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Numerical Solution of Nonlinear Partial Differential Equation by Legendre Multiwavelet Method

Science

KEYWORDS : Legendre multiwavelet, nonlinear partial differential equations, Galerkin method.

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ABSTRACT

In this work, the Legendre multiwavelet basis with considering the standard Galerkin method has been applied to give the approximate solution of nonlinear partial differential equations (NPDE's). The properties of the Legendre multiwavelet presented. These properties together with the standard Galerkin method are then utilized to reduce nonlinear partial differential equations to the solution of an algebraic system. Numerical results and comparison with exact solution given to demonstrate the applicability and efficiency of the method.

1. Introduction

In 1807, Joseph Fourier developed a method for representing a signal with a series of coefficients based on an analysis function. He laid the mathematical basis from which the wavelet theory is developed. The first to mention wavelets was Alfred Haar in 1909 in his PhD thesis. In the 1930's, Paul Levy found the scale-varying Haar basis function superior to Fourier basis functions. Jean Morlet and Alex Grossman again derive the transformation method of decomposing a signal into wavelet coefficients and reconstructing the original signal in 1981. In 1986, Stephane Mallat and Yves Meyer developed a multiresolution analysis using wavelets. They mentioned the scaling function of wavelets for the first time, it allowed researchers and mathematicians to construct their own family of wavelets using the derived criteria. Around 1998, Ingrid Daubechies used the theory of multiresolution wavelet analysis to construct her own family of wavelets. Her set of wavelet orthonormal basis functions have become the cornerstone of wavelet applications today. Wavelet analysis can be performed in several ways, a continuous wavelet transform, a discretized continuous wavelet transform and a true discrete wavelet transform. The application of wavelet analysis becomes more widely spread as the analysis technique becomes more generally known. The fields of application vary from science, engineering, medicine to finance. Types of wavelets are Haar Wavelets (orthogonal in L_2, compact Support, scaling function is symmetric wavelet function is antisymmetric, Infinite support in frequency domain), Shannon Wavelet (orthogonal, localized in frequency domain, easy to calculate, infinite support and slow decay), Meyer Wavelets (Fourier transform of father function) and Daubishes wavelets (orthogonal in L_2, compact support, zero moments of father-function). Nonlinear partial differential equations appear in many branches of physics, engineering and applied mathematics, including nonlinear optics, quantum field theory, fluid mechanics, elasticity theory and condensed matter physics. Studying nonlinear partial differential equations (NPDEs) is very important. These equations are often too complicated to be solved exactly and even if an exact solution is obtained, the required calculations may be too complicated. Very recently, many powerful methods have been presented, such as the Adomian decomposition method [1-5], the homotopy perturbation method (HPM) [6-9], and the differential transform method [10-13]. The Legendre wavelets is successfully applied for solving differential, integral and integro-differential equations is thoroughly considered in [14-20]. The aim of this work is to present the Legendre multiwavelet for approximating the solution of a nonlinear partial differential equations (NPDE's). The method consists of expanding the solution by Legendre multiwavelet with unknown coefficients. The properties of Legendre multiwavetlet together with the Galerkin method are then utilized to evaluate the unknown coefficients and find an approximate solution to the (NPDE's). The article is organized as follows: In Section II, we describe the basic formulation of wavelets and Legendre multiwavelet required for our subsequent development. Section III is devoted to the solution of some examples of nonlinear partial differential equations (NPDE's) by using Legendre multiwavelet. In Section IV, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples. Section V consists of a brief summary.

2. Properties of Legendre multiwavelets

2.1 Wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously we have the following family of continuous wavelets [14-20]

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R} \ , a \neq 0. \tag{1}$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, $a_0 > 1, b_0 > 0$ and $n, k \in \mathbb{N}$ we have the following family of discrete wavelets:

$$\psi_{k,n}(t) = |a|^{\frac{k}{2}}\psi\left(a_0{}^k t - nb_0\right), \tag{2}$$

where $\psi_{k,n}(t)$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$ then $\psi_{k,n}(t)$ forms an orthonormal basis [14-20].

