Homework 22 Solutions

1. (Based on *Stewart 11.8 #6*) Consider the function $f(x,y) = e^{xy}$, and the constraint $x^3 + y^3 = 16$.

(a) Use Lagrange multipliers to find the coordinates (x, y) of any points on the constraint where the function f could attain a maximum or minimum.

Solution: We wish to find the extreme values of the function $f(x,y) = e^{xy}$ subject to the constraint $g(x,y) = x^3 + y^3 = 16$. Lagrange multipliers tells us that $\nabla f = \lambda \nabla g$ or $\langle ye^{xy}, xe^{xy}\rangle = \langle 3\lambda x^2, 3\lambda y^2\rangle$, and so we get the system

$$\begin{aligned} ye^{xy} &= 3\lambda x^2 \\ xe^{xy} &= 3\lambda y^2 \\ x^3 + y^3 &= 16. \end{aligned}$$

Note that if either x or y is zero, then $x = y = 0$, which contradicts $x^3 + y^3 = 16$, so we can assume that $x \neq 0$ and $y \neq 0$. Then

$$\lambda = \frac{ye^{xy}}{3x^2} = \frac{xe^{xy}}{3y^2},$$

from which we get $x^3 = y^3$ and so $x = y$. Since $x^3 + y^3 = 16$, we get $2x^3 = 16$ or $x = y = 2$. Thus the point $(2, 2)$ is a point on the constraint where the function f could attain a maximum or minimum.

(b) For each point you found in part (a), is the point a maximum, a minimum, both or neither? Explain your answer carefully. What are the minimum and maximum values of f on the constraint? Please explain your answers carefully.

Solution: We can see that there is no minimum value, since we can choose points satisfying the constraint $x^3+y^3 = 16$ that make $f(x,y) = e^{xy}$ arbitrarily close to 0 (but never equal 0). More specifically, if x is given, then $y = \sqrt[3]{16 - x^3}$. If x grows positive without bound ($x \to +\infty$), then y grows *negative* without bound ($y \to -\infty$) and so e^{xy} approaches zero ($xy \to -\infty$ and so $e^{xy} \to 0$). So there is *no* minimum value of f and the maximum value is $f(2,2) = e^4$.

(c) The extreme value theorem, which we covered last week, guarantees that under the right circumstances, we are guaranteed to find absolute minima and maxima for a function f on a certain constraint. Please explain why parts (a) and (b) don't violate the extreme value theorem.

Solution: The extreme value theorem states that if a real-valued function f is continuous in a closed and bounded region, then f must attain its maximum and minimum value, each at least once. This isn't satisfied when considering the lower bound, since the constraint approaches 0 but never reaches it. So we haven't broken any rules of the extreme value theorem.

In Figure 1, we show the constraint $g(x,y) = 16$ and a number of level curves $f(x,y)$. These level curves are all

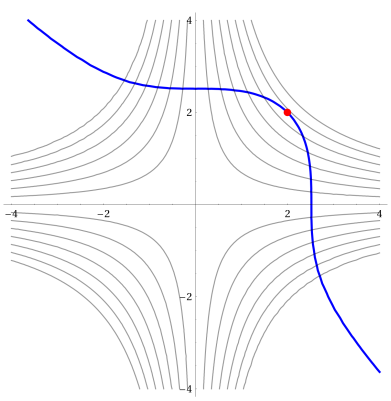


Figure 1: The constraint and level curves for Problem 1

powers of 2. In the first and third quadrant they are positive powers, increasing as we come out from the origin. In the second and fourth quadrants they are negative powers of 2, decreasing as we come out from the origin. The maximum is shown as a red dot; the value $f(2,2) = e^4 \approx 54.6$ puts it between the level curves $f = 32$ and $f = 64$.

2. (*Stewart 11.8 #10*) Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = x^2y^2z^2$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

Solution: We wish to find the extreme values of the function $f(x, y, z) = x^2y^2z^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 1$. Lagrange multipliers tells us that $\nabla f = \lambda \nabla g$ or $\langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2 \rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$, and so we get the system

$$\begin{aligned} 2xy^2z^2 &= 2\lambda x \\ 2yx^2z^2 &= 2\lambda y \\ 2zx^2y^2 &= 2\lambda z \\ x^2 + y^2 + z^2 &= 1 \end{aligned}$$

Let's consider two cases: $\lambda = 0$ or $\lambda \neq 0$.

