Convergence Analysis of Legendre Wavelets Method for Solving Fredholm Integral Equations

S. G. Venkatesh[a]

venkamaths@gmail.com
corresponding author

S. K. Ayyaswamy[b]

sjcayya@yahoo.co.in

S. Raja Balachandar[c]

srbala09@gmail.com
[a,b,c]Department of Mathematics, School of Humanities and sciences
SASTRA University
Thanjavur-613401, Tamilnadu, India

Abstract

In this paper, we study the Legendre wavelets for the solution of linear, nonlinear and singular Fredholm integral equations of second kind using approximation technique. The properties of Legendre wavelets together with the Gaussian integration method are used to reduce the problem to the solution of algebraic equations. The main purpose of this article is to discuss the theoretical analysis of Legendre wavelet approximation method namely, the uniqueness of solution, the convergence analysis for the solution of second kind Fredholm integral equations. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique.

1 Introduction

Integral equations of various types play an important role in many fields of science and engineering. The most frequently investigated integral equations

are Fredholm linear equation and its nonlinear counterparts [1,2,3,4]. In recent years, many different orthonormal basis functions, such as Fourier functions [1],wavelets [2,3,4] etc. have been used to approximate the solution of these integral equations. However, the most attractive one among them especially for large scale problems may be the wavelet bases, in which the kernel can be represented as a sparse matrix. This is mainly due to its excellent 'locality' and high order vanishing moment properties. In the conventional wavelet based methods for solving integral equations, sooner or later one needs to compute the inner products of wavelets or the associated scaling functions with regular square integrable functions [2,3,4]. Because many types of wavelets are not necessarily smooth, special quadrature rules are required [2]. This may sometimes be difficult and time consuming, especially for those problems involving partial support and singular wavelet integrals [2]. In the last two decades, wavelets were used for solving integral equations and because of the MRA property; they produced some good approximations [5-10]. Ulo Lepik [11] discussed the solution of nonlinear Fredholm integral equations using Haar wavelet method. J.N.Xiao et al [12] had solved the second kind integral equations by periodic wavelet galerkin method. S.A.Yousefi had identified the solution of Lane-Emden and Emden-Fowler equations using Legendre approximation technique [13,14].

In the present article, we are concerned with the application of Legendre wavelets to find the approximate solution of Fredholm linear integral equation of second kind

$$f(x) - \int_0^1 k(x,t) f(t) dt = g(x) \tag{1}$$

and Fredholm nonlinear integral equation of second kind

$$f(x) - \int_0^1 k(x,t) h[f(t)] dt = g(x) \tag{2}$$

where f(x) is an unknown function. The Legendre wavelet method (LWM) [13] consists of reducing the given integral equations to a system of simultaneous linear and nonlinear equations. The properties of Legendre wavelets together with the Gaussian integration formula are then utilized to evaluate the unknown coefficients and find an approximate solution to (1) and (2).

The organization of the paper is as follows: In section 2, we describe the basic formulation of wavelets and Legendre wavelets which are required for our subsequent development. Section 3 is devoted to the solution of (1) and (2) by using integral operator and Legendre wavelets. In section 4, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples. Concluding remarks are given in the final section.

2 Properties of Legendre wavelets

2.1 Wavelets and Legendre wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter 'a' and the translation parameter 'b' vary continuously, we have the following family of continuous wavelets as:

$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), a, b \in R, a \neq 0$

If we restrict the parameters 'a' and 'b' to discrete values as $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, $a_0 > 1$, $b_0 > 0$ and n, k are positive integers, we have the following family of discrete wavelets: $\psi_{k,n}(t) = |a|^{-\frac{1}{2}} \psi\left((a_0)^k t - nb_0\right)$ where $\psi_{k,n}(t)$ forms an orthonormal basis.

