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A Factorization Strategy for Tensor Robust PCA

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In this file, proofs of the technical theorems and lemmas are given.

Appendix A. Proof of Lemma 1 and Theorem 1

Proof of Lemma 1. Let the full t-SVD of $\mathcal{X}$ be $\mathcal{X} = \mathcal{U} * \underline{\boldsymbol{\Lambda}} * \mathcal{V}^\top$, where $\mathcal{U}, \mathcal{V} \in \mathbb{R}^{r\times r\times d_3}$ are orthogonal tensors and $\underline{\boldsymbol{\Lambda}} \in \mathbb{R}^{r\times r\times d_3}$ is f-diagonal. Then

$$\|\mathcal{X}\|_\star = \left\|\overline{\mathcal{U} * \underline{\boldsymbol{\Lambda}} * \mathcal{V}^\top}\right\|_* = \left\|\overline{\mathcal{U}} \cdot \overline{\underline{\boldsymbol{\Lambda}}} \cdot \overline{\mathcal{V}^\top}\right\|_* = \left\|\overline{\underline{\boldsymbol{\Lambda}}}\right\|_*. \tag{22}$$

Then $\mathcal{P} * \mathcal{X} * \mathcal{Q}^\top = (\mathcal{P} * \mathcal{U}) * \underline{\boldsymbol{\Lambda}} * (\mathcal{Q} * \mathcal{V})^\top$. Since

$$\begin{aligned} (\mathcal{P} * \mathcal{U})^\top * (\mathcal{P} * \mathcal{U}) &= \mathcal{U}^\top * \mathcal{P}^\top * \mathcal{P} * \mathcal{U} = \mathcal{I}, \\ (\mathcal{Q} * \mathcal{V})^\top * (\mathcal{Q} * \mathcal{V}) &= \mathcal{V}^\top * \mathcal{Q}^\top * \mathcal{Q} * \mathcal{V} = \mathcal{I}, \end{aligned} \tag{23}$$

we obtain that

$$\begin{aligned} \|\mathcal{P} * \mathcal{X} * \mathcal{Q}^\top\|_\star &= \left\|\overline{\mathcal{P} * \mathcal{X} * \mathcal{Q}^\top}\right\|_* \\ &= \left\|\overline{(\mathcal{P} * \mathcal{U}) * \underline{\boldsymbol{\Lambda}} * (\mathcal{Q} * \mathcal{V})^\top}\right\|_* \\ &= \left\|\overline{(\mathcal{P} * \mathcal{U})} \cdot \overline{\underline{\boldsymbol{\Lambda}}} \cdot \overline{(\mathcal{Q} * \mathcal{V})^\top}\right\|_* \\ &= \left\|\overline{\underline{\boldsymbol{\Lambda}}}\right\|_*. \end{aligned} \tag{24}$$

Thus, $\|\mathcal{P} * \mathcal{X} * \mathcal{Q}^\top\|_\star = \|\mathcal{X}\|_\star$. □

Proof of Theorem 1. Note that $(\mathcal{P}_* * \mathcal{C}_* * \mathcal{Q}_*^\top, \mathcal{S}_*)$ is a feasible point of Problem (8), then we have

$$\begin{aligned} \|\mathcal{L}^\star\|_\star + \lambda\|\mathcal{S}^\star\|_1 &\le \|\mathcal{P}_* * \mathcal{C}_* * \mathcal{Q}_*^\top\|_\star + \lambda\|\mathcal{S}_*\|_1 \\ &= \|\mathcal{C}_*\|_\star + \lambda\|\mathcal{S}_*\|_1 \end{aligned} \tag{25}$$

By the assumption that $r_{\mathrm{t}}(\mathcal{L}^\star) \le r$, there exists a decomposition $\mathcal{L}^\star = \mathcal{P}^\star * \mathcal{C}^\star * (\mathcal{Q}^\star)^\top$, such that $(\mathcal{P}^\star, \mathcal{C}^\star, \mathcal{Q}^\star, \mathcal{S}^\star)$ is also a feasible point of Problem (11).

