Chapter 4 vector spaces

vector spaces and linear transformations

R^n = n-tuples of real numbers

add multiply by scalars
R^n has dimension n.

All vectors of dimension n look like R^n . There are subspaces of R^n and these look like R^m for m <n. They turn out to be in some sense m-tuples, but that may not be the way they appear. The key is the idea of a basis for a vector space.

A vector space over R is a set with an operation called + and scalar multiplication · . You can add two vectors and you can multiply a vector by a number (In this case the scalars are just real numbers.) One of the main motivations is physics where a vector is a quantity which is supposed to have a direction and a magnitude. (force, acceleration, velocity are vector quantities.) In physics you represent a vector by a directed line segment. The length of the segment gives the magnitude of the vector. The segment is not fixed in space. You add vectors by using the parallelogram law.

Vector addition satisfies many of the usual rules of addition, as does scalar multiplication. v + w = w + v commutative law

(v + w) + z = v + (w + z) associative
a (v+w) = a v + a w (a is a scalar)
a (b v) = (ab) v (associative)
(a + b) v = a v + b v
0 + v = v + 0 = v for all v (0 is the zero vector)
For each v there is -v such that v + (-v) = 0
((-v) + v =0 follows from the commutative law)
1 v = v for every vector v.

In R^n all of these are familiar properties of addition of vectors and multiplication by numbers. So R^n is a vector space over R. (You could talk about C^n, n-tuples of complex numbers. We do not have, in general, a multiplication of two vectors. In R^3 there is a cross product which is a kind of vector multiplication. There is a dot product of vectors in R^n which we will talk about when discuss inner product spaces.

A general kind of question is: given vectors v, w, z, ... what vectors are linear combinations of these? a v + b w + c z + ... What do you get when you take linear combinations of vectors?

(1,2,3) (4,5,6) Is the vector (9,12,15) a linear combination of the other two?

(9,12,15) = x (1,2,3) + y (4,5,6) ???

(x +4 y, 2 x + 5y, 3 x + 6y) =??

(9,12,15)

x+4y =9

2x + 5y =12

3x + 6y =15

x=1, y = 2 is the solution so (9,12,15) is a linear of (1,2,3) and (4,5,6).

A different example: Let V be all polynomials of degree <=10.

f(x) = a +b x + c x^2+ ...+ m x^10

a polynomial of degree 10. We add polynomials the usual way. If we add two polynomials of degree at most 10 we get a polynomial of degree at most 10. We can multiply a polynomial by a number. There is a 0 polynomial. The set of polynomials of degree at most 10 is a vector space with the usual addition and multiplication by numbers. (We can multiply polynomials, but that is not part of the vector space structure.Also, if we multiply polynomials of degree at most 10 we may get a polynomial of degree as much as 20.) The set of all polynomials of degree at most 10 is a vector space.

f(x) = a +b x + c x^2+ ...+ m x^10 ↔ (a,b,c,...,m)

Addition of polynomials corresponds to addition of 11-tuples; multiplication by scalars corresponds to the usual multiplication of polynomials by numbers. From the vector space point of view the set of polynomials of degree at most 10 is "the same" as R^{11} . The set of polynomials of degree at most 10 as a vector space is "the same" as R^{11}.

Suppose you want a polynomial f(x) of degree at most 10 whose values are prescribed at certain points.

Suppose we want a polynomial of degree at most 2: g(x) = a + b x + c x^2.

Suppose I want g(0) = 2, g(1) = 3 and g(2) =5.

a =2; a +b + c = 3; a + 2b + 4 c = 5

There is a unique solution you find a unique g(x) with these conditions.

Suppose we want a polynomial h(x) of degree at most 2 such that

h(1) = 2; h'(1) =3; h''(1) = 5.

h(x)= a +b x + c x^2

a + b + c = 2; b + 2c = 3; 2 c = 5.

We wind up with a system of 3 simultaneous equations (linear) in 3 unknowns.

(a f + b g)' = a f' + b g'

If f and g are polynomials of degree at most 10, and a and b are numbers, then a f+ b is a m of degree at most 10 and its derivative is a polynomial of degree at most 9. Differentiation is a linear map. Here we are talking about linear maps. We could represent everything by vectors and matrices.

f(x) =1

g(x) = x

h(x) = x^2

a f(x) + b g(x) + c h(x)

= a + b x + c x^2

We can write down equations for a , b and c to give whatever values to the expression and first two derivatives.

We are led to systems of simultaneous equations. Differentiation is a linear function: (f+g) '= f' + g'

Integration: I(x^n) = x^(n+1) /(n+1)

Integration is a linear function. I(f+g) = I(f) + I(g). However, the integral of a polynomial of degree 10 is a polynomial of degree 11.

If we think of integration (with 0 constant term) as a map from polynomials to polynomials it is a linear map.

Suppose we look at all functions of the form a sin(x) + b cos(x) + c e^x. We can add two such functions and we can multiply such a function by a number. The set of such functions is a vector space. This is an example of a 3-dimensional vector space. You can differentiate such a function:

f(x) = a sin(x) + b cos(x) + c e^x

f'(x) = a cos(x) - b sin(x) + c e^x.

f' is another function of the same kind. Differentiation is linear. (f+g)'=f'+g'

I(a sin(x) + b cos(x) + c e^x) = -a cos(x) + b sin(x) + c e^x This is linear.

You can evaluate these functions at points: f(π) = -b + c e^π.

This is also a linear map.

f(x) = a sin(x) + b cos(x) + c e^x $\longleftrightarrow$(a,b,c)

linear maps from R^3 to R^3 correspond to differentiation and integration.

You could consider all polynomials (of arbitrary degree.) That is a vector space of infinite dimension.

These are all examples of vector spaces. Differentiation and integration are linear functions. All sorts of situations end up being problems in vector spaces, with linear equations.

matrices systems of simultaneous equations

dimension the dimension of R^10 is 10.

the dimension of the space of polynomials of degree at most 10 is 11.

the dimension of the space of all polynomials is infinite.

linear maps systems of simultaneous equations

Any two vector spaces of the same dimension are, in some sense, the same.

Solving linear problems in R^n enables us to solve linear problems much more generally.

In R^3 a plane through the origin is actually a 2-dimensional vector space. This plane is a subspace of R^3.

All sorts of problems involving vector spaces can be reduced to solving systems of simultaneous equations. If V is a vector space of dimension n then in some sense it is R^n. There may be natural ways of representing it.

The dimension of a vector space is well-defined. basis. linear transformation .