Commun Nonlinear Sci Numer Simulat 15 (2010) 2284–2292

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Solving a nonlinear fractional differential equation using Chebyshev wavelets

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ARTICLE INFO

Article history:
Received 20 August 2009
Received in revised form 17 September 2009
Accepted 17 September 2009
Available online 22 September 2009

Keywords:
Operational matrix
Chebyshev wavelets
Fractional calculus
Nonlinear fractional differential equations

ABSTRACT

Chebyshev wavelet operational matrix of the fractional integration is derived and used to solve a nonlinear fractional differential equations. Some examples are included to demonstrate the validity and applicability of the technique.

1. Introduction

The use of fractional differential and integral operators in mathematical models has become increasingly widespread in recent years. Several forms of fractional differential equations have been proposed in standard models, and there has been significant interest in developing numerical schemes for their solution. These methods include Laplace transforms [1], Fourier transforms [2], eigenvector expansion [3], Adomian decomposition method (ADM) [4,5], Variational Iteration Method (VIM) [6,7], Fractional Differential Transform Method (FDTM) [8,9], Fractional Difference Method (FDM) [10] and Power Series Method [11]. But, few papers reported application of wavelet to solve the fractional order differential equations [12,13].

In view of successful application of wavelet operational matrix in system analysis [14,15], system identification [16,17], optimal control [18–20] and numerical solution of integral and differential equations [21–26], together with the characteristic of wavelet functions, we hold that they should be applicable to solve the fractional order systems.

So my purpose is to introduce the method to solve multi-order arbitrary differential equations, which include the linear and nonlinear differential equations.

Similar to the integer-order case, firstly, the underlying fractional differential equation is converted into a fractional integral equation via fractional integration; subsequently, the various signals involved in the fractional integral equation are approximated by representing them as linear combinations of the wavelet functions and truncating them at optimal levels; finally, the integral equation is converted to an algebraic equation by introducing the wavelet operational matrix of the fractional integration. Therefore, there are some questions to be answered:

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doi:10.1016/j.cnsns.2009.09.020

(1) How to derive Chebyshev wavelet operational matrix of the fractional integration.
(2) How to analyze the fractional differential equations via Chebyshev wavelet operational matrices of the fractional integration.

The paper is organized as follows: I begin by introducing some necessary definitions and mathematical preliminaries of the fractional calculus theory which are required for establishing our results. In Section 3, after describing the basic formulation of wavelets and Chebyshev wavelets, I derive Chebyshev wavelet operational matrix of the fractional integration. In Section 4, I present three examples to show the efficiency and simplicity of the method.

2. Preliminaries and notations

I give some necessary definitions and mathematical preliminaries of the fractional calculus theory which are used further in this paper. The Riemann–Liouville fractional integration of order $\alpha > 0$ is defined as [1]

$$(I^{\alpha}f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau, \tag{1}$$

$$(I^0 f)(t) = f(t), \tag{2}$$

and its fractional derivative of order $\alpha > 0$ is normally used:

$$(D^{\alpha}f)(t) = \left(\frac{d}{dt}\right)^n (I^{n-\alpha}f)(t) \quad (n-1 < \alpha \leqslant n), \tag{3}$$

where n is an integer. For Riemann–Liouvilles definition,one has

$$I^{\alpha}t^{v} = \frac{\Gamma(v+1)}{\Gamma(\alpha+v+1)}t^{\alpha+v}. \tag{4}$$

The Riemann–Liouville derivative have certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce now a modified fractional differential operator D^{α} proposed by Caputo.

$$(D^{\alpha}f)(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,d\tau \quad (n-1 < \alpha \leqslant n), \tag{5}$$

where n is an integer. Caputos integral operator has an useful property:

$$(I^{\alpha}D^{\alpha}f)(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{t^k}{k!} \quad (n-1 < \alpha \leqslant n), \tag{6}$$

where n is an integer.

For more details on the mathematical properties of fractional derivatives and integrals see [1,27].

3. Chebyshev wavelet operational matrix of the fractional integration

3.1. Chebyshev wavelet

Wavelets are a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously we have the following family of continuous wavelets as [23].

$$\psi_{a,b}(t) = |a|^{-1/2}\psi\left(\frac{t-b}{a}\right), \quad a, b \in R, \ a \neq 0. \tag{7}$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k},\ b = nb_0a_0^{-k},\ a_0 > 1,\ b_0 > 0$, where n and k are positive integers, the family of discrete wavelets are defined as

$$\psi_{k,n}(t) = |a_0|^{k/2}\psi(a_0^k t - nb_0), \tag{8}$$

where $\psi_{k,n}(t)$ from a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1,\ \psi_{k,n}(t)$ forms as orthogonal basis.

