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Numerical solution of Fredholm integral equations of the first kind by using linear Legendre multi-wavelets ☆

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Abstract

In this paper, we suggest an efficient method for solving Fredholm integral equations of the first kind. The continuous Legendre multi-wavelets constructed on $[0,1]$ are utilized as a basis in Galerkin method to reduce the solution of linear integral equations to a system of algebraic equations. Illustrative examples are included to demonstrate the efficiency and the application of the technique.

Keywords: Fredholm integral equations of the first kind; Multi-wavelets; Galerkin method; Algebraic equations

1. Introduction

Many inverse problems in science and engineering lead to the solution of integral equations of the first kind, namely

$$\int_0^1 k(x,t)y(t)\mathrm{d}t = f(x), \quad x \in [0,1], \tag{1}$$

where $f(x)$ and $k(x,t)$ are known functions and $y(x)$ is the unknown function to be determined. In general, equations of this form are ill-posed for a given k and f; Eq. (1) may have no solution.

There have been several numerical methods for approximating the solution of the first kind integral equations. As we know, it is important to select a suitable basis function in numerical methods for integral equations. Many kinds of basis functions have been proposed, such as triangular basis function, pulse basis function, polynomial basis function, spline and B-spline basis function. Recently, wavelet basis function has been proposed to solve Fredholm integral equations numerically. One of the most attractive proposals made in the recent years was an idea connected to the application of wavelets as basis functions in the method of moments [6]. The wavelet technique allows the creation of very fast algorithms when compared to the

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algorithms ordinarily used. Various wavelet basis are applied. In addition to the conventional Duabechies wavelets, Wavelet moment method [4], Haar wavelets [5], linear B-splines [6], Walsh functions [7] have been used. Recently, Maleknejad et al. [3] used Haar wavelet-collocation method and [1] used Legendre wavelets for the first kind Fredholm equations.

In this paper, we present the application of the linear Legendre multi-wavelets as basis functions in Galerkin's method for numerical solution of the first kind Fredholm integral equations. The method is tested with the numerical examples.

2. The linear Legendre multi-wavelets and its properties

2.1. The linear Legendre multi-wavelets

In this section, we describe the linear Legendre multi-wavelet method. Khellat [2] used this method to solve optimal control problem. For constructing the linear Legendre multi-wavelets, at first we describe scaling functions $\phi_0(x)$ and $\phi_1(x)$ as follows:

$$\phi_0(x) = 1, \phi_1(x) = \sqrt{3}(2x-1), \quad 0 \leqslant x < 1.$$

Let $\phi(x) := [\phi_0(x), \phi_1(x)]^{\mathrm{T}}$ satisfy the following vector refinement equations:

$$\phi(x) = \sum_{n\in Z} a_n \phi(2x-n), \quad x \in R.$$

Let $\psi(x) = [\psi^0(x), \psi^1(x)]^{\mathrm{T}}$ be mother wavelet function, then

$$\psi(x) = \sum_{n\in Z} (-1)^n a_{1-n} \phi(2x-n).$$

We obtain the linear Legendre mother wavelets as

$$\psi^0(x) = \begin{cases} -\sqrt{3}(4x-1), & 0 \leqslant x < \frac{1}{2}, \\ \sqrt{3}(4x-3), & \frac{1}{2} \leqslant x < 1. \end{cases} \tag{2}$$

$$\psi^1(x) = \begin{cases} 6x-1, & 0 \leqslant x < \frac{1}{2}, \\ 6x-5, & \frac{1}{2} \leqslant x < 1. \end{cases} \tag{3}$$

With dilation and translation, we obtain the linear Legendre multi-wavelets:

$$\psi^j_{kn}(x) = \begin{cases} 2^{\frac{k}{2}} \psi^j(2^k x - n), & \frac{n}{2^k} \leqslant x < \frac{n+1}{2^k}, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where $n = 0, 1, \ldots, 2^k - 1, k$ is any nonnegative integer, $j = 1, 2$. The family $\{\psi^j_{kn}(x)\}$ forms an orthonormal basis for $L^2(R)$.

2.2. Function approximation

A function $f(t)$ defined over $[0, 1)$ may be expanded as

$$f(t) = c_0\phi_0(t) + c_1\phi_1(t) + \sum_{k=0}^{\infty}\sum_{j=0}^{1}\sum_{n=0}^{\infty} c^j_{kn}\psi^j_{kn}(t), \tag{5}$$

where

$$c^j_{kn} = (f(t),\ \psi^j_{kn}(t)), \tag{6}$$

in which $(.,.)$ denotes the inner product. If the infinite series in Eq. (5) is truncated, then it can be written as

$$f(t) \simeq c_0\phi_0(t) + c_1\phi_1(t) + \sum_{k=0}^{M}\sum_{j=0}^{1}\sum_{n=0}^{2^k-1} c^j_{kn}\psi^j_{kn}(t) = C^{\mathrm{T}}\Psi(t), \tag{7}$$

where C and $\Psi(t)$ are matrices given by

$$C = [c_0, c_1, c_{00}^0, c_{00}^1, \dots c_{M0}^0, c_{M1}^0, \dots, c_{M(2^M-1)}^0, c_{M0}^1, c_{M1}^1, \dots, c_{M(2^M-1)}^1]^{\mathrm{T}}, \tag{8}$$

$$\Psi(t) = [\phi_0, \phi_1, \psi_{00}^0, \psi_{00}^1, \dots, \psi_{M0}^0, \psi_{M1}^0, \dots, \psi_{M(2^M-1)}^0, \psi_{M0}^1, \psi_{M1}^1, \dots, \psi_{M(2^M-1)}^1]^{\mathrm{T}}. \tag{9}$$

We relabel these functions as follows: let $g_i := \phi_{i-1},\ i = 1, 2$, and let $g_{2^{(k+1)}+i} := \psi_{kn}^j$ for $j = 1, 2$, $k = 0, 1, \dots$ and $i = 1, 2, \dots, 2^{k+1}$.