If $\lambda = 0$, then the product of x, y and z is zero. This means that one, two or all three of these variables are zero. It can't be all three because of the constraint equation, so we end up with the points

$$(x, y, z) = (0, 0, \pm 1),\ (0, \pm 1, 0),\ (\pm 1, 0, 0),\ \left(0, \pm\tfrac{1}{\sqrt{2}}, \pm\tfrac{1}{\sqrt{2}}\right)\ \left(\pm\tfrac{1}{\sqrt{2}}, 0, \pm\tfrac{1}{\sqrt{2}}\right)\ \text{and}\ \left(\pm\tfrac{1}{\sqrt{2}}, \pm\tfrac{1}{\sqrt{2}}, 0\right).$$

At all of these points, the value of $f(x, y, z) = x^2y^2z^2$ is zero. (Note that this is the absolute minimum, because f can never be negative.)

On the other hand, if $\lambda \neq 0$, then none of x, y and z are zero. Solving for λ in the first three equations gives us $\lambda = y^2z^2 = x^2z^2 = x^2y^2$. In this case we find that $x^2 = y^2 = z^2 = \dfrac{1}{3}$. Thus the maximum value of f is $\dfrac{1}{27}$ at these eight points

$$(x, y, z) = \left(\pm\tfrac{1}{\sqrt{3}}, \pm\tfrac{1}{\sqrt{3}}, \pm\tfrac{1}{\sqrt{3}}\right).$$

3. (*Stewart 11.8 #18*) Find the extreme values of the function $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ on the (closed, bounded) region described by the inequality $x^2 + y^2 \leq 16$.

Solution: We need to use Lagrange multipliers for the boundary and partial derivatives to determine critical points from the interior. Let's consider the interior first. The gradient of $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ is $\nabla f = \langle 4x - 4, 6y \rangle = \langle 0, 0 \rangle$, which vanishes only at $(x, y) = (1, 0)$. This critical point of f lies in the region $x^2 + y^2 < 16$, and at this point the value of f is $f(1, 0) = -7$.

On the boundary, we find the extreme values of $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ subject to the constraint $g(x, y) = x^2 + y^2 = 16$ using Lagrange multipliers. Then $\nabla f = \lambda \nabla g$ gives us the system

$$\begin{aligned} 4x - 4 &= 2\lambda x, \\ 6y &= 2\lambda y \\ x^2 + y^2 &= 16. \end{aligned}$$

From the second equation, we have either $y = 0$ or $y \neq 0$. If $y = 0$, then $x = \pm 4$ from the constraint. If $y \neq 0$, then the second equation gives $\lambda = 3$. The first equation is then $4x - 4 = 2 \cdot 3x$, from which we find $x = -2$ and (from the constraint) $y = \pm 2\sqrt{3}$. Thus on the boundary, the extreme points must happen at (some subset of) the points $(\pm 4, 0)$ and $(-2, \pm 2\sqrt{3})$.

Now we compare the value of f at each of the points we've found in the previous two paragraphs. It's easy if we make small table:

Point (x, y)	$f(x, y) = 2x^2 + 3y^2 - 4x - 5$
$(1, 0)$	7
$(4, 0)$	11
$(-4, 0)$	43
$(-2, 2\sqrt{3})$	47
$(-2, -2\sqrt{3})$	47

.

Thus the maximum value of $f(x, y)$ on the disk $x^2 + y^2 \leq 16$ is $f(-2, \pm 2\sqrt{3}) = 47$, and the minimum value is $f(1, 0) = -7$.

4. (*Stewart 11.8 #38*) Use Lagrange multipliers to solve the following problem. The base of an aquarium with given volume V is made of slate and the sides are made of glass. (There is no top on the aquarium.) If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of the materials. Your answers will be given in terms of V.

