Simulating Fractional Derivatives using Matlab

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Abstract—**Fractional calculus has recently attracted much attention in the literature. In particular, fractional derivatives are widely discussed and applied in many areas. However, it is still hard to develop numerical methods for fractional calculus. In this paper, based on Fourier series and Taylor series technique, we provide some numerical methods for computing and simulating fractional derivatives by using Matlab. Some numerical examples are also presented.**

Index Terms—**fractional derivative, fractional integral, Matlab, Fourier series, Taylor series.**

I. Introduction

Fractional calculus is a branch of mathematical analysis that studies the possibility of taking real number powers or complex number powers of the differentiation operator. Generally speaking, $\dfrac{d^n y}{dx^n}$ represents the n th derivative of y with regard to x. What does it mean if we take n to a fractional number? This is a very important and interesting question asked by many mathematicians. In the history, fractional calculus has long been a pure theoretical problem. However, in the recent years, calculus has been successfully applied to many areas such as automatic control and signal processing (see [1-12]). Despite the applications of fractional calculus, it is hard to develop numerical methods for fractional derivatives due to its complex definitions. In this paper, we provide several numerical methods for computing fractional derivatives. We arrange the paper as follows, section 1 is the introduction of fractional calculus, section 2 is devoted to the numerical methods and its Matlab code and numerical examples, section 3 is the conclusion.

We review some basics for fractional calculus first. In the literature, there are various definitions of fractional calculus. We list some major ones below.

Grunwald-Letnikov definition,

$$ {}_a D_t^\alpha f(x) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=o}^{[(x-a)/h]} (-1)^j \binom{\alpha}{j} f(x - jh) \text{ ,(1)} $$

where $\binom{\alpha}{j}$ is a binomial coefficient and a the initial value. In a sense, Grunwald-Letnikov fractional derivative is a kind of generalization of integer derivative by taking integer difference to fractional case.

The classical form of fractional calculus is given by the Riemann–Liouville integral and the corresponding derivative is calculated using Lagrange's rule for differential operators. Computing n-th order derivative over the integral of order $n - \alpha$, the α order derivative is obtained. It is important to remark that n is the nearest integer bigger than α.

Riemann–Liouville fractional integral,

$$ {}_a J_t^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - \tau)^{\alpha-1} f(\tau) d\tau , \qquad (2) $$

where $0 < \alpha < 1$ and a is the initial value. The initial value a is usually set to be 0.

Starting from the Riemann–Liouville fractional integral, we come to

Riemann–Liouville fractional derivative,

$$ {}_a D_x^\gamma f(x) = D^n J^{n-\gamma} f(x) $$
$$ = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dx^n} \Big[\int_a^x (x-\tau)^{n-\gamma-1} f(\tau) d\tau \Big] \text{ ,(3)} $$

where $n - 1 < \gamma \le n$.

By contrast, the Grünwald–Letnikov derivative starts with the derivative instead of the integral.

Another option for defining fractional derivatives is Caputo fractional derivative. It was introduced by M. Caputo in 1990. Caputo's definition is often preferred in solving differential equations because it is not necessary to define the fractional order initial conditions. Caputo's definition is illustrated as follows.

$$ {}_a D_x^\alpha f(x) = \frac{1}{\Gamma(1-\gamma)} \int_a^x (x-\tau)^{-\gamma} f^{(m+1)}(\tau) d\tau \text{ ,(4)} $$

where $\alpha = m + \gamma$, and $0 < \gamma \le 1$, m is an integer.

It is proved that Grunwald-Letnikov fractional derivative is identical to Caputo fractional derivative for the majority of analytic functions. The slight difference between the two appears when dealing with constant

function. As a matter of fact, for a constant, the Caputo fractional derivative is zero while its Riemann–Liouville fractional derivative is not zero. In the literature, Caputo fractional derivative is usually used to handle initial value fractional ODE.

We list some well known properties of fractional derivative (See [13]).

a. If α is an integer, the fractional derivative is just identical to the traditional derivative. In this sense, fractional calculus is a kind of interpolation of the traditional calculus.

b. fractional operator satisfies the linearity .e.g.

$$_aD_t^\alpha[\lambda f(t) + \mu g(t)] = \lambda\,_aD_t^\alpha f(t) + \mu\,_aD_t^\alpha g(t)\ ,\ (5)$$

where λ and μ are arbitrary real numbers.

c. $_aD_t^\alpha[_aD_t^\beta f(t)] = \,_aD_t^\beta[_aD_t^\alpha f(t)] = \,_aD_t^{\alpha+\beta} f(t)$;

The property c means all fractional operators form a semi-group.