Moreover, since $(\mathcal{C}_*, \mathcal{S}_*)$ is a global optimal solution to Problem (11), then we have that

$$\|\mathcal{C}_*\|_\star + \lambda\|\mathcal{S}_*\|_1 \le \|\mathcal{C}^\star\|_\star + \lambda\|\mathcal{S}^\star\|_1. \tag{26}$$

By $\mathcal{L}^\star = \mathcal{P}^\star * \mathcal{C}^\star * (\mathcal{Q}^\star)^\top$, we have

$$\|\mathcal{L}^\star\|_\star = \|\mathcal{P}^\star * \mathcal{C}^\star * (\mathcal{Q}^\star)^\top\|_\star = \|\mathcal{C}^\star\|_\star. \tag{27}$$

Thus, we deduce

$$\|\mathcal{C}_*\|_\star + \lambda\|\mathcal{S}_*\|_1 \le \|\mathcal{L}^\star\|_\star + \lambda\|\mathcal{S}^\star\|_1. \tag{28}$$

According to Eqs. (25) and (28), we further have

$$\|\mathcal{C}_*\|_\star + \lambda\|\mathcal{S}_*\|_1 = \|\mathcal{L}^\star\|_\star + \lambda\|\mathcal{S}^\star\|_1. \tag{29}$$

In this way, $(\mathcal{P}_* * \mathcal{C}_* * \mathcal{Q}_*^\top, \mathcal{S}_*)$ is also the optimal solution to the TRPCA Problem (8). □

Appendix B. Proof of Lemma 2

Proof of Lemma 2.

$$\begin{aligned}\|\mathcal{P} * \mathcal{A} - \mathcal{B}\|_{\mathrm{F}}^2 &= \|\mathcal{P} * \mathcal{A} - \mathcal{B}\|_{\mathrm{F}}^2 \\ &= \|\mathcal{P} * \mathcal{A}\|_{\mathrm{F}}^2 + \|\mathcal{B}\|_{\mathrm{F}}^2 - 2\left\langle \mathcal{P} * \mathcal{A}, \mathcal{B}\right\rangle\end{aligned} \tag{30}$$

Since $\mathcal{P}^\top * \mathcal{P} = \mathcal{I}_r$, we have that

$$\begin{aligned}\|\mathcal{P} * \mathcal{A}\|_{\mathrm{F}}^2 &= \frac{1}{d_3}\|\overline{\mathcal{P} * \mathcal{A}}\|_{\mathrm{F}}^2 = \frac{1}{d_3}\|\overline{\mathcal{P}} \cdot \overline{\mathcal{A}}\|_{\mathrm{F}}^2 \\ &= \frac{1}{d_3}\mathrm{Tr}\Big((\overline{\mathcal{P}} \cdot \overline{\mathcal{A}})^H(\overline{\mathcal{P}} \cdot \overline{\mathcal{A}})\Big) = \frac{1}{d_3}\mathrm{Tr}\Big((\overline{\mathcal{A}}^H\overline{\mathcal{A}}\Big) \\ &= \frac{1}{d_3}\|\overline{\mathcal{A}}\|_{\mathrm{F}}^2 = \|\mathcal{A}\|_{\mathrm{F}}^2\end{aligned} \tag{31}$$

Also, we have

$$\begin{aligned}\left\langle \mathcal{P} * \mathcal{A}, \mathcal{B}\right\rangle &= \frac{1}{d_3}\left\langle \overline{\mathcal{P} * \mathcal{A}}, \overline{\mathcal{B}}\right\rangle = \frac{1}{d_3}\left\langle \overline{\mathcal{P}} \cdot \overline{\mathcal{A}}, \overline{\mathcal{B}}\right\rangle \\ &= \frac{1}{d_3}\left\langle \overline{\mathcal{P}}, \overline{\mathcal{B}} \cdot \overline{\mathcal{A}}^H\right\rangle = \frac{1}{d_3}\sum_{k=1}^{d_3}\left\langle \widetilde{\mathcal{P}}^{(k)}, \widetilde{\mathcal{X}}^{(k)}\right\rangle,\end{aligned} \tag{32}$$

where $\mathcal{X} = \mathcal{B} * \mathcal{A}^\top$ and $\widetilde{\mathcal{X}} = \mathrm{fft}_3(\mathcal{X})$.

According to the trace inequality of Von Neuman, the inequality reaches its maximum when matrices $\widetilde{\mathcal{P}}^{(k)} \in \mathbb{C}^{d_1 \times r}$ and $\widetilde{\mathcal{X}}^{(k)} \in \mathbb{C}^{d_1 \times r}$ have the same right and left singular vectors.

