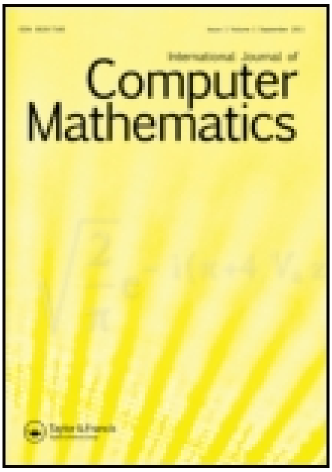

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M. R.A. Darani [a], H. Adibi [a] & M. Lakestani [b]

[a] Faculty of Mathematics and Computer Science, Amirkabir University of Technology, No. 424, Hafez Ave., Tehran, Iran

[b] Faculty of Mathematical Science, University of Tabriz, 19 Bahman blvd, Tabriz, Iran

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M.R.A. Darani[a], H. Adibi[a] and M. Lakestani[b]*

[a] *Faculty of Mathematics and Computer Science, Amirkabir University of Technology, No. 424, Hafez Ave., Tehran, Iran;* [b] *Faculty of Mathematical Science, University of Tabriz, 19 Bahman blvd, Tabriz, Iran*

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This paper is concerned with the construction of a biorthogonal multiwavelet basis in the unit interval to form a biorthogonal flatlet multiwavelet system. The system is then used to solve nonlinear ordinary differential equations. The biorthogonality and high vanishing moment properties of this system result in efficient and accurate solutions. Finally, numerical results for some test problems with known solutions are presented and the absolute errors are compared with the errors resulting from B-spline bases.

Keywords: operational matrix of derivative; flatlet; oblique multiwavelets; biorthogonal system

2000 AMS Subject Classifications: 65T50; 65L10; 65M70

1. Introduction

The advantages of multiwavelets, as extensions from scalar wavelets and their promising features have resulted in an increasing trend to study them. Features such as orthogonality, compact support, symmetry, high-order vanishing moments and the simple structure make multiwavelets useful both in theory and in applications such as signal compression and denoising [7,9,14,16]. A multiwavelet basis can be successfully used for representing differential operator modelling to solve partial differential equations [1,2,4]. In fact, the use of multiwavelet basis leads to sparse representation for a wide class of differential integral and integro-differential operators due to moments of the simple functions involved. In some works such as [3], representations of operators are constructed by using multiwavelets with the goal of developing adaptive solvers for linear and nonlinear partial differential equations. The use of operator modelling converts differential equations to systems of algebraic equations. A biorthogonal multiwavelet system (BMS) with multiplicity m consists of a pair of m biorthogonal multiscaling functions and a corresponding pair of m biorthogonal multiwavelets. It is known that short support and high vanishing moments are the

*Corresponding author. Email: lakestani@tabrizu.ac.ir

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two most important features of a BMS. The dual multiwavelets constructed in this paper provide a high order of polynomial reproduction and offer specified approximation order near the borders. Moreover, their important smoothness properties transform into polynomial reproductions in the discrete setting. In fact, the functions in the dual basis are smoother and have a higher polynomial reproduction than the flatlet refinable vector. For increasing m, the polynomial reproduction grows and this might be the reason why the absolute errors in the numerical experiments decrease as m increases: Since the true solution of the examples are power series, a higher m might allow for a higher order Taylor approximation of the solution. We refer the readers to [5,7,10] for more information about construction and samples of BMS and wavelets. Using multiwavelet basis provides a better approximation for problems having smooth solutions. It should be noted that despite the fact that smoothness is a suitable feature for problems such as signal processing and image decomposition, thus far, non-smooth wavelets have provided desirable results in certain numerical methods [3,6].

In this paper, we use flatlet multiwavelets with multiplicity m and will present an algebraic tool to extend them to the BMS. Also we derive an algorithm to compute the operational matrix of the derivative (OMD) for solving ordinary differential equations of the general form

$$y''(x) = f(x, y(x)), \quad x \in [0, 1], \tag{1}$$

$$y'(0) = y_0, \quad y'(1) = y_1. \tag{2}$$

Here f, is a known function, y_0 and y_1 are given real numbers and y is the unknown function to be found.

The existence of solution of Equation (1) with Neumann boundary conditions is studied in [11] using the quasi-linearization method. Also solving differential equations by using the wavelet method has been discussed in many papers [12,13,15]. For this purpose, different approaches such as the finite-element method, boundary element method, Galerkin and collocation methods are used. Wavelets and multiwavelets can be separated into three distinct types, orthogonal, semi-orthogonal and non-orthogonal(oblique). Here, we use oblique multiwavelet which will be discussed next. In this article, the functions are approximated by dual flatlet multiwavelets. Then the primal flatlet multiwavelets are used to obtain the coefficients of the expansions. This paper is organized as follows: In Section 2, we describe the formulations of the flatlet scaling functions and multiwavelets on $[0, 1]$. In Section 3, we construct biorthogonal basis for flatlet multiwavelets and discuss their existence. Also we derive the OMD required for our subsequent development. In Section 4, the proposed method is used to solve the second-order ordinary differential equations. In Section 5, we report our computational results and demonstrate the accuracy of the proposed numerical scheme by presenting numerical examples. Section 6 ends this paper with a brief conclusion.

2. Flatlet multiwavelet system

A flatlet multiwavelet system [8] with multiplicity $m + 1$ consists of $m + 1$ scaling functions and $m + 1$ wavelets defined on $[0, 1]$. The simplest example ($m = 0$) for the flatlet family is identical to the Haar wavelets. To construct higher-order flatlet multiwavelet system, we can follow the same procedures as Haar wavelets. The scaling functions in this system are defined as a set of $m + 1$ unit constant functions $\phi_0(x), \ldots, \phi_m(x)$ defined by

$$\phi_i(x) = \begin{cases} 1 & \dfrac{i}{m+1} \leq x < \dfrac{i+1}{m+1}, \\ 0 & \text{otherwise} \end{cases}, \quad i = 0, 1, \ldots, m. \tag{3}$$

Let $m+1$ functions $\psi_0(x), \ldots, \psi_m(x)$ be flatlet wavelets corresponding to flatlet scaling functions defined on [0, 1]. We construct corresponding wavelets by using a two-scale relation which will be introduced next. First for simplicity, we put flatlet scaling functions and wavelets into two vector functions

$$\Phi(x) = \begin{bmatrix} \phi_0(x) \\ \vdots \\ \phi_i(x) \\ \vdots \\ \phi_m(x) \end{bmatrix}, \quad \Psi(x) = \begin{bmatrix} \psi_0(x) \\ \vdots \\ \psi_i(x) \\ \vdots \\ \psi_m(x) \end{bmatrix}. \tag{4}$$

