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Application of the CAS Wavelet in Solving Fredholm-Hammerstein Integral Equations of the Second Kind with Error Analysis

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Abstract: In this paper, we present a computational method for solving Fredholm-Hammerstein integral equations of the second kind. The method utilizes CAS wavelets constructed on the unit interval as basis in the Galerkin method and reduces the solution of the Hammerstein integral equation to the solution of a nonlinear system of algebraic equations. Error analysis is presented for this method. The properties of CAS wavelets are used to make the wavelet coefficient matrices sparse, which eventually leads to the sparsity of the coefficient matrix of the system obtained. Finally, numerical examples are presented to show the validity and efficiency of the technique.

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Key words: Fredholm-Hammerstein integral equation . CAS wavelet . error analysis . sparse matrix . nonlinear system of algebraic equation

INTRODUCTION

Many problems of mathematical physics, fluid mechanics and electrochemical machining are reduced to the solution of integral equations. These equations also occur as reformulations of other mathematical problems such as partial differential equations and ordinary differential equations. Integral equations are usually difficult to solve analytically and therefore, it is required to obtain approximate solutions [5, 7, 8, 10, 16, 17]. In recent years, several simple and accurate methods based on orthogonal basic functions, including wavelets, have been used to estimate the solution of integral equations. The main advantage of using an orthogonal basis is that it reduces the problem to the solution of a system of algebraic equations. Overall, there are so many different families of orthogonal functions which can be used in this method so that i is sometimes difficult to select the most suitable one. Since 1991, the wavelet technique has been applied to the solution of integral equations [4, 12, 15, 19, 20, 27]. Wavelets, as very well localized functions, are considerably useful for solving integral equations and provide accurate solutions. Also, the wavelet technique allows the creation of very fast algorithms when compared with the algorithms ordinarily used.

In various fields of science and engineering, we meet with a large class of integral equations which are called Fredholm-Hammerstein integral equations of the second kind, namely

$$u(x) - \lambda\int_0^1 K(x,y)\phi(y,u(y))dy = f(x), \ \ 0 \le x \le 1 \qquad (1)$$

where $f \in L^2[0,1]$ and $K \in L^2([0,1]\times[0,1])$ are known functions, $\lambda \in \mathbb{R}$ is a constant, ϕ is given continuous function which is nonlinear in u and u is the unknown function to be determined.

Several methods have been proposed for numerical solution of these types of integral equation. In [22], the Petrov-Galerkin method and the iterated Petrov-Galerkin method have been applied to solve a class of nonlinear integral equations. Lardy [25] described a variation of the Nystrom method for Hammerstein integral equations of the second kind. B-spline wavelets are applied to solve Fredholm-Hammerstein integral equations of the second kind in [26]. Lepik [15] studied a computational method for solving Fredholm-Hammerstein integral equations of the second kind by using Haar wavelets. In [8], a method for numerical solution of nonlinear Fredholom integral equations utilizing positive definite functions is proposed by Alipanah and Dehghan. Legendre wavelets are used to

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approximate the solution of linear and nonlinear integral equations with weakly singular kernels in [20]. Authors of [28] introduced a numerical method for solving nonlinear Fredholm integral equations of the second kind using Sinc-collocation method.

The main purpose of this article is to present a numerical method for solving Eq. (1) by using CAS wavelets. The properties of CAS wavelets are used to convert Eq. (1) into a nonlinear system of algebraic equations. This system may be solved by using an appropriate numerical method, such as Newtons iteration method. We will notice that, these wavelets make the wavelet coefficient matrices sparse and accordingly leads to the sparsity of the coefficient matrix of the final system and provide accurate solutions. Also, we discuss on the convergence of the CAS wavelets and the error analysis for the presented method.

Furthermore, the CAS wavelets have been used to approximate the solution of linear integral equations [13], Volterra integral equations of the second kind [18], integro-differential equations [32], nonlinear Volterra integro-differential equations of arbitrary order [29], nonlinear Fredholm integro-differential equations of fractional order [30] and optimal control systems by time-dependent coefficients [31].

The outline of the paper is as follows: In Section 2, we review some properties of CAS wavelets and approximate the function $f(x)$ and also the kernel function K(x,y) by these wavelets. Section 3 is devoted to present a computational method for solving Eq. (1) utilizing CAS wavelets and approximate the unknown function u(x). We provide the error analysis for the method and the convergence theorem of the CAS wavelet bases in Section 4. In Section 5, the sparsity of the wavelet coefficient matrix is studied. Numerical examples are given in Section 6. Finally, we conclude the article in Section 7.

