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## ON SEIFFERT-LIKE MEANS

ALFRED WITKOWSKI

ABSTRACT. We investigate the representation of homogeneous, symmetric means in the form

$$M(x, y) = \frac{x - y}{2f\left(\frac{x-y}{x+y}\right)}.$$

This allows for a new approach to comparing means. As an example, we provide optimal estimate of the form

$$(1 - \mu)\min(x, y) + \mu\max(x, y) \leq M(x, y) \leq (1 - \nu)\min(x, y) + \nu\max(x, y)$$

and

$$M\left(\frac{x+y}{2} - \mu\frac{x-y}{2}, \frac{x+y}{2} + \mu\frac{x-y}{2}\right) \leq N(x, y) \leq M\left(\frac{x+y}{2} - \nu\frac{x-y}{2}, \frac{x+y}{2} + \nu\frac{x-y}{2}\right)$$

for some known means.

## 1. INTRODUCTION, DEFINITIONS AND NOTATION

Looking at the two means introduced by Seiffert in [13]

$$P(x, y) = \begin{cases} \frac{x - y}{2 \arcsin \frac{x-y}{x+y}} & x \neq y \\ x & x = y \end{cases},$$

and in [14]

$$T(x, y) = \begin{cases} \frac{x - y}{2 \arctan \frac{x-y}{x+y}} & x \neq y \\ x & x = y \end{cases},$$

as well as two other Seiffert-like means involving inverse hyperbolic function: introduced in [11]

$$M(x, y) = \begin{cases} \frac{x - y}{2 \operatorname{arsinh} \frac{x-y}{x+y}} & x \neq y \\ x & x = y \end{cases},$$

and well known logarithmic mean

$$L(x, y) = \begin{cases} \frac{x - y}{2 \operatorname{artanh} \frac{x-y}{x+y}} = \frac{x - y}{\log x - \log y} & x \neq y \\ x & x = y \end{cases},$$

it begets the following questions:

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**Question 1.1.** For positive  $x, y$  and a positive function  $f : (0, 1) \rightarrow \mathbb{R}$  let

$$(1) \quad S_f(x, y) = \begin{cases} \frac{|x-y|}{2f\left(\frac{|x-y|}{x+y}\right)} & x \neq y, \\ x & x = y \end{cases}.$$

Under what assumptions on  $f$  the function  $S_f$  is a mean, i.e. satisfies  $\min(x, y) \leq S_f(x, y) \leq \max(x, y)$  for all  $x, y > 0$ ?

**Question 1.2.** What means  $M$  can be represented in the form  $S_f$ ?

The aim of this paper is to answer the two question stated above and explore the subject. In particular, we introduce a metric and an algebraic structure on the set of means. Then we show how the representation of a mean in the form  $S_f$  can be used to investigate properties of  $M$ . In particular, we offer a simple criterium for Schur convexity of  $M$  and criteria for finding optimal bounds of the form

$$(1-\mu)K(x, y) + \mu N(x, y) \leq M(x, y) \leq (1-\nu)K(x, y) + \nu N(x, y),$$

$$M\left(\frac{x+y}{2} - \mu\frac{x-y}{2}, \frac{x+y}{2} + \mu\frac{x-y}{2}\right) \leq N(x, y) \leq M\left(\frac{x+y}{2} - \nu\frac{x-y}{2}, \frac{x+y}{2} + \nu\frac{x-y}{2}\right)$$

where  $K < M < N$  are homogeneous means.

Denote by  $\mathcal{M}$  the set of functions  $M : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  satisfying the following conditions:

- (A)  $M$  is symmetric, i.e.  $M(x, y) = M(y, x)$  for all  $x, y \in \mathbb{R}_+$ .
- (B)  $M$  is positively homogeneous of order 1, i.e. for all  $\lambda > 0$  holds  $M(\lambda x, \lambda y) = \lambda M(x, y)$ .
- (C)  $M$  "lies in between", i.e.  $\min(x, y) \leq M(x, y) \leq \max(x, y)$ .

Elements of  $\mathcal{M}$  shall be called means. The set of strict means, i.e. means satisfying the condition

- (D)  $\min(x, y) < M(x, y) < \max(x, y)$  whenever  $x \neq y$ ,

will be denoted by  $\mathcal{M}^\circ$ .

By  $\mathcal{S}$  we shall denote the set of functions  $f : (0, 1) \rightarrow \mathbb{R}$  satisfying

$$(2) \quad \frac{z}{1+z} \leq f(z) \leq \frac{z}{1-z}.$$

We shall call them Seiffert functions.

By  $\mathcal{S}^\circ$  we shall denote the set of these Seiffert functions for which both inequalities in (2) are strict.

Note two important properties of Seiffert functions:

$$(3) \quad \lim_{z \rightarrow 0} f(z) = 0, \text{ and } \lim_{z \rightarrow 0} \frac{f(z)}{z} = 1.$$

The set of all real functions  $f : (0, 1) \rightarrow \mathbb{R}$  satisfying the condition  $|f(z)| \leq 1$  for all  $z$  will be denoted by  $\mathcal{B}$  and its subset with strict inequality by  $\mathcal{B}^\circ$ . The set  $\mathcal{B}$  is a complete metric space with metric  $d_{\mathcal{B}}(f, g) = \sup_z |f(z) - g(z)|$ .

## 2. ANSWERS TO THE QUESTIONS

The next theorem gives complete (and rather surprising) answer to questions stated in the previous section.

**Theorem 2.1.** *The mapping  $f \rightarrow S_f$  is a one-to-one correspondence between  $\mathcal{S}$  and  $\mathcal{M}$  that transforms  $\mathcal{S}^\circ$  onto  $\mathcal{M}^\circ$ .*

*Proof.* It is obvious that  $S_f$  is symmetric and positively homogeneous.  $S_f$  is defined for all positive  $x \neq y$ , since  $0 < \frac{|x-y|}{x+y} < 1$ . Suppose that  $x < y$  and let  $z = \frac{y-x}{x+y}$ . Then, the inequalities (2) read

$$\frac{y-x}{2y} \leq f\left(\frac{y-x}{x+y}\right) \leq \frac{y-x}{2x}.$$

This is equivalent to  $x \leq S_f(x, y) \leq y$  and shows that  $S_f$  satisfies (C). Also note that for  $f \in \mathcal{S}^\circ$  the inequalities above are strict, which means that  $S_f$  belongs to  $\mathcal{M}^\circ$ .

Conversely, for  $M \in \mathcal{M}$  we have

$$\begin{aligned} M(x, y) &= \frac{x+y}{2} M\left(\frac{x+y-(y-x)}{x+y}, \frac{x+y+(y-x)}{x+y}\right) \\ &= \frac{y-x}{2 \frac{z}{M(1+z, 1-z)}}, \end{aligned}$$

and the function

$$(4) \quad f_M(z) = \frac{z}{M(1+z, 1-z)}$$

in the denominator of the right-hand side is a Seiffert function, because  $1-z \leq M(1+z, 1-z) \leq 1+z$ . Again, if  $M$  is strict, then so is  $f_M$ .  $\square$

The next corollary follows immediately from the above proof.

