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Fractal dimension

Definition

Zeros of the Brownian

The lower bound the fractal dimens

The upper bound for the fractal dimension

Simulation results

## Zeros of Brownian motion

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MAP575 Foundations of probability and applications

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Fractal dimension

Zeros of the Brownian motion

the fractal dimension

The upper bound for
the fractal dimension

Simulation results

#### 1 Fractal dimension

- Definition
- Comparison
- 2 Zeros of the Brownian motion
  - The lower bound for the fractal dimension
  - The upper bound for the fractal dimension
- 3 Simulation results

Peng-Wei CHEN

Fractal dimension

Definition

Zeros of t

Brownian motion

the fractal dimension

the fractal dimension

Simulation results

# Section 1

# Fractal dimension

## Minkowski dimension

Zeros of Brownian motion

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Fractal

Definition

Zeros of the Brownian motion

The lower bound for the fractal dimension The upper bound for

Simulation results

#### Definition

Let  $N(K, \varepsilon)$  be the minimal number of open balls of diameter  $\varepsilon$  needed to cover a subset K of a metric space.

We define the upper Minkowski dimension as

$$\overline{\dim}_{\mathcal{M}}(K) = \limsup_{\varepsilon \to 0} \frac{\log N(K,\varepsilon)}{\log 1/\varepsilon}$$

with  $\liminf$  for the lower Minkowski dimension. Finally, we define  $\dim_{\mathcal{M}}(K)$  the Minkowski dimension if these two values agree.

- Equal with two equivalent metric.
- $N(k,\varepsilon) \sim C\varepsilon^{-\dim_{\mathcal{M}}(K)}$
- Box counting dimension



# Minkowski dimension: examples

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Fractal dimension

Definition

Zeros of the Brownian motion

The lower bound for the fractal dimension. The upper bound for the fractal dimension

Simulation results

# $\dim_{\mathcal{M}}(K) = \lim_{\varepsilon \to 0} \frac{\log N(K, \varepsilon)}{\log 1/\varepsilon}$

#### Exemples

For 
$$\frac{1}{n^2} < \varepsilon \le \frac{1}{(n+1)^2}$$
,  $0, \frac{1}{n}, \frac{1}{n+1} \cdots \in [0, \frac{1}{n}]$ ,  $n \le N(K, \varepsilon) \le 2n + 1$ .

## Hausdorff dimension

Zeros of Brownian motion

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Fractal dimension

dimensio Definition

Zeros of the Brownian

Brownian motion

The lower bound

The upper bound for

Simulation

#### **Definition**

We define the Hausdorff dimension of K as

$$\dim(K) = \inf \left\{ \alpha : \inf \left\{ \sum_{i} |U_{i}|^{\alpha} : K \subset \bigcup_{i} U_{i} \right\} = 0 \right\}$$

$$\mathcal{H}^{\alpha}_{\infty}(K)$$

# Hausdorff dimension: examples

Zeros of Brownian motion

Peng-We CHEN

Fractal dimensio

Definition

Zeros of th

Zeros of the Brownian motion

the fractal dimension The upper bound for the fractal dimension

Simulation results

## Examples

- $\epsilon 2^{-n}$
- For  $\alpha > 1$ ,  $\mathcal{H}^{\alpha}_{\infty}(K) = 0$ .

# Mass Distribution Principle

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Fractal dimension Definition

Zeros of the Brownian motion

The lower bound for the fractal dimension The upper bound for the fractal dimension

Simulation

### Lemma (Mass Distribution Principle)

If  $\mu(B(x,r)) \leq Cr^{\alpha}$  for a strictly positive Borel measure  $\mu$  on E with a certain C and  $\alpha$ , then

$$\dim(E) \geq \alpha$$

$$(U_i)_{i \in \mathbb{N}}$$
 so that  $E \subset \bigcup_i U_i$ ,  $(r_i)_{i \in \mathbb{N}}$  so that  $|(U_i)| < r_i$ .

$$\mu(U_i) \leq \mu(B(x_i, r_i)) \leq Cr_i^{\alpha}$$

$$\sum_{i} |U_{i}|^{\alpha} \geq \sum_{i} \frac{\mu(U_{i})}{C} \geq \frac{\mu(E)}{C} > 0$$