Now the two-scale relations for the flatlet multiwavelet system may be expressed as

$$\Phi(x) = \mathbf{P} \begin{bmatrix} \Phi(2x) \\ \Phi(2x-1) \end{bmatrix}, \quad \Psi(x) = \mathbf{Q} \begin{bmatrix} \Phi(2x) \\ \Phi(2x-1) \end{bmatrix}, \tag{5}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are $(m+1) \times 2(m+1)$ matrices. Rewriting the two-scale relations (5) in the matrix form, yields

$$\begin{bmatrix} \Phi(x) \\ \Psi(x) \end{bmatrix} = \begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} \begin{bmatrix} \Phi(2x) \\ \Phi(2x-1) \end{bmatrix}, \tag{6}$$

which is called the reconstruction relation. Also the coefficients matrix in Equation (6) is called the reconstruction matrix (RCM) which is invertible. Because of the simplicity of the flatlet scaling functions, the matrix $\mathbf{P}$ in the two-scale relations (5) is obtained as

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & & & & & 0 \\ & & 1 & 1 & & & \\ & & & & \ddots & & \\ 0 & & & & & 1 & 1 \end{bmatrix}. \tag{7}$$

For computing the $2(m+1)^2$ entries of matrix $\mathbf{Q}$, we need $2(m+1)^2$ independent conditions. There are many possibilities in choosing the conditions to be used that result in different flatlet multiwavelet systems with different properties. In this sequel, we use the $((m+1)(m+2))/2$ orthonormality conditions

$$\int_0^1 \psi_i(x)\psi_j(x)\,\mathrm{d}x = \delta_{i,j}, \quad i, j = 0, 1, \ldots, m, \tag{8}$$

and also $((m+1)(3m+2))/2$ vanishing moment conditions

$$\int_{-\infty}^{\infty} \psi_i(x)x^j\,\mathrm{d}x = 0, \quad i = 0, 1, \ldots, m, \quad j = 0, 1, \ldots, m+i. \tag{9}$$

By using Equations (3) and (5), Equation (9) can be written as the following system of linear equations

$$\sum_{k=0}^{2(m+1)} \{(k+1)^{j+1} - (k)^{j+1}\} q_{j,k} = 0, \quad j = 0, \ldots, m+i. \tag{10}$$

By solving Equations (8) and (10), the unknown matrix $\mathbf{Q}$ and so $\Psi(x)$ are obtained. As an example, for the first-order flatlet basis functions

$$\phi_0(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2}, \\ 0 & \text{otherwise} \end{cases}$$
$$\phi_1(x) = \begin{cases} 1 & \frac{1}{2} \le x < 1, \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

we get

$$\mathbf{Q} = \pm \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} & \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{bmatrix}. \tag{12}$$

Which implies that there are two associated multiwavelets. A simple form of multiwavelets for the above example may be given as

$$\psi_0(x) = \sqrt{2} \begin{cases} \frac{1}{2} & 0 \le x < \frac{1}{4}, \\ \frac{-1}{2} & \frac{1}{4} \le x < \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} \le x < 1, \\ 0 & \text{otherwise} \end{cases}, \quad \psi_1(x) = \sqrt{10} \begin{cases} \frac{1}{10} & 0 \le x < \frac{1}{4}, \\ \frac{-3}{10} & \frac{1}{4} \le x < \frac{1}{2}, \\ \frac{3}{10} & \frac{1}{2} \le x < \frac{3}{4}, \\ \frac{-1}{10} & \frac{3}{4} \le x < 1, \\ 0 & \text{otherwise}. \end{cases} \tag{13}$$

Also the second-order flatlet multiwavelet system is

$$\phi_i(x) = \begin{cases} 1 & \frac{i}{3} \le x < \frac{i+1}{3}, \\ 0 & \text{otherwise} \end{cases}, \quad i = 0, 1, 2,$$

$$\psi_0(x) = \sqrt{10} \begin{cases} \frac{1}{6} & 0 \le x < \frac{1}{6}, \\ \frac{-7}{30} & \frac{1}{6} \le x < \frac{1}{3}, \\ \frac{-2}{15} & \frac{1}{3} \le x < \frac{1}{2}, \\ \frac{2}{15} & \frac{1}{2} \le x < \frac{2}{3}, \\ \frac{7}{30} & \frac{2}{3} \le x < \frac{5}{6}, \\ \frac{-1}{6} & \frac{5}{6} \le x < 1, \\ 0 & \text{otherwise} \end{cases}, \quad \psi_1(x) = \sqrt{14} \begin{cases} \frac{1}{14} & 0 \le x < \frac{1}{6}, \\ \frac{-3}{14} & \frac{1}{6} \le x < \frac{1}{3}, \\ \frac{1}{7} & \frac{1}{3} \le x < \frac{2}{3}, \\ \frac{-3}{14} & \frac{2}{3} \le x < \frac{5}{6}, \\ \frac{1}{14} & \frac{5}{6} \le x < 1, \\ 0 & \text{otherwise} \end{cases},$$

$$\psi_2(x) = \sqrt{14} \begin{cases} \frac{-1}{42} & 0 \le x < \frac{1}{6}, \\ \frac{5}{42} & \frac{1}{6} \le x < \frac{1}{3}, \\ \frac{-5}{21} & \frac{1}{3} \le x < \frac{1}{2}, \\ \frac{5}{21} & \frac{1}{2} \le x < \frac{2}{3}, \\ \frac{-5}{42} & \frac{2}{3} \le x < \frac{5}{6}, \\ \frac{1}{42} & \frac{5}{6} \le x < 1, \\ 0 & \text{otherwise}. \end{cases} . \tag{14}$$

3. Biorthogonal flatlet multiwavelet system

Let $\tilde{\Phi}(x)$ and $\tilde{\Psi}(x)$ be dual scaling and wavelet vector functions in biorthogonal flatlet multiwavelet system (BFMS), respectively, as

$$\tilde{\Phi}(x) = \begin{bmatrix} \tilde{\phi}_0(x) \\ \vdots \\ \tilde{\phi}_i(x) \\ \vdots \\ \tilde{\phi}_m(x) \end{bmatrix}, \quad \tilde{\Psi}(x) = \begin{bmatrix} \tilde{\psi}_0(x) \\ \vdots \\ \tilde{\psi}_i(x) \\ \vdots \\ \tilde{\psi}_m(x) \end{bmatrix}. \tag{15}$$

