HAAR WAVELET ALGORITHM FOR SOLVING CERTAIN DIFFERENTIAL, INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT

In recent years, there has been greater attempt to find wavelet based solutions of numerical equations using wavelet methods. In this paper, Haar wavelet algorithm has been developed to reduce the complexity in construction of wavelet based operational matrices. The operational matrices are very powerful for finding the derivatives of Haar wavelet functions. Moreover, Haar wavelet collocation techniques used to find solutions of problems concerning differential, integral and integro-differential equations. Wavelet solutions are in good agreement with analytic ones. Finally, we have analyzed the errors.

Keywords: Haar Wavelets, Operational Matrix, Haar Collocation Technique, Multiresolution Analysis

1 INTRODUCTION

For dynamical systems which involve abrupt variations or in case of functions having short interval of existence, the global support of orthogonal functions like block pulse, Laguerre, Legendre, Chebyshev and Fourier on the whole interval $A \le t \le B$ are a drawback. Chen and Hsiao fill the gap by developing Haar based algorithm for solving differential equations in 1997. In numerical analysis this technique is used to reduce differential equations to algebraic one in linear matrix system.

The basic idea relies on the integral of a basic vector $\varphi(t)$ such that $\int_0^x \varphi(t)dt \cong P\varphi(x),$ where $\varphi(t) = [\varphi_{(0)}(t), \varphi_{(1)}(t), \varphi_{(2)}(t), ..., \varphi_{(m-1)}(t)]^T$ in which elements $\varphi_{(0)}(t), i = 0,1,2,....,m-1$ are the basic orthogonal functions defined on $[A, B]$. The matrix P is uniquely defined.

Wavelet analysis allows representing a function or signal in terms of a set of orthonormal basis functions called wavelets, which are localized both in time and scale. From a continuous function $\psi(x)$ called mother wavelet, wavelet family is formed by translation and dilation of

$\psi(x)=2^{j/2}\psi(2^j x-k),$ where j,k are non-negative integers. Selecting a suitable $\psi(x)$ various wavelet families are obtained. Wavelets introduced by Ingrid Daubechies in 1998 are quite frequently used for solving differential problems, due to differentiability and having minimum size support of these wavelets.

A drawback of Daubechies wavelets is that they do not have an explicit expression, and therefore analytical differentiation or integration is not possible. Thereby complicating the solutions of differential equations where integrals of the type

$$\int_A^B G(t,\psi_{jk},\psi'_{jk},\psi''_{jk})dt$$

are to be computed. G is generally a nonlinear function. To evaluate these integrals, concept of connection coefficients is introduced. But the process of evaluating these coefficients is very complicated, and must be carried out separately for different types of integrals. Besides, it can be done only for some simple types of nonlinearities, mainly for quadratic nonlinearity. Other wavelets like Symlet and Coiflet suffer this disadvantage.

The first attempt to apply differential and integral equations was made in 1990s. Based on Haar function, a rectangular pulse pair, introduced by Alfred Haar in (1910), Chen and Hsiao(1997), applied successful application of Haar wavelet in solving differential equations. Haar function infect is the Daubechies wavelet of order one. Certainly, it is the simplest of the orthonormal wavelets with compact support. As short coming, Haar wavelets are not continuous; their derivatives do not exist at the point of discontinuity. Thereby direct application of Haar wavelet is not possible in solving differential equations. Two possibilities are generally worked out to have the applications of Haar wavelets.

i) Regularization of piecewise constant Haar function by interpolation splines as introduced by Cattani(2001).But this complicates the solution process and simplicity of Haar function is lost.

ii) Chen and Hsiao(1997) have given a concept in which function (required solution) corresponding to highest derivative of differential equation is expanded into Haar series. The other derivatives are obtained through Haar series. The simplicity of Haar wavelet is preserved in this technique. Since then the solutions of dynamical systems by Haar wavelet took tremendous growth.