**Corollary 2.1.** *For arbitrary  $M \in \mathcal{M}$  and  $f \in \mathcal{S}$  the identities*

$$f_{S_f}(z) = z, \quad M(x, y) = S_{f_M}(x, y)$$

*hold.*

Very important property of the mappings between the set of means and Seiffert functions is their antimonotonicity.

**Corollary 2.2.** *The following conditions are equivalent*

- i For all  $x, y > 0$ ,  $M(x, y) \leq N(x, y)$ ,
- ii For each  $0 < z < 1$ ,  $f_M(z) \geq f_N(z)$ .

*Also the following conditions are equivalent*

- iii For all  $x, y > 0$ ,  $x \neq y$ ,  $M(x, y) < N(x, y)$ ,
- iv For each  $0 < z < 1$ ,  $f_M(z) > f_N(z)$ .

In many cases comparing the Seiffert functions is much easier than comparing the means, so the antimonotonicity provides a new tool for proving inequalities between means.

The following corollary is a trivial consequence of Theorem 2.1.

**Corollary 2.3.** *For the means  $\min$  and  $\max$  hold*

$$(5) \quad f_{\min}(z) = \frac{z}{1-z}, \quad f_{\max}(z) = \frac{z}{1+z}.$$

The pair of mappings  $f \rightarrow S_f$  and  $M \rightarrow f_M$  establishes a one-to-one and antimonotone with respect to corresponding partial orders, correspondence between the family of homogeneous symmetric means and the family  $\mathcal{S}$ .

Let us transform (2) as follows:

$$(6) \quad \begin{aligned} \frac{1-z}{z} &\leq \frac{1}{f(z)} \leq \frac{1+z}{z} \\ -1 &\leq \frac{1}{f(z)} - \frac{1}{z} \leq 1. \end{aligned}$$

This enables us to define a metric on the set of Seiffert functions by

$$(7) \quad d_S(f, g) = \sup_{0 < z < 1} \left| \frac{1}{f(z)} - \frac{1}{g(z)} \right|.$$

**Lemma 2.1.** *The space  $(\mathcal{S}, d_S)$  has the following properties*

- (1)  $(\mathcal{S}, d_S)$  is a complete metric space and  $\text{diam } \mathcal{S} = 2$ ,
- (2)  $(\mathcal{S}, d_S)$  is a unit ball centered at the identity function  $\text{id}(z) = z$ .

*Proof.* Completeness follows, since the convergence in  $d_S$  implies the pointwise convergence, hence the limit function satisfies (6). The same inequality implies (2).  $\square$

Clearly, the metric on  $\mathcal{S}$  induces the metric on  $\mathcal{M}$  as

$$d_{\mathcal{M}}(M, N) = d_S(f_M, f_N).$$

Thus our space of means is a unit ball centered at the arithmetic mean.

To obtain the explicit formula for  $d_{\mathcal{M}}(M, N)$ , write

$$\begin{aligned} d_{\mathcal{M}}(M, N) &= d_S(f_M, f_N) = \sup_{0 < z < 1} \left| \frac{1}{f_M(z)} - \frac{1}{f_N(z)} \right| \\ &= \sup_{x \neq y} \frac{2}{|x-y|} \left| \frac{|x-y|}{2f_M(z)} - \frac{|x-y|}{2f_N(z)} \right| = 2 \sup_{x \neq y} \left| \frac{M(x, y) - N(x, y)}{x-y} \right|. \end{aligned}$$

For more properties of this metric, see [7].

As an application, let us prove the following result

**Theorem 2.2.** *For  $M, N \in \mathcal{M}$  satisfying  $d_{\mathcal{M}}(M, N) < 2$ , there exists a unique mean  $K \in \mathcal{M}$  such that for all  $x, y$*

$$K(x, y) = K(M(x, y), N(x, y)).$$

Note that this result is known in case  $M, N$  are strict means. There exist means that satisfy  $d_{\mathcal{M}}(M, N) < 2$  and are not strict, and there are strict means with  $d_{\mathcal{M}}(M, N) = 2$ .

*Proof.* Define the mapping  $\Phi : \mathcal{S} \rightarrow \mathcal{S}$  by  $\Phi(X)(x, y) = X(M(x, y), N(x, y))$ .

$$\begin{aligned} \frac{1}{f_{\Phi(X)}(z)} &= \frac{X(M(1+z, 1-z), N(1+z, 1-z))}{z} = X\left(\frac{1}{f_M(z)}, \frac{1}{f_N(z)}\right) \\ &= \frac{\frac{1}{f_M(z)} - \frac{1}{f_N(z)}}{2f_X\left(\frac{f_M^{-1}(z) - f_N^{-1}(z)}{f_M^{-1}(z) + f_N^{-1}(z)}\right)} = \frac{1}{2} \left( \frac{1}{f_M(z)} - \frac{1}{f_N(z)} \right) \frac{1}{f_X(u)}, \end{aligned}$$

where  $u = \frac{f_N(z) - f_M(z)}{f_M(z) + f_N(z)}$ . Thus

$$\frac{1}{f_{\Phi(X)}(z)} - \frac{1}{f_{\Phi(Y)}(z)} = \frac{1}{2} \left( \frac{1}{f_M(z)} - \frac{1}{f_N(z)} \right) \left( \frac{1}{f_X(u)} - \frac{1}{f_Y(u)} \right),$$

which yields

$$d_{\mathcal{M}}(\Phi(X), \Phi(Y)) \leq \frac{1}{2} d_{\mathcal{M}}(M, N) d_{\mathcal{M}}(X, Y).$$

Applying the Banach fixed point theorem, we complete the proof.  $\square$

### 3. GROUP STRUCTURE

In this section we define a group structure in  $\mathcal{M}^\circ$ . Obviously we express the group action in terms of Seiffert functions. It follows from formula (2) that the operator  $\mathcal{A} : \mathcal{S}^\circ \rightarrow \mathcal{B}^\circ$  defined by

$$\mathcal{A}(f)(z) = \frac{1}{f(z)} - \frac{1}{z}$$

is an isometry. Let  $\gamma : (-1, 1) \rightarrow \mathbb{R}$  be an odd, continuous bijection such that  $\gamma(0) = 0$ . We define the addition on  $\mathcal{S}^\circ$  by

$$(f \oplus g)(z) = \mathcal{A}^{-1}(\gamma^{-1}(\gamma(\mathcal{A}(f)(z)) + \gamma(\mathcal{A}(g)(z)))).$$

One can easily check that  $(\mathcal{S}^\circ, \oplus)$  is an abelian group, and the identity function is the neutral element.

The mapping  $M \rightarrow f_M$  allows us to transfer the group structure to the set of strict means by setting

$$M \oplus N = S_{f_M \oplus f_N}.$$

Clearly, the arithmetic mean becomes the neutral element, and for every  $M$  we have  $M \oplus (2A - M) = A$ .

