Probabilistic Machine Learning:
2. Statistical refresher

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Apropriate agreements to propagate his ideas has been acquired.

Expectation of (Real) Random Variables

- Expectation:

$$\mathbb{E}(X) = \begin{cases} \sum_i x_i f_X(x_i) & \text{if } X \in \{x_1, \dots, x_k\} \subset \mathbb{R} \\ \int_{-\infty}^{\infty} x f_X(x) dx & \text{if } X \quad \text{continuous} \end{cases}$$

Example

- Bernoulli, $f_X(x) = p^x(1-p)^{1-x}$, for $x \in \{0, 1\}$
  $\mathbb{E}(X) = 0(1-p) + 1p = p$
- Binomial, $f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$, for $x \in \{0, \dots, n\}$
  $\mathbb{E}(X) = np$
- Gaussian, $f_X(x) = \mathcal{N}(x; \mu, \sigma^2)$
  $\mathbb{E}(X) = \mu$

- Linearity of expectation: $\mathbb{E}(\alpha X + \beta Y) = \alpha \mathbb{E}(X) + \beta \mathbb{E}(Y), \quad \alpha, \beta \in \mathbb{R}$
  $\mathbb{E}(c) = c$

Expectation of Functions of RVs

- $$\mathbb{E}(g(X)) = \begin{cases} \sum_i g(x_i) f_X(x_i) & \text{if } X \quad \text{discrete, } g(x_i) \in \mathbb{R} \\ \int_{-\infty}^{\infty} g(x) f_X(x) dx & \text{if } X \quad \text{continuous} \end{cases}$$

Example

- variance, $var(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$
- Bernoulli variance, $\mathbb{E}(X^2) = \mathbb{E}(X) = p$, thus $var(X) = p(1 - p)$
- Gaussian variance, $\mathbb{E}((X - \mu)^2) = \sigma^2$

The importance of the Gaussian

The importance of the Gaussian

Take n independent RVs $X_1, \ldots, X_n$, with $\mathbb{E}[X_i] = \mu_i$ and $var(X_i) = \sigma_i^2$

- Their sum, $Y_n = \sum_{i=1}^{n} X_i$ satisfies:

$$\mathbb{E}[Y_i] = \sum_{i=1}^{n} \mu_i \equiv \mu$$

$$var(Y_n) = \sum_{i=1}^{n} \sigma_i^2 \equiv \sigma^2$$

- Let $Z_n = \frac{Y_n - \mu}{\sigma}$, thus $\mathbb{E}[Z_n] = 0$ and $var(Z_n) = 1$
- Central limit theorem: under mild conditions:

$$lim_{n \to \infty} Z_n \sim \mathcal{N}(0, 1)$$

Two (or More) Random Variables

- Joint pmf of two discrete RVs: $f_{X,Y}(x,y) = \mathbb{P}(X = x \wedge Y = y)$
  Extends trivially to more than two RVs.
- Joint pdf of two continuous RVs: $f_{X,Y}(x,y)$, such that

$$\mathbb{P}((X,Y) \in A) = \int\int_A f_{X,Y}(x,y)dxdy, \quad A \in \sigma(\mathbb{R}^2)$$

  Extends trivially to more than two RVs.
- Marginalization:

$$f_Y(y) = \begin{cases} \sum_x f_{X,Y}(x,y) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx & \text{if } X \text{ continuous} \end{cases}$$

- Independence: $(X \perp\!\!\!\perp Y) \iff f_{X,Y} = f_X(x)f_Y(y) \implies \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$

Conditionals and Bayes' Theorem

- Conditional pmf:

$$f_{X|Y}(x|y) = \mathbb{P}(X = x|Y = y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

- Conditional pdf:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

...the meaning is technically delicate

- Bayes' theorem:

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)}$$

- Also valid in the mixed case (e.g., X continuous, Y discrete)

Joint, Marginal, and Conditional Probabilities: An Example

- A pair of binary variables $X, Y \in \{0, 1\}$, with joint pmf:

$f_{X,Y}(x, y)$	$Y = 0$	$Y = 1$
$X = 0$	1/5	2/5
$X = 1$	1/10	3/10

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$X = 0$	1/5	2/5
$X = 1$	1/10	3/10

