ISyE 6739 – Summer 2017

Homework #1 (Modules 1.1–1.5) — Solutions

1. Suppose we define the following line segments: $U = [0, 2]$, $A = [0.5, 1]$, and $B = [0.5, 1.5]$. What are $\overline{\overline{A}}$, $\overline{A \cup B}$, $A \cup \overline{B}$, $\overline{A \cap B}$, and $\overline{A} \cap B$?

   Solution: First of all, $\overline{\overline{A}} = A = [0.5,\ 1]$. $\square$

   Now, note that $A \subseteq B$. Then

   $\overline{A \cup B} = [0, \frac{1}{2}) \cup (\frac{3}{2}, 2]$. $\square$

   $A \cup \overline{B} = [0, 1] \cup (\frac{3}{2}, 2]$. $\square$

   $\overline{A \cap B} = [0, \frac{1}{2}) \cup (1, 2]$. $\square$

   $\overline{A} \cap B = (1, \frac{3}{2}]$. $\square$

2. Prove DeMorgan's Laws. You can use Venn diagrams or argue mathematically.

   Solution: Let's prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

   Proof:

$$\begin{aligned} x \in \overline{A \cup B} \quad & \text{iff} \quad x \notin A \cup B \\ & \text{iff} \quad x \notin A \text{ and } x \notin B \\ & \text{iff} \quad x \in \overline{A} \text{ and } x \in \overline{B} \\ & \text{iff} \quad x \in \overline{A} \cap \overline{B} \quad \square \end{aligned}$$

   Now let's prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

   Proof:

$$\begin{aligned} x \in \overline{A \cap B} \quad & \text{iff} \quad x \notin A \cap B \\ & \text{iff} \quad x \notin A \text{ or } x \notin B \\ & \text{iff} \quad x \in \overline{A} \text{ or } x \in \overline{B} \\ & \text{iff} \quad x \in \overline{A} \cup \overline{B} \quad \square \end{aligned}$$

3. Prove Bonferroni's inequality: $\mathsf{Pr}(A \cap B) \geq \mathsf{Pr}(A) + \mathsf{Pr}(B) - 1$.

   Solution: First of all, we know that

$$\mathsf{Pr}(A \cup B) = \mathsf{Pr}(A) + \mathsf{Pr}(B) - \mathsf{Pr}(A \cap B),$$

so that

$$\mathsf{Pr}(A \cap B) = \mathsf{Pr}(A) + \mathsf{Pr}(B) - \mathsf{Pr}(A \cup B).$$

Since $\mathsf{Pr}(A \cup B) \leq 1$, the result follows. $\square$

4. A box contains 4 marbles (two reds, one green, and one yellow).

   (a) Consider an experiment that consists of taking one marble from the box, then replacing it in the box, and then drawing a second marble from the box. What is the sample space?
   Solution: $S = \{RR, RG, RY, GR, GG, GY, YR, YG, YY\}$. $\square$

   (b) Repeat the above when the second marble is drawn *without* replacing the first marble.
   Solution: $S = \{RR, RG, RY, GR, GY, YR, YG\}$. $\square$

5. Let E, F, and G be 3 events. Suppose that $U = E \cup F \cup G$. Find expressions for the following events:

   (a) only G occurs.
   Solution: $\overline{E} \cap \overline{F} \cap G$ $(= \overline{E} \cap \overline{F},$ since $U = E \cup F \cup G)$. $\square$

   (b) F and G occur, but not E.
   Solution: $\overline{E} \cap F \cap G$. $\square$

   (c) at least one event occurs.
   Solution: $E \cup F \cup G$ $(= U)$. $\square$

   (d) at least two events occur.
   Solution: $(E \cap F) \cup (E \cap G) \cup (F \cap G)$. $\square$

   (e) all three events occur.
   Solution: $E \cap F \cap G$. $\square$

   (f) none occur.
   Solution: $\overline{E \cup F \cup G}$ $(= \emptyset)$, since "none" is the opposite of "at least one". $\square$

