MTH 230

HW 4

Esteban Morales

1.) If $a \cdot c \mid b \cdot c \wedge c \neq 0, \Rightarrow a \mid b$

$a \cdot c \mid b \cdot c \Leftrightarrow \exists k \ (a \cdot c) \cdot k = b \cdot c, \ k \in \mathbb{Z}, \overset{\boxed{c \neq 0}}{\Rightarrow} a \cdot k = b$

$\Leftrightarrow a \mid b$. by definition of divisibility

2.) find gcd(123, 76) using e.e.a. Use result to find $s, t \in \mathbb{Z}$ s.t. $123s + 76t = 1$

$$\left.\begin{aligned}
123 &= (1) \cdot 76 + 47 \\
76 &= (1) \cdot 47 + 29 \\
47 &= (1) \cdot 29 + 18 \\
29 &= (1) \cdot 18 + 11 \\
18 &= (1) \cdot 11 + 7 \\
11 &= (1) 7 + 4 \\
7 &= 1 \cdot 4 + 3 \\
4 &= 1 \cdot 3 + 1 \\
3 &= 1 \cdot 3 + 0
\end{aligned}\right\} \Rightarrow \gcd(123, 76) = 1$$

$1 = 4 - (1)3$

$1 = 4 - (1)(7 - 4(1)) = 4 - (7 - 4) = 4 - 7 + 4 = (2)4 - (1)7$

$1 = 2(11 - (1)7) - 7 = 2(11 - 7) - 7 = 2(11) - 3(7)$

$1 = -3(18 - 11) + 2(11) = -3(18) + 5(11)$

MTH 230 HW 4 Esteban Morales ②

$1 = 5(29-18) - 3(18) = 5(29) - 8(18)$

$1 = -8(47-29) + 5(29) = -8(47) + 13(29)$

$1 = 13(76-47) - 8(47) = 13(76) - 21(47)$

$1 = -21(123-76) + 13(76) = -21(123) + 34(76) = 1$

Thus we have that for $s = -21$, $t = 34$

$123s + 76t = 1$

3.) $\mathcal{F} = \{\overset{f_0}{0}, \overset{f_1}{1}, \overset{f_2}{1}, \overset{f_3}{2}, \overset{f_4}{3}, \overset{f_5}{5}, \overset{f_6}{8}, \overset{f_7}{13} \ldots\}$

Show:

$$\sum_{i=0}^{n} f_i = f_{n+2} - 1$$

Pf: base case for $n=2$, $P(0)$ " $\sum_{i=0}^{0} f_i = f_{0+2} - 1$

$\Leftrightarrow f_0 = f_2 - 1 \Leftrightarrow 0 = 1 - 1 \Leftrightarrow 0 = 0$ ✓

(IH) Assume for some $k \in \mathbb{Z}$, $P(k)$ that is, $\sum_{i=0}^{k} f_i = f_{k+2} - 1$

$\Rightarrow$ Want to show $\sum_{i=0}^{k+1} f_i = f_{(k+1)+2} - 1$

by def of $f_n = f_{n-1} + f_{n-2}$

$$\sum_{i=0}^{k} f_i = f_{k+2} - 1 \Leftrightarrow \sum_{i=0}^{k} f_i + f_{k+1} = \overbrace{f_{k+2} - 1 + f_{k+1}}^{f_{k+3} \text{ by def}} \Rightarrow \sum_{i=0}^{k+1} f_i = f_{(k+1)+2} - 1$$ ✓

thus by PMI $P(n)$ holds for all $n \in \mathbb{N}$

for $k+1 \Rightarrow$ ... $a \in B(k+1$ □

MTH 230 HW 4

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4.) $\forall n \geq 18$, $\exists m, k \in \mathbb{Z}_{\geq 0}$

s.t. $n = 4m + 7k$

base Case, $n = 18$ $P(18)$ " $18 = 4m + 7k$ "

$\Leftrightarrow 18 = 4(1) + 7(2) \Leftrightarrow 18 = 18$ ✓

~~take~~

base case $n = 19$ $P(19)$ " $19 = 4m + 7k$ " $\Leftrightarrow 19 = 4 \cdot 3 + 7 \cdot 1$