Legendre wavelets $\psi_{n,m}(t) = \psi(k, \hat{n}, m, t)$ have four arguements: $\hat{n} = 2n - 1, n = 1, 2, 3, ..., 2^{k-1}$, k can assume any positive integer, m is the order of Legendre polynomials and t is the normalized time. They are defined on the interval $[0, 1)$ as

$$\psi_{n,m}(t) = \left\{ \begin{array}{ll} \sqrt{m + \frac{1}{2}} 2^{\frac{k}{2}} P_m\left(2^k t - \hat{n}\right) & , for \frac{\hat{n}-1}{2^k} \leq t \leq \frac{\hat{n}+1}{2^k} \\ 0 & , otherwise \end{array} \right\} \quad (3)$$

where $m = 0, 1, 2, ..., M - 1, n = 1, 2, 3, ..., 2^{k-1}$. The coefficient $\sqrt{m + \frac{1}{2}}$ is for orthonormality, the dilation parameter is $a = 2^{-k}$ and translation parameter is $b = \hat{n}2^{-k}$. Here $P_m(t)$ are well-known Legendre polynomials of order m which are defined on the interval [-1,1], and can be determined with the aid of the following recurrence formulae:

$P_0(t) = 1, \ P_1(t) = t, \ P_{m+1}(t) = \left(\frac{2m+1}{m+1}\right) t \ P_m(t) - \left(\frac{m}{m+1}\right) P_{m-1}(t),$
where m=1,2,3,...

2.2 Function Approximation

A function $f(t)$ defined over [0,1) may be expanded as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(t) \quad (4)$$

where $c_{nm} = \langle f(t), \psi_{nm}(t) \rangle$, in which $\langle ., . \rangle$ denotes the inner product. If the infinite series in Eq.(4) is truncated, then Eq.(4) can be written as $f(t) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(t) = C^T \psi(t)$ where C and $\psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$C = [c_{10}, c_{11}, \cdots, c_{1M-1}, c_{20}, c_{21}, \cdots, c_{2M-1}, \cdots, c_{2^{K-1}0}, \cdots, c_{2^{K-1}M-1}]^T \quad (5)$$

$$\psi\left(t\right) = \left[\psi_{10}\left(t\right), \cdots, \psi_{1M-1}\left(t\right), \psi_{20}\left(t\right), \cdots, \psi_{2M-1}\left(t\right), \cdots, \psi_{2^{K-1}0}\left(t\right), \cdots, \psi_{2^{K-1}M-1}\left(t\right)\right]^{T} \tag{6}$$

3   Legendre Wavelet scheme for second kind Fredholm integral equations

Consider the integral equation given in Eq.(1) and Eq. (2)

$f\left(x\right) = g\left(x\right) + \int_0^1 k\left(x, t\right) f\left(t\right) dt$ and $f\left(x\right) = g\left(x\right) + \int_0^1 k\left(x, t\right) h\left[f\left(t\right)\right] dt$

$$\text{Let } f\left(x\right) = C^T \psi\left(x\right) \tag{7}$$

Hence the above equations becomes

$$C^T \psi\left(x\right) = g\left(x\right) + \int_0^1 k\left(x, t\right) C^T \psi\left(t\right) dt \text{ and } C^T \psi\left(x\right) = g\left(x\right) + \int_0^1 k\left(x, t\right) h\left[C^T \psi\left(t\right)\right] dt \tag{8}$$

We now collocate Eq.(8) at $2^{k-1}M$ points x_i as

$$C^T \psi\left(x_i\right) = g\left(x_i\right) + \int_0^1 k\left(x, t\right) C^T \psi\left(x_i\right) dt \tag{9}$$

$$C^T \psi\left(x_i\right) = g\left(x_i\right) + \int_0^1 k\left(x, t\right) \left[C^T \psi\left(x_i\right)\right] dt \tag{10}$$

Suitable collocation points are zeros of Chebyshev polynomials

$x_i = cos\left(\left(2i + 1\right)\pi / 2^k M\right), \quad i = 1, 2, ..., 2^{k-1}M.$

In order to use the Gaussian integration formula for Eq.(9)and Eq.(10), we transfer the intervals [0,1] into the interval [-1,1] by means of the transformation $\tau = 2t - 1$.