Note that according to the biorthogonality conditions in BFMS, we must have

$$\langle \tilde{\phi}_i, \phi_j \rangle = \int_0^1 \tilde{\phi}_i(x)\phi_j(x)\,\mathrm{d}x = \delta_{i,j}, \tag{16}$$

$$\langle \tilde{\psi}_i, \psi_j \rangle = \int_0^1 \tilde{\psi}_i(x)\psi_j(x)\,\mathrm{d}x = \delta_{i,j}, \tag{17}$$

$$\langle \tilde{\psi}_i, \phi_j \rangle = \int_0^1 \tilde{\psi}_i(x)\phi_j(x)\,\mathrm{d}x = 0, \quad i, j = 0, 1, \dots, m. \tag{18}$$

Now we introduce $\tilde{\phi}_i(x)$ and $\tilde{\psi}_i(x)$, $i = 0, 1, \dots, m$, as polynomials and piecewise polynomials of degree m, respectively, by

$$\tilde{\phi}_i(x) = \begin{cases} a_{i1} + a_{i2}x + \cdots + a_{i,m+1}x^m & 0 \le x < 1, \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

$$\tilde{\psi}_i(x) = \begin{cases} b^1_{i1} + b^1_{i2}x + \cdots + b^1_{i,m+1}x^m & 0 \le x < \frac{1}{2}, \\ b^2_{i1} + b^2_{i2}x + \cdots + b^2_{i,m+1}x^m & \frac{1}{2} \le x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Based on biorthogonality conditions (16)–(18), we show that coefficients $a_{i,j}$, $b^1_{i,j}$ and $b^2_{i,j}$, $i = 0, \dots, m$, and $j = 1, \dots, m+1$, in Equations (19) and (20) are uniquely determined.

LEMMA 3.1 *Let $A = [a_{i,j}]_{n\times n}$ be a square matrix with $a_{i,j} = p_{i-1}(j)$, where p is a polynomial of exact degree $i-1$, then A is invertible.*

Proof Let A be singular. If so, there exists a non-zero vector $\mathbf{w} = [w_1, w_2, \dots, w_n] \in \mathbb{R}^n$ such that

$$\sum_{i=1}^{n} w_i a_{i,j} = 0, \quad j = 1, 2, \dots, n.$$

Therefore the polynomial $P_{n-1}(j) = \sum_{i=1}^{n} w_i p_{i-1}(x)$ is of exact degree $n-1$ and satisfies

$$P_{n-1}(j) = \sum_{i=1}^{n} w_i p_{i-1}(j) = 0, \quad j = 1, 2, \dots, n.$$

Hence, P_{n-1} has n zeros. This means $P \equiv 0$ and we have a contradiction. ■

THEOREM 3.2 *For oblique flatlet multiwavelets, The dual functions defined in Equations* (19) *and* (20) *are uniquely determined.*

Proof First we prove the theorem for dual functions in Equation (19). Equations (15) and (19) result in

$$\tilde{\Phi}(x) = \begin{bmatrix} \tilde{\phi}_0(x) \\ \vdots \\ \tilde{\phi}_i(x) \\ \vdots \\ \tilde{\phi}_m(x) \end{bmatrix} = \mathbf{A} \begin{bmatrix} 1 \\ \vdots \\ x^i \\ \vdots \\ x^m \end{bmatrix} \cdot \chi_{[0,1]}(x), \tag{21}$$

where χ is the characteristic function and $\mathbf{A} = (a_{ij})$ is the matrix of unknown coefficients in Equation (19). If so, Equation (16) can be written in the following matrix form

$$\langle \tilde{\Phi}, \Phi^{\mathrm{T}} \rangle = \int_0^1 \tilde{\Phi}(x)\Phi^{\mathrm{T}}(x)\,\mathrm{d}x = I, \tag{22}$$

where I is the identity matrix with rank $m+1$. Now, according to Equation (21) we have

$$\begin{aligned} I = \langle \tilde{\Phi}, \Phi^{\mathrm{T}} \rangle &= \int_0^1 \mathbf{A}[1, \dots, x^i, \dots, x^m]^{\mathrm{T}} \Phi^{\mathrm{T}}(x)\chi_{[0,1]}(x)\,\mathrm{d}x \\ &= \int_0^1 \mathbf{A}[1, \dots, x^i, \dots, x^m]^{\mathrm{T}} \Phi^{\mathrm{T}}(x)\,\mathrm{d}x \\ &= \mathbf{A} \int_0^1 [1, \dots, x^i, \dots, x^m]^{\mathrm{T}} \Phi^{\mathrm{T}}(x)\,\mathrm{d}x, \end{aligned}$$

so

$$I = \mathbf{A} \begin{bmatrix} \int_0^1 \phi_0(x)\,\mathrm{d}x & \int_0^1 \phi_1(x)\,\mathrm{d}x & \cdots & \int_0^1 \phi_m(x)\,\mathrm{d}x \\ \int_0^1 x\phi_0(x)\,\mathrm{d}x & \int_0^1 x\phi_1(x)\,\mathrm{d}x & \cdots & \int_0^1 x\phi_m(x)\,\mathrm{d}x \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^1 x^m\phi_0(x)\,\mathrm{d}x & \int_0^1 x^m\phi_1(x)\,\mathrm{d}x & \cdots & \int_0^1 x^m\phi_m(x)\,\mathrm{d}x \end{bmatrix}. \tag{23}$$

Applying Equation (3) to the ijth entry of Equation (23) yields

$$\begin{aligned} \int_0^1 x^{i-1}\phi_{j-1}(x)\,\mathrm{d}x &= \int_{(j-1)/(m+1)}^{j/(m+1)} x^{i-1}\mathrm{d}x \\ &= \left[\frac{x^i}{i}\right]_{(j-1)/(m+1)}^{j/(m+1)} \\ &= \frac{1}{i}\left(\left(\frac{j}{m+1}\right)^i - \left(\frac{j-1}{m+1}\right)^i\right). \end{aligned}$$

So, Equation (23) can be written as

$$I = \mathbf{A} \cdot \mathbf{D}_1 \cdot \mathbf{D}_2 \cdot \mathbf{C},$$

where

$$\mathbf{D}_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{m+1} \end{bmatrix}, \quad \mathbf{D}_2 = \begin{bmatrix} (m+1)^{-1} & 0 & \cdots & 0 \\ 0 & (m+1)^{-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (m+1)^{-(m+1)} \end{bmatrix}$$

and

$$\mathbf{C} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 3 & \cdots & (m+1)^2 - m^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2^m - 1^m & \cdots & (m+1)^{m+1} - m^{m+1} \end{bmatrix}.$$

Here, the diagonal matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ are invertible and $\mathbf{C}$ is a square matrix whose ijth entry can be obtained as $q_{i-1}(j)$, where

$$q_{i-1}(x) = x^i - (x-1)^i, \quad x \in IR.$$

Now according to the Lemma 3.1, $\mathbf{C}$ is invertible. Therefore

$$\mathbf{A} = \mathbf{C}^{-1} \cdot \mathbf{D}^{-1},$$

is determined uniquely.