It is clear that if  $g_n$  converges pointwise to  $g$ , then  $f \oplus g_n$  converges pointwise to  $f \oplus g$ . Unfortunately, the addition is not continuous with respect to the metric  $d_{\mathcal{S}}$ . To show this, consider the following example: for natural  $n$  let  $z_n = \gamma^{-1}(n)$  and let  $g_n, g, f \in \mathcal{B}^\circ$  be defined by

$$g_n(z) = \begin{cases} z_{n+1} & \text{if } z = z_n, \\ z & \text{otherwise,} \end{cases} \quad g(z) = z, \quad f(z) = -z.$$

We have  $d_{\mathcal{B}}(g_n, g) = |z_{n+1} - z_n| \rightarrow 0$ ,  $\gamma^{-1}(\gamma(f(z)) + \gamma(g(z))) = 0$  and  $\gamma^{-1}(\gamma(f(z_n)) + \gamma(g(z_n))) = \gamma^{-1}(1)$ . Consequently,  $d_{\mathcal{M}}(f \oplus g_n, f \oplus g) \not\rightarrow 0$ .

### 4. NEW SEIFFERT-LIKE MEANS

Seiffert introduced two means corresponding to arcsin and arctan. Two other means mentioned in the introduction come from their hyperbolic companions. Now, we shall show, that also sin, tan, sinh and tanh are Seiffert functions. To this end, we use two lemmata.

**Lemma 4.1.** *The inequalities*

$$(8) \quad t > \operatorname{arsinh} t > \arctan t > \tanh t > \frac{t}{1+t}$$

hold for all  $t > 0$ . Moreover,

$$(9) \quad \operatorname{arsinh} t > \sin t$$

holds for  $0 < t < \pi/2$  and

$$(10) \quad \sin t > \arctan t$$

is valid for  $0 < t < 1$ .

*Proof.* Since  $\cosh^2 t > \left(1 + \frac{t^2}{2}\right)^2 > 1 + t^2$ , integrating from 0 to  $t$  the inequalities

$$1 > \frac{1}{\sqrt{1+t^2}} > \frac{1}{1+t^2} > \frac{1}{\cosh^2 t},$$

we obtain first three inequalities in (8). To prove the last one, observe that the graph of the convex function  $\cosh t$  and the straight line  $1 + t$  intersect at two points:  $t = 0$  and  $t = t_0 > 0$ . Thus  $\frac{1}{\cosh^2 t} - \frac{1}{(1+t)^2}$  is positive for  $0 < t < t_0$  and negative for  $t > t_0$ . Therefore, the function

$$h(t) = \tanh t - \frac{t}{1+t} = \int_0^t \frac{1}{\cosh^2 t} - \frac{1}{(1+t)^2} dt$$

increases from  $h(0) = 0$  to  $h(t_0)$  and then decreases to  $h(\infty) = 0$ , hence is nonnegative. This completes the proof of the rightmost inequality in (8).

To prove (9), note that for  $0 < t < \pi/2$  the inequalities

$$\begin{aligned} (1+t^2) \cos^2 t &= (1+t^2) \frac{\cos 2t + 1}{2} < (1+t^2) \frac{1 - 2t^2 + 2t^4/3 + 1}{2} \\ &= 1 - \frac{1}{3}t^4(2 - t^2) < 1 \end{aligned}$$

hold. Thus  $\frac{1}{\sqrt{1+t^2}} > \cos t$  and we obtain (9) by integration.

To prove (10), observe that for  $0 < t < 1$

$$(1+t^2) \cos t > (1+t^2)(1-t^2/2) = 1 + \frac{t^2(1-t^2)}{2} > 1,$$

and apply the same argument as above. □

Lemma 4.1 shows that the following inequalities hold if  $x \neq y$

$$(11) \quad \frac{x+y}{2} \leq M(x, y) \leq \frac{x-y}{2 \sin \frac{x-y}{x+y}} \leq T(x, y) \leq \frac{x-y}{2 \tanh \frac{x-y}{x+y}} < \max(x, y).$$

Another set of means follows from the next lemma.

**Lemma 4.2.** *The inequalities*

$$(12) \quad t < \sinh t < \tan t < \operatorname{artanh} t < \frac{t}{1-t}$$

hold for all  $0 < t < 1$ . Moreover,

$$(13) \quad \sinh t < \arcsin t < \operatorname{artanh} t$$

holds for  $0 < t < 1$ . The functions  $\arcsin t$  and  $\tan t$  are not comparable in  $0 < t < 1$ .

*Proof.* The inequalities (12) follow from (8) and the fact, that the graph of an inverse function is symmetric with respect to the main diagonal. The same argument applied to (9) implies the first inequality in (13), while the second inequality can be obtained by integration of  $\frac{1}{\sqrt{1-t^2}} < \frac{1}{1-t^2}$ .

It follows from (10) and the remark about the graph of an inverse function that  $\arcsin t < \tan t$  for  $t < \sin 1$ , while  $\arcsin 1 > \tan 1$ . □

Lemma 4.2 implies the following chains of inequalities between means

$$(14) \quad \frac{x+y}{2} > \frac{x-y}{2 \sinh \frac{x-y}{x+y}} > P(x, y) > L(x, y) > \min(x, y)$$

$$(15) \quad \frac{x+y}{2} > \frac{x-y}{2 \sinh \frac{x-y}{x+y}} > \frac{x-y}{2 \tan \frac{x-y}{x+y}} > L(x, y) > \min(x, y).$$

So all hyperbolic and inverse offspring of sine and tangent functions forms Seiffert-like means. We shall see in a while that both of them are much more fertile.

## 5. INTEGRAL TRANSFORMATION

**Theorem 5.1.** *If  $f \in \mathcal{M}$  is concave and*

$$(16) \quad I(f)(z) = \int_0^z \frac{f(t)}{t} dt,$$

*then  $I(f)$  is also concave and  $f(z) \leq I(f)(z) \leq z$  for all  $z \in (0, 1)$ .*

*Similarly, if  $f$  is convex, then  $I(f)$  is also convex and  $f(z) \geq I(f)(z) \geq z$ .*

*Proof.* First, note that setting  $f(0) = 0$  we extend  $f$  to a concave function on  $[0, 1)$ . Inequalities (2) imply  $f'(0) = 1$  so  $f(z) \leq z$  by concavity. Moreover, the divided difference  $f(t)/t$  decreases which yields concavity of  $I(f)$ , and implies

$$f(z) = \int_0^z \frac{f(z)}{z} dt \leq \int_0^z \frac{f(t)}{t} dt \leq \int_0^z 1 dt = z.$$

The proof in case of convex function is similar. □

The operator  $I$  is monotone on the set of functions where it exists, and because  $f_{\max}$  is concave and  $f_{\min}$  is convex, we obtain the following corollary.