Eq. (9) and eq.(10) may then be written as

$C^T \psi\left(x_i\right) = g\left(x_i\right) + \frac{x_i}{2} \int_{-1}^1 k\left(x, \frac{1}{2}\left(\tau + 1\right)\right) C^T \psi\left(\frac{1}{2}\left(\tau + 1\right)\right) d\tau$ and

$C^T \psi\left(x_i\right) = g\left(x_i\right) + \frac{x_i}{2} \int_{-1}^1 k\left(x, \frac{1}{2}\left(\tau + 1\right)\right) h\left(C^T \psi\left(\frac{1}{2}\left(\tau + 1\right)\right)\right) d\tau$

By using the Gaussian integration formula, we get

$$C^T \psi\left(x_i\right) = g\left(x_i\right) + \frac{x_i}{2} \sum_{j=1}^{s} w_j k\left(x, \frac{1}{2}\left(\tau + 1\right)\right) C^T \psi\left(\frac{1}{2}\left(\tau + 1\right)\right) d\tau \tag{11}$$

$$C^T \psi\left(x_i\right) = g\left(x_i\right) + \frac{x_i}{2} \sum_{j=1}^{s} w_j k\left(x, \frac{1}{2}\left(\tau + 1\right)\right) h\left(C^T \psi\left(\frac{1}{2}\left(\tau + 1\right)\right)\right) d\tau \tag{12}$$

where τ_j is s zeros of Legendre polynomials P_{s+1} and w_j are the corresponding weights. Here the weight w_j can be identified with the help of the formula $w_j = \int_{-1}^1 \prod_{j=o, j \neq i}^s \left(\frac{\tau - \tau_j}{\tau_i - \tau_j}\right) d\tau$

4 Existence of uniqueness and Convergence analysis:

In this section, we discuss the theoretical analysis of uniqueness and convergence of our approach.

Theorem 4.1 *Uniqueness theorem*

Eq.(1)and (2) have unique solution whenever $0 < \alpha < 1$, where $\alpha = L$

Proof:

Eq.(1) and (2) can be written in the form $u(x) = g(x) + \int_0^1 F(t, u(t))\, dt$ where $f(x) = u(x)$. $F(t, u(t))$ includes the nonlinear term and is Lipschitz continuous with $|F(u) - F(v)| \leq L\,|u - v|$.

Let u and u^* be two different solutions for Eq.(1).

$$|u - u^*| = \left| \left(g(x) + \int_0^1 F(t, u(t))\, dt\right) - \left(g(x) + \int_0^1 F(t, u^*(t))\, dt\right) \right|$$

$$\Rightarrow |u - u^*| = \left| \int_0^1 [F(u) - F(u^*)]\, dt \right| \leq \int_0^1 |F(u) - F(u^*)|\, dt \leq L\,|u - u^*|.$$

This implies that $|u - u^*|(1 - L) \leq 0$ where $\alpha = L$.

As $0 < \alpha < 1$, $|u - u^*| = 0$, implies $u = u^*$ and this completes the proof.

Theorem 4.2 *Convergence theorem*

The series solution Eq.(4) using LWM converges towards u(x).

Proof:

Let $L^2(R)$ be the Hilbert space and $\psi_{k,n}(t) = |a|^{-\frac{1}{2}} \psi\left(a_0{}^k t - n b_0\right)$ where $\psi_{k,n}(t)$ form a basis of $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$, $\psi_{k,n}(t)$ forms an orthonormal basis.

Let $u(x) = \sum_{i=1}^{M-1} C_{1i}\psi_{1,i}(x)$ where $C_{1i} = \langle u(x), \psi_{1,i}(x)\rangle$ for k=1 and $\langle . \rangle$ represents an inner product. Let us denote $\psi_{1,i}(x)$ as $\psi(x)$ and Let $\alpha_j = \langle u(x), \psi(x)\rangle$.

Define the sequence of partial sums S_n of $(\alpha_j \psi(x_j))$; Let S_n and S_m be arbitrary partial sums with $n \geq m$. We are going to prove that S_n is a Cauchy sequence in Hilbert space.

Let $S_n = \sum_{j=1}^n \alpha_j \psi(x_j)$. Now $\langle u(x), S_n\rangle = \left\langle u(x), \sum_{j=1}^n \alpha_j \psi(x_j)\right\rangle = \sum_{j=1}^n |\alpha_j|^2$. We will claim that $\|S_n - S_m\|^2 = \sum_{j=m+1}^n |\alpha_j|^2$ for $n > m$.

Now $\left\|\sum_{j=m+1}^n \alpha_j \psi(x_j)\right\|^2 = \left\langle \sum_{j=m+1}^n \alpha_i \psi(x_i), \sum_{j=m+1}^n \alpha_j \psi(x_j)\right\rangle = \sum_{j=m+1}^n |\alpha_j|^2$ for $n > m$.

i.e $\left\|\sum_{j=m+1}^n \alpha_j \psi(x_j)\right\|^2 = \sum_{j=m+1}^n |\alpha_j|^2$ for $n > m$.