2.2 Legendre multiwavelets [21]

Legendre multiwavelets $\psi_{nm}(t) = \psi(k, n, m, t)$ have four arguments; $n, n = 0, 1, 2, \dots, 2^k - 1, k$ can assume any positive integer, m is the order for Legendre polynomials and t is the normalized time. They are defined on the interval [0, 1):

$$\psi_{nm}(t) = \begin{cases} \sqrt{2m+1}\, 2^{\frac{k}{2}}\, P_m\left(2^k t - n\right), & \text{for } \dfrac{n}{2^k} \le t \le \dfrac{n+1}{2^k} \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

where $m = 0, 1, \dots, M - 1$, M nonnegative integer and $n = 0, 1, 2, \dots, 2^k - 1$. The coefficient $\sqrt{2m+1}$ is for orthonormality, $P_m(t)$ are the well-known shifted Legendre polynomials of order m which are defined on the interval [0, 1], and can be determined with the aid of the following recurrence formula:

$P_0(t) = 1, \qquad P_1(t) = 2t - 1,$

$$P_{m+1}(t) = \left(\frac{2m+1}{m+1}\right)(2t-1)P_m(t) - \left(\frac{m}{m+1}\right)P_{m-1}(t), m = 1,2,3, \dots . \tag{4}$$

Also the two-dimensional Legendre multiwavelet are defined as [10]:

$$\psi_{n_1 m_1 n_2 m_2}(x,t) = \begin{cases} A\, P_{m_1}\left(2^{k_1}x - n_1\right)P_{m_2}\left(2^{k_2}t - n_2\right), & \text{for } \dfrac{n_1}{2^{k_1}} \le x \le \dfrac{n_1+1}{2^{k_1}} \\ & \quad\ \ \dfrac{n_2}{2^{k_2}} \le t \le \dfrac{n_2+1}{2^{k_2}}\ , \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

where $A = \sqrt{(2m_1+1)(2m_2+1)}\, 2^{\frac{k_1+k_2}{2}}$, n_1 and n_2 are defined similarly to n, k_1 and k_2 can assume any positive integer, m_1 and m_2 are the order for Legendre polynomials and $\psi_{n_1 m_1 n_2 m_2}(x,t)$ forms a basis for $L^2([0, 1]\times[0, 1])$.

2.3 Function Approximation

A function f(x, t) defined over [0, 1] × [0, 1] can be expand as [21]:

$$f(x,t) = \sum_{n=1}^{\infty}\sum_{i=0}^{\infty}\sum_{l=1}^{\infty}\sum_{j=0}^{\infty} c_{n,i,l,j}\, \psi_{n,i}(x)\psi_{l,j}(t). \tag{6}$$

If the infinite series in equation (6) is truncated, then equation (6) can written as:

$$f(x,t) = \sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{N}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M} c_{n,i,l,j}\, \psi_{n,i}(x)\psi_{l,j}(t) = \Psi^T(x)F\,\Psi(t) \tag{7}$$

Where $\Psi(x)$ and $\Psi(t)$ are $2^{k_1}(M_1+1)\times 1$ and $2^{k_2}(M_2+1)\times 1$ matrices, respectively given by

$$\Psi(x) = \left[\psi_{10}(x), \dots, \psi_{1M_1}(x), \dots, \psi_{20}(x), \dots, \psi_{2M_1}(x), \dots, \psi_{(2^{k_1}-1)0}(x), \dots, \psi_{(2^{k_1}-1)M_1}(x)\right],$$

$$\Psi(t) = \left[\psi_{10}(t), \dots, \psi_{1M_1}(t), \dots, \psi_{20}(t), \dots, \psi_{2M_1}(t), \dots, \psi_{(2^{k_1}-1)0}(t), \dots, \psi_{(2^{k_1}-1)M_1}(t)\right]. \tag{8}$$

Also, F is a $2^{k_1}(M_1+1)\times 2^{k_2}(M_2+1)$ matrix whose elements can be calculated from

$$\int_0^1\int_0^1 \psi_{ni}(x)\phi_{lj}(t)f(x,t)dtdx, \tag{9}$$

with, $n = 0, 1, \dots, 2^{k_1} - 1, i = 0, \dots, M_1, l = 0, 1, \dots, 2^{k_2} - 1, j = 0, \dots, M_2$.

