Analysis Done Right. (title w.i.p.)

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September 11, 2024

Section Chapter 1. Insert motivation ex. 1 (bolzanos existence problem)

Mathematicians wanted a way to show that there is an existence even if we do not have a direct way of finding.

Section Absolute Value and Why it's so useful. We will begin by describing the properties of absolute value as a refresher and a warm up to some "baby" proofs.

The absolute value function is critical in our study of analysis as it is the first way that we will define a distance metric in one dimensional space. But more on distance later.

1.) Positive Definite. $\mid x \mid \geq 0$ and $\mid x \mid = 0 \iff x = 0$

This first property is quite useful as it makes sure that our result will never be negative, which is extremely important when taking distance into consideration as negative distance has no inherent meaning.

2.) Multiplicative Property. $\mid x * y \mid = \mid x \mid * \mid y \mid$ Multiplication works nicely with absolute value.

3.) Symmetry Property. $\mid x \mid = \mid -x \mid$
This property is a consequence of the positive definiteness property and is useful in some of the forthcoming proofs.

Corollary(with a short proof): $\mid a - b \mid = \mid -(a-b) \mid = \mid b - a \mid \Rightarrow$
$\mid a - b \mid = \mid b - a \mid$
This corollary shows us that order is not important when subtracting inside the absolute value.

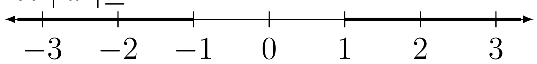
4.) Triangle Innequality $\mid a + b \mid \leq \mid a \mid + \mid b \mid$
A useful result that will help us with almost all of our limit proofs in the upcoming chapter.

5.) Second Definition $\mid x \mid = \sqrt{x^2}$

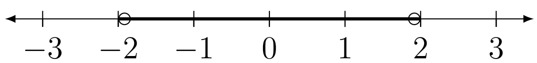
"Or $\mid x \mid = \sqrt[n]{x}$ where $n = 2k, k \in \mathbb{Z}$

" 6. Disjunctive Innequality let $k > 0, \mid x \mid \geq k \iff x \geq k \vee x \leq -k$
Take this example for k = 1 to help visualize the range of possibilites of x.

let $\mid x \mid \geq 1$



7.) Conjunctive Innequality let $k > 0 \mid x \mid \leq k \iff x > -k \wedge x < k$
Take another example where k = 2
Consider $\mid x \mid \leq 2$



8. Division Property. $\mid \frac{a}{b} \mid = \frac{|a|}{|b|}$ where b $\neq$ 0 Another useful property that we will use extensively in the following chapter.

9.Absorption $\mid\mid x \mid\mid = \mid x \mid$

Proof. Take $\mid\mid x \mid\mid = \mid x \mid$, *since* $\mid x \mid \geq 0$, by def. □

10. Exponential Property. $\mid x^k \mid = \mid x \mid^k$

11.) Connectedness let $k > 0, \mid x \mid = k \iff x = k \vee x = -k$

These are all the properties of absolute value that we are going to be using in this text. We will commence the following page with proofs for all of these properties which again will serve as a nice warmup for what's to come.

Proof. 1.) **Proof of Positive Definite**
Recall : $\mid x \mid \geq 0$ and $\mid x \mid = 0 \iff x = 0$

Case 1: assume $x \geq 0, so \mid x \mid = x$
Case 2: $x < 0 so \mid x \mid = -1 * x \ Thus \mid x \mid > 0 \Rightarrow \mid x \mid \geq 0$
Assume that $\mid x \mid = 0$ and show that $x = 0$
We will now prove the biconditional going in the forwards direction Assume towards a contradiction that $x \neq 0$
Case 1:$x > 0 then \mid x \mid = x > 0$
Case 2:$x < 0 \Rightarrow \mid x \mid = -1 * x, where - 1 * x > 0, thus \mid x \mid > 0$
Which contracts our assumption that $\mid x \mid = 0, thusx \nless 0$
Thus we have shown that $\mid x \mid = 0 \Rightarrow x = 0$
Reverse Direction of the biconditional
if $ifx = 0 \Rightarrow \mid x \mid = 0, bydef.$ So our conjunction portion is true.

