Solving Differential Equation Using Finite Element Method

Piotr Magiera

February 2022

1 Introduction

The aim of this paper is to use Finite Element Method to solve the equation

$$\frac{d^2u}{dx^2} = 4\pi G\rho(x)$$

with following assumptions:

$$u(0) = 5,$$
$$u(3) = 4,$$
$$\rho(x) = \begin{cases} 0 & x \in \langle 0, 1\rangle \\ 1 & x \in (1, 2\rangle \\ 0 & x \in (2, 3\rangle \end{cases},$$
$$u : \langle 0, 3\rangle \to \mathbb{R},$$
$$G = 6.67 * 10^{-11}.$$

2 Weak form of the equation

Let $U = \{f \in H^1(0, 3) : f(0) = f(3) = 0\}$. Function u satisfies the equation if and only if for any $v \in U$

$$\int_0^3 u''v dx = \int_0^3 4\pi G\rho v dx.$$

Using integration by parts we obtain

$$u'v\Big|_0^3 - \int_0^3 u'v'dx = 4\pi G \int_0^3 \rho v dx.$$

Since $v \in U$, $u'v\Big|_0^3 = 0$. Moreover, $\rho(x) \neq 0 \iff x \in \langle 1, 2\rangle$ and $\rho(x) = 1$ for $x \in \langle 1, 2\rangle$, hence

$$-\int_0^3 u'v'dx = 4\pi G \int_1^2 v dx. \tag{1}$$

Let us define functions $\bar{u}$ and w such that

$$u = \bar{u} + w \tag{2}$$

and $w \in U$. Therefore $w(0) = w(3) = 0$, which implies $\bar{u}(0) = u(0) = 5$ and $\bar{u}(3) = u(3) = 4$. According to the above remark, we define

$$\bar{u}(x) = 5 - \frac{x}{3}. \tag{3}$$

Applying (2) and (3) we can rewrite (1) as

$$\int_0^3 (\frac{1}{3} - w')v'dx = 4\pi G \int_1^2 vdx$$

which is equivalent to weak form of the main equation

$$-\int_0^3 w'v'dx = 4\pi G \int_1^2 vdx - \frac{1}{3}\int_0^3 v'dx. \tag{4}$$

3 Discretization

We will denote by n the number of subdomains the $\langle 0, 3\rangle$ set will be divided in and by $h = \frac{3}{n}$ the length of each subdomain. Furthermore, let:

$$\forall i \in \{0, ..., n\} : x_i = h * i,$$

$$\forall i \in \{1, ..., n-1\} : e_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{for } x \in (x_{i-1}, x_i\rangle \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & \text{for } x \in (x_i, x_{i+1}) \\ 0 & \text{otherwise} \end{cases}.$$

The idea behind discretization is to approximate infinite-dimensional linear space U with finite-dimensional space $V \subset U$. Since we have Dirichlet boundary conditions on both sides of the domain, let $V = Lin\{e_1, ..., e_{n-1}\}$. Now we have replaced the equation (4) with its finite-dimensional version where $w, v \in V$.

4 Matrix form of the problem

The fact that $w \in V$ implies that $w = \sum_{i=1}^{n-1} \alpha_i e_i$. With the notation $B(w, v) = -\int_0^3 w'v'dx$, $\bar{L}(v) = 4\pi G \int_1^2 vdx - \frac{1}{3}\int_0^3 v'dx$, we can write (4), taking $v = e_j$ for $j \in \{1, ..., n-1\}$, in the form

$$\forall j \in \{1, ..., n-1\} : B(\sum_{i=1}^{n-1} \alpha_i e_i, e_j) = \bar{L}(e_j),$$

which is equivalent to

$$\forall j \in \{1, ..., n-1\} : \sum_{i=1}^{n-1} \alpha_i B(e_i, e_j) = \bar{L}(e_j). \tag{5}$$

The set of equations (5) can be represented in matrix form as

$$\begin{bmatrix} B(e_1, e_1) & B(e_2, e_1) & \cdots & B(e_{n-1}, e_1) \\ B(e_1, e_2) & B(e_2, e_2) & \cdots & B(e_{n-1}, e_2) \\ \vdots & \vdots & \ddots & \vdots \\ B(e_1, e_{n-1}) & B(e_2, e_{n-1}) & \cdots & B(e_{n-1}, e_{n-1}) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} = \begin{bmatrix} \bar{L}(e_1) \\ \bar{L}(e_2) \\ \vdots \\ \bar{L}(e_{n-1}) \end{bmatrix}. \tag{6}$$

5 Simplification of the matrix form

It is easily seen that $B(e_i, e_j) \neq 0 \iff |i-j| \leq 1$. Since

$$e_i'(x) = \begin{cases} \frac{1}{h} & x \in (x_{i-1}, x_i) \\ -\frac{1}{h} & x \in (x_i, x_{i+1}) \\ 0 & x \in (-\infty, x_{i-1}) \cup (x_{i+1}, +\infty) \end{cases},$$

we get constant values:

$$a = B(e_i, e_i) = -\int_{x_{i-1}}^{x_{i+1}} \frac{1}{h^2} dx,$$

$$b = B(e_i, e_j) = \int_{x_i}^{x_{i+1}} \frac{1}{h^2} dx \text{ for } |i-j| = 1.$$

Obviously

$$\forall i \in \{1, ..., n-1\} : \int_0^3 e_i' dx = 0 \implies \bar{L}(e_i) = 4\pi G \int_1^2 v dx.$$

We can now rewrite (6) as

$$\begin{bmatrix} a & b & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ b & a & b & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & b & a & b & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & b & a & b & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & b & a & b \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & b & a \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} = \begin{bmatrix} \bar{L}(e_1) \\ \bar{L}(e_2) \\ \vdots \\ \bar{L}(e_{n-1}) \end{bmatrix}. \tag{7}$$

6 Plot of the solution

In the figure 1 we can see plot of the solution obtained using Python script.

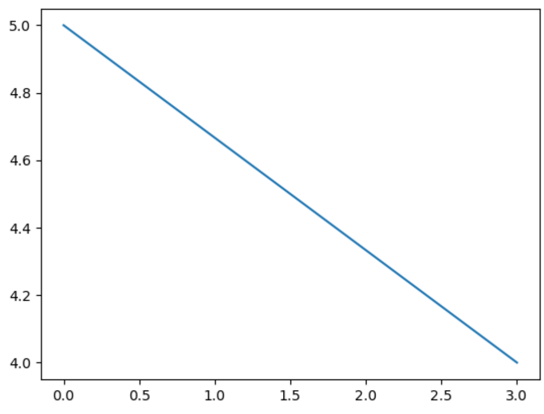

Figure 1: Plot of $y = u(x)$