With the pioneering work by Hsiao and Wang (2001), there has been increasing attempt to solve the systems of ordinary, integral and integro-differential equations via Haar wavelet, in mathematical context. Here we present a brief review of progress in this field by different scholars.

Ordinary differential equations

In 1997, Chen and Hsiao gave solution of lumped parameter system. Subsequently, Wang et al.(2001) studied time delayed system. Lepik(2007) worked on linear, nonlinear, Robertson and singular perturbation problems and boundary value ODE while Zhi et al.(2009) studied finite, length beam system.

Segmentation method for solving linear and non linear ODE was developed by Lepik (2007). Linear and nonlinear stiff systems were handled by Hsiao-Wang(2001). A thorough analysis of Haar applications in linear and nonlinear boundary problems with convergence analysis was treated by Islam(2010).

Integral and Integro-differential equations

For numerical solution of linear integral equations traditional quadrature formula methods and spline approximations are used. The methods require solving system of linear equations. The process is complicated as it involves big matrices requiring a huge number of arithmetic operations and a large storage capacity. The fruitful way is to convert fully populated transform matrix to sparse matrix and wavelet basis is the one possibility. Lepik et al. (2007&2009) studied Fredholm and Volterra type linear integral, weakly singular Volterra, nonlinear Volterra and integro-differential equations.

Multiresolution Analysis (MRA)

Family of Haar wavelets utilizes the concept of MRA(Daubechies,1998).
The increasing sequence $\left\{V_j\right\}_{j\in Z}$ of subset of $L^2(R)$ with scaling function φ is called MRA if it satisfies the conditions

$$(i)\bigcap_{-\infty}^{\infty}V_j=\left\{0\right\},\bigcup_{-\infty}^{\infty}V_j=L^2(R)$$

$$(ii)\, f(x)\in V_0 \text{ iff } f(2^j x)\in V_j\ \forall j\in Z$$

$$(iii)\left\{\varphi(x-k),k\in Z\right\} \text{ is an orthonormal basis for } V_0$$

By above definition of MRA, the sequence $\left\{\varphi_{j,k}(x)=2^{j/2}\varphi(2^j x-k)\right\}_{-\infty}^{\infty}$ forms an orthogonal basis for V_j. For mother wavelet $\psi(x)$, $\psi(x)=\sum_{k=0}^{N-1}a_k\varphi(2x-k).$

Section 2 is devoted to mathematical formulation of Haar wavelets and operational matrix. Here we develop an algorithm which will make easy computation of various entries of latter. Section 3 represents the function in terms of series formulation of Haar wavelets. We formulate and demonstrate three important applications concerning differential, integral and integro-differential equations. Section 4 concludes the finding; while last section is the set of references used.

2 HAAR WAVELETS AND OPERATIONAL MATRICES

A Haar function, called Haar scaling, is a function with magnitude unity in the interval $[0,1]$. Let $h_1(x)=1$ for $0\le x\le 1$. Take the second generated curve $h_2(x)$ as a square pulse obtained from $h_1(x)$ after compression of [0,1] into two halves $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$.The curve $h_2(x)$ is called Haar wavelet. All the other subsequent curves are generated from $h_2(x)$ with the two operations of translation and dilation.

$$h_1(x) = \begin{cases} 1 & for\ x \in [0,1] \\ 0 & elsewhere \end{cases}$$

In nut shell,

$$h_i(x) = \begin{cases} 1 & for\ x \in [\xi_1(i), \xi_2(i)] \\ -1 & for\ x \in [\xi_2(i), \xi_3(i)] \\ 0 & elsewhere \end{cases} \tag{2.1}$$

$$i = 2^j + k + 1,\ j \geq 0,\ 0 \leq k \leq 2^j - 1$$

Here $\xi_1 = \dfrac{k}{m}, \xi_2 = \dfrac{k+0.5}{m}$ and $\xi_3 = \dfrac{k+1}{m}$, $m = 2^j$, $j = 0,1,2,....,j.J$ is the maximum level of resolution. $k = 0,1,2,....,m-1,$ the translation parameter.