□

Proof. 2.) Proof for multiplicative property.
Recall : $| x * y |=| x | * | y |$
$| ab |=| a | * | b | \; a, b \in \mathbb{R}$

Case 1, one real (a or b) is zero. Without loss of generality, let a be zero.
thus, $| ab |=| 0 * b |=| 0 |= 0$
Case 2. They have different signs(a * b < 0) Without loss of generality let a < 0 and b > 0.
$| a * b |=$ -1 * a * b since they have different signs, where$-1 * a * b > 0$
Case 3: They both have the same sign (a * b > 0) Without loss of generality let a,b < 0
$| a * b |$ = -1 * -1 * a * b = a * b ☐

Proof. 3. **Proof of Symmetry.**
Recall : $| x |=| -x |$
$| -x |=| -1 * x |=| -1 | * | x |$, by the multiplicative property.
$1* | x |=| x |$

☐

Proof. Proof of Corrollary
$| a - b |=| -1 * (b - a) |=| b - a |$

☐

Proof. **4.) Triangle Innequality**
Recall : $| a \pm b |\leq| a | + | b |$
Note : $\forall x, x \leq| x |$ and $x \geq -1* | x |$
Let $a, b \in \mathbb{R}$ We can observe
$-1* | a |\leq a \leq| a |$
$-1* | b |\leq b \leq| b |$
if we sum these two compound innequalities we obtain:
$-| a | - | b |\leq a + b \leq| a | + | b |\Rightarrow -1(| a | + | b |) \leq a + b \leq| a | + | b |$
If we look closely we can see a resemblance to the conjunctive innequality. We can see that our "k" is $| a | + | b |$ Thus we can say that $| a + b |\leq| a | + | b |$ This can be expanded upon to include $| a - b |\leq| a | + | b |$ as we can use the 3.) Symmetry Property inside the left hand side of our inequality.

☐

5. Will not be proven as it is an equivalent definition of the absoulte value

Proof. **6. Proof of the disjunctive innequality** let $k \in \mathbb{R}+ \; | x |\leq \iff x \geq \vee x \leq -k$
Forward Direction of Biconditional:
Assume that $| x |\geq k$ and we need to show that $x \geq k \vee x \leq -k$
Case 1: Assume $x \geq 0$ so $| x |= x$ and $| x |\geq k, thus x \geq k$
Case 2: Asuume $x < 0$ so $| x |= -1 * x$ thus, $| x |\geq k \Rightarrow -1 * x \geq k \Rightarrow x \leq -k$
Now we look at the backwards direction.
Assume $x \geq k \vee x \leq -k$ We need to show that $| x |\geq k$
Case 1: $x \geq k > 0 \Rightarrow| x |= x \geq k$
Case 2: $x \leq k < 0 \Rightarrow| x |= -1 * x \geq k \Rightarrow| x |\geq k$ Since we have demonstrated that both

implications are true, their conjunct must be true and we have proven the equivalence of the disjunctive innequality.

$\square$

Proof. **7.) Conjunctive Innquality.** Recall : let $k > 0 \mid x \mid \leq k \iff x > -k \wedge x < k$

Since we have another biconditional statement, we will first prove the forwards direction
$\Rightarrow$
Case 1: assume $\mid x \mid \leq k$ and $x \geq 0$
$x \geq 0 \Rightarrow x > -k$ by hypothesis, k is positive. Also $\mid x \mid = x$ but $\mid x \mid \leq k \Rightarrow x \leq k$
so $-k \leq x \wedge x \leq k$
Case 2: assume $x < 0$
$x < 0 \Rightarrow \mid x \mid = -1 * x$, but $\mid x \mid \leq k \Rightarrow -1 * x \leq k \Rightarrow -k \leq x$
Note that since $x < 0 \Rightarrow x \leq k$ since we assumed k to be positive.
Thus $x \leq k \wedge x \geq -k$
Now we will prove the backwards direcition
$\Leftarrow$
Assume that $-k \leq x \leq k$ we need to show that $\mid x \mid \leq k$
Case 1: assume that $x \geq 0$ thus $\mid x \mid = x \leq k$ by hypothesis
Case 2: assume that $x < 0$ so $\mid x \mid = -1 * x$. We know that $x \geq -k$ and consequently $\mid x \mid = -1 * x \leq k \Rightarrow \mid x \mid \leq k$

Thus we have proved both directions of the biconditional.

$\square$

Proof. **8.) Division Property**
Recall : $\mid \frac{a}{b} \mid = \frac{|a|}{|b|}$ where b $\neq 0$
We first show that $\mid \frac{1}{b} \mid = \frac{1}{|b|}$ where $b \neq 0$
Case 1: $b \leq 0 \mid \frac{1}{b} \mid = \frac{1}{|b|} = \frac{1}{b}$
Case 2: $b < 0, \mid \frac{1}{b} \mid = \frac{1}{-1*b} = \frac{1}{|b|}$

Now we use a clever trick to complete the proof.
$\mid \frac{a}{b} \mid = \mid a * \frac{1}{b} \mid = \mid a \mid * \mid \frac{1}{b} \mid = \mid a \mid * \frac{1}{|b|} = \frac{|a|}{|b|}$ $\square$

The proof of 9.) was included with its definition due to its short length.