**Corollary 5.1.** *If  $f$  is a Seiffert function such that  $I(f)$  exists, then  $I(f)$  is also a Seiffert function.*



In particular we conclude that the following functions are Seiffert functions

$$\begin{aligned}
\text{Si}_n(z) &= \begin{cases} \sin z & n = 0 \\ \int_0^z \frac{\text{Si}_{n-1}(t)}{t} dt & n > 0 \end{cases}, \\
\text{ASi}_n(z) &= \begin{cases} \arcsin z & n = 0 \\ \int_0^z \frac{\text{ASi}_{n-1}(t)}{t} dt & n > 0 \end{cases}, \\
\text{Ti}_n(z) &= \begin{cases} \tan z & n = 0 \\ \int_0^z \frac{\text{Ti}_{n-1}(t)}{t} dt & n > 0 \end{cases}, \\
\text{ATi}_n(z) &= \begin{cases} \arctan z & n = 0 \\ \int_0^z \frac{\text{ATi}_{n-1}(t)}{t} dt & n > 0 \end{cases}, \\
\text{SHi}_n(z) &= \begin{cases} \sinh z & n = 0 \\ \int_0^z \frac{\text{SHi}_{n-1}(t)}{t} dt & n > 0 \end{cases}, \\
\text{ASHi}_n(z) &= \begin{cases} \text{arsinh } z & n = 0 \\ \int_0^z \frac{\text{ASHi}_{n-1}(t)}{t} dt & n > 0 \end{cases}, \\
\text{THi}_n(z) &= \begin{cases} \tanh z & n = 0 \\ \int_0^z \frac{\text{THi}_{n-1}(t)}{t} dt & n > 0 \end{cases}, \\
\text{ATHi}_n(z) &= \begin{cases} \text{artanh } z & n = 0 \\ \int_0^z \frac{\text{ATHi}_{n-1}(t)}{t} dt & n > 0 \end{cases}.
\end{aligned}$$

Since all functions mentioned in Lemma 4.1 are concave, we have the following grid of inequalities between means:

$$\begin{array}{ccccccc}
\cdots & < & \frac{x-y}{2 \text{ASHi}_2 \frac{x-y}{x+y}} & < & \frac{x-y}{2 \text{ASHi}_1 \frac{x-y}{x+y}} & < & \frac{x-y}{2 \text{arsinh } \frac{x-y}{x+y}} = M(x, y) \\
& & \wedge & & \wedge & & \wedge \\
\cdots & < & \frac{x-y}{2 \text{Si}_2 \frac{x-y}{x+y}} & < & \frac{x-y}{2 \text{Si}_1 \frac{x-y}{x+y}} & < & \frac{x-y}{2 \sin \frac{x-y}{x+y}} \\
\frac{x+y}{2} & < & \wedge & & \wedge & & \wedge \\
\cdots & < & \frac{x-y}{2 \text{ATi}_2 \frac{x-y}{x+y}} & < & \frac{x-y}{2 \text{ATi}_1 \frac{x-y}{x+y}} & < & \frac{x-y}{2 \arctan \frac{x-y}{x+y}} = T(x, y) \\
& & \wedge & & \wedge & & \wedge \\
\cdots & < & \frac{x-y}{2 \text{THi}_2 \frac{x-y}{x+y}} & < & \frac{x-y}{2 \text{THi}_1 \frac{x-y}{x+y}} & < & \frac{x-y}{2 \tanh \frac{x-y}{x+y}}
\end{array}$$

The horizontal lines are granted by Theorem 5.1 and vertical ones follow from Lemma 4.1.

Before providing a similar picture for the other four functions, recall that there is no comparison between arcsine and tangent functions. Nevertheless, the operator  $I$  quickly rectifies this irregularity.

**Lemma 5.1.** *For  $0 < z < 1$  we have*

$$\text{ASi}_1(z) = \int_0^z \frac{\arcsin t}{t} dt < \int_0^z \frac{\tan t}{t} dt = \text{Ti}_1(z).$$

*Proof.* Let  $q(t) = \arcsin t - \tan t$ . As shown in the proof of Lemma 4.2, for  $t < \sin 1 \approx 0.841$  the inequality  $q(t) < 0$  holds. For  $t > \pi/4 \approx 0.785$  we have  $q'(t) = \frac{\cos^2 t - \sqrt{1-t^2}}{\sqrt{1-t^2} \cos^2 t}$ ,  $q'(\pi/4) < 0$  and  $q'(1) > 0$ . Since  $\cos^2 t$  is convex and  $\sqrt{1-t^2}$  concave, their graphs intersect exactly in one point. Thus  $q(t)$  changes sign exactly once in the interval  $(0, 1)$ . This implies that the function  $u(z) = \int_0^z q(t)/t dt$  has exactly one local minimum, and since  $u(z) = 0$  and  $u(1) \approx -0.016$ , it is negative, which completes the proof.  $\square$

Now Theorem 5.1 together with Lemmata 4.2 and 5.1 yield

$$\begin{array}{ccccccc}
\frac{x+y}{2} & > & \frac{x-y}{2 \sinh \frac{x-y}{x+y}} & > & \frac{x-y}{2 \text{SHi}_1 \frac{x-y}{x+y}} & > & \frac{x-y}{2 \text{SHi}_2 \frac{x-y}{x+y}} & > & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
P(x, y) & = & \frac{x-y}{2 \arcsin \frac{x-y}{x+y}} & > & \frac{x-y}{2 \text{ASi}_1 \frac{x-y}{x+y}} & > & \frac{x-y}{2 \text{ASi}_2 \frac{x-y}{x+y}} & > & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \frac{x-y}{2 \tan \frac{x-y}{x+y}} & > & \frac{x-y}{2 \text{Ti}_1 \frac{x-y}{x+y}} & > & \frac{x-y}{2 \text{Ti}_2 \frac{x-y}{x+y}} & > & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
L(x, y) & = & \frac{x-y}{2 \operatorname{artanh} \frac{x-y}{x+y}} & > & \frac{x-y}{2 \text{ATHi}_1 \frac{x-y}{x+y}} & > & \frac{x-y}{2 \text{ATHi}_2 \frac{x-y}{x+y}} & > & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & 
\end{array}$$

## 6. SCHUR CONVEXITY

Given a symmetric, convex set  $D \subset \mathbb{R}^2$ , a partial order in  $D$  is defined by

$$(17) \quad (x_1, y_1) \prec (x_2, y_2) \Leftrightarrow x_1 + y_1 = x_2 + y_2 \text{ and } \max(x_1, y_1) \leq \max(x_2, y_2).$$

A symmetric function  $h : D \rightarrow \mathbb{R}$  is called Schur-convex if it preserves this partial order, i.e. if  $(x_1, y_1) \prec (x_2, y_2)$  yields  $h(x_1, y_1) \leq h(x_2, y_2)$ , and Schur-concave if the partial order gets reversed.