Chebyshev wavelets $\psi_{nm}(t)$, on the interval $[0, 1)$ are defined as [23]

$$\psi_{n,m} = \begin{cases} 2^{k/2}\widetilde{T}_m(2^k t - 2n + 1), & \frac{n-1}{2^{i-1}} \leqslant t < \frac{n}{2^{i-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

where

$$\widetilde{T}_m = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0, \\ \sqrt{\frac{2}{\pi}}T_m(t), & m > 0, \end{cases} \tag{10}$$

and $m = 0, 1, \dots, M-1$, $n = 1, 2, \dots, 2^{k-1}$, k is any positive integer and $T_m(t)$ are Chebyshev ploynomials of the first kind of degree m which are orthogonal with respect to the weight function $\omega(t) = 1/\sqrt{1-t^2}$, on the interval $[-1, 1]$ and $T_m(t)$ can be determined by the following recurrence formula:

$$T_0(t) = 1, \quad T_1(t) = t, \quad sT_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \quad m = 1, 2, 3, \dots. \tag{11}$$

A function $f(t)$ defined over $[0, 1)$ may be expanded as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t), \tag{12}$$

where $c_{nm} = \langle f(t),\ \psi_{nm}(t)\rangle$, in which $\langle , \rangle$ denotes the inner product.

If the infinite series in Eq. (12) is truncated, then Eq. (12) can be written as

$$f(t) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\Psi(t), \tag{13}$$

where T indicates transposition, C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$\begin{aligned} C &\triangleq \left[c_{10}, c_{11}, \dots, c_{1M-1}, c_{20}, \dots, c_{2M-1}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}M-1}\right]^T \\ \Psi(t) &\triangleq \left[\psi_{10}, \psi_{11}, \dots, \psi_{1M-1}, \psi_{20}, \dots, \psi_{2M-1}, \dots, \psi_{2^{k-1}0}, \dots, \psi_{2^{k-1}M-1}\right]^T \end{aligned} \tag{14}$$

Taking the collocation points as following:

$$t_i = \frac{(2i-1)}{2^k M}, \quad i = 1, 2, \dots, 2^{k-1}M. \tag{15}$$

We define the Chebyshev wavelet matrix$\Phi_{m\times m}$ as:

$$\Phi_{m\times m} \triangleq \left[\ \Psi\left(\tfrac{1}{2m}\right) \quad \Psi\left(\tfrac{3}{2m}\right) \quad \cdots \quad \Psi\left(\tfrac{2m-1}{2m}\right)\ \right]. \tag{16}$$

For example, when $M = 3$ and $k = 2$ the Chebyshev wavelet is expressed as

$$\Phi_{6\times 6} = \begin{bmatrix} 2.2568 & 2.2568 & 2.2568 & 0 & 0 & 0 \\ 1.0638 & 9.5746 & 18.0854 & 0 & 0 & 0 \\ -2.4823 & 54.2562 & 201.7761 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.2568 & 2.2568 & 2.2568 \\ 0 & 0 & 0 & 1.0638 & 9.5746 & 18.0854 \\ 0 & 0 & 0 & -2.4823 & 54.2562 & 201.7761 \end{bmatrix}.$$

3.2. Operational matrix of the fractional integration

The integration of the vector $\Psi(t)$ defined in Eq. (14) can be obtained as

$$\int_0^t \Psi(\tau)\,d\tau \approx P\Psi(t), \tag{17}$$

where P is the $2^{k-1}M \times 2^{k-1}M$ operational matrix for integration [23].

Our purpose is to derive the Chebyshev wavelet operational matrix of the fractional integration. For this purpose, we rewrite Riemann–Liouville fractional integration, as following:

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t), \tag{18}$$

where $\alpha \in R$ is the order of the integration, $\Gamma(\alpha)$ is the Gamma function and $t^{\alpha-1} * f(t)$ denotes the convolution product of $t^{\alpha-1}$ and $f(t)$. Now if $f(t)$ is expanded in Chebyshev wavelets, as shown in Eq. (12), the Riemann–Liouville fractional integration becomes

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t) \approx C^T \frac{1}{\Gamma(\alpha)}\left\{t^{\alpha-1} * \Psi(t)\right\}. \tag{19}$$

Thus if $t^{\alpha-1} * f(t)$ can be integrated, then expanded in Chebyshev wavelets, the Riemann–Liouville fractional integration is solved via the Chebyshev wavelets.