Proof. **10.) Exponential Property**
Recall : $\mid x^k \mid = \mid x \mid^k$
We can observe that by definition of exponents $\mid x^k \mid = \mid x * x * x ... * x \mid$ (multiply x, k times). By our 2.) Multiplicative Property, this is equivalent to $\mid x \mid * \mid x \mid * \mid x \mid ... * \mid x \mid$ again, k times. But this is equal to $\mid x \mid^k$ $\square$

Definition of Distance Function.
d is a distance function between two points of a space (a set) V if and only if
$d : V \times V \longrightarrow S \subseteq \mathbb{R}$
Let $a, b \in V$ and $d(a, b) \in \mathbb{R} \wedge d(a, b) \geq 0$
Properties of a Distance Function:
1.) Positive Definite : $d(a, b) \geq 0 \wedge d(a, b) = 0 \iff a = b$
2.) Symmetry : $d(a, b) = d(b, a)$
3.) Triangle Innequality : Let $a, b, c \in V$ then $d(a, b) \leq d(a, c) + d(b, c)$
With this construction, we can now prove our first important theorem.

Theorem 1. The absolute value of the difference of two reals is a distance function.

Proof. let $a, b, c \in \mathbb{R}$ then,
1.) $\mid a - b \mid \geq 0$ and $\mid a - b \mid = 0 \iff a = b$
This was proven as property 1.) of absolute value already.
2.) $\mid a - b \mid = \mid b - a \mid$
This was proven as property 3.) of absolute value already.
3.) Triangle Innequality : $\mid a - b \mid \leq \mid a - c \mid + \mid b - c \mid$ We know that $\mid a - b \mid = \mid a - b + c - c \mid = \mid a - c + c - b \mid \leq \mid a - c \mid + \mid c - b \mid$, by the triangle innequality.
by symmetry: $\mid a - c \mid + \mid b - c \mid$. Thus, $\mid a - b \mid \leq \mid a - c \mid + \mid b - c \mid$

$\square$

Section Ideas!. For any $x, y, \epsilon \in \mathbb{R}$
$x + y > \epsilon > 0$
Divide and Conquer's
1.) $(x > \frac{\epsilon}{2} \wedge y > \frac{\epsilon}{2}) \Rightarrow x + y > \epsilon$
2.) $(x < \frac{\epsilon}{2} \wedge y < \frac{\epsilon}{:} wq2) \Rightarrow x + y < \epsilon$
3.) $(x < \frac{\epsilon}{b} \wedge y < b) \Rightarrow x * y < \frac{\epsilon}{b} * b \Rightarrow x * y < \epsilon$
4.) $(x > \frac{\epsilon}{b} \wedge y > b) \Rightarrow x * y > \epsilon$
Transitivity of the Order Relations $\leq$ and $\geq$
5.) $(x \geq a > b) \Rightarrow x > b$
6.) $(x \leq a < b) \Rightarrow x < b$

Definition of Types of δ neighborhoods of x_0
We will primarily focus on two types of delta neighborhoods in this text. The first will be a Type I neighborhood or standard. The second will be a Type II or punctured neighborhood. We will see why momentarily.
Type 1, A := $\{x \in \mathbb{R} | x_0 - \delta < x < x_0 + \delta, \delta > 0\}$
For an element to be in this neighborhood we must consider the following argument
$x \in A$ if and only if $x \in (x_0 - \delta, x_0 + \delta) \iff x_0 - \delta < x < x_0 + \delta \iff -\delta < x + x_0 < +\delta$
$\mid x - x_0 \mid < \delta$

Type II : B:= $\{x \in \mathbb{R} | x_0 - \delta < x < x_0 + \delta, x \neq x_0, \delta > 0\}$
The only difference between our Type I and Type II intervals is that our Type II interval excludes the point x_0 from the interval whereas our Type I includes x_0

Definition of Def. of an Accumulation point of a set.
Given a metric space V and a distance function d, we say x_0 is an accumulation point of V if and only if every Type II,punctured δ neighborhood around x_0 intersects the set V.
In $\mathbb{R}$: Left Accumulation point of $S \subseteq \mathbb{R}$
we say x_0 is a left accumulation point of $\mathbb{R}$ if and only if
$\forall \delta > 0 (x_0 - \delta, x_0) \cap s \neq \emptyset$
Right Accumulation point of $S \subseteq \mathbb{R}$
$\forall \delta > 0 (x_0, x_0 + \delta) \cap s \neq \emptyset$
An Accumulation Point of $S \subseteq \mathbb{R}$
x_0 is an accumulation point of the set S if and only if
$\forall \delta > 0 (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta) \cap s \neq \emptyset$
Again I would like to highlight the fact that the accumulation point is a punctured interval, that is, it does not contain the point x_0