Setting  $c = (x_1 + y_1)/2$ ,  $t_1 = |x_1 - y_1|/2$ ,  $t_2 = |x_2 - y_2|/2$ , we see that  $(x_1, y_1) \prec (x_2, y_2)$  is equivalent to  $t_1 \leq t_2$ , and thus we can say that  $h$  is Schur-convex (resp. concave) if and only if for all  $c$  the function  $h(c+t, c-t)$  increases (resp. decreases) for  $t > 0$ , cf. [9, I.3.A.2.b]. It will be useful to introduce the strict Schur-convexity: it is when the inequality in (17) is strict whenever  $(x_1, y_1)$  is not a permutation of  $(x_2, y_2)$ . In this case all reasoning in this section remains valid with the adverb 'strictly' added to all mentioned properties.

Schur convexity of means is an interesting subject investigated by many mathematicians (see e.g. [1, 15, 6] and the references therein).

In case of a homogeneous symmetric mean, the Schur-convexity condition may be written in a very simple form:  $M(x, y)$  is Schur-convex (resp. concave) if and only if the function  $s(t) = M(1+t, 1-t)$  increases in the unit interval. Let us see how this condition translates into the language of Seiffert functions. We have  $s(t) = M(1+t, 1-t) = \frac{t}{f_M(t)}$ , so we have the following.

**Theorem 6.1.** *A mean  $M$  is Schur-convex (resp. concave) if and only if the function  $f_M(z)/z$  decreases (res. increases).*

Note that if a Seiffert function  $f$  is concave (resp. convex), then its divided difference  $f(z)/z$  decreases (resp. increases), so we have

**Corollary 6.1.** *If  $f \in \mathcal{S}$  is concave (resp. convex), then the mean  $S_f(x, y)$  is Schur-convex (resp. Schur-concave).*

**Corollary 6.2.** *The means*

$$\frac{x-y}{2 \operatorname{Si}_n \frac{x-y}{x+y}}, \quad \frac{x-y}{2 \operatorname{ASHi}_n \frac{x-y}{x+y}}, \quad \frac{x-y}{2 \operatorname{ATi}_n \frac{x-y}{x+y}}, \quad \frac{x-y}{2 \operatorname{THi}_n \frac{x-y}{x+y}}$$

*are strictly Schur-convex, while means*

$$\frac{x-y}{2 \operatorname{Ti}_n \frac{x-y}{x+y}}, \quad \frac{x-y}{2 \operatorname{ATHi}_n \frac{x-y}{x+y}}, \quad \frac{x-y}{2 \operatorname{ASi}_n \frac{x-y}{x+y}}, \quad \frac{x-y}{2 \operatorname{SHi}_n \frac{x-y}{x+y}}$$

*are strictly Schur-concave.*

## 7. MEANS WITH VARYING ARGUMENTS

Denote by  $\widehat{f}(z)$  the function  $f(z)/z$ . For  $0 < t < 1, 0 < z < 1$  and  $f \in \mathcal{S}$  we have the inequalities

$$\frac{z}{1+z} < \frac{1}{t} \frac{tz}{1+tz} \leq \frac{f(tz)}{t} \leq \frac{1}{t} \frac{tz}{1-tz} < \frac{z}{1-z},$$

which shows that  $f_t(z) = f(tz)/t$  are also Seiffert functions. Note that  $\lim_{t \rightarrow 0} f_t(z) = z$ , thus this process defines a homotopy between  $f$  and  $\operatorname{id}$ . It is a matter of simple transformation to verify, that if  $f$  corresponds to a mean  $M$ , then  $f_t$  maps to the mean

$$M_t(x, y) = M \left( \frac{x+y}{2} - t \frac{x-y}{2}, \frac{x+y}{2} + t \frac{x-y}{2} \right).$$

There are numerous papers on comparison between means of the form  $M_t$  and other classical means (see e.g. [2, 3, 4, 5, 8, 10, 16]).

The most popular problem is formulated as follows: given two means  $M, N$  satisfying  $A < M < N$ , find optimal  $p, q$  such that  $N_p \leq M \leq N_q$ . To avoid double subscripts, let  $m$  and  $n$  be the Seiffert functions of  $M$  and  $N$ , respectively. Then, by Corollary 2.2 the inequalities  $N_p < M < N_q$  are equivalent to  $n_p > m > n_q$ . The inequality  $A < N$  implies  $n(z) < z$ . Assume additionally that the function  $\widehat{n}(z)$  is strictly decreasing, (in case of classical means, their Seiffert functions are usually concave, so this condition is satisfied). Then the following inequalities are equivalent:

$$n_p(z) > m(z) > n_q(z) \equiv \frac{n(pz)}{pz} > \frac{m(z)}{z} > \frac{n(qz)}{qz} \equiv p < \frac{\widehat{n}^{-1}(\widehat{m}(z))}{z} < q.$$

Thus we have proven the following theorem.

**Theorem 7.1.** *Let  $M$  and  $N$  be two means with Seiffert functions  $m$  and  $n$ , respectively. Suppose that  $\widehat{n}(z)$  is strictly monotone and let  $p_0 = \inf_z \frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$  and  $q_0 = \sup_z \frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$ .*

*If  $A(x, y) < M(x, y) < N(x, y)$  for all  $x \neq y$  then the inequalities*

$$N_p(x, y) \leq M(x, y) \leq N_q(x, y)$$

*hold if and only if  $p \leq p_0$  and  $q \geq q_0$ .*

*If  $N(x, y) < M(x, y) < A(x, y)$  for all  $x \neq y$  then the inequalities*

$$N_q(x, y) \leq M(x, y) \leq N_p(x, y)$$

*hold if and only if  $p \leq p_0$  and  $q \geq q_0$ .*

To illustrate this theorem, let us consider two examples featuring the contraharmonic mean  $C(x, y) = \frac{x^2+y^2}{x+y}$ .

**Example 7.1.** Let  $M(x, y) = \frac{x^2+xy+y^2}{x+\sqrt{xy}+y}$ . It is known (cf. [17]) that for  $x \neq y$

$$A(x, y) < M(x, y) < C(x, y)$$

holds. We have

$$\widehat{m}(z) = \frac{2 + \sqrt{1-z^2}}{3+z^2}, \quad \widehat{c}(z) = \frac{1}{1+z^2}, \quad \frac{\widehat{c}^{-1}(\widehat{m}(z))}{z} = \sqrt{\frac{1 - \sqrt{1-z^2}}{z^2}}.$$

The function  $1 - \sqrt{1-u}$  is convex, so its divided difference increases from  $1/2$  to  $1$ , therefore we obtain

$$C_{\sqrt{2}/2}(x, y) \leq M(x, y) \leq C(x, y).$$

**Example 7.2.** The contraharmonic mean belongs to the family of Gini means defined in general case by  $G(r, s; x, y) = \left(\frac{x^r+y^r}{x^s+y^s}\right)^{1/(r-s)}$ . They are increasing with respect to parameters  $r, s$ , thus for  $0 < \alpha < 2$  we have

$$A(x, y) = G(1, 0; x, y) < G(1, \alpha; x, y) < G(1, 2; x, y) = C(x, y).$$

Fix  $\alpha$  and let  $M = G(1, \alpha)$ . An easy calculation shows that

$$\frac{\widehat{c}^{-1}(\widehat{m}(z))}{z} = 2^{-\frac{1}{2(\alpha-1)}} \sqrt{\frac{[(1+z)^\alpha + (1-z)^\alpha]^{1/(\alpha-1)} - 2^{1/(\alpha-1)}}{z^2}}.$$

We shall show that the function under the square root increases. To this end, we shall use the following version of de l'Hospital's rule ([12]): if  $f$  and  $g$  are differentiable functions  $g'(x) \neq 0$  and such that  $(f'/g')(x)$  increases (decreases), then so does the divided difference  $\frac{f(x)-f(a)}{g(x)-g(a)}$ .

