Numerical Computation Method in Solving Integral Equation by Using the Second Chebyshev Wavelets

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Abstract— *In this paper, a numerical method for solving the Fredholm and Volterra integral equations is presented. The method is based upon the second Chebyshev wavelet approximation. The properties of the second Chebyshev wavelet are first presented and then operational matrix of integration of the second Chebyshev wavelets basis and product operation matrix of it are derived. The second Chebyshev wavelet approximation method is then utilized to reduce the integral equation to the solution of algebraic equations combining Galerkin method. Some comparative examples are included to demonstrate superiority of operational matrix of the second Chebyshev wavelets to those of Legendre wavelets and CAS wavelets. It shows higher accuracy of the second Chebyshev wavelets method.*

Keywords: The second Chebyshev wavelets, Operational matrix of integration, Product operational matrix, Integral equation

1. Introduction

In recent years, wavelets have found their way into many different fields of science and engineering, particularly, wavelets are very successfully used in signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms[1]. The main advantage of wavelet method for solving the integral equation and differential equation is after discreting the coefficients matrix of algebraic equations is spare[2]. So, the computational cost is low.

Several wavelets methods for approximating the solution of the integral equations and differential equations are known. Haar wavelets method was presented in [3-5]. CAS wavelets method was developed in [6,7]. Harmonic wavelets method of successive approximation was introduced in [8]. In [9,10], E. Babolian applied operational matrix of integration of Chebyshev wavelets basis to the integral equations and differential equations and it was used in solving a nonlinear fractional differential equation in [11]. K. Maleknejad[12] introduced Legendre wavelets method for Fredholm and Volterra integral equations, while in [13] the integral and differential equations were solved by Legendre wavelets.

Here we will construct the second Chebyshev wavelets on the interval $[0, 1]$. The wavelets basis are suitable for numerical solutions of the integral equation.

2. Properties of the second Chebyshev wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function $\psi(x)$ called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously we have the following family of continuous wavelets as[2]

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right),\ a,\ b \in R, \quad a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k},\ b = nb_0a_0^{-k},\ a_0 > 1, \quad b_0 > 0$, we have the following family of discrete wavelets

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}}\psi(a_0^k t - nb_0), \quad k, n \in \mathbb{Z},$$

where $\psi_{k,n}$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$ then $\psi_{k,n}(t)$ form an orthonormal basis.

The second Chebyshev wavelets $\psi_{n,m}(t) = \psi(k, n, m, t)$ involve four arguments, $n = 1, ..., 2^{k-1}$, k is assumed any positive integer, m is the degree of the second Chebyshev polynomials and t is the normalized time. They are defined on the interval $[0, 1)$ as

$$\psi_{nm}(t) = \begin{cases} 2^{\frac{k}{2}}\tilde{\mathrm{U}}_{\mathrm{m}}(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leqslant t < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where

$$\tilde{\mathrm{U}}_{\mathrm{m}}(t) = \sqrt{\frac{2}{\pi}}\mathrm{U}_{\mathrm{m}}(t), \tag{2}$$

and $m = 0, 1, \ldots, M-1$. In Eq. (2) the coefficients are used for orthonormality. Here $U_m(t)$ are the second Chebyshev polynomials of degree m which respect to the weight function $\omega(t) = \sqrt{1-t^2}$ on the interval $[-1, 1]$ and satisfy the following recursive formula

$$U_0(t) = 1, \quad U_1(t) = 2t,$$

$$U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), \quad m = 1, 2, \cdots$$

We should note that in dealing with the second Chebyshev wavelets the weight funtion $\tilde{\omega}(t) = \omega(2t-1)$ have to be dilate and translate as

$$\omega_n(t) = \omega(2^k t - 2n + 1),$$

A function $f(t)$ defined over $[0, 1)$ may be expanded as

$$f(t) = \sum_{n=0}^{\infty} \sum_{m \in z} c_{nm} \psi_{nm}(t), \tag{3}$$

where

$$c_{nm} = (f(t), \psi_{nm}(t))_{\omega_n} = \int_0^1 \omega_n(t) \psi_{nm}(t) f(t)\,\mathrm{d}t, \tag{4}$$

in which $(\cdot\ ,\ \cdot)$ denotes the inner product in $L^2_{\omega_n}[0, 1]$. If the infinite series in Eq. (3) is truncated, then it can be written as

