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A Physical Interpretation for the Fractional Derivative in Levy Diffusion

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Abstract—To the authors' knowledge, previous derivations of the fractional diffusion equation are based on stochastic principles [1], with the result that physical interpretation of the resulting fractional derivatives has been elusive [2]. Herein, we develop a *fractional flux law* relating solute flux at a given point to what might be called the complete (two-sided) fractional derivative of the concentration distribution at the same point. The fractional derivative itself is then identified as a typical superposition integral over the spatial domain of the Levy diffusion process. While this interpretation does not obviously generalize to all applications, it does point toward the search for superposition principles when attempting to give physical meaning to fractional derivatives. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

As detailed in several recent monographs [2–4], numerous applications for fractional calculus have been found in various physics, engineering, and financial areas. However, as also noted by these authors, a clear physical interpretation of the fractional derivative has been elusive. Podlubny [2] concludes that “the complete theory of fractional differential equations, especially the theory of boundary value problems for fractional differential equations, can be developed only with the use of both left and right derivatives”. As shown below, this conclusion is key for developing a relatively simple physical interpretation for the fractional spatial derivative that appears in the derivation of diffusion equations based on Levy motion rather than Brownian motion. Such equations are of much current interest in the fields of physics and hydrology [1,5–10]. The result of our analysis is the development of a *fractional flux law* relating solute flux at a given point to what might be called the complete (two-sided) fractional derivative of the concentration distribution

at the same point. The fractional derivative itself is then identified as a typical superposition integral over the spatial domain of the Levy diffusion process.

2. LEFT- AND RIGHT-HAND FRACTIONAL DERIVATIVES

In our development, we will use the left and right versions of the Riemann-Liouville fractional derivative [2]. For a spatial variable x , in the domain $a \leq x \leq b$, and a function $f(x, t)$ of space x , and time $t \geq 0$, one may define the left and right fractional derivatives as

$${}_a D_x^p f(x) = \frac{1}{\Gamma(n-p)} \left(\frac{\partial^n}{\partial x^n} \right) \int_a^x (x-\xi)^{n-p-1} f(\xi) d\xi \quad \text{and} \quad (1)$$

$${}_x D_b^p f(x) = \frac{1}{\Gamma(n-p)} \left(\frac{\partial^n}{\partial x^n} \right) \int_x^b (x-\xi)^{n-p-1} f(\xi) d\xi. \quad (2)$$

In the above, ${}_a D_x^p f(x)$ and ${}_x D_b^p f(x)$ are the left and right fractional derivatives, respectively, around the point x , p is the order of the fractional derivative, and n is an integer indicating the order of the conventional partial differentiation applied to the integral. Various combinations of n and p give all orders of fractional (and integer) differentiation.

3. NON-FICKIAN DIFFUSION

The governing equation for Levy diffusion of a solute at concentration $C(x, t)$, with conventional advective transport at velocity v , is given by Benson *et al.* [9,10] as

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial x} \left(vC - K \left(\frac{1}{2} \right) {}_a D_x^{\alpha-1} C - K \left(\frac{1}{2} \right) {}_x D_b^{\alpha-1} C \right), \quad (3)$$

with $n = 1$ in the fractional derivatives, α ($1 < \alpha \leq 2$ for the present application) representing the Levy index in the symmetric Levy probability density function (PDF), and K representing the (constant) fractional diffusion coefficient. As written, equation (3) is a direct generalization of the Gaussian (Brownian motion) case and reduces to this case when $\alpha = 2$.

Equation (3) represents the usual conservation statement that at each point the negative of the divergence of the solute flux, composed of the advective flux and the fractional diffusive flux, is equal to the change of concentration with time. Thus, implicit in (3) is a fractional flux law that becomes the classical Fick's law when $\alpha \rightarrow 2$. For the symmetric Levy case, this law is given by

$$q = -\frac{K}{2} ({}_a D_x^{\alpha-1} C + {}_x D_b^{\alpha-1} C) = \frac{-K}{2\Gamma(2-\alpha)} \frac{\partial}{\partial x} \left[\int_a^x \frac{C(\xi, t) d\xi}{(x-\xi)^{\alpha-1}} + \int_x^b \frac{C(\xi, t) d\xi}{(\xi-x)^{\alpha-1}} \right]. \quad (4)$$

Using the Fourier transform, it is easy to show that equation (4) reduces to the classical Fick's law ($q = -K \frac{dC}{dx}$) when α becomes 2 [2].