To prove the uniqueness of dual functions in Equation (20) we must have

$$\tilde{\Psi}(x) = \tilde{\mathbf{Q}} \begin{bmatrix} \tilde{\Phi}(2x) \\ \tilde{\Phi}(2x-1) \end{bmatrix}, \tag{24}$$

in which $\tilde{\mathbf{Q}}$ is a $(m+1) \times 2(m+1)$ matrix. Rewriting Equations (17) and (18) in the matrix form, yields

$$\begin{cases} \langle \Phi, \tilde{\Psi}^{\mathrm{T}} \rangle = 0, \\ \langle \Psi, \tilde{\Psi}^{\mathrm{T}} \rangle = I. \end{cases} \tag{25}$$

Substituting Equation (24) into Equation (25), we get

$$\begin{cases} \langle \Phi, [\tilde{\Phi}^{\mathrm{T}}(2\cdot), \tilde{\Phi}^{\mathrm{T}}(2\cdot -1)]\tilde{\mathbf{Q}}^{\mathrm{T}} \rangle = 0, \\ \langle \Psi, [\tilde{\Phi}^{\mathrm{T}}(2\cdot), \tilde{\Phi}^{\mathrm{T}}(2\cdot -1)]\tilde{\mathbf{Q}}^{\mathrm{T}} \rangle = I. \end{cases} \tag{26}$$

Which by using Equation (5) results in

$$\begin{cases} \langle \mathbf{P}[\Phi(2\cdot), \Phi(2\cdot -1)], [\tilde{\Phi}^{\mathrm{T}}(2\cdot), \tilde{\Phi}^{\mathrm{T}}(2\cdot -1)]\tilde{\mathbf{Q}}^{\mathrm{T}} \rangle = 0, \\ \langle \mathbf{Q}[\Phi(2\cdot), \Phi(2\cdot -1)], [\tilde{\Phi}^{\mathrm{T}}(2\cdot), \tilde{\Phi}^{\mathrm{T}}(2\cdot -1)]\tilde{\mathbf{Q}}^{\mathrm{T}} \rangle = I. \end{cases} \tag{27}$$

Finally, using Equation (22) we get in matrix form, the system of linear equations

$$\begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} \tilde{\mathbf{Q}} = \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{28}$$

Now since the coefficient matrix RCM in Equation (28) is invertible so $\tilde{\mathbf{Q}}$ is determined uniquely. ■

For example, the dual multiscaling functions and multiwavelets corresponding to Equations (11) and (13) are computed as

$$\tilde{\phi}_0(x) = \begin{cases} 3-4x & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases},$$

$$\tilde{\phi}_1(x) = \begin{cases} -1+4x & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases},$$

$$\tilde{\psi}_0(x) = \begin{cases} 2\sqrt{2}(1-4x) & 0 \le x < \frac{1}{2} \\ -2\sqrt{2}(3-4x) & \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise} \end{cases},$$

$$\tilde{\psi}_1(x) = \begin{cases} \sqrt{10}(1-4x) & 0 \le x < \frac{1}{2} \\ \sqrt{10}(3-4x) & \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

Also the dual multiwavelets corresponding to Equation (14) are computed as

$$\tilde{\phi}_0(x) = \begin{cases} \frac{11}{2} - 18x + \frac{27}{2}x^2 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases},$$

$$\tilde{\phi}_1(x) = \begin{cases} \frac{-7}{2} + 27x - 27x^2 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases},$$

$$\tilde{\phi}_2(x) = \begin{cases} 1 - 9x + \frac{27}{2}x^2 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases},$$

$$\tilde{\psi}_0(x) = \begin{cases} \sqrt{10}\left(\frac{7}{4} - \frac{33}{2}x + 27x^2\right) & 0 \le x < \frac{1}{2} \\ -\sqrt{10}\left(\frac{49}{4} - \frac{75}{2}x + 27x^2\right) & \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise} \end{cases},$$

$$\tilde{\psi}_1(x) = \begin{cases} \sqrt{14}\left(\frac{9}{4} - \frac{45}{2}x + \frac{81}{2}x^2\right) & 0 \le x < \frac{1}{2} \\ \sqrt{14}\left(\frac{81}{4} - \frac{117}{2}x + \frac{81}{2}x^2\right) & \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise} \end{cases},$$

$$\tilde{\psi}_2(x) = \begin{cases} -\sqrt{14}\left(1 - 12x + 27x^2\right) & 0 \le x < \frac{1}{2} \\ \sqrt{14}\left(16 - 42x + 27x^2\right) & \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

3.1 *Function approximation*

A function $f(x)$ defined in [0,1] may be approximated by the flatlet scaling functions as

$$f(x) \simeq \sum_{i=0}^{m} \sum_{k=0}^{2^J-1} c_{i,k}\phi_i(2^J x - k),$$

for some m and J, where

$$c_{i,k} = \int_0^1 f(x)\tilde{\phi}_i(2^J x - k)\,\mathrm{d}x \quad i = \mathbf{0}, \mathbf{1}, \ldots, m, \quad k = \mathbf{0}, \mathbf{1}, \ldots, 2^J - 1. \tag{31}$$

Also, we can approximate $f(x)$ by the flatlet multiwavelet as

$$f(x) \simeq \sum_{i=0}^{m} c_i' \phi_i(x) + \sum_{i=0}^{m} \sum_{l=0}^{J} \sum_{k=0}^{2^l-1} d_{i,l,k} \psi_i(2^l x - k), \tag{32}$$

or by dual of the flatlet multiwavelet

$$f(x) \simeq \sum_{i=0}^{m} \tilde{c}_i' \tilde{\phi}_i(x) + \sum_{i=0}^{m} \sum_{l=0}^{J} \sum_{k=0}^{2^l-1} \tilde{d}_{i,l,k} \tilde{\psi}_i(2^l x - k), \tag{33}$$

where

$$\begin{aligned}
c_i' &= \int_0^1 f(x)\tilde{\phi}_i(x)\,\mathrm{d}x, \\
d_{i,l,k} &= \int_0^1 f(x)\tilde{\psi}_i(2^l x - k)\,\mathrm{d}x, \\
\tilde{c}_i' &= \int_0^1 f(x)\phi_i(x)\,\mathrm{d}x, \\
\tilde{d}_{i,l,k} &= \int_0^1 f(x)\psi_i(2^l x - k)\,\mathrm{d}x, \quad i = 0, \ldots, m; \; l = 0, \ldots, J; \; k = 0, \ldots, 2^l - 1.
\end{aligned}$$