The Laplace transform for fractional integral is as follows:

$$L[J_t^\alpha f(t)] = s^{-\gamma}L[f(t)];$$

The Laplace transform for fractional derivative is

$$L[_0D_t^\alpha f(t)] = s^\alpha L[f(t)] - \sum_{k=1}^{n-1} s^k[_0D_t^{\alpha-k-1}f(t)]_{t=0}\ .$$

The Fourier transform for fractional calculus can be expressed as

$$F[_{-\infty}D_t^\alpha f(t)] = (j\omega)^\alpha F[f(t)],$$

where α can either be negative meaning integral or positive meaning fractional number.

II. MATLAB NUMERICAL METHODS FOR FRACTIONAL CALCULUS

A. Computing Fractional Derivative by using Fourier Series

A periodic function defined on $[-L, L]$ can be transformed to Fourier series, e.g.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x\right).$$

where

$$\begin{cases} a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\dfrac{n\pi x}{L}dx,\ n = 0,1,2... \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\dfrac{n\pi x}{L}dx,\ n = 1,2... \end{cases}\ .\ (6)$$

If $x \in (a,b)$, we get $L = (b-a)/2$. By letting $u = \hat{x} + L + a$, we transform $f(\hat{x})$ to a function defined on $(-L, L)$, then expand it into Fourier Series in the form of (6).

For $\sin(x)$ and $\cos(x)$ function, we have

$$\frac{d^k}{dx^k}\sin(ax) = a^k\sin\left(ax + \frac{k\pi}{2}\right),$$

and

$$\frac{d^k}{dx^k}\cos(ax) = a^k\cos\left(ax + \frac{k\pi}{2}\right).$$

It can be proved that the above conclusion still holds if k is fractional number (see [14]). Therefore, we have

$$D^\gamma f(x) = \frac{a_0}{\Gamma(1-\gamma)}x^{-\gamma} +$$

$$\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^\gamma\left(a_n\cos\left(\frac{n\pi}{L}x + \frac{\gamma\pi}{2}\right) + b_n\sin\left(\frac{n\pi}{L}x + \frac{\gamma\pi}{2}\right)\right)$$

According to the above conclusion, if γ is a fraction which is bigger than 1, we can transform it into $n + \beta$ where n is a positive integer and $0 < \beta < 1$. In this case, we can compute the n th derivative first, then compute its β th derivative.

So we develop the following procedure:

STEP1: Compute the Fourier Series. The matlab code is as below ,

```
function [A, B, F]=fourier(f, x, p, a, b)
If nargin==3, a= -pi; b=pi;
end
L= (b-a)/2;
if a+b,
f=subs(f, x, x+L+a);
end
A=int (f, x, -L,L)/L;B=[]; F=A/2;
for n=1:p
An=int(f*cos(n*pi*x/L), x, -L, L)/L;
bn=int(f*cos(n*pi*x/L), x, -L, L)/L;
A=[A, an];B=[B, bn]
F=F+an*cos(n*pi*x/L)+ bn*sin(n*pi*x/L);
end
if a+b, F=subs(F, x, x-L-a);
end
```

STEP 2:
Based on the above algorithm, we have the following Matlab code to calculate the fractional derivative.

```
function F=fdiff (A, B, t, gam, a, b)
A0=a(1);A=A(2:end);
n=length(B);L=(b-a)/2;
If gam>=0, F=0;
else, F=a0*t^gam/gam;end
for i=1:n,an=i*pi/L;bn=gam*pi/2;
F=F+an^gam*(A(i)*cos9an*t+bn)+
B(i)*sin(an*t+bn));
end
If a+b,
F=subs(F,t,t-L-a);
end
```

Example 1 :
Let test function be

$$f(x) = x(x - \pi)(x - 2\pi).$$

We get its Fourier series by Matlab code as follows
STEP1:
syms x; f=x;
f=x(x-pi)(x-2*pi)*
*[A,B,F]= fourier (f,x,20,0,2*pi);*

