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FREDHOLM-VOLTERRA INTEGRAL EQUATION WITH SINGULAR KERNEL

M. A. DARWISH

ABSTRACT. The purpose of this paper is to obtain the solution of Fredholm-Volterra integral equation with singular kernel in the space $L_2(-1,1) \times C(0,T)$, $0 \leq t \leq T < \infty$, under certain conditions. The numerical method is used to solve the Fredholm integral equation of the second kind with weak singular kernel using the Toeplitz matrices. Also, the error estimate is computed and some numerical examples are computed using the MathCad package.

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Key word and phase : Fredholm-Volterra integral equation, singular integral operator, Toeplitz matrices.

1. Introduction

Singular integral equations have received considerable interest in the mathematical literature, because of their many field of applications. For example in applied mathematics arise in areas as diverse as the theory of elasticity, viscoelasticity, hydrodynamics and others. The solution of these equations can be obtained analytically by using Cauchy method [14], the potential theory method [3], the orthogonal polynomials method [7], the Fourier transformation method [8] and Krein's method [9]. More recently, numerical solution of these equations is a much studied subject of numerous works, since analytical methods on practical problems often fail. For examples, Galerkin method [6], fast method [10], block by block method [4, 11], Nyström method [4] and Toeplitz matrices method [13]. The theory of singular integral equations has assumed various technique and has increasing important in many fields of applications. Many

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problems of the quasi-static displacement in the theory of elasticity and viscoelasticity can be reduced to the Fredholm-Volterra integral equation

$$(1)\quad \mu\psi(x,t)+\int_{-1}^{1}\psi(\xi,t)k(\xi,x)\,d\xi+\int_{0}^{t}\psi(x,\tau)F(t,\tau)\,d\tau = \\ =\pi[\gamma(t)+\beta(t)x-f(x)],$$

$|x|\leq 1,\ \ 0\leq t\leq T<\infty$, under the conditions

$$(2)\qquad P(t)=\int_{-1}^{1}\psi(\xi,t)\,d\xi,\quad M(t)=\int_{-1}^{1}\xi\psi(\xi,t)\,d\xi.$$

Here $f(x)\in L_2(-1,1)$, $\gamma(t)$ and $\beta(t)\in C(0,T)$, while $F(t,\tau)$ is a continuous function in the interval $0\leq t\leq T<\infty$, and the kernel of the Fredholm integral in Eq.(1) has the Karlman form, $k(x,y)=|y-x|^{-\nu},\ \ 0<\nu<1$. In [1] the general solution of Eq.(1) under the conditions (2) where the kernel has the logarithmic form is obtained. Eq. (1) is of the first kind when $\mu=0$ and of the second kind when $\mu\neq 0$. Our main object in this paper is to obtain the general solution of the boundary value problem of Fredholm-Volterra equation with Karlman kernel in the class $L_2(-1,1)\times C(0,T)$ for certain conditions of $\gamma(t)$ and $\beta(t)$. This paper is organized as follows. In §2 the conditions under which the solution of Eq.(1) exists are given. In §3 the general solution of Eq.(1) under the conditions (2) is established in a series form. In §4 the numerical solution of Fredholm integral equation of the second is obtained. Finally, in §5 some numerical examples are computed.

2. The Integral Operator

Consider the integral operator

$$(3)\qquad K\phi=\int_{-1}^{1}k(x,y)\,\phi(y)\,dy,$$

$K:\ L_2(-1,\ 1)\longrightarrow\ L_2(-1,\ 1)$. We assume through the time interval $0\leq t\leq T<\infty$ the following conditions:

(1) $k(x,y)\in C([-1,1]\times[-1,1])$.

(2) The discontinuous kernel satisfies $\{\int_{-1}^{1}\int_{-1}^{1}k^2(x,y)\,dx\,dy\}^{\frac{1}{2}=c<\infty}$.