$\Leftrightarrow 19 = 19$ ✓

base Case $n = 20$ $P(20)$ " $20 = 4m + 7k$ " $\Leftrightarrow 20 = 4(5) + 7(0)$

$\Leftrightarrow 20 = 20$ ✓

base Case $n = 21$ $P(21)$ " $21 = 4m + 7k$ " $\Leftrightarrow 21 = 4(0) + 7(3)$

$\Leftrightarrow 21 = 21$ ✓

(IH) Using Strong Induction assume $P(K)$ $\forall K \geq 18$

Want to Show $P(K+1)$

$\exists m, k$ s.t. $K+1 = 4m + 7k$

If $K+1 \equiv 0 \mod(4)$: $\exists m$ s.t. $K+1 = 4 \cdot m + 7(0)$ ✓

If $K+1 \equiv 1 \mod(4)$: $(K+1) - 1 = p \cdot q \Rightarrow$ by Strong Induction $K-3 \equiv 1 \mod 4$

and we Know $\exists m, k$ s.t. $K-3 = 4m + 7k$. If we incremat m by 1

$\Rightarrow K+1 = 4(m+1) + 7k$ ✓

If $K+1 \equiv 2 \mod (4)$: $\Rightarrow 4 \mid K-1$

- by Strong Induction for $K-3$, $\exists x, m$ s.t. $K-3 = 4m + 7x$

$\Rightarrow K-3 \equiv 2 \mod (4)$ and $K+1 = 4(m+1) + 7x$

If $K+1 \equiv 3 \mod (4)$:

by (IH) $K-3 \equiv 3 \mod (4) \Rightarrow K+1 = 4(m+1) + 7K$

thus for all possibilities, we can use our Strong Inductive Hypothesis and Guarantee that we can sum up to the next natural number. Thus by PMI, $P(K+1)$ holds and $\exists_{m,x} \; \forall n \geq 18$ $n = 4m + 7x$

5.) We cannot assume that the "last K horses" of our K+1 horses have the same color as we only know that our first K horses are of the same color. Our Inductive hypothesis does not hold for the K+1 step. the horses do not "overlap"

6.) $$B(n) = \begin{cases} B(1) = \{0\} = \{\emptyset\} \\ B(n) = \{m \mid (m-1) \in B(\frac{n}{2})\} & \text{if } n = 2k \\ B(n) = \{m \mid (m-1) \in B(\frac{n-1}{2})\} \cup \{0\} & \text{if } n = 2k+1 \end{cases}$$

a.) $B(113) = \{m : m-1 \in B(56)\} \cup \{0\} \mid m = \{6, 5, 4, 0\}$

$B(56) = \{m : m-1 \in B(28)\} \mid m = \{5, 4, 3\}$

$B(28) = \{m : m-1 \in B(14)\} \mid m = \{4, 3, 2\}$

$B(14) = \{m : m-1 \in B(7)\} \mid m = \{3, 2, 1\}$

$B(7) = \{m : m-1 \in B(3)\} \cup \{0\} \mid m = \{2, 1, 0\}$

$B(3) = \{m : m-1 \in B(1)\} \cup \{0\} \mid m = \{1, 0\}$

$B(1) := 0$

$\Rightarrow \quad 113 = 2^6 + 2^5 + 2^4 + 2^0$ ✓

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b.) pf: base Case $n = 1$, by def, algorithm terminates Since $B(1) := 0$ ✓

Assume that for $B(K)$, $B(K)$ terminates and ever $B(K-n)$ terminates. Want to show $B(K+1)$ terminates

If $K+1 \equiv 0 \bmod (2) \Rightarrow B(K+1) := \underset{m \mid m-1 \in}{B}(\frac{K+1}{2})$, where $\frac{K+1}{2} = q \in \mathbb{Z}$ and by Strong Induction Hypothesis, $B(q)$ terminates. thus $B(K+1)$ terminates as well.

If $K+1 \equiv 1 \mod(2)$ then $B(K+1) := m, m-1 \in B\left(\frac{(K+1)-1}{2}\right) \cup \{0\}$

note $\frac{(K+1)-1}{2} = p \in \mathbb{Z}$. $B(p)$ terminates by our Strong Induction Hypothesis. thus $B(K+1)$ also terminates.

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c.) $\forall n>0, \quad n = \sum_{a \in B(n)} 2^a$

Pf: base Case $n=1$ $\quad P(1)$ " $1 = \sum_{\{0\}} 2^0 \iff 1 = 1$ ✓

(IH) Assume that for some K, $P(K)$ Holds true

that is $K = \sum_{a \in B(K)} 2^a$

Show that $K+1 = \sum_{a \in B(K+1)} 2^a$

we know : $K = \sum_{a \in B(K)} 2^a \implies K+1 = \left(\sum_{a \in B(K)} 2^a\right) + \underbrace{1}_{\text{Disappears}} \iff K+1 = \sum_{a \in B(K+1)} 2^a$

note: If $K \equiv 0 \mod(2) \implies B(K) = B(K+1) \cup \{0\} \implies 2^0 = 1$

If $K \equiv 1 \mod(2) \implies K+1 \equiv 0 \mod(2) \implies 0 \in B(K)$

and by def $p = \frac{K+1}{2}$, where $p = \sum_{a \in B(p)} 2^a \implies$ we can multiply p by 2 and receive our unique sum of Powers of 2 for $K+1 \implies K+1 = \sum_{a \in B(K+1)} 2^a$ □