We perform SVD on its first $\lceil \frac{d_3+1}{2} \rceil$ frontal slices $\widetilde{\mathcal{X}}^{(k)} \in \mathbb{C}^{d_1 \times}$ as follows

$$\widetilde{\mathcal{X}}^{(k)} = \boldsymbol{U}^{(k)}\boldsymbol{S}^{(k)}(\boldsymbol{V}^{(k)})^H, \ \forall k = 1, \cdots, \lceil \frac{d_3+1}{2} \rceil, \tag{33}$$

where $\boldsymbol{U}^{(k)} \in \mathbb{C}^{d_3 \times r}$ is a column-orthogonal matrix,$\boldsymbol{V}^{(k)} \in \mathbb{C}^{r \times r}$ is an orthogonal matrix, $\boldsymbol{S}^{(k)} = \mathrm{diag}(\sigma_1^{(k)}, ..., \sigma_r^{(k)})$, and $\sigma_1^{(k)} \geq \sigma_2^{(k)} \geq ... \geq \sigma_r^{(k)} \geq 0$ are the singular values of $\widetilde{\mathcal{X}}^{(k)}$. Using the relationships between FFT and t-SVD [13], we have that for all $k > \lceil \frac{d_3+1}{2} \rceil$, the frontal slice $\widetilde{\mathcal{X}}^{(k)}$ also has an SVD as

$$\begin{aligned}\widetilde{\mathcal{X}}^{(k)} &= \mathrm{conj}\big(\widetilde{\mathcal{X}}^{(d_3-k+2)}\big) \\ &= \mathrm{conj}\big(\boldsymbol{U}^{(d_3-k+2)}\big)\boldsymbol{S}^{(d_3-k+2)}\mathrm{conj}\big((\boldsymbol{V}^{(d_3-k+2)}\big)^H.\end{aligned} \tag{34}$$

Then, we construct a semi-orthogonal tensor $\mathcal{U} \in \mathbb{R}^{d_1 \times r \times d_3}$ and orthogonal tensor $\mathcal{V} \in \mathbb{R}^{r \times r \times d_3}$ as a pair of "singular vector tensors" of $\mathcal{X}$:

$$\widetilde{\mathcal{U}}^{(k)} = \begin{cases} \boldsymbol{U}^{(k)}, & k \leq \lceil \dfrac{d_3+1}{2} \rceil \\ \mathrm{conj}(\boldsymbol{U}^{(d_3-k+2)}), & k > \lceil \dfrac{d_3+1}{2} \rceil \end{cases} \tag{35}$$

and

$$\widetilde{\mathcal{V}}^{(k)} = \begin{cases} \boldsymbol{V}^{(k)}, & k \leq \lceil \frac{d_3+1}{2} \rceil \\ \mathrm{conj}(\boldsymbol{V}^{(d_3-k+2)}), & k > \lceil \frac{d_3+1}{2} \rceil \end{cases}. \tag{36}$$

Further, we construct $\mathcal{P} \in \mathbb{R}^{d_1 \times r \times d_3}$ by

$$\widetilde{\mathcal{P}}^{(k)} = \widetilde{\mathcal{U}}^{(k)} (\widetilde{\mathcal{V}}^{(k)})^H, \quad \forall k \leq d_3. \tag{37}$$

Thus we have $\mathcal{P}^\top * \mathcal{P} = \mathcal{I}$. Also, according to the trace inequality of Von Neuman, the left hand side of Eq. (32) get its maximum and thus Problem (14) get its minimum. □

Appendix C. Proof of Theorem 2

Before proving Theorem 2, we need the following lemmas.