F =
*3/2000*sin(20*x)+3/250*sin(10*x)+12*sin(x)+12/156 25*sin(25*x)+1/486*sin(18*x)+1/2250*sin(30*x)+4/112 5*sin(15*x)+12/2197*sin(13*x)+12/1331*sin(11*x)+4/9 *sin(3*x)+3/4394*sin(26*x)+3/2*sin(2*x)+3/686*sin(14 *x)+12/125*sin(5*x)+1/18*sin(6*x)+4/3087*sin(21*x)+ 3/2662*sin(22*x)+4/6561*sin(27*x)+3/1024*sin(16*x)+ 3/16*sin(4*x)+4/243*sin(9*x)+3/128*sin(8*x)+3/5488* sin(28*x)+12/4913*sin(17*x)+1/1152*sin(24*x)+12/243 89*sin(29*x)+1/144*sin(12*x)+12/343*sin(7*x)+12/121 67*sin(23*x)+12/6859*sin(19*x).*

By matlab code,
ezplot(F, [0,1]),

we get the figure of the Fourier series of $f(x)$ as below .
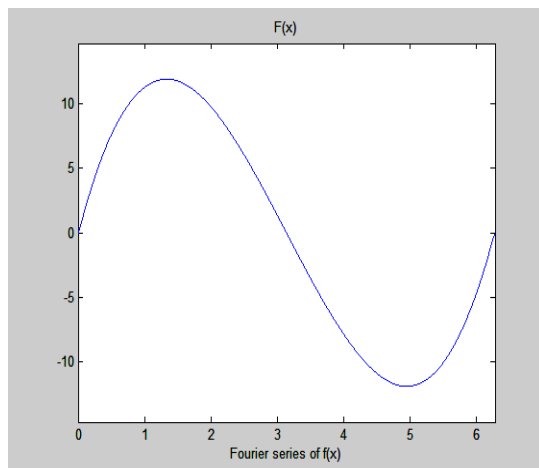
Fig. 1 (Fourier series approximation to $f(x)$)

STEP2:
F0.5= fdiff(A,B,x,0.5,0,1)
F0.5 =
*3/2*2^(1/2)*sin(2*x+1/4*pi)+3/8*sin(4*x+1/4*pi)+3/ 256*sin(16*x+1/4*pi)-*
*12*sin(x+1/4*pi)+3/250*10^(1/2)*sin(10*x+1/4*pi)-*
*12/4913*17^(1/2)*sin(17*x+1/4*pi)-*
*12/24389*29^(1/2)*sin(29*x+1/4*pi)-*
*12/1331*11^(1/2)*sin(11*x+1/4*pi)+1/162*2^(1/2)*sin(18*x+1/4*pi)+1/18*6^(1/2)*sin(6*x+1/4*pi)-*
*12/6859*19^(1/2)*sin(19*x+1/4*pi)+3/1000*5^(1/2)*sin (20*x+1/4*pi)-4/3087*21^(1/2)*sin(21*x+1/4*pi)-*
*4/9*3^(1/2)*sin(3*x+1/4*pi)+1/72*3^(1/2)*sin(12*x+1/ 4*pi)+3/2662*22^(1/2)*sin(22*x+1/4*pi)-*
*12/343*7^(1/2)*sin(7*x+1/4*pi)-*
*12/12167*23^(1/2)*sin(23*x+1/4*pi)-*
*12/2197*13^(1/2)*sin(13*x+1/4*pi)+3/64*2^(1/2)*sin(8 *x+1/4*pi)-*
*12/3125*sin(25*x+1/4*pi)+3/4394*26^(1/2)*sin(26*x+1 /4*pi)+3/686*14^(1/2)*sin(14*x+1/4*pi)+1/576*6^(1/2)*

*sin(24*x+1/4*pi)+1/2250*30^(1/2)*sin(30*x+1/4*pi)- 12/125*5^(1/2)*sin(5*x+1/4*pi)- 4/2187*3^(1/2)*sin(27*x+1/4*pi)+3/2744*7^(1/2)*sin(2 8*x+1/4*pi)-4/1125*15^(1/2)*sin(15*x+1/4*pi)- 4/81*sin(9*x+1/4*pi)*

