MTH 224O - Spring 2024

Solutions of Practice Problems for Exam 1

Problem 1: (a) Let an outcome be the result of 6 coin tosses (order important). Then, the sample space has 2^6 equally likely outcomes. There are $\binom{6}{3}$ outcome with exactly 3 heads and 3 tails (# of ways to choose 3 heads out of 6 throws). So, the probability is $\frac{\binom{6}{3}}{2^6}$.

(b) Let an outcome be the 4 chosen cards (order not important). Then, the sample space has $\binom{52}{4}$ equally likely outcome. There are 13^4 outcome with exactly 4 different suites (13 ways of choosing a heart, 13 ways of choosing a spade, 13 ways of choosing a club, and 13 ways of choosing a diamond). So, the probability is $\frac{13^4}{\binom{52}{4}}$.

(c) We have 8^2 possibilities for the first rook, then we delete its row and column, and we are left with 7 by 7 squares, so 7^2 for the second rook. We continue similarly, and hence there are $8^2 \cdot 7^2 \cdots 2^2 \cdot 1^2$ possibilities. Since all the rooks are the same, we need to divide this number by their 8! permutations. We are left with $\frac{(8!)^2}{8!} = 8!$ possibilities. There are $\binom{64}{8}$ ways to place 8 rooks anywhere on the board, and therefore the probability we are looking for is $\frac{8!}{\binom{64}{8}}$.

Problem 2: (a) $N = i$ if we get $i-1$ tails and then get a head. Let T_i be the event of getting tails in the i-th toss, so that $T_i^c$is the event of getting heads in i-th toss. Then,

$$\mathbb{P}(N = i) = \mathbb{P}(T_1 \cap T_2 \cap \cdots \cap T_{i-1} \cap T_i^c) = \mathbb{P}(T_1) \times \cdots \times \mathbb{P}(T_{i-1}) \times \mathbb{P}(T_i^c) = \frac{1}{2^i},$$

in which we used that fact that $T_1, \ldots, T_i$ are independent events.

(b) By Bayes' formula:

$$\mathbb{P}\left(N = 2 \,|\, S = 4\right) = \frac{\mathbb{P}\left(S = 4 \,|\, N = 2\right) \cdot \mathbb{P}\left(N = 2\right)}{\sum_{i=1}^{4} \mathbb{P}\left(S = 4 \,|\, N = i\right) \mathbb{P}\left(N = i\right)}.$$

We calculate: $\mathbb{P}\left(S = 4 \,|\, N = 2\right) = \frac{|\{(2,2),(1,3),(3,1)\}|}{6^2} = \frac{3}{36} = \frac{1}{12}$. $\mathbb{P}\left(N = 2\right) = 2^{-2} = \frac{1}{4}$. Similarly:

$$\begin{aligned}
\mathbb{P}\left(S = 4 \,|\, N = 1\right) \cdot \mathbb{P}\left(N = 1\right) &= \frac{1}{6} \cdot \frac{1}{2} \\
\mathbb{P}\left(S = 4 \,|\, N = 2\right) \cdot \mathbb{P}\left(N = 2\right) &= \frac{1}{12} \cdot \frac{1}{4} \\
\mathbb{P}\left(S = 4 \,|\, N = 3\right) \cdot \mathbb{P}\left(N = 3\right) &= \frac{|\{(1,1,2),(1,2,1),(2,1,1)\}|}{6^3} \cdot \frac{1}{8} = \frac{3}{6^3 \cdot 8} \\
\mathbb{P}\left(S = 4 \,|\, N = 4\right) \cdot \mathbb{P}\left(N = 4\right) &= \frac{1}{6^4} \cdot \frac{1}{16}.
\end{aligned}$$

Finally, we get

$$\mathbb{P}\left(S = 4\right) = \frac{1}{12} + \frac{1}{12 \cdot 4} + \frac{3}{216 \cdot 8} + \frac{1}{1296 \cdot 16} = \frac{2197}{20736}$$

$$\mathbb{P}\left(N=2 \,|\, S=4\right)=\frac{\frac{1}{4\cdot 12}}{\frac{2197}{20736}}\approx 0.1966.$$

(c) We have

$$\mathbb{P}\left(N=even\right)=\sum_{i=1}^{\infty}\mathbb{P}\left(N=2i\right)=\sum_{i=1}^{\infty}2^{-2i}=\sum_{i=1}^{\infty}4^{-i}=\frac{1}{1-\frac{1}{4}}-1=\frac{1}{3}.$$

