Division **M** | *Mathematical Olympiads* **January 15, 2014** *for Elementary & Middle Schools* | Contest **3**

3A *Time: 3 minutes*

At its last competition, the 12 students of the Centerville Math Team had the following distribution of scores: 1 student scored 5 points, 3 each scored 4 points, 4 each scored 3 points, 1 student scored 2 points, 1 student scored 1 point, and 2 students scored 0 points. To the nearest tenth, what was the mean (average) individual score?

3B *Time: 5 minutes*

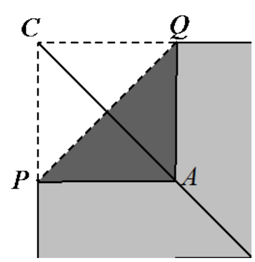
Write as a fraction in lowest terms: $\dfrac{\left(\frac{2}{9}\right)^3+\left(\frac{2}{9}\right)^2}{\left(\frac{1}{3}-\frac{1}{9}\right)^2\times 1\frac{5}{6}}$.

3C *Time: 5 minutes*

Find the greatest prime factor of the sum of 10! and 8!.
[Note: $n! = n\times(n-1)\times(n-2)\times\ldots\times 2\times 1$ and is read as "*n* factorial".
For example: 6! = 6 × 5 × 4 × 3 × 2 × 1.]

3D *Time: 6 minutes*

A square sheet of paper has an area of 60 square units. The paper is folded so that corner *C* lands on point *A* as shown. The area of the lightly shaded L-shaped region is three times the area of the folded over darkly shaded region, ΔPAQ. The length of the "crease" $\overline{PQ}$ is $\sqrt{N}$. Find *N*.



3E *Time: 7 minutes*

Two trains, each traveling at 60 kph [1 kilometer/minute] are on parallel tracks heading toward one another. One train is one-half kilometer long and the other train is two-thirds kilometer long. The fronts of the two trains pass the same point *A* at the same time. How many <u>seconds</u> does it take for the two trains to completely pass each other?

Please fold over on line. Write answers on back.

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Contest **3**

3A *Student Name and Answer*

3B *Student Name and Answer*

3C *Student Name and Answer*

3D *Student Name and Answer*

$N =$

3E *Student Name and Answer*

Seconds

Please fold over on line. Write answers in these boxes.

SOLUTIONS AND ANSWERS

3A METHOD 1: *Strategy: Use the definition of mean (average).*

Add the scores and divide by 12: (5 + 4 + 4 + 4 + 3 + 3 + 3 + 3 + 2 + 1 + 0 + 0)÷12 = 32÷12. **Since 32 ÷ 12 = 2.666…, the mean to the nearest tenth is 2.7.**

METHOD 2: *Strategy: Use a quick way to add the scores.*

1(5) + 3(4) + 4(3) + 1(2) + 1(1) + 2(0) = 32 and 32/12 = 2.666… = 2.7 to the nearest tenth.

FOLLOW-UP: If there were two additional mathletes who had identical scores, what would each of them scored to raise the mean to be exactly 3? [5 each]

3A answer: 2.7

3B METHOD 1: *Strategy: Factor and simplify numerator and denominator separately.*

$$\frac{\left(\frac{2}{9}\right)^3+\left(\frac{2}{9}\right)^2}{\left(\frac{1}{3}-\frac{1}{9}\right)^2\times 1\frac{5}{6}}=\frac{\left(\frac{2}{9}\right)^2\left(\frac{2}{9}+1\right)}{\left(\frac{2}{9}\right)^2\left(\frac{11}{6}\right)}=\frac{\left(\frac{11}{9}\right)}{\left(\frac{11}{6}\right)}=\frac{11}{9}\times\frac{6}{11}=\frac{2}{3}$$

The fraction in lowest terms is 2/3.

METHOD 2: *Strategy: Simplify numerator and denominator and then divide.*

Numerator: $\left(\frac{2}{9}\right)^3+\left(\frac{2}{9}\right)^2=\frac{8}{729}+\frac{4}{81}=\frac{8}{729}+\frac{36}{729}=\frac{44}{729}$.