The index $i = m + k + 1$. Maximum of i is $M = 2m = 2^{j+1}$. The collocation points $x_l = \dfrac{l-0.5}{2m}, l = 1,2,3,....,2m$ are obtained by discretizing Haar function $h_i(x)$ by dividing the interval [0,1] into $2m$ parts of equal length $\Delta t = \dfrac{1}{2m}$ to get coefficient matrix H of order $2m \times 2m$.

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Notice that Haar wavelets are orthogonal, i.e.

$$\int_0^1 h_i(x) h_l(x) dx = \begin{cases} \dfrac{1}{m} & for\ i = l \\ 0 & for\ i \neq l \end{cases}$$

The operational matrix P which is a $2m$ square matrix is define by

$$P_{1,i}(x) = \int_0^x h_i(t) dt \tag{2.2}$$

Often, we need the integrals

$$P(x) = \underbrace{\int_A^x \int_A^x .... \int_A^x}_{\alpha-times} h_i(t)\, dt^\alpha = \frac{1}{(\alpha-1)!} \int_A^x (x-t)^{\alpha-1} h_i(t) dt \tag{2.2a}$$

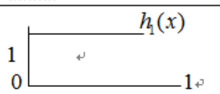
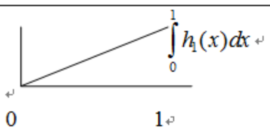
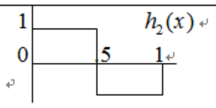
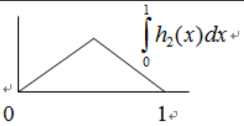
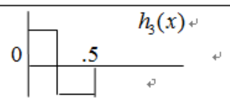
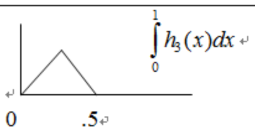
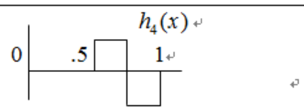
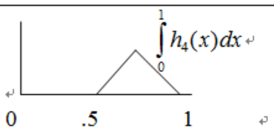
$\alpha = 2,3,....,n$ and $i = 1,2....2m.$

The case $\alpha = 1$, corresponds to function $P_{1,i}(x)$. Taking into account (2.1) these integrals can be calculated analytically; by doing it we obtain

$$P_{\alpha,i}(x) = \begin{cases} 0 & for\ x < \xi_1(i) \\ \frac{1}{\alpha!}[x-\xi_1(i)]^{\alpha} & for\ x \in [\xi_1(i), \xi_2(i)] \\ \frac{1}{\alpha!}\left\{[x-\xi_1(i)]^{\alpha} - 2[x-\xi_2(i)]^{\alpha}\right\} & for\ x \in [\xi_2(i), \xi_3(i)] \\ \frac{1}{\alpha!}\left\{[x-\xi_1(i)]^{\alpha} - 2[x-\xi_2(i)]^{\alpha} + [x-\xi_3(i)]^{\alpha}\right\} & for\ x > \xi_3(i) \end{cases} \tag{2.3}$$

These formulas hold for $i > 1$ In case $i = 1$ we have $\xi_1 = A, \xi_2 = \xi_3 = B$ and $P_{\alpha,1}(x) = \frac{1}{\alpha!}(x-A)^{\alpha}$. Here $A = 0, B = 1$. The resulting $P_{1,i}(x)$ is a triangular shape, whereas $P_{2,i}(x)$ is the parabolic one.

Table gives the graph of first four Haar wavelets and corresponding integrals.

S. No.	Haar wavelets	Corresponding integral
1.		
2.		
3.		
4.		

The pattern of design of the matrices P is similar to Haar wavelet and is based on the algorithm(2.3).

Below we develop an algorithm based on (2.3) which would reduce computational and procedural complexity in comparison to what has been developed in Chen and Hsiao (1997).

