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Numerical solution of linear Fredholm integral equation by using hybrid Taylor and Block-Pulse functions

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Abstract

In this paper, we use a combination of Taylor and Block-Pulse functions on the interval [0,1], that is called Hybrid functions, to estimate the solution of a linear Fredholm integral equation of the second kind. We convert the integral equation to a system of linear equations, and by using numerical examples we show our estimation have a good degree of accuracy.

Keywords: Block-Pulse functions; Fredholm integral equation; Operational matrix; Product operation; Taylor polynomials

1. Introduction

In recent years, many different basic functions have used to estimate the solution of integral equations, such as orthonormal bases and wavelets [3–6]. In this paper we use a simple bases, a combination of Block-Pulse functions on $[0, 1]$, and Taylor polynomials, that is called the Hybrid Taylor Block-Pulse functions, to solve the linear Fredholm integral equation of the second kind. One of the advantages of this method is that, the coefficients of expansion of

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each function in this bases, could be compute directly without estimation. The method is used to solve some examples and the numerical results are represented in the last section.

1.1. Hybrid Taylor Block-Pulse functions

Definition. A set of Block-Pulse functions $b_i(\lambda)$, $i = 1, 2, \ldots, m$ on the interval $[0, 1]$ is defined as follows:

$$b_i(\lambda) = \begin{cases} 1 & \frac{i-1}{m} \leqslant \lambda < \frac{i}{m}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

The Block-Pulse functions on $[0, 1]$ are disjoint, that is, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, m$ we have $b_i(t)b_j(t) = \delta_{ij}b_i(t)$, also these functions have the property of orthogonality on $[0, 1]$, see [3].

Consider Taylor polynomials $T_m(t) = t^m$ on the interval $[0, 1]$. The Hybrid Taylor Block-Pulse functions are defined as follows.

Definition. For $m = 0, 1, 2, \ldots, M-1$ and $n = 0, 1, 2, \ldots, N$ the Hybrid Taylor Block-Pulse functions is defined as

$$b(n, m, t) = \begin{cases} T_m(Nt - (n-1)) & \frac{n-1}{N} \leqslant t < \frac{n}{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Suppose we estimate function $f(t) \in L^2[0, 1]$, using Taylor polynomials of the order $M - 1$, on the interval $[a, b]$, then using Taylor's Residual Theorem, the truncation error is

$$e_n(t) = f(t) - \sum_{i=0}^{M-1} \frac{(t-a)^i}{i!} f^{(i)}(a) = \frac{(t-a)^M}{M!} f^{(M)}(\xi),$$

where ξ lies between a and t. Then

$$\|e_n(t)\|_\infty \leqslant \frac{(b-a)^M}{M!} \|f^{(M)}(t)\|_\infty. \tag{1.3}$$

If we use the Hybrid Taylor Block-Pulse functions on the interval $[0, 1]$, then for ith sub interval $\left[\frac{i-1}{N}, \frac{i}{N}\right]$, we will have [3]

$$\|e_n(t)\|_\infty \leqslant \frac{1}{N^M M!} \|f^{(M)}(t)\|_\infty, \tag{1.4}$$

where the infinity norm is computed on the ith sub interval. It shows that the error improves while N and M are increased.

1.2. The operational matrix

If $B(t) = [b(1,0,t), b(1,1,t), \dots, b(1,M-1,t), b(2,0,t), \dots, b(N,M-1,t)]^{\mathrm{T}}$, be the vector function of Hybrid Taylor and Block-Pulse functions on $[0,1]$, the integration of this vector $B(t)$ follows:

$$\int_0^t B(t')\,\mathrm{d}t' \simeq PB(t), \tag{1.5}$$

where P is an $MN \times MN$ matrix, that is called the operation matrix for Hybrid Taylor and Block-Pulse functions [3].

Suppose that E_i, $i = 1,2,\dots,M$ be the operation matrix of Taylor polynomials on ith sub interval $\left[\frac{i-1}{N}, \frac{i}{N}\right]$, then the operation matrix P has the following form [6]:

$$P = \begin{bmatrix} E_1 & H_{12} & \dots & H_{1N} \\ 0 & E_2 & \dots & H_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & E_N \end{bmatrix}, \tag{1.6}$$

where H_{lj} is an $M \times M$ matrix and is defined as follows [3,6]:

$$H_{lj} = \frac{1}{N} \begin{bmatrix} 1 & 0 & \dots & 0 \\ \frac{1}{2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{M} & 0 & \dots & 0 \end{bmatrix} \tag{1.7}$$

also E_i on the ith interval is defined as follows:

$$E_i = \frac{1}{N} \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{M-1} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}. \tag{1.8}$$

1.3. The product operation matrix

The following property of the product of two Hybrid Taylor and Block-Pulse vector functions will also be used:

$$B(t)B^{\mathrm{T}}(t)C \simeq \widetilde{C}^{\mathrm{T}} B(t), \tag{1.9}$$

where C is a given MN column vector and $\widetilde{C}$ is an $MN \times MN$ matrix, which is called the product operation matrix of Hybrid Taylor and Block-Pulse functions.