Denominator: $\left(\frac{1}{3}-\frac{1}{9}\right)^2\times 1\frac{5}{6}=\left(\frac{2}{9}\right)^2\times\frac{11}{6}=\frac{4}{81}\times\frac{11}{6}=\frac{44}{486}$.

Divide: $\frac{44}{729}\div\frac{44}{486}=\frac{44}{729}\times\frac{486}{44}=\frac{2}{3}$

3B answer: $\frac{2}{3}$

3C *Strategy: Apply the definition of factorial.*

Since 10! = 10 × 9 × 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1 and 8! = 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1, we can deduce that 10! = 10 × 9 × 8!.

10! + 8! = 10 × 9 × 8! + 8!
= 8!(10 × 9 + 1)
= 8!(91)
= 8! × 7 × 13

Every prime factor of 8! is less than 8. **The greatest prime factor is 13.**

3C answer: 13

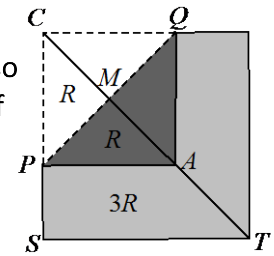
3D: 48

3E: 35

3D METHOD 1: *<u>Strategy</u>: Apply area formulas.*

Let K equal the area of ΔPAQ. Then the area of square $CPAQ$ is $2K$ and the areaa of the L-shaped region is $3K$. The total area of the big square is 60 so $60 = 2K + 3K = 5K$. K is 12 and the area of square $CPAQ = 24$. Since the area of a square is s^2, where s represents the length of a side of the square, $s=\sqrt{24}$. Use the Pythagorean theorem, $\left(\sqrt{24}\right)^2+\left(\sqrt{24}\right)^2=(PQ)^2$ so $(PQ)^2=48$ and $PQ=\sqrt{48}$. **Therefore N = 48.**

METHOD 2: *<u>Strategy</u>: Apply the concept that the ratio of areas of similar triangle equals the ratio of the squares of the corresponding sides.*



Let the area of $\Delta PMA = R$. Then the area of ΔCMP is also R and the area of quadrilateral $PATS$ is $3R$. Both ΔPMA and ΔCST are right isosceles triangles so they are similar to each other and the ratio of their areas equals the ratio of the squares of a pair of corresponding sides. That is $\dfrac{A_{\Delta PMA}}{A_{\Delta CST}}=\dfrac{(MP)^2}{(ST)^2}$. Thus

$\dfrac{R}{5R}=\dfrac{(MP)^2}{\left(\sqrt{60}\right)^2}$ and $(MP)^2=\dfrac{60}{5}=12$ so $MP=\sqrt{12}$. It follows that $PQ=2\sqrt{12}=\sqrt{4}\sqrt{12}=\sqrt{48}$.

Consequently N = 48.

3E METHOD 1: *<u>Strategy</u>: Use the formula distance = rate $\times$ time.*

Since the shorter train is 1/2 kilometer long and it travels at 1 km/min, it will take 1/2 minute to completely pass point A. The longer train is 2/3 km long so it still has an additional 1/6 km (2/3 – 1/2 = 1/6) before it too passes point A. Since both trains are traveling at 1 km/min and each has to travel 1/2 of the 1/6 km they must each travel 1/12 km. This will take an additional 1/12 of a minute or 5 seconds. **The total amount of time needed for the two trains to pass is 1/2 minute + 5 seconds or 35 seconds.**

METHOD 2: *<u>Strategy</u>: Consider one train moving at 0 km/h and the other at 120 km/h.*

If we keep one train from moving and let the other move at the sum of their rates, 120 km/h or 2kn per minute (1 km per so seconds), it will take the same amount of time for the one train to pass the other. The total distance the one train must move is 1/2 km + 2/3 km = 7/6 km. This takes 7/6 of 30 seconds or 35 seconds.