3. Solution of nonlinear partial differential equations

Consider the following nonlinear partial differential equations with independent variables x and t a dependent variables

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = 0, \tag{10}$$

with initial condition $u(x, 0) = f_1(x), \qquad u_t(x, 0) = f_2(x)$

where, $L = \frac{\partial^2}{\partial t^2}$, R is a linear operator and Nu(x, t) is the nonlinear term.

Volume : 3 | Issue : 2 | February 2014 • ISSN No 2277 - 8179

A Galerkin approximation to (10) constructed as follows. The approximation u_{NM} is sought in the form of the truncated series:

$$u_{NM}(x,t) = \sum_{n=1}^{2^{k_1}}\sum_{i=0}^{N}\sum_{l=1}^{2^{k_2}}\sum_{j=0}^{M} t^2\, a_{n,i,l,j}\, \psi_{n,i}(x)\psi_{l,j}(t) + w(x,t), \tag{11}$$

where $w(x,0) = f_1(x)$, $w_t(x,0) = f_2(x)$, and ψ_{ij} are Legendre multiwavelet basis.

In (11), $w_1(x,t)$and $w_2(x,t)$ are not unique. We can have different choice. We choose

$$w(x,t) = f_1(x) + tf_2(x), \quad . \tag{12}$$

Now we have $u_{NM}(x,0) = f_1(x)$, $\frac{\partial u_{NM}(x,0)}{\partial t} = f_2(x)$, this approximation provides greater flexibility in which to impose initial conditions. The expansion coefficient $c_{n,i,l,j}$ determined by Galerkin equations:

$$\langle F(u_{NM}), \psi_{n,i}\psi_{l,j}\rangle = 0, \tag{13}$$

where $\langle . \rangle$ denotes inner product defined as

$$\langle F(u_{NM}), \psi_{n,i}\psi_{l,j}\rangle = \int_0^1\int_0^1 F(u_{NM})(x,t)\psi_{ni}(x)\psi_{lj}(t)dtdx, \tag{14}$$

Galerkin equations (21) gives a system of $2^{k_1-1}(N+1) \times 2^{k_2-1}(M+1)$ nonlinear equations, which can be solved for the elements of $a_{n,i,l,j}$, $b_{n,i,l,j}$, $i = 0,1,\ldots N$, $j = 0,1,\ldots,M, n = 1,2,\ldots,2^{k_1}, l = 1,2,\ldots,2^{k_2}$ using suitable method and get the approximate solution (10).

4. ILLUSTRATIVE EXAMPLES

Example 4.1 Consider the Symmetrical-regular long wave equation [22]

$$u_{tt} + u_{xx} + (u^2)_{tx} + u_{ttxx} = 0, \tag{15}$$

With initial conditions

$$u(x,0) = \frac{33}{16} + \frac{3}{32}\tanh(-\frac{1}{4}x)^2, \quad u_t(x,0) = -\frac{3}{256}\operatorname{sech}\left(\frac{x}{4}\right)^2 \tanh(\frac{x}{4}) \tag{16}$$

We applied the method presented in this article $k_1 = k_2 = 0$ and $M = N = 1$ and solved Eq. (15). From eq. (12) we have

$$w(x,t) = \frac{33}{16} + \frac{3}{32}\tanh\left(-\frac{1}{4}x\right)^2 - \frac{3\,t}{256}\operatorname{sech}\left(\frac{x}{4}\right)^2 \tanh\left(\frac{x}{4}\right),$$

and from Eq. (13) we obtain

$$a_{0,0,0,0} = 0.0001247718276, \qquad a_{0,0,0,1} = 0.00009364392,$$
$$a_{0,1,0,0} = -0.00003022767119, \qquad a_{0,1,0,1} = -0.00001788604233 \tag{17}$$

Thus from (11) we have

$$u_{NM}(x,t) = -0.0000387264644t^2 + 0.0004317083086t^3 + 0.0000026045294t^2x - 0.000214632508t^3x$$
$$+2.0625 + 0.09375\tanh\left(\frac{x}{4}\right)^2 - 0.01171875\tanh\left(\frac{x}{4}\right)t + 0.01171875\tanh\left(\frac{x}{4}\right)^3 t \tag{18}$$

The exact solution $u(x,t) = \frac{33}{16} + \frac{3}{32}\tanh\left(\frac{1}{4}x - \frac{1}{16}t\right)^2$ and approximate solution (18) plotted in Fig.1 and Fig.2. Absolute errors between the exact solution and the approximant solution (18) as shown in Table 1.

