ISyE 6739 — Summer 2017

Homework #4 (Modules 2.6–2.8) — Solutions

1. Suppose $X \sim \text{Unif}(1,3)$. Find the p.d.f. of $Z = e^X$.

   Solution: The c.d.f. of Z is

$$\begin{aligned} G(z) &= \mathsf{Pr}(Z \le z) \\ &= \mathsf{Pr}(e^X \le z) \\ &= \mathsf{Pr}(X \le \ell\text{n}(z)) \\ &= \int_1^{\ell\text{n}(z)} f(x)\,dx \quad (\text{if } 1 \le \ell\text{n}(z) \le 3) \\ &= (\ell\text{n}(z) - 1)/2. \end{aligned}$$

   Now you can get the p.d.f.

$$g(z) = \frac{d}{dz}G(z) = \begin{cases} 0 & \text{if } z < e \text{ or } z > e^3 \\ \frac{1}{2z} & \text{if } e \le z \le e^3 \end{cases} \qquad \diamondsuit$$

2. Suppose X has p.d.f. $f(x) = 2xe^{-x^2}$, $x \ge 0$. Find the distribution of $Z = X^2$.

   Solution: The c.d.f. of Z is

$$\begin{aligned} G(z) &= \mathsf{Pr}(Z \le z) \\ &= \mathsf{Pr}(X^2 \le z) \\ &= \mathsf{Pr}(-\sqrt{z} \le X \le \sqrt{z}) \\ &= \mathsf{Pr}(0 \le X \le \sqrt{z}) \quad (\text{since } X \ge 0) \\ &= \int_0^{\sqrt{z}} 2xe^{-x^2}\,dx \\ &= 1 - e^{-z}. \end{aligned}$$

   Thus, Z is Exp(1). $\diamondsuit$

3. (Hines, et al., 4–1). A refrigerator manufacturer subjects his finished products to a final inspection. Of interest are two categories of defects: scratches or flaws in the porcelain finish, and mechanical defects. The number of each type of defects is a random variable. The results of inspecting 50 refrigerators are shown in the following joint p.m.f. table, where X represents the occurrence of finish defects and Y represents the occurrence of mechanical defects.

$Y\backslash X$	0	1	2	3	4	5
0	11/50	4/50	2/50	1/50	1/50	1/50
1	8/50	3/50	2/50	1/50	1/50	
2	4/50	3/50	2/50	1/50		
3	3/50	1/50				
4	1/50					

(a) Find the marginal distributions of X and Y.
Solution: Let's re-write the table, this time including the marginals.

$Y\backslash X$	0	1	2	3	4	5	$f_Y(y)$
0	11/50	4/50	2/50	1/50	1/50	1/50	20/50
1	8/50	3/50	2/50	1/50	1/50		15/50
2	4/50	3/50	2/50	1/50			10/50
3	3/50	1/50					4/50
4	1/50						1/50
$f_X(x)$	27/50	11/50	6/50	3/50	2/50	1/50	

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(b) Find the conditional distribution of mechanical defects, given that there are no finish defects.
Solution:

$$f(y|X=0) = \frac{f(0,y)}{f_X(0)} = \frac{f(0,y)}{27/50} = \begin{cases} 11/27 & \text{if } y=0 \\ 8/27 & \text{if } y=1 \\ 4/27 & \text{if } y=2 \\ 3/27 & \text{if } y=3 \\ 1/27 & \text{if } y=4 \\ 0 & \text{otherwise} \end{cases}$$

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(c) Find the conditional distribution of finish defects, given that there are no mechanical defects.
Solution:

$$f(x|Y=0) = \frac{f(x,0)}{f_Y(0)} = \frac{f(x,0)}{20/50} = \begin{cases} 11/20 & \text{if } x=0 \\ 4/20 & \text{if } x=1 \\ 2/20 & \text{if } x=2 \\ 1/20 & \text{if } x=3,4,5 \\ 0 & \text{otherwise} \end{cases}$$