Consider $T(t) = [T_0(t), T_1(t), \dots, T_{M-1}(t)]^{\mathrm{T}}$ and $A = [a_0, a_1, \dots, a_{M-1}]^{\mathrm{T}}$, where $T_i(t), i = 0, 1, \dots, M-1$ is ith Taylor polynomial, then

$$T(t)T^{\mathrm{T}}(t)A \simeq \widetilde{A}T(t),$$

where $\widetilde{A}$ is an $M \times M$ matrix, and defined as follows:

$$\widetilde{A} = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{M-2} & a_{M-1} \\ 0 & a_0 & a_1 & \dots & a_{M-3} & a_{M-2} \\ 0 & 0 & a_0 & \dots & a_{M-4} & a_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_0 & a_1 \\ 0 & 0 & 0 & \dots & 0 & a_0 \end{bmatrix}. \tag{1.10}$$

The product operation matrix for Hybrid Taylor and Block-Pulse functions is defined as follows:

$$B(t)B^{\mathrm{T}}(t)C \simeq \widetilde{C}^{\mathrm{T}}B(t),$$

such that

$$\widetilde{C} = \begin{bmatrix} \widetilde{C}_1 & 0 & \dots & 0 \\ 0 & \widetilde{C}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \widetilde{C}_N \end{bmatrix} \tag{1.11}$$

and $\widetilde{C}_i$ is the operation matrix of transferred Taylor polynomials on the ith sub interval and $\widetilde{C}_1 = \widetilde{C}_2 = \cdots = \widetilde{C}_N$ [1,3,6].

2. Function approximation

A function $f \in L^2[0, 1)$ can be approximated as

$$f(t) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} c(n, m)b(n, m, t) = C^{\mathrm{T}}B(t), \tag{2.1}$$

where $B(t)$ is the vector function defined before and $c(n, m)$ is defined as follows [3]:

$$c(n, m) = \frac{1}{N^m m!} \left(\frac{\mathrm{d}^m f(t)}{\mathrm{d}t^m} \right) \bigg|_{t=\frac{n-1}{N}} \tag{2.2}$$

for $n = 1, 2, \dots, N$ and $m = 0, 1, \dots, M-1$.

We can also approximate the function $k(t,s) \in L^2([0,1] \times [0,1])$ as follows:

$$k(t,s) = B^{\mathrm{T}}(t)KB(s), \tag{2.3}$$

where K is an $MN \times MN$ matrix that

$$K_{ij} = \frac{1}{N^{u+v}u!v!}\left(\frac{\partial^{i+j}k(t,s)}{\partial t^i \partial s^j}\right)\Bigg|_{(t,s)=\left(\frac{i}{N},\frac{j}{N}\right)} \tag{2.4}$$

for $i,j = 0,1,\dots,MN-1$, $u = i - \left[\frac{i}{N}\right]N$, $v = j - \left[\frac{j}{N}\right]N$.

We also define the matrix D as follows

$$D = \int_0^1 B(t)B^{\mathrm{T}}(t)\,\mathrm{d}t. \tag{2.5}$$

For the Hybrid Taylor and Block-Pulse functions, D has the following form:

$$D = \begin{bmatrix} D_1 & 0 & \dots & 0 \\ 0 & D_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_N \end{bmatrix},$$

where D_i is defined as follows:

$$D_i = \frac{1}{N}\int_0^1 T(t)T^{\mathrm{T}}(t)\,\mathrm{d}t.$$

3. Fredholm integral equation of the second kind

Consider the following integral equation:

$$q(t)y(t) = x(t) + \lambda\int_0^1 k(t,s)y(s)\,\mathrm{d}s, \tag{3.1}$$

where $x \in L^2[0,1]$, $q \in L^2[0,1]$, $k \in L^2([0,1]\times[0,1])$ and y is an unknown function [2].

Let approximate x,q,y and k by (2.1)–(2.4) as follows:

$$x(t) \simeq X^{\mathrm{T}}B(t), \quad q(t) \simeq Q^{\mathrm{T}}B(t), \quad y(t) \simeq Y^{\mathrm{T}}B(t), \quad k(t,s) \simeq B^{\mathrm{T}}(t)KB(s).$$

With substituting in (3.1)

$$Q^{\mathrm{T}}B(t)B^{\mathrm{T}}(t)Y = B^{\mathrm{T}}(t)X + \lambda\int_0^1 B^{\mathrm{T}}(t)KB(s)B^{\mathrm{T}}(s)Y\,\mathrm{d}s$$

with (1.7) we have

$$\begin{aligned} B^{\mathrm{T}}(t)\widetilde{Q}^{\mathrm{T}}Y &= B^{\mathrm{T}}(t)X + \lambda B^{\mathrm{T}}(t)K\left(\int_0^1 B(s)B^{\mathrm{T}}(s)\,\mathrm{d}s\right)Y \\ &= B^{\mathrm{T}}(t)X + \lambda B^{\mathrm{T}}(t)KDY = B^{\mathrm{T}}(t)(X + \lambda KDY), \end{aligned}$$

then

$$\widetilde{Q}^{\mathrm{T}}Y = (X + \lambda KDY) \Rightarrow (\widetilde{Q}^{\mathrm{T}} - \lambda KD)Y = X. \tag{3.2}$$

This is a linear system of equations that gives the numerical solution of the integral equation, y_n.

