# Probabilistic Machine Learning: 5. Dirichlet-multinomial model

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The presentation was inspired by Chapter 3 of Kevin Murphy book "Machine Learning A Probabilistic Perspective", 2012, MIT.

Apropriate agreements to propagate his ideas has been acquired.

# Already: coin toss problem

We have covered:

- inferring a distribution over a discrete variable drawn from a finite hypothesis space
- inferring the probability that a coin shows up heads
- given a series of discrete observations

Let’s focus now on a dice:

- dice is $K$ sided :)

# Johann Dirichlet (1805–1859)

- German mathematician with French roots
- with the support of Humboldt and Gauss, Dirichlet was offered a teaching position at the University of Breslau (1827-1828)
- in 1842 obtained a full professor position at the University of Breslau
- Dirichlet distribution named after him

# Dirichlet distribution

- multivariate generalization of the beta distribution
- parameters:
  - $K > 2$ - categories
  - $\alpha_1, \ldots, \alpha_K$ - concentration parameters ($\alpha_i > 0$)
- support: $x_1, \ldots, x_K$ where $x_i \in (0, 1)$ and $\sum_{i=1}^{K} x_i = 1$

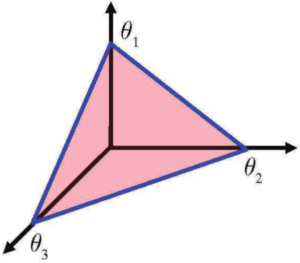
# Dirichlet distribution

About the support ($x$):

- over the probability simplex
- $S_K = \{x : 0 \leq x_k \leq 1, \sum_{k=1}^{K} x_k = 1\}$

Probability distribution over $X$

- $Dir(x|\alpha) = \frac{1}{\mathrm{B}(\boldsymbol{\alpha})} \prod_{k=1}^{K} x_k^{\alpha_k - 1}$
- with normalizing factor:
  $\mathrm{B}(\boldsymbol{\alpha}) = \frac{\prod_{k=1}^{K} \Gamma(\alpha_k)}{\Gamma\left(\sum_{k=1}^{K} \alpha_k\right)}$



# Dirichlet distribution

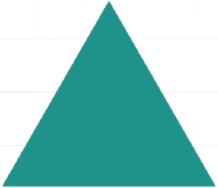


Figure: Dir(1,1,1)

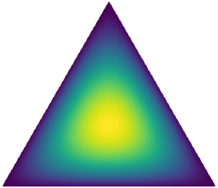


Figure: Dir(2,2,2)

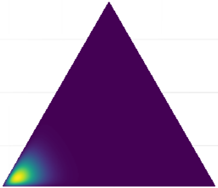


Figure: Dir(20,2,2)

# Dirichlet distribution

- mean:

$$\mathbb{E}(X_i) = \frac{\alpha_i}{\sum_k \alpha_k}$$

- mode:

$$x_i = \frac{\alpha_i - 1}{\sum_{k=1}^{K} \alpha_k - K}, \quad \alpha_i > 1$$

# Recipe of specifying the model

Define

- likelihood
- prior

and derive

- posterior
- posterior predictive

# Likelihood

Let's suppose:

- we observe $N$ dice rolls
- $\mathcal{D} = \{x_1, x_2, \ldots, x_N\}$
- $x_i \in \{1, \ldots, K\}$

Assuming the data is iid, likelihood has the form:

$$\mathcal{L}(\mathcal{D} \mid \theta) = \prod_{k=1}^{K} \theta_k^{N_k}$$

The number of times event $k$ occured is given by sufficient statistics: $N_k = \sum_{i=1}^{N} \mathbb{I}(y_i = k)$

# What about prior?

- need of prior from $K$-dimensional probability simplex
- the best should be conjugate one
- fortunatelly, Dirichlet distribution satisfies both criteria

$$Dir(\theta|\alpha) = \frac{1}{\mathrm{B}(\boldsymbol{\alpha})} \prod_{k=1}^{K} \theta_k^{\alpha_k - 1} \mathbb{I}(x \in S_K)$$

**Conjugate priors**

When the prior and the posterior have the same form, we say that the prior is a conjugate prior for the corresponding likelihood.

# Posterior

**Multiply the likelihood by the prior:**

$$
\begin{aligned}
p(\theta|D) &\propto p(\mathcal{D} \mid \theta)p(\theta) \\
&\propto \prod_{k=1}^{K} \theta_k^{N_k} \theta_k^{\alpha_k - 1} = \prod_{k=1}^{K} \theta_k^{\alpha_k + N_k - 1} \\
&= Dir(\theta \mid \alpha_1 + N_1, \ldots, \alpha_k + N_K)
\end{aligned}
$$

- posterior is obtained by adding the prior hyper-parameters (pseudo-counts) $\alpha_k$ to the empirical counts $N_k$
- MAP estimate is given by $\hat{\theta}_k = \frac{N_k + \alpha_k - 1}{N + \sum_{k=1}^{K} \alpha_k - K}$

# Posterior predictive distribution

How to make prediction of future observable data?

Predicting the probability of single toss under posterior:

$$
\begin{aligned}
p(X = j|\mathcal{D}) &= \int p(X = j|\theta)p(\theta|\mathcal{D})d\theta \\
&= \int p(X = j|\theta_j)\left[\int p(\theta_{-j}, \theta_j|\mathcal{D})d\theta_{-j}\right]d\theta_j \\
&= \int \theta_j p(\theta_j|\mathcal{D})d\theta_j = \mathbb{E}(\theta_j|\mathcal{D}) \\
&= \frac{\alpha_j + N_j}{\sum_{k=1}^{K} \alpha_k + N_k}
\end{aligned}
$$

# Example: language model using bag of words

- application of Bayesian smoothing using the Dirichlet-multinomial model is to language modeling
- predict which words might occur next in a sequence

Let's assume:

- the *i*'th word, $X_i \in \{1, \ldots, K\}$, is sampled independently from all the other words using a $Cat(\theta)$ distribution
- Cat - categorical distribution (Multinouli, generalization of Bernouli)
  - when the *i*-th outcome is obtained, the *i*-th entry of the random variable *X* takes value 1, while all other entries take value 0, e.g. [0,1,0,0,0,0,0]
- called: **bag of word model**

# Example: language model using bag of words

## Example

Mary had a little lamb, little lamb, little lamb,
Mary had a little lamb, its fleece as white as snow

## Vocabulary

| mary | lamb | little | big | fleece | white | black | snow | rain | unk |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |