Chapter 8

Simplification

8.1 Reciprocal of a Number

The reciprocal of the nonzero number N is $\frac{1}{N}$. Observe that the product of a nonzero number and its reciprocal is always $1 : N \cdot \frac{1}{N} = \frac{N}{N} = 1$.
EXAMPLES: The reciprocal of 2 by definition is $\frac{1}{2}$. By definition, the reciprocal of $\frac{1}{2}$ is $\frac{1}{\frac{1}{2}}$ which is equal to $\frac{1 \cdot 2}{\frac{1}{2} \cdot 2} = \frac{2}{1} = 2$. Similarly, the reciprocal of $\frac{2}{3}$ is $\frac{1}{\frac{2}{3}}$ which likewise is equal to $\frac{1 \cdot \frac{3}{2}}{\frac{2}{3} \cdot \frac{3}{2}} = \frac{3}{2}$.

RULE: The reiprocal of the nonzero fraction $\frac{a}{b}$ is equivalent to $\frac{b}{a}$

Another name for reciprocal is *multiplicative inverse*.

8.2 Complex Fractions

A complex fraction is a fraction whose numerator or denominator (or both) contains a fraction. A complex fraction can be simplified and expressed as an equivalent simple fraction.

EXAMPLE: $\frac{5}{\frac{3}{4}} = 5 \cdot \frac{4}{3} = \frac{20}{3}$. Note that instead of dividing by $\frac{3}{4}$, we multiply the numerator by the reciprocal of $\frac{3}{4}$. (This is commonly referrred to as "invert and multiply".) The following shows why this is mathematically correct:

$$\frac{5}{\frac{3}{4}} = \frac{5 \cdot \frac{4}{3}}{\frac{3}{4} \cdot \frac{4}{3}} = \frac{5 \cdot \frac{4}{3}}{1} = 5 \cdot \frac{4}{3}$$

8.3 Extended Fractions

In this book, an example of an "extended fraction" (or continued fractions) is $\frac{1}{5+\frac{1}{5+\frac{1}{5}}}$ ... fraction can be written as a simple fraction by simplifying it "from the bottom up":

$$\frac{1}{5+\frac{1}{5+\frac{1}{5}}} = \frac{1}{5+\frac{1}{\frac{26}{5}}} = \frac{1}{5+\frac{5}{26}} = \frac{1}{\frac{135}{26}} = \frac{26}{135}$$

8.4 Factors of a Number

A factor of a number N is a number which divides N exactly.
Example: the factors of 12 are 1, 2, 3, 4, 6, and 12. Notice that every number N has
1 among its factors. Recall that when a number is greater than 1 and has just itself
factors, the number is *prime*.

8.5 Pairing Factors of a Number

When all the different factors of a given number N are listed, they usually can be
that the product of each pair is N. Consider the factors of 30.

$$\underbrace{1 \quad 30} \quad \underbrace{2 \quad 15} \quad \underbrace{3 \quad 10} \quad \underbrace{5 \quad 6}$$

In the above diagram, notice that the eight different factors of 30 can be paired so
product of each pair is 30 : $1 \cdot 30, 2 \cdot 15, 3 \cdot 10, 5 \cdot 6$.

Consider the different factors of 36.

$$\underbrace{1 \quad 36} \quad \underbrace{2 \quad 18} \quad \underbrace{3 \quad 12} \quad \underbrace{4 \quad 9} \quad \underbrace{6}$$

The above pairing reveals that there is an odd number of factors. Since 36 is a square
6 must be paired with itself to yield the product 36.

RULE: A square number always has an odd number of different factors.

8.6 Factoring Completely

A number is factored completely when it is expressed as a product of primes.
EXAMPLE: $90 = 2 \cdot 3 \cdot 3 \cdot 5$. The right side, $2 \cdot 3 \cdot 3 \cdot 5$, is the *complete factori*
90. The factors of 90 can be written as 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, and 9
$1, 2, 3, 5, 2 \cdot 3, 3 \cdot 3, 2 \cdot 5, 3 \cdot 5, 2 \cdot 3 \cdot 3, 2 \cdot 3 \cdot 5, 3 \cdot 3 \cdot 5$, and $2 \cdot 3 \cdot 3 \cdot 5$)

8.7 Exponents

Suppose the complete factorization of a number is $7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$. It is convinent to write this as 7^5. Here the small 5 placed to the upper right of 7 tells us that 7^5 represent a product in which 7 appears as a factor 5 times. The small number 5 in this case is called the *exponent* and the number 7 which appears as a factor is called the *base*. Clearly, the factors of 7^5 are $1, 7, 7 \cdot 7, 7 \cdot 7 \cdot 7, 7 \cdot 7 \cdot 7 \cdot 7, 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$. The factors of 7^5 can also be represented using the exponents: $1, 7^1, 7^2, 7^3, 7^4$, and 7^5. From these *exponential forms*, one can see that 7^5 has 6 different factors.

RULES: If P is a prime number, then P^N has $N + 1$ different factors.

8.8 How many Factors Does a Number Have?

EXAMPLE: How many different factors does 400 have?

Factor 400 completely in exponential form:
$400 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 = 2^4 \cdot 5^2$
2^4 has 5 different factors and 5^2 has 3 different factors
The number 400 will have $5 \cdot 3 = 15$ different factors

A listing of the different factors is instructive. The factors of 2^4 are $1, 2, 4, 8, 16$; the factors of 5^2 are $1, 5, 25$. The different factors of 400 are presented in the following tables:

In Factored Form			Same results simplified		
$1 \cdot 1$	$1 \cdot 5$	$1 \cdot 25$	1	5	25
$2 \cdot 1$	$2 \cdot 5$	$2 \cdot 25$	2	10	50
$4 \cdot 1$	$4 \cdot 5$	$4 \cdot 25$	4	20	100
$8 \cdot 1$	$8 \cdot 5$	$8 \cdot 25$	8	40	200
$16 \cdot 1$	$16 \cdot 5$	$16 \cdot 25$	16	80	400

The table at the left shows each of the 15 factors as a product of two factors. Observe the pattern of the entries in this table both vertically and horizontally. The table at the right shows each of the 15 factors in standard form. These entries can be used to make the following ordered list of the 15 factors: 1, 2, 4, 5, 8, 10, 16, 20, 25, 40, 50, 80, 100, 200, 400.

RULE: If $N = P^a \cdot Q^b \cdot R^c \cdot \ldots$ where $P, Q, R, \ldots$ are primes and $a, b, c, \ldots$ are exponents, then N has $(a + 1) \cdot (b + 1) \cdot (c + 1) \cdot \ldots$ factors.

8.9 Factors, Multiples, and Divisibility

If F is a factor of N, then *F divides N exactly, N is divisble by F, and N is a multiple of F.*

EXAMPLE. The following statements are equivalent.

1. 6 is a factor of 30
2. 6 divides 30 exactly
3. 30 is divisible by 6
4. 30 is a multiple of 6