Solution: We can write the cost function as $C(x, y, z) = 5xy + 2xz + 2yz$ and our constraint as $g(x, y, z) = xyz = V$. Since $\nabla C = \langle 5y + 2z, 5x + 2z, 2x + 2y\rangle$ and $\nabla g = \langle yz, xz, xy\rangle$, the equation $\nabla f = \lambda \nabla g$ becomes

$$\begin{aligned} 5y + 2z &= \lambda yz, && (1)\\ 5x + 2z &= \lambda xz, && (2)\\ 2x + 2y &= \lambda xy, && (3)\\ xyz &= V && (4) \end{aligned}$$

Now subtracting equation (2) from equation (1) gives us $\lambda z(y - x) = 5(y - x)$. So either $x = y$ or $x - y \neq 0$, in which case $\lambda z = 5$. This second case is impossible: if $\lambda z = 5$, then $z \neq 0$ so we can solve for λ as $\lambda = \frac{5}{z}$. Plugging this into (1) gives $5y + 2z = 5y$, or $z = 0$. But we've already seen that z *can't* be zero, since $\lambda z = 5$. So we must have $x = y$.

Since $x = y$, equations 1 and (2) are the same, and we can write the above system as

$$\begin{aligned} 5y + 2z &= \lambda yz, && (1')\\ 4y &= \lambda y^2, && (3')\\ y^2 z &= V && (4') \end{aligned}$$

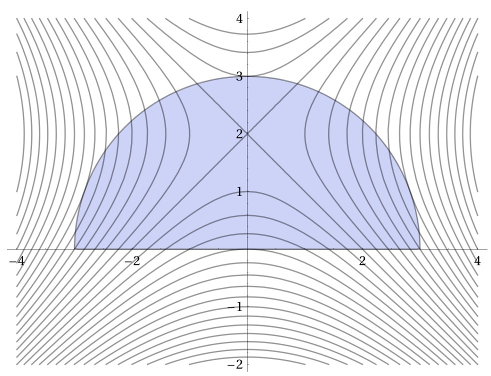
From equation ($3'$), we see that either $y = 0$ (which it can't, since $y^2 z = V$) or $\lambda y = 4$. Plugging this last equation into ($1'$), we get $5y + 2z = 4z$, or $5y = 2z$. Hence $z = \frac{5}{2}y$, from which equation ($4'$) becomes $\frac{5}{2}y^3 = V$. Thus the dimensions which minimize cost are $x = y = \sqrt[3]{\frac{2}{5}V}$ units, $z = \frac{5}{2}\sqrt[3]{\frac{2}{5}V} = V^{1/3}\left(\frac{5}{2}\right)^{2/3}$ units.

5. Let $\mathcal{R}$ be the (closed and bounded) region described by $\{x^2 + y^2 \leq 9 \text{ and } y \geq 0\}$.

(a) Use Mathematica as follows to plot the region $\mathcal{R}$ and the level sets of f together:

```
region = RegionPlot[x^2+y^2 <= 9 && y >= 0, {x,-4,4}, {y,-2,4}, AspectRatio->3/4]
levels = ContourPlot[x^2-(y-2)^2, {x,-4,4}, {y,-2,4}, ContourShading->False, Contours->Range[-20,20]]
Show[region,levels]
```

Solution: Here's a Mathematica sketch with the additional options `Axes -> True, Frame -> False`.



(b) Find the absolute minimum and maximum values of the function $f(x, y) = x^2 - (y - 2)^2$ on the region $\mathcal{R}$, and the points at which those extrema occur. Please explain carefully what you are doing in your solution.

Solution: We need to use Lagrange multipliers for the boundary and partial derivatives to determine critical points from the interior. Let's consider the interior first. The gradient of $f(x, y) = x^2 - (y - 2)^2$ is $\nabla f = \langle 2x, -2(y - 2)\rangle$, which is the zero vector $\mathbf{0}$ only when $x = 0$ and $y = 2$. Thus $(x, y) = (0, 2)$ is the only critical point of f.

