Zeros of Brownian motion

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MAP575 Foundations of probability and applications

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Section 1

Fractal dimension

Minkowski dimension

Definition

Let $N(K, \varepsilon)$ be the minimal number of open balls of diameter ε needed to cover a subset K of a metric space.
We define the upper Minkowski dimension as

$$\overline{dim}_{\mathcal{M}}(K) = \limsup_{\varepsilon \to 0} \frac{\log N(K, \varepsilon)}{\log 1/\varepsilon}$$

with lim inf for the lower Minkowski dimension. Finally, we define $dim_{\mathcal{M}}(K)$ the Minkowski dimension if these two values agree.

- Equal with two equivalent metric.
- $N(k, \varepsilon) \sim C\varepsilon^{-dim_{\mathcal{M}}(K)}$
- Box counting dimension

Minkowski dimension : examples

$$\dim_{\mathcal{M}}(K) = \lim_{\varepsilon \to 0} \frac{\log N(K, \varepsilon)}{\log 1/\varepsilon}$$

Exemples

	Finite set	$[0, 1]$	$\{0\} \cup \{1, \frac{1}{2}, \ldots, \}$
Minkowski	0	1	$\frac{1}{2}$

For $\frac{1}{n^2} < \varepsilon \leq \frac{1}{(n+1)^2}$, $0, \frac{1}{n}, \frac{1}{n+1} \cdots \in [0, \frac{1}{n}]$,
$n \leq N(K, \varepsilon) \leq 2n + 1$.

Hausdorff dimension

Definition

We define the Hausdorff dimension of K as

$$\dim(K) = \inf \left\{ \alpha : \underbrace{\inf \left\{ \sum_i |U_i|^\alpha : K \subset \bigcup_i U_i \right\}}_{\mathcal{H}^\alpha_\infty(K)} = 0 \right\}$$

Hausdorff dimension : examples

Examples

	Finite set	[0, 1]	$\{0\} \cup \{1, \frac{1}{2}, \dots, \}$
Hausdorff	0	1	0

- $\varepsilon 2^{-n}$
- For $\alpha > 1$, $\mathcal{H}^{\alpha}_{\infty}(K) = 0$.

Mass Distribution Principle

Lemma(Mass Distribution Principle)

If $\mu(B(x,r)) \leq Cr^{\alpha}$ for a strictly positive Borel measure μ on E with a certain C and α, then

$$\dim(E) \geq \alpha$$

$(U_i)_{i\in\mathbb{N}}$ so that $E \subset \bigcup_i U_i$, $(r_i)_{i\in\mathbb{N}}$ so that $|(U_i)| < r_i$.

$$\mu(U_i) \leq \mu\left(B(x_i, r_i)\right) \leq Cr_i^{\alpha}$$

$$\sum_i |U_i|^{\alpha} \geq \sum_i \frac{\mu(U_i)}{C} \geq \frac{\mu(E)}{C} > 0$$

Comparison between the two dimensions

Definition

$$\lim_{\varepsilon \to 0} \frac{\log N(K,\varepsilon)}{\log 1/\varepsilon} \text{ and } \inf\{\alpha : \mathcal{H}^{\alpha}_{\infty}(K) = 0\}$$

Examples

	Finite set	$[0,1]$	$\{0\} \cup \{1, \frac{1}{2}, \ldots, \}$
Minkowski	0	1	$\frac{1}{2}$
Hausdorff	0	1	0

Inequality

Proposition

$\dim(K) \le \underline{\dim}_{\mathcal{M}}(K) \le \overline{\dim}_{\mathcal{M}}(K)$

$B_i = B(x_i, \frac{\varepsilon}{2})$, $N(K, \varepsilon)$ balls.
For $\alpha > \underline{\dim}_{\mathcal{M}}$,
$\sum_{i=1}^{N(K,\varepsilon)} |B_i|^\alpha = N(K, \varepsilon) \left(\frac{\varepsilon}{2}\right)^\alpha \le \frac{1}{2^\alpha} \varepsilon^{\alpha - \underline{\dim}_{\mathcal{M}}} \to 0.$

Section 2

Zeros of the Brownian motion

The fractal dimension of zeros of the Brownian motion

Notation

We note $Z_B = \{t \in [0, +\infty) : B(t) = 0\}$ the set of zeros of the Brownian motion B_t.

Theorem

With probability 1, $\dim(Z_B) = \frac{1}{2}$.

$\frac{C}{\sqrt{\varepsilon}}$ balls.

$\sum_i |U_i|^\alpha = 0$ for $\alpha > \frac{1}{2}$.

Lower bound

Lemma

With probability 1,

$$\dim\{t \in [0,1] : Y(t) = 0\} \geq 1/2$$

where $Y(t) = \max_{0 \leq s \leq t} B_s - B_t$.