# Comparison between the two dimensions

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Fractal dimension

Definition Comparison

Zeros of the Brownian motion

The lower bound for the fractal dimension The upper bound fo

Simulation

#### Definition

$$\lim_{\varepsilon \to 0} \frac{\log N(K,\varepsilon)}{\log 1/\varepsilon} \text{ and inf } \{\alpha: \mathcal{H}^\alpha_\infty(K) = 0\}$$

## Examples

	Finite set	[0, 1]	$\{0\} \cup \{1, \frac{1}{2}, \dots, \}$
Minkowski	0	1	$\frac{1}{2}$
Hausdorff	0	1	Ō

# Inequality

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Fractal dimension

Comparison

Zeros of the Brownian motion

the fractal dimensio

The upper bound for
the fractal dimensio

Simulation

## Proposition

$$\dim(K) \leq \underline{\dim}_{\mathcal{M}}(K) \leq \overline{\dim}_{\mathcal{M}}(K)$$

$$B_i = B(x_i, \frac{\varepsilon}{2}), N(K, \varepsilon)$$
 balls.

For 
$$\alpha > \underline{\dim}_{\mathcal{M}}$$
,

$$\sum_{i=1}^{N(K,\varepsilon)} |B_i|^{\alpha} = N(K,\varepsilon) \left(\frac{\varepsilon}{2}\right)^{\alpha} \leq \frac{1}{2^{\alpha}} \varepsilon^{\alpha - \dim M} \to 0.$$

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Fractal

o com

Comparison

Zeros of the Brownian

motion

The lower bound

The upper bound for the fractal dimension

Simulation results

# Section 2

## Zeros of the Brownian motion

# The fractal dimension of zeros of the Brownian motion

Zeros of Brownian motion

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Fractal
dimension
Definition

Zeros of the Brownian motion

The lower bound for the fractal dimension The upper bound for

Simulation

#### Notation

We note  $Z_B = \{t \in [0, +\infty) : B(t) = 0\}$  the set of zeros of the Brownian motion  $B_t$ .

#### **Theorem**

With probability 1, dim $(Z_B) = \frac{1}{2}$ .

$$\frac{C}{\sqrt{s}}$$
 balls

$$\sum_{i=1}^{\sqrt{\varepsilon}} |U_i|^{\alpha} = 0 \text{ for } \alpha > \frac{1}{2}.$$

## Lower bound

Zeros of Brownian motion

Peng-We

Fractal dimensio

Zeros of the Brownian

The lower bound for the fractal dimension

Simulation results

#### Lemma

With probability 1,

$$\dim\{t \in [0,1]: Y(t) = 0\} \ge 1/2$$

where  $Y(t) = \max_{0 \le s \le t} B_s - B_t$ .

Let  $M(t) = \max_{0 \le s \le t} B_s$ ,  $\mu([a,b]) = M(b) - M(a)$  define a measure with support strictly positive on  $\{t \in [0,1] : Y(t) = 0\}$ .

$$M(b)-M(a) \leq \max_{0 \leq h \leq b-a} B_{a+h} - B_a \leq C_{\alpha}(b-a)^{\alpha}$$

# Lower bound

Zeros of Brownian motion

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Fractal dimension Definition Comparison

Brownian
motion
The lower bound for the fractal dimension

Simulation results

#### Lemma

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$$M(b)-M(a) \leq \max_{0 \leq h \leq b-a} B_{a+h} - B_a \leq C_{\alpha}(b-a)^{\alpha}$$

## (Admitted), Hölder continuous

The Brownian motion is almost surely  $\alpha$ -Hölder continuous for  $\alpha < 0.5$ .

