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Numerical solution of Abel's integral equation by using Legendre wavelets

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Abstract

A numerical method for solving Abel's integral equation as singular Volterra integral equations is presented. The method is based upon Legendre wavelet approximations. The properties of Legendre wavelet are first presented. These properties are then utilized to reduce the singular Volterra integral equations to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.
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1. Introduction

Wavelets theory is a relatively new and an emerging area in mathematical research. It has been applied in a wide range of engineering disciplines; particularly, wavelets are very successfully used in signal analysis for waveform representations and segmentations, time-frequency analysis and fast algorithms for easy implementation [1]. Wavelets permit the accurate representation of a

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variety of functions and operators. Moreover wavelets establish a connection with fast numerical algorithms [2].

In 1823, Abel, when generalizing the tautochrone problem derived the equation

$$\int_0^x \frac{\phi(t)}{\sqrt{x-t}}\mathrm{d}t = f(x),$$

where $f(x)$ is a given function and $\phi(t)$ is an unknown function. This equation is a particular case of a linear Volterra of the first kind.

Abel's problem is the following: Find a curve in the vertical XoY so that a material point, having started its motion at a point of the curve with ordinate x without initial velocity and moving along the curve under the action of gravity without friction, will reach the axis oX in time $t = f(x)/\sqrt{2g}$ (g is the acceleration in free falling).

Abel's equation is one of the integral equations derived directly from a concrete problem of mechanics or physics (without passing through a differential equation). Historically, Abel's problem is the first one to lead to the study of integral equations.

The generalized Abel's integral equations on a finite segment appeared for the first time in the paper of Zeilon [3].

Several numerical methods for approximating the solution of integral equations are known. For Fredholm–Hammerstein integral equations, the classical method of successive approximations was introduced in [4]. A variation of the Nystrom method was presented in [5]. A collocation type method was developed in [6]. In [7], Brunner applied a collocation-type method to nonlinear Volterra–Hammerstein integral equations and integro-differential equations, and discussed its connection with the iterated collocation method. Guoqiang [8] introduced and discussed the asymptotic error expansion of a collocation-type method for Volterra–Hammerstein integral equations. The methods in [6,8] transform a given integral equation into a system of nonlinear equations, which has to be solved with some kind of iterative method. A numerical solution of weakly singular Volterra integral equations was introduced in [9]. However, very few references have been found in technical literature dealing with integral equations.

In this paper we use Legendre wavelets for solving singular Volterra integral equations of the form,

$$\text{First kind:}\quad \int_0^x \frac{y(t)}{\sqrt{x-t}}\mathrm{d}t = f(x) \tag{1}$$

and

$$\text{Second kind:}\quad y(x) = f(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}}\mathrm{d}t, \tag{2}$$

where $f(x)$, is in $L^2(R)$ on the interval $0 \leqslant x,\ t \leqslant 1$.

Our method consists of reducing Eqs. (1) and (2) to a set of algebraic equations by expanding $y(x)$ with unknown coefficients. The properties of these wavelets are then utilized to evaluate the unknown coefficients. The paper is organized as follows: In Section 2 we describe the basic formulation of wavelets and Legendre wavelets required for our subsequent development. Section 3 is devoted to the solution of Eqs. (1) and (2). In Section 4, we report our numerical finding and demonstrate the accuracy of the proposed numerical scheme by considering numerical examples.

2. Properties of Legendre wavelets

2.1. Wavelets and Legendre wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously we have the following family of continuous wavelets as [10]:

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), \quad a,b \in \mathbb{R}, \;\; a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}, b = nb_0a_0^{-k}, a_0 > 1, b_0 > 0$ and n, and k positive integers, we have the following family of discrete wavelets:

$$\psi_{k,n}(t) = |a_0|^{k/2}\psi(a_0^k t - nb_0),$$

where $\psi_{k,\,n}(t)$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$ then $\psi_{k,n}(t)$ forms an orthonormal basis [10].