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4. (Hines, et al., 4–4). Consider a situation in which the surface tension and acidity of a chemical product are measured. These variables are coded such that surface tension is measured on a scale $0 \le X \le 2$, and acidity is measured on a scale $2 \le Y \le 4$. The probability density function of (X, Y) is

$$f(x,y) \;=\; \begin{cases} k(6-x-y) & \text{if } 0 \le x \le 2,\ 2 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the appropriate value of k.
   Solution: Setting
   $$\int_2^4 \int_0^2 k(6-x-y)\,dx\,dy \;=\; 1,$$
   we find that $k = 1/8$. $\diamondsuit$

   (b) Calculate the probability that $X < 1,\ Y < 3$.
   Solution:
   $$\mathsf{Pr}(X < 1, Y < 3) \;=\; \int_2^3 \int_0^1 (1/8)(6-x-y)\,dx\,dy \;=\; 3/8. \quad \diamondsuit$$

   (c) Calculate the probability that $X + Y \le 4$.
   Solution: This is a little tough; pay attention to the limits.
   $$\mathsf{Pr}(X + Y \le 4) \;=\; \int_2^4 \int_0^{4-y} (1/8)(6-x-y)\,dx\,dy \;=\; 2/3. \quad \diamondsuit$$

   (d) Find the probability that $X < 1.5$.
   Solution:
   $$f_X(x) \;=\; \int_{\Re} f(x,y)\,dy \;=\; \int_2^4 (1/8)(6-x-y)\,dy \;=\; \frac{3-x}{4}, \quad 0 < x < 2.$$
   This implies that
   $$\mathsf{Pr}(X < 1.5) \;=\; \int_0^{1.5} \frac{3-x}{4}\,dx \;=\; 0.844. \quad \diamondsuit$$

   (e) Find the marginal densities of both X and Y.
   Solution: $f_X(x)$ is given above. Similarly,
   $$f_Y(y) \;=\; \int_{\Re} f(x,y)\,dx \;=\; \int_0^2 (1/8)(6-x-y)\,dx \;=\; \frac{5-y}{4}, \quad 2 < y < 4. \quad \diamondsuit$$

5. Suppose that $f(x,y) = cxy^2$ for $0 < x < y^2 < 1$ and $0 < y < 1$.

(a) Find c.

Solution:

$$1 \;=\; \int\int_{\Re^2} f(x,y)\,dx\,dy \;=\; \int_0^1 \int_0^{y^2} cxy^2\,dx\,dy \;=\; c/14.$$

This immediately implies that $c = 14$. $\diamondsuit$

(b) Find the marginal p.d.f. of X, $f_X(x)$.

Solution:

$$f_X(x) \;=\; \int_{\Re} f(x,y)\,dy \;=\; \int_{\sqrt{x}}^1 14xy^2\,dy \;=\; \frac{14}{3}(x - x^{5/2}), \quad 0 < x < 1. \quad \diamondsuit$$

(c) Find the marginal p.d.f. of Y, $f_Y(y)$.

Solution:

$$f_Y(y) \;=\; \int_{\Re} f(x,y)\,dx \;=\; \int_0^{y^2} 14xy^2\,dx \;=\; 7y^6, \quad 0 < y < 1. \quad \diamondsuit$$

(d) Find $\mathsf{E}[X]$.

Solution:

$$\mathsf{E}[X] \;=\; \int_{\Re} x f_X(x)\,dx \;=\; \int_0^1 \frac{14}{3}(x^2 - x^{7/2})\,dx \;=\; \frac{14}{27}. \quad \diamondsuit$$

(e) Find $\mathsf{E}[Y]$.

Solution:

$$\mathsf{E}[Y] \;=\; \int_{\Re} y f_Y(y)\,dy \;=\; \int_0^1 7y^7\,dy \;=\; \frac{7}{8}. \quad \diamondsuit$$

(f) Find the conditional p.d.f. of X given $Y = y$, $f(x|y)$.

Solution:

$$f(x|y) \;=\; \frac{f(x,y)}{f_Y(y)} \;=\; \frac{2x}{y^4},\; 0 < x < y^2 < 1. \quad \diamondsuit$$

6. **Mathemusical Bonus:** What is the largest prime number to be found in the lyrics of a song from the Top-40 era?

   Solution: Tommy Tutone's song "Jenny" mentions the prime number 8675309. www.youtube.com/watch?v=6WTdTwcmxyo