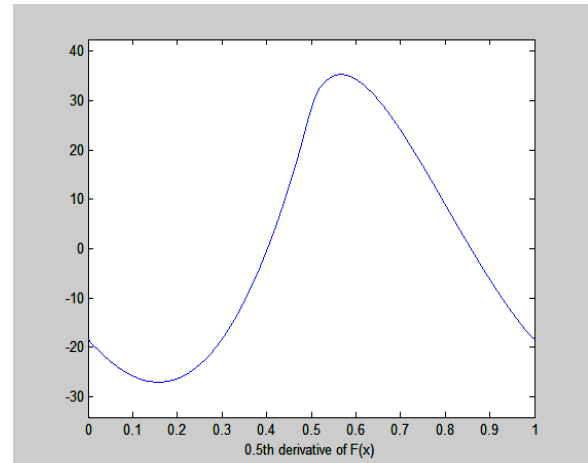By ezplot(F0.5,[0,1]), we get the 0.5^{th} derivative of $f(x)$ as below.

Fig. 2 (0.5^{th} derivative of $f(x)$)

By
F2= fdiff(A,B,x,0.7,0,1)
F3= fdiff(A,B,x,0.8,0,1)
F4=fdiff(A,B,x,0.9,0,1)
*ezplot(F2,[0,2*pi]);*
hold on
*ezplot(F3,[0,2*pi]);*
*ezplot(F4,[0,2*pi]);*

we can see the 0.7^{th}, 0.8^{th} and 0.9^{th} derivative of $f(x)$ in figure 3.
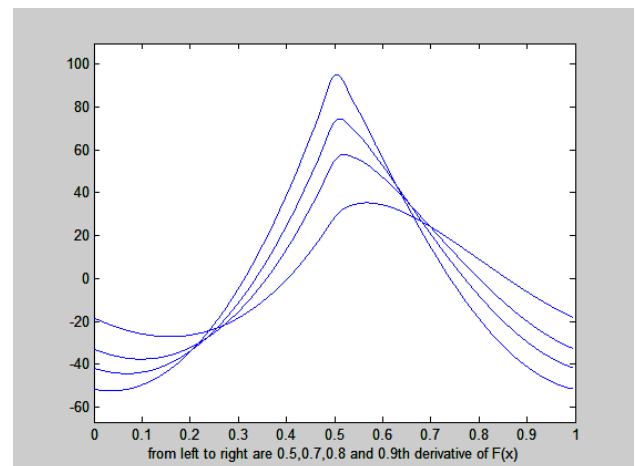

Fig. 3 (0.7^{th}, 0.8^{th} and 0.9^{th} derivatives of $f(x)$)

Example2 :
Test function is $f(x) = x(1 - x)$
STEP1:
syms x; f=x;
f=x(1-x)*
*[A,B,F]=fourier(f,x,20,0,2*pi);*
Its Fourier series is

F=

1/6+1/pi^2*cos(2*pi*(x-1/2))-1/4/pi^2*cos(4*pi*(x-1/2))+1/9/pi^2*cos(6*pi*(x-1/2))-1/16/pi^2*cos(8*pi*(x-1/2))+1/25/pi^2*cos(10*pi*(x-1/2))-1/36/pi^2*cos(12*pi*(x-1/2))+1/49/pi^2*cos(14*pi*(x-1/2))-1/64/pi^2*cos(16*pi*(x-1/2))+1/81/pi^2*cos(18*pi*(x-1/2))-1/100/pi^2*cos(20*pi*(x-1/2))+1/121/pi^2*cos(22*pi*(x-1/2))-1/144/pi^2*cos(24*pi*(x-1/2))+1/169/pi^2*cos(26*pi*(x-1/2))-1/196/pi^2*cos(28*pi*(x-1/2))+1/225/pi^2*cos(30*pi*(x-1/2))-1/256/pi^2*cos(32*pi*(x-1/2))+1/289/pi^2*cos(34*pi*(x-1/2))-1/324/pi^2*cos(36*pi*(x-1/2))+1/361/pi^2*cos(38*pi*(x-1/2))-1/400/pi^2*cos(40*pi*(x-1/2))

By ezplot(F,[0,1]), we can see the figure of Fourier series of $x(1-x)$ in fig 4.

