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Numerical solution of Fredholm integral equations by using CAS wavelets

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Abstract

A numerical method for solving the Fredholm integral equations is presented. The method is based upon CAS wavelet approximations. The properties of CAS wavelet are first presented. CAS wavelet approximations method are then utilized to reduce the Fredholm integral equations to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Keywords: CAS wavelets; Fredholm integral equations; Numerical method

1. Introduction

Wavelets theory is a relatively new and an emerging area in mathematical research. It has been applied in a wide range of engineering disciplines; particularly, wavelets are very successfully used in signal analysis for waveform representation and segmentations, time–frequency analysis and fast algorithms for easy implementation [1]. Wavelets permit the accurate representation of a variety of functions and operators. Moreover wavelets establish a connection with fast numerical algorithms [2].

Several numerical methods for approximating the solution of integral equations are known. For Fredholm–Hammerstein integral equations, the classical method of successive approximations was introduced in [3]. A variation of the Nystrom method was presented in [4]. A collocation type method was developed in [5]. In [6], Yalcinbas applied a Taylor polynomial solution to Volterra–Fredholm integral equations.

In the present article, we are concerned with the application of CAS wavelets to the numerical solution of a Fredholm integral equation of the form

$$y(x) = f(x) + \lambda \int_0^1 K(x,t)y(t)\mathrm{d}t \quad 0 \leqslant x, t \leqslant 1, \tag{1}$$

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where $f(x)$ and the kernels $K(x,t)$ are assumed to be in $L^2(R)$ on the interval $0 \leqslant x, t \leqslant 1$. We assume that Eq. (1) has a unique solution y to be determined.

In this paper, we introduce a new numerical method to solve Fredholm integral equations. The method consists of reducing the integral equations to a set of algebraic equations by expanding the solution as CAS wavelets with unknown coefficients. The CAS wavelets are first given. The product operational matrix and orthonormality property of CAS wavelets basis are then utilized to evaluate the coefficients of CAS wavelets expansion.

The article is organized as follows: In Section 2, we describe the basic formulation of wavelets and CAS wavelets required for our subsequent development. Section 3 is devoted to the solution of Eq. (1) by using CAS wavelets. In Section 4, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples.

2. Properties of CAS wavelets

2.1. Wavelets and CAS wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously we have the following family of continuous wavelets as [7]

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, \ a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, $a_0 > 1$, $b_0 > 0$ and n, and k positive integers, we have the following family of discrete wavelets:

$$\psi_{k,n}(t) = |a_0|^{k/2}\psi\left(a_0^k t - nb_0\right),$$

where $\psi_{k,n}(t)$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$ then $\psi_{k,n}(t)$ forms an orthonormal basis [7].

CAS wavelets $\psi_{nm}(t) = \psi(k,n,m,t)$ have four arguments $n = 0, 1, 2, \ldots, 2^k - 1, k$ can assume any nonnegative integer, m is any integer and t is the normalized time. They are defined on the interval $[0, 1)$ as

$$\psi_{nm}(t) = \begin{cases} 2^{\frac{k}{2}}\mathrm{CAS}_m(2^k t - n), & \text{for } \frac{n}{2^k} \leqslant t < \frac{n+1}{2^k}, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where

$$\mathrm{CAS}_m(t) = \cos(2m\pi t) + \sin(2m\pi t). \tag{3}$$

The dilation parameter is $a = 2^{-k}$ and translation parameter is $b = n\ 2^{-k}$.

The set of CAS wavelets are an orthonormal basis.

2.2. Function approximation

A function $f(t)$ defined over $[0, 1)$ may be expanded as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m\in Z} c_{nm}\psi_{nm}(t), \tag{4}$$

where $c_{nm} = (f(t), \psi_{nm}(t))$, in which $(.,.)$ denotes the inner product. If the infinite series in Eq. (4) is truncated, then Eq. (4) can be written as

$$f(t) \simeq \sum_{n=1}^{2^k}\sum_{m=-M}^{M} c_{nm}\psi_{nm}(t) = C^{\mathrm{T}}\Psi(t), \tag{5}$$

where C and $\Psi(t)$ are $2^k(2M+1)\times 1$ matrices given by

$$C=\left[c_{1(-M)},c_{1(-M+1)},\dots,c_{1(M)},c_{2(-M)},\dots,c_{2(M)},\dots,c_{2^k(-M)},\dots,c_{2^k(M)}\right]^{\mathrm{T}}, \tag{6}$$

$$\Psi(t)=\left[\psi_{1(-M)}(t),\psi_{1(-M+1)}(t),\dots,\psi_{1M}(t),\psi_{2(-M)}(t),\dots,\psi_{2M}(t),\dots,\psi_{2^k(-M)}(t),\dots,\psi_{2^kM}(t)\right]^{\mathrm{T}}. \tag{7}$$