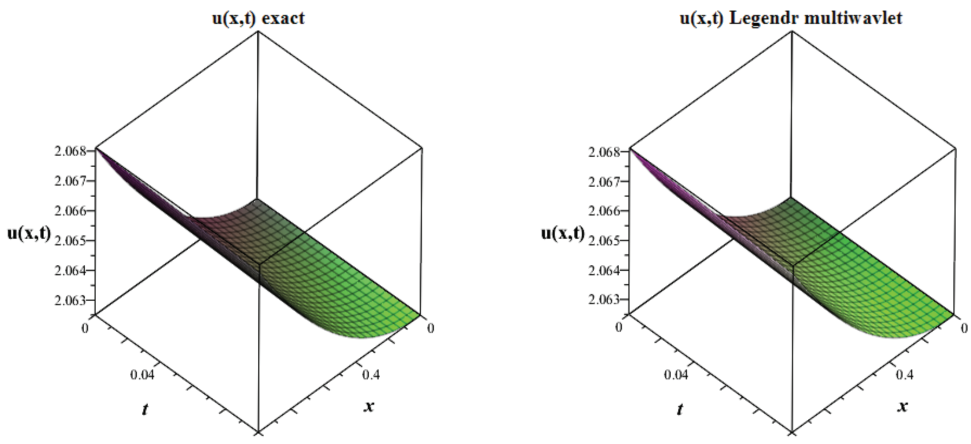

Fig.1 Exact and approximate solutions of u(x, t) for 0 ≤ x, t≤ 1

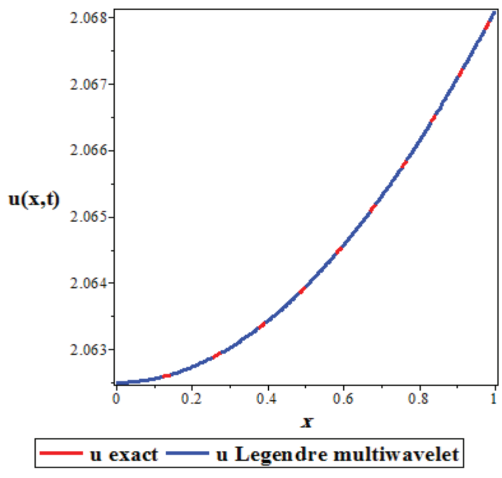

Fig.2 Exact and approximate solutions of u(x, t) for 0 ≤ x≤ 1, t = 0.1

Table 1 Exact and approximate solution of example 1 and the error

x ,t=0.1	Exact	Legendre multiwavelet	$\lvert u_{ex} - u_{Leg} \rvert$
0.1	2.062532951	2.062529323	3.617×10^{-6}
0.2	2.062679215	2.062675593	3.628×10^{-6}
0.3	2.062941723	2.062938125	3.621×10^{-6}
0.4	2.063319171	2.063315616	3.596×10^{-6}
0.5	2.063809692	2.063806196	3.554×10^{-6}
0.6	2.064410877	2.064407457	3.495×10^{-6}
0.7	2.065119801	2.065116471	3.420×10^{-6}
0.8	2.065933051	2.065929826	3.329×10^{-6}
0.9	2.066846759	2.066843651	3.225×10^{-6}
1.0	2.067856646	2.067853667	3.108×10^{-6}

Example 4.2 Consider the Klien–Gordon problem [23, 24]

$$u_{tt} - u_{xx} + u^2 = 6xt(x^2 - t^2) + x^6t^6, \tag{19}$$

with initial conditions $u(x,0) = u_t(x,0) = 0$.