$$f(t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(t) = C^T \Psi(t), \tag{5}$$

where C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$\begin{aligned} C = [&c_{10}, c_{11}, \ldots, c_{1(M-1)}, c_{20}, \ldots, c_{2(M-1)}, \ldots, \\ &c_{2^{k-1}0}, \ldots, c_{2^{k-1}(M-1)}]^T \end{aligned} \tag{6}$$

and

$$\begin{aligned} \Psi(t) = [&\psi_{10}, \psi_{11}, \ldots, \psi_{1(M-1)}, \psi_{20}, \ldots, \psi_{2(M-1)}, \\ &\ldots, \psi_{2^{k-1}0}, \ldots, \psi_{2^{k-1}(M-1)}]^T. \end{aligned} \tag{7}$$

Similarly, a function $k(x, t) \in L^2_{\omega_n}([0, 1] \times\ [0, 1])$ may be approximated as

$$k(x, t) = \Psi(x)^T K \Psi(t), \tag{8}$$

where K is $2^{k-1}M \times 2^{k-1}M$ matrix with

$$K_{ij} = (\psi_i(x), (k(x, t), \psi_j(t))). \tag{9}$$

3. Operational matrix of integration and product operation matrix

In this section we will first derive the operational matrix P of integration[14−16] which plays a great role in dealing with the problem of integro-differential equations and Volterra integral equations. First we construct the 6×6 matrix P for $k = 2$ and $M = 3$. In this case, the six basis functions are given by

$$\left.\begin{aligned} \psi_{10}(t) &= 2\sqrt{\tfrac{2}{\pi}}, \\ \psi_{11}(t) &= 2\sqrt{\tfrac{2}{\pi}}(8t-2), \\ \psi_{12}(t) &= 2\sqrt{\tfrac{2}{\pi}}(64t^2 - 32t + 3), \end{aligned}\right\} \quad 0 \leqslant t < \frac{1}{2}, \tag{10}$$

$$\left.\begin{aligned} \psi_{20}(t) &= 2\sqrt{\tfrac{2}{\pi}}, \\ \psi_{21}(t) &= 2\sqrt{\tfrac{2}{\pi}}(8t-6), \\ \psi_{22}(t) &= 2\sqrt{\tfrac{2}{\pi}}(64t^2 - 96t + 35), \end{aligned}\right\} \quad \frac{1}{2} \leqslant t < 1. \tag{11}$$

By integrating (10) and (11) from 0 to t and representing it to the matrix form, we obtain

$$\begin{aligned} \int_0^t \psi_{10}(t')\,\mathrm{d}t' &= \begin{cases} 2\sqrt{\dfrac{2}{\pi}}t, & 0 \leqslant t < \frac{1}{2}, \\ \sqrt{\dfrac{2}{\pi}}, & \frac{1}{2} \leqslant t < 1. \end{cases} \\ &= \frac{1}{4}\psi_{10}(t) + \frac{1}{8}\psi_{11}(t) + \frac{1}{2}\psi_{20}(t) \\ &= \left[\frac{1}{4}, \frac{1}{8}, 0, \frac{1}{2}, 0, 0\right] \Psi_6(t), \end{aligned}$$

$$\begin{aligned} \int_0^t \psi_{11}(t')\,\mathrm{d}t' &= \begin{cases} 4\sqrt{\dfrac{2}{\pi}}(2t^2 - t), & 0 \leqslant t < \frac{1}{2}, \\ 0, & \frac{1}{2} \leqslant t < 1, \end{cases} \\ &= -\frac{3}{16}\psi_{10}(t) + \frac{1}{16}\psi_{12}(t) \\ &= \left[-\frac{3}{16}, 0, \frac{1}{16}, 0, 0, 0\right] \Psi_6(t). \end{aligned}$$