Algorithm

(2.3) can be expressed in notational form as:

$$\begin{bmatrix} P_{\alpha,1} \\ P_{\alpha,2} \\ . \\ . \\ P_{\alpha,2M} \\ \end{bmatrix}$$

Wherein

$$I.\, P_{\alpha,1}(x) = \frac{1}{\alpha!}x^{\alpha} \qquad \textit{for } x \in [0,1] \tag{2.4.1}$$

$$II.\, P_{\alpha,i}(x) = \begin{cases} \frac{1}{\alpha!}x^{\alpha} & \textit{for } x \in [\xi_1, \xi_2] \\ \frac{1}{\alpha!}\left[\left(\frac{2l-1}{4M}+\frac{1}{2m}\right)^{\alpha} - 2\left(\frac{2l-1}{4M}\right)^{\alpha}\right] & \textit{for } x \in [\xi_2, \xi_3] \end{cases} \tag{2.4.2a,b}$$

$M = \max\{m\}$ and $l = 1, 2, 3, \ldots, \frac{M}{m}, 2 \le i \le 2M$.

Explanation of Algorithm

Let $x - \xi_2 = \frac{2l-1}{4M}$.

We find $x = \frac{2l-1}{4M} + \xi_2$.

Now $x - \xi_1 = \frac{2l-1}{4M} + \xi_2 - \xi_1 = \frac{2l-1}{4M} + \frac{1}{2m}$.

Using the values of $x-\xi_1$ and $x-\xi_2$ in 3^{rd} term of (2.3), we establish the above algorithm (2.4.2b). Rest is obvious.

Let $\alpha = 1$ and $M = 2$. Then

$$P_4 = \frac{1}{8}\begin{bmatrix} 1 & 3 & 5 & 7 \\ 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Corresponding to $M = 4$,

$$P_8 = \frac{1}{16}\begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 \\ 1 & 3 & 5 & 7 & 7 & 5 & 3 & 1 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

If $\alpha = 2$ and $M = 2$,

$$P_4 = \frac{1}{2!8^2}\begin{bmatrix} 1 & 3^2 & 5^2 & 7^2 \\ 1 & 3^2 & 23 & 31 \\ 1 & 7 & 0 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

Corresponding to $M = 4$,

$$P_8 = \frac{1}{2!16^2}\begin{bmatrix} 1 & 3^2 & 5^2 & 7^2 & 9^2 & 11^2 & 13^2 & 15^2 \\ 1 & 3^2 & 5^2 & 7^2 & 79 & 103 & 119 & 122 \\ 1 & 3^2 & 23 & 31 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3^2 & 23 & 31 \\ 1 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

Remarkable that Chen and Hsiao consider integral of Haar wavelets as:

$$\int_0^t h_m(t)dt \cong P_{m\times m} h_m(t), \qquad t \in [0,1) \tag{2.5}$$

which is, however, different than in Lepik (2007) as explained in Section2,*m*-square matrix *P*, called operational matrix of integration, is given by

$$P_{m\times m} = \frac{1}{2m}\begin{bmatrix} 2mP_{\left(\frac{m}{2}\right)\times\left(\frac{m}{2}\right)} & -H_{\left(\frac{m}{2}\right)\times\left(\frac{m}{2}\right)} \\ H^{-1}_{\left(\frac{m}{2}\right)\times\left(\frac{m}{2}\right)} & O_{\left(\frac{m}{2}\right)\times\left(\frac{m}{2}\right)} \end{bmatrix}, \quad P_{1\times 1} = \left[\frac{1}{2}\right]$$

where $H_{m\times m} = \left[h_m(t_0), h_m(t_1), ..., h_m(t_{m-1})\right]^T$, $\frac{i}{m} \le t_i \le \frac{i+1}{m}$.

