ISyE 6739 — Summer 2017

Homework #3 (Covers Modules 2.1–2.5) — Solutions

1. A die is rolled 5 times. Let X denote the number of times that you see a 4, 5, or 6.

   (a) What's the distribution of X?
   Solution: $X \sim \text{Bin}(5, 1/2)$. $\diamondsuit$

   (b) Find $\mathsf{Pr}(X = 4)$.
   Solution:
   $$\mathsf{Pr}(X=4) \;=\; \binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{5-4} \;=\; 5/32. \quad \diamondsuit$$

2. Suppose $X \sim \text{Pois}(2)$. Find $\mathsf{Pr}(X > 3)$.
   Solution:
   $$\mathsf{Pr}(X>3) \;=\; 1 - \mathsf{Pr}(X \le 3) \;=\; 1 - \sum_{k=0}^{3} e^{-2}2^k/k! \;=\; 0.1429. \quad \diamondsuit$$

3. Suppose X has the following discrete distribution.

x	-1	0	2	3
$\mathsf{Pr}(X=x)$	0.2c	0.3	0.2	0.1

   (a) Find the value of c that will make the p.m.f. sum to 1.
   Solution: Note that
   $$1 \;=\; \sum_x \mathsf{Pr}(X=x) \;=\; 0.2c + 0.3 + 0.2 + 0.1.$$
   This implies that $c = 2$. $\diamondsuit$

   (b) Find the c.d.f. $F(x)$ for all x.
   Solution: We have
   $$F(x) \;=\; \mathsf{Pr}(X \le x) \;=\; \begin{cases} 0 & \text{if } x < -1 \\ 0.4 & \text{if } -1 \le x < 0 \\ 0.7 & \text{if } 0 \le x < 2 \\ 0.9 & \text{if } 2 \le x < 3 \\ 1.0 & \text{if } x \ge 3 \end{cases} \quad \diamondsuit$$

(c) Calculate $\mathsf{E}[X]$.
Solution:

$$\mathsf{E}[X] \;=\; \sum_x x\mathsf{Pr}(X=x) \;=\; 0.3. \quad \diamondsuit$$

(d) Calculate $\mathsf{Var}(X)$.
Solution:

$$\mathsf{E}[X^2] \;=\; \sum_x x^2\mathsf{Pr}(X=x) \;=\; 2.1.$$

This implies that $\mathsf{Var}(X) = \mathsf{E}[X^2] - (\mathsf{E}[X])^2 = 2.01$. $\diamondsuit$

(e) Calculate $\mathsf{Pr}(1 \le X \le 2)$.
Solution: Since X can only equal $-1, 0, 2, 3$, we have $\mathsf{Pr}(1 \le X \le 2) = \mathsf{Pr}(X = 2) = 0.2$. $\diamondsuit$

4. Suppose that X is the lifetime of a lightbulb and that $X \sim \text{Exp}(2/\text{year})$.

(a) What's the probability that the bulb will survive at least a year, $\mathsf{Pr}(X > 1)$?
Solution: If $X \sim \text{Exp}(\lambda)$, we know from class that $\mathsf{Pr}(X > t) = e^{-\lambda t}$. Thus, in this problem, we find that $\mathsf{Pr}(X > 1) = e^{-2}$. $\diamondsuit$

(b) Suppose the bulb has already survived a year. What's the probability that it will survive another year, i.e., $\mathsf{Pr}(X > 2 \,|\, X > 1)$?
Solution: By conditional probability, we have

$$\mathsf{Pr}(X > 2 \,|\, X > 1) \;=\; \frac{\mathsf{Pr}(X > 2 \,\cap\, X > 1)}{\mathsf{Pr}(X > 1)} \;=\; \frac{\mathsf{Pr}(X > 2)}{\mathsf{Pr}(X > 1)} \;=\; \frac{e^{-4}}{e^{-2}} \;=\; e^{-2}.$$

Note that this is the same answer as in (a), and is an example of what is called the memoryless property (which we will talk about later). $\diamondsuit$

5. Suppose that X is continuous with p.d.f. $f(x) = cx^2$, $0 \le x \le 1$.

(a) Find the value of c that will make the p.d.f. integrate to 1.
Solution: Note that

$$1 \;=\; \int_{\Re} f(x)\,dx \;=\; \int_0^1 cx^2\,dx \;=\; c/3.$$

This implies that $c = 3$. $\diamondsuit$

(b) Find the c.d.f. $F(x)$ for all x.
Solution:

$$F(x) \;=\; \mathsf{Pr}(X \le x) \;=\; \begin{cases} 0 & \text{if } x < 0 \\ x^3 & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x \ge 1 \end{cases} \qquad \diamondsuit$$