Definition of Interior points. We will now define interior points of a set, which behave similarly to an accumulation point.
x_0 is an interior point of the set S $\iff \exists \delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq S$
Again, be sure to take note that an interior point does indeed include the point x_0

Definition of Boundary Points.
we say x_0 is a boundary point of a set S if and only if every set containing x_0 non-trivially intersects S and its complement. Note: x_0 does not need to intersect S
We define the complement of S $\subseteq \mathbb{R}$ as follows. if $S = (0, 1)$ then $S' = (-\infty, 0] \cup [1, +\infty)$

Section Limits. x_0 is our quintessential accumulation point of the domain of an arbitrary funcion f(x). $\epsilon \in \mathbb{R}^+, \epsilon > 0$. Epsilon represents a radial distance around some number $L \in \mathbb{R}$ on the y axis of a two dimensional graph. $\delta \in \mathbb{R}^+$ represents a radius around x_0

Lemma The Very Useful Lemma
This result is extremely useful for the upcoming section on limits and thus has been dubbed The Very Useful Lemma. Let $a, b \in \mathbb{R}, if \mid a - b \mid < \epsilon$ for all $\epsilon > 0$ then a = b.
At first, this result might look a little strange so lets try to unpack it. First we notice that a and b can be any numbers of our choosing but they must satisfy the condition that their distance(see theorem 1 on page 5) must be less than a positive number of our choosing ϵ. If for all ϵ that we pick, we can make the distance of these two functions less than ϵ our two numbers a and b are equal.
Let us now prove this rigorously.

Proof. Assuming towards a contradiction, assume that $a \neq b$ and that $a > b$ without loss of generality. Then $\mid a - b \mid = a - b > 0$ Let's pick a clever ϵ where $\epsilon = a - b$ But by our initial assumtion of the conditional, we know that $\mid a - b \mid < \epsilon = a - b$ But then we observe that $a - b < a - b \Rightarrow a < a$ which is a contradiction. Thus a = b.

□

Definition of 1.) Real Limits at real accumulation points

$$\lim_{x \to x_0} f(x) = L \in \mathbb{R} \iff \forall \epsilon > 0 \exists \delta > 0, 0 < \mid x - x_0 \mid < \delta \Rightarrow \mid f(x) - L \mid < \epsilon$$

This definition of limits has plaugued undergraduates for years and is known to be particularly tricky to fully digest due to the nested quantifiers in the statement.
Our definition merely tries to asset that if we pick some radius around the limit(a limit is a real value on the y axis for our purposes) then we can find the existence of a radius δ around the x-axis such that all of the values of f(x) that come from delta are in our radius of epsilon

Definition of Real Limits at $+\infty$

$$\lim_{x \to \infty} f(x) \iff \forall \epsilon > 0 \exists \delta > 0, (x > \delta \Rightarrow \mid f(x) - L \mid < \epsilon)$$

Notice that with our accumulation point at $+\infty$ our condition for δ is that our x is always greater than δ which is to say, for any number you pick for δ I'll choose an x larger than that so that we can consider our x at $+\infty$

Definition of Real Limits at -∞

$$\lim_{x \to -\infty} f(x) = L \in \mathbb{R} \iff \forall \epsilon > 0 \exists \delta > 0, (x < -\delta \Rightarrow \mid f(x) - L \mid < \epsilon)$$

At this point, we can start observing the similarities between our limit definitions and what changes. Note that here, since our x is approaching -∞ we make our x smaller than -δ for any delta we pick.

Definition of Infinite Limits at real accumulation points.

$$\lim_{x \to x_0} f(x) = +\infty \iff \forall \epsilon > 0 \exists \delta > 0, (x < -\delta \Rightarrow \mid f(x) - L \mid < \epsilon)$$

Definition of Infinite Limits at positive infinity

$$\lim_{x \to \infty} f(x) = +\infty \iff \forall \epsilon > 0 \exists \delta > 0, (x < -\delta \Rightarrow \mid f(x) - L \mid < \epsilon)$$

Definition of Infinite limits at negative infinity

$$\lim_{x \to -\infty} f(x) = +\infty \iff \forall \epsilon > 0 \exists \delta > 0, (x < -\delta \Rightarrow \mid f(x) - L \mid < \epsilon)$$

Example 1.) Proof of a Real limit at a real accumulation point.
Prove that $\lim_{x \to 0} x^2 + x^3 = 0$