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Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration

E. Babolian [a,*], F. Fattahzadeh [b]

[a] *Department of Mathematics, Teacher Training University, Tehran, Iran*
[b] *Department of Mathematics, Islamic Azad University (Central Tehran Branch) Tehran, Iran*

Abstract

In this paper we present operational matrix of integration (OMI) of Chebyshev wavelets basis and the product operation matrix (POM) of it. Some comparative examples are included to demonstrate the superiority of operational matrix of Chebyshev wavelets to those of Legendre wavelets.

Keywords: Chebyshev wavelets; Legendre wavelets; Operational matrix of integration; Product operation matrix

1. Introduction

Orthogonal functions and polynomials receive attention in dealing with various problems [1–7]. One of those is differential or integral equation. The main characteristic of using orthogonal basis is that it reduces these problems to solving a system of linear algebraic equations, by truncated approximation series

$$y(t) \simeq y_N(t) = \sum_{i=0}^{N-1} c_i \phi_i(t)$$

and using the operational matrix of integration and the product operation matrix, to eliminate the integral operations [7]. The elements $\phi_0(t), \phi_1(t), \ldots, \phi_{N-1}(t)$, are the orthonormal basis functions defined on a certain interval $[a, b]$. Here we choose $\phi_i(t)$, as Chebyshev wavelets on $[0, 1]$. Legendre wavelets and its operational matrix of integration have been introduced and used in variational problems in [1,8].

* Corresponding author.
E-mail addresses: babolian@saba.tmu.ac.ir (E. Babolian), fattahzadeh@iauctb.ac.ir (F. Fattahzadeh).

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2. Chebyshev wavelets and their properties

2.1. Wavelets and Chebyshev wavelets

Wavelets have been very successfully used in many scientific and engineering fields. They constitute a family of functions constructed from dilation and transformation of a single function called the mother wavelet $\psi(x)$, we have the following family of continuous wavelets as [9]:

$$\psi_{a,b}(x) = |a|^{-1/2}\psi\left(\frac{x-b}{a}\right), \quad a,b \in \mathbb{R}, \ a \neq 0.$$

Chebyshev wavelets $\psi_{n,m} = \psi(k,n,m,t)$, have four arguments, $n = 1,2,\ldots,2^{k-1}$, k can assume any positive integer, m is degree of Chebyshev polynomials of the first kind and t denotes the time.

$$\psi_{n,m}(t) = \begin{cases} 2^{k/2}\widetilde{T}_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leqslant t < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where

$$\widetilde{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0, \\ \sqrt{\frac{2}{\pi}}T_m(t), & m > 0, \end{cases} \tag{2}$$

and $m = 0,1,\ldots,M-1$, $n = 1,2,\ldots,2^{k-1}$. In Eq. (2) the coefficients are used for orthonormality. Here $T_m(t)$, are Chebyshev polynomials of the first kind of degree m which are orthogonal with respect to the weight function $\omega(t) = 1/\sqrt{1-t^2}$, on $[-1,1]$, and satisfy the following recursive formula:

$$T_0(t) = 1, \quad T_1(t) = t, \quad T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \quad m = 1,2,\ldots$$

We should note that in dealing with Chebyshev wavelets the weight function $\omega(x)$ have to be dilated and translated as

$$\omega_n(t) = \omega(2^k t - 2n + 1),$$

to get orthogonal wavelets.

2.2. Function approximation

A function $f(t) \in \mathscr{L}^2_{\widetilde{\omega}}[0,1]$, (where $\widetilde{\omega}(t) = \omega(2t-1)$) may be expanded as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t), \tag{3}$$

where $c_{n,m} = (f(t), \psi_{n,m}(t))_{\omega_n}$, in which $(\cdot,\cdot)$ denotes the inner product in $\mathscr{L}^2_{\omega_n}[0,1]$. If we consider truncated series in (3), we obtain

$$f(t) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) = \mathbf{C}^{\mathrm{T}}\Psi(t), \tag{4}$$

where $\mathbf{C}$ and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$\begin{aligned} C &= [c_1, c_2, \ldots, c_{2^{k-1}}]^{\mathrm{T}}, \\ \Psi(t) &= [\psi_1, \psi_2, \ldots, \psi_{2^{k-1}}]^{\mathrm{T}}, \end{aligned} \tag{5}$$

and

$$c_i = [c_{i0}, c_{i1}, \ldots, c_{i,M-1}],$$
$$\psi_i(t) = [\psi_{i0}(t), \psi_{i1}(t), \ldots, \psi_{i,M-1}(t)],$$
for $i = 1,2,\ldots,2^{k-1}$.