From Bessel's inequality, we have $\sum_{j=m+1}^n |\alpha_j|^2$ is convergent and hence $\left\|\sum_{j=m+1}^n \alpha_j \psi(x_j)\right\|^2 \to 0$ as $m, n \to \infty$. i.e $\left\|\sum_{j=m+1}^n \alpha_j \psi(x_j)\right\| \to 0$ and $\{S_n\}$ is a cauchy sequence and it converges to say 's'.

We assert that $u(x) = s$.

Now $\langle s - u(x), \psi(x_i)\rangle = \langle s, \psi(x_i)\rangle - \langle u(x), \psi(x_i)\rangle$

$$= \langle Lt_{n\to\infty} \ S_n, \psi(x_j)\rangle \ - \ \alpha_j = \alpha_j - \alpha_j$$
$$\Rightarrow \langle s - u(x), \psi(x_i)\rangle = 0 \ \text{ and hence } u(x) = s \ \text{ and } \sum_{j=1}^{n} \alpha_j \psi(x_j) \text{ converges}$$

to *u(x)* and this completes the proof.

The error bound for the above function approximation is elucidated in the following lemma,given in [14].

Lemma 4.3 *Suppose that the function* $f : [0,1] \to \ R$ *is m times continuously differentiable,* $f \in C^m [0,1]$. *Then* $C^T\psi$ *approximate f with an mean error bound as follows:*
$$\left\| f - C^T\psi \right\| \le \tfrac{1}{m!2^{mk}} \ sup_{x\in[0,1]} \left| f^{(m)(x)} \right|.$$
Proof: See [14]

5   Test problems

We provide three numerical examples discussed in [12] to validate the Legendre wavelet method presented in this paper.

Problem 1.

Consider the following linear integral equation of second kind

$$f(x) - \int_0^1 sin(4\pi x + 2\pi t)f(t)dt = cos(2\pi x) + \frac{1}{2}sin(4\pi x) \qquad (13)$$

We apply the method presented in this paper and solve Eq.(13) with K=1 and M=5.

Let $f(x) = C^T\psi(x)$

$$C^T\psi(x) = \int_0^1 sin(4\pi x + 2\pi t)C^T\psi(x)dx + cos(2\pi x) + \frac{1}{2}sin(4\pi x) \qquad (14)$$

Solving Eq.(14), we get $c_{10} = 1$; $c_{11} = \frac{-\pi^2}{\sqrt{3}}$; $c_{12} = \frac{-\pi^2}{3\sqrt{5}}$; $c_{13} = 0$; $c_{14} = \frac{\pi^4}{105\sqrt{9}}$

By using Eq.(7), we have $f(x) = c_{10}\psi_{10} + c_{11}\psi_{11} + c_{12}\psi_{12} + c_{13}\psi_{13} + c_{14}\psi_{14}$

Hence $f(x) = 1 - 2\pi^2 x^2 - \frac{2}{3}\pi^4 x^4 + \frac{4}{45}\pi^6 x^6 + \cdots$ and the closed form solution is given by $f(x) = cos(2\pi x)$, which is the exact solution.

Problem 2.

Consider the nonlinear integral equation of the second kind

$$f(x) - \int_0^1 sin(2\pi x)tf^2(t)dt = \frac{3}{4}sin(2\pi x) \qquad (15)$$

We apply the method presented in this paper and solve Eq.(15) with K=1 and M=6.

Let $f(x) = C^T\psi(x)$.

$C^T\psi(x) = \int_0^1 sin(2\pi x)t[C^T\psi(x)]^2 dx + cos(2\pi x) + \frac{3}{4}sin(2\pi x)$

Simplifying, we get $c_{10} = 0$; $c_{11} = 114.3227$; $c_{12} = 25.1220$; $c_{13} = 6.8041760$; $c_{14} = 0.9709$; $c_{15} = 0.0061$

By using Eq.(7), we have $f(x) = c_{10}\psi_{10} + c_{11}\psi_{11} + c_{12}\psi_{12} + c_{13}\psi_{13} + c_{14}\psi_{14} + c_{15}\psi_{15}$

Hence $f(x) = 2\pi x - \frac{4}{3}\pi^3 x^3 + \frac{4}{15}\pi^5 x^5 + \cdots$ and the closed form solution is given by $f(x) = sin(2\pi x)$, which is the exact solution.