Also, we define a m-set of Block Pulse Functions (BPF) as:

$$b_i(t) = \begin{cases} 1, & i/m \leqslant t < (i+1)/m, \\ 0, & \text{otherwise}, \end{cases} \tag{20}$$

where $i = 0, 1, 2, \ldots, (m-1)$.

The functions $b_i(t)$ are disjoint and orthogonal. That is

$$b_i(t)b_l(t) = \begin{cases} 0, & i \neq l, \\ b_i(t), & i = l. \end{cases} \tag{21}$$

$$\int_0^1 b_i(\tau)b_l(\tau)\,d\tau = \begin{cases} 0, & i \neq l, \\ 1/m, & i = l. \end{cases} \tag{22}$$

Similarly, Chebyshev wavelets may be expanded into an m-term block pulse functions (BPF) as

$$\Psi_m(t) = \Phi_{m\times m}B_m(t). \tag{23}$$

where $B_m(t) \triangleq [\, b_0(t) \quad b_1(t) \quad \cdots \quad b_i(t) \quad \cdots \quad b_{m-1}(t)\,]^T$

In Ref. [28], Kilicman and Al Zhour have given the Block Pulse operational matrix of the fractional integration F^α as following:

$$(I^\alpha B_m)(t) \approx F^\alpha B_m(t), \tag{24}$$

where

$$F^\alpha = \frac{1}{m^\alpha}\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdot s & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{25}$$

with $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$.

Next, we derive the Chebyshev wavelet operational matrix of the fractional integration. Let

$$\left(I^\alpha \Psi_m\right)(t) \approx P^\alpha_{m\times m}\Psi_m(t), \tag{26}$$

where matrix $P^\alpha_{m\times m}$ is called the Chebyshev wavelet operational matrix of the fractional integration.

Using Eqs. (23) and (24), we have

$$\left(I^\alpha \Psi_m\right)(t) \approx \left(I^\alpha \Phi_{m\times m}B_m\right)(t) = \Phi_{m\times m}\left(I^\alpha B_m\right)(t) \approx \Phi_{m\times m}F^\alpha B_m(t). \tag{27}$$

From Eqs. (26) and (27) we get

$$P^\alpha_{m\times m}\Psi_m(t) = P^\alpha_{m\times m}\Phi_{m\times m}B_m(t) = \Phi_{m\times m}F^\alpha B_m(t). \tag{28}$$

Then, the Chebyshev wavelet operational matrix of the fractional integration $P^\alpha_{m\times m}$ is given by

$$P^\alpha_{m\times m} = \Phi_{m\times m}F^\alpha \Phi^{-1}_{m\times m}. \tag{29}$$

The fractional integration of the function t was selected to verify the correctness of matrices $P^\alpha_{m\times m}$. That is because the fractional integration of the function $f(t) = t$ is easily obtained as following $(I^\alpha f)(t) = \frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1}$, which is easily used to compare the result obtained by the proposed method. When $\alpha = 0.5$, $m = 48$ $(M = 3,\ k = 5)$, the comparison results for the fractional integration is shown in Fig. 1.

4. Applications and results

In this section, we will use the Chebyshev wavelet operational matrices of the fractional integration to solve nonlinear fractional (arbitrary) order differential equation. These examples are considered because closed form solutions are available for them, or they have also been solved using other numerical schemes. This allows one to compare the results obtained using this scheme with the analytical solution or the solutions obtained using other schemes.

Example 1. Following Odibat and Momani [29], we consider fractional Riccati equation

$$D^\alpha y(t) = 2y(t) - [y(t)]^2 + 1, \quad 0 < \alpha \leqslant 1, \quad 0 \leqslant t < 5 \tag{30}$$

subject to the initial state $y(0) = 0$, which is studied by Odibat and Momani [29] by using the modified homotopy perturbation method. Here we use the Chebyshev wavelet operational matrices of the fractional integration to solve it.