# Lower bound

Zeros of Brownian motion

Peng-We

Fractal dimension Definition

Zeros of the Brownian motion The lower bound for the fractal dimension

Simulation results

#### Lemma

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$$\dim\{t \in [0,1]: Y(t) = 0\} \ge 1/2$$

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Let  $M(t) = \max_{0 \le s \le t} B_s$ ,  $\mu([a, b]) = M(b) - M(a)$  define a measure with support strictly positive on  $\{t \in [0, 1] : Y(t) = 0\}$ .

$$M(b)-M(a) \leq \max_{0 \leq h \leq b-a} B_{a+h} - B_a \leq C_{\alpha}(b-a)^{\alpha}$$

#### Recall: Mass Distribution Principle

$$\mu(B(x,r)) \leq Cr^{\alpha} \to \dim(E) \geq \alpha$$

## The counter of a set

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Fractal

dimensior

Compariso

Zeros of th Brownian

motion

the fractal dimension

The upper bound for
the fractal dimension

Simulation

### The counter

We note

$$N_m(A) = \sum_{k=1}^{2} 1_{A \cap \left[\frac{k-1}{2^m}, \frac{k}{2^m}\right] 
eq \emptyset}$$

## The counter of a set

Zeros of Brownian motion

Peng-We

Fractal dimension Definition

Brownian
motion
The lower bound for

The upper bound for the fractal dimension

Simulation results

#### Lemma

Suppose A is a closed random subset of [0,1] such that

$$\mathbb{E}[N_m(A)] \le c2^{m\alpha}$$

for some  $c, \alpha > 0$ . Then  $\dim_{\mathcal{M}}(A) \leq \alpha$ .

$$\mathbb{E}\left[\sum_{m=1}^{\infty} \frac{N_m(A)}{2^{m(\alpha+\varepsilon)}}\right] = \sum_{m=1}^{\infty} \frac{\mathbb{E}N_m(A)}{2^{m(\alpha+\varepsilon)}} < \infty$$

$$\sum_{m=1}^{\infty} \frac{N_m(A)}{2^{m(\alpha+\varepsilon)}} < \infty \text{ a.s., } \lim\sup_{m\to\infty} \frac{N_m(A)}{2^{m(\alpha+\varepsilon)}} = 0$$

$$\overline{\dim}_{\mathcal{M}}(A) = \limsup_{\varepsilon \to 0} \frac{\log N(A, \varepsilon)}{\log 1/\varepsilon} \leq \limsup_{m \to \infty} \frac{\log(2^{m(\alpha + \varepsilon)})}{\log(2^m)} = \alpha + \varepsilon$$

# The number of zeros in an interval

Zeros of Brownian motion

Peng-W

Fractal dimensior

Brownian motion

The lower bound fo

The upper bound for the fractal dimension

Simulation

#### Lemma

For any  $a, \varepsilon > 0$  we have

$$\mathbb{P}(\exists t \in (a, a + \varepsilon) : B(t) = 0) \leq C\sqrt{\frac{\varepsilon}{a + \varepsilon}}$$

Consider the event 
$$A = \{|B(a + \varepsilon)| \le \sqrt{\varepsilon}\}.$$

We have 
$$\mathbb{P}(A) = \mathbb{P}\left(|B(1)| \le \sqrt{\frac{\varepsilon}{a+\varepsilon}}\right) \le 2\sqrt{\frac{\varepsilon}{a+\varepsilon}}$$

And

$$\begin{array}{ll} \mathbb{P}(A) & \geq \mathbb{P}\left(A \text{ and } 0 \in B_{[a,a+\varepsilon[}\right) \\ & \geq \min_{a \leq t \leq a+\varepsilon} \mathbb{P}\left(A|B_t=0\right) \mathbb{P}\left(0 \in B_{[a,a+\varepsilon[}\right) \end{array}\right) \end{array}$$

Since  $\mathbb{P}(|B_1| \leq 1) > 0$ , we have

$$\mathbb{P}\left(0 \in B_{[a,a+\varepsilon[}\right) \leq \frac{2}{c} \sqrt{\frac{\varepsilon}{a+\varepsilon}}$$

## The theorem

Zeros of Brownian motion

Peng-We

Fractal dimensio

Zeros of the Brownian

The lower bound for the fractal dimensi

The upper bound for the fractal dimension

Simulation

#### Theorem

With probability 1,  $\dim(Z_B) = \frac{1}{2}$ .