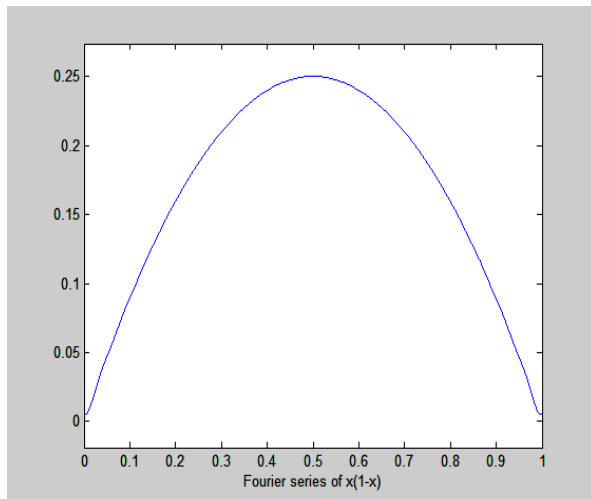Fig. 4 (Fourier series approximation to $x(1-x)$)

By Matlab code,
F1=fdiff(A,B,x,0.5,0,1),
We get the 0.5th derivative of $x(1-x)$,
F1 =

5644425081792261/2251799813685248/pi^2*cos(2*pi*x+1/4*pi)-3991211251234741/4503599627370496/pi^2*cos(4*pi*x+1/4*pi)+1222053877647539/2533274790395904/pi^2*cos(6*pi*x+1/4*pi)-5644425081792261/18014398509481984/pi^2*cos(8*pi*x+1/4*pi)+3155329544198077/14073748835532800/pi^2*cos(10*pi*x+1/4*pi)-2304326890293041/13510798882111488/pi^2*cos(12*pi*x+1/4*pi)+7466872530178867/55169095435288576/pi^2*cos(14*pi*x+1/4*pi)-3991211251234741/36028797018963968/pi^2*cos(16*pi*x+1/4*pi)+1058329702836049/11399736556781568/pi^2*cos(18*pi*x+1/4*pi)-8924619670322873/112589990684262400/pi^2*cos(20*pi*x+1/4*pi)+4680110038394005/68116944363978752/pi^2*cos(22*pi*x+1/4*pi)-

1222053877647539/20266198323167232/pi^2*cos(24*pi*x+1/4*pi)+5087816013229255/95138542128201728/pi^2*cos(26*pi*x+1/4*pi)-5279876200345031/110338190870577152/pi^2*cos(28*pi*x+1/4*pi)+683148885646777/15832967439974400/pi^2*cos(30*pi*x+1/4*pi)-5644425081792261/144115188075855872/pi^2*cos(32*pi*x+1/4*pi)+2909070101014387/81346268269379584/pi^2*cos(34*pi*x+1/4*pi)-374176054803257/11399736556781568/pi^2*cos(36*pi*x+1/4*pi)+1537717407869923/50806233296273408/pi^2*cos(38*pi*x+1/4*pi)-3155329544198077/112589990684262400/pi^2*cos(40*pi*x+1/4*pi).

By *ezplot(F1,[0,1]),* we get the 0.5th derivative which is shown in figure 5.
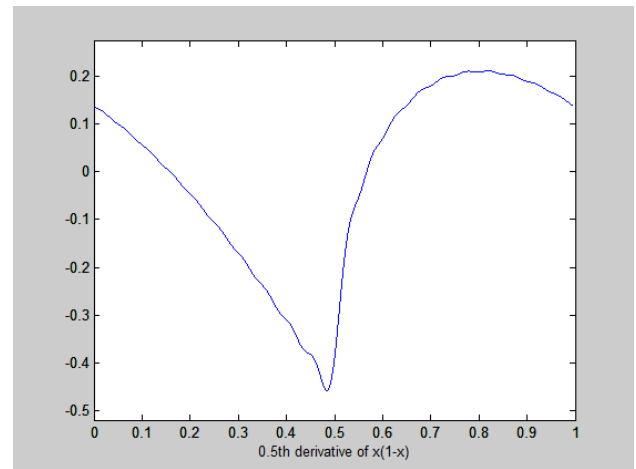

Fig. 5 (0.5th derivatives of $x(1-x)$)