Let  $f(x) = [(1+z)^\alpha + (1-z)^\alpha]^{1/(\alpha-1)}$  and  $g(x) = z^2$ . We intend to show that  $(f'/g')(z)$  increases in  $(0, 1)$ . We have

$$\begin{aligned} \frac{f'}{g'}(z) &= [(1+z)^\alpha + (1-z)^\alpha]^{(2-\alpha)/(\alpha-1)} \times \frac{\alpha}{2(\alpha-1)} \frac{(1+z)^{\alpha-1} - (1-z)^{\alpha-1}}{z} \\ &=: h_1(z) \times \frac{\alpha}{2(\alpha-1)} \frac{h_2(z)}{z}. \end{aligned}$$

Case 1:  $1 < \alpha < 2$ . The power function  $z^\alpha$  is convex, so  $(1+z)^\alpha + (1-z)^\alpha$  increases and so does  $h_1(z)$ . The function  $h_2(z)$  is positive and convex, so its divided difference increases, thus  $f'/g'$  increases being a product of positive increasing functions.

Case 2:  $0 < \alpha < 1$ . The power function  $z^\alpha$  is concave, so  $(1+z)^\alpha + (1-z)^\alpha$  decreases, thus  $h_1(z)$  increases. The function  $h_2(z)$  is negative and concave, so its divided difference decreases, and the negative factor  $\frac{\alpha}{2(\alpha-1)}$  turns it into positive increasing. Thus again  $f'/g'$  increases.

Therefore

$$\sqrt{\frac{\alpha}{2}} = \lim_{z \rightarrow 0} \frac{\widehat{c}^{-1}(\widehat{m}(z))}{z} \leq \frac{\widehat{c}^{-1}(\widehat{m}(z))}{z} \leq \frac{\widehat{c}^{-1}(\widehat{m}(1))}{1} = 1$$

and we obtain the optimal inequalities

$$C \left( \frac{x+y}{2} + \sqrt{\frac{\alpha}{2}} \frac{x-y}{2}, \frac{x+y}{2} - \sqrt{\frac{\alpha}{2}} \frac{x-y}{2} \right) \leq G(1, \alpha; x, y) \leq C(x, y).$$

Using the same technique as in Example 7.2 we obtain the following results for power means  $G(0, \alpha; x, y)$ :

**Example 7.3.** Let  $N(x, y) = G(0, 2; x, y)$  denote the root-mean square. For  $1 < \alpha < 2$  the following inequalities

$$N \left( \frac{x+y}{2} + p \frac{x-y}{2}, \frac{x+y}{2} - p \frac{x-y}{2} \right) \leq G(0, \alpha; x, y) \leq N \left( \frac{x+y}{2} + q \frac{x-y}{2}, \frac{x+y}{2} - q \frac{x-y}{2} \right)$$

hold if and only if  $p \leq \sqrt{\alpha-1}$  and  $q \geq \sqrt{4^{1-1/\alpha}-1}$ .

**Example 7.4.** For  $-1 < \alpha < 1$  the following inequalities

$$H(x, y) \leq G(0, \alpha; x, y) \leq H \left( \frac{x+y}{2} + q \frac{x-y}{2}, \frac{x+y}{2} - q \frac{x-y}{2} \right)$$

hold if and only if  $q \leq \sqrt{\frac{1-\alpha}{2}}$ .

## 8. APPROXIMATION BY CONVEX COMBINATION OF MEANS

Suppose all positive distinct  $x, y$  three means  $K, M, N$  satisfy for inequalities  $K(x, y) < M(x, y) < N(x, y)$ . Our goal is to determine the best possible constants  $\mu$  and  $\nu$  such that the inequalities

$$(18) \quad (1-\mu)K(x, y) + \mu N(x, y) \leq M(x, y) \leq (1-\nu)K(x, y) + \nu N(x, y)$$

are valid for all  $x, y$ . In terms of Seiffert functions, the inequalities (18) have the form

$$\frac{1-\mu}{f_K(z)} + \frac{\mu}{f_N(z)} \leq \frac{1}{f_M(z)} \leq \frac{1-\nu}{f_K(z)} + \frac{\nu}{f_N(z)},$$

which is equivalent to

$$(19) \quad \mu \leq \frac{\frac{1}{f_M(z)} - \frac{1}{f_K(z)}}{\frac{1}{f_N(z)} - \frac{1}{f_K(z)}} \leq \nu.$$

Thus we have the following result.

**Theorem 8.1.** *For three means satisfying  $K < M < N$  the inequalities (18) hold if and only if*

$$\mu \leq \inf_{0 < z < 1} \frac{\frac{1}{f_M(z)} - \frac{1}{f_K(z)}}{\frac{1}{f_N(z)} - \frac{1}{f_K(z)}} \text{ and } \sup_{0 < z < 1} \frac{\frac{1}{f_M(z)} - \frac{1}{f_K(z)}}{\frac{1}{f_N(z)} - \frac{1}{f_K(z)}} \leq \nu.$$

Let us illustrate this result by finding the optimal bounds of eight Seiffert-like means discussed in Section 4 by convex combination of min and max. In this case the inequalities (19) read

$$\mu \leq \frac{1}{2} \left( \frac{1}{f(z)} - \frac{1}{z} + 1 \right) \leq \nu,$$

thus we have to find the upper and lower bound of the function  $\frac{1}{f(z)} - \frac{1}{z}$ . In all eight cases we can write  $f(z) = z + cz^3 + \dots$ , so  $\lim_{z \rightarrow 0} \frac{1}{f(z)} - \frac{1}{z} = 0$ .