PROPERTIES OF CAS WAVELETS

CAS wavelets: Wavelets consist of a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets [3]

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}}\,\psi\left(\frac{t-b}{a}\right), a,b \in \mathbb{R}, a \neq 0 \tag{2}$$

If we restrict the parameters a and b to discrete values

$$a = a_0^{-k}, b = n b_0 a_0^{-k}, a_0 > 1, b_0 > 0$$

where n and k are positive integers, then we have the following family of discrete wavelets

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}}\,\psi(a_0^k t - n b_0) \tag{3}$$

where $\psi_{k,n}(t)$ form a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2$, $b_0 = 1$ then $\psi_{k,n}(t)$ forms an orthonormal basis [3, 13]. The CAS wavelets,

$$\psi_{nm}(t) = \psi(k,n,m,t)$$

have four arguments; $n = 1,2,\ldots,2^k$, k is any non-negative integer, m is any integer and t is the normalized time. The orthonormal CAS wavelets are defined on the interval [0, 1) by [13, 18]

$$\psi_{nm}(t) = \begin{cases} 2^{k/2}\,CAS_m(2^k t - n + 1), & \frac{n-1}{2^k} \le t < \frac{n}{2^k} \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

where

$$CAS_m(t) = \cos(2m\pi t) + \sin(2m\pi t) \tag{5}$$

Remark 1: Note that for m =0, the CAS wavelets have the following form:

$$\psi_{n0}(t) = 2^{k/2} B_n(t) = 2^{k/2} \begin{cases} 1, & \frac{n-1}{2^k} \le t < \frac{n}{2^k} \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

where $\{B_n(t)\}_{n=1}^{2^k}$ are a basis set that are called the block pulse functions (BPFs) over the interval [0, 1) [24].

Function approximation: A function $f(x) \in L^2[0,1]$ may be expanded as

$$f(x) = \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} c_{nm} \psi_{nm}(x) \tag{7}$$

where

$$c_{nm} = \langle f(x), \psi_{nm}(x) \rangle \tag{8}$$

in which <.,.> denotes the inner product. The series (7) is truncated as

$$f(x) \simeq P_{k,M}(f(x)) = \sum_{n=1}^{2^k} \sum_{m=-M}^{M} c_{nm} \psi_{nm}(x) = C^t \Psi(x) \tag{9}$$

where C and $\Psi(x)$ are two vectors given by

$$C = [c_{1(-M)}, c_{1(-M+1)}, \ldots, c_{1M}, c_{2(-M)}, \ldots, c_{2M}, \ldots, c_{(2^k)(-M)}, \ldots, c_{(2^k)M}]^t = [c_1, c_2, \ldots, c_{2^k(2M+1)}]^t \quad (10)$$

and

$$\Psi(x) = [\psi_{1(-M)}(x), \psi_{1(-M+1)}(x), \ldots, \psi_{1M}(x), \psi_{2(-M)}(x), \ldots, \psi_{2M}(x), \ldots, \psi_{(2^k)(-M)}(x), \ldots, \psi_{(2^k)M}(x)]^t = [\psi_1(x), \psi_2(x), \ldots, \psi_{2^k(2M+1)}(x)]^t \quad (11)$$

Similarly, by considering

$$i = n(2M+1) - M + m, \; j = n'(2M+1) - M + m'$$

we approximate $K(x,y) \in L^2([0,1]\times[0,1])$ as

$$K(x,y) \simeq \sum_{i=1}^{2^k(2M+1)} \sum_{j=1}^{2^k(2M+1)} K_{ij}\psi_i(x)\psi_j(y) \quad (12)$$

or in the matrix form

$$K(x,y) \simeq \Psi^t(x)\mathbf{K}\Psi(y) \quad (13)$$

where

$$\mathbf{K} = [K_{ij}]_{1\le i,j\le 2^k(2M+1)}$$

ith the entries

$$K_{ij} = \langle \psi_i(x), \langle K(x,y), \psi_j(y) \rangle\rangle \quad (14)$$

Let $L_N(x)$ be the shifted Legendre polynomial of order N on [0, 1]. Then the Legendre-Gauss Lobatto nodes are

$$x_0 = 0 < x_1 < \ldots < x_{N-1} < x_N = 1$$

and x_p, p = 1..N-1 are the zeros of $L'_N(x)$, where $L'_N(x)$ is the derivative of $L_N(x)$ with respect to $x\in[0,1]$. No explicit formulas are known for the points x_p and so they are computed numerically using subroutines [33].

Also we approximate the integral of f on [0, 1] as:

$$\int_0^1 f(s)w(s)ds \simeq \sum_{p=1}^{N} w_p f(x_p) \quad (15)$$

where x_p are the Legendre-Gauss-Lobatto nodes and the weights w_p given in

$$w_p = \frac{2}{N(N+1)[L_N(x_p)]^2}, \quad p = 0,\ldots,N \quad (16)$$

Analytical calculating c_is and K_{ij}s to approximate the one or two dimensional functions via CAS wavelets take long times. But using the quadrature rule (15) can decrease this time and make commodious methods to calculate inner products. Therefore we have

$$c_{n,m} = \int_0^1 f(x)\psi_{n,m}(x)dx \simeq \sum_{p=1}^{N} w_p f(x_p)\psi_{n,m}(x_p) \quad (17)$$

and

$$K_{ij} = \int_0^1\int_0^1 \psi_{n,m}(x)\psi_{n',m'}(y)K(x,y)dxdy \simeq \sum_{q=1}^{N}\sum_{p=1}^{N} w_p w_q \psi_{n,m}(x_p)\psi_{n',m'}(x_q)K(x_p,x_q) \quad (18)$$

