MTH 224, Spring 2024

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Lecture 14

Section 4.3: independent random variables
Section 3.1: continuous random variable and their probability density functions (pdf)

14.1. Independent random variables

DEFINITION 14.1. Random variables X, Y are said to be independent if for every $A, B \subseteq \mathbb{R}^n$,

$$\mathbb{P}\left((X \in A) \cap (Y \in B)\right) = \mathbb{P}\left(X \in A\right) \cdot \mathbb{P}\left(Y \in B\right).$$

In other words, if the events $\{X \in A\}$ and $\{Y \in B\}$ are independent for every $A, B \subseteq \mathbb{R}^n$. If X, Y are both discrete random variables, then X and Y are independent if and only if for every x, y

$$p_{X,Y}\left(x, y\right) = p_X\left(x\right) \cdot p_Y\left(y\right).$$

More generally, $X_1, \ldots, X_N$ are independent if $p_{X_1 \ldots X_N}(x_1, \ldots x_N) = p_{X_1}(x_1) \times \cdots \times p_{X_N}(x_N)$.

REMARK 14.2. Random variables that are not independent are also called dependent random variables.

REMARK 14.3. If X, Y are independent then $p_{X|Y}\left(x|y\right) = p_X\left(x\right)$ and $p_{Y|X}\left(y|x\right) = p_Y\left(y\right)$. Show these!

EXERCISE 14.4. Roll two fair dice. X = largest number, Y = smallest number. Are X, Y independent?

SOLUTION. $\mathbb{P}\left(X = 1, Y = 2\right) = 0 \neq \mathbb{P}\left(X = 1\right) \cdot \mathbb{P}\left(Y = 2\right)$ which means that X and Y are not independent.

EXAMPLE 14.5. Let X, Y be independent. Show that $\mathbb{E}\left[XY\right] = \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right]$.

SOLUTION. We have:

$$\begin{aligned}\mathbb{E}\left[XY\right] &= \sum_x \sum_y xyp_{XY}(x, y) = \sum_x \sum_y xyp_X(x)p_Y(y)\\ &= \sum_x xp_X(x) \sum_y yp_Y(y) = \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right].\end{aligned}$$

REMARK 14.6. In general, if $X_1, \ldots, X_n$ are independent, then $\mathbb{E}[X_1 \cdots X_n] = \mathbb{E}[X_1] \times \cdots \times \mathbb{E}[X_N]$.

EXAMPLE 14.7. Show that if X, Y are independent, then $\text{Cov}\left(X, Y\right) = 0$ i.e., X, Y are uncorrelated.

SOLUTION. $\text{Cov}\left(X, Y\right) = \mathbb{E}\left[XY\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right] = \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right] = 0$.

EXAMPLE 14.8. Show that if X, Y are independent, then $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.

SOLUTION. Since $\text{Cov}(X, Y) = 0$ by the previous example, we have $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) = \text{Var}(X) + \text{Var}(Y)$.

REMARK 14.9. However, $\operatorname{Cov}(X, Y) = 0$ does not mean that X and Y are independent! In other words, X, Y can be uncorrelated and dependent. For example, let X be a random variable taking values $-1, 0$, and 1 with equal probability (of 1/3) and define $Y = X^2 = \begin{cases} 1, & \text{with probability} \frac{2}{3} \\ 0, & \text{with probability} \frac{1}{3} \end{cases}$. Clearly, X and Y are not independent. But, they are uncorrelated as $\mathbb{E}[XY] = \mathbb{E}[X] = 0$. So, $\operatorname{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0 - 0 \times \mathbb{E}[Y] = 0$.

EXAMPLE 14.10. Let $X \sim \operatorname{NB}(r, p)$. Show that $\mathbb{E}[X] = \frac{r}{p}$, $\operatorname{Var}(X) = r \cdot \frac{1-p}{p^2}$.