In the same way, the 0.5th ,0.7th and 0.9th derivative are shown in figure 6..
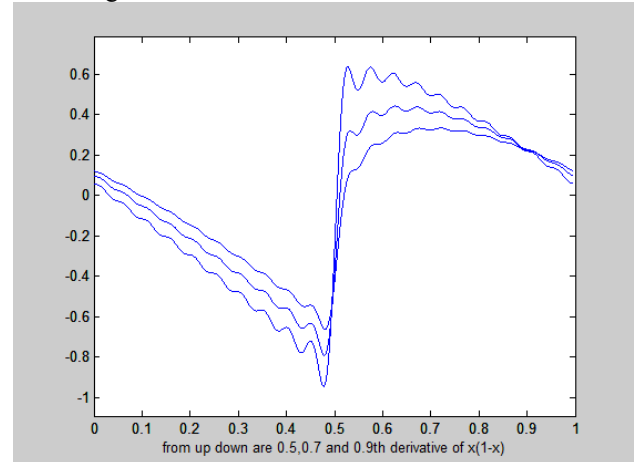

Fig. 6 (0.5th ,0.7th and 0.9th derivatives of $x(1-x)$)

B.   . Taylor Series Method

We now look at fractional derivative from another perspective. For integer derivative, we have

$$(x^n)' = nx^{n-1}, \ (x^n)'' = n(n-1)x^{n-2},$$

and

$$(x^n)^{(k)} = \frac{n!}{(n-k)!} x^{n-k}.$$

Starting from this point, we can go to the conclusion

$$(x^n)^{(\alpha)} = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha},$$

where α can either be a fractional or an arbitrary real number. So we have

$$x^{(\alpha)} = \frac{\Gamma(2)}{\Gamma(2-\alpha)} x^{1-\alpha},$$

where α is either be a fractional number or an arbitrary real number. In particular, we have

$$(x)^{(\frac{1}{2})} = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})} \sqrt{x}.$$

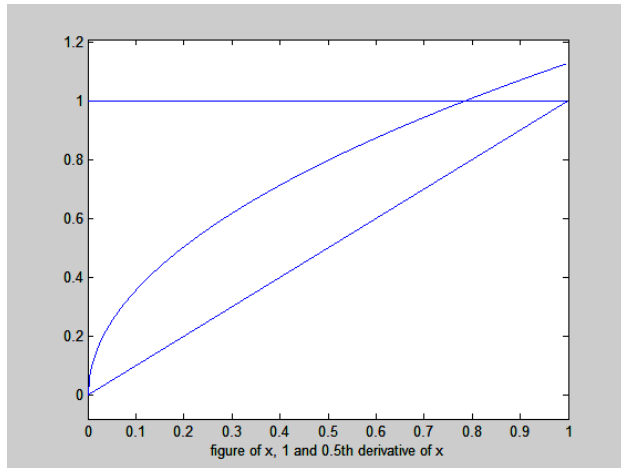The figure of $(x)^{(\frac{1}{2})}$ is as below figure 7.

Fig. 7 (0.5^{th} derivatives of x)

From the above conclusion, we see that we can get the fractional derivative of a function through computing its fractional derivatives of its Taylor series.

Example1 :

By Taylor series method, we can get the derivative of $f(x) = x$, the figure of $x^{(\alpha)}, \alpha \in [0,1]$ is as below.
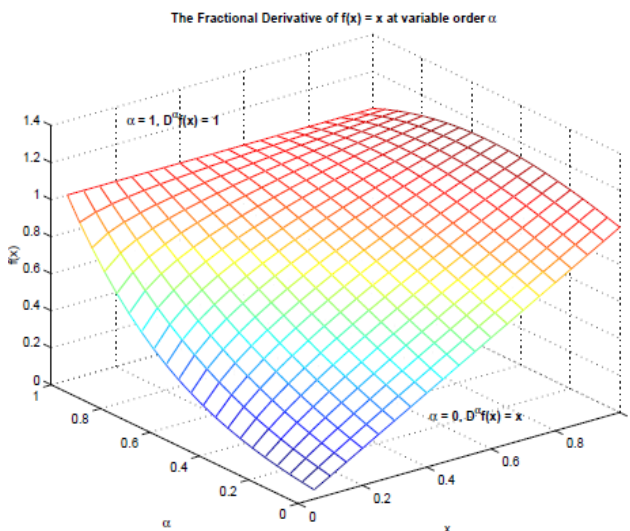

Fig. 8 (α derivatives of x)

Example2 :

For $f(x) = \sin x$

STEP1: Compute its Taylor series:
syms x
 f=sin(x)
 taylor(f,x,10)
 ans =
 x-1/6*x^3+1/120*x^5-1/5040*x^7+1/362880*x^9