Legendre wavelets $\psi_{\mathrm{nm}}(t) = \psi(k,n,m,t)$ have four arguments; translation argument $n = 1,2,3,\ldots,2^{k-1}$, dilation argument k can assume any positive integer, m is the order for Legendre polynomials and t is the normalized time [11]. They are defined on the interval $[0,1)$ as

$$\psi_{nm}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}2^{\frac{k}{2}}P_m(2^k t - n), & \text{for } \frac{n-1}{2^k} \leqslant t < \frac{n}{2^k} \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

where $m = 0,1,\ldots,M-1$, $n = 1,2,3,\ldots,2^{k-1}$. In Eq. (2) the coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, the dilation parameter is $a = 2^{-k}$ and translation parameter is $b = n2^{-k}$. Here, $P_m(t)$ are the well-known shifted Legendre polynomials of order m which are orthogonal with respect to the weight function $w(t) = 1$ on the interval $[0,1]$.

2.2. Function approximation

A function $f(t)$ defined over $[0,1)$ may be expanded as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t), \tag{4}$$

where $c_{nm} = (f(t), \psi_{\mathrm{nm}}(t))$, in which $(\cdot,\cdot)$ denotes the inner product.

If the infinite series in Eq. (4) is truncated, then Eq. (4) can be written as

$$f(t) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^{\mathrm{T}}\Psi(t), \tag{5}$$

where C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$C = [c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}M-1}]^{\mathrm{T}},$$
$$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \ldots, \psi_{1M-1}(t), \psi_{20}(t), \ldots, \psi_{2M-1}(t), \ldots,$$
$$\psi_{2^{k-1}0}(t), \ldots, \psi_{2^{k-1}M-1}(t)]^{\mathrm{T}}.$$

3. Solution of singular Volterra integral equation

In this section we solve singular Volterra integral Eqs. (1) and (2) by using Legendre wavelets. Using Eq. (5) we approximate $y(x)$ and $f(x)$ as

$$y(x) = C^{\mathrm{T}}\Psi(x), \quad f(x) = F^{\mathrm{T}}\Psi(x), \tag{6}$$

where coefficients of F are known.

Then from Eqs. (1) , (2) and (6) we have

$$\text{First kind:}\ \ F^{\mathrm{T}}\psi(x) = \int_0^x \frac{C^{\mathrm{T}}\psi(t)}{\sqrt{x-t}}\,\mathrm{d}t, \tag{7}$$

$$\text{Second kind:}\ \ C^{\mathrm{T}}\psi(x) = F^{\mathrm{T}}\psi(x) + \int_0^x \frac{C^{\mathrm{T}}\psi(t)}{\sqrt{x-t}}\,\mathrm{d}t. \tag{8}$$

Since the basis of Legendre wavelets $\psi(t)$ are polynomial it sufficient to calculate $\int_0^x \frac{t^n}{\sqrt{x-t}}\mathrm{d}t$.

We have

$$\int_0^x \frac{t^n}{\sqrt{x-t}}\,\mathrm{d}t = \frac{\sqrt{\pi}x^{(\frac{1}{2}+n)}\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}. \tag{9}$$

Now by (9) let

$$\int_0^x \frac{\psi(t)}{\sqrt{x-t}}\mathrm{d}t = S\psi(x), \tag{10}$$

where S is $2^{k-1}M \times 2^{k-1}M$ matrices. Then from (7) , (8) and (10) we have

$$\text{First kind:}\ \ F^{\mathrm{T}}\psi(x) = C^{\mathrm{T}}S\psi(x),$$

$$\text{Second kind:}\ \ C^{\mathrm{T}}\psi(x) = F^{\mathrm{T}}\psi(x) + C^{\mathrm{T}}S\psi(x)$$

and then

$$\text{First kind:}\ \ F^{\mathrm{T}} = C^{\mathrm{T}}S, \tag{11}$$

$$\text{Second kind:}\ \ C^{\mathrm{T}} = F^{\mathrm{T}} + C^{\mathrm{T}}S, \tag{12}$$

Eqs. (11) and (12) are a linear systems in terms of C and the solution is

$$\text{First kind:}\ \ C^{\mathrm{T}} = F^{\mathrm{T}} \cdot S^{-1},$$

$$\text{Second kind:}\ \ C^{\mathrm{T}} = F^{\mathrm{T}} \cdot (I-S)^{-1},$$

thus $u(x) = C^{\mathrm{T}}\psi(x)$ is solution of (1) and (2).