   (g) at most one occurs.
   Solution:

$$\begin{aligned}
\text{“at most one”} &= \text{“none”} \cup \text{“exactly one”} \\
&= (\overline{E \cup F \cup G}) \cup [(E \cap \overline{F} \cap \overline{G}) \cup (\overline{E} \cap F \cap \overline{G}) \cup (\overline{E} \cap \overline{F} \cap G)] \\
&= (\overline{F} \cap \overline{G}) \cup (\overline{E} \cap \overline{G}) \cup (\overline{E} \cap \overline{F}) \quad (\text{since } U = E \cup F \cup G). \quad \square
\end{aligned}$$

(h) at most two occur.
Solution: $\overline{E \cap F \cap G}$, since "at most 2" is the opposite of "all 3". $\square$

6. How many 5-letter words can be formed from the alphabet if we require

(a) The 2nd letter to be a vowel (a, e, i, o, u)?
Solution: $26 \cdot 5 \cdot 26 \cdot 26 \cdot 26 \;=\; 2{,}284{,}880$. $\square$

(b) Exactly one vowel?
Solution:

$$\text{(Place vowel)}\cdot\text{(choose vowel)}\cdot\text{(choose 4 consonants)} \;=\; \binom{5}{1} 5\cdot(21)^4 \;=\; 4{,}862{,}025. \quad \square$$

(c) At least one vowel?
Solution:

$$\text{(Total \# words)} - \text{(Words with 0 vowels)} \;=\; (26)^5 - (21)^5 \;=\; 7{,}797{,}275. \quad \square$$

7. As if you have nothing better to do, toss a die 6000 times. (I suppose you can do this in Excel.) How many times do each of the numbers come up? Approximately how many would you expect?

Solution: You'd expect about 1000 occurrences of each number (but results may vary). Let me know what you get. $\square$

8. Toss a coin until you see two H's in a row. How many tries did it take? Now toss a coin until you see the sequence HT. How long did that take? Thoughts?

Solution: I wasn't really looking for you to actually solve anything here, but I'll write down some interesting comments anyway — even though we're still a couple classes away from being able to do this rigorously. For instance, let's see how long it takes for you to get to HT (at least, on average). To do so, you first have to get to H, which you can do in any of the following ways: H (1 toss), TH (2 tosses), TTH (3 tosses), etc. It turns out that it takes an average of 2 tosses until the first H appears. By exactly the same reasoning, it takes an average of two *more* tosses until a T appears. So then you have a series of TTT$\cdots$THHH$\cdots$HT, where the first HT takes an average of 4 tosses to appear.

By similar, but more-complicated reasoning, you can show that it takes an average of 6 tries to get to HH; and it takes an average of only 3 tries to get to either HH *or* HT. $\square$

9. (Bonus) Three dice are tossed. What is the probability that the same number appears on exactly two of the three dice?

   Solution: There are actually a buncha ways to do this problem. For instance, let X equal the number of times a 1 appears on the three tosses. Then, as we will have learned by now, $X \sim \text{Bin}(3, 1/6)$, so that

$$\mathsf{Pr}(X = 2) \;=\; \binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right).$$

   Thus, the prob that *some* number appears twice is

$$6 \cdot \mathsf{Pr}(X = 2) \;=\; 6 \cdot \binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) \;=\; 0.417 \quad \square$$

10. (Bonus) Wedding invitations. Put 4 names on index cards and accompanying envelopes. Randomly place each of the index cards into an envelope. What's the probability that you'll get at least one correct match?

    Solution: We will learn pretty soon that this is a special case of what's called the "envelope problem". You can enumerate out all of the possibilities, and you will eventually see that the answer is

$$\mathsf{Pr}(\text{at least one match}) \;=\; 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} \;=\; 0.625. \quad \square$$