STEP2**:** Compute its Taylor series fractional derivative:
From the above conclusion, we have

$$f^{(\frac{1}{2})}(x) = \frac{1}{\Gamma(\frac{3}{2})} x^{\frac{1}{2}} - \frac{1}{\Gamma(\frac{7}{2})} x^{\frac{5}{2}} + \frac{1}{\Gamma(\frac{11}{2})} x^{\frac{9}{2}}$$
$$- \frac{1}{\Gamma(\frac{15}{2})} x^{\frac{13}{2}} + \frac{1}{\Gamma(\frac{19}{2})} x^{\frac{17}{2}} - \frac{1}{\Gamma(\frac{23}{2})} x^{\frac{21}{2}},$$

$$f^{(0.25)}(x) = \frac{1}{\Gamma(1.75)} x^{0.75} - \frac{1}{\Gamma(3.75)} x^{2.75} + \frac{1}{\Gamma(5.75)} x^{4.75}$$
$$- \frac{1}{\Gamma(7.75)} x^{6.75} + \frac{1}{\Gamma(9.75)} x^{8.75} - \frac{1}{\Gamma(11.75)} x^{10.75}$$

and

$$f^{(0.9)}(x) = \frac{1}{\Gamma(1.1)} x^{0.1} - \frac{1}{\Gamma(3.1)} x^{2.1} + \frac{1}{\Gamma(5.1)} x^{4.1}$$
$$- \frac{1}{\Gamma(7.1)} x^{6.1} + \frac{1}{\Gamma(9.1)} x^{8.1} - \frac{1}{\Gamma(11.1)} x^{10.1}.$$

By matlab code,
f0.5=(1/gamma(3/2))*x^(1/2)-(1/gamma(7/2))*x^(5/2)+(1/gamma(11/2))*x^(9/2)-(1/gamma(15/2))*x^(13/2)+(1/gamma(19/2))*x^(17/2)-(1/gamma(23/2))*x^(21/2),
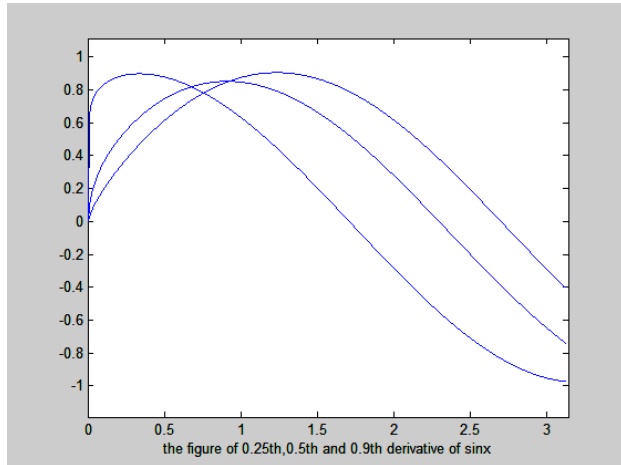
We have the 0.5^{th} derivative of $\sin(x)$
f0.5 =

5081767996463981/4503599627370496*x^(1/2)-1355138132390395/4503599627370496*x^(5/2)+2753296522951913/144115188075855872*x^(9/2)-2464489195369545/4611686018427387904*x^(13/2)+1237076929440399/147573952589676412928*x^(17/2)-1587427037276903/18889465931478580854784*x^(21/2).

In the same way, we can also compute the 0.25^{th} and 0.9^{th} derivative of $\sin(x)$.

By *ezplot(f0.5, [0,pi])*, we get the figure of the 0.5^{th} derivative of $\sin(x)$.

Fig. 9 (0.25th, 0.5th and 0.9th derivatives of $\sin(x)$)

From the above figure, we can see the 0.9th derivative of sin(x) is quite close to $\sin'(x) = \cos(x)$.