4. Error estimation

Consider the following Fredholm integral equation of the second kind:

$$y(t) = \lambda \int_0^1 k(t,s)y(s)\,\mathrm{d}s + x(t).$$

Let $e_n(t) = y(t) - y_n(t)$ be the error function, where $y_n(t)$ is the estimation of the true solution $y(t)$. Then

$$y_n(t) = \lambda \int_0^1 k(t,s)y_n(s)\,\mathrm{d}s + x(t) + H_n(t), \tag{4.1}$$

where $H_n(t)$ is the perturbation function that depends only on $y_n(t)$, and is given with

$$H_n(t) = y_n(t) - \lambda \int_0^1 k(t,s)y_n(s)\,\mathrm{d}s - x(t). \tag{4.2}$$

With (4.1) and (4.2) we have

$$e_n(t) - \lambda \int_0^1 k(t,s)e_n(s)\,\mathrm{d}s = -H_n(t). \tag{4.3}$$

This is a Fredholm integral equation of the second kind. We can solve this integral equation, using the method mentioned before as an estimation of the error function of the method.

5. Numerical examples

Consider the following three examples. We solve them with different M and N's, using the method represented before. The result improves when we use larger M and N's as shown in tables.

Example 1

$$y(t) = \int_0^1 (t+s)y(s)\,\mathrm{d}s + \mathrm{e}^t + (1-e)t - 1.$$

With exact solution $y(t) = \mathrm{e}^t$. Table 1 shows the numerical results of Example 1.

Example 2

$$y(t) = \int_0^1 \left(s^2t - (3/2)st^2\right)y(s)\,\mathrm{d}s + (3/4)t^2 - \frac{4}{3}\ln(2)t + \frac{5}{9}t.$$

With exact solution $y(t) = 2\ln(t+1)$. Table 2 shows the numerical results of Example 2.

Example 3

$$y(t) = -\frac{1}{3}\int_0^1 \mathrm{e}^{2t-(5/3)s}y(s)\,\mathrm{d}s + \mathrm{e}^{2t+(1/3)}.$$

With exact solution $y(t) = \mathrm{e}^{2t}$. Table 3 shows the numerical results of Example 3.

Table 1

M	N	$\lVert y - y_n \rVert_\infty$	Cond$(I-KD)$
3	10	1.383039×10^{-3}	102.5481
3	20	1.777834×10^{-4}	114.2248
3	40	2.255183×10^{-5}	122.7133
3	80	2.840509×10^{-6}	128.4338
4	10	1.383039×10^{-3}	130.2973
4	20	1.777834×10^{-4}	144.1830
4	40	2.255183×10^{-5}	154.0536
4	80	2.840509×10^{-6}	160.6114

Table 2

M	N	$\lVert y - y_n \rVert_\infty$	Cond$(I-KD)$
3	10	6.145928×10^{-3}	2.2860
3	20	1.527655×10^{-3}	2.5913
3	40	3.808887×10^{-4}	2.7694
3	80	9.509965×10^{-5}	2.8656
4	10	4.557614×10^{-3}	2.5172
4	20	1.139760×10^{-3}	2.8776
4	40	2.849499×10^{-4}	3.0860
4	80	7.123710×10^{-5}	3.1981

Table 3

M	N	$\|y - y_n\|_\infty$	$\mathrm{Cond}(I - KD)$
3	10	3.005651×10^{-2}	6.0920
3	20	8.668870×10^{-3}	7.0734
3	40	2.316608×10^{-3}	7.6292
3	80	5.981580×10^{-4}	7.9248
4	10	2.892984×10^{-2}	7.1086
4	20	7.351252×10^{-3}	8.3168
4	40	1.847061×10^{-3}	9.0048
4	80	4.625381×10^{-4}	9.3718

6. Conclusion

The Block-Pulse functions have the property of orthogonality on the interval $[0, 1]$, then we can combine various bases with Block-Pulse functions to produce Hybrid functions, with the property of semi-orthogonality (orthogonality on disjoint sub intervals). The Hybrid Taylor and Block-Pulse functions are used to solve the Fredholm integral equation. The same approach can be used to solve other problems. The numerical examples shows that the accuracy improves with increasing the M and N, then for better results, using the larger M, specially larger N is recommended.

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