4. PHYSICAL INTERPRETATION OF THE $\alpha - 1$ FRACTIONAL DERIVATIVE

Consider the two integrals in equation (4) given by

$$\frac{\partial}{\partial x} \int_a^x \frac{C(\xi, t) d\xi}{(x-\xi)^{\alpha-1}} \quad \text{and} \quad \frac{\partial}{\partial x} \int_x^b \frac{C(\xi, t) d\xi}{(\xi-x)^{\alpha-1}}. \quad (5)$$

Applying Leibniz' rule to each integral yields

$$\frac{\partial}{\partial x} \int_a^x \frac{C(\xi, t) d\xi}{(x-\xi)^{\alpha-1}} = \int_a^x \frac{\partial}{\partial x} \left[\frac{C(\xi, t) d\xi}{(x-\xi)^{\alpha-1}} \right] + \lim_{\xi \rightarrow x} \left[\frac{C(x, t)}{(x-\xi)^{\alpha-1}} \right] \quad \text{and} \quad (6a)$$

$$\frac{\partial}{\partial x} \int_x^b \frac{C(\xi, t) d\xi}{(\xi-x)^{\alpha-1}} = \int_x^b \frac{\partial}{\partial x} \left[\frac{C(\xi, t) d\xi}{(\xi-x)^{\alpha-1}} \right] - \lim_{\xi \rightarrow x} \left[\frac{C(x, t)}{(\xi-x)^{\alpha-1}} \right]. \quad (6b)$$

In the Appendix, it is shown rigorously that the two limits in equation (6) cancel. If the partial differentiation of the integrands is then performed, one may write

$$q = \frac{-K(1-\alpha)}{2\Gamma(2-\alpha)} \left[\int_a^x \frac{C(\xi, t) d\xi}{(x-\xi)^\alpha} - \int_x^b \frac{C(\xi, t) d\xi}{(\xi-x)^\alpha} \right]. \quad (7)$$

Differentiating equation (7) with respect to x yields

$$dq_\xi(x, t) = -\frac{K(1-\alpha)}{2\Gamma(2-\alpha)} \left[\frac{C(\xi, t) d\xi}{(x-\xi)^\alpha} \right] + \frac{K(1-\alpha)}{2\Gamma(2-\alpha)} \left[\frac{C(\xi, t) d\xi}{(\xi-x)^\alpha} \right]. \quad (8)$$

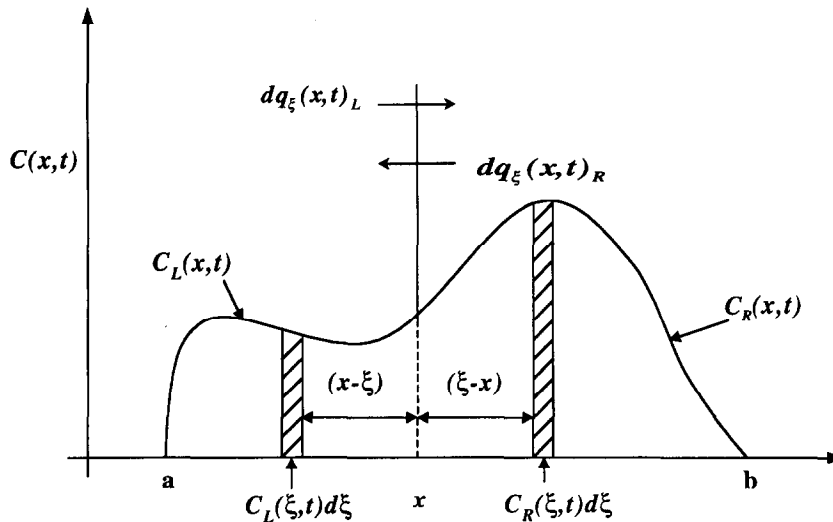


Figure 1. Diagram illustrating the superposition principle underlying the fractional derivative in Levy diffusion.