(c) Calculate $\mathsf{E}[X]$.
Solution:

$$\mathsf{E}[X] \;=\; \int_{\Re} x f(x)\, dx \;=\; \int_0^1 3x^3\, dx = 3/4. \quad \diamondsuit$$

(d) Calculate $\mathsf{Var}(X)$.
Solution: Similarly,

$$\mathsf{E}[X^2] \;=\; \int_{\Re} x^2 f(x)\, dx \;=\; \int_0^1 3x^4\, dx \;=\; 3/5.$$

This implies that $\mathsf{Var}(X) = \mathsf{E}[X^2] - (\mathsf{E}[X])^2 = 3/80$. $\quad\diamondsuit$

(e) Calculate $\mathsf{Pr}(0 \le X \le 1/2)$.
Solution: $\mathsf{Pr}(0 \le X \le 1/2) = F(1/2) - F(0) = 1/8$. $\quad\diamondsuit$

6. Let $\mathsf{E}[X] = -4$, $\mathsf{Var}(X) = 5$, and $Z = -4X + 7$. Find $\mathsf{E}[-3Z]$ and $\mathsf{Var}(-3Z)$.
Solution: We have

$$\mathsf{E}[-3Z] \;=\; \mathsf{E}[12X - 21] \;=\; 12\mathsf{E}[X] - 21 \;=\; -69. \quad \diamondsuit$$

and

$$\mathsf{Var}(-3Z) \;=\; \mathsf{Var}(12X - 21) \;=\; 144\mathsf{Var}(X) \;=\; 720. \quad \diamondsuit$$

7. When a machine is adjusted properly, 50% of the items it produces are good and 50% are bad. However, the machine is *improperly* adjusted 10% of the time; in this case, 25% of the items it makes are good and 75% are bad. Suppose that 5 items produced by the machine are selected at random and inspected. If 4 of these items are good (and 1 is bad), what's the probability that the machine was adjusted properly at the time? **Hint:** Try Bayes Theorem using Binomial conditional probabilities.

Solution: Let X be the number of good items (out of 5). Further, define the following events.

$$\begin{aligned} P &= \text{"machine is properly adjusted", which implies } X \sim \text{Bin}(5, 1/2) \\ I &= \text{"machine is improperly adjusted", which implies } X \sim \text{Bin}(5, 1/4) \end{aligned}$$

Now Bayes implies that

$$\begin{aligned} \mathsf{Pr}(P|X=4) &= \frac{\mathsf{Pr}(X=4|P)\mathsf{Pr}(P)}{\mathsf{Pr}(X=4|P)\mathsf{Pr}(P)+\mathsf{Pr}(X=4|I)\mathsf{Pr}(I)} \\ &= \frac{\binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1(0.9)}{\binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1(0.9)+\binom{5}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^1(0.1)} \\ &= 0.9897. \quad \diamondsuit \end{aligned}$$

8. Suppose that $X \sim \text{Unif}(-1, 6)$. Compare the upper bound on the probability $\mathsf{Pr}(|X-\mu| \geq 1.5\sigma)$ obtained from Chebychev's inequality with the exact probability.

Solution: By Chebychev, we get the following bound.

$$\mathsf{Pr}(|X-\mu| \geq 1.5\sigma) \leq \frac{1}{(1.5)^2} = 4/9. \quad \diamondsuit$$

Now let's get the exact probability. First of all, recall that if $X \sim \text{U}(a, b)$, then $\mathsf{E}[X] = (a+b)/2$ and $\mathsf{Var}(X) = (a-b)^2/12$. Since $X \sim \text{U}(-1, 6)$, we have $\mathsf{E}[X] = 5/2$ and $\sigma^2 = \mathsf{Var}(X) = 49/12$. Thus, we can make the following exact calculations.

$$\begin{aligned} \mathsf{Pr}(|X-\mu| \geq 1.5\sigma) &= \mathsf{Pr}\left(\left|X-\frac{5}{2}\right| \geq 1.5 \cdot \sqrt{49/12}\right) \\ &= 1-\mathsf{Pr}\left(\left|X-\frac{5}{2}\right| < 3.03\right) \\ &= 1-\mathsf{Pr}\left(-3.03 < X-\frac{5}{2} < 3.03\right) \\ &= 1-\mathsf{Pr}\left(-0.53 < X < 5.53\right) \\ &= 1-\int_{-0.53}^{5.53} f(x)\,dx \\ &= 1-\frac{1}{7}(5.53+0.53) = 0.1343. \quad \diamondsuit \end{aligned}$$

9. What Zombies song is the theme song of the S-Town podcast?

   Solution: "A Rose For Emily".
   https://www.youtube.com/watch?v=LF55LNrHBSw ♢