(3) The Potential function $\phi(x)$ is continuous in $x\in[-1,1]$ and satisfies

$$\{\int_{-1}^{1}|\phi(y)|^2\,dy\}^{1/2}\leq\ A||\phi||_2,$$

where $||.||_2$ denotes the L_2 norm and A is a constant.

(4) The potential function $\phi(x)$ satisfies the Lipschitz condition with respect to the second argument, $|\phi(x,y_1)-\phi(x,y_2)| \leq L|y_1-y_2|$.

The continuity of the integral operator, $K\phi = \int_{-1}^{1} k(x,y)\ \phi(y)\ dy$, in $L_2(-1,1)$ can be proved. For taking $x_1,\ x_2 \in L_2(-1,1)$, we have

$$\begin{aligned} & |\int_{-1}^{1} k(x_1,y)\ \phi(y)\ dy - \int_{-1}^{1} k(x_2,y)\ \phi(y)\ dy| \\ \leq\ & (\int_{-1}^{1} \phi^2(y)\ dy)^{1/2}(\int_{-1}^{1}[k^2(x_1,y)-k^2(x_2,y)]\ dy)^{1/2} \\ & \leq\ ||\phi||_2\ g(x_1,x_2), \end{aligned}$$

where $g(x_1,x_2) \to 0$ as $x_1 \to x_2$.

The normality of the integral operator can be proved as follows:-

$$\begin{aligned} ||K\phi||_2 &= [\int_{-1}^{1} dx(\int_{-1}^{1} k(x,y)\ \phi(y)\ dy)^2]^{1/2} \\ &\leq [\int_{-1}^{1} dx \int_{-1}^{1} \phi^2(y)\ dy \int_{-1}^{1} k^2(x,y)\ dy]^{1/2} \\ &= C\ ||\phi||_2. \end{aligned}$$

So, $||K|| \leq C$.

Theorem 1. *The integral operator (3) is a positive compact self-adjoint operator. So we may write $K\phi = \sum \alpha_n \phi_n$, where α_n and ϕ_n are the eigenvalues and the eigenfunctions of the integral operator, respectively.*

3. Solution of the Problem

The Fredholm integral equation of the second kind with Karlman kernel can be obtained from the boundary value problem (1) and (2). For this, let $t=0$ in (1) and (2), we obtain

(4)

$$\mu\psi(x,0) + \int_{-1}^{1} \psi(\xi,0)k(\xi,x)\ d\xi\ =\ \pi[\gamma(0)+\beta(0)x-f(x)],\ |x| \leq 1,$$

and

$$P(0) = \int_{-1}^{1} \psi(\xi,0)\ d\xi, \quad M(0) = \int_{-1}^{1} \xi\psi(\xi,0)\ d\xi, \tag{5}$$

respectively. Rewriting Eq.(4) in the form

$$\mu\psi'(x) + \int_{-1}^{1} \psi'(\xi)k(\xi, x)\, d\xi \;=\; \pi\, f'(x),\ |x| \le 1, \tag{6}$$

where $\psi'(x) = \psi(x,0)$ and $f'(x) = [\gamma(0) + \beta(0)x - f(x)]$. Eq. (6) is a Fredholm integral equation of the second kind. In §4, the numerical solution of this equation with Karlman, $k(x,y) = |y-x|^{-\nu}$, $0 < \nu < 1$, kernel has been established.

Now, we approximate the solution of Eq.(1) when $\gamma(t)$ and $\beta(t) \in C(0,\ T)$ in the form

$$\psi(x,\ t) = \psi_0(x,\ t) + \psi_1(x,\ t) = \psi_j(x,\ t), \tag{7}$$

where $\psi_j(x,\ t)$, $j = 0,\ 1$, is defined, respectively, from the solutions of the following integral equations

$$\begin{aligned} (8)\quad & \mu[\psi_j(x,\ t) - \psi_j(x,\ 0)] + \int_{-1}^{1} [\psi_j(\xi,\ t) - \psi_j(\xi,\ 0)]k(\xi, x)\, d\xi\ + \\ & + \int_0^t \psi_j(x,\tau)F(t,\tau)\, d\tau \;=\; \pi\delta_j[\gamma(t) + \beta(t)x - \gamma(0) - \beta(0)x], \\ & j = 0,\ 1,\ |x| \le 1,\ \ 0 \le t \le T < \infty,\ \delta_j = \begin{cases} 1 & j = 0, \\ 0 & j = 1. \end{cases} \end{aligned}$$