$$H_{m\times m}^{-1}=\frac{1}{m}H_{m\times m}^{T}diag(r),\ r=\left[1.1.2.2.4.4....,\underbrace{\frac{m}{2},\frac{m}{2},....,\frac{m}{2}}_{\frac{m}{2}elements}\right]^{T}\ \text{for}\, m>2.$$

If $(QH)_{il}=\int_0^{t_l}\int_0^{t}h_i(t)dtdt.$ Then for $j=1, m=2.$

$$QH=\frac{1}{128}\begin{bmatrix}1 & 9 & 25 & 49\\ 1 & 9 & 23 & 31\\ 1 & 7 & 0 & 0\\ 0 & 0 & 1 & 7\end{bmatrix}$$

which is the same as obtained through (2.2a).

Notice that $P^2\neq Q$.

3 FUNCTION APPROXIMATION AND APPLICATIONS

As Haar wavelets are orthogonal; this means that any square integrable function over [0,1] can be expanded into Haar wavelets series as:

$$y(x)=\sum_{i=1}^{\infty}a_ih_i(x) \tag{3.1}$$

where $a_i^{'}s$ are Haar wavelet coefficients.

If $y(x)$ be piecewise constant, then sum can be terminated to finite term, that is

$$y(x_l)=\sum_{i=1}^{2M}a_ih_i(x_l)=a^TH \tag{3.1a}$$

$a^T=\left\{a_1,a_2,....,a_{2M}\right\}$, $H=\left\{h_1(x),h_2(x),....,h_{2M}(x)\right\}^T$.

Solutions of boundary value problems can be considered as approximations (3.1a).

Norm of error function $v(l)=y_{app}(x_l)-y_{ex}(x_l)$ is defined by

$$\left\|v\right\|_p=\left(\sum_{i=1}^{2M}\left|v(l)\right|^p\right)^{1/p}$$

Following two error estimates are applied:

(i) Local estimates $\delta_j = \left\| \frac{v}{y_{ex}} \right\|_\infty = \underset{1 \le l \le 2M}{Max} \left| \frac{y(x_l)}{y_{ex}(x_l)} - 1 \right|$

(ii) Global estimates $\sigma_j = \frac{\|v\|_2}{2M}$

However, we prefer the absolute error estimation

$$e_j = \underset{1 \le l \le 2M}{Max} \left| y_{app}(x_l) - y_{ex}(x_l) \right|.$$

To demonstrate the applicability of Haar wavelets, we focus on the following numerical problems, using our algorithm (2.4.1 & 2.4.2a,b) for P (not as in (2.5)) and utilizing MATLAB software.

3.1 Application in Solving Linear ODEs

Consider nth order Linear ODE:

$$Ly(x) = f(x), \qquad A \le x \le B,\ L \cong \text{ Differential operator} \tag{3.1b}$$

Step 1: $y^n(x) = \sum_{i=1}^{2M} a_i h_i(x)$

Step2: For $\alpha < n$

$$y^n(x) = \sum_{i=1}^{2M} a_i P_{n-\alpha,i}(x) + \sum_{\sigma=1}^{n-\alpha-1} \frac{1}{\sigma!}(x-A)^\sigma y_0^{(\alpha+\sigma)}$$

Step3:Substitute various derivatives as obtained in steps 1 & 2 in the equation (3.1) and calculate $a_i^{'}$s to get the numerical solution.

Example 3.1

Consider the second order initial value linear ODE:

$$y^{''}(x) + y(x) = u(x), \quad y(0) = y^{'}(0) = 0, \qquad x \in [0,1] \tag{3.1.1}$$

Follow the procedural steps to obtain system in matrix form $a[H+P] = f$. Computer simulations are carried out for $j = 3$.The solution is as shown in the Figure 3.1