Now let's evaluate the function at the boundary. We wish to extremize the function $f(x, y) = x^2 - (y-2)^2$ subject to one of two constraints: $g(x, y) = x^2 + y^2 = 9$ (for the semicircle) and $h(x, y) = y = 0$ (for the base of the half-disc). Thus we really have two Lagrange multipliers problems: $\nabla f = \lambda \nabla g$ and $\nabla f = \lambda \nabla h$.
We'll start with the first Lagrange multipliers problem. We get the system

$$2x = 2\lambda x, \tag{5}$$
$$-2(y-2) = 2\lambda y, \tag{6}$$
$$x^2 + y^2 = 9, \qquad (y \geq 0) \tag{7}$$

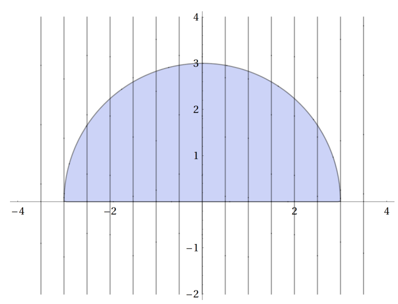
Let's consider the individual cases. From (5), either $x = 0$ (in which case $y = 3$ from the boundary, not -3) or $x \neq 0$. If $x \neq 0$, then $\lambda = 1$. Plugging this into equation (6), we get $-2y + 4 = 2y$ or $y = 1$. From equation (7), this gives two points $(x, y) = (\pm 2\sqrt{2}, 1)$. Thus this Lagrange multipliers problem has given us three candidate points: $(x, y) = (0, 3)$ and $(\pm 2\sqrt{2}, 1)$.
The other Lagrange multipliers problem involves solving $\nabla f = \lambda \nabla h$, which is $\langle 2x, -2(y-2) \rangle = \lambda \langle 0, 1 \rangle$. This becomes the system of equations

$$2x = 0, \tag{8}$$
$$-2(y-2) = \lambda, \tag{9}$$
$$y = 0, \qquad (y \geq 0) \tag{10}$$

Clearly equations (8) and (10) give us only the point $(x, y) = (0, 0)$.
Now we evaluate all the points:

point (x, y)	$f(x, y)$	Classification
$(0, 2)$	0	
$(\pm 2\sqrt{2}, 1)$	7	maximum
$(0, 3)$	-1	
$(0, 0)$	-4	minimum
$(\pm 3, 0)$	5	

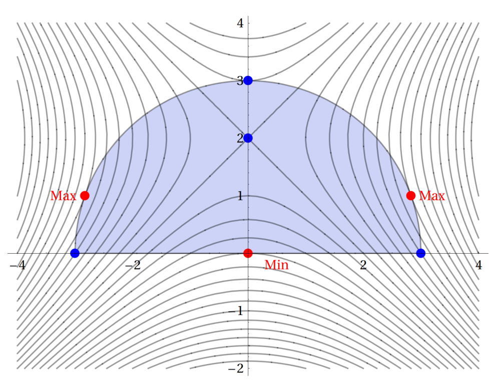
Notice that we've also included the points $(x, y) = (\pm 3, 0)$, since these are the endpoints of our boundary. You could easily come up with examples where the maximum and minimum values of a function occurs only at the endpoints of our boundaries $g(x, y)$ =constant and $h(x, y)$ =constant. One simple example is $f(x, y) = x$, which gives us this picture:



The region $\mathcal{R}$ with contours of $f(x, y) = x$

(c) On your printout from part (a), indicate where the minima and maxima are located – just draw them in.

Solution: Here we've drawn (and labeled) the minima and maxima in red. In blue we've indicated the other points we checked as well:



6. (**Optional:** 4 extra credit points) Let $\mathcal{E}$ be the (closed and bounded) solid region described by $\{x^2 + y^2 + z^2 \le 9 \text{ and } z \ge 0\}$. Find the absolute minimum and maximum of the function $f(x, y, z) = x^2 - y^2 + (z - 2)^2$ on the region $\mathcal{E}$. Please explain your solution very carefully.

Solution: Here we use the same process as Problem 5 to solve this. That is, we find the critical points on the interior and use Lagrange multipliers on the boundary to find additional points. In the previous problem we included two additional points because they were the edges of our boundaries. This time, we'll have to worry about the "edge" circle $x^2 + y^2 = 9$ when $z = 0$.