The expressions (32) and (33) may be written in the following matrix forms

$$f(x) \simeq \Theta^{\mathrm{T}} \cdot \mathbf{f}, \tag{34}$$

and

$$f(x) \simeq \tilde{\Theta}^{\mathrm{T}} \cdot \tilde{\mathbf{f}}, \tag{35}$$

respectively, where

$$\Theta(x) = \begin{bmatrix} \phi_0(x) \\ \vdots \\ \phi_m(x) \\ \psi_0(x) \\ \vdots \\ \psi_i(2^l x - k) \\ \vdots \\ \psi_m(2^J x - 2^J + 1) \end{bmatrix}, \quad \tilde{\Theta}(x) = \begin{bmatrix} \tilde{\phi}_0(x) \\ \vdots \\ \tilde{\phi}_m(x) \\ \tilde{\psi}_0(x) \\ \vdots \\ \tilde{\psi}_i(2^l x - k) \\ \vdots \\ \tilde{\psi}_m(2^J x - 2^J + 1) \end{bmatrix}, \tag{36}$$

and $\mathbf{f}, \tilde{\mathbf{f}}$ are N-vectors as

$$\begin{aligned}
\mathbf{f} &= [c_0', \ldots, c_m', d_{0,0,0}, \ldots, d_{i,l,k}, \ldots, d_{m,J,2^J-1}]^{\mathrm{T}}, \\
\tilde{\mathbf{f}} &= [\tilde{c}_0', \ldots, \tilde{c}_m', \tilde{d}_{0,0,0}, \ldots, \tilde{d}_{i,l,k}, \ldots, \tilde{d}_{m,J,2^J-1}]^{\mathrm{T}},
\end{aligned}$$

in which $N = 2^J(m+1)$.

Also, a two variable function $g(x, y)$ can be approximated by the flatlet multiwavelet

$$g(x, y) \simeq \tilde{\Theta}^{\mathrm{T}}(y)\mathbf{G}\Theta(x), \tag{37}$$

where

$$[G]_{i,j} = \int_0^1 \int_0^1 g(x, y)\tilde{\theta}_i(x)\theta_j(y)\,\mathrm{d}x\,\mathrm{d}y, \quad i, j = 1, 2, \ldots, N.$$

Note that $\theta_i(x)$ and $\tilde{\theta}_i(x)$, $i = 1, 2, \ldots, N$, are ith component of $\Theta(x)$ and $\tilde{\Theta}(x)$, respectively.

Next, we use representation (35) for approximating the unknown function. In fact, these basis functions use piecewise polynomials of degree m and have higher order approximations comparing with flatlet multiwavelets.

3.2 *The OMD*

In our method, it is crucial to express the expansion of $f'(x)$ in terms of coefficients of the expansion of $f(x)$. This can be done by using the OMD. By considering Equation (35) let

$$f'(x) \simeq \tilde{\Theta}^{\mathrm{T}} \cdot \dot{\mathbf{f}}, \tag{38}$$

if so, the OMD $\mathbf{D}$ connects two coefficient vectors $\mathbf{f}$ and $\dot{\mathbf{f}}$ by

$$\dot{\mathbf{f}} = \mathbf{D} \cdot \mathbf{f}. \tag{39}$$

So, by using OMD, we can obtain the coefficients of expansion $f'(x)$ from the coefficients of expansion $f(x)$. Let $d_{i,j}$ be the ijth entry of $\mathbf{D}$. If so, by using Equations (38) and (39), we have

$$d_{i,j} = \int_0^1 \theta_i(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\theta}_j(x)\right)\mathrm{d}x, \quad i, j = 1, 2, \ldots, N. \tag{40}$$

Also, by using Equations (38) and (39) we can obtain the kth derivative of f in the expansion (35) as

$$f^{(k)}(x) \simeq \tilde{\Theta}(x)^{\mathrm{T}} \cdot \mathbf{D}^k \cdot \mathbf{f}. \tag{41}$$

Now, we suggest a practical way for determining the entire elements of OMD. Note that from definitions of $\Theta(x)$ and $\tilde{\Theta}(x)$, we have

$$\int_0^1 \Theta(x)\tilde{\Theta}(x)^T\,\mathrm{d}x = \mathbf{I}, \tag{42}$$

where $\mathbf{I}$ is the identity matrix with rank N. According to Equation (36),

$$\theta_i(x) = \begin{cases} \phi_{i-1}(x) & i = 1, 2, \ldots, m+1, \\ \psi_{j'}(2^n x - l) & i = m+2, m+3, \ldots, N, \end{cases} \tag{43}$$

where

$$\begin{aligned} j' &= (i-1) \bmod (m+1), \\ n &= [\log_2 k], \\ k &= \left[\frac{i-1}{m+1}\right], \\ l &= k - 2^n, \end{aligned} \tag{44}$$

and $[x]$ is the greatest integer less than or equal to x. Note that $\theta_i(x)$'s and $\tilde{\theta}_i(x)$'s are piecewise differentiable functions. For obtaining the elements of OMD, we consider two following cases.

Case 1 $i = 1, \dots, m+1$. In this case from Equation (43) we have

$$\theta_i(x) = \phi_{i-1}(x). \tag{45}$$

Substituting Equation (45) into Equation (40) yields

$$\begin{aligned} d_{i,j} &= \int_0^1 \phi_{i-1}(x) \left(\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\theta}_j(x)\right) \mathrm{d}x \\ &= \int_{(i-1)/(m+1)}^{i/(m+1)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\theta}_j(x)\right) \mathrm{d}x. \end{aligned} \tag{46}$$

Case 2 $i = m+2, m+3, \dots, N$. In this case from Equation (43) we get

$$\theta_i(x) = \psi_{j'}(2^n x - l), \tag{47}$$

where j', n, l are defined in Equation (44). By using Equation (5), Equation (47) gets replaced by

$$\theta_i(x) = \sum_{u=1}^{m+1} q_{j'+1,u}\phi_{u-1}(2(2^n x - l)) + \sum_{u=m+1}^{2m+2} q_{j'+1,u}\phi_{u-1}(2(2^n x - l) - 1). \tag{48}$$

Substituting Equation (48) into Equation (40) yields

$$\begin{aligned} d_{i,j} &= \int_0^1 \sum_{u=1}^{m+1} q_{j'+1,u}\phi_{u-1}(2(2^n x - l)) \left(\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\theta}_j(x)\right) \mathrm{d}x \\ &\quad + \int_0^1 \sum_{u=m+1}^{2m+2} q_{j'+1,u}\phi_{u-1}(2(2^n x - l) - 1) \left(\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\theta}_j(x)\right) \mathrm{d}x. \end{aligned}$$

So we have

$$\begin{aligned} d_{i,j} &= \sum_{u=1}^{m+1} q_{j'+1,u} \int_0^1 \phi_{u-1}(2(2^n x - l)) \left(\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\theta}_j(x)\right) \mathrm{d}x \\ &\quad + \sum_{u=m+1}^{2m+2} q_{j'+1,u} \int_0^1 \phi_{u-1}(2(2^n x - l) - 1) \left(\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\theta}_j(x)\right) \mathrm{d}x. \end{aligned}$$

Finally, by using Equation (3) we get

$$d_{i,j} = \sum_{u=1}^{m+1} q_{j'+1,u} \int_{S(u-1,n,2l)}^{S(u,n,2l)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\theta}_j(x)\right) \mathrm{d}x + \sum_{u=m+1}^{2m+2} q_{j'+1,u} \int_{S(u-1,n,2l+1)}^{S(u,n,2l+1)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\theta}_j(x)\right) \mathrm{d}x, \tag{49}$$

where $S(u, n, l) = 2^{-(n+1)}(u/(m+1) + l)$.