The integration of the vector $\Psi(t)$, can be obtained as

$$\int_0^t \Psi(s)\,\mathrm{d}s = P\Psi(t). \tag{6}$$

Our main purpose is to obtain the matrix P.

2.3. Operational matrix of integration (OMI)

In this section we give the structure of OMI for Chebyshev wavelets. Because of the properties of Chebyshev polynomials (see [10]) we have,

$$\int_0^t \psi_{10}(s)\,\mathrm{d}s = \begin{cases} 2^{-k/2}\frac{1}{\sqrt{\pi}}\{T_1(2^k t-1)+1\}, & 0 \leqslant t < \frac{1}{2^{k-1}}, \\ \frac{1}{\sqrt{\pi}}2^{-k/2+1}, & t \geqslant \frac{1}{2^{k-1}}, \end{cases} \tag{7}$$

$$\int_0^t \psi_{11}(s)\,\mathrm{d}s = \begin{cases} \frac{1}{4}2^{-k/2}\sqrt{\frac{2}{\pi}}\{T_0(2^k t-1)+T_2(2^k t-1)-2\}, & 0 \leqslant t < \frac{1}{2^{k-1}}, \\ 0, & t \geqslant \frac{1}{2^{k-1}}, \end{cases} \tag{8}$$

$$\int_0^t \psi_{1r}(s)\,\mathrm{d}s = \begin{cases} 2^{-k/2-1}\sqrt{\frac{2}{\pi}}\{\frac{1}{r+1}T_{r+1}(2^k t-1) & \\ -\frac{1}{r-1}T_{r-1}(2^k t-1)-\frac{(-1)^{r+1}}{r+1}+\frac{(-1)^{r-1}}{r-1}\}, & 0 \leqslant t < \frac{1}{2^{k-1}}, \\ 2^{-k/2-1}\sqrt{\frac{2}{\pi}}\{\frac{1-(-1)^{r+1}}{r+1}-\frac{1-(-1)^{r-1}}{r-1}\}, & t \geqslant \frac{1}{2^{k-1}}, \end{cases} \tag{9}$$

for $r = 2,3,\ldots,M-1$.

Now we obtain the structure of P of Eq. (6). We denote the matrix P of Eq. (6) with $M \times M$ block matrices P_{ij},

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1,2^{k-1}} \\ P_{21} & P_{22} & \cdots & P_{2,2^{k-1}} \\ \vdots & \vdots & \ddots & \vdots \\ P_{2^{k-1},1} & P_{2^{k-1},2} & \cdots & P_{2^{k-1},2^{k-1}} \end{bmatrix}. \tag{10}$$

Using Eq. (7) we can obtain the first row of matrix P_{1j}, for $j = 1,2,\ldots,2^{k-1}$. We can express the right-hand side of Eq. (7) by means of the basis functions ψ_{ij} as

$$2^{-k/2}\frac{1}{\sqrt{\pi}}[T_1(2^k t-1)+1] = \sum_{j=0}^{M-1}\psi_{1j}(t)p_{1j}^{11} + \sum_{j=0}^{M-1}\psi_{2j}(t)p_{1j}^{12} + \cdots + \sum_{j=0}^{M-1}\psi_{2^{k-1},j}(t)p_{1j}^{1,2^{k-1}} 0 \leqslant t < 1/2^{k-1}, \tag{11}$$

where p_{1j}^{1l} is the $(1,j)$ entry of block matrix P_{1l}. Also we have similar identity for $t \geqslant \frac{1}{2^{k-1}}$. Now we have to insert $p_{1j}^{11} = 0$, for $j = 2,3,\ldots,M-1$ because the degree of $\psi_{1j}, j = 2,3,\ldots,M-1$, are greater than two, but the left-hand side of (11) is a polynomial of first degree. Also ψ_{ij}, for $i = 2,3,\ldots,2^{k-1}$, and $j = 0,1,\ldots,M-1$, are zero in $\left[0,\frac{1}{2^{k-1}}\right]$, so we get the first row of P_{1j} as

$$\begin{aligned} &P_{11}^1 = [2^{-k}, 2^{-k-1/2}, 0, 0, \ldots, 0], \\ &P_{1j}^1 = [2^{-k+1}, 0, 0, 0, \ldots, 0], \\ &\quad j = 2,\ldots,2^{k-1}. \end{aligned} \tag{12}$$

