ISyE 6739 — Summer 2017

Homework #5 (Modules 2.9–2.12) — Solutions

1. Suppose that $f(x,y) = 14xy^2$ for $0 < x < y^2 < 1$ and $0 < y < 1$.

   (a) Find the marginal p.d.f. of X, $f_X(x)$.

   Solution:
   $$f_X(x) \;=\; \int_{\Re} f(x,y)\,dy \;=\; \int_{\sqrt{x}}^{1} 14xy^2\,dy \;=\; \frac{14}{3}(x - x^{5/2}), \quad 0 < x < 1. \quad \diamondsuit$$

   (b) Find the marginal p.d.f. of Y, $f_Y(y)$.

   Solution:
   $$f_Y(y) \;=\; \int_{\Re} f(x,y)\,dx \;=\; \int_0^{y^2} 14xy^2\,dx \;=\; 7y^6, \quad 0 < y < 1. \quad \diamondsuit$$

   (c) Find $\mathsf{E}[X]$.

   Solution:
   $$\mathsf{E}[X] \;=\; \int_{\Re} x f_X(x)\,dx \;=\; \int_0^1 \frac{14}{3}(x^2 - x^{7/2})\,dx \;=\; \frac{14}{27}. \quad \diamondsuit$$

   (d) Find the conditional p.d.f. of X given $Y = y$, $f(x|y)$.

   Solution:
   $$f(x|y) \;=\; \frac{f(x,y)}{f_Y(y)} \;=\; \frac{2x}{y^4},\ 0 < x < y^2 < 1. \quad \diamondsuit$$

   (e) Find the conditional expectation, $\mathsf{E}[X|y]$.

   Solution:
   $$\mathsf{E}[X|y] \;=\; \int_{\Re} x f(x|y)\,dx \;=\; \int_0^{y^2} \frac{2x}{y^4}\,dx \;=\; \frac{2y^2}{3}. \quad \diamondsuit$$

   (f) Find the "double" conditional expectation, $\mathsf{E}[\mathsf{E}[X|Y]]$.

   Solution:
   $$\mathsf{E}[\mathsf{E}[X|Y]] \;=\; \int_{\Re} \mathsf{E}[X|y] f_Y(y)\,dy \;=\; \int_0^1 \frac{2y^2}{3} 7y^6\,dy \;=\; \frac{14}{27}. \quad \diamondsuit$$

2. (Hines, et al., 4–8.) Consider the probability distribution of the discrete random vector (X, Y), where X represents the number of orders for aspirin in August in the neighborhood drugstore and Y represents the number of orders in September. The joint distribution is shown in the following table.

$Y \backslash X$	51	52	53	54	55
51	0.06	0.05	0.05	0.01	0.01
52	0.07	0.05	0.01	0.01	0.01
53	0.05	0.10	0.10	0.05	0.05
54	0.05	0.02	0.01	0.01	0.03
55	0.05	0.06	0.05	0.01	0.03

(a) Find the marginal distributions.

Solution: After we add up the usual stuff, we get the following marginals:

z	51	52	53	54	55
$f_X(z)$	0.28	0.28	0.22	0.09	0.13
$f_Y(z)$	0.18	0.15	0.35	0.12	0.20

$\diamondsuit$

(b) Find the expected sales in September, given that sales in August were either 51, 52, 53, 54, or 55, respectively.

Solution:

$$\mathsf{E}[Y|X = x] \;=\; \sum_y y f_{Y|X}(y|x) \;=\; \frac{1}{f_X(x)} \sum_y y f(x, y).$$

For example, we get

$$\mathsf{E}[Y|X = 51] \;=\; \frac{1}{0.28}[(51)(0.06) + (52)(0.07) + \cdots + (55)(0.05)].$$

Similarly, we get the following table.

x	51	52	53	54	55
$\mathsf{E}[Y|X = x]$	52.86	52.96	53	53	53.46

$\diamondsuit$

3. (Hines, et al., 4–9). Assume that X and Y are coded scores of two intelligence tests, and the p.d.f. of (X, Y) is given by

$$f(x,y) \;=\; \begin{cases} 6x^2y & \text{if } 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value of the score on test #2 given the score on test #1. Then find the expected value of the score on test #1 given the score on test #2.

Hints: After the usual algebra and calculus, you get

$$\mathsf{E}[X|Y = y] \;=\; 3/4$$

and

$$\mathsf{E}[Y|X = x] \;=\; 2/3.$$

Note that the answers don't depend on y or x, respectively! This is because X and Y are *independent* (Why?) $\diamondsuit$

4. (Hines, et al., 4–31.) Given the following joint p.d.f.'s, determine whether or not X and Y are independent.