In a view of Theorem 2.1, and for $t \ge 0$, we write

$$\psi_j(x,\ t) = \sum_{k=1}^{\infty} \left[a_{2k}^{(j)}(t)\ \psi_{2k}(x) + a_{2k-1}^{(j)}(t)\ \psi_{2k-1}(x)\right] \tag{9}$$

and

$$\alpha_k \int_{-1}^{1} \psi_k(\xi)\ k(x,\ \xi)\ d\xi = \psi_k(x),\ |x| \le 1,\ k \ge 1, \tag{10}$$

where $\psi_{2k}(x)$ and $\psi_{2k-1}(x)$ stand for the even and the odd functions, respectively. Using (8) and (9) in (10), we obtain the following integral equations

$$a_k^{(1)}(t) + \beta_k \int_0^t a_k^{(1)}(\tau)\ F(t,\ \tau)\ d\tau = a_k^{(1)}(0), \tag{11}$$

$$a_{2k}^{(0)}(t) + \beta_{2k} \int_0^t a_{2k}^{(0)}(\tau)\ F(t,\ \tau)\ d\tau = \pi\beta_{2k}b_{2k}[\gamma(t) - \gamma(0)] \tag{12}$$

and

(13)
$$a_{2k-1}^{(0)}(t) + \beta_{2k-1} \int_0^t a_{2k-1}^{(0)}(\tau)\, F(t,\, \tau)\, d\tau = \pi\beta_{2k-1}b_{2k-1}[\beta(t) - \beta(0)],$$

where $\sum_{k=1}^{\infty} b_{2k}\psi_{2k} = 1$, $\sum_{k=1}^{\infty} b_{2k-1}\psi_{2k-1} = x$, $k \geq 1$, $0 \leq t \leq T < \infty$, $a_k^{(0)}(0) \equiv 0$ and $\beta_k = \alpha_k(1 + \mu\alpha_k)^{-1}$.

Equations (11)-(13) represent the Volterra integral equations of the second kind with continuous kernel, $F(t,\, \tau)$. These equations have solutions in $C(0,\, T)$ (see [15]). The constant $a_k^{(1)}(0)$, $k \geq 1$, can be obtained from Eqs.(7) and (4). So the general solution of Eqs.(1)-(3) can be adapted in the form

$$\psi(x,\, t) = \sum_{k=1}^{\infty} \left[a_k^{(0)}(t) + a_k^{(1)}(t)\right] \psi_k(x). \tag{14}$$

Lemma 1. [2] *Let $\{x_n\}$ be a countably infinite orthonormal set in a Hilbert space. Then the infinite series $\sum_{n=1}^{\infty} \alpha_n\, x_n$ (where α_n's are scalars) converges uniformly if and only if the series of real numbers $\sum_{n=1}^{\infty} |\alpha_n|^2$ converges.*

By the above lemma, the series (14) is uniformly convergent in $L_2(-1,1)$ for $t \in [0,T]$, $T > 0$ because the series $\sum_{k=1}^{n} \left[a_k^{(0)}(t) + a_k^{(1)}(t)\right]^2$ converges as $n \to \infty$.