Similarly we have

$$\begin{aligned} \int_0^t \psi_{12}(t')\,\mathrm{d}t' &= \frac{1}{12}\psi_{10}(t) - \frac{1}{24}\psi_{11}(t) \\ &= \left[\frac{1}{12}, -\frac{1}{24}, 0, 0, 0, 0\right] \Psi_6(t), \end{aligned}$$

$$\begin{aligned} \int_0^t \psi_{20}(t')\,\mathrm{d}t' &= \frac{1}{4}\psi_{20}(t) + \frac{1}{8}\psi_{21}(t) \\ &= \left[0, 0, 0, \frac{1}{4}, \frac{1}{8}, 0\right] \Psi_6(t), \end{aligned}$$

$$\begin{aligned} \int_0^t \psi_{21}(t')\,\mathrm{d}t' &= -\frac{3}{16}\psi_{20}(t) + \frac{1}{16}\psi_{22}(t) \\ &= \left[0, 0, 0, -\frac{3}{16}, 0, \frac{1}{16}\right] \Psi_6(t), \end{aligned}$$

$$\begin{aligned} \int_0^t \psi_{22}(t')\,\mathrm{d}t' &= \frac{1}{12}\psi_{20}(t) - \frac{1}{24}\psi_{21}(t) \\ &= \left[0, 0, 0, \frac{1}{12}, -\frac{1}{24}, 0\right] \Psi_6(t). \end{aligned}$$

Thus

$$\int_0^t \Psi_6(t')\,\mathrm{d}t' = P_{6\times6}\Psi_6(t), \tag{12}$$

where

$$\Psi_6(t) = [\psi_{10}, \psi_{11}, \psi_{12}, \psi_{20}, \psi_{21}, \psi_{22}]^T$$

and

$$P_{6\times6} = \frac{1}{4}\begin{bmatrix} 1 & \frac{1}{2} & 0 & 2 & 0 & 0 \\ -\frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & 0 \end{bmatrix}.$$

In Eq. (12) the subscript of $P_{6\times 6}$ and Ψ_6 denote the dimension. In fact the matrix $P_{6\times 6}$ can be written as

$$P_{6\times 6} = \frac{1}{4}\begin{bmatrix} L_{3\times 3} & F_{3\times 3} \\ O_{3\times 3} & L_{3\times 3} \end{bmatrix},$$

where

$$L_{3\times 3} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ -\frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{3} & -\frac{1}{6} & 0 \end{bmatrix}, \qquad F_{3\times 3} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For general case, we have

$$\int_0^t \Psi(t')\,\mathrm{d}t' = P\Psi(t), \tag{13}$$

where P is a $2^{k-1}M \times 2^{k-1}M$ matrix for integration and is given as

$$P = \frac{1}{2^k}\begin{bmatrix} L & F & F & \cdots & F & F \\ O & L & F & \cdots & F & F \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & L & F \\ O & O & O & \cdots & O & L \end{bmatrix},$$

where F and L are $M \times M$ matrices given by

$$F = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$L = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \cdots & 0 \\ -\frac{3}{4} & 0 & \frac{1}{4} & \cdots & 0 \\ \frac{1}{3} & -\frac{1}{6} & 0 & \cdots & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{8} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{M-2}\frac{1}{M-1} & 0 & 0 & \cdots & \frac{1}{2(M-1)} \\ (-1)^{M-1}\frac{1}{M} & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then we will obtain the product operation matrix, which is important for solving Volterra integral equations.

Let

$$\Psi(t)\Psi^T(t)C \simeq \tilde{C}\Psi(t), \tag{14}$$

where $\widetilde{C}$ is $2^{k-1}M \times 2^{k-1}M$ product operation matrix. To illustrate the calculation procedure we choose $k = 2, M = 3$. Thus we have

$$C = [c_{10}, c_{11}, c_{12}, c_{20}, c_{21}, c_{22}]^T, \tag{15}$$

$$\Psi(t) = [\psi_{10}, \psi_{11}, \psi_{12}, \psi_{20}, \psi_{21}, \psi_{22}]^T, \tag{16}$$

where, the six basis functions are given by Eq. (10) and Eq. (11).