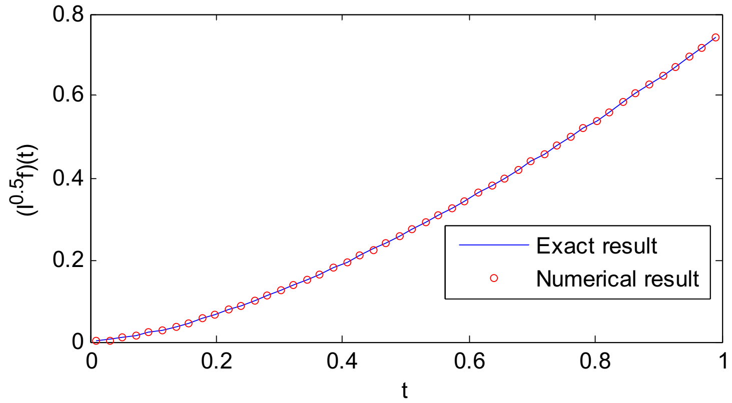

Fig. 1. Integration of 0.5-order of the function $f(t) = t$.

Let

$$D^{\alpha}y(t) = K_m^T \Psi_m(t) \tag{31}$$

together with the initial states, then we have

$$y(t) = K_m^T P_{m\times m}^{\alpha} \Psi_m(t). \tag{32}$$

Since $\Psi_m(t) = \Phi_{m\times m} B_m(t)$, from Eq. (32) we have

$$y(t) = K_m^T P_{m\times m}^{\alpha} \Phi_{m\times m} B_m(t). \tag{33}$$

Let

$$K_m^T P_{m\times m}^{\alpha} \Phi_{m\times m} = [a_1, a_2, \ldots, a_m] \tag{34}$$

and using Eq. (20),we have

$$[y(t)]^2 = [a_1 b_1(t) + a_2 b_2(t) + \cdots + a_m b_m(t)]^2 = [a_1^2, a_2^2, \ldots, a_m^2] B_m(t). \tag{35}$$

Substituting Eqs. (31), (32) and (35) into Eq. (30), we have

$$K_m^T \Phi_{m\times m} B_m(t) + [a_1^2, a_2^2, \ldots, a_m^2] B_m(t) - 2K_m^T P_{m\times m}^{\alpha} \Phi_{m\times m} B_m(t) - [1, 1, \ldots, 1] B_m(t) = 0. \tag{36}$$

This is a nonlinear system of algebraic equations, here we use the Matlab function fsolve to solve Eq. (36). The numerical solution, for $m = 96$, is shown in Fig. 2. The exact solution of this problem, when $\alpha = 1$, is

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right),$$

and we can observe that, as $t \to \infty$, $y(t) \to 1 + \sqrt{2}$. From Fig. 2 we can see the numerical solution is in very good agreement with the exact solution when $\alpha = 1$. Therefore, we hold that the solution for $\alpha = 0.5$ and $\alpha = 0.75$ is also credible. Numerical results with comparison to Ref. [29] is given in Table 1 on the interval $[0,1]$.

The difference between our result and the result in Ref. [29] is obvious. However, at given conditions, we hold that our results are better for $\alpha = 0.5$ and $\alpha = 0.75$. That is because only the fourth-order term of the homotopy perturbation solution were used in evaluating the approximate solutions in Ref. [29].

In order to assess the advantages and the accuracy of the Chebyshev wavelets method for solving nonlinear fractional differential equations, we use our method to solve another nonlinear fractional differential equation, whose exact solutions are known.

Example 2. Following El-Mesiry et al. [30], we consider the nonlinear fractional differential equation

$$aD^{2.0}y(t) + bD^{\alpha_2}y(t) + cD^{\alpha_1}y(t) + e[y(t)]^3 = f(t), \quad 0 < \alpha_1 \leqslant 1, \quad 1 < \alpha_2 \leqslant 2 \tag{37}$$

and

$$f(t) = \frac{2a}{\Gamma(2)}t + \frac{2b}{\Gamma(4-\alpha_2)}t^{3-\alpha_2} + \frac{2c}{\Gamma(4-\alpha_1)}t^{3-\alpha_1} + e\left[\frac{1}{3}t^3\right]^3$$

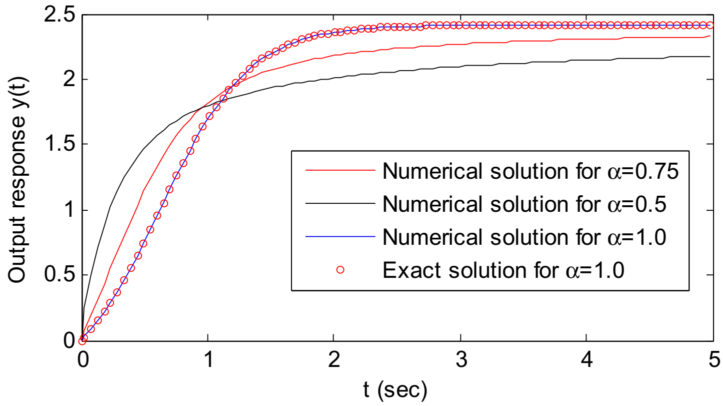

Fig. 2. Numerical solution and exact solution of $\alpha = 1$ for $m = 96$.