2.3. The product operational matrix of the CAS wavelet

Let

$$\Psi(t)\Psi^{\mathrm{T}}(t)C\simeq\widetilde{C}\Psi(t), \tag{8}$$

where $\widetilde{C}$ is a $2^k(2M+1)\times 2^k(2M+1)$ product operational matrix. To illustrate the calculation procedure we choose $M=1$ and $k=1$. Thus we have

$$C=\left[c_{1(-1)},c_{10},c_{11},c_{2(-1)},c_{20},c_{21}\right]^{\mathrm{T}}, \tag{9}$$

$$\Psi(t)=\left[\psi_{1(-1)}(t),\psi_{10}(t),\psi_{11}(t),\psi_{2(-1)}(t),\psi_{20}(t),\psi_{21}(t)\right]^{\mathrm{T}}. \tag{10}$$

In Eq. (10) we have

$$\left.\begin{array}{l}\psi_{1(-1)}(t)=2^{1/2}(\cos(4\pi t)-\sin(4\pi t))\\ \psi_{10}(t)=2^{1/2}\\ \psi_{11}(t)=2^{1/2}(\cos(4\pi t)+\sin(4\pi t))\end{array}\right\}\quad 0\leqslant t<\frac{1}{2}, \tag{11}$$

$$\left.\begin{array}{l}\psi_{2(-1)}(t)=2^{1/2}(\cos(4\pi t)-\sin(4\pi t))\\ \psi_{20}(t)=2^{1/2}\\ \psi_{21}(t)=2^{1/2}(\cos(4\pi t)+\sin(4\pi t))\end{array}\right\}\quad \frac{1}{2}\leqslant t<1, \tag{12}$$

we also get

$$\Psi(t)\Psi^{\mathrm{T}}(t)=\begin{pmatrix}\psi_{1(-1)}\psi_{1(-1)} & \psi_{1(-1)}\psi_{10} & \psi_{1(-1)}\psi_{11} & \psi_{1(-1)}\psi_{2(-1)} & \psi_{1(-1)}\psi_{20} & \psi_{1(-1)}\psi_{21}\\ \psi_{10}\psi_{1(-1)} & \psi_{10}\psi_{10} & \psi_{10}\psi_{11} & \psi_{10}\psi_{2(-1)} & \psi_{10}\psi_{20} & \psi_{10}\psi_{21}\\ \psi_{11}\psi_{1(-1)} & \psi_{11}\psi_{10} & \psi_{11}\psi_{11} & \psi_{11}\psi_{2(-1)} & \psi_{11}\psi_{20} & \psi_{11}\psi_{21}\\ \psi_{2(-1)}\psi_{1(-1)} & \psi_{2(-1)}\psi_{10} & \psi_{2(-1)}\psi_{11} & \psi_{2(-1)}\psi_{2(-1)} & \psi_{2(-1)}\psi_{20} & \psi_{2(-1)}\psi_{21}\\ \psi_{20}\psi_{1(-1)} & \psi_{20}\psi_{10} & \psi_{20}\psi_{11} & \psi_{20}\psi_{2(-1)} & \psi_{20}\psi_{20} & \psi_{20}\psi_{21}\\ \psi_{21}\psi_{1(-1)} & \psi_{21}\psi_{10} & \psi_{21}\psi_{11} & \psi_{21}\psi_{2(-1)} & \psi_{21}\psi_{20} & \psi_{21}\psi_{21}\end{pmatrix}. \tag{13}$$