Theorem 2. *The solution of the integral equations (1)-(2), where $\gamma(t)$, $\beta(t) \in C(0,\, T)$, is given by a uniformly convergent series in the class $l_2(-1,\, 1) \times C(0,\, T)$.*

Now, if we write

$$\gamma(t) = \gamma_0(t) + \gamma_1(t), \quad \beta(t) = \beta_0(t) + \beta_1(t), \tag{15}$$

the integral equations (8) become

$$\begin{gathered}(16)\ \mu[\psi_j(x,\, t) - \psi_j(x,\, 0)] + \int_{-1}^{1} [\psi_j(\xi,\, t) - \psi_j(\xi,\, 0)]k(\xi, x)\, d\xi + \\ + \int_0^t \psi_j(x,\tau)F(t,\tau)\, d\tau \;= \pi[\gamma_j(t) + \beta_j(t)x - \gamma_j(0) - \beta_j(0)x], \\ j = 0,\, 1,\, |x| \leq 1,\;\; 0 \leq t \leq T < \infty.\end{gathered}$$

Let j=0 in Eq.(16), then for the values of $\gamma_0(t)$ and $\beta_0(t)$, Eq.(16) has a solution $\psi_0(x,\ t) = \psi_0(x)$, which is independent of t. If

$$(17)\quad \int_0^t F(t,\ \tau)\,d\tau = \gamma_0(t) - \gamma_0(0),\ \int_0^t F(t,\ \tau)\,d\tau = \beta_0(t) - \beta_0(0),$$

then, we have

$$\psi_0(x) = \psi(A + Bx), \tag{18}$$

where A and B are constants. Also, if we write $\gamma_1(t)$ and $\beta_1(t)$ in the form

$$\gamma_1(t) = \sum_{k=1}^{\infty} \gamma_{2k}\, a_{2k}(t), \quad \beta_1(t) = \sum_{k=1}^{\infty} \gamma_{2k-1}\, a_{2k-1}(t), \tag{19}$$

the integral equation (16) takes the form, $j = 1$,

$$\alpha_n \int_{-1}^{1} k(\xi,\ x)\, h_n(\xi)\, d\xi - h_n(x) = f_n{}'(x),\ |x| \le 1,\ n \ge 1, \tag{20}$$

where

$$h_n(x) = \frac{1}{\pi\gamma_n\alpha_n}\ \psi_n(x)\ \ and\ \ f_n{}'(x) = \begin{cases} 1, & n = 2m, \\ x, & n = 2m-1. \end{cases} \tag{21}$$

under the conditions

$$\begin{aligned} P(t) &= P_0 + \pi \sum_{k=1}^{\infty} P_k \gamma_{2k} \alpha_{2k} a_{2k}(t), \qquad P_0 = 2\pi D, \\ M(t) &= M_0 + \pi \sum_{k=1}^{\infty} M_k \gamma_{2k-1} \alpha_{2k-1} a_{2k-1}(t), \quad M_0 = \pi C, \\ P_k &= \int_{-1}^{1} h_{2k}(x)\,dx = 0,\ \ M_k = \int_{-1}^{1} x h_{2k-1}(x)\,dx = 0, \qquad k \ge 1. \end{aligned} \tag{22}$$

4. Fredholm integral equation

Now, our attention comes to obtain the solution of Fredholm integral equation of the second kind (6). In [1], the solution of Eq.(6) is obtained by using Legendre polynomials. Rewrite Eq.(6) in the form

$$\phi(x) + \lambda \int_{-1}^{1} k(x,y)\,\phi(y)\,dy = g(x),\ |x| \le 1. \tag{23}$$

where $\phi = \mu\psi'$, $g = \pi\ f'$ and $\lambda = \frac{1}{\mu}$; $\mu \neq 0$. The integral term of Eq.(23) can be written in the form

$$\int_{-1}^{1} k(x,y)\ \phi(y)\ dy = \sum_{n=-N}^{N-c} \int_{nh}^{(n+c)h} k(x,y)\ \phi(y)\ dy, \tag{24}$$

where $n = -N,\ -N+1,\ \ldots,\ N-c$; $c = 1,\ 2,\ \ldots,\ N-1$ and $h = \frac{1}{N}$. Abdou and others, [13], obtained the solution of Eq.(23) by using Toeplitz matrix method in the case $c = 1$. Here, we use the same method with $c = 2$. In this case the integral equation (23) takes the form

$$\phi(x) + \lambda \sum_{n=-N}^{N-2} \int_{nh}^{(n+2)h} k(x,y)\ \phi(y)\ dy = g(x),\ |x| \leq 1. \tag{25}$$