Now we give the second row of P_{1j}. So, we consider the right-hand side of Eq. (8) and express it as the basis functions ψ_{ij}, for interval $0 \leqslant t < 1/2^{k-1}$

$$\begin{aligned} &\frac{1}{4}2^{-k/2}\sqrt{\frac{2}{\pi}}[T_0(2^k t-1)+T_2(2^k t-1)-2] \\ &\quad = \sum_{j=0}^{M-1}\psi_{1j}(t)p_{2j}^{11} + \sum_{j=0}^{M-1}\psi_{2j}(t)p_{2j}^{12} + \cdots + \sum_{j=0}^{M-1}\psi_{2^{k-1},j}(t)p_{2j}^{1,2^{k-1}}, \quad 0 \leqslant t < 1/2^{k-1}, \end{aligned} \tag{13}$$

where p_{2j}^{1l} is the $(2,j)$ entry of block matrix P_{2l}. For the interval $t \geqslant \frac{1}{2^{k-1}}$ of Eq. (8) we have

$$0=\sum_{j=0}^{M-1}p_{2j}^{11}\times\left\{\text{functions that are zero for } t\geqslant\frac{1}{2^{k-1}}\right\}+\sum_{j=0}^{M-1}0\times\left\{\text{functions that are nonzero for } t\geqslant\frac{1}{2^{k-1}}\right\},\quad t\geqslant\frac{1}{2^{k-1}},\tag{14}$$

that yield,

$$\begin{aligned}&P_{11}^2=\left[\frac{-\sqrt{2}}{4}2^{-k},0,\frac{1}{4}2^{-k},0\ldots,0\right],\\&P_{1j}^2=[0,0,0,0,\ldots,0],\\&\quad j=2,\ldots,2^{k-1}.\end{aligned}\tag{15}$$

Now by means of Eq. (9) we obtain rth ($r\geqslant 2$) row of matrices $P_{1j}, j=1,2,\ldots,2^{k-1}$. We can express the right-hand side of Eq. (9) as the basis functions ψ_{ij}, for the first interval we have

$$\begin{aligned}&2^{-k/2-1}\sqrt{\frac{2}{\pi}}\left[\frac{T_{r+1}(2^kt-1)}{r+1}-\frac{T_{r-1}(2^kt-1)}{r-1}-\frac{(-1)^{r+1}}{r+1}+\frac{(-1)^{r-1}}{r-1}\right]\\&\quad=p_{r0}^{11}2^{k/2}\frac{1}{\sqrt{\pi}}T_0(2^kt-1)+\sum_{j=0}^{M-1}\text{coefficients}\times\{\text{functions of degree less than } r-1\}\\&\qquad+p_{r,r-1}^{11}2^{k/2}\sqrt{\frac{2}{\pi}}T_{r-1}(2^kt-1)+p_{r,r}^{11}2^{k/2}\sqrt{\frac{2}{\pi}}T_r(2^kt-1)+p_{r,r+1}^{11}2^{k/2}\sqrt{\frac{2}{\pi}}T_{r+1}(2^kt-1)\\&\qquad+\sum_{j=0}^{M-1}\text{coefficients}\times\{\text{functions of degree greater than } r+1\}+\sum_{j=0}^{M-1}\text{coefficients}\\&\qquad\times\{\text{functions that are zero for } 0\leqslant t<1/2^{k-1}\},\end{aligned}\tag{16}$$