Also from Equation (36), we have

$$\tilde{\theta}_i(x) = \begin{cases} \tilde{\phi}_{i-1}(x) & i = 1, 2, \ldots, m+1, \\ \tilde{\psi}_{j'}(2^n x - l) & i = m+2, m+3, \ldots, N, \end{cases} \tag{50}$$

where j', n, l, satisfy Equation (44) and according to Equations (19) and (20), $\tilde{\theta}_j(x)$, $j = 1, 2, \ldots, N$, are piecewise differentiable functions. So we can express their support by

$$\textbf{supp}\, \tilde{\theta}_j(x) = E_j^1 \bigcup E_j^2 = \Xi_j, \quad j = 1, 2, \ldots, N \tag{51}$$

where E_j^1 and E_j^2 are subintervals of [0, 1], in each of which $\tilde{\theta}_j$ is continuous. Note that we can determine the support of $\tilde{\theta}_j$'s in the same way as expressed in Equations (45) and (47). Applying Equation (51) in Equation (46) yields

$$d_{i,j} = \int_{\Omega_{j,1}} \left(\frac{\mathrm{d}}{\mathrm{d}x} \tilde{\theta}_j(x) \right) \mathrm{d}x, \quad i, j = 1, 2, \ldots, N.$$

Also by applying Equation (51) in Equation (49), we have

$$\begin{aligned} d_{i,j} &= \sum_{u=1}^{m+1} q_{j'+1,u} \int_{\Omega_{j,2}} \left(\frac{\mathrm{d}}{\mathrm{d}x} \tilde{\theta}_j(x) \right) \mathrm{d}x \\ &\quad + \sum_{u=m+1}^{2m+2} q_{j'+1,u} \int_{\Omega_{j,3}} \left(\frac{\mathrm{d}}{\mathrm{d}x} \tilde{\theta}_j(x) \right) \mathrm{d}x, \quad i, j = 1, 2, \ldots, N, \end{aligned} \tag{52}$$

where

$$\begin{aligned} \Omega_{j,1} &= \left[\frac{i-1}{m+1}, \frac{i}{m+1} \right] \bigcap \Xi_j = [\omega_{j,1}^1, \omega_{j,2}^1], \\ \Omega_{j,2} &= [S(u-1, n, 2l), S(u, n, 2l)] \bigcap \Xi_j = [\omega_{j,1}^2, \omega_{j,2}^2], \\ \Omega_{j,3} &= [S(u-1, n, 2l+1), S(u, n, 2l+1)] \bigcap \Xi_j = [\omega_{j,1}^3, \omega_{j,2}^3]. \end{aligned} \tag{53}$$

So we can compute $d_{i,j}$ for two cases, above, as below

$$d_{i,j} = \tilde{\theta}_j(x)|_{\Omega_{j,1}}, \quad i = 1, 2, \ldots, m+1, \tag{54}$$

$$d_{i,j} = \sum_{u=1}^{m+1} q_{j'+1,u} (\tilde{\theta}_j(t)|_{\Omega j,2}) + \sum_{u=m+1}^{2m+2} q_{j'+1,u} (\tilde{\theta}_j(t)|_{\Omega j,3}), \quad i = m+2, m+3, \ldots, N, \tag{55}$$

where $j = 1, 2, \ldots, N$. In the above relations we use this notation

$$f(x)|_{[a,b]} = f(b) - f(a).$$

Therefore, all elements of OMD are computed by adding the pointwise values of $\tilde{\theta}_i$'s which is easier than explicitly calculating in Equation (40).

4. Solving the ordinary differential equations

In this section, we solve Equation (1) by utilizing BFMS. For this purpose, we use Equation (35), to approximate the functions involved by

$$y(x) = \tilde{\Theta}(x)^{\mathrm{T}} \cdot \mathbf{y}, \tag{56}$$

$$f(x, y(x)) = \tilde{\Theta}(x)^{\mathrm{T}} \cdot \mathbf{f}, \tag{57}$$

where **y** and **f** are unknown vectors.

Now, by using Equation (41) and substituting Equations (56) and (57) into Equations (1) and (2), respectively, we get

$$\tilde{\Theta}(x)^{\mathrm{T}} \cdot \mathbf{D}^2 \cdot \mathbf{y} = \tilde{\Theta}(x)^{\mathrm{T}} \cdot \mathbf{f}, \tag{58}$$

$$\tilde{\Theta}(0)^{\mathrm{T}} \cdot \mathbf{D} \cdot \mathbf{y} = y_0, \tag{59}$$

$$\tilde{\Theta}(1)^{\mathrm{T}} \cdot \mathbf{D} \cdot \mathbf{y} = y_1. \tag{60}$$

Also expressions (56) and (57) yield

$$f(x, \tilde{\Theta}(x)^{\mathrm{T}} \cdot \mathbf{y}) = \tilde{\Theta}(x)^{\mathrm{T}} \cdot \mathbf{f}. \tag{61}$$

In order to find the solution in Equation (1), we collocate Equations (58) and (61) in $N-2$ evenly spaced nodes $x_i = (i/N), i = 2, 3, \ldots, N-1$ and N evenly spaced nodes $x_i = (i-1/N-1), i = 1, 2, \ldots, N$, respectively. The above equations together with Equations (59) and (60), make a system of $2N$ linear equations with $2N$ unknowns. By solving this system we obtain unknown vectors **f**, **y** and consequently $y(x)$. Here, we use the Newton iteration method, for solving the system of nonlinear equations (61). Some numerical examples are stated in the next section.