Lemma 4. *[10] Let $\|\cdot\|$ denote any norm with dual norm $\|\cdot\|^*$. If $\boldsymbol{y} \in \partial \|\boldsymbol{x}\|$, then it holds that $\|\boldsymbol{y}\|^* \leq 1$.*

Lemma 5. *The sequence $\{\mathcal{Y}_t\}, \{\mathcal{Y}_t^1\}, \{\mathcal{Y}_{t+1}^2\}, \{\mathcal{Y}_{t+1}^3\}$ in Algorithm 1 are bounded.*

Proof. First, according to the optimality of $\mathcal{S}_{t+1}$ in Problem 18, we have that

$$\mathbf{0} \in \lambda \partial \|\mathcal{S}_{t+1}\|_1 + \mu_t (\mathcal{P}_{t+1} * \mathcal{C}_{t+1} * \mathcal{Q}_{t+1}{}^\top + \mathcal{S}_{t+1} - \mathcal{M} + \mathcal{Y}_t/\mu_t),$$

which means

$$-\mathcal{Y}_{t+1} \in \lambda \partial \|\mathcal{S}_{t+1}\|_1 \Rightarrow \|\mathcal{Y}_{t+1}\|_\infty \leq \lambda.$$

Thus, $\{\mathcal{Y}_t\}$ is a bounded sequence.

Then, according to the optimality of $\mathcal{Q}_{t+1}$ to Problem 16, we obtain

$$\|\mathcal{Y}_{t+1}^2\|_{\mathrm{F}} \leq \|\mathcal{Y}_{t+1}^3\|_{\mathrm{F}}.$$

Next, the optimality of $\mathcal{P}_{t+1}$ to Problem (13) leads to

$$\begin{aligned} \|\mathcal{Y}_{t+1}^3\|_{\mathrm{F}} \leq & \|\mathcal{P}_t * \mathcal{C}_t * \mathcal{Q}_t{}^\top + \mathcal{S}_t - \mathcal{M} + \mathcal{Y}_t/\mu_t\|_{\mathrm{F}} \\ = & \|\mathcal{Y}_t/\mu_{t-1} - \mathcal{Y}_{t-1}/\mu_{t-1} + \mathcal{Y}_t/\mu_t\|_{\mathrm{F}}. \end{aligned}$$

Since the boundedness of $\{\mathcal{Y}_t\}$ leads to the boundedness of $\{\mathcal{Y}_t^3\}$. Then $\{\mathcal{Y}_t^2\}$ is also bounded.

Using the optimality of $\mathcal{C}_{t+1}$ to Problem (17), we further have that

$$\mathbf{0} \in \partial \|\mathcal{C}_{t+1}\|_\star + \mu_t \big(\mathcal{C}_{t+1} + \mathcal{P}_{t+1}{}^\top * (\mathcal{S}_t - \mathcal{M} + \mathcal{Y}_t/\mu_t) * \mathcal{Q}_{t+1}\big),$$

which means

$$-{\mathcal{P}_{t+1}}^{\top}\mathcal{Y}^1_{t+1}\mathcal{Q}_{t+1} \in \partial\|\mathcal{C}_{t+1}\|_{\star} \Rightarrow \|{\mathcal{P}_{t+1}}^{\top}\mathcal{Y}^1_{t+1}\mathcal{Q}_{t+1}\| \le 1.$$

Let $\mathcal{P}^{\perp}_{t+1} = \mathcal{I} - \mathcal{P}_{t+1}$ and $\mathcal{Q}^{\perp}_{t+1} = \mathcal{I} - \mathcal{Q}_{t+1}$. Note that we have

$$\|(\mathcal{P}^{\perp}_{t+1})^{\top}\mathcal{Y}^1_{t+1}\mathcal{Q}_{t+1}\| = \|(\mathcal{P}^{\perp}_{t+1})^{\top}\mathcal{Y}^3_{t+1}\mathcal{Q}_{t+1}\| \le \|\mathcal{Y}^3_{t+1}\|.$$

Thus, $\{(\mathcal{P}^{\perp}_{t+1})^{\top}\mathcal{Y}^1_{t+1}\mathcal{Q}_{t+1}\}$ is bounded. Similarly, sequences $\{(\mathcal{P}^{\perp}_{t+1})^{\top}\mathcal{Y}^1_{t+1}\mathcal{Q}^{\perp}_{t+1}\}$ and $\{{\mathcal{P}_{t+1}}^{\top}\mathcal{Y}^1_{t+1}\mathcal{Q}^{\perp}_{t+1}\}$ are also bounded. In this way, $\{\mathcal{Y}^1_t\}$ is also bounded. □

Equipped with the above two lemmas, we are able to prove Theorem 2.