SOLUTION OF HAMMERSTEIN INTEGRAL EQUATIONS

In this section, the CAS wavelet method is used for solving Fredholm-Hammerstein integral equations of the second kind in the form (1). For this aim, we let

$$z(x) = \phi(x,u(x)) \quad (19)$$

and approximate functions $f(x), u(y), z(x)$ and $K(x, y)$ in the matrix forms

$$f(x) \simeq F^t\Psi(x) \quad (20)$$

$$u(x) \simeq U^t\Psi(x) \quad (21)$$

$$z(x) \simeq Z^t\Psi(x) \quad (22)$$

$$K(x,y) \simeq \Psi^t(x)\mathbf{K}\Psi(y) \quad (23)$$

By substituting (20), (21), (22) and (23) into Eq. (1), we obtain

$$\Psi^t(x)U \simeq \lambda\int_0^1 \Psi^t(x)\mathbf{K}\Psi(y)\Psi^t(y)Zdy + \Psi^t(x)F \quad (24)$$

$$= \lambda\Psi^t(x)\mathbf{K}\left(\int_0^1 \Psi(y)\Psi^t(y)dy\right)Z + \Psi^t(x)F \quad (25)$$

Now, we define the residual $R_{k,M}(x)$ as

$$R_{k,M}(x) = \Psi^t(x)U - \lambda\Psi^t(x)\mathbf{K}\left(\int_0^1 \Psi(y)\Psi^t(y)dy\right)Z - \Psi^t(x)F \quad (26)$$

By using the orthonormality of the CAS wavelets on [0,1] implies that

$$\int_0^1 \Psi(y)\Psi^t(y)dy = \mathbf{I} \quad (27)$$

where $\mathbf{I}_{2^k(2M+1)\times 2^k(2M+1)}$ is the identity matrix. So, we have

$$R_{k,M}(x) = \Psi^t(x)U - \lambda\Psi^t(x)\mathbf{K}Z - \Psi^t(x)F \quad (28)$$

Our aim is to compute $u_1, u_2, \ldots, u_{2^k(2M+1)}$ such that $R_{k,M}(x) \equiv 0$, but in general, it is not possible to choose such u_i, $i = 1,2,\ldots,2^k(2M+1)$. In this work, utilizing the Galarkin method, $R_{k,M}(x)$ is made as small as possible such that

$$< \psi_{m,n}(x), R_{k,M}(x) >= 0 \quad (29)$$

where n = 1,2,…,2k and m = -M, -M+1…,M.

Now by taking inner product $<\Psi(x),.>$ upon both sides of Eq. (28) and using Eq. (27) we obtain

$$U = \lambda\mathbf{K}Z + F \quad (30)$$

Thus, we have

$$u(x) \simeq \Psi^t(x)(\lambda\,\mathbf{K}Z + F) \quad (31)$$

By substituting (22) and (31) in Eq. (19), we get

$$\Psi^t(x)Z = \phi(x, \Psi^t(x)(\lambda\,\mathbf{K}Z + F)) \quad (32)$$

By evaluating Eq. (32) at collocation points $\{x'_j\}_{j=1}^{2^k(2M+1)}$, where

$$x'_j = \frac{(j - \frac{1}{2})}{2^k(2M+1)}$$

we obtain the following nonlinear system of algebraic equations

$$\begin{gathered} \Psi^t(x'_j)\,Z = \phi(x'_j, \Psi^t(x'_j)(\lambda\mathbf{K}Z + F)) \\ j = 1,2,\ldots,2^k(2M+1) \end{gathered} \quad (33)$$

for unknowns

$$Z = [z_1, z_2, \ldots, z_{2^k(2M+1)}]^t$$

Then, by computing Z, we can obtain the vector U from Eq. (30) and acquire the approximate solution of Eq. (1), by using (21).

ERROR ANALYSIS

This section covers the convergence analysis of the proposed method. At first, we indicate that the CAS wavelet expansion of a function *f*(x), with bounded second derivative, converges uniformly to *f*(x). But before that, for ease reference, we present the following lemma:

Lemma 4.1: If the CAS wavelet expansion of a continuous function *f*(x) converges uniformly, then the CAS wavelet expansion converges to the function *f*(x).

Proof: Let

$$g(x) = \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} c_{nm}\psi_{nm}(x) \quad (34)$$

where

$$c_{nm} = < f(x), \psi_{nm}(x) >$$

Multiplying both sides of (34) by $\psi_{pq}(x)$, in which p and q are fixed and then integrating termwise, justified by uniformly convergence, on [0,1], we have

$$\int_0^1 g(x)\psi_{pq}(x)dx = \int_0^1 \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} c_{nm}\psi_{nm}(x)\psi_{pq}(x)dx \quad (35)$$

$$= \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} c_{nm}\int_0^1 \psi_{nm}(x)\psi_{pq}(x)dx = c_{pq} \quad (36)$$

Thus $< g(x), \psi_{nm}(x) >= c_{nm}$ for $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Consequently *f*, g have same Fourier expansions with the CAS wavelet basis and therefore $f(x) = g(x)$; $(0 \le x \le 1)$ [2].