So we get

$$\Psi(t)\Psi^T(t) = \begin{bmatrix} \psi_{10}\psi_{10} & \psi_{10}\psi_{11} & \psi_{10}\psi_{12} & \psi_{10}\psi_{20} & \psi_{10}\psi_{21} & \psi_{10}\psi_{22} \\ \psi_{11}\psi_{10} & \psi_{11}\psi_{11} & \psi_{11}\psi_{12} & \psi_{11}\psi_{20} & \psi_{11}\psi_{21} & \psi_{11}\psi_{22} \\ \psi_{12}\psi_{10} & \psi_{12}\psi_{11} & \psi_{12}\psi_{12} & \psi_{12}\psi_{20} & \psi_{12}\psi_{21} & \psi_{12}\psi_{22} \\ \psi_{20}\psi_{10} & \psi_{20}\psi_{11} & \psi_{20}\psi_{12} & \psi_{20}\psi_{20} & \psi_{20}\psi_{21} & \psi_{20}\psi_{22} \\ \psi_{21}\psi_{10} & \psi_{21}\psi_{11} & \psi_{21}\psi_{12} & \psi_{21}\psi_{20} & \psi_{21}\psi_{21} & \psi_{21}\psi_{22} \\ \psi_{22}\psi_{10} & \psi_{22}\psi_{11} & \psi_{22}\psi_{12} & \psi_{22}\psi_{20} & \psi_{22}\psi_{21} & \psi_{22}\psi_{22} \end{bmatrix}.$$

Expanding each product by wavelet basis we have

$$\Psi(t)\Psi^T(t) = 2\sqrt{\frac{2}{\pi}}\begin{bmatrix} \psi_{10} & \psi_{11} & \cdots & 0 \\ \psi_{11} & \psi_{10}+\psi_{12} & \cdots & 0 \\ \psi_{12} & \psi_{11} & \cdots & 0 \\ 0 & 0 & \cdots & \psi_{22} \\ 0 & 0 & \cdots & \psi_{21} \\ 0 & 0 & \cdots & \psi_{20}+\psi_{22} \end{bmatrix}.$$

By using the vector C, the $\tilde{C}$ is

$$\tilde{C} = 2\sqrt{\frac{2}{\pi}}\begin{bmatrix} \tilde{C}_1 & O \\ O & \tilde{C}_2 \end{bmatrix},$$

where $\tilde{C}_i (i = 1, 2)$ are 3×3 matrices given by

$$\tilde{C}_i = \begin{bmatrix} c_{i0} & c_{i1} & c_{i2} \\ c_{i1} & c_{i0}+c_{i2} & c_{i1} \\ c_{i2} & c_{i1} & c_{i0}+c_{i2} \end{bmatrix}.$$

4. Solving linear integral equation

First, consider the following integral equation

$$y(x) = \int_0^1 k(x,t)y(t)\,\mathrm{d}t + f(x), \qquad x \in [0,1], \tag{17}$$

where $f(x) \in L^2_\omega([0,1], k(x,t) \in L^2_\omega([0,1] \times \ [0,1])$ are known and $y(t)$ is the unknown function to be determined. If we approximate f, y and k by the way mentioned before

$$y(x) = \Psi(x)^T C, \quad f(x) = \Psi(x)^T F, \\ k(x,t) = \Psi(x)^T K \Psi(t). \tag{18}$$

Substitute Eq. (18) into Eq. (17), we have

$$\begin{aligned} \Psi(x)^T C &= \Psi(x)^T F + \int_0^1 \Psi(x)^T K \Psi(t)\Psi(t)^T C \mathrm{d}t \\ &= \Psi(x)^T F + \Psi(x)^T K \left(\int_0^1 \Psi(t)\Psi(t)^T \mathrm{d}t\right) C \\ &= \Psi(x)^T (F + KDC), \end{aligned}$$

then

$$(I - KD)C = F, \tag{19}$$

where

$$D = \int_0^1 \Psi(t)\Psi(t)^T\,\mathrm{d}t.$$

Then, for the following Volterra integral equation

$$y(x)-\int_0^x k(x,t)y(t)\,\mathrm{d}t=f(x),\qquad x\in[0,1],\tag{20}$$

with Eq. (5), Eq. (9), Eq. (13) and Eq. (14) we have

$$\begin{aligned}\int_0^x k(x,t)y(t)\,\mathrm{d}t &\simeq \int_0^x \Psi(x)^T K\Psi(t)\Psi(t)^T C\mathrm{d}t\\ &= \Psi(x)^T K\left(\int_0^x \Psi(t)\Psi(t)^T C\mathrm{d}t\right)\\ &= \Psi(x)^T K\int_0^x \tilde{C}\Psi(t)\mathrm{d}t\\ &= \Psi(x)^T K\tilde{C}P\Psi(x).\end{aligned}$$