Proof of Theorem 2. First, according to the process of Algorithm 1, we have the following chain of inequalities of the Lagrangian:

$$\begin{aligned}
&L_{\mu_t}(\mathcal{P}_{t+1},\mathcal{C}_{t+1},\mathcal{Q}_{t+1},\mathcal{S}_{t+1},\mathcal{Y}_t)\\
&\le L_{\mu_t}(\mathcal{P}_{t+1},\mathcal{C}_{t+1},\mathcal{Q}_{t+1},\mathcal{S}_t,\mathcal{Y}_t)\\
&\le L_{\mu_t}(\mathcal{P}_{t+1},\mathcal{C}_t,\mathcal{Q}_{t+1},\mathcal{S}_t,\mathcal{Y}_t)\\
&\le L_{\mu_t}(\mathcal{P}_{t+1},\mathcal{C}_t,\mathcal{Q}_t,\mathcal{S}_t,\mathcal{Y}_t)\\
&\le L_{\mu_t}(\mathcal{P}_t,\mathcal{C}_t,\mathcal{Q}_t,\mathcal{S}_t,\mathcal{Y}_t)\\
&\le L_{\mu_{t-1}}(\mathcal{P}_t,\mathcal{C}_t,\mathcal{Q}_t,\mathcal{S}_t,\mathcal{Y}_{t-1}) + \frac{\mu_t+\mu_{t-1}}{2\mu_{t-1}^2}\|\mathcal{Y}_t-\mathcal{Y}_{t-1}\|_{\mathrm{F}}^2\\
&\le L_{\mu_0}(\mathcal{P}_1,\mathcal{C}_1,\mathcal{Q}_1,\mathcal{S}_1,\mathcal{Y}_0) + \sum_{s=1}^{t}\frac{\mu_s+\mu_{s-1}}{2\mu_{s-1}^2}\|\mathcal{Y}_s-\mathcal{Y}_{s-1}\|_{\mathrm{F}}^2\\
&\le L_{\mu_0}(\mathcal{P}_1,\mathcal{C}_1,\mathcal{Q}_1,\mathcal{S}_1,\mathcal{Y}_0) + \big(\max_s\|\mathcal{Y}_s-\mathcal{Y}_{s-1}\|_{\mathrm{F}}^2\big)\sum_{s=1}^{t}\frac{\mu_s+\mu_{s-1}}{2\mu_{s-1}^2}
\end{aligned}$$

Note that the quantity $\max_s\|\mathcal{Y}_s-\mathcal{Y}_{s-1}\|_{\mathrm{F}}^2$ in the above inequality is bounded, since $\{\mathcal{Y}_t\}$ is bounded. Recall the update of μ_t in Algorithm 1 $\mu_t = \rho\mu_{t-1} = \rho^t\mu_0$, then we show the quantity $\sum_{t=1}^{\infty}\frac{\mu_t+\mu_{t-1}}{2\mu_{t-1}^2}$ is also bounded, since

$$\sum_{t=1}^{\infty}\frac{\mu_t+\mu_{t-1}}{2\mu_{t-1}^2} = \frac{\rho+1}{2\mu_0}\sum_{t=1}^{\infty}\frac{1}{\rho^{t-1}} = \frac{\rho(\rho+1)}{2\mu_0(\rho-1)}.$$

Thus, $L_{\mu_{t-1}}(\mathcal{P}_t,\mathcal{C}_t,\mathcal{Q}_t,\mathcal{S}_t,\mathcal{Y}_{t-1})$ is bounded. Note that

$$\begin{aligned}
&L_{\mu_t}(\mathcal{P}_{t+1},\mathcal{C}_{t+1},\mathcal{Q}_{t+1},\mathcal{S}_{t+1},\mathcal{Y}_t)\\
&= \|\mathcal{C}_t\|_{\star} + \lambda\|\mathcal{S}_t\|_1 + \big\langle\mathcal{Y}_{t-1}, \mathcal{P}_t * \mathcal{C}_t * \mathcal{Q}_t^{\top} + \mathcal{S}_t - \mathcal{M}\big\rangle\\
&\qquad + + \frac{\mu_{t-1}}{2}\|\mathcal{P}_t * \mathcal{C}_t * \mathcal{Q}_t^{\top} + \mathcal{S}_t - \mathcal{M}\|_{\mathrm{F}}^2\\
&= \|\mathcal{C}_t\|_{\star} + \lambda\|\mathcal{S}_t\|_1 + \left\langle\mathcal{Y}_{t-1}, \frac{\mathcal{Y}_t-\mathcal{Y}_{t-1}}{\mu_{t-1}}\right\rangle\frac{\mu_{t-1}}{2}\|\frac{\mathcal{Y}_t-\mathcal{Y}_{t-1}}{\mu_{t-1}}\|_{\mathrm{F}}^2\\
&= \|\mathcal{C}_t\|_{\star} + \lambda\|\mathcal{S}_t\|_1 + \frac{1}{2\mu_{t-1}}\|\mathcal{Y}_t-\mathcal{Y}_{t-1}\|_{\mathrm{F}}^2.
\end{aligned}$$

