MAP 552 exercice B3

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Question 1

We can use the theorem 6.12. Since φ in $\mathbb{H}^2$. We only need to show that, for all $t \leq 0$, we have $\int_0^t \frac{\varphi_s}{||\varphi_s||_n} \frac{\varphi_s^T}{||\varphi_s||_n} ds = tI_n = t$ (n = 1).

$$= \int_0^t \frac{\varphi_s \varphi_s^T}{|\varphi_s^1|^2 + \cdot + |\varphi_s^n|^2} ds = \int_0^t 1 ds = t$$

Therefore, W_t is a Brownian motion by theorem 6.12..

Question 2

We apply the Ito's formula on

$$X_t = \sum_{i=1}^n \left(x_i + B_t^i\right)^2, (x_1, \dots, x_n) \in \mathbb{R}^n$$

With $f(t,x) = ||z+x||_2^2, z = (x_1, \dots, x_n)$, we have the equation

$$f(t, B_t) = f(0,0) + \int_0^t Df(u, B_u) \cdot dB_u + \int_0^t \left(f_t + \frac{1}{2} Tr\left[D^2 f\right]\right)(u, B_u) du$$

We have

$$\begin{aligned} \partial_t f(t,x) &= 0 \\ Df(t,x) &= 2(z+x) \\ D^2 f(t,x) &= 2I_n,\ I_n \text{ is the diagonal matrix} \end{aligned}$$

, so

$$f(t, B_t) = f(0,0) + \int_0^t 2(z+B_s) \cdot dB_s + \int_0^t (0+n)dt = ||z||_2^2 + \int_0^t n dt + \int_0^t 2B_s \cdot dB_s$$

We identify

$$\begin{aligned} X_0 &= ||z||_2^2 \\ \phi_s &= 2(z+B_s) \\ \psi_s &= n \end{aligned}$$

For $\mathbb{E}\left[\int_0^T ||\phi_s||_n^2 ds\right] < +\infty$, by Fubini's theorem :

$$= \int_0^T \mathbb{E}\left[||2(z+B_s)||_n^2\right] ds = 4 \int_0^T z \cdot z + ns ds = 4Tz \cdot z + 2nT^2 < \infty \text{ for T} > 0$$

because $B_s \sim N(0, sI_n)$ in distribution.

Question 3

We define

$$d\beta_t = \frac{1}{\sqrt{X_t}}(B_t + z) \cdot dB_t, \beta_0 = 0$$

Therefore, we have

$$\beta_t = \int_0^t \frac{(B_t + z) \cdot dB_t}{\sqrt{X_t}}$$

Define $\varphi_s = z + B_t$, then $||\varphi_s||_n = \sqrt{X_t}$. By question 1, (β_t) is thus a Brownian motion.