also for $t\geqslant\frac{1}{2^{k-1}}$, we have

$$\begin{aligned}&2^{-k/2-1}\sqrt{\frac{2}{\pi}}\left[\frac{1-(-1)^{r+1}}{r+1}-\frac{1-(-1)^{r-1}}{r-1}\right]\\&\quad=\sum_{j=0}^{M-1}\text{coefficients}\times\left\{\text{functions that are zero for } t\geqslant\frac{1}{2^{k-1}}\right\}+p_{r0}^{12}2^{k/2}\frac{1}{\sqrt{\pi}}\\&\qquad+\sum_{j=0}^{M-1}\text{coefficients}\times\{\text{functions of degree greater than one}\}+p_{r0}^{13}2^{k/2}\frac{1}{\sqrt{\pi}}\\&\qquad+\sum_{j=0}^{M-1}\text{coefficients}\times\{\text{functions of degree greater than one}\}+\cdots+p_{r0}^{1,2^{k-1}}2^{k/2}\frac{1}{\sqrt{\pi}}\\&\qquad+\sum_{j=0}^{M-1}\text{coefficients}\times\{\text{functions of degree greater than one}\},\end{aligned}\tag{17}$$

but we have to insert the coefficients in summations (16) and (17) zero, in order to have equality in both sides. Therefore we get,

$$p_{r0}^{1l}=2^{-k-1/2}\left(\frac{1-(-1)^{r+1}}{r+1}-\frac{1-(-1)^{r-1}}{r-1}\right)\quad\text{for } l=2,3,\ldots,2^{k-1},\ \ r\geqslant 2\tag{18}$$

also from Eq. (16), we get

$$p_{r0}^{1l} = 2^{-k-1/2}\left(\frac{(-1)^{r-1}}{r-1} - \frac{(-1)^{r+1}}{r+1}\right), \quad p_{r,r-1}^{1l} = \frac{-2^{-k-1}}{r-1}, \quad p_{r,r+1}^{1l} = \frac{2^{-k-1}}{r+1}. \tag{19}$$

In fact we have proved:

Lemma 1

$$I_1 = \int_0^t \psi_1(x)\,\mathrm{d}x = L\psi_1(t) + F\psi_2(t) + F\psi_3(t) + \cdots + F\psi_{2^{k-1}}(t), \tag{20}$$

where

$$L = P_{11}, \quad F = P_{1j}, \; j = 2,3,\ldots,2^{k-1},$$

i.e.

$$L = 2^{-k}\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \cdots & 0 \\ \frac{-\sqrt{2}}{4} & 0 & \frac{1}{4} & 0 & 0 & \cdots & 0 \\ \frac{-\sqrt{2}}{3} & \frac{-1}{2} & 0 & \frac{1}{6} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{r-1}}{r-1} - \frac{(-1)^{r+1}}{r+1}\right) & 0 & 0 & \cdots & \frac{-1}{2(r-1)} & 0 & \frac{1}{2(r+1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{M-2}}{M-2} - \frac{(-1)^{M}}{M}\right) & 0 & 0 & 0 & \cdots & \frac{-1}{2(M-2)} & 0 \end{bmatrix}, \tag{21}$$

and

$$F = 2^{-k}\begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \frac{-2\sqrt{2}}{3} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{r+1}}{r+1} - \frac{1-(-1)^{r-1}}{r-1}\right) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{M}}{M} - \frac{1-(-1)^{M-2}}{M-2}\right) & 0 & \cdots & 0 \end{bmatrix}. \tag{22}$$

Similar to Eqs. (7)–(9) we can obtain $\int_0^t \psi_{nm}(x)\,\mathrm{d}x$, $m = 0,1,\ldots,M-1$. Since the vector $\psi_n(t)$, $n = 2,3,\ldots,2^{k-1}$ of Eq. (5) is exactly the translated of vector $\psi_1(t)$ to the interval $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}]$ we have

$$\int_0^t \psi_{n0}(s)\,\mathrm{d}s = \begin{cases} 2^{-k/2}\frac{1}{\sqrt{\pi}}\{T_1(2^k t - 2n + 1) + 1\}, & \frac{n-1}{2^{k-1}} \leqslant t < \frac{n}{2^{k-1}}, \\ \frac{1}{\sqrt{\pi}}2^{-k/2+1}, & t \geqslant \frac{n}{2^{k-1}}, \end{cases} \tag{23}$$

$$\int_0^t \psi_{n1}(s)\,\mathrm{d}s = \begin{cases} \frac{1}{4}2^{-k/2}\sqrt{\frac{2}{\pi}}\{T_0(2^k t - 2n + 1) + T_2(2^k t - 2n + 1) - 2\}, & \frac{n-1}{2^{k-1}} \leqslant t < \frac{n}{2^{k-1}}, \\ 0, & t \geqslant \frac{n}{2^{k-1}}, \end{cases} \tag{24}$$

