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Expansion method for linear Fredholm integral equations of second kind by Chebyshev, Legendre and Shannon wavelats and the comparison of their numerical results.

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Abstract

In this paper, we solve linear Fredholm integral equations of second kind by Chebyshev,Legendre and Shannon wavelets bases.For this,we estimate the solution of linear Fredholm integral equations by these wavelets and construct Galerkin system.To compare the efficiency of every wavelet,we consider numerical examples, whose exact solutions are available.Finally we proceed to the numerical solutions of integral equations of each wavelet.

Keywords: Chebyshev wavelet, Legendre wavelet, Shannon wavelet, expansion method, Galerkin system.

1 Introduction

The main characteristic of orthogonal basis is that it reduces solving of linear integral equation to solving a system of linear algebraic equations. Galerkin method uses orthogonal functions and we used Chebyshev,Legendre and Shannon wavelets for solving linear Fredholm integral equations. We truncated $x(t)$ by $x_N(t)$:

$$x(t) \approx x_N(t) = \sum_{i=0}^{N-1} c_i \varphi_i(t) \tag{1}$$

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In relation (1)the elements $\varphi_0(t)$,$\varphi_1(t)$,...,$\varphi_{N-1}(t)$ are the orthogonal basis functions defined on a certain interval[a,b]. Here we consider $\varphi_i(t)$ as Chebyshev,Legendre and Shannon wavelets. The main characteristic of wavelet bases (and therefore Chebyshev,Legendre and Shannon wavelets) which leads to a sparse matrix is that:

1)The vanishing moment property

2)Having small interval of support.

As least we compare results of numerical solution of linear Fredholm integral equation by Chebyshev,Legendre and Shannon wavelets to show the efficiency of each wavelet.

2 Chebyshev wavelets and the structure the Chebyshev-Galerkin system

2.1 Chebyshev wavelets and their properties

$T_m(t)$ are Chebyshev polynomials of the first kind of degree m which are orthogonal with respect to the weight function w(t)=$1/\sqrt{1-t^2}$,on the interval [-1,1] wich are presented as the following formulas:

$$T_0(t) = 1, T_1(t) = t, T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t) \qquad m = 1, 2, 3, \dots \tag{2}$$

Chebyshev wavelets are composed of Chebyshev polynomials $\psi_{n,m} = \psi(k, n, m, t)$ in which $n = 1, 2, \dots, 2^{k-1}$, k is any positive integer,m is the degree of Chebyshev polynomials of the first kind and t denotes the time[3]:

$$\psi_{n,m} = \begin{cases} 2^{k/2\widetilde{T_m}(2^kt-2n+1)} & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

where

$$\widetilde{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}} & \text{m=0,} \\ \sqrt{\frac{2}{\pi}}T_m(t) & \text{m}\neq\text{0.} \end{cases} \tag{4}$$

2.2 Function approximation

A function $x(t) \in L^2[a,b]$ can be expanded as

$$x(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t) \tag{5}$$

where $c_{n,m} = (x(t), \psi_{n,m}(t))$ are coefficients of wavelets. If we truncated series in (5),we obtain

$$x(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t)$$

Let

$$C = [C_1, C_2, \ldots, C_{2^{k-1}}]^T \tag{6}$$

$$\psi(t) = [\psi_1, \psi_2, \ldots, \psi_{2^{k-1}}]^T \tag{7}$$

and

$$c_i = [c_{i0}, c_{i1}, \ldots, c_{i,M-1}]$$

$$\psi_i(t) = [\psi_{i0}, \psi_{i1}, \ldots, \psi_{i,M-1}] \qquad fori = 1, 2, \ldots, 2^{k-1}$$

therefore we have:

$$x(t) \approx C^T\psi(t) \tag{8}$$

where C and ψ,are $2^{k-1}M \times 1$ matrices.

2.3 Fredholm integral equations via CWG

consider the following Fredholm integral equations of sekond kind:

$$x(s) - \int_0^1 k(s,t)x(t)dt = y(s) \tag{9}$$

where $y(s) \in L^2[0,1], K \in L^2[0,1] \times [0,1]$ and $x(t)$ is the unknown function.Now we approximate x and k by Chebyshev wavelets:

$$x(t) \approx C^T\psi(t), \quad k(s,t) \approx \psi^T(s)k\psi(t), \quad y(s) \approx y^T\psi(s) \tag{10}$$

by substituted (10) in (9) we have:

$$\psi^T(s)C - \int_0^1 \psi^T(s)k\psi(t)\psi^T(t)Cdt = \psi^T(s)y$$

but

$$\int_0^1 \psi^T(s)k\psi(t)\psi^T(t)Cdt = \psi^T(s)k(\int_0^1 \psi(t)\psi^T(t)dt)C = \psi^T(s)kC$$

therefore

$$(I-k)C = y \tag{11}$$

We obtain moments of matrix k and y by the inner products:

$$y_{i,l} =< y, \psi_{il} >= \int_0^1 y(s)\psi_{il}(s)w_l(s)ds = \int_{l-1/2^{k-1}}^{l/2^{k-1}} y(s)2^{k/2}\widetilde{T}_i(2^k s - 2l + 1)w(2^k - 2l + 1)ds$$