Expanding each product by CAS wavelet basis we get

$$\Psi(t)\Psi^{\mathrm{T}}(t)\simeq\begin{pmatrix}\sqrt{2}\psi_{10} & \sqrt{2}\psi_{1(-1)} & 0 & 0 & 0 & 0\\ \sqrt{2}\psi_{1(-1)} & \sqrt{2}\psi_{10} & \sqrt{2}\psi_{11} & 0 & 0 & 0\\ 0 & \sqrt{2}\psi_{11} & \sqrt{2}\psi_{10} & 0 & 0 & 0\\ 0 & 0 & 0 & \sqrt{2}\psi_{20} & \sqrt{2}\psi_{2(-1)} & 0\\ 0 & 0 & 0 & \sqrt{2}\psi_{2(-1)} & \sqrt{2}\psi_{20} & \sqrt{2}\psi_{21}\\ 0 & 0 & 0 & 0 & \sqrt{2}\psi_{21} & \sqrt{2}\psi_{20}\end{pmatrix}. \tag{14}$$

By using the vector C in Eq. (9) the 6×6 matrix $\widetilde{C}$ in Eq. (8) is

$$\widetilde{C}=\sqrt{2}\begin{pmatrix}\widetilde{C}_1 & 0\\ 0 & \widetilde{C}_2\end{pmatrix},$$

where $\widetilde{C}_i$, $i = 1, 2$ are 3×3 matrices given by

$$\widetilde{C}_i = \begin{pmatrix} c_{i0} & c_{i(-1)} & 0 \\ c_{i(-1)} & c_{i0} & c_{i1} \\ 0 & c_{i1} & c_{i0} \end{pmatrix}.$$

3. Solution of the Fredholm integral equations

Consider the Fredholm integral equations given in Eq. (1). In order to use CAS wavelets, we first approximate $y(x)$ as

$$y(x) = C^{\mathrm{T}}\Psi(x), \tag{15}$$

$$f(x) = d^{\mathrm{T}}\Psi(x), \tag{16}$$

and

$$K(x,t) = \Psi(x)^{\mathrm{T}}K\Psi(t), \tag{17}$$

where C, and $\Psi(x)$ are defined similarly to Eqs. (6) and (7). Also K is $2^k(2M+1) \times 2^k(2M+1)$ matrices where the elements of K calculated as follows:

$$\int_0^1 \int_0^1 \Psi_{ni}(x)\Psi_{lj}(t)K(x,t)\mathrm{d}t\mathrm{d}x,$$

where

$$n = 1, \ldots, 2^k, \quad i = -M, \ldots, M, \quad l = 1, \ldots, 2^k, \quad j = -M, \ldots, M.$$

Then we have

$$C^{\mathrm{T}}\psi(x) = d^{\mathrm{T}}\Psi(x) + \lambda \int_0^1 \Psi(x)^{\mathrm{T}}K\Psi(t)\psi(t)^{\mathrm{T}}C\mathrm{d}t. \tag{18}$$

Thus with the orthonormality of CAS wavelets we have

$$\psi(x)^{\mathrm{T}}C = \Psi(x)^{\mathrm{T}}d + \lambda\Psi(x)^{\mathrm{T}}KC. \tag{19}$$

Eq. (19) is a linear systems interms of C and the answer is

$$C = (I - K)^{-1}d.$$

where I is identity matrix.

4. Illustrative examples

We applied the method presented in this paper and solved three examples. This method differs from the Taylor series method used in [6], Adomian approach considered in [8] and collocation-type method approximation given in [5] and thus could be used as a basis for comparison.

4.1. Example 1

Consider the Fredholm integral equation

$$y(x) = \cos(4\pi x) + \int_0^1 ty(t)\mathrm{d}t. \tag{20}$$

We applied the method presented in this paper and solved Eq. (20) with $k = 1$ and $M = 1$. We have

$$d=\left[\frac{1}{2\sqrt{2}},0,\frac{1}{2\sqrt{2}},\frac{1}{2\sqrt{2}},0,\frac{1}{2\sqrt{2}}\right]^{\mathrm{T}}$$

