HW 3

MTH 230

1.) $x^2 + y^2 = 11$

prove for all $x, y \in \mathbb{Z}$. $x^2 + y^2 \neq 1$

$x^2 = 11 - y^2$, $x^2 \geq 0$ by properties of even exponentiation

$\Rightarrow 11 - y^2 \geq 0 \Rightarrow 11 \geq y^2$, $y^2 \leq 11$ $\boxed{\text{w.l.o.g.}}$

- proof by Cases :

(1) let $x = 0$, $0^2 + y^2 = 11 \Rightarrow y^2 = 11 \Rightarrow y = \sqrt{11} \notin \mathbb{Z}$

(2) let $x = 1$, $1^2 + y^2 = 11 \Rightarrow y^2 = 10 \Rightarrow y = \sqrt{10} \notin \mathbb{Z}$

(3) let $x = 2$, $2^2 + y^2 = 11 \Rightarrow y^2 = 7 \Rightarrow y = \sqrt{7} \notin \mathbb{Z}$

(4) let $x = 3$, $3^2 + y^2 = 11 \Rightarrow y^2 = 2 \Rightarrow y = \sqrt{2} \notin \mathbb{Z}$

2.) $\sum_{n=1}^{N} (2n-1) = N^2$, $\forall N \in \mathbb{Z}$, $N \geq 1$ $\quad 2(k+1) - 1$

(Base case), $n = 1$

$2n - 1 = N^2 \Leftrightarrow 2 \cdot 1 - 1 = 1^2 \Leftrightarrow 1 = 1$ ✓

(IH) Assume that $\forall k \in \mathbb{Z}$, $1 \leq k < N$

$\sum_{n=1}^{k} 2n-1 = k^2$, Now we must demonstrate equality for $k+1$

If $\sum_{n=1}^{k} 2n-1 = 1 + 3 + \ldots 2k-1 = k^2$

$\Rightarrow \left(\sum_{n=1}^{k} 2n-1\right) + (k+1) = 1 + 3 + \ldots (2k-1) + 2k+1 = k^2 \cancel{+ (k+1)}$

$\Rightarrow \sum_{n=1}^{k} (2n-1) + (k+1) = \cancel{(2k+2-1) + k^2 = k^2 + (2k) + 1}$

$= k^2 + k + 1 = 1 + 3 + \ldots (2k-1) + (2k+1)$

$\Leftrightarrow (k+1)^2 = 1 + 3 + \ldots (2k-1) + 2k+1$

thus, by the principle of mathematical Induction, it follows that $\forall n \geq 1$, $\sum_{i=1}^{n} 2i-1 = n^2$

3) Prove for all n $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(base case) for $n=1$, $\sum_{i=1}^{1} i^2 = 1^2 \iff \frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1$

(IH) assume that for some k, $1 \le k < n$

$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ holds true

$$\Rightarrow \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$\Rightarrow \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)\left(k(2k+1) + 6(k+1)\right)}{6} = \frac{(k+1)\left(2k^2+7k+6\right)}{6}$$

$$= \frac{(k+1)(2k+3)(k+2)}{6} \checkmark$$

thus, by the Principle of Mathematical Induction, for all $n \in \mathbb{N}$, $n \ge 1$ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ holds true

4.) Prove that every Integer can be written as $3k$, $3k+1$, or $3k-1$ for an Integer k.

Using modular arithmetic, $n = 3k \equiv 0 \bmod (3)$

$n = 3k+1 \equiv 1 \bmod(3)$ and ($\cancel{n = 3k+2 \equiv 2 \bmod(3)}$)

and $n = 3k-1 \equiv 2 \bmod(3)$

let us examine the cases of

Case 1 (

by the Euclidean Division theorem,

If

Prove that Every ~~number~~ Integer n, can be $3k$, $3k-1$, $3k+1$

Base Cases: for $n=0$, $n=3k$, $k=0$ ✓

for $n=1$, $n=3k+1$, $k=0$ ✓

for $n=2$, $n=3k-1$, $k=1$ ✓

Assume that for $n=m$, $n=m-1$, $n=m-2$, the statement holds. We want to show that this holds for $n=m+1$

if $m=3k$ then $n=m+1 \Rightarrow n=3k+1=3k+1=m+1$ ✓ $\equiv 1 \bmod(3)$

if $m=3k+1$ then $m+1=3k+2=3(k+1)-1$

$\Rightarrow$ $m+1$ is $3q-1$ and thus $\equiv 2 \bmod(3)$

if $m=3k-1$ then $m+1=3k \Rightarrow \equiv 0 \bmod(3)$

thus by mathematical Induction this property holds for all integers.

5.) let $p,q \in \mathbb{Z}$, show $\exists s \in \mathbb{N}$

such that (a) $\exists t \in \mathbb{N}$, $ps=qt$

pf:

~~Assume that~~

We want to show that $ps=qt$ which we can equivalently state as

want: $\frac{s}{t}=\frac{q}{p}$; let s be $\frac{lcm(p,q)}{p}$ and let t be the $\frac{lcm(p,q)}{q}$. back substituting yields

$\frac{lcm(p,q)}{p} \cdot p = \frac{lcm(p,q)}{q} \cdot q \iff lcm(p,q) = lcm(p,q)$ $\square$

5.)

(b) $p \cdot s' = q t'$, let $S := \{(s', t') \in \mathbb{Z} \mid p \cdot s' = q \cdot t', \; p, q \in \mathbb{Z}\}$

~~Sube~~ can observe that $S \neq \emptyset$ as
there exist multiple trivial solutions.
Since S is a set of well founded sets /
by the well ordering principle there exists
a minimal element for where $ps' = qt'$ and $s' \geq s$ □

6.) Show that any positive Integer can be
written as a sum of unique powers of 3, but
allowing both positive and negative signs

base case, for $n = 1$, $3^0 = 1$ ✓

(IH) Assume for any positive Integer less than n
can be expressed as a sum of unique powers
of 3, with coefficients 1 or -1.

consider a Positive Integer n,
we can always find the closest power of 3
that is less than or equal to n
if $3^k = n$, we are done.
if $3^k < n$ consider $\frac{n}{3^k}$ where $n = 3^k \cdot q + r$

$0 \leq r < 3^k$

Case 1

$r = 0$

$3^k \mid n$ by (IH) n can be written as a sum and differen of powers and 3.

Case 2: $0 < r < 3^k$

by (IH) r can be expressed as a unique sum of powers of by the two following cases

Either $0 < r < \frac{3^k}{2}$ (1) or $\frac{3^k}{2} < r < 3^k$ (2)

If (1) $n = 3^k \cdot q + r$, use (IH) express r

If (2) let $s = 3^k - r$

$0 < s < \frac{3^k}{2}$, $n = 3^k \cdot q + r$, $n = 3^k \cdot q + 3^k - s$

$n = 3^k(q+1) - s$

by (IH) s can be expressed as a sum of unique powers of 3 all less then 3^k with coefficients 1 or -1

by Principle of Mathematical Induction