Now let

$$u = 2^k s - 2l + 1$$

$$y_{i,l} = 2^{-k/2}\int_{-1}^1 y(2^{-k}(u + 2l - 1))\widetilde{T}_i(u)w(u)du$$

$$\approx 2^{-k/2}\frac{\pi}{N}\sum_{j=0}^{N} y(2^{-k}(cos(\frac{j\pi}{N} + 2l - 1)))cos(\frac{ji\pi}{N})\delta_i \tag{12}$$

$$\delta_i = \begin{cases} \sqrt{\frac{1}{\pi}} & \text{i=0,} \\ \sqrt{\frac{2}{\pi}} & \text{i}\neq\text{0.} \end{cases}$$

and

$$k_{i,l} = 2^{-k}\int_{-1}^1\int_{-1}^1 \widetilde{T}_l(u)\widetilde{T}_l(v)w(u)w(v)k(2^{-k}(v + 2i - 1), 2^{-k}(u + 2i - 1))dudv \tag{13}$$

3 Legendre wavelets and the structure the Legendre-Galerkin system

3.1 Legendre wavelets and their properties

Legendre polynomials of order m,$L_m(t)$,which are orthogonal with respect to the weight function $w(t) = 1$ and derived from formula:

$$L_0(t) = 1, \quad L_1(t) = t, \quad L_{m+1}(t) = \tfrac{2m+1}{m+1} t L_m(t) - \tfrac{m}{m+1} L_{m-1}(t) \qquad m = 1, 2, 3, \ldots$$

Legendre wavelets are composed of Legendre polynomials $\psi_{m,n} = \psi(k, n, m, t)$ in which $n = 1, 2, \ldots$, $k = 2, 3, \ldots$, $m = 0, 1, \ldots, M-1$ is the degree of Legendre polynomials, M is any positive integer.They are defind on the interval $[0, 1)$ as follows[6]:

$$\psi_{n,m}(t) = \begin{cases} (2m+1)^{1/2}\ 2^{k/2} L_m(2^k t - 2n + 1) & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

3.2 Function approximation

A function $x(t) \in L^2[a, b]$ can be expanded as

$$x(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t) \tag{15}$$

where $c_{n,m} = (x(t), \psi_{n,m}(t))$ are coefficients of wavelets.

If we truncated series in (15),we obtain

$$x(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(t) \approx C^T \psi(t)$$

C and ψ define in $(6), (7)$.

3.3 Fredholm integral equations via LWG

Approximate x, k and y by Legendre wavelets therefore we have:

$$(I - k)C = y \tag{16}$$

$$y_{i,l} =< y, \psi_{i,l} >= \int_0^1 y(s)\psi_{i,l}(s)w(s)ds = \int_{i-1/2^{k-1}}^{i/2^{k-1}} y(s)(2l+1)^{1/2}\ 2^{k/2}L_l(2^k s - 2i + 1)ds$$

now let

$$u = 2^k s - 2i + 1$$

$$y_{i,l} = (2l+1)^{1/2}2^{-k/2}\int_{-1}^{1} y(2^{-k}(u+2i-1))L_l(u)du \tag{17}$$

and

$$k_{i,l} = (2l+1)^{1/2}\ 2^{-k}\int_{-1}^{1}\int_{-1}^{1} L_l(u)L_l(v)k(2^{-k}(v+2i-1), 2^{-k}(u+2i-1))dudv \tag{18}$$

4 Shannon wavelets and the structure the Shannon-Galerkin system

4.1 Shannon wavelets and their properties

Shannon (sinc) wavelet,an important function,is defined as below:

$$sinc(x) = \begin{cases} \frac{sin(x\pi)}{x\pi} & \text{x}\neq 0, \\ 1 & \text{x=0.} \end{cases} \tag{19}$$

Shannon wavelet properties[5]:

1.The Shannon scaling function is smooth.This means that the function and its entire derivative exist and continuous.

2.Shannon function does not have compact support.

3.The Shannon function is even.

4.Set $h^{-1/2}sinc(\frac{x-ih}{h})$ is orthogonal for every $h > 0$.

Classical Shannon sampling theorem[7]

Let f be a continuos member of $L^2(R)$ whose Fourier transform vanishes outside of the cube $I = [-1/2, 1/2]$.Then f is uniquely determined by its values on Z:

$$f(x) = \sum_{k\in Z} f(k)sinc(x-k) \tag{20}$$

4.2 Fredholm integral equations via SWG

Now we use Shannon classic theorem for approximate functions in Fredholm integral equations of second kind[5]:

$$x(s) \approx x_N(s) = \sum_{i=-(N+1)}^{i=N+1} x_i sinc(\frac{s-ih}{h})$$

$$y(s) \approx y_N(s) = \sum_{i=-(N+1)}^{i=N+1} y_i sinc(\frac{s-ih}{h})$$

$$k_j(s) = \int_{a)}^{b} k(s,t) sinc(\frac{t-jh}{h})dt \qquad -(N+1) \leq j < (N+1)$$

$$k_j(s) \approx k_{j,N}(s) = \sum_{i=-(N+1)}^{i=N+1} k_{ij} sinc(\frac{s-ih}{h}) \quad j = -(N+1), -N, \ldots, N, N+1$$

where

$$x_i = x(ih), \qquad y_i = y(ih), \qquad k_{ij} = k_j(ih)$$

Now we of 4^{th} properties of Shannon wavelet therefore we have:

$$x_i - \sum_{j=-(N+1)}^{N+1} k_{ij} x_j = y_i \qquad i = -(N+1), -N, \ldots, N, (N+1)$$

5 Expansion error

We use choose a threshold $\varepsilon_0 > 0$, and get the system of linear equations whose matrix is sparse.