We applied the method presented in this article and solved Eq. (19) with $k_1 = k_2 = 0$ and$M = N = 1$. The exact solution $u(x,t) = x^3t^3$and approximate solution of Eq. (19) plotted in Fig. 3 and Fig.4. Table 2 show the absolute error obtained by our method and the method in [23].

Volume : 3 | Issue : 2 | February 2014 • ISSN No 2277 - 8179

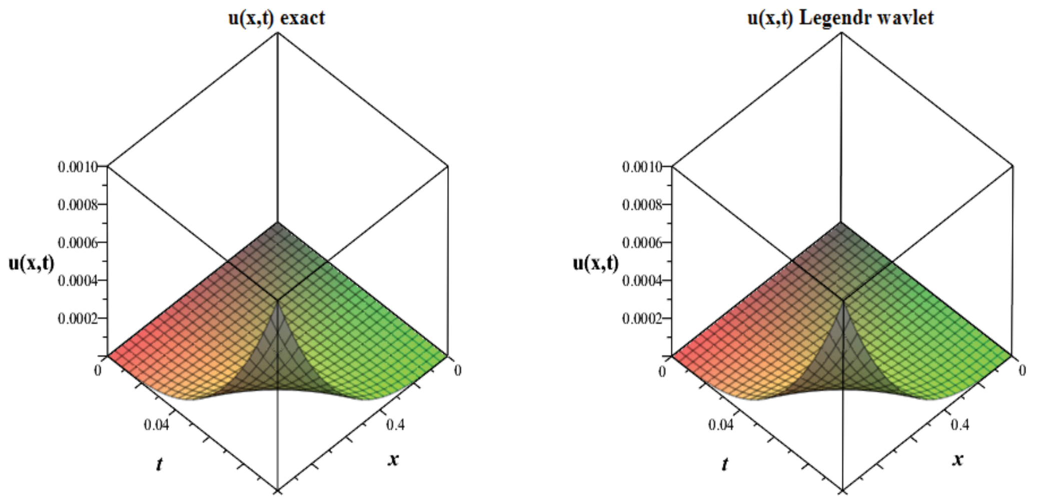

Fig.3 Exact and approximate solutions of u(x, t) for 0 ≤ x, t≤ 1

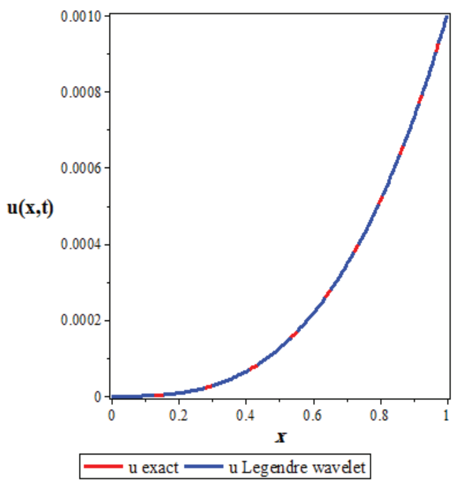

Fig.4 Exact and approximate solutions of u(x, t) for 0 ≤ x≤ 1, t = 0.1

Table 2 Exact and approximate solution of example 2 and the error

x ,t=1	Error e* in [23]	Error e in [23]	Error in Presented method
0.1	1×10^{-6}	5×10^{-6}	1×10^{-9}
0.3	2×10^{-6}	6×10^{-6}	1×10^{-8}
0.5	2×10^{-6}	7×10^{-6}	2×10^{-8}
0.7	2×10^{-6}	7×10^{-6}	2×10^{-8}
0.9	2×10^{-6}	7×10^{-6}	8×10^{-9}

Example 4.3 Consider the Symmetric Regularized Long Wave (SRLW) equation [25]

$$u_{tt} + u_{xx} + uu_{xt} + u_x u_t + u_{xxtt} = 0, \qquad (20)$$

with initial conditions

$$u(x,0) = -\frac{81}{8} + \frac{3}{40} sech\left(\frac{x}{4}\right)^2 \quad and \quad u_t(x,0) = -\frac{3}{800}\mathrm{Sech}\left(\frac{x}{4}\right)^2 \tanh\left(\frac{x}{4}\right).$$

We applied the method presented in this article and solved Eq. (20) with $k_1 = k_2 = 0$ and $M = N = 1$. The exact solution $u(x,t) = -\frac{81}{8} + \frac{3}{40}\mathrm{Sech}\left(\frac{1}{4}\left(\frac{t}{10} + x\right)\right)^2$ and approximate solution of Eq. (20) plotted in Fig. 5 and Fig.6. Absolute errors between the exact solution and the approximant solution as shown in Table 3.