Then, the sequence $\left\{\|\mathcal{C}_t\|_\star + \lambda\|\mathcal{S}_t\|_1\right\}$ is bounded.

According to the orthogonal invariance of TNN given in Lemma 1, we have

$$\|\mathcal{P}_t * \mathcal{C}_t * \mathcal{Q}_t^\top\|_\star = \|\mathcal{C}_t\|_\star.$$

Then, we obtain that $(\mathcal{C}_t, \mathcal{P}_t * \mathcal{C}_t * \mathcal{Q}_t^\top, \mathcal{S}_t)$ is bounded.

According to the process of Algorithm 1, we have

$$\begin{aligned} \mathcal{S}_{t+1} - \mathcal{S}_t &= \mu_t^{-1}(\mathcal{Y}_{t+1} - \mathcal{Y}_{t+1}^1) \\ \mathcal{C}_{t+1} - \mathcal{C}_t &= \mu_t^{-1}(\mathcal{P}_{t+1} * (\mathcal{Y}_{t+1}^1 - \mathcal{Y}_{t+1}^2) * \mathcal{Q}_{t+1}^\top) \end{aligned}$$

and the following relationships

$$\begin{aligned} \mathcal{P}_{t+1} * \mathcal{C}_{t+1} * \mathcal{Q}_{t+1}^\top - \mathcal{P}_t * \mathcal{C}_t * \mathcal{Q}_t^\top &= \mu_t^{-1}\big(\mathcal{Y}_{t+1}^1 + \mathcal{Y}_{t-1} - (1+\rho)\mathcal{Y}_t\big) \\ \mathcal{P}_{t+1} * \mathcal{C}_{t+1} * \mathcal{Q}_{t+1}^\top + \mathcal{S}_{t+1} - \mathcal{M} &= \mu_t^{-1}(\mathcal{Y}_{t+1} - \mathcal{Y}_t). \end{aligned}$$

By the update of $\mu_t = \rho\mu_{t-1}$ with $\rho = 1.1$ in Algorithm 1, we have the fact that $\lim_{t\to\infty} \mu_t = +\infty$. Combing the above with the boundedness of $\mathcal{Y}_t$ and $\mathcal{Y}_t^i,\ i = 1, 2, 3$, we have

$$\begin{aligned} \mu_t^{-1}(\mathcal{Y}_{t+1} - \mathcal{Y}_{t+1}^1) &\to \mathbf{0} \\ \mu_t^{-1}(\mathcal{Y}_{t+1}^1 - \mathcal{Y}_{t+1}^2) &\to \mathbf{0} \\ {\mu_t}^{-1}\big(\mathcal{Y}_{t+1}^1 + \mathcal{Y}_{t-1} - (1+\rho)\mathcal{Y}_t\big) &\to \mathbf{0} \\ \mu_t^{-1}(\mathcal{Y}_{t+1} - \mathcal{Y}_t) &\to \mathbf{0}. \end{aligned} \tag{38}$$

Then $\{\mathcal{S}_t\}, \{\mathcal{C}_t\}, \{\mathcal{P}_t * \mathcal{C}_t * \mathcal{Q}_t^\top\}$ are Cauchy sequences, and

$$\|\mathcal{P}_{t+1} * \mathcal{C}_{t+1} * \mathcal{Q}_{t+1}^\top + \mathcal{S}_{t+1} - \mathcal{M}\|_\infty \le \varepsilon. \tag{39}$$