$$\int_0^t \psi_{nm}(s)\,\mathrm{d}s \begin{cases} 2^{-k/2-1}\sqrt{\frac{2}{\pi}}\left\{\frac{1}{r+1}T_{r+1}(2^k t - 2n + 1) - \frac{1}{r-1}T_{r-1}(2^k t - 2n + 1) - \frac{(-1)^{r+1}}{r+1} + \frac{(-1)^{r-1}}{r-1}\right\}, & \frac{n-1}{2^{k-1}} \leqslant t < \frac{n}{2^{k-1}}, \\ 2^{-k/2-1}\sqrt{\frac{2}{\pi}}\left\{\frac{1-(-1)^{r+1}}{r+1} - \frac{1-(-1)^{r-1}}{r-1}\right\}, & t \geqslant \frac{n}{2^{k-1}}, \end{cases} \tag{25}$$

for $m = 2,3,\ldots,M-1$.

Theorem 1

$$\begin{aligned}
I_1 &= \int_0^t \psi_1(x)\,\mathrm{d}x = L\psi_1(t) + F\psi_2(t) + F\psi_3(t) + \cdots + F\psi_{2^{k-1}}(t),\\
I_2 &= \int_0^t \psi_2(x)\,\mathrm{d}x = O\psi_1(t) + L\psi_2(t) + F\psi_3(t) + \cdots + F\psi_{2^{k-1}}(t),\\
&\vdots\\
I_{2^{k-1}} &= \int_0^t \psi_{2^{k-1}}(x)\,\mathrm{d}x = O\psi_1(t) + O\psi_2(t) + O\psi_3(t) + \cdots + L\psi_{2^{k-1}}(t),
\end{aligned} \tag{26}$$

where O, denotes the zero matrix and the matrices L and F are introduced in Eqs. (21) and (22).

Proof. From Eqs. (23)–(25), I_2, is the same as I_1, with this obvious difference that Chebyshev functions are translated to the next interval, therefore the coefficient matrix P_{21}, have to be zero matrix. Also the matrices P_{2j}, for $j = 2, 3, \ldots, 2^{k-1}$, will be shifted the first row i.e. $P_{22} = L$, $P_{2j} = F$, for $j = 3, \ldots, 2^{k-1}$; this process can be continued to obtain the last row, and therefore it yields the structure of matrix OMI,

$$P = \begin{bmatrix} L & F & F & \cdots & F \\ O & L & F & \cdots & F \\ \vdots & O & \ddots & \ddots & \vdots \\ & & & & F \\ O & O & \cdots & O & L \end{bmatrix}. \qquad \square \tag{27}$$

3. The product operation matrix (POM)

In this section we will obtain the product operation matrix, which is important for solving differential and integral equations.

As we know the support of ψ_{nm}, the entries of vector $\psi_n(t)$, are $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$, therefore

$$\psi_i(t)\psi_j^t(t) = O_{M\times M}, \quad i \neq j,$$

hence

$$\Psi(t)\Psi^t(t) = \mathrm{diag}(\psi_1(t)\psi_1^t(t), \psi_2(t)\psi_2^t(t), \ldots, \psi_{2^{k-1}}(t)\psi_{2^{k-1}}^t(t)). \tag{28}$$

Now we would like to write the entries of symmetric matrices $\psi_i(t)\psi_i^t(t)$, $i = 1, 2, \ldots, 2^{k-1}$, as a linear combination of entries of vector $\Psi(t)$, or simply as $\psi_i(t)$. By definition of Chebyshev polynomials, [10], we have

$$\int_{-1}^{1} T_m^2(x)T_k(x)\omega(x)\,\mathrm{d}x = \begin{cases} \pi & m = k = 0, \\ \frac{\pi}{4} & 2m = k \neq 0, \\ \frac{\pi}{2} & m \neq 0,\ k = 0, \\ 0 & \text{otherwise}, \end{cases} \tag{29}$$