Theorem 4.1: A function $f(x) \in L^2[0,1]$, with bounded second derivative, say $|f''(x)| \le \gamma$, can be expanded as an infinite sum of the CAS wavelets and the series converges uniformly to *f*(x), that is

$$f(x) = \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} c_{nm}\psi_{nm}(x) \quad (37)$$

Furtheremore, we have

$$\|P_{k,M}f - f\|_\infty \le \frac{\gamma}{\pi^2}\sum_{n=2^k+1}^{\infty}\sum_{m=M+1}^{\infty}\frac{1}{n^{\frac{5}{2}}m^2}, \quad x \in [0,1] \quad (38)$$

Proof. From Eq. (8), it follows that

$$\begin{aligned} c_{nm} &= \int_0^1 f(x)\psi_{nm}(x)dx \\ &= \int_{\frac{n-1}{2^k}}^{\frac{n}{2^k}} 2^{\frac{k}{2}} f(x)\,CAS_m(2^k x - n + 1)dx \end{aligned} \quad (39)$$

By substituting 2^k x-n+1 = t in (39), yields

$$c_{nm} = \frac{1}{2^{\frac{k}{2}}}\int_0^1 f\left(\frac{t+n-1}{2^k}\right)CAS_m(t)dt \quad (40)$$

$$=\frac{1}{2^{\frac{k}{2}}}\int_0^1 f\left(\frac{t+n-1}{2^k}\right)d\left(\frac{\sin(2m\pi t)-\cos(2m\pi t)}{2m\pi}\right) \tag{41}$$

$$=\frac{1}{2^{\frac{k}{2}}}f\left(\frac{t+n-1}{2^k}\right)\frac{\sin(2m\pi t)-\cos(2m\pi t)}{2m\pi}\Bigg]_0^1 \tag{42}$$

$$-\frac{1}{2^{\frac{3k}{2}}(2m\pi)}\int_0^1 f'\left(\frac{t+n-1}{2^k}\right)\big(\sin(2m\pi t)-\cos(2m\pi t)\big)dt \tag{43}$$

$$=\frac{1}{2^{\frac{3k}{2}}(2m\pi)}\int_0^1 f'\left(\frac{t+n-1}{2^k}\right)d\left(\frac{CAS_m(t)}{2m\pi}\right) \tag{44}$$

$$=\frac{1}{2^{\frac{3k}{2}}(2m\pi)^2}f'\left(\frac{t+n-1}{2^k}\right)CAS_m(t)\Bigg]_0^1 \tag{45}$$

$$-\frac{1}{2^{\frac{5k}{2}}(2m\pi)^2}\int_0^1 f''\left(\frac{t+n-1}{2^k}\right)CAS_m(t)dt \tag{46}$$

Thus, we get

$$|c_{nm}|^2=\left|\frac{1}{2^{\frac{5k}{2}}(2m\pi)^2}\int_0^1 f''\left(\frac{t+n-1}{2^k}\right)CAS_m(t)dt\right|^2 \tag{47}$$

$$\le\left(\frac{1}{2^{\frac{5k}{2}}(2m\pi)^2}\right)^2\int_0^1\left|f''\left(\frac{t+n-1}{2^k}\right)\right|^2dt\int_0^1|CAS_m(t)|^2\,dt \tag{48}$$

$$\le\left(\frac{N}{2^{\frac{5k}{2}}(2m\pi)^2}\right)^2\int_0^1|CAS_m(t)|^2\,dt \tag{49}$$

from orthonormality of CAS wavelets, we know that $\int_0^1|CAS_m(t)|^2\,dt=1.$ Since $n\le 2^k$, we have

$$|c_{nm}|\le\frac{\gamma}{4\pi^2 n^{\frac{5}{2}}m^2} \tag{50}$$

Hence, the series $\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}}c_{nm}$ is absolutely convergent. On the other hand, we have

$$\left|\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}}c_{nm}\psi_{nm}(x)\right|\le\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}}|c_{nm}\|\psi_{nm}(x)| \le 2\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}}|c_{nm}|<\infty \tag{51}$$

Accordingly, utilizing Lemma 4.1, the series $\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}}c_{nm}\psi_{nm}(x)$ converges to $f(x)$ uniformly.

Moreover, we conclude that

$$P_{k,M}f(x)-f(x)_\infty=2\sum_{n=2^k+1}^{\infty}\sum_{m=M+1}^{\infty}c_{nm}\psi_{nm}(x)_\infty\le 4\sum_{n=2^k+1}^{\infty}\sum_{m=M+1}^{\infty}|c_{nm}|$$

Now, from (50), we obtain

$$P_{k,M}f(x)-f(x)_\infty\le\frac{\gamma}{\pi^2}\sum_{n=2^k+1}^{\infty}\sum_{m=M+1}^{\infty}\frac{1}{n^{\frac{5}{2}}m^2} \tag{52}$$

This completes the proof.