The integral of the right hand side of Eq.(24) may be written in the form

$$\int_{a}^{a+2h} k(x,y)\ \phi(y)\ dy = A(x)\ \phi(a) + B(x)\ \phi(a+h) + C(x)\phi(a+2h) + R, \tag{26}$$

where $A(x)$, $B(x)$ and $C(x)$ are arbitrary functions to be determined, R is the estimated error of order $O(h^4)$, and $a = nh$. In order to determine $A(x)$, $B(x)$ and $C(x)$, we put $\phi(x) = 1$, $\phi(x) = x$ and $\phi(x) = x^2$ and solve the three resultant equations, $R = 0$. Then we have

$$A(x) = \frac{1}{2h^2}[(a^2 + 3ah + 2h^2)\ I_1(x) - (2a+3h)\ I_2(x) + I_3(x)], \tag{27}$$

$$B(x) = \frac{1}{h^2}[-a(a+2h)\ I_1(x) + 2(2+h)\ I_2(x) - I_3(x)], \tag{28}$$

$$C(x) = \frac{1}{2h^2}[a(a+h)\ I_1(x) - (2a+h)\ I_2(x) + I_3(x)], \tag{29}$$

where $I_1(x)$, $I_2(x)$ and $I_3(x)$ are given by

$$I_1(x) = \int_{a}^{a+2h} k(x,y)\ dy, \tag{30}$$

$$I_2(x) = \int_{a}^{a+2h} k(x,y)\ y\ dy, \tag{31}$$

$$I_3(x) = \int_a^{a+2h} k(x,y)\, y^2\, dy. \tag{32}$$

By putting $x = mh$ and $a = nh$, $-N \leq m \leq N$, $-N \leq n \leq N-1$, in (26). Then (23) takes the form

$$\Phi(mh) + \lambda\, a_{n,m}\, \Phi(nh) = G(mh), \tag{33}$$

Φ is a vector of $2N+2$ elements. Here, the matrix $a_{n,m}$ is defined as follows

$$a_{n,m} = \begin{cases} A_N & n = -N \\ A_{-N+1} + B_N & n = -N+1 \\ A_n + B_{n-1} + C_{n-2} & -N < n < N \\ B_{N-1} + C_{N-2} & n = N \\ C_{N-1} & n = N+1. \end{cases} \tag{34}$$

The square matrix $a_{n,m}$ can be written in the form

$$a_{n,m} = a'_{n,m} + \begin{pmatrix} d_{-N,-N} & d_{-N,-N+1} & \cdots & d_{-N,N+1} \\ e_{-N+1,-N} & e_{-N+1,-N+1} & \cdots & e_{-N+1,N+1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ r_{N,-N} & r_{N,-N+1} & \cdots & r_{N,N+1} \\ s_{N+1,-N} & s_{N+1,-N+1} & \cdots & s_{N+1,N+1} \end{pmatrix} \tag{35}$$

The matrix $a'_{n,m}$ is the Toeplitz matrix of order $2N+2$, where $-N \leq n,m \leq N+1$, and the elements of the second matrix are zeros except the elements of the first two rows and last two rows. Substituting from Eq.(35) about $a_{n,m}$ into Eq.(34), we obtain

$$\Phi(nh) + \lambda \left[a'_{n,m} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \right. \tag{36}$$

$$\left. \cdot \begin{pmatrix} d_{-N,-N} & d_{-N,-N+1} & \cdots & d_{-N,N+1} \\ e_{-N+1,-N} & e_{-N+1,-N+1} & \cdots & e_{-N+1,N+1} \\ r_{N,-N} & r_{N,-N+1} & \cdots & r_{N,N+1} \\ s_{N+1,-N} & s_{N+1,-N+1} & \cdots & s_{N+1,N+1} \end{pmatrix} \right] \Phi(nh) = G(nh).$$