(a) $g(x, y) = 4xye^{-(x^2+y^2)}$, $x > 0$, $y > 0$.
Solution: Since (i) there are no funny limits and (ii) you can factor $g(x, y) = (4xe^{-x^2})(ye^{-y^2})$, we see that X and Y are independent. $\diamondsuit$

(b) $f(x, y) = 3x^2y^{-3}$, $0 < x < y < 1$.
Solution: Funny limits imply *not* independent. $\diamondsuit$

(c) $f(x, y) = 6(1 + x + y)^{-4}$, $x > 0$, $y > 0$.
Solution: Can't factor $f(x, y) = g(x)h(y)$ implies *not* independent. $\diamondsuit$

5. (Hines, et al., 4–19.) Let X and Y have joint p.d.f. $f(x, y) = 2$, $0 < x < y < 1$. Find the correlation between X and Y.

Solution: I won't go through all of the tedious calculations, but here are the highlights.

$$f_X(x) \;=\; \int_x^1 2\,dy \;=\; 2(1-x), \quad 0 < x < 1$$

and

$$f_Y(y) \;=\; \int_0^y 2\,dx \;=\; 2y, \quad 0 < y < 1.$$

Then you get (in the usual way)

$$\mathsf{E}[X] \;=\; 1/3, \quad \mathsf{Var}(X) \;=\; 1/18, \quad \mathsf{E}[Y] \;=\; 2/3, \quad \mathsf{Var}(Y) \;=\; 1/18.$$

Further,

$$\mathsf{E}[XY] \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dx\,dy \;=\; \int_0^1\int_0^y 2xy\,dx\,dy \;=\; 1/4.$$

This finally implies that

$$\rho \;=\; \frac{\mathsf{E}[XY]-\mathsf{E}[X]\mathsf{E}[Y]}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)}} \;=\; 0.5. \quad \diamondsuit$$

6. (Hines, et al., 4–21). Consider the data from Hines, et al., 4–1, reproduced below.

$Y\backslash X$	0	1	2	3	4	5
0	11/50	4/50	2/50	1/50	1/50	1/50
1	8/50	3/50	2/50	1/50	1/50	
2	4/50	3/50	2/50	1/50		
3	3/50	1/50				
4	1/50					

Are X and Y independent? Find the correlation.

Solution: After the usual manipulations, get $\rho = -0.1355$. So X and Y are *not* independent. $\diamondsuit$

7. Let $\mathsf{Var}(X) = \mathsf{Var}(Y) = 20$, $\mathsf{Var}(Z) = 30$, $\mathsf{Cov}(X,Y) = 2$, $\mathsf{Cov}(X,Z) = -3$, and $\mathsf{Cov}(Y,Z) = -4$. Find $\mathsf{Corr}(X,Z)$ and $\mathsf{Var}(X-2Y+5Z)$.

Solution:

$$\mathsf{Corr}(X,Z) \;=\; \frac{\mathsf{Cov}(X,Z)}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Z)}} \;=\; -0.1225$$

and

$$\begin{aligned}\mathsf{Var}(X-2Y+5Z) \;&=\; \mathsf{Var}(X)+4\mathsf{Var}(Y)+25\mathsf{Var}(Z)\\ &\qquad -2\cdot 2\mathsf{Cov}(X,Y)+2\cdot 5\mathsf{Cov}(X,Z)-2\cdot 10\mathsf{Cov}(Y,Z)\\ &=\; 892. \quad \diamondsuit\end{aligned}$$

8. Suppose $X \sim \text{Exp}(\lambda)$. Use the m.g.f. of X to find $\mathsf{E}[X^k]$.

   Solution: By class notes, the m.g.f. of the $\text{Exp}(\lambda)$ is $M_X(t) = \frac{\lambda}{\lambda - t}$ for $\lambda > t$. Therefore,

   $$\mathsf{E}[X^k] \;=\; \frac{d^k}{dt^k} M_X(t)\Big|_{t=0} = \frac{k!}{\lambda^k},$$

   where the final answer follows after a little elbow grease. $\Diamond$.

9. (Hines, et al., 4–18.) Let X and Y be two random variables such that $Y = a + bX$. Show that the moment generating function of Y is $M_Y(t) = e^{at} M_X(bt)$.

   Solution:

   $$M_Y(t) \;=\; \mathsf{E}[e^{tY}] \;=\; \mathsf{E}[e^{t(a+bX)}] \;=\; e^{at}\mathsf{E}[e^{(bt)X}] \;=\; e^{at} M_X(bt). \quad \Diamond$$