Now, we proceed by discussing the convergence of the presented method. For this purpose, let the operators K and $\hat{\mathbf{T}}$ be respectively

$$\mathbf{K}(u(x))=\lambda\int_0^1 K(x,y)\phi(y,u(y))dy \tag{53}$$

$$\hat{\mathbf{T}}(u(x))=\mathbf{K}(u(x))+f(x) \tag{54}$$

for all $u\in L^2[0,1]$ and $x\in[0,1]$, then Eq. (1) can be written as the operator equation

$$u(x)=\hat{\mathbf{T}}(u(x)) \tag{55}$$

The Galerkin-CAS wavelet scheme of Eq. (1) is [21, 22]

$$u_{k,M}=P_{k,M}\mathbf{K}u_{k,M}+P_{k,M}f \tag{56}$$

where $u_{k,M}$ is the approximate solution which obtained by the presented method. Now, we defined the operator $T_{k,M}$ as follows

$$\mathbf{T}_{k,M}(u_{k,M}(x))=P_{k,M}\mathbf{K}(u_{k,M}(x))+P_{k,M}(f(x)) \tag{57}$$

Therefore, Eq. (56) can be written as

$$u_{k,M}=\mathbf{T}_{k,M}u_{k,M} \tag{58}$$

In order to study the convergence of $u_{k,M}$ to u_0, where u_0 is the exact solution of Eq. (1), we first make some suitable assumptions on K, g and f [21].

(1) $f\in C[0,1]$

(2) $\phi(x, y)$ is continuous on $[0,1]\times\mathbb{R}$ and there is $C_1>0$ s.t $|\phi(x,u_1)-\phi(x,u_2)|\le C_1|u_1-u_2|$ for all $u_1, u_2\in\mathbb{R}\$$.

(3) There is a constant C_2 such that the partial derivative $\phi^{(0,1)}$ of ϕ with respect to the second variable satisfies

$$|\phi^{(0,1)}(x,u_1)-\phi^{(0,1)}(x,u_2)|\leq C_1|u_1-u_2| \text{ for all } u_1,u_2\in\mathbb{R}.$$

(4) For $u\in C[0,1], \phi(.,u(.)),\phi^{(0,1)}(.,u(.))\in C[0,1]$.

Under the above assumptions, we can present the following theorem:

Theorem 4.2: Let $u_0\in[0,1]$ be solution of the equation (1). Assume that 1 is not an eigenvalue of $K'(u_0)$, where $K'(u_0)$ denote the Frechet derivative of K at u_0. Then the Galerkin-wavelet approximation equation (56) has, for each sufficiently large $u_{k,M}$, a unique solution $u_{k,M}$ in some ball of radius δ centered at u_0, $B(u_0,\delta)$. Further, there exists 0<q<1, independent of n, such that if

$$\alpha_{k,M}=\|(\mathbf{I}-\mathbf{T}'_{k,M}(u_0))^{-1}(\mathbf{T}_{k,M}(u_0)-\hat{\mathbf{T}}_{k,M}(u_0)\|_\infty \tag{59}$$

then

$$\frac{\alpha_{k,M}}{1+q}\leq\|u_{k,M}-u_0\|_\infty\leq\frac{\alpha_{k,M}}{1-q} \tag{60}$$

and

$$\|u_{k,M}-u_0\|_\infty\leq C\|P_{k,M}u_0-u_0\|_\infty \tag{61}$$

where C is a constant independed of k and M.

Proof: [21].

Remark 2: As a conclusion from Theorems 4.2 and 4.1, if the function $u_0(x)\in L^2[0,1]$, with bounded second derivative, say $|u_{0''}(x)|\leq\gamma_0$, then the proposed method is convergence and also we have

$$e_{k,M\infty}=\|u_{k,M}-u_0\|_\infty\leq C\frac{\gamma_0}{\pi^2}\sum_{n=2^k+1}^{\infty}\sum_{m=M+1}^{\infty}\frac{1}{n^{\frac{5}{2}}m^2} \tag{62}$$

SPARSE REPRESENTATION OF THE MATRIX K

We proceed by discussing the sparsity of the matrix K, as an important issue for increasing the computation speed.

Theorem 5.1: Suppose that K_{ij} is the CAS wavelet coefficient of the continuous kernel K(x,y). If

$$\left|\frac{\partial^4K(x,y)}{\partial x^2\partial y^2}\right|\leq\delta$$

where N is a positive constant, then we have

$$|K_{ij}|\leq\frac{\delta}{16\pi^4(nn')^{\frac{5}{2}}(mm')^2} \tag{63}$$

for $n,n'\in\mathbb{N}$ and $m,m'\in\mathbb{Z}$.