3. Solution of Fredholm integral equations of the first kind

In this section, the wavelets in previous section are used to solve integral equations. The classic Galerkin method for Fredholm integral equations of the first kind given in (1) consists of seeking and approximate solution of the form

$$y_n(x) = \sum_{i=1}^{n} c_i g_i(x), \quad x \in [0, 1). \tag{10}$$

Substituting (10) into (1), we find that

$$\int_0^1 k(x, t) y_n(t) \mathrm{d}t - f(x) = r_n(x), \quad x \in [0, 1),$$

where $r_n(x)$ is the residual such that $r_n(x) = 0$ for $y_n(x) = y(x)$. Our goal is to compute $c_1, c_2, \dots c_n$ such that $r_n(x) \equiv 0$. However, in general, it is not possible to choose such $c_i, i = 1, 2, \dots, n$. This method aims at making $r_n(x)$ as small as possible. In this method, we compute $c_i,\ i = 1, 2, \dots, n$. such that

$$(r_n, g_i(x)) = 0, \quad i = 1, 2, \dots, n.$$

Thus we obtain the system of algebraic equations:

$$\mathrm{AC} = \mathrm{F}$$

for unknowns $c = [c_1, c_2, \dots c_n]^{\mathrm{T}}$, where

$$A_{ij} = \left(\int_0^1 k(x, t) g_i(t) \mathrm{d}t, g_j \right), \quad F_i = (f, g_i),$$

in which $(.,.)$ denotes the inner product.

4. Illustrative examples

In this section, we applied the method presented in the paper for solving Eq. (1) and some examples below. The computations associated with the examples were performed using Mathematica 5.2.

Example 1.1. Consider the first king integral equation:

$$\int_0^1 \sin(xt) y(t) \mathrm{d}t = \frac{\sin x - x \cos x}{x^2}. \tag{11}$$

The exact solution for this problem is $y(x) = x$.

The numerical solution for (11) is obtained by the method of Section 3 with $M = 1$. Table 1 shows that the numerical result of the example is better than the result in [1] where y and $\tilde{y}$ denote the exact solution and the numerical solution, respectively.

Example 1.2. As the second example consider the following integral equation:

$$\int_0^1 \mathrm{e}^{x^2 t} y(t) \mathrm{d}t = \frac{\mathrm{e}^{x^2+1} - 1}{x^2 + 1}. \tag{12}$$

The exact solution for this problem is $y(x) = \mathrm{e}^x$.

Table 1
Comparison of Legendre multi-wavelets method and Legendre wavelets method

x	$\lvert y-\tilde{y}\rvert$	
	Legendre multi-wavelets method	Legendre wavelets method
0.0	3.23061947e-05	1.18306642e-04
0.1	8.45545414e-06	3.54275591e-05
0.2	1.53952865e-05	1.19338968e-05
0.3	9.57037368e-06	2.37777265e-05
0.4	5.94402353e-07	1.03929644e-07
0.5	1.98352666e-06	1.12102530e-05
0.6	8.74168553e-07	5.36233190e-06
0.7	2.35189552e-07	1.42808164e-06
0.8	4.45641330e-08	5.92497748e-07
0.9	1.56195279e-08	6.99406276e-07

Table 2
Comparison of Legendre multi-wavelets method and Legendre wavelets method

x	$\lvert y-\tilde{y}\rvert$	
	Legendre multi-wavelets method	Legendre wavelets method
0.0	3.29068541e-04	3.31979851e-04
0.1	2.09841152e-04	2.46913447e-04
0.2	2.22902691e-04	2.80285455e-04
0.3	4.64012750e-04	3.79013553e-04
0.4	7.72142295e-04	8.00568182e-04
0.5	6.14408828e-03	3.1366571e-02
0.6	5.76060230e-04	3.66885505e-04
0.7	9.01336409e-04	1.38639650e-03
0.8	2.06038193e-04	1.05102508e-03
0.9	5.42307180e-04	4.11549773e-03

We solve (12) using the method with $M = 1$ in Section 3. Table 2 shows that the numerical result of the example is better than the result in [1] where y and $\tilde{y}$ denote the exact solution and the numerical solution, respectively.

5. Conclusion

The aim of present work is to propose an efficient method for solving Fredholm integral equation of first kind. The present method reduces an integral equation into a set of algebraic equations. Legendre multi-wavelets are well behaved basic functions that are orthonormal on $[0,1]$. The example express this method more efficient and faster than the Legendre wavelets, especially the ordinary ones [8].

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