Then

$$\Psi^T(x)C=f(x)+\Psi(x)^T K\tilde{C}P\Psi(x).\tag{21}$$

By evaluating this equation in $2^{k-1}M$ points $\{x_i\}_{i=1}^{2^{k-1}M}$ in interval $[0,1)$ we have a system of linear equations

$$\Psi(x_i)^T C=f(x_i)+\Psi(x_i)^T K\tilde{C}P\Psi(x_i).\tag{22}$$

In calculating the elements of matrices of Galerkin method we often need to calculate the inner products of functions and the second Chebyshev wavelets basis. Here we discuss some formulae. By using p-points closed Gauss Chebyshev quadrature rule we have

$$\begin{aligned}&(f,\psi_{n,m})_{\omega_n}\\ &=\int_0^1 \omega_n(t)\psi_{n,m}(t)f(t)\,\mathrm{d}t\\ &=\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t)2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}\mathrm{U}_m(2^k t-2n+1)\omega(2^k t-2n+1)\,\mathrm{d}t\\ &=2^{-\frac{k}{2}}\sqrt{\frac{2}{\pi}}\int_{-1}^1 f\left(\frac{t+2n-1}{2^k}\right)\mathrm{U}_m(t)\omega(t)\,\mathrm{d}t\\ &\simeq 2^{-\frac{k}{2}}\sqrt{\frac{2}{\pi}}\frac{\pi}{p+1}\sum_{l=1}^{p} f\left(\frac{\cos(l\pi/(p+1))+2n-1}{2^k}\right)\\ &\sin\left[\frac{(m+1)l\pi}{p+1}\right]\sin\frac{l\pi}{p+1}.\end{aligned}$$

for $n=1,\cdots,2^{k-1}$, $m=0,1,\cdots,M-1$.

5. Numerical examples

For showing efficiency of our numerical method, we consider the following examples.

Example 1. Consider the Fredholm integral equation of the second kind

$$y(x)-x\int_0^1 t^2 y(t)\,\mathrm{d}t=\sin x-x(\cos 1+2\sin 1-2),\tag{23}$$

with the exact solution $y(x)=\sin x$. Table 1 shows the comparison of the absolute error between exact solution and approximate solution for $k=2, M=3$ among Legendre wavelets(Leg for short), CAS wavelets and the second Chebyshev wavelets(Che for short) methods. Where y and y_n in the Table 1 denote the exact solution and the numerical solution, respectively.

Table 1: Numerical results of Example 1

x_r	$\lvert y-y_n\rvert$		
	Che	Leg	CAS
0.0	0.001269	0.001013	0.123699
0.2	0.000235	0.000280	0.008219
0.4	0.000199	0.000358	0.008096
0.6	0.000181	0.000284	0.008002
0.8	0.000160	0.000219	0.006031
1.0	0.000936	0.000734	0.080036

Example 2. Consider the following equation

$$y(x)-x\int_0^1 ty(t)\,\mathrm{d}t=e^x-x,\tag{24}$$

with exact solution $y(x)=e^x$. Table 2 shows the comparison of the absolute error between exact solution and approximate solution for $k=2, M=4$ among Legendre wavelets(Leg for short), CAS wavelets and the second Chebyshev wavelets(Che for short) methods.

Table 2: Numerical results of Example 2

x_r	$\lvert y-y_n\rvert$		
	Che	Leg	CAS
0.0	0.000064	0.000047	0.140991
0.2	0.000007	0.000011	0.012311
0.4	0.000015	0.000020	0.004524
0.6	0.000025	0.000032	0.003513
0.8	0.000015	0.000019	0.023426
1.0	0.000114	0.000081	0.303905

Example 3. Consider the following Volterra integral equation[10,12]

$$y(x)-x\int_0^x (xt^2-t)y(t)\,\mathrm{d}t=-\frac{3}{4}x^6+\frac{1}{3}x^5+x^4-\frac{1}{2}x^3+3x-1,\tag{25}$$

with exact solution $y(x)=3x-1$. Table 3 shows the comparison of the absolute error between exact solution and approximate solution for $k=2, M=3$ among Legendre wavelets(Leg for short), Chebyshev wavelets[10] and the second Chebyshev wavelets(Che for short) methods.