**Example 8.1.** Let  $f(z) = \sin z$ . The function  $\cos z$  is concave in  $(0, \pi/2)$  thus so are  $\cos(tz)$  and  $\int_0^1 \cos(tz)dt = \frac{\sin z}{z}$ . Since the reciprocal of a positive concave function is convex,  $\frac{z}{\sin z}$  is convex and consequently, its divided difference

$$\frac{1}{z} \left( \frac{z}{\sin z} - 1 \right) = \frac{1}{\sin z} - \frac{1}{z}$$

increases from 0 to  $1/\sin(1) - 1$ . Thus

$$\frac{x+y}{2} \leq \frac{x-y}{2 \sin \frac{x-y}{x+y}} \leq \left( 1 - \frac{1}{2 \sin 1} \right) \min(x, y) + \frac{1}{2 \sin 1} \max(x, y).$$

**Example 8.2.** Let  $f(z) = \arcsin z$ . To investigate monotonicity of  $\frac{1}{\arcsin z} - \frac{1}{z}$ , set  $z = \sin t$  and use the result of the previous example to see, that the range of this function is  $(2/\pi - 1, 0)$ . Thus,

$$\frac{1}{\pi} \min(x, y) + \left( 1 - \frac{1}{\pi} \right) \max(x, y) \leq P(x, y) \leq \frac{x+y}{2}.$$

**Example 8.3.** Let  $f(z) = \tan z$ . Then,

$$\left( \frac{1}{\tan z} - \frac{1}{z} \right)' = \frac{1}{z^2} - \frac{\tan'(z)}{\tan^2(z)} = \frac{\sin^2 z - z^2}{z^2 \sin^2 z} < 0,$$

so  $\frac{1}{\tan z} - \frac{1}{z}$  decreases from 0 to  $1/\tan(1) - 1$  and consequently

$$\left( 1 - \frac{1}{2 \tan 1} \right) \min(x, y) + \frac{1}{2 \tan 1} \max(x, y) \leq \frac{x-y}{2 \tan \frac{x-y}{x+y}} \leq \frac{x+y}{2}.$$

**Example 8.4.** In case  $f(z) = \arctan z$ , we set  $z = \tan t$  and use the previous example to get

$$\frac{x+y}{2} \leq T(x, y) \leq \left( 1 - \frac{2}{\pi} \right) \min(x, y) + \frac{2}{\pi} \max(x, y).$$

**Example 8.5.** The story of hyperbolic tangent is quite similar to that of tangent, the only difference is that  $\sinh z \geq z$ .

$$\frac{x+y}{2} \leq \frac{x-y}{2 \tanh \frac{x-y}{x+y}} \leq \left( 1 - \frac{1}{2 \tanh 1} \right) \min(x, y) + \frac{1}{2 \tanh 1} \max(x, y).$$

**Example 8.6.** As shall be expected, for the inverse hyperbolic tangent we shall use Example 8.5 and the substitution  $t = \tanh z$ . Nevertheless, it is wise to note that  $\lim_{z \rightarrow 1} \operatorname{artanh} z = \infty$ , so in this case there is only trivial lower bound

$$\min(x, y) \leq L(x, y) \leq \frac{x+y}{2}.$$

**Example 8.7.** An attempt to use the same approach as in Example 8.1 fails for  $f(z) = \sinh z$ , because  $\sinh z/z$  is convex and the reciprocal of a convex function may not be convex.

$$\begin{aligned} (20) \quad \left( \frac{1}{\sinh z} - \frac{1}{z} \right)' &= \frac{\sinh^2 z - z^2 \cosh z}{z^2 \sinh^2 z} \\ &= \frac{\cosh z}{\sinh^2 z} \left( \frac{\sinh z \cosh^{-1/2} z}{z} + 1 \right) \left( \frac{\sinh z \cosh^{-1/2} z}{z} - 1 \right). \end{aligned}$$

Let  $q(z) = \sinh z \cosh^{-1/2} z$ . Then  $q''(z) = \frac{1}{4} \cosh^{-3/5} z \sinh z (\cosh^2 z - 3)$ , hence  $q$  is concave in  $(0, \operatorname{arcosh} \sqrt{3})$ . Since  $\operatorname{arcosh} \sqrt{3} \approx 1.14$ , we conclude that  $q(z)/z$  decreases in the interval  $(0, 1)$ , and thus the expression (20) is negative. Therefore we have

$$\left(1 - \frac{1}{2 \sinh 1}\right) \min(x, y) + \frac{1}{2 \sinh 1} \max(x, y) \leq \frac{x - y}{2 \tan \frac{x-y}{x+y}} \leq \frac{x + y}{2}.$$

And finally

**Example 8.8.** For  $f(z) = \operatorname{arsinh} z$ , the substitution  $z = \sinh t$  converts  $\frac{1}{\operatorname{arsinh} z} - \frac{1}{z}$  into  $\frac{1}{t} - \frac{1}{\sinh t}$  and monotonicity follows from the previous example, since  $1 > z > t$ . The optimal inequalities then are

$$\frac{x + y}{2} \leq \frac{x - y}{2 \operatorname{arsinh} \frac{x-y}{x+y}} \leq \left(1 - \frac{1}{2 \operatorname{arsinh} 1}\right) \min(x, y) + \frac{1}{2 \operatorname{arsinh} 1} \max(x, y).$$

## 9. MISCELLANEA

In this section, we collect some facts about means and Seiffert functions, that might be useful for future investigations.

This surprising result follows from the results of Section 7.

**Theorem 9.1.** *If  $M$  is a mean satisfying  $M(x, y) \leq A(x, y)$ , and  $M_{1/2}(x, y) = M(\frac{3x+y}{4}, \frac{x+3y}{4})$ , then  $M_{1/2}^2/A$  is a mean.*

Note that in general  $M^2/A$  is not a mean (take the harmonic mean as an counterexample).

*Proof.* Let  $f$  be the Seiffert mean of  $M$ . Then the function  $2f(z/2)$  is the Seiffert function of  $M_{1/2}$ . Consider the function  $g(z) = 4f^2(z/2)/z$ . Since  $z \leq f(z)$ , we have  $z \leq g(z)$  and

$$g(z) \leq \frac{4}{z} \frac{z^2/4}{(1 - z/2)^2} < \frac{z}{1 - z}.$$

Thus  $g$  is a Seiffert function and its corresponding mean is

$$\frac{|x - y|z}{8f^2(z/2)} = \left(\frac{|x - y|}{2 \cdot 2f(z/2)}\right)^2 \frac{2}{x + y} = \frac{M_{1/2}^2(x, y)}{A(x, y)}.$$

□

This result may be generalized as follows

**Corollary 9.1.** *If  $M$  is a mean satisfying  $M(x, y) \leq A(x, y)$ , and  $M_t(x, y) = M(\frac{x+y}{2} + t\frac{x-y}{2}, \frac{x+y}{2} - t\frac{x-y}{2})$ , then  $M_t^{1/t} A^{1-1/t}$  is also a mean.*

To prove this, it is enough to show that  $g(z) = z[g(tz)/(tz)]^{1/t}$  is a Seiffert function. We leave the details to the reader.

**Remark 9.1.** It is easy to see that  $z + az^3$  is a Seiffert function if and only if  $-1/2 \leq a \leq 1/2$ .

**Theorem 9.2.** *The inequalities*

$$(21) \quad A(x, y) \leq \frac{|x - y|}{2 \sin \frac{x-y}{x+y}} \leq A(x, y) \frac{6A^2(x, y)}{5A^2(x, y) + G^2(x, y)}$$

hold.