With the aid of Figure 1, equation (8) may be used as the basis for developing a superposition concept that underlies the fractional derivative when applied to Levy diffusion. Since the variance of Levy motion is unbounded, a mass concentration at a finite distance $|x - \xi|$ from the point x can cause a diffusive flux at point x . Mass to the left of x ($\xi < x$) will cause a positive flux, and that to the right of x ($\xi > x$) will cause a negative flux. The sum of these two fluxes will be the net flux. Let a negative constant be defined as

$$B(\alpha) = \frac{1-\alpha}{2\Gamma(2-\alpha)}.$$

Then, equation (8) may be written as

$$dq_\xi(x, t) = dq_\xi(x, t)_L + dq_\xi(x, t)_R = -K \left[\frac{B(\alpha)C_L(\xi, t) d\xi}{(x-\xi)^\alpha} \right] + K \left[\frac{B(\alpha)C_R(\xi, t) d\xi}{(\xi-x)^\alpha} \right], \quad (9)$$

with $C_L(\xi)$ the mass concentration distribution to the left of x , and $C_R(\xi)$ the mass concentration distribution to the right. Thus, for a one-dimensional system of unit cross-sectional area, one can interpret $C_L(\xi) d\xi$ as an infinitesimal mass at location ξ to the left of x that produces the infinitesimal Levy diffusive flux, $dq_\xi(x, t)_L$, at position x , and $C_R(\xi) d\xi$ the infinitesimal mass to the right that produces the diffusive flux $dq_\xi(x, t)_R$. Then, the function $B(\alpha)/(x-\xi)^\alpha$ is a function analogous to a Green's function that represents, for a given α value, the attenuation of the diffusive flux by the distance between x and ξ . The total flux at location x is obtained by

integrating $dq_\xi(x, t)$ over the subdomains (a, x) and (x, b) . Thus, for the Levy diffusion problem, one can view the fractional derivative as

$$\begin{aligned} & \frac{1}{2} (aD_x^{\alpha-1}C(x, t) + xD_b^{\alpha-1}C(x, t)) \\ &= \frac{1}{2} \left[\frac{1}{\Gamma(2-\alpha)} \frac{\partial}{\partial x} \int_a^x \frac{C(\xi, t) d\xi}{(x-\xi)^{\alpha-1}} + \frac{1}{\Gamma(2-\alpha)} \frac{\partial}{\partial x} \int_x^b \frac{C(\xi, t) d\xi}{(\xi-x)^{\alpha-1}} \right] \\ &= -\frac{1}{K} \left[\int_a^x dq_\xi(x, t)_L + \int_x^b dq_\xi(x, t)_R \right] = -\frac{q(x, t)}{K}. \end{aligned} \quad (10)$$

5. AN ALTERNATE DEFINITION OF THE $\alpha - 1$ FRACTIONAL DERIVATIVE

Equation (10) is made more compact if what one might call a two-sided fractional derivative is defined as

$$aD_b^{\alpha-1}C(x, t) \equiv \frac{1}{2} \left[\frac{1}{\Gamma(2-\alpha)} \frac{\partial}{\partial x} \int_a^x \frac{C(\xi, t) d\xi}{(x-\xi)^{\alpha-1}} + \frac{1}{\Gamma(2-\alpha)} \frac{\partial}{\partial x} \int_x^b \frac{C(\xi, t) d\xi}{(\xi-x)^{\alpha-1}} \right]. \quad (11)$$

This would be analogous to defining the conventional first derivative as

$$\frac{df}{dx} \equiv \lim_{(\Delta x \rightarrow 0)} \frac{1}{2} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{f(x) - f(x - \Delta x)}{\Delta x} \right], \quad (12)$$

which is actually more general than the common definition since it gives a way of defining the slope at a point where the slope is discontinuous. With such a definition, one can write the fractional flux law as

$$q = -K (aD_b^{\alpha-1}C(x, t)), \quad (13a)$$

which yields the fractional diffusion equation

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x} K (aD_b^{\alpha-1}C(x, t)), \quad (13b)$$

and a clear physical meaning in terms of a superposition integral is given for the two-sided fractional derivative. While this meaning does not obviously generalize to all other applications and fractional derivatives, it does point toward the search for superposition principles when attempting to give physical meaning to fractional derivatives.