We have

$$\mathbb{E}\left[N_m(Z_B)\right] = \sum_{k=1}^{2^m} \mathbb{P}\left(\exists t \in \left[\frac{k-1}{2^m}, \frac{k}{2^m}\right] : B(t) = 0\right)$$

$$\leq C \sum_{k=1}^{2^m} \frac{1}{\sqrt{k}}$$

$$< \tilde{C}2^{\frac{m}{2}}$$

#### Lemma

For any  $a, \varepsilon > 0$  we have

$$\mathbb{P}(\exists t \in (\frac{k-1}{2^m}, \frac{k-1}{2^m} + \frac{1}{2^m}) : B(t) = 0) \leq C\sqrt{\frac{\frac{1}{2^m}}{\frac{k}{2^m}}}$$

# The theorem

Zeros of Brownian motion

Peng-We

Fractal dimension Definition

Zeros of the Brownian

motion

The lower bound for the fractal dimension

The upper bound for the fractal dimension

## Theorem

With probability 1,  $\dim(Z_B) = \frac{1}{2}$ .

We have

$$\mathbb{E}\left[N_m(Z_B)\right] = \sum_{k=1}^{2^m} \mathbb{P}\left(\exists t \in \left[\frac{k-1}{2^m}, \frac{k}{2^m}\right] : B(t) = 0\right)$$

$$\leq C \sum_{k=1}^{2^m} \frac{1}{\sqrt{k}}$$

$$\leq \tilde{C} 2^{\frac{m}{2}}$$

#### Lemma

Suppose A is a closed random subset of [0,1] such that

$$\mathbb{E}[N_m(A)] < c2^{m\alpha}$$

for some  $c, \alpha > 0$ . Then  $\overline{\dim}_{\mathcal{M}}(A) \leq \alpha$ .

Combine with the lower bound, we have thus  $dim(Z_B) = \frac{1}{2}$  als/23

Peng-We CHEN

Fractal

dimension

Comparison

Zeros of th Brownian

Brownian motion

the fractal dimension

Simulation results

# Section 3

Zeros of Brownian motion

Peng-We CHEN

Fractal

Definition

Zeros of th

Zeros of the Brownian

The lower bound for the fractal dimension

the fractal dimension

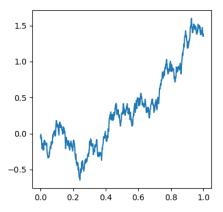


Figure: The brownian motion path

Zeros of Brownian motion

Peng-We

Fractal dimension Definition

Zeros of the Brownian motion

The lower bound for the fractal dimension.

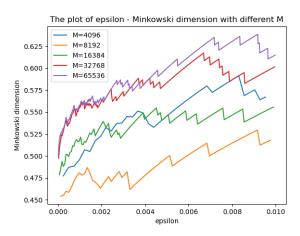


Figure: Some examples with different length of step.

Zeros of Brownian motion

Peng-We

Fractal dimensior

Zeros of th

motion

The lower bound for the fractal dimension

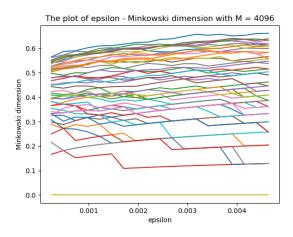


Figure: Some examples with same length of step.

Zeros of Brownian motion

Peng-We

Fractal dimension

Comparison Zeros of the

Brownian motion

the fractal dimensio

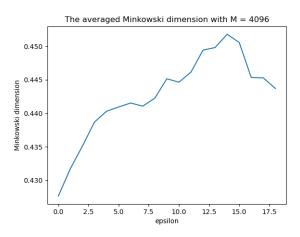


Figure: The distribution of the average Minkowski dimension.

Zeros of Brownian motion

Simulation results

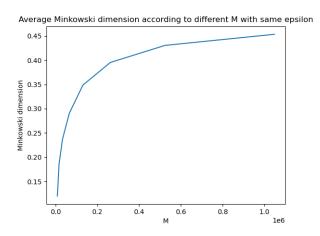


Figure: Average Minkowski dimension according to different M with same epsilon 4 D > 4 A > 4 B > 4 B >

# Thanks!

Zeros of Brownian motion

Peng-We

Fractal

dimension

Definition

Comparison

Zeros of th Brownian

Brownian motion

The lower bound for the fractal dimension