Supplemental material to

# The Crane Operator's Tricks and other Shenanigans with a Pendulum

## I. DAMPING THE TORSION BALANCE

The differential equation of the torsion balance subjected to an external torque $n(t)$ is given by

$$\ddot{\varphi}_t + \omega_0^2 \varphi_t = \frac{n(t)}{I}, \tag{1}$$

where $\omega_0^2 = \kappa/I$, $I$ is the moment of inertia of the pendulum, $\kappa$ is the torsional spring constant of the pendulum restoring force, and $\phi$ is the angular deflection. In the Laplace domain, it is

$$\frac{\Phi_t(s)}{N(s)} = \frac{1}{I}\frac{1}{s^2+\omega_0^2}. \tag{2}$$

The torque acting on the pendulum is proportional to $\sin\big(\left(\varphi_s - \varphi_t\right)/\varphi_{\mathrm{norm}}\big)$.

The torque does not change linearly but rather proportional to a cosine. To calculate the response, we need to combine three functions with different time shifts. They are

$$f_1(t) = u(t), \tag{3}$$

$$f_2(t) = u(t)\cos\left(\frac{t}{\tau}\frac{\pi}{2}\right), \text{ and} \tag{4}$$

$$f_3(t) = u(t)\sin\left(\frac{t}{\tau}\frac{\pi}{2}\right), \tag{5}$$

where $u(t)$ denotes the Heaviside step function, and $\tau$ is the duration of one move.

The external torque in the time domain with a total amplitude of $n_a$ and the moves starting at $t_1$ and $t_3$ is

$$n(t) = \frac{n_a}{2}\Big(f_1(t-t_1) - f_2(t-t_1) - f_3(t-t_2) + f_3(t-t_3) - f_2(t-t_4) + f_1(t-t_4)\Big). \tag{6}$$

The first move starts at $t_1$ and is completed at $t_2 = t_1 + \delta t$. The second move starts at $t_3$ and ends at $t_4 = t_3 + \delta t$. Consistent with the main text, the duration of the move is abbreviated by $\delta t$.

Figure 1 shows the torque for $n_a = 1 \times 10^{-8}\,\mathrm{N\,m}$, $t_1 = 30\,\mathrm{s}$, $t_3 = 70\,\mathrm{s}$, and $\delta t = 19.2\,\mathrm{s}$.

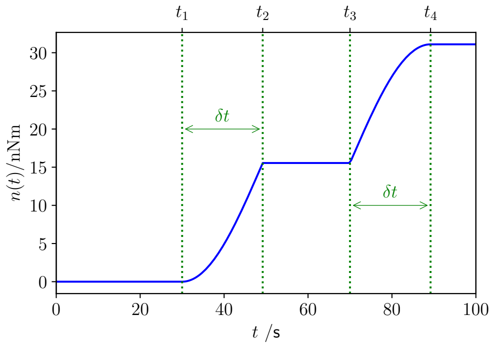


FIG. 1. The torque as a function of time, according to equation 6.

In the $s$ domain using the abbreviation $\nu = \pi/(2\tau)$, the torque is given by

$$N(s) = \frac{n_A}{2}\left(e^{-st_1}\frac{1}{s} - e^{-st_1}\frac{s}{s^2+\nu^2} - e^{-st_2}\frac{\nu}{s^2+\nu^2} + e^{-st_3}\frac{\nu}{s^2+\nu^2} - e^{-st_4}\frac{s}{s^2+\nu^2} + e^{-st_4}\frac{1}{s}\right) \quad (7)$$

Including the unit pulse that makes the pendulum swing at $t = 0$ through the equilibrium position with $\dot{\phi}(0) = v_0$ and multiplying with

$$\frac{\Phi_t(s)}{N(s)} = \frac{1}{I}\frac{1}{s^2+\omega_o^2}. \quad (8)$$

yields,

$$\begin{aligned}\Phi_t(s) = \frac{v_0}{s^2+\omega_o^2} + \frac{n_A}{2I}\Bigg[ & e^{-st_1}\left(\frac{1}{\omega_0^2 s} - \frac{s}{\omega_o^2\left(\omega_o^2+s^2\right)}\right) + e^{-st_4}\left(\frac{1}{\omega_0^2 s} - \frac{s}{\omega_o^2\left(\omega_o^2+s^2\right)}\right)\\
& - e^{-st_1}\left(\frac{s}{(\nu^2+\omega_o^2)(\omega_o^2+s^2)} - \frac{s}{(\nu^2+\omega_o^2)(\nu^2+s^2)}\right)\\
& - e^{-st_4}\left(\frac{s}{(\nu^2+\omega_o^2)(\omega_o^2+s^2)} - \frac{s}{(\nu^2+\omega_o^2)(\nu^2+s^2)}\right)\\
& - e^{-st_2}\left(\frac{\nu}{(\nu^2-\omega_o^2)(\omega_o^2+s^2)} - \frac{\nu}{(\nu^2-\omega_o^2)(\nu^2+s^2)}\right)\\
& + e^{-st_3}\left(\frac{\nu}{(\nu^2-\omega_o^2)(\omega_o^2+s^2)} - \frac{\nu}{(\nu^2-\omega_o^2)(\nu^2+s^2)}\right)\Bigg]\end{aligned} \quad (9)$$

