Division M

Mathematical Olympiads for Elementary and Middle Schools

February 8, 2012

Contest 4

4A ***Time: 3 minutes***

In the 4×4 grid shown, the numbers 1, 2, 3, and 4 can appear exactly once in each row, each column, and each shaded 2×2 grid. Given the entries in five boxes, what number should replace X?

	4	1	
			2
3			X
		3	

4B ***Time: 4 minutes***

Express in lowest terms:

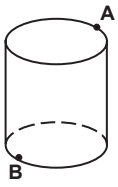
$$\frac{5 - 10 + 15 - 20 + \cdots - 490 + 495 - 500}{7 - 14 + 21 - 28 + \cdots - 686 + 693 - 700}$$

4C ***Time: 5 minutes***

The sum of three whole numbers, *A*, *B*, and *C*, is 32. *C* is 10 more than *A*. *B* differs from one of the other numbers by 3 and the other by 7. Find *B*.

4D ***Time: 6 minutes***

In the tin can shown, the height is 8 cm and the circumference is 12 cm. Points A and B lie "opposite" each other on the two rims. Find the shortest distance along the surface of the can from A to B.



4E ***Time: 7 minutes***

Two different primes are randomly selected from the first nine prime numbers. What is the probability that their sum is 30? *(Answer must be expressed in fraction form.)*

Please fold over on line. Write answers on back.

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Contest **4**

4A *Student Name and Answer*

4B *Student Name and Answer*

4C *Student Name and Answer*

B =

4D *Student Name and Answer*

cm

4E *Student Name and Answer*

Please fold over on line. Write answers in these boxes.

Division M — Mathematical Olympiads for Elementary and Middle Schools — February 8, 2012 — Contest 4

SOLUTIONS AND ANSWERS

4A *Strategy: Use reasoning.*

	4	1	A
		B	2
3	C	D	X
		3	

First fill in the boxes labeled A and B which must contain the numbers 3 and 4. A is not 4, so A must be 3 and B must be 4. Next fill in box D with a 2. Then C is 1 or 4. Since C cannot be 4, C is 1 and **the box marked with an X contains a 4.**

Follow-Up: *Create your own 4 × 4 Sudoku-type puzzle by filling in the answers and then erasing some of the numbers. Make sure that the numbers you leave allow only 1 solution.*

4B **METHOD 1:** *Strategy: Factor the numerator and denominator.*

Factor out the common factor in both the numerator and denominator and then cancel to get the following.

$$\frac{5\,(1 - 2 + 3 - 4 + \ldots - 98 + 99 - 100)}{7\,(1 - 2 + 3 - 4 + \ldots - 98 + 99 - 100)} = \mathbf{\frac{5}{7}}$$

METHOD 2: *Strategy: Pair numbers in both numerator and denominator.*

$$\frac{(5 - 10) + (15 - 20) + \ldots + (495 - 500)}{(7 - 14) + (21 - 28) + \ldots + (693 - 700)}$$

$$= \frac{(-5) + (-5) + (-5) + \ldots + (-5)}{(-7) + (-7) + (-7) + \ldots + (-7)} = \frac{-250}{-350} = \frac{5}{7}$$

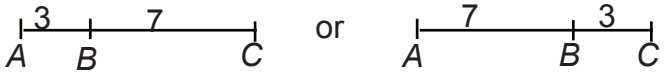
METHOD 3: *Strategy: Look for a pattern in the partial sums.*

Start with $\frac{5}{7}$, next $\frac{5-10}{7-14}$, then $\frac{5-10+15}{7-14+21}$, and so on. In each case the fraction equals $\frac{5}{7}$. In the given fraction both the numerator and denominator contain the same number of terms, 100, so the value remains at $\frac{5}{7}$.

Follow-Up: *Explain why the successive fractions listed in Method 3 are equivalent.*

4C *Strategy: Set up and solve possible equations.*

C is 10 more than *A*. Also, *B*. is 3 away from one number and 7 away from the other number. Then $A < B < C$.

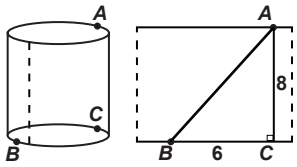


Either $B = A + 3$ or $B = A + 7$. Substitute each possibility into the equation $A + B + C = 32$ to get $A + (A + 3) + (A + 10) = 32$ or $A + (A + 7) + (A + 10) = 32$. In the first case, $A = 19/3$ and the second equation results in $A = 5$. *A* is a whole number so $A = 5$ and $\boldsymbol{B = 12}$.

4A	**4**
4B	$\mathbf{\frac{5}{7}}$
4C	**12**
4D	**10**
4E	$\mathbf{\frac{3}{36}}$ *or equivalent*

4D *Strategy: Remove the label from the can.*

Cut open the cylinder along the dotted line and unroll it to get a rectangle with *A* on the top edge and *B* on the bottom edge. Then connect *A* and *B* with a straight line segment. Notice that *AC* = 8 cm and *BC* = 6 cm. (*BC* is one-half the circumference). Apply the Pythagorean Theorem or recognize the Pythagorean triple 6-8-10 to get ***AB* = 10 cm.**



4E *Strategy: List the primes; find the pairs that add to 30; list all the possible pairs.*

The first nine prime numbers are: 2, 3, 5, 7, 11, 13, 17, 19, and 23.
The only pairs that have a sum of 30 are {7, 23}, {11, 19}, and {13, 17}.

There are several ways to count all possible pairs of primes.

- Method 1: The prime 2 can be paired with each of the 8 other primes listed. 3 has already been paired with 2, so there are an additional 7 pairs for 3. Similarly, there are 6 additional pairs for 5, and so on. In all there are 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36 pairs of primes.
- Method 2: There are 9 possible values for the first of the two primes. Each of these may be paired with any of the 8 remaining values, a total of 9×8 = 72. This, however, counts each pair twice (e. g. 2 paired with 3 and 3 paired with 2). The total number of pairs is 72÷2 = 36.
- Method 3: List all pairs of primes in an orderly manner.

The probability that the sum of the 2 primes 30 is $^{3}/_{36} = {}^{1}/_{12}$.

Follow-Ups: *(1) Two different primes are selected from the first 9 prime numbers. What is the probability that their sum is odd?* [$^{2}/_{9}$] *(2) Two primes, not necessarily different, are selected from the first 9 prime numbers. What is the probability that their sum is odd?* [$^{16}/_{81}$] *(3) Two different primes are selected from the first 20 prime numbers. What is the probability that their sum is 30?* [$^{3}/_{190}$]

NOTE: *Other Follow-Up problems related to some of the above can be found in our books "Math Olympiad Contest Problems for Elementary and Middle Schools" and "Creative Problem Solving in School Mathematics."*