SOLUTION. One can write $X = X_1 + \cdots + X_r$ where $X_i \sim \operatorname{G}(p)$ are independent (why?). Since we already know that $\mathbb{E}[X_i] = \frac{1}{p}$ and $\operatorname{Var}[X_i] = \frac{1-p}{p^2}$, we obtain that

$$\mathbb{E}[X] = \sum_{i=1}^{r} \mathbb{E}[X_i] = \frac{r}{p}$$

and

$$\operatorname{Var}(X) = \operatorname{Var}(\sum_{i=1}^{r} X_i) = \sum_{i=1}^{r} \operatorname{Var}(X_i) + \underbrace{\sum_{i \neq j} \sum \operatorname{Cov}(X_i, X_j)}_{=0} = \frac{r(1-p)}{p^2}.$$

14.2. Continuous distributions

- Discrete random variables cannot be used in certain applications in which measurements are in a continuous set, such as lifetime of a machine, travel times between two points, or temperatures. In other scenario, we may even replace discrete measurement by continuous measurements just to simplify the model, for example price of a stock.
- For these types of experiments, we use the notion of a continuous random variable.
- In a continuous model, $\mathbb{P}(X = x) = 0$ so a pmf would not make sense . Instead we have the **probability density function (pdf)**.

DEFINITION 14.11. A random variable X is said to be continuous if there exists a function $f_X(x) \geq 0$ defined on $\mathbb{R}$ such that

$$\mathbb{P}(X \in B) = \int_B f_X(x)\, dx$$

for each $B \subseteq \mathbb{R}$. The function $f_X(x)$ is called pdf (probability density function, or just "density") of X.

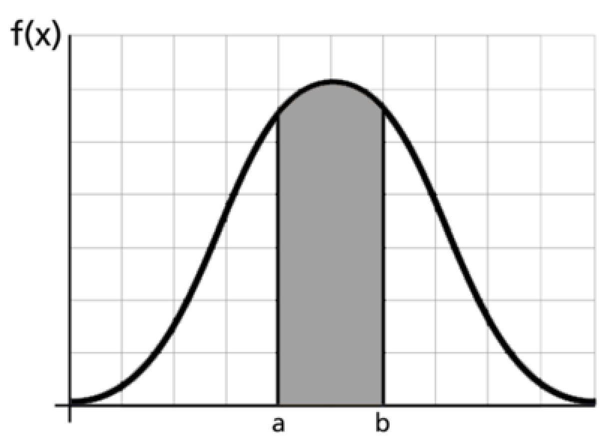

FIGURE 1. Example of pdf. Here the area is $\mathbb{P}(X \in [a, b])$

Remark 14.12. We have:

- For $B = [a, b]$, we have that $\mathbb{P}(a \le X \le b) = \int_a^b f_X(x)\,dx$.
- For any $a \in \mathbb{R}$, we have that $\mathbb{P}(X = a) = \int_a^a f_X(x)\,dx = 0$.
- By the probability axiom: $\int_{-\infty}^{\infty} f_X(x)\,dx = \mathbb{P}(-\infty < X < \infty) = 1$.

Example 14.13. Suppose that X is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} C\left(4x - 2x^2\right), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}.$$

What is the value of C ?

Solution. Since $f(x)$ is a pdf, we must have

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f\,dx = C \int_0^2 \left(4x - 2x^2\right) dx \\ &= C\left(4 \cdot \left.\frac{x^2}{2}\right|_0^2 - 2 \cdot \left.\frac{x^3}{3}\right|_0^2\right) = C\left(8 - 2 \cdot \frac{8}{3}\right) = C \cdot \frac{8}{3}. \end{aligned}$$

Hence $C = \frac{3}{8}$. Therefore, the pdf of X is $f(x) = \frac{3}{8}\left(4x - 2x^2\right)$, for $x \in (0, 2)$. It is implied that $f(x) = 0$ for $x \in (-\infty, 0] \cup [2, +\infty)$.