and

$$K_{6\times 6}=\begin{bmatrix} 0 & \frac{1}{8\pi} & 0 & 0 & \frac{1}{8\pi} & 0\\ 0 & 0.125 & 0 & 0 & 0.125 & 0\\ 0 & \frac{-1}{8\pi} & 0 & 0 & \frac{-1}{8\pi} & 0\\ 0 & \frac{1}{8\pi} & 0 & 0 & \frac{1}{8\pi} & 0\\ 0 & 0.375 & 0 & 0 & 0.375 & 0\\ 0 & \frac{-1}{8\pi} & 0 & 0 & \frac{-1}{8\pi} & 0 \end{bmatrix}$$

and then

$$C=(I-K)^{-1}d=\left[\frac{1}{2\sqrt{2}},0,\frac{1}{2\sqrt{2}},\frac{1}{2\sqrt{2}},0,\frac{1}{2\sqrt{2}}\right]^{\mathrm{T}}.$$

Therefore

$$y(x)=C^{\mathrm{T}}\Psi(x)=\cos(4\pi x),$$

which is the exact solution.

4.2. Example 2

Consider the Fredholm integral equation by

$$y(x)=\frac{1}{4}-x+\int_0^1(3t-6x^2)y(t)\mathrm{d}t \tag{21}$$

which has the exact solution $y(x)=x^2-x$. We applied the CAS wavelets approach and solved Eq. (21). We have

$$d=\left[\frac{-1}{2M\pi},\ldots,\frac{-1}{4\pi},\frac{-1}{2\pi},\frac{-1}{4},\frac{1}{2\pi},\frac{1}{4\pi},\ldots,\frac{1}{2M\pi}\right]^{\mathrm{T}}$$

and the answer is

$$C=\left[\frac{1}{2M^2\pi^2},\ldots,\frac{1}{8\pi^2},\frac{1}{2\pi^2},\frac{-1}{6},\frac{1}{2\pi^2},\frac{1}{8\pi^2},\ldots,\frac{1}{2M^2\pi^2}\right]^{\mathrm{T}}.$$

Thus

$$y(x)=\lim_{M->\infty}C^T\Psi(x)=x^2-x$$

which is the exact solution.

4.3. Example 3

Consider the nonlinear Fredholm integral equation by

$$y(x)=\frac{1}{4}x+\cos(2\pi x)-\int_0^1(xt)y^2(t)\mathrm{d}t. \tag{22}$$

Let $y(x)=C^{\mathrm{T}}\psi(x)$, $\frac{1}{4}x+\cos(2\pi x)=d^{\mathrm{T}}\Psi(x)$, $x=h^{\mathrm{T}}\Psi(x)$ and $t=h^{\mathrm{T}}\Psi(t)$. Thus Eq. (22) becomes

$$C^{\mathrm{T}}\Psi(x)=d^{\mathrm{T}}\Psi(x)-\int_0^1\Psi^{\mathrm{T}}(x)hh^{\mathrm{T}}\Psi(t)C^{\mathrm{T}}\Psi(t)\Psi^{\mathrm{T}}(t)C\mathrm{d}t.$$

Using product operational matrix (8) we have

$$\Psi^{\mathrm{T}}(x)C = \Psi^{\mathrm{T}}(x)d - \Psi^{\mathrm{T}}(x)h \int_0^1 h^{\mathrm{T}} \Psi(t) \Psi^{\mathrm{T}}(t) \widetilde{C} C \mathrm{d}t$$

and by orthonormality we have

$$\Psi^{\mathrm{T}}(x)C = \Psi^{\mathrm{T}}(x)d - \Psi^{\mathrm{T}}(x)hh^{\mathrm{T}} \widetilde{C} C$$

and then

$$C - d + hh^{\mathrm{T}} \widetilde{C} C = 0.$$

The unknown vector C obtain with solution of the above algebraic system. For $M = 1$ and $k = 0$ we have

$$c_{1(-1)} = 0.5, \quad c_{10} = 0, \quad c_{11} = 0.5$$

and then $y(x) = \cos(2\pi x)$ which is exact solution.

5. Conclusion

The aim of present work is to develope an efficient and accurate method for solving the Fredholm integral equations. The present method reduces an integral equation into a set of algebraic equations. The integration of the product of two CAS wavelet function vectors is an identity matrix, hence making CAS wavelet computationally attractive. It is also shown that the CAS wavelets provide an exact solution. Illustrative examples are included to demonstrate the validity and applicability of the technique.

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