ISyE 6739 — Summer 2017

Homework #6 (Modules 3.1–3.3) — Solutions

1. (Hines et al., 5–2. binomial.) Six independent trips to the moon are planned, each of which has estimated success probability 0.95. What's the probability that at least 5 will be successful?

Solution:

$$\begin{aligned}\mathsf{Pr}(X \geq 5) &= \sum_{x=5}^{6} \binom{6}{x} (0.95)^x (0.05)^{6-x} \\ &= 6(0.95)^5(0.05) + (0.95)^6 \\ &= 0.9672. \quad \diamondsuit\end{aligned}$$

2. (Hines et al., 5–6. binomial m.g.f.) Find the mean and variance of the binomial using the m.g.f.

Solution:

$$\begin{aligned}M_X(t) &= \mathsf{E}[e^{tX}] = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\ &= (pe^t + q)^n, \quad \text{where } q = 1-p.\end{aligned}$$

$$\mathsf{E}[X] = M_X'(0) = [n(pe^t+q)^{n-1}pe^t]|_{t=0} = np.$$

$$\begin{aligned}\mathsf{E}[X^2] &= M_X''(0) = np[e^t(n-1)(pe^t+q)^{n-2}(pe^t) + (pe^t+q)^{n-1}e^t]|_{t=0} \\ &= (np)^2 - np^2 + np.\end{aligned}$$

$$\mathsf{Var}(X) = \mathsf{E}[X^2] - (\mathsf{E}[X])^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p) = npq. \quad \diamondsuit$$

3. (Hines et al., 5–9. geometric.) The probability of a successful firing of a cruise missile is 0.95. Assuming independent tests, what's the prob that the first failure occurs with the fifth missile?

Solution: $\mathsf{Pr}(X = 5) = (0.95)^4(0.05) = 0.0407. \quad \diamondsuit$

4. (Hines et al., 5–30. Poisson.) Phone calls arrive at a switchboard according to a Pois(10/hour) process. The current system can handle up to 20 calls in an hour without becoming overloaded. What's the probability of an overload in the next hour?

Solution:

$$\begin{aligned}\mathsf{Pr}(X > 20) = \mathsf{Pr}(X \geq 21) &= \sum_{x=21}^{\infty} \frac{e^{-10}(10)^x}{x!} \\ &= 1 - \mathsf{Pr}(X \leq 20) = 1 - \sum_{x=0}^{20} \frac{e^{-10}(10)^x}{x!} \\ &= 0.002. \quad \diamondsuit\end{aligned}$$

5. (Hines et al., 6–13. exponential.) The time to failure of a TV is exponential with a mean of 3 years. A company offers insurance for the first year of usage. On what percentage of policies will the company have to pay claims?

Solution: Let X = Life Length.

$$\mathsf{E}(X) = \frac{1}{\lambda} = 3 \quad \Rightarrow \quad \lambda = \frac{1}{3},$$

so

$$\mathsf{Pr}(X < 1) \;=\; 1 - e^{-1/3} \;=\; 0.283.$$

Thus, 28.3% of policies result in a claim. $\diamondsuit$

6. (Hines et al., 6–16. exponential.) A transistor has an exponential time-to-failure distribution with a mean-time-to-failure of 20,000 hours. Suppose that the transistor has already lasted 20,000 hours. What's the probability that it fails by 30,000 hours?

Solution:

$$\mathsf{Pr}(X > x+s | X > x) \;=\; \mathsf{Pr}(X > s) \;=\; \mathsf{Pr}(X > 10000) \;=\; e^{-10000/20000} \;=\; 0.6064,$$

so $\mathsf{Pr}(X < 30000 | X > 20000) \;=\; 0.3936.$ $\diamondsuit$

7. (Hines et al., 7–1(a)–(e). normal.) Suppose Z is standard normal. Find

(a) $\mathsf{Pr}(0 < Z < 2)$.

(b) $\mathsf{Pr}(-1 < Z < 1)$.

(c) $\mathsf{Pr}(Z < 1.65)$.

(d) $\mathsf{Pr}(Z > -1.96)$.

(e) $\mathsf{Pr}(|Z| > 1.5)$.

Solution:

(a) $\mathsf{Pr}(0 \le Z \le 2) = \Phi(2) - \Phi(0) = 0.97725 - 0.5 = 0.47725$.

(b) $\mathsf{Pr}(-1 \le Z \le 1) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.68268$.

(c) $\mathsf{Pr}(Z \le 1.65) = \Phi(1.65) = 0.95053$.

(d) $\mathsf{Pr}(Z \ge -1.96) = \Phi(1.96) = 0.9750$.

(e) $\mathsf{Pr}(|Z| \ge 1.5) = 2[1 - \Phi(1.5)] = 0.1336$. $\diamondsuit$

8. (Hines et al., 7–3(a). normal.) Find c such that $\Phi(c) = 0.94062$.

   Solution: From the back of the book, $c = \Phi^{-1}(0.94062) = 1.56$. $\diamondsuit$

9. (Hines et al., 7–5(a). normal.) If $X \sim N(80, 10^2)$, find $\mathsf{Pr}(X < 100)$.

   Solution: $\mathsf{Pr}(X \le 100) = \Phi\left(\frac{100 - 80}{10}\right) = \Phi(2) = 0.97725$. $\diamondsuit$

10. (Hines et al., 7–7. normal.) A manager requires job applicants to take a test and score a 500. The test scores are normally distributed with a mean of 485 and standard deviation of 30. What percent of applicants pass?

    Solution: $\mathsf{Pr}(X > 500) = 1 - \Phi\left(\frac{500 - 485}{30}\right) = 1 - \Phi(0.5) = 0.30854$, i.e., 30.854%. $\diamondsuit$

11. **Mathemusical Bonus:** Suppose that a, k, and e are all nonzero. Use Beatles lyrics to prove that $m = t$.

Solution: According to "The End", the Beatles state that "The love you take is equal to the love you make." Canceling all of the similar terms and dividing by ake, we obtain the desired result. $\diamondsuit$