ISyE 6739 — Summer 2017

Homework #8 (Modules 4.1–4.3) — Solutions

Most of the following problems are from Hines, et al. (a "+" indicates otherwise).

1. (Hines, et al. 8–1.) Elementary data analysis. The shelf life of a high-speed photographic film is being investigated by the manufacturer. The following data are available (in days).

126	129	134	141
131	132	136	145
116	128	130	162
125	126	134	129
134	127	120	127
120	122	129	133
125	111	147	129
150	148	126	140
130	120	117	131
149	117	143	133

Construct a histogram and comment on the properties of the data.

Solution: $\bar{x} = 131.30$, $s^2 = 113.85$, $s = 10.67$. □

2. (Hines, et al. 8–25.) Interesting algebra questions.

(a) Consider the quantity $\sum_{i=1}^{n}(x_i - a)^2$. For what value of a is this minimized?

(b) $^+$Now consider the quantity $\sum_{i=1}^{n}|x_i - a|$. Comment on the value of a that minimizes this fella.

Solution:

(a) Differentiate and you eventually get $a = \bar{x}$. □

(b) Now think about the median. □

3. (Hines, et al. 10–1. MSE.) Suppose we have a random sample of size $2n$ from a population denoted X, and $\mathsf{E}[X] = \mu$ and $\mathsf{Var}(X) = \sigma^2$. Let

$$\bar{X}_1 = \frac{1}{2n}\sum_{i=1}^{2n} X_i \quad \text{and} \quad \bar{X}_2 = \frac{1}{n}\sum_{i=1}^{n} X_i$$

be two estimators of μ. Which is the better estimator of μ? Explain your choice.

Solution: Both estimators are unbiased. Now, $\mathsf{Var}(\bar{X}_1) = \sigma^2/2n$ while $\mathsf{Var}(\bar{X}_2) = \sigma^2/n$. Since $\mathsf{Var}(\bar{X}_1) < \mathsf{Var}(\bar{X}_2)$, $\bar{X}_1$ is a more efficient estimator than $\bar{X}_2$. □

4. (Hines, et al. 10–4. MSE.) Suppose that $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\theta}_3$ are estimators of θ. We know that $\mathsf{E}[\hat{\theta}_1] = \mathsf{E}[\hat{\theta}_2] = \theta$, $\mathsf{E}[\hat{\theta}_3] \neq \theta$, $\mathsf{Var}(\hat{\theta}_1) = 12$, $\mathsf{Var}(\hat{\theta}_2) = 10$, and $\mathsf{E}[(\hat{\theta}_3 - \theta)^2] = 6$. Compare these three estimators. Which do you prefer and why?

Solution: Since $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased, we have $\mathsf{MSE}(\hat{\theta}_1) = \mathsf{Var}(\hat{\theta}_1) = 12$, $\mathsf{MSE}(\hat{\theta}_2) = \mathsf{Var}(\hat{\theta}_2) = 10$, and $\mathsf{MSE}(\hat{\theta}_3) = \mathsf{E}[(\hat{\theta}_3 - \theta)^2] = 6$. Therefore, $\hat{\theta}_3$ is the best estimator because it has the smallest MSE. □

5. (Hines, et al. 10–13. Geometric MSE.) Let $X_1, \ldots, X_n$ be an i.i.d. sample of geometric random variables with parameter p.

   (a) Find the maximum likelihood estimator of p.

   (b) $^+$Suppose you have $n = 5$ observations, 3,5,1,2,7. What is the resulting MLE?

   (c) $^+$What is the MLE of p^2?

Solution:

(a) The likelihood function is

$$L(p) \;=\; \prod_{i=1}^{n}(1-p)^{x_i-1}p = p^n(1-p)^{\Sigma x_i - n}.$$

Then

$$\ell\text{n}\, L(p) \;=\; n\,\ell\text{n}\, p + (\sum_{i=1}^{n} x_i - n)\ell\text{n}\,(1-p).$$

From $d\,\ell\text{n}\, L(p)/dp = 0$, we obtain

$$\frac{n}{p} - \frac{\sum_{i=1}^{n} x_i - n}{1-p} \;=\; 0,$$

so that

$$\hat{p} \; = \; \frac{n}{\sum_{i=1}^{n} x_i} \; = \; 1/\bar{x}. \quad \Box$$

(b) $\hat{p} = 1/\bar{x} = 0.278.$ $\Box$

(c) By the Invariance Property of MLE's, $\widehat{p^2} = \hat{p}^2 = 1/\bar{x}^2 = 0.0773.$ $\Box$

6. (Bernoulli MLE.) Let $X_1, \ldots, X_n$ be an i.i.d. sample of Bernoulli random variables with parameter p.

   (a) Find the maximum likelihood estimator of p, based on a sample of size n.

   (b) Now find a MOM estimator. Are there multiple MOM estimators?

   Solution:

   (a) Start with

   $$L(p) \; = \; \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} \; = \; p^{\sum_{i=1}^{n} x_i}(1-p)^{n-\sum_{i=1}^{n} x_i}.$$

   After the usual manipulations, you'll find that the MLE is simply $\hat{p} = \bar{x}.$ $\Box$

   (b) Since $\mathsf{E}[X_i] = p$, a MOM estimator is $\bar{x}.$ $\Box$

   On the other hand, you may recall that $\mathsf{E}[X_i^k] = p$ for any positive integer k, so other MOM estimators are of the form $\frac{1}{n}\sum_{i=1}^{n} x_i^k.$ $\Box$

7. (Hines, et al. 9–5. Normal distribution.) A population of power supplies for a personal computer has an output voltage that is normally distributed with a mean of 5.00 V and a standard deviation of 0.10 V. A random sample of eight power supplies is selected. Specify the sampling distribution of $\bar{X}$.

   Solution: $N(\mu, \sigma^2/n) = N(5, (0.10)^2/8) = N(5, 0.00125).$ $\Box$

8. (Hines, et al. 9–23(a). χ^2 quantile.) Find $\chi^2_{0.95,8}$.

   Solution: 2.73. $\Box$

9. (Hines, et al. 9–24(a). t quantile.) Find $t_{0.25,10}$.

   Solution: 0.700. $\square$

10. (Hines, et al. 9–25(a). F quantile.) Find $F_{0.25,4,9}$.

    Solution: 1.63. $\square$