CHAPTER 5. INVERSE FUNCTIONS

Section 5.3. The Natural Exponential Function.

Recall that $y = \ln x$ is an increasing function with domain $(0, \infty)$, and range $(-\infty, \infty)$. Therefore, $y = \ln x$ is a one-to-one function and has an inverse function denoted $y = e^x$ that is such that

$$\boxed{y = e^x, \quad \text{if and only if} \quad x = \ln y.}$$

- **Properties of $y = e^x$.**
  - The domain of $y = e^x$ is $(-\infty, \infty)$
  - The range of $y = e^x$ is $(0, \infty)$.
  - $e^{\ln x} =$ x if x is in $(0, \infty)$.
  - $\ln(e^x) =$ x if x is in $(-\infty, \infty)$.

$e^{\ln x} = x$

$x \longrightarrow \ln x$

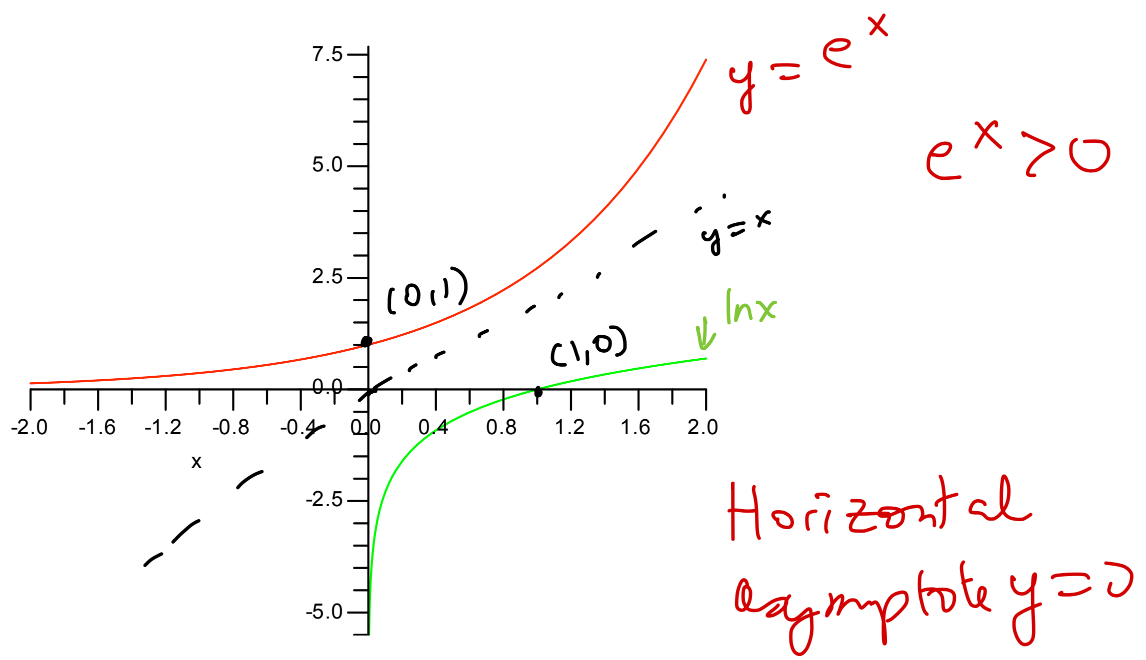
$(x>0$

- **Graph of $y = e^x$.**

  Since we know that the graphs of a function and its inverse are symmetric with respect to $y = x$, we can obtain the graph of $y = \ln x$ from the graph of $y = e^x$.