Proof: Note that

$$|K_{ij}|=\left|\int_0^1\int_0^1\psi_{n,m}(x)\psi_{n',m'}(y)K(x,y)dxdy\right| = 2^k\left|\int_{\frac{n-1}{2^k}}^{\frac{n}{2^k}}\int_{\frac{n'-1}{2^k}}^{\frac{n'}{2^k}}K(x,y)CAS_m(2^kx-n+1)CAS_{m'}(2^ky-n'+1)dydx\right|$$

Now, let $2^kx-n+1=t$ and $2^ky-n'+1=s$, then

$$|K_{ij}|=\frac{1}{2^k}\left|\int_0^1\int_0^1K(\frac{t+n-1}{2^k},\frac{s+n'-1}{2^k})CAS_m(t)CAS_{m'}(s)dsdt\right|$$

Similar to the proof of theorem 4.1, we obtain

$$|K_{ij}|\leq\frac{1}{2^{5k}(2m\pi)^2(2m'\pi)^2}\left|\int_0^1\int_0^1\frac{\partial^4K(\frac{t+n-1}{2^k},\frac{s+n'-1}{2^k})}{\partial t^2\partial s^2}CAS_m(t)CAS_{m'}(s)dsdt\right|$$

$$\leq\frac{\delta}{16\pi^4 2^{5k}(mm')^2}\left(\int_0^1\left|CAS_m(t)\right|^2dt\right)^{\frac{1}{2}}\left(\int_0^1\left|CAS_{m'}(s)\right|^2ds\right)^{\frac{1}{2}}\leq\frac{\delta}{16\pi^4(nn')^{\frac{5}{2}}(mm')^2}$$

As a conclusion from Theorem 5.1, when I or j→∞ then $|K_{ij}|\to 0$ and accordingly by increasing k or M, we can make K sparse. For this purpose, we choose a threshold ε_0 and define

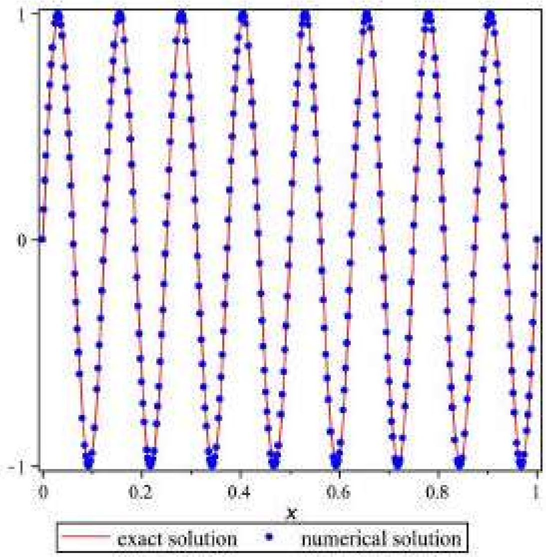

Fig. 1: Approximate solution for example 6.1 with $k = 4$, $M = 2$, $\varepsilon_0 = 10^{-14}$

$$\overline{\mathbf{K}} = [\overline{K}_{ij}]_{2^{k-1}M \times 2^{k-1}M} \tag{64}$$

where

$$\text{bar}K_{ij} = \begin{cases} K_{ij}, & |K_{ij}| \geq \varepsilon_0 \\ 0, & \text{otherwise} \end{cases} \tag{65}$$

Obviously, $\overline{\mathbf{K}}$ is a sparse matrix. Now, we rewrite Eq. (31) as follows

$$U = \lambda\overline{\mathbf{K}}Z + F \tag{66}$$

We can use Eq. (66) instead of Eq. (31).

NUMERICAL EXAMPLES

In order to test the validity of the present method, three examples are solved and the numerical results are compared with their exact solution [14, 15, 26]. It is seen that good agreements are achieved, as dilation parameter $a = 2^k$ decreases. In addition, in Examples 6.2 and 6.3, our results are compared with numerical results in [15]. All routines are written in Maple and run on a Pentium 4 PC Laptop with 2.10 GHz of CPU and 4 GB of RAM.

Example 6.1: As the first example let,

$$u(x) - 2\int_0^1 \frac{1}{1+u^2(y)}dy = \sin(16\pi x) - \sqrt{2}, \ 0 \leq x \leq 1 \tag{67}$$

with the exact solution $u_{ex}(x) = \sin(16\pi x)$.

Table 1: Some numerical results for example 6.1

x	Exact solution	Approximate solution $k = 2$, $M = 2$, $\varepsilon_0 = 10^{-5}$	Approximate solution $k = 3$, $M = 2$, $\varepsilon_0 = 10^{4}$
0.0	0.0000000000	-0.0000026141	-0.0000000673
0.1	-0.9510565165	-0.9510799315	-0.9510564852
0.2	-0.5877852524	-0.5877994622	-0.5877853479
0.3	0.5877852524	0.5877989018	0.5877854270
0.4	0.9510565165	0.9510787301	0.9510566007
0.5	0.0000000000	0.0000012856	-0.0000001770
0.6	-0.9510565165	-0.9510804363	-0.9510567362
0.7	-0.5877852524	-0.5878000485	-0.5877853274
0.8	0.5877852524	0.5877993699	0.5877852326
0.9	0.9510565165	0.9510799408	0.9510563837
1.0	0.0000000000	0.0000035962	-0.0000001075

Table 1 shows the numerical results for this example with $k = 4$, $M = 2$, $\varepsilon_0 = 10^{-5}$ and $k = 3$, $M = 2$, $\varepsilon_0 = 10^{-4}$. Also, the approximate solution for $k = 3$, $M = 2$, $\varepsilon_0 = 10^{-14}$ is graphically shown in Fig. 1 which agrees with the exact solution. Here, we employ the double 10-point Legendre-Gauss-Lobatto quadrature rule for numerical integration and note that increasing the number of integration points does not improve results.