because

$$\begin{aligned}
T_r(x)T_s(x) &= \frac{1}{2}\{T_{r+s}(x) + T_{|r-s|}(x)\},\\
T_m^2(x)T_k(x) &= \frac{1}{2}\{T_{2m}(x)T_k(x) + T_0(x)T_k(x)\},
\end{aligned} \tag{30}$$

and

$$\int_{-1}^{1} T_i(x)T_j(x)\omega(x)\,\mathrm{d}x = \begin{cases} \frac{\pi}{2} & i = j \neq 0, \\ \pi & i = j = 0, \\ 0 & i \neq j, \end{cases}$$

moreover

$$\int_{-1}^{1} T_m^3(x)\omega(x)\,\mathrm{d}x = \begin{cases} \pi & m = 0, \\ 0 & m \neq 0, \end{cases}$$

and if $m \neq k \neq r$, then

$$\int_{-1}^{1} T_m(x)T_k(x)T_r(x)\omega(x)\,\mathrm{d}x = \begin{cases} \frac{\pi}{4} & m+k=r \text{ or } |m-k| = r, \\ 0 & \text{otherwise.} \end{cases}$$

Also

$$\begin{aligned}
&\psi_{i0}\psi_{il} = \frac{2^{k/2}}{\sqrt{\pi}}\psi_{il}, \quad l = 0,1,\ldots,M-1, \\
&\psi_{il}\psi_{ir} \simeq \frac{2^{k/2}}{\sqrt{\pi}}\psi_{i0} + \frac{2^{k/2-1/2}}{\sqrt{\pi}}\psi_{i,l+r}, \quad r = l \neq 0, \\
&\psi_{il}\psi_{ir} \simeq \frac{2^{k/2-1/2}}{\sqrt{\pi}}(\psi_{i,|l-r|} + \psi_{i,l+r}), \quad r \neq l, \;\; r,l \neq 0 \\
&\text{for } l+r \leqslant M-1 \quad \text{for } i = 1,2,\ldots,2^{k-1}
\end{aligned} \tag{31}$$

for $l+r > M-1$ the second terms in the above formulas have been deleted. The first equation is obvious from explicit formula of ψ_{i0}. For simplicity we consider our result for $i = 1$, we know for $i = 2,\ldots,2^{k-1}$, the relations are the same as this case because the only difference is that the integrals are obtained in the next interval with the translated of the same functions. Also the above formula can be obtained by inserting explicit formula for functions involved

$$\psi_{il}\psi_{ir} \simeq \sum_{v=0}^{M-1} p_v^{lr}\psi_{iv}, \quad l,r = 0,1,\ldots,M-1,$$

where

$$p_v^{lr} = (\psi_{il}\psi_{ir}, \psi_{iv})_{\omega_i}.$$

Now if we consider the vector c_i (of Eq. (5)), we can write

$$\psi_i(t)\psi_i^t(t)c_i \simeq \widetilde{c_i}\psi_i(t), \tag{32}$$

where

$$\tilde{c_i} = \frac{2^{k/2}}{\sqrt{\pi}} \begin{bmatrix}
c_{i0} & c_{i1} & c_{i2} & c_{i3} & \cdots & c_{i,M-2} & c_{i,M-1} \\
c_{i1} & c_{i0} + \frac{1}{\sqrt{2}}c_{i2} & \frac{1}{\sqrt{2}}(c_{i1}+c_{i3}) & \frac{1}{\sqrt{2}}(c_{i2}+c_{i4}) & \cdots & \frac{1}{\sqrt{2}}(c_{i,M-3}+c_{i,M-1}) & \frac{1}{\sqrt{2}}c_{i,M-2} \\
c_{i2} & \frac{1}{\sqrt{2}}(c_{i1}+c_{i3}) & c_{i0} + \frac{1}{\sqrt{2}}c_{i4} & \frac{1}{\sqrt{2}}(c_{i1}+c_{i5}) & \cdots & \frac{1}{\sqrt{2}}c_{i,M-4} & \frac{1}{\sqrt{2}}c_{i,M-3} \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\
\cdots & \cdots & \cdots & c_{i0} + \frac{1}{\sqrt{2}}c_{i\mu} & c_{i1} + \frac{1}{\sqrt{2}}c_{i,\mu+1} & \cdots & \frac{1}{\sqrt{2}}c_{i,v} \\
\cdots & \cdots & \cdots & c_{i1} + \frac{1}{\sqrt{2}}c_{i,\mu+1} & c_{i0} & \vdots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & c_{i0} & \frac{1}{\sqrt{2}}c_{i1} \\
c_{i,M-1} & \frac{1}{\sqrt{2}}c_{i,M-2} & \cdots & \cdots & \cdots & \frac{1}{\sqrt{2}}c_{i1} & c_{i0}
\end{bmatrix} \tag{33}$$