APPENDIX

RIGOROUS BASIS FOR EQUATION (7)

Let $f(x)$ be the function

$$f(x) = \int_a^x \frac{C(\xi)}{(x-\xi)^{\alpha-1}} d\xi + \int_x^b \frac{C(\xi)}{(\xi-x)^{\alpha-1}} d\xi. \quad (i)$$

This is the term in brackets in equation (4) with the dependence on time being suppressed. We want to show that in a suitable sense f has a derivative, and it is equal to

$$g(x) = (1-\alpha) \left[\int_a^x \frac{C(\xi)}{(x-\xi)^\alpha} d\xi - \int_x^b \frac{C(\xi)}{(\xi-x)^\alpha} d\xi \right]. \quad (ii)$$

For the case without boundaries, that is, $b = -\infty$, $a = \infty$, this matter is most clearly treated by considering the Fourier transform $\hat{f}(w)$ of f , and the Fourier transform $\hat{g}(w)$ of g . Elementary calculations show that

$$iw\hat{f}(w) = \hat{g}(w), \quad (\text{iii})$$

so in a distributional sense g is the derivative of f .

When

$$-\infty < a < b < \infty,$$

essentially the same argument can be made using principal values for the integrals. More precisely, for $\varepsilon > 0$, define

$$f_\varepsilon(x) = \int_a^{x-\varepsilon} \frac{C(\xi)}{(x-\xi)^{\alpha-1}} d\xi + \int_{x+\varepsilon}^b \frac{C(\xi)}{(\xi-x)^{\alpha-1}} d\xi \quad (\text{iv})$$

and

$$g_\varepsilon(x) = (1-\alpha) \left[\int_a^{x-\varepsilon} \frac{C(\xi)}{(x-\xi)^\alpha} d\xi - \int_{x+\varepsilon}^b \frac{C(\xi)}{(\xi-x)^\alpha} d\xi \right]. \quad (\text{v})$$

Then,

$$\frac{df_\varepsilon(x)}{dx} = \frac{C(x-\varepsilon)}{\varepsilon^{\alpha-1}} - \frac{C(x+\varepsilon)}{\varepsilon^{\alpha-1}} + g_\varepsilon(x).$$

Assuming $C(\cdot)$ is a continuous function, then

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \left[\frac{df_\varepsilon}{dx} - g_\varepsilon \right] = 0.$$

Thus, if the integral (ii) defining g is taken in the sense of principal values, i.e.,

$$g(x) = \lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} (1-\alpha) \left[\int_a^{x-\varepsilon} \frac{C(\xi)}{(x-\xi)^{\alpha-1}} d\xi - \int_{x+\varepsilon}^b \frac{C(\xi)}{(\xi-x)^{\alpha-1}} d\xi \right],$$

then the argument leading from equations (4) to (7) is valid.

REFERENCES

1. M.M. Meerschaert, D.A. Benson and B. Baumer, Multidimensional advection and fractional dispersion, *Phys. Rev. E* **59**, 5026–5028, (1999).
2. I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, (1999).
3. K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, New York, (1993).
4. S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach, Amsterdam, (1993).
5. A. Compte, Stochastic foundations of fractional dynamics, *Phys. Rev. E* **53**, 4191–4193, (1996).
6. A. Compte, Continuous time random walks in moving fluids, *Phys. Rev. E* **55**, 6821–6831, (1997).
7. A.S. Chavez, A fractional diffusion equation to describe Levy flights, *Phys. Lett. A* **239**, 13–16, (1998).
8. R. Metzler and T.F. Nonnenmacher, Fractional diffusion, waiting-time distributions, and Cattaneo-type equations, *Phys. Rev. E* **57**, 6409–6414, (1998).
9. D.A. Benson, S.W. Wheatcraft and M.M. Meerschaert, Application of a fractional advection-dispersion equation, *Water Resour. Res.* **36**, 1403–1412, (2000).
10. D.A. Benson, S.W. Wheatcraft and M.M. Meerschaert, The fractional order governing equation of Levy motion, *Water Resour. Res.* **36**, 1413–1423, (2000).