Let $M(t) = \max_{0 \leq s \leq t} B_s$,
$\mu([a,b]) = M(b) - M(a)$ define a measure with support strictly positive on $\{t \in [0,1] : Y(t) = 0\}$.

$$M(b) - M(a) \leq \max_{0 \leq h \leq b-a} B_{a+h} - B_a \leq C_\alpha (b-a)^\alpha$$

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$$M(b) - M(a) \leq \max_{0 \leq h \leq b-a} B_{a+h} - B_a \leq C_\alpha (b-a)^\alpha$$

(Admitted), Hölder continuous

The Brownian motion is almost surely α-Hölder continuous for $\alpha < 0.5$.

Lower bound

Lemma

With probability 1,

$$\dim\{t \in [0,1] : Y(t) = 0\} \geq 1/2$$

where $Y(t) = \max_{0\leq s\leq t} B_s - B_t$.

Let $M(t) = \max_{0\leq s\leq t} B_s$,
$\mu([a,b]) = M(b) - M(a)$ define a measure with support strictly positive on $\{t \in [0,1] : Y(t) = 0\}$.

$$M(b) - M(a) \leq \max_{0\leq h\leq b-a} B_{a+h} - B_a \leq C_\alpha (b-a)^\alpha$$

Recall : Mass Distribution Principle

$\mu(B(x,r)) \leq Cr^\alpha \rightarrow \dim(E) \geq \alpha$

The counter of a set

The counter

We note

$$N_m(A) = \sum_{k=1}^{2^m} 1_{A \cap [\frac{k-1}{2^m}, \frac{k}{2^m}] \neq \emptyset}$$

The counter of a set

Lemma

Suppose A is a closed random subset of $[0,1]$ such that

$$\mathbb{E}[N_m(A)] \le c2^{m\alpha}$$

for some $c, \alpha > 0$. Then $\overline{\dim}_{\mathcal{M}}(A) \le \alpha$.

$$\mathbb{E}\left[\sum_{m=1}^{\infty} \frac{N_m(A)}{2^{m(\alpha+\varepsilon)}}\right] = \sum_{m=1}^{\infty} \frac{\mathbb{E}N_m(A)}{2^{m(\alpha+\varepsilon)}} < \infty$$

$\sum_{m=1}^{\infty} \frac{N_m(A)}{2^{m(\alpha+\varepsilon)}} < \infty$ a.s., $\limsup_{m\to\infty} \frac{N_m(A)}{2^{m(\alpha+\varepsilon)}} = 0$

$$\overline{\dim}_{\mathcal{M}}(A) = \limsup_{\varepsilon\to 0} \frac{\log N(A,\varepsilon)}{\log 1/\varepsilon} \le \limsup_{m\to\infty} \frac{\log(2^{m(\alpha+\varepsilon)})}{\log(2^m)} = \alpha + \varepsilon$$

The number of zeros in an interval

Lemma

For any $a, \varepsilon > 0$ we have

$$\mathbb{P}(\exists t \in (a, a+\varepsilon) : B(t) = 0) \leq C\sqrt{\frac{\varepsilon}{a+\varepsilon}}$$

Consider the event $A = \{|B(a+\varepsilon)| \leq \sqrt{\varepsilon}\}$.
We have $\mathbb{P}(A) = \mathbb{P}\left(|B(1)| \leq \sqrt{\frac{\varepsilon}{a+\varepsilon}}\right) \leq 2\sqrt{\frac{\varepsilon}{a+\varepsilon}}$
And

$$\begin{aligned}\mathbb{P}(A) &\geq \mathbb{P}\left(A \text{ and } 0 \in B_{[a,a+\varepsilon[}\right)\\ &\geq \min_{a \leq t \leq a+\varepsilon} \mathbb{P}\left(A | B_t = 0\right) \mathbb{P}\left(0 \in B_{[a,a+\varepsilon[}\right)\end{aligned}$$

Since $\mathbb{P}\left(|B_1| \leq 1\right) > 0$, we have

$$\mathbb{P}\left(0 \in B_{[a,a+\varepsilon[}\right) \leq \frac{2}{c}\sqrt{\frac{\varepsilon}{a+\varepsilon}}$$

The theorem

Theorem

With probability 1, $\dim(Z_B) = \frac{1}{2}$.

We have

$$\begin{aligned}\mathbb{E}\left[N_m(Z_B)\right] &= \sum_{k=1}^{2^m} \mathbb{P}\left(\exists t \in [\tfrac{k-1}{2^m}, \tfrac{k}{2^m}] : B(t) = 0\right) \\ &\leq C \sum_{k=1}^{2^m} \frac{1}{\sqrt{k}} \\ &\leq \tilde{C} 2^{\frac{m}{2}}\end{aligned}$$

Lemma

For any $a, \varepsilon > 0$ we have

$$\mathbb{P}(\exists t \in (\frac{k-1}{2^m}, \frac{k-1}{2^m} + \frac{1}{2^m}) : B(t) = 0) \leq C\sqrt{\frac{\frac{1}{2^m}}{\frac{k}{2^m}}}$$

The theorem

Theorem

With probability 1, $\dim(Z_B) = \frac{1}{2}$.