C. . Grunwald-Letnikov Method:

An efficient method to calculate fractional derivative is Grunwald-Letnikov method, e.g.

$$_aD_t^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=o}^{[(t-a)/h]} (-1)^j \binom{\alpha}{j} f(t - jh)$$

We can get good approximation to fractional derivative if h is sufficiently small. And it can be proved that the accuracy of this method is $o(h)$. The Matlab code is as below.

```
function dy=gdiff(y,x,gam)
h=x(2)-x(1);dy(1)=0;y=y(:);x=x(:);
w=1;
for j=2:length(x),w(j)=w(j-1)*(1-(gam+1)/(j-1));
end
for i=2 :length(x),dy(i)=w(1:i)*[y(i:-1:1)]/h^gam;
end
```

By the Matlab code
t=0:0.001:pi;y=sin(t);dy=gdiff(y,t,0.9);plot(t,dy)
t=0:0.001:pi;y=sin(t);dy=gdiff(y,t,0.9);plot(t,dy);
hold on;
t=0:0.001:pi;y=sin(t);dy=gdiff(y,t,0.1);plot(t,dy);
*t=0:0.001:pi;y=sin(t);dy=gdiff(y,t,0.5);plot(t,dy);*

We get 0.1th, 0.5th and 0.9th derivative of $\sin(x)$ as below figure 10.
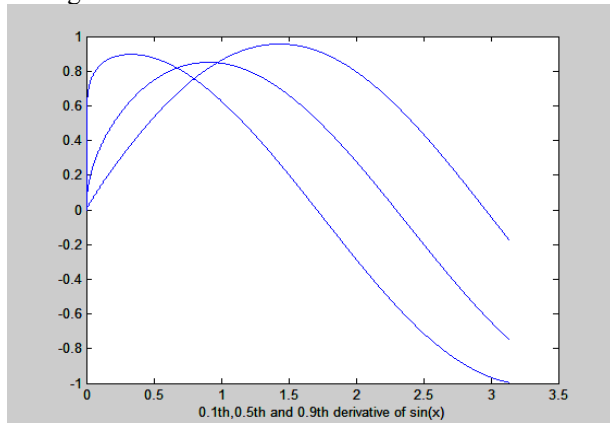
Fig.10   (0.1th, 0.5th and 0.9th derivatives of $\sin(x)$)

By the Matlab code
t=0:0.001:5;y=exp(t);plot(t,y)
t=0:0.001:3;y=exp(t);plot(t,y)
hold on;
t=0:0.001:3;y=exp(t);dy=gdiff(y,t,0.3);plot(t,dy)
t=0:0.001:3;y=exp(t);dy=gdiff(y,t,0.5);plot(t,dy)
t=0:0.001:3;y=exp(t);dy=gdiff(y,t,0.7);plot(t,dy)

We get the 0.3th, 0.5th and 0.7th derivative of e^x which are shown in figure 11. And we can see 0.3th, 0.5th and 0.7th derivative of e^x is almost identical to e^x which is a well known property in classic calculus.
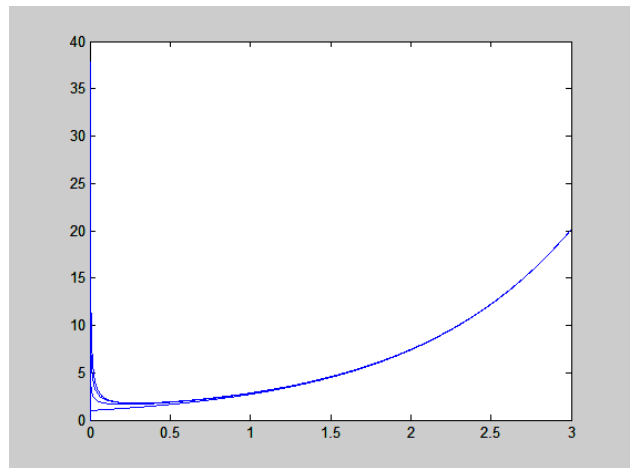
Fig. 11 (0.3th, 0.5th and 0.7th derivatives of e^x)

In calculus, the exponent function e^z satisfies the property $(e^z)' = e^z$. From the figure 11, we find that $(e^z)^{(\alpha)} = e^z$ still holds where α can be fractional numbers..

III. CONCLUSION

Fractional calculus is becoming an important tool in many academic fields. But it is not easy to numerically compute fractional integrals and fractional derivatives due to the lack of numerical methods. However, based on Fourier series and Taylor series theory and by using mathematical soft ware such as Matlab, we can efficiently compute fractional derivatives. From the numerical examples presented in this paper, we can see the more accuracy the Fourier series and Taylor series approximation is, the more fractional derivative we can get.

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