5. Numerical examples

The BFMS is now applied to some examples with known exact and numerical solutions [12]. The absolute values of errors of solutions at a selection of chosen points, are tabulated. Also, computations were carried out for different m's. From the tables, we can observe the convergence

Table 1. Absolute values of error for Example 1.

t_i	B-spline wavelet [12]	BFMS for $m=5$, $J=2$	BFMS for $m=6$, $J=3$
0.0	1.3×10^{-7}	2.4×10^{-4}	2.0×10^{-7}
0.1	5.9×10^{-6}	2.4×10^{-4}	4.2×10^{-10}
0.2	5.6×10^{-6}	2.3×10^{-4}	2.0×10^{-7}
0.3	5.2×10^{-6}	2.2×10^{-4}	3.9×10^{-7}
0.4	2.2×10^{-6}	2.0×10^{-4}	5.8×10^{-7}
0.5	4.4×10^{-7}	1.8×10^{-4}	3.5×10^{-7}
0.6	4.0×10^{-7}	1.7×10^{-4}	7.3×10^{-8}
0.7	1.5×10^{-6}	1.7×10^{-4}	7.4×10^{-8}
0.8	1.1×10^{-6}	1.6×10^{-4}	7.8×10^{-8}
0.9	1.5×10^{-6}	1.5×10^{-4}	7.8×10^{-8}
1.0	9.4×10^{-6}	1.5×10^{-4}	7.9×10^{-8}

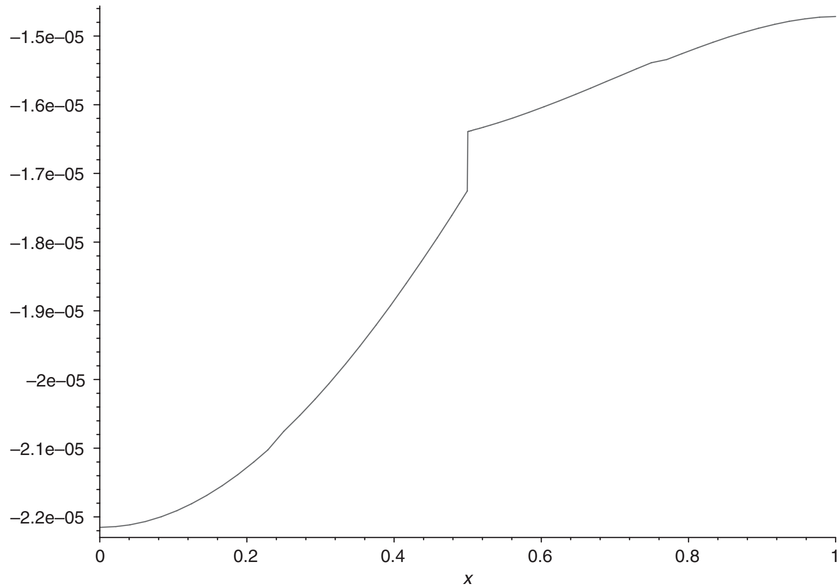

Figure 1. The error function for Example 1, for $m = 5$, $J = 3$.

of numerical solutions as m is increased. Furthermore, the main advantages of the method are its simplicity and small computations costs which result from the sparsity of the associated matrices and also small number of the coefficients of wavelet representations.

Example 1 Consider the following problem, [16]

$$y''(x) = (4x^2 - 2)y(x) \quad x \in [0, 1],$$
$$y'(0) = 0, \quad y'(1) = -\frac{2}{\mathrm{e}}, \tag{62}$$

with the exact solution $y(x) = \mathrm{e}^{-x^2}$. Table 1 shows the absolute values of error in some points. Results are compared with [16]. Also the error function for $m = 5$, $J = 3$ is shown in Figure 1. In this case, the maximum absolute value of error is 2.2×10^{-5}.

Table 2. Absolute values of error for Example 2.

t_i	B-spline wavelet [12]	BFMS for $m = 6$, $J = 2$	BFMS for $m = 7$, $J = 3$
0.0	5.6×10^{-6}	1.9×10^{-6}	7.4×10^{-8}
0.1	2.6×10^{-5}	1.9×10^{-6}	7.6×10^{-8}
0.2	1.7×10^{-5}	2.0×10^{-6}	8.2×10^{-8}
0.3	1.6×10^{-5}	2.2×10^{-6}	9.1×10^{-8}
0.4	1.4×10^{-5}	2.6×10^{-6}	1.0×10^{-7}
0.5	1.2×10^{-5}	4.0×10^{-6}	1.4×10^{-7}
0.6	1.0×10^{-5}	4.3×10^{-6}	1.5×10^{-7}
0.7	7.2×10^{-6}	3.9×10^{-6}	1.4×10^{-7}
0.8	5.3×10^{-6}	3.7×10^{-6}	1.3×10^{-7}
0.9	5.5×10^{-6}	3.5×10^{-6}	1.3×10^{-7}
1.0	1.6×10^{-6}	3.4×10^{-6}	1.3×10^{-7}

Example 2 The nonlinear problem, [12]

$$y''(x) = 2y(x)^3 \quad x \in [0, 1],$$
$$y'(0) = -1, \quad y'(1) = -\frac{1}{4}, \tag{63}$$

has the exact solution $y(x) = (1/x + 1)$. The absolute values of error in some points are shown in Table 2.

Example 3 Consider the following nonlinear problem

$$y''(x) = -e^{-2y(x)} \quad x \in [0, 1],$$
$$y'(0) = 1, \quad y'(1) = \frac{1}{2}. \tag{64}$$

Table 3. Absolute values of error for Example 3.

t_i	BFMS for $m = 4$, $J = 3$	BFMS for $m = 5$, $J = 2$	BFMS for $m = 6$, $J = 2$
0.0	8.8×10^{-5}	6.6×10^{-6}	2.2×10^{-8}
0.1	8.8×10^{-5}	7.7×10^{-6}	2.2×10^{-8}
0.2	8.9×10^{-5}	8.9×10^{-6}	2.3×10^{-8}
0.3	9.3×10^{-5}	1.0×10^{-5}	2.3×10^{-8}
0.4	9.6×10^{-5}	1.1×10^{-5}	2.5×10^{-8}
0.5	1.1×10^{-5}	1.9×10^{-5}	5.5×10^{-9}
0.6	1.2×10^{-4}	1.9×10^{-5}	2.5×10^{-8}
0.7	1.1×10^{-4}	1.8×10^{-5}	2.5×10^{-8}
0.8	1.1×10^{-4}	1.7×10^{-5}	2.4×10^{-8}
0.9	1.1×10^{-4}	1.7×10^{-5}	2.4×10^{-8}
1.000	1.1×10^{-4}	1.6×10^{-5}	2.4×10^{-8}