We have

$$\begin{aligned} \mathbb{E}\left[N_m(Z_B)\right] &= \sum_{k=1}^{2^m} \mathbb{P}\left(\exists t \in [\tfrac{k-1}{2^m}, \tfrac{k}{2^m}] : B(t) = 0\right) \\ &\leq C \sum_{k=1}^{2^m} \frac{1}{\sqrt{k}} \\ &\leq \tilde{C} 2^{\frac{m}{2}} \end{aligned}$$

Lemma

Suppose A is a closed random subset of $[0, 1]$ such that

$$\mathbb{E}[N_m(A)] \leq c 2^{m\alpha}$$

for some $c, \alpha > 0$. Then $\overline{\dim}_{\mathcal{M}}(A) \leq \alpha$.

Combine with the lower bound, we have thus $dim(Z_B) = \frac{1}{2}$ a.s.

Section 3

Simulation results

Some simulations for the Minkowski dimension

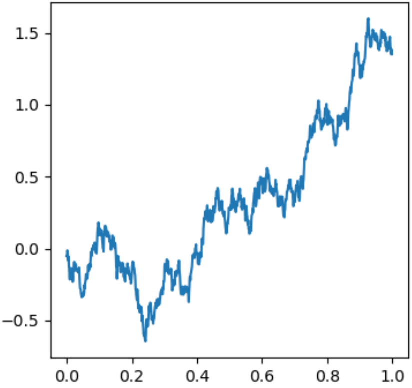

Figure: The brownian motion path

Some simulations for the Minkowski dimension

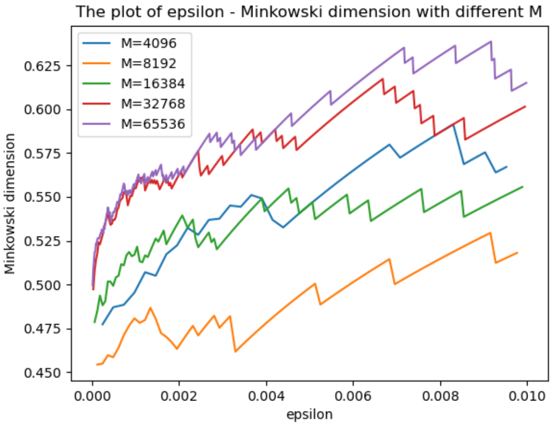

Figure: Some examples with different length of step.

Some simulations for the Minkowski dimension

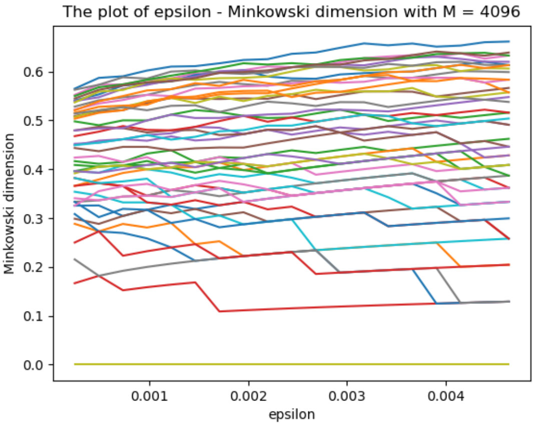

Figure: Some examples with same length of step.

Some simulations for the Minkowski dimension

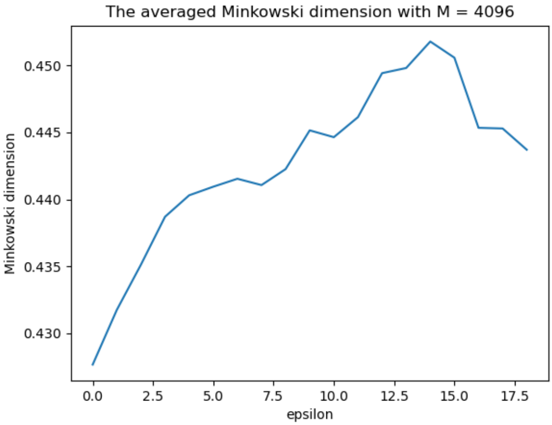

Figure: The distribution of the average Minkowski dimension.

Some simulations for the Minkowski dimension

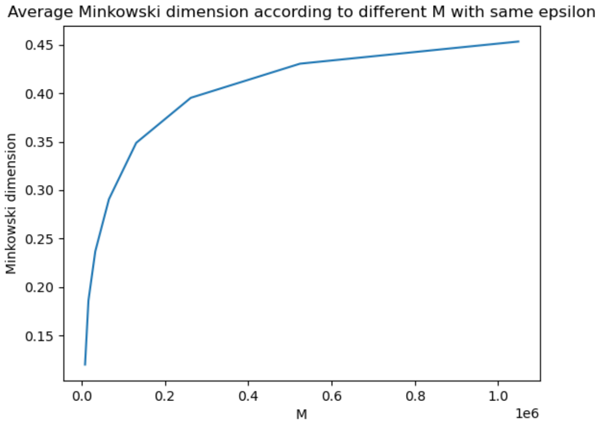

Figure: Average Minkowski dimension according to different M with same epsilon

Thanks!