Formula (36) can be written in the form

$$b_{n,m}\Phi(nh) + \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \tag{37}$$

$$\cdot \begin{pmatrix} d_{-N,-N} & d_{-N,-N+1} & \cdots & d_{-N,N+1} \\ e_{-N+1,-N} & e_{-N+1,-N+1} & \cdots & e_{-N+1,N+1} \\ r_{N,-N} & r_{N,-N+1} & \cdots & r_{N,N+1} \\ s_{N+1,-N} & s_{N+1,-N+1} & \cdots & s_{N+1,N+1} \end{pmatrix} \Phi(nh) = G(nh),$$

where $b_{n,m} = I + \lambda\, a'_{n,m}$, I is the identity matrix of order $2N+2$. Finally, we have

$$\Phi = Z^{-1}.G(nh), \tag{38}$$

where

$$Z = (b_{n,m} + \lambda\, M)\,,\ |b_{n,m} + \lambda\, M| \neq 0, \tag{39}$$

and

(40)

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_{-N,-N} & d_{-N,-N+1} & \cdots & d_{-N,N+1} \\ e_{-N+1,-N} & e_{-N+1,-N+1} & \cdots & e_{-N+1,N+1} \\ r_{N,-N} & r_{N,-N+1} & \cdots & r_{N,N+1} \\ s_{N+1,-N} & s_{N+1,-N+1} & \cdots & s_{N+1,N+1} \end{pmatrix}.$$

The error estimate is determined from Eq.(26) by letting $\phi(x) = x^3$,

$$R = \mid \int_a^{a+2h} k(x,y)\ \phi(y)\ dy - (A(x)\phi(a) - B(x)\phi(a+h) - C(x)\phi(a+2h)\ \mid \leq \beta h^4, \tag{41}$$

where β is a constant.

5. Examples

Here, some numerical examples are given with different values of N. We consider the kernel $k(x,y) = |x-y|^{-\nu}$, $\nu = 0.5$. The error is given in Tab.1 (see Fig. 1), with $N = 6$ and in Tab.2 (see Fig. 2) with $N = 8$.

x	-1	-0.5	0	0.5	1
R	0.228	0.087	0.008	6.726^{-6}	1.249^{-6}

Tab.1

x	-1	-0.75	-0.5	-0.25	0.25	0.5	0.75	1
R	0.087	0.047	0.021	0.007	$5.921.10^{-6}$	$7.271.10^{-7}$	$3.312.10^{-7}$	$1.989.10^{-7}$

Tab.2

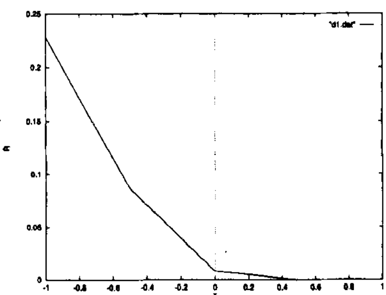

Fig.1 Errors estimate where $\nu = 0.5$ and $N = 6$

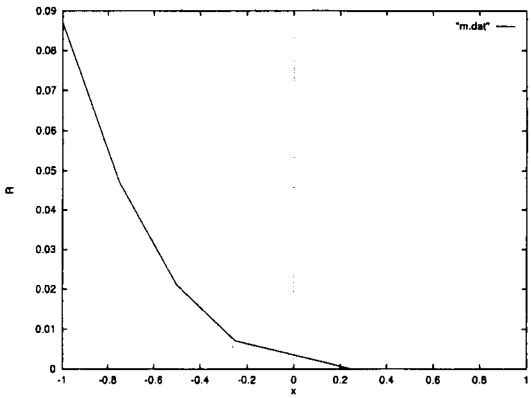

Fig.2 Errors estimate where $\nu = 0.5$ and $N = 8$

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Mohamed Abdalla Darwish
Mathematisches Seminar II
Universität Kiel
D24098 Kiel, Deutschland
e-mail: mdh@numerik.uni-kiel.de

Permanent address
Department of Mathematics
Faculty of Education
Damanhour branch
Alexandria University
Damanhour, Egypt