(d) By LTP, we have:

$$\begin{aligned}\mathbb{P}\left(S=4 \,|\, N=even\right)&=\frac{\mathbb{P}\left(S=4 \text{ and } N=even\right)}{\mathbb{P}\left(N=even\right)}=\\&=\frac{\mathbb{P}\left(S=4 \,|\, N=2\right)\cdot\mathbb{P}\left(N=2\right)+\mathbb{P}\left(S=4 \,|\, N=4\right)\cdot\mathbb{P}\left(N=4\right)}{1/3}\\&=3\cdot\left(\frac{1}{12}\cdot\frac{1}{4}+\frac{1}{6^4}\cdot\frac{1}{16}\right)\approx 0.063.\end{aligned}$$

Problem 3: (a) For $n=1,\ldots,10$, let $A_n=$ the n-th boy is next to a girl. Then, $\mathbb{1}_{A_1}+\cdots+\mathbb{1}_{A_{10}}$ is the number of boys who stand next to a girl, and

$$\mathbb{E}[\mathbb{1}_{A_1}+\cdots+\mathbb{1}_{A_{10}}]=\mathbb{P}(A_1)+\cdots+\mathbb{P}(A_{10}).$$

For any $n=1,\ldots,10$, define $B_n=$ n-th boy is at the start or end of the line. By LTP, we have

$$\mathbb{P}(A_n)=\mathbb{P}(B_n)\mathbb{P}(A_n|B_n)+\mathbb{P}(B_n^c)\mathbb{P}(A_n|B_n^c)=\frac{2}{20}\frac{10}{19}+\frac{18}{20}\left(1-\frac{9}{19}\frac{8}{18}\right)=\frac{29}{38}.$$

Finally, $\mathbb{E}[\mathbb{1}_{A_1}+\cdots+\mathbb{1}_{A_{10}}]=\mathbb{P}(A_1)+\cdots+\mathbb{P}(A_{10})=10\times\frac{29}{38}=\frac{145}{19}\approx 7.63$.

Alternative solution: For $i=1,\ldots,20$, let $A_i'=$ a boy is standing in the i-th position and next to a girl. Then, $\mathbb{1}_{A_1'}+\cdots+\mathbb{1}_{A_{20}'}$ is the number of boys who stand next to a girl, and we have

$$\mathbb{E}[\mathbb{1}_{A_1'}+\cdots+\mathbb{1}_{A_{20}'}]=\mathbb{P}(A_1')+\cdots+\mathbb{P}(A_{20}').$$

We have

$$\mathbb{P}(A_1')=\mathbb{P}(A_{20}')=\frac{1}{2}\times\frac{10}{19}=\frac{5}{19},$$

and for $i=2,\ldots,19$, we have

$$\mathbb{P}(A_i')=\frac{1}{2}\left(1-\frac{9}{19}\frac{8}{18}\right)=\frac{15}{38}.$$

Finally, $\mathbb{E}[\mathbb{1}_{A_1'}+\cdots+\mathbb{1}_{A_{20}'}]=\mathbb{P}(A_1')+\cdots+\mathbb{P}(A_{20}')=2\times\frac{5}{19}+18\times\frac{15}{38}=\frac{145}{19}\approx 7.63$.

(b) For $n=1,\ldots,10$, let $A_n=$ the n-th boy is next to a girl. Then, $\mathbb{1}_{A_1}+\cdots+\mathbb{1}_{A_{10}}$ is the number of boys who stand next to a girl, and

$$\mathbb{E}[\mathbb{1}_{A_1}+\cdots+\mathbb{1}_{A_{10}}]=\mathbb{P}(A_1)+\cdots+\mathbb{P}(A_{10}).$$

We have $\mathbb{P}(A_n) = 1 - \frac{9}{19}\frac{8}{18} = \frac{15}{19}$. Finally, $\mathbb{E}[\mathbb{1}_{A_1} + \cdots + \mathbb{1}_{A_{10}}] = \mathbb{P}(A_1) + \cdots + \mathbb{P}(A_{10}) = 10 \times \frac{15}{19} = \frac{150}{19} \approx 7.89$.

Alternative solution: For $i = 1, \ldots, 20$, let A'_i = a boy is standing in the i-th position and next to a girl. Then, $\mathbb{1}_{A'_1} + \cdots + \mathbb{1}_{A'_{20}}$ is the number of boys who stand next to a girl, and we have

$$\mathbb{E}[\mathbb{1}_{A'_1} + \cdots + \mathbb{1}_{A'_{20}}] = \mathbb{P}(A'_1) + \cdots + \mathbb{P}(A'_{20}).$$

For $i = 1, \ldots, 20$, we have $\mathbb{P}(A'_i) = \frac{1}{2}\left(1 - \frac{9}{19}\frac{8}{18}\right) = \frac{15}{38}$. Finally, $\mathbb{E}[\mathbb{1}_{A'_1} + \cdots + \mathbb{1}_{A'_{20}}] = \mathbb{P}(A'_1) + \cdots + \mathbb{P}(A'_{20}) = 20 \times \frac{15}{38} = \frac{150}{19} \approx 7.89$.