Example 6.2: In this example, we solve integral equation

$$u(x) = e^x - 0.382\sin(x) - 0.301\cos(x) + \int_0^1 \sin(x+y)\ln(u(y))dy, \ 0 \leq x \leq 1 \tag{68}$$

by the present method, in which the exact solution is $u_{ex}(x) = e^x$ [26].

Table 2 shows the numerical results for this example with $k = 3$, $M = 1$, $\varepsilon_0 = 10^{-5}$ and $k = 4$, $M = 2$, $\varepsilon_0 = 10^{-4}$ and results are compared with [15]. Also, the approximate solution for $K = 2,3,4,5$ and $M = 1$ are graphically shown in Fig. 2. It is seen the numerical results are improved, as parameter k increases. All computations performed using the 10-point Legendre-Gauss-Lobatto quadrature rule to approximate the integration numerically.

Example 6.3: As our final example, we consider the weakly singular integral equation

$$u(x) - \int_0^1 |x-y|^{-1/2}[u(y)]^2 dy = f(x), \ 0 \leq x \leq 1 \tag{69}$$

where

$$f(x) = [x(1-x)]^{\frac{1}{2}} + \frac{16}{15}x^{\frac{5}{2}} + 2x^2(1-x)^{\frac{1}{2}} + \frac{4}{3}x(1-x)^{\frac{3}{2}} + \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{4}{3}x^{\frac{3}{2}} - 2x(1-x)^{\frac{1}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}}$$

