MTH 230 HW 5 Esteban Morales

1.)

(a) If $3p^2 = q^2$, $p, q \in \mathbb{Z}$ Show that $3|p \wedge 3|q$

from our assumption, we can see Immediately that $3|q^2$ ($\exists p^2 \in \mathbb{Z} : 3 \cdot p^2 = q^2$)

If $3|q^2 \Rightarrow 3|q$ by the fundamental theorem of Arithmetic (unique prime factorization) we know that this is true. ($3|q \cdot q \Rightarrow 3|q$ since 3 is prime) so $q = 3k$, $k \in \mathbb{Z}$

$\Rightarrow 3p^2 = q^2 \Rightarrow 3p^2 = (3k)^2 \Leftrightarrow 3p^2 = 9k^2 \Leftrightarrow p^2 = 3k^2$

$\Rightarrow 3|p^2$ and by similar reasoning as before, $3|p$

(b) Assume there are positive Integers p and q with $3p^2 = q^2$. $q, p \in \mathbb{N}$

Use (a) and the well ordering principle

Assume (RAA) that these natural numbers p, q exist.

$p = 3k$ and $q = 3m$ by (a)

$\Rightarrow 3(3k)^2 = (3m)^2 = 3 \cdot 9k^2 = 9m^2 \Rightarrow \boxed{3k^2 = m^2}$ ☆

$\Rightarrow 3|m^2 \Rightarrow 3|m$ (by previous reasoning 1.(a)) $\Rightarrow m = 3v$, $v \in \mathbb{N}$

$\Rightarrow$ ☆: $3k^2 = (3v)^2 \Rightarrow 3k^2 = 9v^2 \Rightarrow k^2 = 3v^2 \Rightarrow 3|k^2 \Rightarrow 3|k$

$\Rightarrow k = 3x$, $x \in \mathbb{N}$ $\Rightarrow$ this process can repeat Indefinitely,

by the W.O. Principle there exist a least element in a subset of $\mathbb{N}$.

thus our assumption leads to a Contradiction and $\nexists q, p \in \mathbb{N}$ $3p^2 = q^2$

$\square$

2.) find all Integer Solutions to $15s + 25t = 10$

$25 = (1)15 + 10$
$15 = (1)10 + 5$
$10 = (2)5 + 0$

$\Rightarrow \quad 10 = (2)5$

$\Rightarrow 10 = 2(15-10) = 2\cdot(15) - 2(10)$

$\Rightarrow 10 = 2\cdot(15) - 2(25-15)$

$\Rightarrow 10 = 4(15) - 2(25)$

$s_0 = 4 \;, \; t_0 = -2$

$$\Rightarrow \quad s = 4 + \tfrac{25}{5}q$$
$$t = -2 - \tfrac{15}{5}q$$

3.) Solve the congruences

(a) $\quad 5x + 4 \equiv 7 \pmod 9 \iff 5x - 3 = 9k, \; k \in \mathbb{Z}$

$\iff 9 \mid 5x - 3 \iff 5x - 9k = 3$

$9 = (1)5 + 4$
$5 = (1)4 + 1$
$4 = 4(1) + 0$
$\gcd(9,5) = 1$

$1 = 5 - 4(1) \Rightarrow 1 = 5 - (1)(9-5)$

$\Rightarrow 1 = 2(5) - 9(1)$

$\Rightarrow 3 = 3(2(5) - 9(1)) \Rightarrow 3 = 6(5) - 27(1)$
$= 6(5) - 9(3)$

$\Rightarrow x_0 = 6 \;, \; k_0 = 3$

General Solution: $\quad x = 6 + 9p \;, \; p \in \mathbb{Z}$
$K = 3 + 5p$

3.)

(b) $7x \equiv 1 \pmod{11} \iff 7x - 1 = 11 \cdot k \ , k \in \mathbb{Z}$

$\Rightarrow \quad 7x - 11k = 1$

$11 = 7(1) + 4$

$7 = 4(1) + 3$

$4 = 3(1) + 1 \qquad\qquad 1 = 4 - 3(1) \Rightarrow 1 = 4 - (7 - 4(1)) = 2(4) - 7$

$3 = 3(1) + 0 \qquad\qquad 1 = 2(11 - 7) - 7 = 2(11) - 3(7)$

$x_0 = 3 \qquad k_0 = 2$

$x = 3 + 11n \ , n \in \mathbb{Z}$

$k = 2 + 7n$

4.) $\forall n \in \mathbb{Z}, \exists! k \in \mathbb{Z}$ s.t. $|n-7k| \leq 3$

by the Euclidean Division Algorithm

$\forall n \in \mathbb{Z}, \exists! k, r \in \mathbb{Z}$ s.t. $n = 7k + r$, $\boxed{0 \leq r < 7}$

We wish to claim that the max of r is 3 and the min is -3

$r \in \{-3, -2, -1, 0, 1, 2, 3\}$

cases:

If $r = 0, 1, 2, 3$ then our innequality holds trivially, that is

$n = 7k + [0, 1, 2, 3]$ then we are done

If $r = 4$

$n = 7k + 4 \Rightarrow n = 7(k+1) - 3 \Rightarrow \boxed{r = -3}$

$n = 7k + 7 - 3$

$= 7k + 4$

If $r = 5$, $n = 7k + 5$, $n = (7(k+1) - 2) \Rightarrow \boxed{r = -2}$

$\Rightarrow n = 7k + 7 - 2$

$n = 7k + 5$

If $r = 6$, $n = 7k + 6$, $n = 7(k+1) - 1 \Rightarrow \boxed{r = -1}$

$= 7k + 7 - 1$

$7k + 6$

by the Euclidean Division Algorithm, this remainder is unique as we can take our $7(k+1)$ remainders and continuously transpose them over by 1 so our uniqueness will still hold, It would just be shifted over.

5.)

$$6s + 10t + 15v = 1$$

$\cancel{q = 6s + 10t}$ $\quad q = 6s + 15v \quad\quad q - 6s = 15v$

$\cancel{10 = 6 + 4}$

$\cancel{6 = 4(1) + 2}$

$\cancel{4 = 2(2) + 0}$

$\Rightarrow 15 = 2(6) + 3$

$6 = 2(3) + 0$

$q3 = q(15 - 2(6))$ $\quad\Rightarrow 1 = 5 - 2(2)$

$q =$

$s_{q_0} = 1q$

$v_{q_0} = -2vq$

$$10t + 3q = 1$$

$10 = 3(3) + 1$ $\quad 1 = 10 - 3(3)$ $\quad t = 1 + 3k$

$3 = 3(1) + 0$ $\quad q = 3 + 10k, \; k \in \mathbb{Z}$

$t_0 = 1$

$3q_0 = 3$

$S = 1q$

$U = -2q$

6.) $N = 250943$. take the last digit, double it, and subtract from N with $_{n-1}$ digits. ① $2 \cdot 3 = 6$ $\quad 25094 - 6 = 25088$

② 25088, $8 \cdot 2 = 16$, $\quad 2508 - 16 = 2492$

③ $2 \cdot 2 = 4$ $\quad 249 - 4 = 245$ ④ $5 \cdot 2 = 10$, $\quad 24 - 10 = 14$

14 is Divisible by 7 so 250943 is also divisible by 7.

Pf: $N = 10\alpha + b \Rightarrow N = 10\alpha + b \equiv 3a + b \pmod 7$

$\Rightarrow N \equiv 3\alpha + b \pmod 7 \Rightarrow 3\alpha + b - 2b = 3\alpha - b \equiv 0 \pmod 7$ $\square$