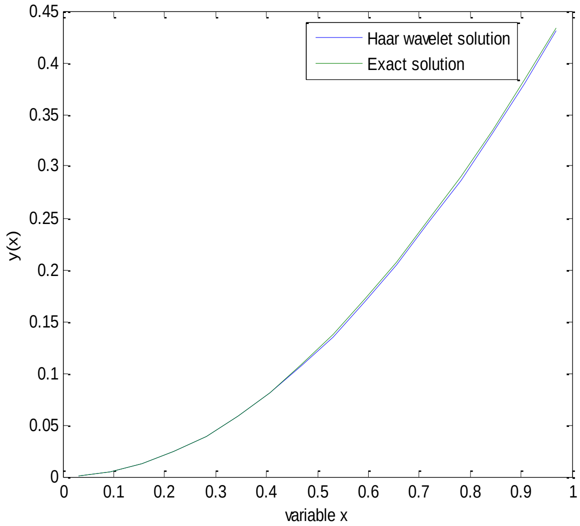


Figure 3.1 illustrate the comparison of Haar solution with the exact solution $1-\cos(x)$.

Maximum error for $j=3$ is 0.003.

Example 3.2

To solve the two point boundary value problem:

$$y''(x)=y'(x)+y(x)+e^{x}(1-2x)=u(x),\ y(0)=1, y(1)=3e,\ x\in[0,1]$$

The procedural steps reduce the equation in the matrix form $a[H-P-Q]=f$. Computer simulations are carried out for j=3.The solutions are shown in the Figure 3.2

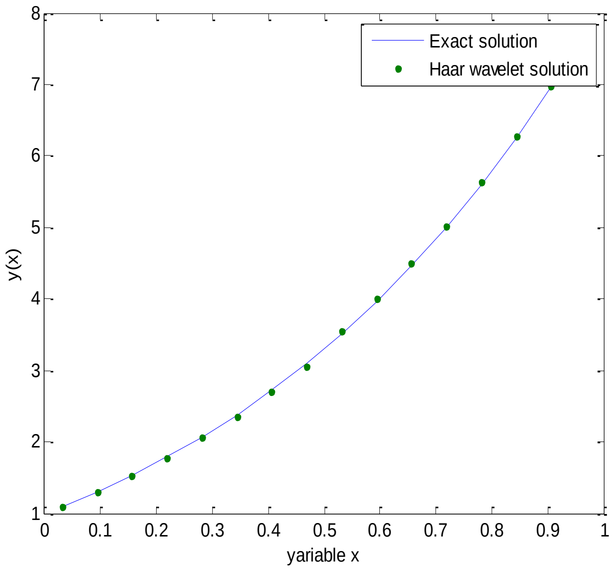


Figure 3.2 illustrate the comparison of Haar solution with the exact solution $e^{x}(1+2x)$

Maximum error for $j=3$ is 0.03.

Example 3.3

To find the solution of two point Neumann boundary value problem:

$y'' - xy' + y = -x\cos(x),\ y'(0) = 2$ and $y(1) = 1.8415$

The problem reduces to $a[H - xP + Q] = f$.

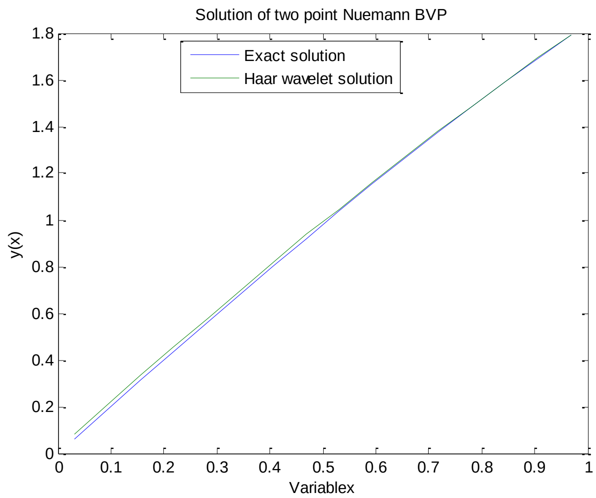


Figure 3.3 depicts comparison of Haar wavelet solution with the exact solution $x + \sin(x)$ and maximum error for $j = 3$ is 0.02.