Table 3: Numerical results of Example 3

x_r	$\lvert y-y_n\rvert$		
	Che	Leg	method in [10]
0.0	0.003479	0.000000	0.0000e-1
0.2	0.000182	0.000401	0.0234e-1
0.4	0.000124	0.001107	0.1084e-1
0.6	0.001746	0.002979	0.1743e-1
0.8	0.000055	0.003141	0.3524e-1
1.0	0.019876	0.009363	0.5923e-1

The results of Example 1 and Example 2 show that the second Chebyshev wavelets method is the same or sightly better than the Legendre case and is more better than the CAS wavelets method. Because CAS wavelets is a period wavelets, it is suitable for the periodic problems. The table of example 3 shows that the degree of accuracy of the second Chebyshev wavelets operational matrix method used for solving the Volterra integral equation is better than the Chebyshev wavelets and Legendre wavelets operational matrix method.

6. Conclusions

The second Chebyshev wavelets operational matrix of integration and its product operational matrix have been obtained in general and used for solving the integral equations. The present method reduces an integral equation into a set of algebraic equations. Some examples are included to demonstrate the superiority of our method. Moreover, the method in this paper can also be used for nonlinear integral equations and integro-differential equations.

References

[1] G. Beylkin, R. Coifman, and V. Rokhlin, "Fast wavelet transforms and numerical algorithms I," *Commun. Pure Appl. Math.*, vol. 44, pp. 141-183, Mar. 1991.

[2] Q. B. Fan, *Wavelet Analysis*. Wuhan, China: Wuhan University Press, 2008.

[3] K. Maleknejad, F. Mirzaee, "Using rationalized Haar wavelet for solving linear integral equations," *Appl. Math. Comp.*, vol. 160, pp. 579-587, Jan. 2005.

[4] E. Babolian, A. Shahsavaran, "Numerical solution of nonlinear Fredholm integral equations of the second kind using Haar wavelets," *Journal of Computational and Applied Mathematics*, vol. 225, pp. 87-95, Mar. 2009.

[5] Ü. Lepik, "Numerical solution of differential equations using Haar wavelets," *Mathematics and Computers in Simulation*, vol. 68, pp. 127-143, Apr. 2005.

[6] S. Yousefi, A. Banifatemi, "Numerical solution of Fredholm integral equations by using CAS wavelets," *Appl. Math. Comp.*, vol. 183, pp. 458-463, Dec. 2006.

[7] D. F. Han, X. F. Shang, "Numerical solution of integro-differential equations by using CAS wavelet operational matrix of integration," *Appl. Math. Comp.*, vol. 194, pp. 460-466, Dec. 2007.

[8] C. Cattani, A. Kudreyko, "Harmonic wavelet method towards solution of the Fredholm type integral equations of the second kind," *Appl. Math. Comp.*, vol. 215, pp. 4164-4171, Feb. 2010.

[9] E. Babolian, F. Fattahzadeh, "Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration," *Appl. Math. Comp.*, vol. 188, pp. 417-426, May. 2007.

[10] E. Babolian, F. Fattahzadeh, "Numerical computation method in solving integral equations by using Chebyshev wavelet operational matrix of integration," *Appl. Math. Comp.*, vol. 188, pp. 1016-1022, May. 2007.

[11] Y. L. Li, "Solving a nonlinear fractional differential equation using Chebyshev wavelets ," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, pp. 2284-2292, Sept. 2010.

[12] K. Maleknejad, M. T. Kajani, Y. Mahmoudi, "Numerical solution of linear Fredholm and Volterra integral equation of the second kind by using Legendre wavelets," *Kybernetes, Int. J. Sys. Math.*, vol. 32, pp. 1530-1539, 2003.

[13] X. Y. Zheng, X. F. Yang, "Techniques for solving integral and differential equations by Legendre wavelets," *International Journal of Systems Science.*, vol. 40, pp. 1127-1137, Nov. 2009.

[14] M. Razzaghi, S. Yousefi, " The Legendre wavelets operational matrix of integration," *Int. J. Syst. Sci.*, vol. 32, pp. 495-502, 2001.

[15] M. Razzaghi, S. Yousefi, "Legendre wavelets method for the solution of nonlinear problems in the calculus of variations," *Mathematical and Computer Modelling*, vol. 34, pp. 45-54, July. 2001.

[16] M. T. Kajani, A. H. Vencheh, and M. Ghasemi, "The Chebyshev wavelets operational matrix of integration and product operation matrix," *International Journal of Computer Mathematics*, vol. 86, pp. 1181-1125, July. 2009.