Let

$$\overline{k_{ij}} = \begin{cases} k_{ij} & \mid k_{ij} \mid \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\overline{k} = (\overline{k_{ij}})$$

Then we solve following system of linear equation:

$$(I - \overline{k})C = y \tag{21}$$

For showing efficiency of numerical methods, we consider the following examples.We note that,[4]:

$$\| e_p \|_2 = (\int_{-1}^{1} e_p^2(t)dt)^{1/2} \approx (\frac{1}{p}\sum_{i=0}^{p} e^2(s_i))^{1/2} \tag{22}$$

$$e(s_i) = x(s_i) - x_p(s_i) \qquad i = 0, 1, \ldots, p$$

Such that $x_p(s_i)$ and $x(s_i)$ are respectively the approximate and exact solution of integral equation.

6 Numerical examples

In Tables 1-6 ε_0 is the threshold,$\varepsilon = \| e_p \|_2$ and n is the order of matrix $A = I - \overline{k}$.

Example 6.1 $x(s) = e^s - \frac{e^{s+1}-1}{s+1} + \int_0^1 e^{st}x(t)dt \qquad exactsolution: \ x(s) = e^s$

a)Chebyshev wavelets

Table 1:Absolute error of exact and approximate of Example 1 by Chebyshev wavelets

M	k	ε_0	ε	n
2	2	10^{-3}	8.2×10^{-3}	2
3	2	10^{-3}	3.3×10^{-6}	6
3	3	10^{-3}	4.1×10^{-8}	12
4	3	10^{-3}	2.3×10^{-10}	16
5	3	10^{-3}	5.1×10^{-13}	20

b)Legendre wavelets

Table 2: Absolute error of exact and approximate of Example 1 by Legendre wavelets

M	k	ε_0	ε	n
2	2	10^{-3}	6.3×10^{-3}	2
3	2	10^{-3}	8.1×10^{-5}	6
3	3	10^{-3}	1.1×10^{-7}	12
4	3	10^{-3}	7.1×10^{-9}	16
5	3	10^{-3}	2.3×10^{-12}	20

c)Shannon wavelets

Table 3: Absolute error of exact and approximate of Example 1 by Shannon wavelets

N	ε_0	ε	n
2	10^{-3}	2.6×10^{-4}	7
3	10^{-3}	5.7×10^{-6}	9
4	10^{-3}	8.9×10^{-9}	11
5	10^{-3}	9.2×10^{-11}	13
6	10^{-3}	3.3×10^{-13}	15
7	10^{-3}	9.4×10^{-17}	17

Example 6.2 $x(s) = sins - s + \int_0^{\pi/2} stx(t)dt$ *exact solution:* $x(s) = sins$

a)Chebyshev wavelets

Table 4:Absolute error of exact and approximate of Example 2 by Chebyshev wavelets

M	k	ε_0	ε	n
2	2	10^{-3}	2.9×10^{-3}	2
3	2	10^{-3}	3.1×10^{-5}	6
3	3	10^{-3}	3.7×10^{-7}	12
4	3	10^{-3}	1.9×10^{-9}	16
5	3	10^{-3}	6.8×10^{-11}	20

b)Legendre wavelets

Table 5: Absolute error of exact and approximate of Example 2 by Legendre wavelets

M	k	ε_0	ε	n
2	2	10^{-3}	2.9×10^{-2}	2
3	2	10^{-3}	1.6×10^{-5}	6
3	3	10^{-3}	6.6×10^{-6}	12
4	3	10^{-3}	5.1×10^{-8}	16
5	3	10^{-3}	1.8×10^{-11}	20

c)Shannon wavelets

Table 6: Absolute error of exact and approximate of Example 2 by Shannon wavelets

N	ε_0	ε	n
2	10^{-3}	1.2×10^{-3}	7
3	10^{-3}	4.4×10^{-4}	9
4	10^{-3}	2.9×10^{-7}	11
5	10^{-3}	8.8×10^{-10}	13
6	10^{-3}	6.6×10^{-13}	15
7	10^{-3}	6.9×10^{-16}	17

7 Conclusion

In this work,we solved Fredholm integral equations of the second kind by using Chebyshev,Legendre and Shannon wavelets that these are orthonormal functions. An orthonormal bases have the advantage that these guarantees the stability of the matrix equation in Galerkin method.

Numerical examples show that,using Shannon wavelets have the best results in larg orders of matrix $A = I - \overline{k}$.

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