# CHAPTER 6. TECHNIQUES OF INTEGRATION
## Section 6.1. Integration by Parts

Question: We know that $\frac{d}{x}\ln x = \frac{1}{x}$, but what is $\int \ln x \ dx$? Recall the product

rule for derivatives.

$$(fg)' = f'g + fg'$$

or $fg' = (fg)' - f'g$

Integrate $\int f(x)\, g'(x)\, dx = \int (f\cdot g)'(x)\, dx - \int f'(x)\, g(x)\, dx$

Integration by parts (IBP)

$$\boxed{\int f(x)\, g'(x)\, dx = f(x)\cdot g(x) - \int f'(x)\, g(x)\, dx}$$

?

So, we can conclude that

$$\int f(x)g'(x)\ dx = f(x)g(x) - \int f'(x)g(x)\ dx$$

or if $f(x) = u$ and $g(x) = v$, $f(x)\ g'(x)\,dx$

$$\boxed{\int \overset{''}{u}\, \overset{''}{dv} = uv - \int v\ du}$$

**Exercises:** Evaluate the integral

1. $\int \underbrace{\ln x}_{u}\, \underbrace{dx}_{dv} = x\ln x - \int x\,\frac{1}{x}\,dx = x\ln x - \int 1\,dx$

$u = \ln x \quad dv = dx$

$du = \frac{1}{x}\,dx\ ,\ v = x$

$$\boxed{\int \ln x\, dx = x\ln x - x + C}$$

IBP $\int u\,dv = uv - \int v\,du$

? ?

2. $\int \tan^{-1} x \; dx$

$u = \tan^{-1} x \qquad dv = dx$

$du = \frac{1}{1+x^2}\,dx \qquad v = x$

$\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx$

$\frac{x}{1+x^2}\,dx = \frac{1}{2}\,du \qquad 1+x^2 = u$

$u = 1+x^2 \qquad du = 2x\,dx$

$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u|$

$$\boxed{\int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + C}$$

3. $\int x^3 e^x \; dx$ (Tabulation method.)

( $\int P(x)\,a^x\,dx$ $\quad$ P is a polynomial

$u = P(x) \qquad dv = a^x\,dx$

$u = x^3 \qquad dv = e^x\,dx$

$du = 3x^2\,dx \qquad v = e^x$ $\quad \int v\,du$

$\int x^3 e^x\,dx = x^3 e^x - 3\int x^2 e^x\,dx$

IBP $\quad u = x^2 \qquad dv = e^x\,dx$

$du = 2x\,dx \qquad v = e^x$

$\int x^3 e^x\,dx = x^3 e^x - 3\left[x^2 e^x - 2\int x e^x\,dx\right]$

IBP $\quad u = x \qquad du = e^x\,dx$

$du = dx \qquad dv = e^x$

$\int x^3 e^x\,dx = x^3 e^x - 3\left[x^2 e^x - 2\left(x e^x - \int e^x\,dx\right)\right]$

$$\boxed{\int x^3 e^x\,dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C}$$

# Tabulation Method

$$\int \underbrace{x^3}_{u}\, \underbrace{e^x\, dx}_{dv} = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$$

| + $u$ All derivative | + $dv$ all antiderivative |
|---|---|
| $x^3$ | $e^x$ |
| $3x^2$ | $e^x$ |
| $6x$ | $e^x$ |
| $6$ | $e^x$ |
| $0$ | $e^x$ |

Can be applied to $\int P(x)\sin(ax)\,dx$

$\int P(x)\cos(bx)\,dx$

$$\int u\,dv = uv - \int v\,du$$

4. $\displaystyle\int e^x \underbrace{\cos x}_{u}\ dx$

$u = \cos x \qquad dv = e^x\,dx$

$du = -\sin x\,dx \qquad v = e^x$

$$\int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx$$

$u = \sin x \qquad dv = e^x\,dx$

$du = \cos x\,dx \qquad v = e^x$

$$\int e^x \cos x\,dx = e^x \cos x + e^x \sin x - \int e^x \cos x\,dx$$ ← equation

$$2\int e^x \cos x\,dx = e^x \cos x + e^x \sin x$$

$$\int e^x \cos x\,dx = \frac{1}{2}\left(e^x \cos x + e^x \sin x\right) + C$$

In the case of a definite integral, integration by parts can be applied as follows.

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx$$

**Exercises:** Evaluate the integral.

1. $\displaystyle\int_{\pi/4}^{\pi/2} x\csc^2 x\ \ dx$

$u = x \quad dv = \csc^2 x\,dx$

$du = dx \quad v = -\cot x$

$$\int_{\pi/4}^{\pi/2} x\csc^2 x\,dx = -x\cot x\Big|_{\pi/4}^{\pi/2} - \underbrace{\int_{\pi/4}^{\pi/2} \cot x\,dx}_{\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}\,dx}$$

$$= 0 + \frac{\pi}{4}$$

$u = \sin x$ , $du = \cos x\,dx$

$x = \pi/4$ , $u = \frac{\sqrt{2}}{2}$

$x = \pi/2$ , $u = 1$

$$\int_{\sqrt{2}/2}^{1} \frac{1}{u}\,du = \ln|u|\Big|_{\sqrt{2}/2}^{1}$$

$$= 0 - \ln\frac{\sqrt{2}}{2}$$

$$\int_{\pi/4}^{\pi/2} x\csc^2 x\,dx = \frac{\pi}{4} + \ln\left(\frac{\sqrt{2}}{2}\right)$$

At times, an integral may first require a substitution before being able to use integration by parts.

2. $\int_1^4 e^{\sqrt{x}}\, dx$ $\sqrt{x} = t$, $dx = 2t\,dt$

(1) Make a substitution $t = \sqrt{x}$ or $x = t^2$ $dx = 2t\,dt$

$x = 1$ $t = 1$ , $x = 4 \Rightarrow t = 2$

$\rightarrow 2\int_1^2 e^t\, t\, dt$

(2) IBP $u = t$ $dv = e^t dt$

$du = dt$ $v = e^t$

$$2\int_1^2 e^t\, t\, dt = 2\left[te^t\Big|_1^2 - \int_1^2 e^t\, dt\right]$$

$$= 2\left[2e^2 - e - (e^2 - e)\right]$$

$$= 2(e^2) = 2e^2$$

Additional example:

$$\int_1^2 (\ln x)^2 dx$$

IBP $\quad u = (\ln x)^2 \quad dv = dx$

$du = 2\frac{\ln x}{x} dx \quad v = x$

$$\int_1^2 (\ln x)^2 dx = x(\ln x)^2 \Big|_1^2 - 2\int_1^2 \frac{\ln x}{\cancel{x}} \cancel{x}\, dx$$

$$= 2(\ln 2)^2 - 2 \int_1^2 \ln x\, dx$$

$u = \ln x \quad dv = dx$

$du = \frac{1}{x} dx \quad v = x$

$x \ln x \Big|_1^2 - \int_1^2 \frac{1}{x} x\, dx$

$= x \ln x \Big|_1^2 - \int_1^2 dx$

$= 2\ln 2 - x \Big|_1^2$

$= 2\ln 2 - 1$

$$\int_1^2 (\ln x)^2 dx = 2(\ln 2)^2 - 2(2\ln 2 - 1).$$