- Marginals: $f_X(0) = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$
  $f_X(1) = \frac{1}{10} + \frac{3}{10} = \frac{4}{10}$
  $f_Y(0) = \frac{1}{5} + \frac{1}{10} = \frac{3}{10}$
  $f_Y(1) = \frac{2}{5} + \frac{3}{10} = \frac{7}{10}$
- Conditional probabilities:

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  $f_Y(0) = \frac{1}{5} + \frac{1}{10} = \frac{3}{10}$
  $f_Y(1) = \frac{2}{5} + \frac{3}{10} = \frac{7}{10}$
- Conditional probabilities:

$f_{X\|Y}(x\|y)$	$Y=0$	$Y=1$
$X=0$	2/3	4/7
$X=1$	1/3	3/7

$f_{Y\|X}(y\|x)$	$Y=0$	$Y=1$
$X=0$	1/3	2/3
$X=1$	1/4	3/4

An Important Multivariate RV: Multinomial

- Multinomial: $X = (X_1, \ldots, X_K)$, $X_i \in \{0, \ldots, n\}$, such that $\sum_i X_i = n$, X_i denotes no. of outcomes class i occurs in n trials

$$f_X(x_1, \ldots, x_k) = \begin{cases} \binom{n}{x_1 x_2 \ldots x_K} p_1^{x_1} p_2^{x_2} \ldots p_K^{x_K} & \text{if } \sum_{i=1}^K x_i = n \\ 0 & \text{if } \sum_{i=1}^K x_i \neq n \end{cases}$$

$$\binom{n}{x_1 x_2 \ldots x_K} = \frac{n!}{x_1! x_2! \ldots x_K!}$$

- Generalizes the binomial from binary to K-classes.

Example

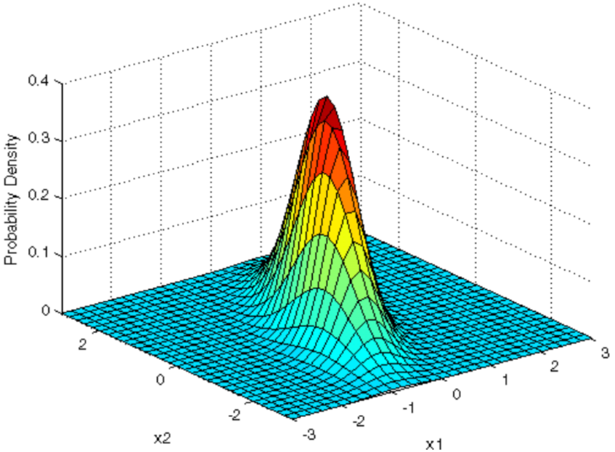
- tossing n independent fair dice, $K = 6$, $p_1 = \ldots = p_6 = 1/6$, x_i = number of outcomes with i dots (of course, $\sum_{i=1}^{K=6} x_i = n$)
- bag of words (BoW) multinomial model with vocabulary of K words and the document that consists of n words in total.

An Important Multivariate RV: Gaussian

- Multivariate Gaussian: $X \in \mathbb{R}^n$

$$f_X(x) = \mathcal{N}(x; \mu, C) = \frac{1}{\sqrt{det(2\pi C)}} exp\Big(-\frac{1}{2}(x-\mu)^T C^{-1}(x-\mu)\Big)$$

- Parameters: vector $\mu \in \mathbb{R}^n$ and matrix $C \in \mathbb{R}^{n\times n}$
- Expected value: $\mathbb{E}(X) = \mu$. Meaning of C: next slide



Covariance, Correlation, and all that...

- Covariance between two RVs:

$$cov(X,Y) = \mathbb{E}\Big[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\Big] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

- Relationship with variance: $var(X) = cov(X,X)$
- Correlation:

$$corr(X,Y) = \rho(X,Y) = \frac{cov(X,Y)}{\sqrt{var(X)}\sqrt{var(Y))}} \in [-1,1]$$

- $X \perp\!\!\!\perp Y \iff f_{X,Y}(x,y) = f_X(x)f_Y(y) \implies cov(X,Y) = 0$

More on Covariances

Let X, Y, W, V are real-valued random variables and a, b, c, d are constant.

- $cov(X, a) = 0$
- $cov(X, X) = var(X)$
- $cov(X, Y) = cov(Y, X)$
- $cov(aX, bY) = abcov(X, Y)$
- $cov(X + a, Y + b) = abcov(X, Y)$
- $cov(aX + bY, cW + dV) = accov(X, W) + adcov(X, V) + bccov(Y, W) + bdcov(Y, V)$