Table 1
Numerical results with comparison to Ref. [29] $m = 192$.

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$		
t	Ours	Ref. [29]	Ours	Ref. [29]	Ours	Ref. [29]	Exact
0.1	0.592756	0.321730	0.310732	0.216866	0.110311	0.110294	0.110295
0.2	0.9331796	0.629666	0.584307	0.428892	0.241995	0.241965	0.241977
0.3	1.1739836	0.940941	0.822173	0.654614	0.395123	0.395106	0.395105
0.4	1.3466546	1.250737	1.024974	0.891404	0.567829	0.568115	0.567812
0.5	1.4738876	1.549439	1.198621	1.132763	0.756029	0.757564	0.756014
0.6	1.5705716	1.825456	1.349150	1.370240	0.953576	0.958259	0.953566
0.7	1.646199	2.066523	1.481449	1.594278	1.152955	1.163459	1.152949
0.8	1.706880	2.260633	1.599235	1.794879	1.346365	1.365240	1.346364
0.9	1.756644	2.396839	1.705303	1.962239	1.526909	1.554960	1.526911
1.0	1.798220	2.466004	1.801763	2.087384	1.689494	1.723810	1.689498

subject to

$$y(0) = y'(0) = 0.$$

The exact solution of this problem is

$$y(t) = \frac{1}{3}t^3$$

. Let

$$D^{2.0}y(t) = K_m^T \Psi_m(t) \tag{38}$$

together with the initial states, then we have

$$D^{\alpha_2}y(t) = K_m^T P_{m\times m}^{2.0-\alpha_2} \Psi_m(t), \tag{39}$$

$$D^{\alpha_1}y(t) = K_m^T P_{m\times m}^{2.0-\alpha_1} \Psi_m(t), \tag{40}$$

$$y(t) = K_m^T P_{m\times m}^{2.0} \Psi_m(t). \tag{41}$$

Since $\Psi_m(t) = \Phi_{m\times m}B_m(t)$, from Eq. (41) we have

$$y(t) = K_m^T P_{m\times m}^{2.0} \Phi_{m\times m}B_m(t). \tag{42}$$

Let

$$K_m^T P_{m\times m}^{2.0} \Phi_{m\times m} = [a_1, a_2, \ldots, a_m] \tag{43}$$

then

$$[y(t)]^3 = [a_1^2, a_2^3, \ldots, a_m^3]B_m(t). \tag{44}$$

Similarly, the input signal $f(t)$ may be expanded by the Chebyshev wavelets as follows:

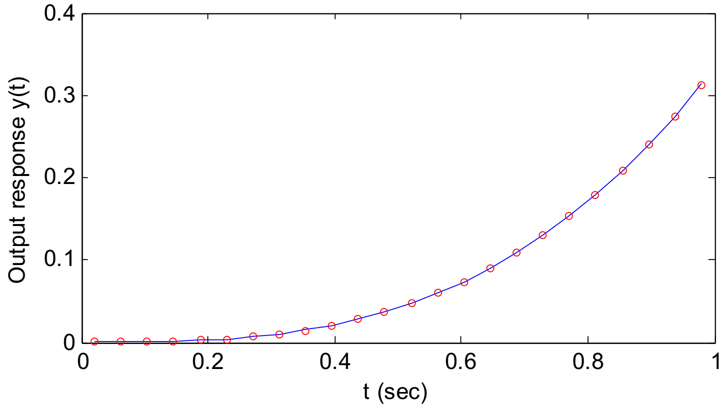

Fig. 3. Numerical and exact solution for $m = 24$.

