)}(0) = -5 \cdot 3!$$

$$f^{(3)}(0) = -30$$