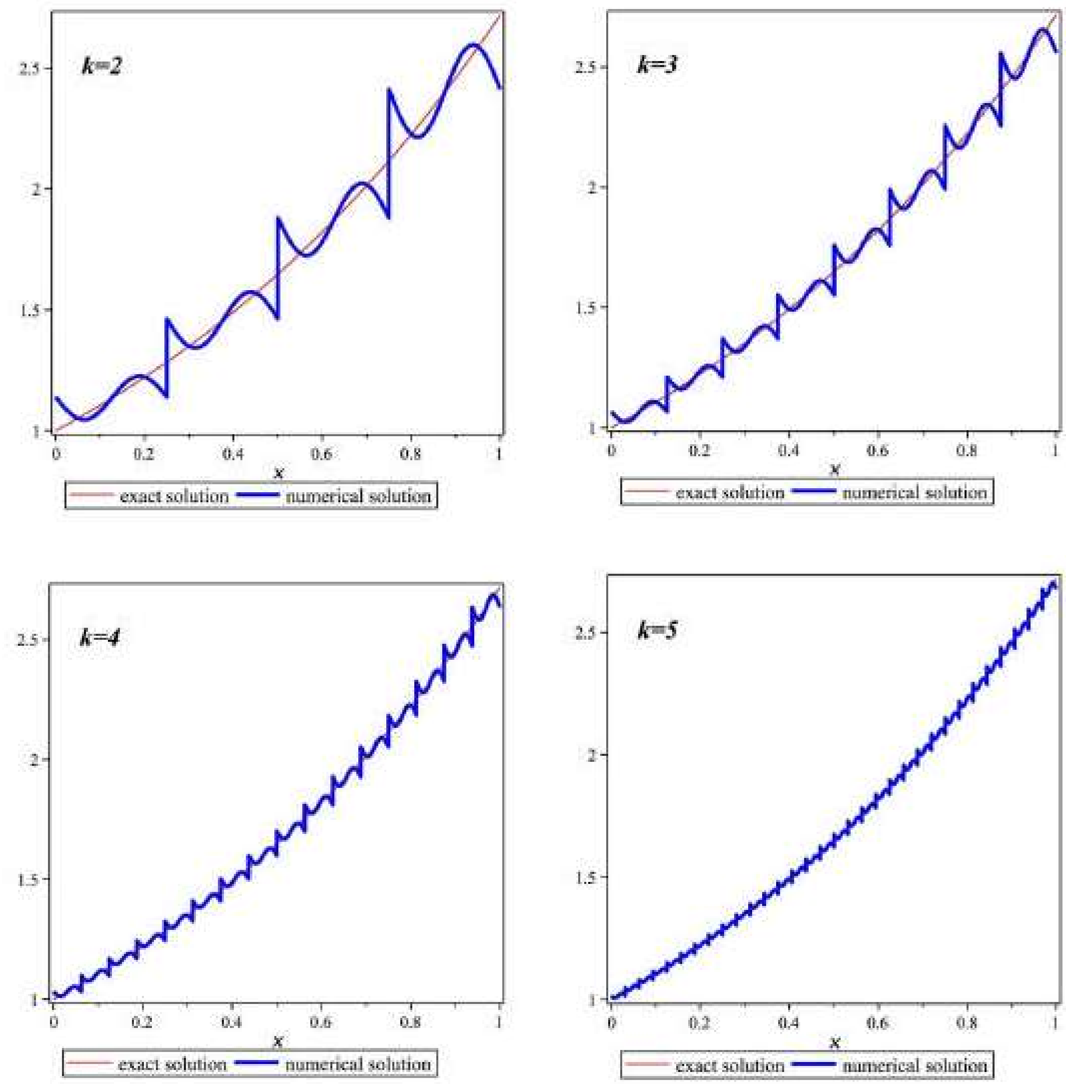

Fig. 2: Approximate solution for Example 6.2 with k = 2,3,4,5, M = 1, $\varepsilon_0 = 10^{-5}$

Table 2: Some numerical results for example 6.2

x	Exact solution	Approximate solution k = 2, M = 2, $\varepsilon_0 = 10^{-5}$	Approximate solution k = 3, M = 2, $\varepsilon_0 = 10^{-4}$	Method in [15] J = 3, M = 8
0.0	1.000000000	1.062849069	1.032166582	1.031911343
0.1	1.105170918	1.052133021	1.100757018	1.098463906
0.2	1.221402758	1.201817441	1.221250143	1.244722676
0.3	1.349858808	1.353779530	1.350148815	1.325000391
0.4	1.491824698	1.520076383	1.497853032	1.501422144
0.5	1.648721271	1.754226644	1.701754993	1.701334180
0.6	1.822118800	1.793120371	1.814841505	1.811060807
0.7	2.013752707	2.018341532	2.013501068	2.052200752
0.8	2.225540928	2.202099016	2.226019073	2.184556329
0.9	2.459603111	2.458492182	2.469542145	2.475426625
1.0	2.718281828	2.554324683	2.635599332	2.635505924

Table 3: Some numerical results for example 6.3

x	Exact solution	Approximate solution $k = 2, M = 2, \varepsilon_0 = 10^{-5}$	Approximate solution $k = 3, M = 2, \varepsilon_0 = 10^{-4}$	Method in [15] J = 3, M = 8
0.0	0.0000000000	0.2059612202	0.1493763423	0.1635059136
0.1	0.3000000000	0.2939927010	0.3078717556	0.2898058405
0.2	0.4000000000	0.4091646056	0.3997102373	0.4128210483
0.3	0.4582575695	0.4539878096	0.4582085957	0.4491607979
0.4	0.4898979486	0.4894564053	0.4888927749	0.4907885809
0.5	0.5000000000	0.4931152718	0.4982964192	0.4986948478
0.6	0.4898979486	0.4894564055	0.4888927764	0.4907885269
0.7	0.4582575695	0.4539878096	0.4582085294	0.4491607874
0.8	0.4000000000	0.4091643224	0.3997102590	0.4128210127
0.9	0.3000000000	0.2939954698	0.3078716438	0.2898058206
1.0	0.0000000000	0.2059639157	0.1493763541	0.1635059241

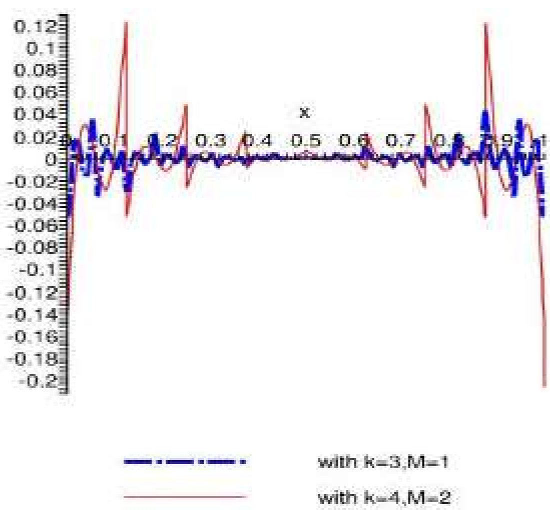

Fig. 3: Error distributions for example 6.3

with the exact solution $u_{ex}(x) = \sqrt{x(1-x)}$ [14, 15].

Table 3 shows the numerical results for this example with k = 3, M = 1, $\varepsilon_0 = 10^{-5}$ and k = 4, M = 2, $\varepsilon_0 = 10^{-4}$ and results are compared with [15]. Also, the error $e(x) = u_{ex}(x) - u_{k,M}(x)$ for k = 3, M = 1 and k = 4, M = 2 is graphically shown in Fig. 3. It is remarkable that the wavelet technique used does not behave as desired at the border points, because $u_{ex}(x)$ is not differentiable at x = 0,1. However, by increasing the parameter k, the behavior improves.

CONCLUSION

In this study we develop an efficient and accurate method for solving nonlinear Fredholm- Hammerstein integral equations of the second kind. The properties of CAS wavelets are used to reduce the problem to the solution of algebraic equations. Error analysis is provided for the new method. However, to obtain better results, using the larger parameter k is recommended. The convergence accuracy of this method was examined for several numerical examples.

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