Example 3.4 (Stoer J, 2002)

To find the solution of two point Neumann boundary value

$-y'' + 400y = -400\cos^2(\pi x) - 2\pi^2\cos(2\pi x), y(0) = 0$ and $y(1) = 0$.

The exact solution is $y(x) = \dfrac{e^{-20}}{1+e^{-20}}e^{20x} + \dfrac{1}{1+e^{-20}}e^{20x} - \cos^2(\pi x)$.

The problem reduces to $a[H - xP + Q] = f$.

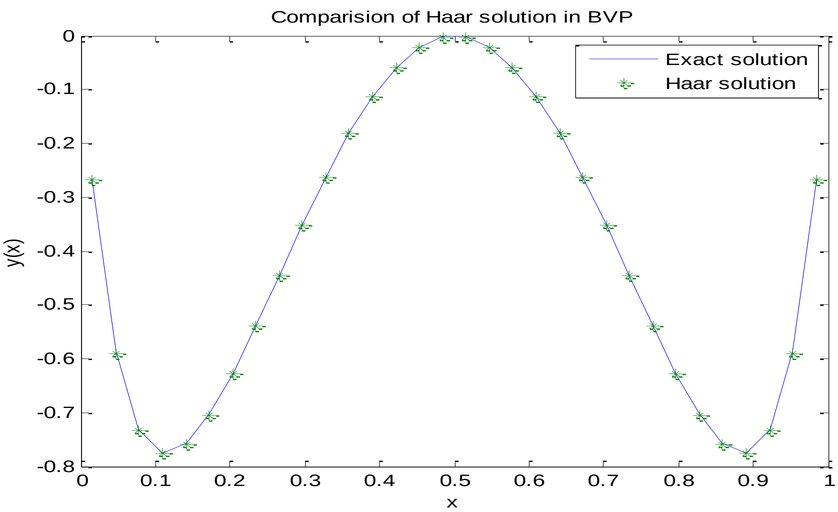


Figure 3.4 depicts comparison of Haar wavelet solution with the exact solution

3.2 Application in Solving Integral Equations(Alpert,1993)

To solve linear Integral equation

$$u(x)-\int_0^1 K(x,t)h_i(t)dt = f(x), \qquad x\in[0,1] \tag{3.2.1}$$

using (3.1) into (3.2.1), we get the matrix system $a\left(H-G\right)=f$.
In it,

$$G_i(x)=\int_0^1 K(x,t)h_i(t)dt=\begin{cases} x+0.5 & for\ i=1 \\ \dfrac{-1}{4m^2} & for\ i>1 \end{cases}$$

Compute a_i's to get the required solution.

Example 3.5

Consider the case $K(x,t)=x+t$ and $f(x)=x^6\log(x)$. The series solution is

$u_{ex}=x^6\log(x)+0.3096x+0.1752$.

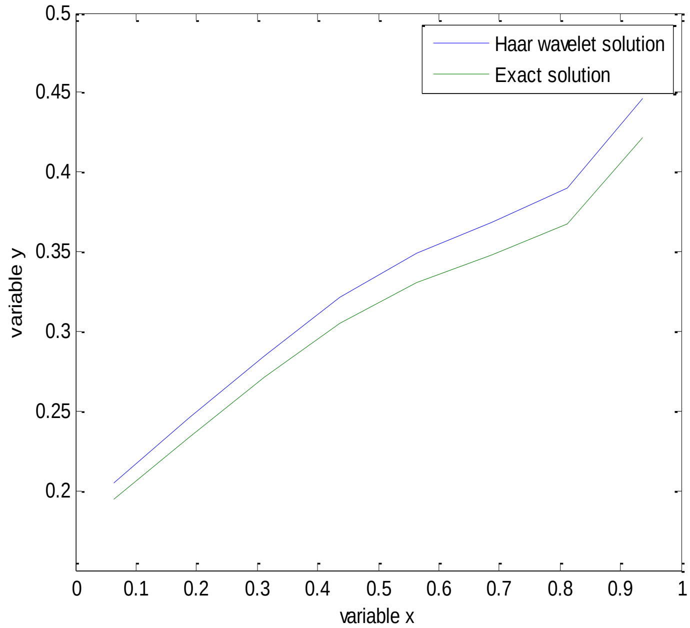


Figure 3.5 depicts comparison of Haar wavelet solution with series solution and maximum error for j = 2 is 0 .02.