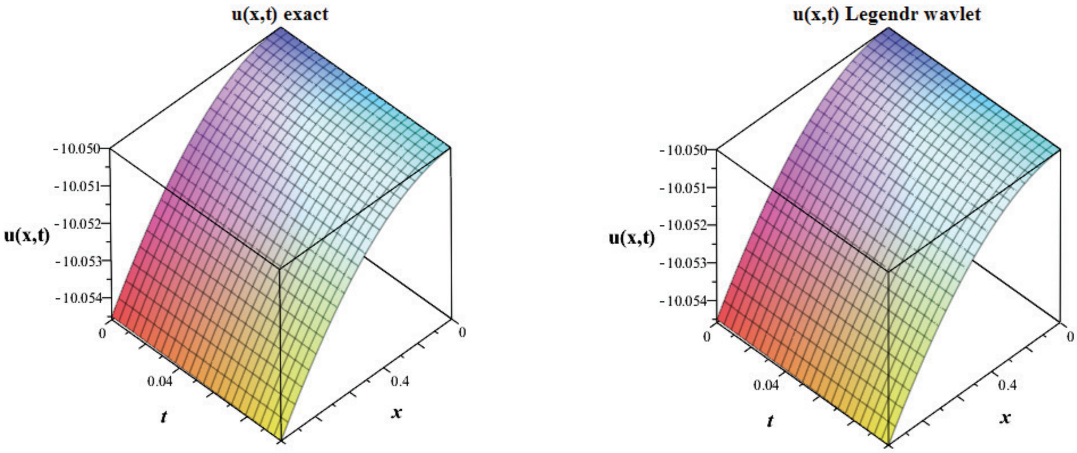

Fig.5 Exact and approximate solutions of u(x, t) for 0 ≤ x, t≤ 1

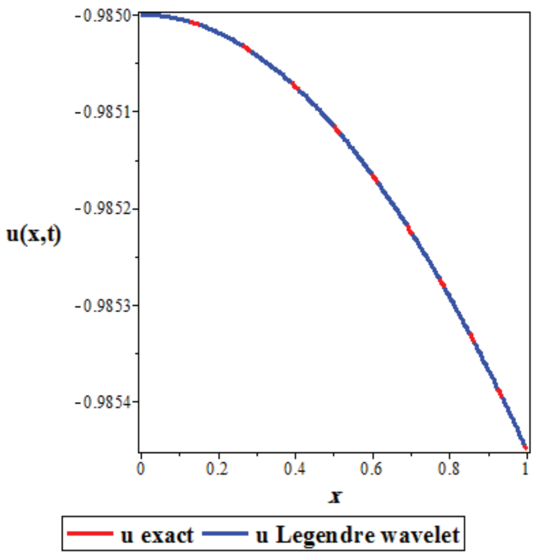

Fig.6 Exact and approximate solutions of u(x, t) for 0 ≤ x≤ 1, t = 0.1

Table 3 Exact and approximate solution of example 3 and the error

x ,t=0.1	Exact	Legendre multiwavelet	$\lvert u_{ex} - u_{Leg} \rvert$
0.1	-10.05005669	-10.05005510	1.600×10^{-6}
0.2	-10.05020634	-10.05020475	1.591×10^{-6}
0.3	-10.05044867	-10.05044710	1.579×10^{-6}
0.4	-10.05078248	-10.05078091	1.565×10-6
0.5	-10.05120613	-10.05120458	1.549×10-6
0.6	-10.05171753	-10.05171600	1.530×10-6
0.7	-10.05231421	-10.05231269	15.10×10-6
0.8	-10.05299331	-10.05299182	1.488×10-6
0.9	-10.05375161	-10.05375014	1.465×10^{-6}
1.0	-10.05458558	-10.05458415	1.440×10^{-6}