*Proof.* The sine function satisfies  $z > \sin z > z - z^3/6$ . It follows from Remark 9.1 that the last function in this chain is a Seiffert function, and a simple calculation shows that its mean is the rightmost mean in (21).  $\square$

For two Seiffert functions  $f_M$  and  $f_N$  and a homogeneous but necessarily symmetric mean  $K$ , the function  $g(z) = K(f_M(z), f_N(z))$  is a Seiffert function corresponding to  $MN/K(N, M)$ . In particular, if  $K$  is  $M, N$  invariant (see Theorem 2.2), then  $MN/K$  is also a mean. Thus the weighted arithmetic mean of two Seiffert functions corresponds to the weighted harmonic mean of means and vice versa.

Now we shall show facts about power series representation of Seiffert functions. The first one is trivial.

**Theorem 9.3.** *If  $0 \leq a_n \leq 1$  for  $n > 1$ , then the function  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  is a Seiffert function.*

*Proof.* Clearly, the series converges for  $0 < z < 1$  and

$$z \leq f(z) \leq \sum_{n=1}^{\infty} z^n = \frac{1}{1-z}.$$

$\square$

The two following theorems concern alternating series.

**Theorem 9.4.** *Let  $1 = a_1 \geq a_2 \geq \dots \geq 0$  be a convex sequence (i.e. satisfying  $2a_k \leq a_{k-1} + a_{k+1}$  for  $k = 2, 3, \dots$ ). Then the function  $f(z) = z + \sum_{n=2}^{\infty} (-1)^{n+1} a_n z^n$  is a Seiffert function.*

*Proof.* Note first, that

$$f(z) = z - z^2(a_2 - a_3 z) - z^4(a_4 - a_5 z) - \dots \leq z.$$

Let  $b_n = a_n - a_{n+1}$ . This sequence decreases monotonically to 0 and we have

$$\begin{aligned} (1+z)f(z) &= z + \sum_{n=2}^{\infty} (-1)^{n+1} a_n z^n + \sum_{n=1}^{\infty} (-1)^{n+1} a_n z^{n+1} \\ &= z + z^2(b_1 - b_2 z) + z^4(b_3 - b_4 z) + \dots \geq z, \end{aligned}$$

thus  $f(z) \geq \frac{z}{1+z}$ .  $\square$

**Corollary 9.2.** *The following functions are Seiffert:*

$$\log(1+z), \quad 2 \frac{z - \log(1+z)}{z}, \quad 3 \frac{\log(1+z) - z + z^2/2}{z^2}, \quad \dots$$

**Theorem 9.5.** *Let  $a_n$ ,  $n \geq 1$  be nonnegative numbers satisfying  $a_1 \leq 1/2, 1 \geq a_2 \geq a_3 \geq \dots$ , then  $f(z) = z - a_1 z^3 + \sum_{n=2}^{\infty} (-1)^n a_n z^{2n+1}$  is a Seiffert function.*

*Proof.* As we know from Remark 9.1 that  $z - a_1 z^3 \geq \frac{z}{1+z}$ , thus

$$f(z) \geq \frac{z}{1+z} + z^5(a_2 - a_3 z^2) + z^9(a_4 - a_5 z^2) + \dots \geq \frac{z}{1+z}.$$

On the other hand

$$f(z) \leq \sum_{n=0}^{\infty} z^{2n+1} < \sum_{n=1}^{\infty} z^n = \frac{z}{1-z},$$

which completes the proof.  $\square$



**Corollary 9.3.** *The following are Seiffert functions:*

$$\sin z, \quad 6 \frac{z - \sin z}{z^2}, \quad 120 \frac{\sin z - z + z^3/6}{z^4}, \quad \dots, \\ 2 \frac{1 - \cos z}{z}, \quad 24 \frac{\cos z - 1 + z^2/z}{z^3}, \quad \dots$$

#### REFERENCES

- [1] Y. Chu and W. Xia. Solution of an open problem for Schur convexity or concavity of the Gini mean values. *Sci. China Ser. A*, 52(10):2099–2106, 2009.
- [2] Y.-M. Chu and S.-W. Hou. Sharp bounds for Seiffert mean in terms of contraharmonic mean. *Abstr. Appl. Anal.*, pages Art. ID 425175, 6, 2012.
- [3] Y.-M. Chu, M.-K. Wang, and W.-M. Gong. Two sharp double inequalities for Seiffert mean. *J. Inequal. Appl.*, pages 2011:44, 7, 2011.
- [4] Y.-M. Chu, M.-K. Wang, and Z.-K. Wang. A best-possible double inequality between Seiffert and harmonic means. *J. Inequal. Appl.*, pages 2011:94, 7, 2011.
- [5] Y.-M. Chu, C. Zong, and G.-D. Wang. Optimal convex combination bounds of Seiffert and geometric means for the arithmetic mean. *J. Math. Inequal.*, 5(3):429–434, 2011.
- [6] V. Čuljak, I. Franjić, G. Roqia, and J. Pečarić. Schur-convexity of averages of convex functions. *J. Inequal. Appl.*, pages Art. ID 581918, 25, 2011.
- [7] B. Farhi. Algebraic and topological structures on the set of mean functions and generalization of the AGM mean. *arXiv:1002.4757 [math.NT]*, Feb. 2010.
- [8] S. Gao, H. Gao, and W. Shi. Optimal convex combination bounds of the centroidal and harmonic means for the Seiffert mean. *International Journal of Pure and Applied Mathematics*, 70(5):701–709, 2011.
- [9] A. W. Marshall, I. Olkin, and B. C. Arnold. *Inequalities: theory of majorization and its applications*. Springer Series in Statistics. Springer, New York, second edition, 2011. ISBN 978-0-387-40087-7. xxviii+909 pp.
- [10] E. Neuman. A one-parameter family of bivariate means. *J. Math. Inequal.*, 7(3):399–412, 2013.
- [11] E. Neuman and J. Sándor. On the Schwab-Borchardt mean. *Math. Pannon.*, 14(2):253–266, 2003.
- [12] I. Pinelis. L’Hospital type rules for monotonicity, with applications. *J. Ineq. Pure & Appl. Math.*, 3(1), 2002.
- [13] H.-J. Seiffert. Werte zwischen dem geometrischen und dem arithmetischen Mittel zweier Zahlen. *Elem. Math.*, 42(4):105–107, 1987.
- [14] H.-J. Seiffert. Aufgabe  $\beta$ 16. *Die Wurzel*, 29:221–222, 1995.
- [15] H.-N. Shi, M. Bencze, S.-H. Wu, and D.-M. Li. Schur convexity of generalized Heronian means involving two parameters. *J. Inequal. Appl.*, pages Art. ID 879273, 9, 2008.
- [16] M.-K. Wang, Y.-M. Chu, and B.-Y. Liu. Sharp inequalities for the Neuman-Sandor mean in terms of arithmetic and contra-harmonic means. *arXiv:1209.5825 [math.CA]*, Sept. 2012.
- [17] A. Witkowski. Monotonicity and convexity of  $S$ -means. *Math. Inequal. Appl.*, 13(1):33–42, 2010.

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