3.3 Application in Solving Integro-Differential Equations(Lepik,2009)

Integro-differential equations are those which contain unknown functions and its derivative under integral sign. For the sake of concreteness, let us consider equation of the following type:

$$u^{'}(x)+\alpha u(x)-\beta\int_0^1 u(x)dx = f(x) \tag{3.3.1}$$

where α and β are constants and $f(x)$ prescribed function. To this equation belongs the initial condition $u(0)=\gamma$.

According to the method suggested by Hsiao, we don't develop into the Haar series the function $u(x)$, but its derivative $u^{'}(x)$

$$u^{'}(x)=\sum_{i=1}^{2M} a_i h_i(x),\; u(x)=\sum_{i=1}^{2M} a_i P_{1,i}(x)+u(0) \tag{3.3.2}$$

where $P_{1,i}(t)=\int_0^x h_i(t)dt$ .Substitution of (3.3.2) into (3.3.1) gives

$$a[H+\alpha P_{1,i}(x)+\beta P_{2,i}(x)] = f-(\alpha+\beta). \tag{3.3.3}$$

Example 3.6

Consider the integro-differential equation in case $\alpha=\beta=1$ and $u(t)=1$

$$\frac{di}{dt}+2i+5\int_0^1 idt = u(t),\; i(0)=0$$

Computer simulations are carried out for $j=3$. The solution is compared with the exact solution $\frac{1}{2}e^{-t}\sin(2t)$ obtained through Laplace transform.

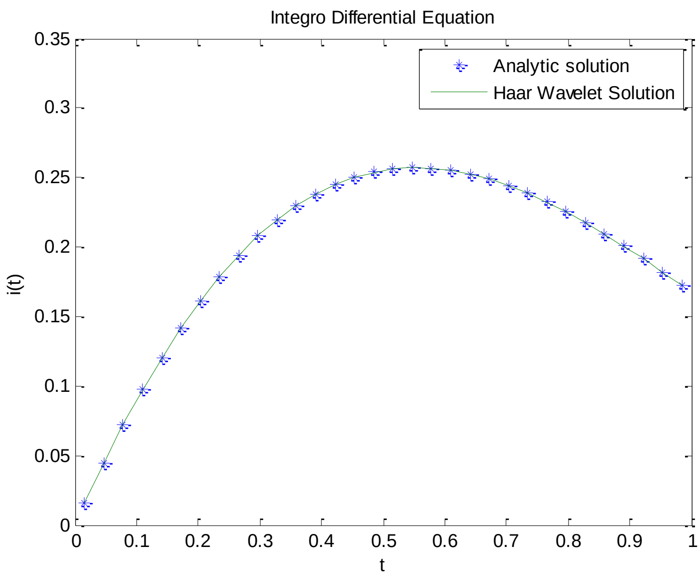


Figure 3.6 shows comparison of Haar wavelet solution with exact solution and the maximum error for $j=4$ is 0.0001.

4 CONCLUSION

The sparseness in Haar wavelets based operational matrices gives precise accuracy in solving numerical equations by Haar collocation method. The algorithm developed facilitates for less time consuming. Moreover, the method has an advantage over wavelet-Galerkin procedure being computationally complicated. Also for small value of resolution a comparatively better solution is obtained. The results are comparable to analytic ones. Better results are expected for comparatively higher value of level of resolution j.

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