Volume : 3 | Issue : 2 | February 2014 • ISSN No 2277 - 8179

Example 4.4 The improved Boussinesq (IB) equation [26]

$$u_{tt} - u_{xx} + uu_{xx} + (u_x)^2 + u_{xxtt} = 0, \tag{21}$$

with initial conditions

$$u(x,0) = -\frac{197}{200} - \frac{3}{400}\tanh\left(\frac{x}{4}\right)^2 \quad and \quad u_t(x,0) = \frac{3}{8000}\tanh\left(\frac{x}{4}\right)\left(1 - \tanh\left(\frac{x}{4}\right)^2\right)$$

We applied the method presented in this article and solved Eq. (21) with $k_1 = k_2 = 0$ and $M = N = 1$. The exact solution $u(x,t) = -0.985 - \frac{3}{400}\tanh[\frac{1}{4}(x - \frac{t}{10})]^2$ and approximate solution of Eq. (21) plotted in Fig. 7 and Fig.8. Absolute errors between the exact solution and the approximant solution as shown in Table 4.

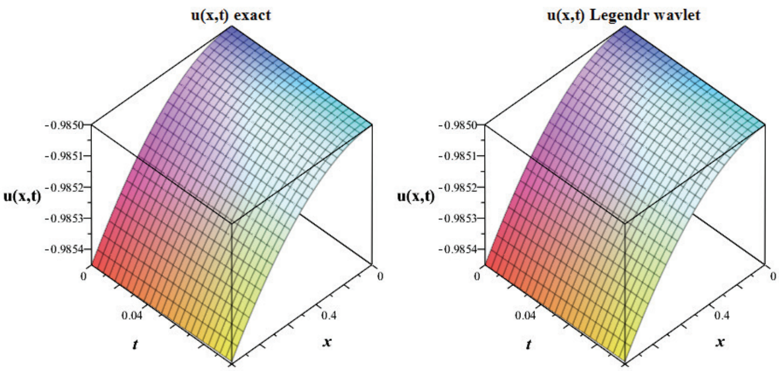

Fig.7. Exact and approximate solutions of u(x, t) for 0 ≤ x, t≤ 1

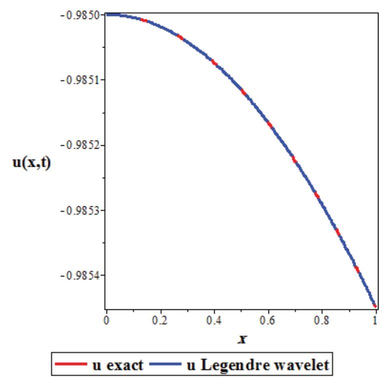

Fig (8). Solutions of u(x, t) for 0 ≤ x≤ 1, t = 0.1

Table 4 Exact and approximate solution of example 4 and the error

x ,t=0.1	Exact	Legendre multiwavelet	$\|u_{ex} - u_{Leg}\|$
0.1	-0.9850037956	-0.9850038204	2.486×10^{-8}
0.2	-0.9850168965	-0.9850169194	2.302×10^{-8}
0.3	-0.9850392841	-0.9850393056	2.140×10^{-8}
0.4	-0.9850708475	-0.9850708675	2.001×10^{-8}
0.5	-0.9851114304	-0.9851114493	1.884×10^{-8}
0.6	-0.9851608341	-0.9851608519	1.787×10^{-8}
0.7	-0.9852188182	-0.9852188352	1.709×10^{-8}
0.8	-0.9852851042	-0.9852851207	1.649×10^{-8}
0.9	-0.9853593777	-0.9853593937	1.605×10^{-8}
1.0	-0.9854412913	-0.9854413071	1.576×10^{-8}

5. Conclusion

In the current work, the Legendr multiwavelet has been applied for solving nonlinear partial differential equations (NPDE's) by reducing the nonlinear partial differential equations (NPDE's) into nonlinear system of algebraic equations and with solving this system, we obtained approximate solution of the problem. In addition, an illustrative example have been included to demonstrate the validity and applicability of the methods. Moreover, only a small number of Legendre multiwavelets are needed to obtain a satisfactory result. The given numerical examples support this claim.

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