Transforming back to the time domain yields for $t > t_4$, we obtain

$$\begin{aligned}\varphi_t(t) = v_0\sin(\omega_o t) + \frac{n_A}{2I}\Bigg[ & \frac{2}{\omega_o^2} - \frac{1}{\omega_o^2}\cos(\omega_o(t-t_1)) - \frac{1}{\omega_o^2}\cos(\omega_o(t-t_4))\\
& - \frac{1}{\nu^2+\omega_o^2}\cos(\omega_o(t-t_1)) + \frac{1}{\nu^2+\omega_o^2}\cos(\nu(t-t_1)) - \frac{1}{\nu^2+\omega_o^2}\cos(\omega_o(t-t_4))\\
& + \frac{1}{\nu^2+\omega_o^2}\cos(\nu(t-t_4)) - \frac{\nu}{\omega_o}\frac{1}{\nu^2-\omega_o^2}\sin(\omega_o(t-t_2)) + \frac{1}{\nu^2-\omega_o^2}\sin(\nu(t-t_2))\\
& + \frac{\nu}{\omega_o}\frac{1}{\nu^2-\omega_o^2}\sin(\omega_o(t-t_3)) - \frac{1}{\nu^2-\omega_o^2}\sin(\nu(t-t_3))\Bigg]\end{aligned} \quad (10)$$

From this equation, the time derivative is calculated. The trigonometric functions are expanded and sorted into the sine and cosine terms. We obtain,

$$\begin{aligned}
\dot{\varphi}_t(t) &= C\cos(\omega_o t) + S\sin(\omega_o t) \quad \text{with} \\
C &= v_0 - \frac{n_a\pi\omega_o\Big(-2\omega_o\tau\cos(\omega_o t_3) + 2\omega_o\tau\cos(\omega_o t_2) + \pi\sin(\omega_o t_1) + \pi\sin(\omega_o t_4)\Big)}{2\kappa(\pi^2 - 4\omega_o^2\delta t^2)} \\
S &= \frac{n_a\pi\omega_o\Big(\pi\cos(\omega_o t_1) + \pi\cos(\omega_o t_4) + 2\omega_o\delta t\sin(\omega_o t_3) - 2\omega_o\delta t\sin(\omega_o t_2)\Big)}{2\kappa(\pi^2 - 4\omega_o^2\delta t^2)}.
\end{aligned} \tag{11}$$

To damp the pendulum to the desired velocity amplitude $v_{\mathrm{des}}$, we minimize the term

$$\left|\sqrt{C^2 + S^2} - v_{\mathrm{des}}\right| \tag{12}$$

as a function of $t_1$ and $t_3$ for given $\delta t$ numerically.

## II. SIMULATION OF THE PENDULUM IN PYTHON

The differential equation for the simple pendulum is

$$m\ddot{x}_m + c\dot{x}_m + \frac{mg}{l}(x_m - x_c) = 0. \tag{13}$$

All variables are defined according to the main paper. With $\omega_o^2 = g/l$ and $\xi = c/(2m\omega_o)$, the equation above can be rewritten as

$$\ddot{x}_m = -2\xi\omega_o\dot{x}_m - \omega_o^2 x_m + \omega_o^2 x_c. \tag{14}$$

A second order differential equation can be reduced to a first order differential equation by introducing the variable

$$x = \begin{pmatrix} x_m \\ \dot{x}_m \end{pmatrix} \tag{15}$$

to obtain

$$\begin{pmatrix} \dot{x}_m \\ \ddot{x}_m \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega_o^2 & -2\xi\omega_o \end{pmatrix}}_{M} \begin{pmatrix} x_m \\ \dot{x}_m \end{pmatrix} + \omega^2 \underbrace{\begin{pmatrix} 0 \\ x_c \end{pmatrix}}_{F} \tag{16}$$

In other words,

$$\dot{x} = Mx + \omega^2 F. \tag{17}$$

This first order differential equation is solved using the Runge-Kutta method.