with

$$\mu = \begin{cases} M-2 & M \text{ even}, \\ M-1 & M \text{ odd}, \end{cases}$$

and

$$v = \begin{cases} M/2 & M \text{ even}, \\ (M-1)/2 & M \text{ odd}, \end{cases}$$

for $i = 1, 2, \ldots, 2^{k-1}$. In fact we have shown that,

$$\Psi\Psi^t C \simeq \widetilde{C}\Psi, \tag{34}$$

where

$$\widetilde{C} = \operatorname{diag}(\widetilde{c_1}, \widetilde{c_2}, \ldots, \widetilde{c}_{2^{k-1}}).$$

4. Numerical examples

Example 1. Consider

$$y(t) + y'(t) = t + 1 + (2/5t^2 - 1/5t - 3/5)u_{1/2}(t), \quad y(0) = 0, \tag{35}$$

where

$$u_c(t) = \begin{cases} 1 & t \geqslant c, \\ 0 & t < c, \end{cases}$$

and the analytic solution is

$$y(t) = \begin{cases} t & 0 \leqslant t < 1/2, \\ 2/5(t^2+1) & 1/2 \leqslant t1. \end{cases}$$

We put

$$y'(t) = C^{\mathrm{T}}\Psi(t), \tag{36}$$

where

$$C = [c_{10}, c_{11}, c_{12}, c_{20}, c_{21}, c_{22}]^{\mathrm{T}}, \tag{37}$$

and

$$\Psi(t) = [\psi_{10}, \psi_{11}, \psi_{12}, \psi_{20}, \psi_{21}, \psi_{22}]^{\mathrm{T}}. \tag{38}$$

using boundary condition in (35) we get

$$y(t) = C^{\mathrm{T}}P\Psi(t). \tag{39}$$

We can also express the functions of right-hand side of (35) as

$$\begin{aligned} t &= \sqrt{\pi}/8[1, 1/\sqrt{2}, 0, 3, 1/\sqrt{2}, 0]^{\mathrm{T}}\Psi(t) = \mathrm{e}^{\mathrm{T}}\Psi(t), \\ 1 &= \sqrt{\pi}/2[1, 0, 0, 1, 0, 0]^{\mathrm{T}}\Psi(t) = d^{\mathrm{T}}\Psi(t), \\ (2/5t^2 - 1/5t - 3/5)u_{1/2}(t) &= \sqrt{\pi}/40[0, 0, 0, -41/4, \sqrt{2}, \sqrt{2}/8]^{\mathrm{T}}\Psi(t) = l^{\mathrm{T}}\Psi(t). \end{aligned} \tag{40}$$

Now by inserting Eqs. (36)–(40) into (35) we get

$$\Psi^{\mathrm{T}}(P^{\mathrm{T}} + I)C = \Psi^{\mathrm{T}}(e + d + l),$$

which is hold for each t in defined interval, therefore we get

$$(P^{\mathrm{T}} + I)C = e + d + l,$$

which yields

$$C = \sqrt{\pi}/2[1, 0, 0, 3/5, \sqrt{2}/10, 0]^{\mathrm{T}}. \tag{41}$$

By virtue of Eq. (39) we get the exact solution.

Example 2. Consider

$$0.25y'(t) + y(t) = 1, \quad y(0) = 0 \tag{42}$$

with the analytic solution $y(t) = 1 - \exp^{(-4t)}$. This problem has been solved by Legendre wavelets [8]. Here we solve it using Chebyshev wavelets, with $k = 2$, $M = 3$. As the Legendre case, first we assume the unknown function $y(t)$ is given by

$$y(t) = C^{\mathrm{T}}\Psi(t), \tag{43}$$

where C and $\Psi(t)$ are as the preceding example. Integrating (42) from 0 to t and using operational matrix, we obtain

$$0.25C^{\mathrm{T}}\Psi(t) + C^{\mathrm{T}}P\Psi(t) = d^{\mathrm{T}}P\Psi(t), \tag{44}$$

where d is given in (40). The above expression is satisfied for all t in the interval $[0, 1)$. Therefore

$$0.25C^{\mathrm{T}} + C^{\mathrm{T}}P = d^{\mathrm{T}}P. \tag{45}$$

This equation can be solved for vector C. Table 1 shows the comparison between the absolute error of exact and approximate solution for various values of M (with $k = 2$) and both methods Legendre and Chebyshev wavelets.