4. Illustrative examples

We applied the method presented in this paper and solved three examples given in [12]. This method differs from the collocation method given in [6,7] and method of [9] and thus could be used as a basis for comparison.

4.1. Example 1

Consider singular Volterra integral equation:

$$y(x) = x^2 + \frac{16}{15}x^{\frac{5}{2}} - \int_0^x \frac{1}{\sqrt{x-t}}y(t)\mathrm{d}t, \tag{13}$$

which has the exact solution $y(x) = x^2$. We applied the Legendre wavelets approach and solved Eq. (13) with $M = 2$ and $k = 0$ and we obtain

$$y(x) = \frac{1}{3}\psi_{00}(x) + \frac{1}{2\sqrt{3}}\psi_{10}(x) + \frac{1}{6\sqrt{5}}\psi_{20}(x) = x^2,$$

which is the exact solution.

4.2. Example 2

Consider singular Volterra integral equation:

$$y(x) = 2\sqrt{x} - \int_0^x \frac{1}{\sqrt{x-t}} y(t)\mathrm{d}t, \tag{14}$$

which has the exact solution $y(x) = 1 - \mathrm{e}^{\pi x}\mathrm{erf}c(\sqrt{\pi x})$, which $\mathrm{erf}c(\sqrt{\pi x})$ is complementary error function and defined by

$$\mathrm{erf}c(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-u^2)\,\mathrm{d}u.$$

We applied the Legendre wavelets approach and solved Eq. (14).

Table 1 presents values of $y(x)$ using the present method with $k = 0, 1$ and $M = 5$ together with the exact values.

4.3. Example 3

Consider singular Volterra integral equation:

$$\frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3) = \int_0^x \frac{1}{\sqrt{x-t}} y(t)\mathrm{d}t, \tag{15}$$

which has the exact solution $y(x) = x^3 - x^2 + 1$. We applied the Legendre wavelets approach and solved Eq. (15) with $M = 3$ and $k = 0$ and we obtain

$$\begin{aligned} y(x) &= \frac{11}{12}\psi_{00}(x) - \frac{1}{20\sqrt{3}}\psi_{10}(x) + \frac{1}{12\sqrt{5}}\psi_{20}(x) + \frac{1}{20\sqrt{7}}\psi_{30}(x) \\ &= x^3 - x^2 + 1, \end{aligned}$$

which is the exact solution.

Table 1
Estimated and exact values of $y(x)$

x	Wavelets $k = 0$, $M = 5$	Wavelets $k = 1$, $M = 5$	Exact
0.0	0.0	0.0	0.0
0.1	0.414032	0.414059	0.414059
0.2	0.508328	0.508352	0.508352
0.3	0.564286	0.564309	0.564309
0.4	0.603321	0.603347	0.603347
0.5	0.632834	0.632868	0.632868
0.6	0.656310	0.656323	0.656323
0.7	0.675572	0.675601	0.675601
0.8	0.691813	0.691842	0.691842
0.9	0.705745	0.705787	0.705787
1.0	0.717912	0.717941	0.717941

5. Conclusion

The aim of present work is to develop an efficient and accurate method for solving singular Volterra integral equations. The problem has been reduced to solving a system of linear algebraic equations.

Wavelets as orthogonal systems have different resolution capability for expanding of function and therefore by increasing of dilation parameter k we get local approximation and this is good for integral equation that has not polynomial solution. In this method we get good approximation with low terms of basis. Illustrative examples are included to demonstrate the validity and applicability of the technique.

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