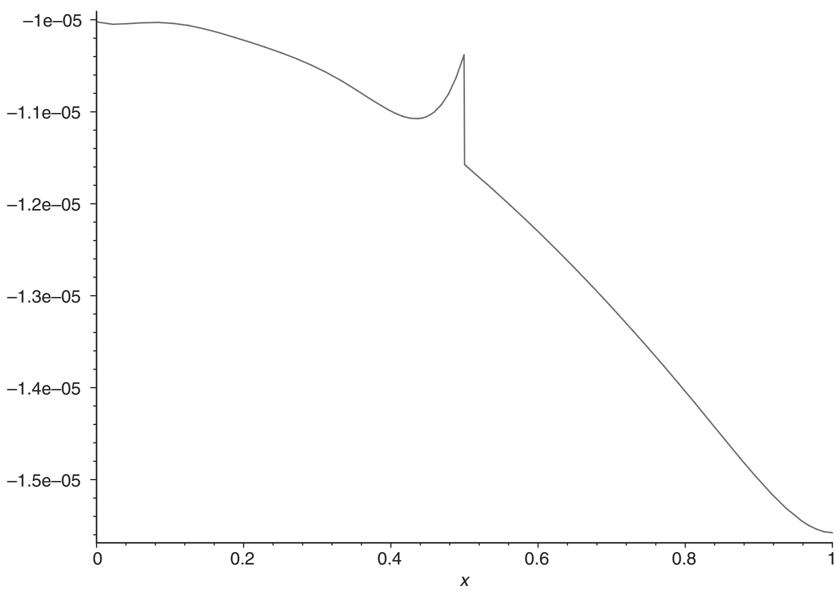

Figure 2. The error function for Example 3, for $m = 5$, $J = 1$.

Table 4. Absolute values of error for Example 4.

t_i	B-spline wavelet [12]	BFMS for $m = 4$, $J = 3$	BFMS for $m = 7$, $J = 1$	BFMS for $m = 7$, $J = 2$
0.0	1.2×10^{-8}	2.8×10^{-5}	4.7×10^{-7}	1.1×10^{-8}
0.2	1.2×10^{-6}	2.6×10^{-5}	4.3×10^{-7}	1.0×10^{-8}
0.4	5.6×10^{-6}	2.0×10^{-5}	3.2×10^{-7}	8.1×10^{-9}
0.6	3.1×10^{-7}	4.5×10^{-5}	1.5×10^{-7}	2.8×10^{-10}
0.8	9.2×10^{-7}	4.8×10^{-5}	1.9×10^{-8}	2.5×10^{-9}
1.0	4.5×10^{-8}	4.2×10^{-5}	1.6×10^{-7}	4.2×10^{-9}

The exact solution is $y(x) = \ln(x + 1)$. The absolute values of error in some points are shown in Table 3. Figure 2, shows the error function for $m = 5$, $J = 1$. The maximum absolute value of error in this case is 1.6×10^{-5}.

Example 4 The problem [12]

$$\begin{aligned} &y''(x) = 2 - 4y(x) \quad x \in [0, 1], \\ &y'(0) = 0, \quad y'(1) = \sin(2), \end{aligned} \tag{65}$$

has the exact solution $y(x) = \sin^2(x)$. The absolute values of error in some points are shown in Table 4.

6. Conclusion

In this paper, we have used the non-orthogonal flatlet multiwavelet and constructed its BFMS. Then we proved that the constructed BFMS exists uniquely. Next, we demonstrated the applicability of BFMS to the differential equation (1). The technique produced accurate results when compared with a set of test problems discussed in [12] and in terms of accuracy, there is no doubt that BFMS technique is superior. We believe that this method may be applied to more complicated problems. This will hopefully be taken up in our future studies.

References

[1] B. Alpert, *A class of basis in L^2 for the sparse representation of integral operators*, SIAM J. Math. Anal. 24(1) (1993), pp. 246–262.
[2] B. Alpert, G. Beylkin, R. Coifman, and V. Rokhlin, *Wavelet-like bases for the fast solutions of second-kind integral equations*, SIAM J. Sci. Statist. Comput. 14(1) (1993), pp. 159–184.
[3] B. Alpert, G. Beylkin, D. Gines, and L. Vozovoi, *Adaptive solution of partial differential equations in multiwavelet bases*, J. Comput. Phys. 182 (2002), pp. 149–190.
[4] A. Cohen, W. Dahmen, and R. DeVore, *Adaptive wavelet methods for elliptic operator equations – convergence rates*, Math. Comp. 70 (2001), pp. 27–75.
[5] L.-H. Cui and Z.-X. Cheng, *A method of construction for biorthogonal multiwavelets system with 2r multiplicity*, Appl. Math. Comput. 167 (2005), pp. 901–918.
[6] W. Dahmen, B. Han, R.Q. Jia, and A. Kunoth, *Biorthogonal multiwavelets on the interval: Cubic Hermite splines*, Constr. Approx. 16(2) (2000), pp. 221–259.
[7] T.N.T. Goodman and S.L. Lee, *Wavelets of multiplicity r*, Trans. Amer. Math. Soc. 342 (1994), pp. 307–324.
[8] S.J. Gortler, P. Schröder, M.F. Cohen, and P. Hanrahan, *Wavelet radiosity*, Computer Graphics Proceedings, Annual Conference Series, Orlando, August 1993, pp. 221–230.
[9] B. Han and Q.T. Jiang, *Multiwavelets on the interval*, Appl. Comput. Harmon. Anal. 12 (2002), pp. 100–127.
[10] F. Keinert, *Wavelets and Multiwavelets*, Chapman and Hall/CRC, A CRC Press Company, Boca Raton, London, New York, Washington, DC, 2003.
[11] R.A. Khan, *Existence and approximation solutions of second order nonlinear Neumann problems*, Electron. J. Differ. Equ. 3 (2005), pp. 1–10.

[12] M. Lakestani and M. Dehghan, *The solution of a second-order nonlinear differential equation with Neumann boundary conditions using semi-orthogonal B-spline wavelets*, Int. J. Comput. Math. 83 (2006), pp. 685–694.
[13] G.Y. Luo, D. Osypow, and M. Irle, *Vibration modelling with fast Gaussian wavelet algorithm*, Adv. Eng. Softw. 33 (2002), pp. 191–197.
[14] V. Strela, P.N. Heller, G. Strang, P. Topiwala, and C. Heil, *The application of multiwavelet filter banks to image processing*, IEEE Trans. Image Process. 8(4) (1999), pp. 548–563.
[15] A. Tangborn and S.Q. Zhang, *Wavelet transform adapted to an approximate Kalman filter system*, Appl. Numer. Math. 33 (2000), pp. 307–316.
[16] X.G. Xia, J.S. Geronimo, D.P. Hardin, and B.E. Suter, *Design of prefilters for discete multiwavelet transform*, IEEE Trans. Signal Process. 44 (1996), pp. 25–35.