Let's find the critical points in the interior first. The gradient of $f(x, y, z) = x^2 - y^2 + (z - 2)^2$ is $\nabla f = \langle 2x, -2y, (2z - 2)\rangle$, which equals $\mathbf{0} = \langle 0, 0, 0\rangle$ only when $x = 0$, $y = 0$ and $z = 2$. So $(x, y, z) = (0, 0, 2)$ is the only critical point of f.

Again the boundary is two Lagrange multipliers problems, one with constraint $g(x, y, z) = x^2 + y^2 + z^2 = 9$ (with $z \ge 0$) and the other with constraint $h(x, y, z) = z = 0$ (with $x^2 + y^2 \le 9$). (In addition, we must also deal with the common edge of these boundaries: $x^2 + y^2 = 9$ and $z = 0$.)

Let's start with the first Lagrange multiplier problem: extremize $f(x, y, z) = x^2 - y^2 + (z - 2)^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 9$ (with $z \ge 0$). The system $\nabla f = \lambda \nabla g$ becomes

$$2x = 2\lambda x, \tag{11}$$
$$-2y = 2\lambda y, \tag{12}$$
$$2z - 4 = 2\lambda z, \tag{13}$$
$$x^2 + y^2 + z^2 = 9, \qquad (z \ge 0) \tag{14}$$

Let's consider two cases: $x = 0$ and $x \ne 0$. If $x \ne 0$, then (11) implies $\lambda = 1$. Then from (13) we get $-4 = 0$, a contradiction. So x must equal zero.

Now we turn to two cases in (12): $y = 0$ and $y \ne 0$. If $y \ne 0$, then $\lambda = -1$ and (13) becomes $2z - 4 = -2z$, from which we get $z = 1$. Thus the last equation becomes $0^2 + y^2 + 1^2 = 9$ and we find $y = \pm 2\sqrt{2}$. On the other hand, if $y = 0$, then our constraint tells us $z = 3$ (not -3, as $z \ge 0$). Thus we get the points $(x, y, z) = (0, \pm 2\sqrt{2}, 1)$ and $(0, 0, 3)$ from this Lagrange multiplier problem.

The second Lagrange multiplier problem involves extremizing $f(x, y, z) = x^2 - y^2 + (z - 2)^2$ subject to the constraint $h(x, y, z) = z = 0$ (with $x^2 + y^2 \le 9$). The system $\nabla f = \lambda \nabla h$ becomes

$$2x = \lambda \cdot 0, \tag{15}$$
$$-2x = \lambda \cdot 0, \tag{16}$$
$$2z - 4 = 2\lambda, \tag{17}$$
$$z = 0, \tag{18}$$

and we get only the point $(x, y, z) = (0, 0, 0)$.

Finally, we look at the value of the function $f(x, y, z) = x^2 - y^2 + (z - 2)^2$ on the boundary curve $x^2 + y^2 + z^2 = 9$, $z = 0$. This is a circle that we can parametrize with $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), 0\rangle$ (where $0 \le t < 2\pi$), in which case $f(\mathbf{r}(t)) = 9\cos^2(t) - 9\sin^2(t) + 4$. Single variable calculus tells us that this has critical points at every multiple of $\pi/2$, from which we get the points $(x, y, z) = (0, \pm 3, 0)$ and $(\pm 3, 0, 0)$.

At long last we evaluate f at all the points we've found:

point (x, y, z)	$f(x, y, z)$	Classification
$(0, 0, 2)$	0	
$(0, \pm 2\sqrt{2}, 1)$	-7	minimum
$(0, 0, 3)$	1	
$(0, 0, 0)$	-4	
$(0, \pm 3, 0)$	-5	
$(\pm 3, 0, 0)$	13	maximum

Thus f has a maximum of 13, which occurs at $(x, y, z) = (\pm 3, 0, 0)$, and a minimum of -7, which occurs at $(x, y, z) = (0, \pm 2\sqrt{2}, 1)$.