- **Limits.**

$$\lim_{x\to\infty} e^x = \infty, \qquad \lim_{x\to-\infty} e^x = 0$$



- **Derivative of $y = e^x$.**

  To find an expression for the derivative of $y = e^x$, implicit differentiation is useful. Let us show that

$$\frac{d}{dx}e^x = e^x. \quad \text{why?}$$

$y = e^x$ is equivalent to $x = \ln y$

$$\frac{d}{dx}y = \frac{d}{dx}e^x \longrightarrow \frac{d}{dx}x = \frac{d}{dx}\ln y(x)$$

$$1 = \frac{1}{y}\frac{dy}{dx}$$

$$\frac{dy}{dx} = y = e^x$$

The Chain rule follows:

$$\boxed{\frac{d}{dx}e^{u(x)} = e^{u(x)}\frac{du}{dx}.}$$

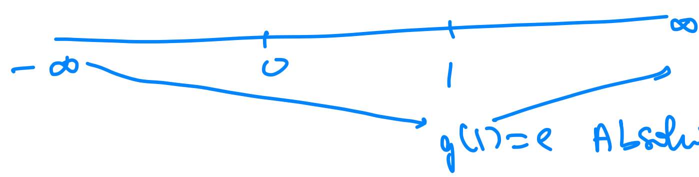
Exercises:

1. Find the absolute minimum of the function $g(x) = \frac{e^x}{x}$. Domain $= (-\infty, 0) \cup (0, \infty)$

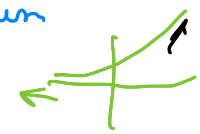
Critical values

$g'(x) = \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$ $\quad g'(x) = 0$ when $x = 1$

If $x > 1$ $g'(x) > 0$, if $x < 1$, $g'(x) < 0$, there is a minimum at $x = 1$



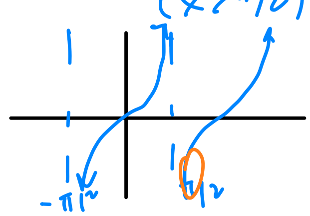
Absolute minimum $g(1) = e$



2. Evaluate the limit. $\lim_{x\to\left(\frac{\pi}{2}\right)^+} e^{\tan x}$.

As $x \to \left(\frac{\pi}{2}\right)^+$ $\quad \tan x \to -\infty$, $e^{\tan x} \to$ "$e^{-\infty}$" $\to 0$

$(x > \pi/2)$



$$\lim_{x\to\left(\frac{\pi}{2}\right)^+} e^{\tan x} = 0$$

$$\lim_{x\to(\pi/2)^-} e^{\tan x} = \infty \qquad \tan x \to \infty$$

- **Integral.**

$$\boxed{\int e^x \; dx = e^x + C,} \quad \text{where } C \text{ is a constant.}$$

Example: Evaluate the integral. $\int \frac{e^{1/x}}{x^2}\, dx$ $\quad 1/x = u$, $\quad \frac{dx}{x^2} = -du$

$u = \frac{1}{x}$ $\qquad du = -\frac{1}{x^2}dx$

$$-\int e^u \, du = -e^u + C$$

Answer: $\int \frac{e^{1/x}}{x^2}\, dx = -e^{1/x} + C$

- **Laws of Exponents.**

  Let x and y be real numbers, and let r be a rational number. Then,

$$\boxed{e^{x+y} = e^x e^y, \quad e^{x-y} = \frac{e^x}{e^y}, \quad (e^x)^r = e^{rx}}$$

- **Additional Exercises:**

  1. Solve the equation for x:

     (a) $e^{-x} = 5$

     $\ln e^{-x} = \ln 5 \Rightarrow -x = \ln 5$

     $x = -\ln 5$

     or $x = \ln\left(\frac{1}{5}\right)$

     (b) $\ln x + \ln(x-1) = 1$ $\quad (x > 1)$

     $\ln[x(x-1)] = 1 \Rightarrow x(x-1) = e$

     $x^2 - x - e = 0$

     $x = \frac{1 \pm \sqrt{1+4e}}{2}$ $\qquad x = \frac{1+\sqrt{1+4e}}{2}$

  2. Evaluate the integral $\int \underbrace{e^x \, dx}_{= du} \sqrt{\underbrace{1+e^x}_{u}}$

     $u = 1 + e^x, \quad du = e^x dx$

     $\int \sqrt{u}\, du = \frac{2}{3} u^{3/2} + C$

     $\int e^x \sqrt{1+e^x}\, dx = \frac{2}{3}(1+e^x)^{3/2} + C$