Problem 3.

Finally, we solve a nonlinear second kind Fredholm integral equation with weakly singular kernel

$$f(x) - \int_0^1 ln\,|x - t|\,f^3(t)dt = g(x) \tag{16}$$

We apply the method presented in this paper and solve Eq.(16) with K=1 and M=5.

Let $f(x) = C^T\psi(x)$ and solving by the method presented above, solution in closed form is $f(x) = \left(x - \frac{1}{2}\right)^{\frac{2}{3}}$, which is the exact solution.

6 Conclusion

In this work, we proposed the Legendre wavelet approximation method (LWM) for solving second kind Fredholm integral equations discussed in [16]. The properties of the Legendre wavelets together with the Gaussian integration method are used to reduce the problem to the solution of algebraic equations. We have also demonstrated the uniqueness and convergence analysis of LWM. Illustrative examples are included to demonstrate the validity and applicability of the technique. These examples reveal that the proposed LWM method is very convenient and in par with other existing methods that gives the exact solution. Furthermore, since the basis of Legendre wavelets are polynomial, the values of integrals for the nonlinear integral equations of the form in Eq.(9)and Eq. (10) are calculated as approximately close to the exact solutions.

References

[1] B.Asady, M.Tavassoli Kajani, A.Hadi Vencheh, A.Heydari, Solving second kind integral equations with hybrid Fourier and block-pulse functions, *Appl.math.Comput.***160** (2005), 517-522.

[2] I.Daubechies, *Ten Lectures on wavelets,*, SIAM, Philadelphia, PA ,1992.

[3] W.Fang, Y.Wang, Y.Xu, An implementation of fast wavelet Galerkin methods for integral equations of the second kind, *J.Sci.comput.***20** (2004), 277-302.

[4] A. Karoui, Wavelets: properties and approximate solution of a second kind integral equation, *Comput. Math. Appl.***46** (2003), 263-277.

[5] K. Maleknejad, M. Yousefi, K. Nouri, Computational methods for integrals involving functions and Daubechies wavelets , *Appl.Math.Comput.***189 (2)** (2007), 1828-1840.

[6] K. Maleknejad, T. Lotfi, Y. Rostami, Numerical computational method in solving Fredholm integral equations of the second kind by using Coifman wavelet, *Appl.Math.Comput.***186 (1)**(2007), 212-218.

[7] K. Maleknejad, A. Khademi, Filter matrix based on interpolation wavelets for solving Fredholm integral equations , *Commun. Nonlinear Sci. Numer. Simul.***16** (2011), 4197-4207.

[8] K. Maleknejad, M. Yousefi, Numerical solution of the integral equation of the second kind by using wavelet bases of Hermite cubic splines ,*Appl. Math.Comput.***183(1)** (2006), 134-141.

[9] K.Maleknejad, M. Tavassoli Kajani, Y. Mahmoudi, Numerical solution of linear Fredholm and volterra integral equation of the second kind by using Legendre wavelets,*Kybernetes.***32: 9/10** (2003),1530 - 1539.

[10] M.Tavassoli kajani, A.hadi vencheh, Solving second kind integral equations with hybrid Chebyshev and block-pulse functions,, *Appl.Math.Comput.***163** (2005), 71-77.

[11] Ulo Lepik and Enn Tamme, Solution of nonlinear Fredholm integral equations via the Haar wavelet method ,*Proc. Estonian Acad. Sci. Phys. Math.***56 (1)**(2007),17-27.

[12] J.Y.Xiao, L.H.Wen, D.Zhang, Solving second kind integral equations by periodic wavelet Galerkin method, *Appl.Math.Comput.***175**(2006), 508-518.

[13] S.A. Yousefi, Legendre wavelets method for solving differential equations of Lane-Emden type, *Appl.Math.Comput.***181** (2006),1417-1422.

[14] S.A. Yousefi, Legendre Scaling function for solving generalized Emden-Fowler Equations, *Int.J. of information and system sciences.***3(2)** (2007), 243-250.