The comparison between two methods shows that the absolute error in the Chebyshev wavelets method is the same or slightly better than the Legendre case.

Example 3. Consider the Bessel differential equation of order zero

$$ty'' + y' + ty = 0, \tag{46}$$

with

$$y(0) = 1, \quad y'(0) = 0. \tag{47}$$

Table 1
Absolute error of exact and approximated solution of Example 2 with Legendre (Lg) and Chebyshev (Chb) wavelets

t	Lg $M = 3$	Chb $M = 3$	Lg $M = 4$	Chb $M = 4$	Lg $M = 5$	Chb $M = 5$
0.0	0.0270	0.0127	0.0038	0.0021	0.4114e−3	0.2038e−3
0.1	0.0108	0.0145	0.0014	0.00132	0.0411e−3	0.0208e−3
0.2	0.0047	0.0038	0.0011	0.0017	0.1114e−3	0.1467e−3
0.9	0.0010	0.0014	0.0002	0.0002	0.0113e−3	0.0052e−3
1.0	0.0036	0.0031	0.0005	0.0003	0.0557e−3	0.0301e−3

Table 2
Absolute error of exact and approximated solution of Example 3 with Legendre (Lg) and Chebyshev (Chb) wavelets

t	Legendre	Chebyshev
0.0	$0.0963 * 10^{-3}$	0.0601−3
0.1	0.0287−3	0.0615−3
0.2	0.0360−3	0.0599−3
0.3	0.0183−3	0.0090−3
0.4	0.0412−3	0.0524−3
0.5	0.2695−3	0.1695−3
0.6	0.0922−3	0.1602−3
0.7	0.0826−3	0.1140−3
0.8	0.0688−3	0.0784−3
0.9	0.1026−3	0.1577−3
1.0	0.2689−3	0.1636−3

The solution is the Bessel function of the first kind as

$$J_0(t) = \sum_{q=0}^{\infty} \frac{(-1)^q}{(q!)^2} \left(\frac{t}{2}\right)^{2q}.$$

If we assume the unknown function $y''(x)$ is given by

$$y''(t) = C^{\mathrm{T}} \Psi(t) \tag{48}$$

and then we use the boundary conditions (47) we get

$$y'(t) = C^{\mathrm{T}} P \Psi(t), \quad y(t) = C^{\mathrm{T}} P^2 \Psi(t) + d^{\mathrm{T}} \Psi(t), \tag{49}$$

where d is given in (40). Substituting (48) and (49) into (46) we obtain

$$\mathrm{e}^{\mathrm{T}} \Psi(t) \Psi^{\mathrm{T}}(t) C + C^{\mathrm{T}} P \Psi(t) + \mathrm{e}^{\mathrm{T}} \Psi(t) \Psi^{\mathrm{T}}(t) P^{2T} C + \mathrm{e}^{\mathrm{T}} \Psi(t) \Psi^{\mathrm{T}}(t) d = 0. \tag{50}$$

Now with using the Product Operation Matrix $\widetilde{C}$ of (34) we get the equation

$$\widetilde{E} C + P^{\mathrm{T}} C + \widetilde{E} P^2 C = -\widetilde{E} d.$$

After finding C we can get the approximated solution by inserting C into (49). Table 2 shows the absolute error in equally divided interval $[0, 1]$ with both Legendre and Chebyshev wavelets methods with $M = 3$, $k = 2$.

5. Conclusion

The Chebyshev wavelets operational matrix of integration and its product operation matrix have been obtained in general and used in some differential equations. Because of weight function of Chebyshev polynomials the absolute error at the end points are less than those of Legendre polynomials. Also these functions have good advantage in dealing with piecewise continuous functions, as are shown.

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