Table 2
Error in $y(t)$ for different values of m.

t	$m = 24$	$m = 48$	$m = 96$	$m = 192$	$m = 384$
0.1	8.195145E−5	2.637545E−5	5.253765E−6	1.627688E−6	3.260279E−7
0.2	2.051792E−4	4.056730E−5	1.265345E−5	2.515861E−6	7.929112E−7
0.3	2.950889E−4	5.839170E−5	1.858048E−5	3.662622E−6	1.158235E−6
0.4	3.054479E−4	9.643199E−5	1.892447E−5	6.042111E−6	1.184173E−6
0.5	5.080055E−4	1.269211E−4	3.171672E−5	7.927087E−6	1.981458E−6
0.6	4.296362E−4	1.392432E−4	2.694677E−5	8.676687E−6	1.681103E−6
0.7	6.384631E−4	1.227641E-4	3.970229E−5	7.648876E−6	2.482836E−6
0.8	7.117552E−4	1.364499E−4	4.459653E−5	8.537701E−6	2.783712E−6
0.9	6.027054E−4	1.967397E−4	3.748356E−5	1.230841E−5	2.343684E−6

$$f(t) = f_m^T \Psi_m(t), \tag{45}$$

where f_m^T is a known constant vector. Substituting Eqs. (38)–(40) and (44) into Eq. (37), together with $\Psi_m(t) = \Phi_{m\times m}B_m(t)$ we have

$$K_m^T\Phi_{m\times m}B_m(t) + K_m^T P_{m\times m}^{2.0-\alpha_2}\Phi_{m\times m}B_m(t) + K_m^T P_{m\times m}^{2.0-\alpha_1}\Phi_{m\times m}B_m(t) + [a_1^3, a_2^3, \ldots, a_m^3]B_m(t) - f_m^T\Phi_{m\times m}B_m(t) = 0. \tag{46}$$

This is a nonlinear system of algebraic equations, here we use the Matlab function fsolve to solve Eq. (46). For $a = 1,\ b = 1,\ c = 1,\ e = 1,\ \alpha_1 = 0.333,\ \alpha_2 = 1.234$, Fig. 3 show a behaviour of the numerical solution for $m = 24$, which is in agreement with the exact solutions, the errors of $y(t)$ at given points for different values of m are shown in Table 2.

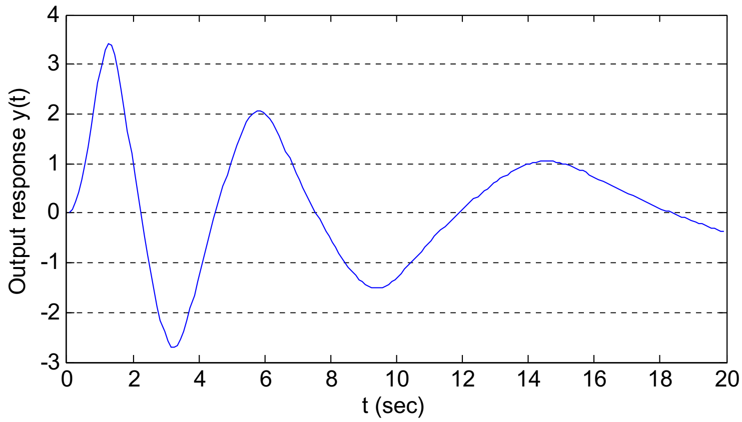

Fig. 4. Numerical and exact solution for $m = 384$.

Example 3. Following El-Mesiry et al. [30], we consider the nonlinear fractional differential equation

$$D^2y(t) + 0.5D^{1.5}y(t) + 0.5y^3(t) = f(t), \quad t > 0, \tag{47}$$

where

$$f(t) = \begin{cases} 8, & 0 \leqslant t \leqslant 1, \\ 0, & t > 1, \end{cases}$$

subject to

$$y(0) = y'(0) = 0.$$

This is a Bagley–Torvik equation where nonlinear term $y^3(t)$ is introduced. This problem was solved in [30]. Fig. 4 shows a behaviour of the numerical solution for $m = 384$. My result is in good agreement with the numerical results obtained by [30]. This demonstrates the importance of my numerical scheme in solving nonlinear multi-order fractional differential equations.

5. Conclusion

We derive Chebyshev wavelet operational matrix of the fractional integration, and use its to solve nonlinear fractional (arbitrary) order differential equation. Several examples are given to demonstrate the powerfulness of the proposed method. Using wavelet operational matrix of the fractional integration to solve the fractional differential equations has several advantages: (1) The method is computer oriented, thus solving higher order differential equation becomes a matter of dimension increasing; (2) The solution is a multi-resolution type and (3) the solution is convergent, even though the size of increment may be large.

Acknowledgements

The author would like to thank the reviewers for their suggestions to improve the quality of the paper.The work was supported by the Foundation of NUIST under Grant (20080305), Foundation of NUIST under Grant (20080153), Foundation of NUIST under Grant (20080256) and in part by Jiangsu Ordinary University Science Research Project under Grant 09KJB510007.

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