□

References

1. Candès, E.J., Li, X., Ma, Y., Wright, J.: Robust principal component analysis? JACM **58**(3), 11 (2011)
2. Fazel, M.: Matrix rank minimization with applications. Ph.D. thesis, PhD thesis, Stanford University (2002)
3. Foucart, S., Rauhut, H.: A mathematical introduction to compressive sensing, vol. 1. Birkhäuser Basel (2013)
4. Friedland, S., Lim, L.: Nuclear norm of higher-order tensors. Mathematics of Computation **87**(311), 1255–1281 (2017)
5. Goldfarb, D., Qin, Z.: Robust low-rank tensor recovery: Models and algorithms. SIAM Journal on Matrix Analysis and Applications **35**(1), 225–253 (2014)
6. Harshman, R.A.: Foundations of the parafac procedure: Models and conditions for an "explanatory" multi-modal factor analysis (1970)
7. Hillar, C.J., Lim, L.: Most tensor problems are np-hard. Journal of the ACM **60**(6), 45 (2009)

8. Huang, B., Mu, C., Goldfarb, D., Wright, J.: Provable models for robust low-rank tensor completion. Pacific Journal of Optimization **11**(2), 339–364 (2015)
9. Kilmer, M.E., Braman, K., Hao, N., Hoover, R.C.: Third-order tensors as operators on matrices: A theoretical and computational framework with applications in imaging. SIAM Journal on Matrix Analysis and Applications **34**(1), 148–172 (2013)
10. Liu, G., Yan, S.: Active subspace: Toward scalable low-rank learning. Neural computation **24**(12), 3371–3394 (2012)
11. Liu, J., Musialski, P., Wonka, P., Ye, J.: Tensor completion for estimating missing values in visual data. IEEE TPAMI **35**(1), 208–220 (2013)
12. Lu, C., Feng, J., Chen, Y., Liu, W., Lin, Z., Yan, S.: Tensor robust principal component analysis: Exact recovery of corrupted low-rank tensors via convex optimization. In: CVPR. pp. 5249–5257 (2016)
13. Lu, C., Feng, J., Liu, W., Lin, Z., Yan, S., et al.: Tensor robust principal component analysis with a new tensor nuclear norm. IEEE TPAMI (2019)
14. Moosmann, F., Stiller, C.: Joint self-localization and tracking of generic objects in 3d range data. In: ICRA. pp. 1138–1144. Karlsruhe, Germany (May 2013)
15. Romera-Paredes, B., Pontil, M.: A new convex relaxation for tensor completion. In: NIPS. pp. 2967–2975 (2013)
16. Tucker, L.R.: Some mathematical notes on three-mode factor analysis. Psychometrika **31**(3), 279–311 (1966)
17. Wang, A., Jin, Z.: Near-optimal noisy low-tubal-rank tensor completion via singular tube thresholding. In: ICDM workshop. pp. 553–560 (2017)
18. Wang, A., Lai, Z., Jin, Z.: Noisy low-tubal-rank tensor completion. Neurocomputing **330**, 267–279 (2019)
19. Wang, A., Wei, D., Wang, B., Jin, Z.: Noisy low-tubal-rank tensor completion through iterative singular tube thresholding. IEEE Access **6**, 35112–35128 (2018)
20. Wu, T., Bajwa, W.U.: A low tensor-rank representation approach for clustering of imaging data. IEEE Signal Processing Letters **25**(8), 1196–1200 (2018)
21. Xie, Y., Tao, D., Zhang, W., Liu, Y., Zhang, L., Qu, Y.: On unifying multi-view self-representations for clustering by tensor multi-rank minimization. International Journal of Computer Vision **126**(11), 1157–1179 (2018)
22. Xu, Y., Hao, R., Yin, W., Su, Z.: Parallel matrix factorization for low-rank tensor completion. Inverse Problems and Imaging **9**(2), 601–624 (2015)
23. Zhang, Z., Aeron, S.: Exact tensor completion using t-svd. IEEE TSP **65**(6), 1511–1526 (2017)
24. Zhang, Z., Ely, G., Aeron, S., Hao, N., Kilmer, M.: Novel methods for multilinear data completion and de-noising based on tensor-svd. In: CVPR. pp. 3842–3849 (2014)
25. Zhou, P., Feng, J.: Outlier-robust tensor pca. In: CVPR (2017)