

# An Approximate Analytic Solution of the Inventory Balance Delay Differential Equation

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**Abstract:** In this paper we present an analytic investigation of the continuous time representation of the inventory balance equation supplemented by an order-up-to replenishment policy. The adopted model parameters describe the startup of a distribution facility facing a constant demand. The exact solution is approximated by a complex exponential function containing integration constants that are dependent on the principal mode of the Lambert W function. We give a detailed review of the solution strategy emphasising some underexposed components, and provide a fair discussion of its limitations. In particular we give a detailed study on the matching of the exact and the approximate analytic solution. We also derive and analyse damped non-oscillatory solutions that have been neglected in the literature. Although the model is prone to the well known permanent inventory deficit, it serves as a solid foundation for future improvements.

**Keywords:** *inventory control, ordering policy, stability, Lambert function.*

## 1. Introduction

“One of the principal reasons used to justify investments in inventories is its role as a buffer to absorb demand variability”; state Baganha and Cohen in 1998 [2]. More than a decade later we find, that modern supply chains composed of distribution units with safety stock are often prone to the instability of orders amplifying upwards in the chain, known as the bullwhip effect. Dejonckheere *et al.* [7] resolve this conundrum by claiming that

“...inventories can have a stabilizing effect on demand variation provided the replenishment decision is designed carefully via common control theory techniques.” *Careful design* translates to the construction and the analysis of the inventory’s mathematical model, serving as the basis of the implemented replenishment policy.

There are different models available in the literature to describe an inventory. The basic concepts of the method exploiting the advances in the field of control theory were laid by Simon [13], Vassian [15], Deizel and Elion [8]. More recently the control theoretic approach was used as a powerful tool in the field of inventory management by John *et al.* [11], Berry *et al.* [3], Disney *et al.* [9], Towill *et al.* [14] and others. The basic philosophy behind the method is threefold. First, the Laplace-transform is applied on the equations governing the dynamics of the inventory, *i.e.* the problem formulated in the time domain is transformed into the operator domain. Second, the transfer function of the system relating the transformed demand and order at a given distribution unit is derived. Third, the tools provided by control theory are exploited, such as stability-, frequency response- and spectral analysis.

A conceptually different line of research is based on stochastic views. The popular paper of Lee *et al.* [12] revealed five causes of the bullwhip effect induced by rational decision making policies, while Chen *et al.* [4] investigated the effect of demand signal processing. The methodology gives insights into the probabilistic nature of inventory control.

The third important family of models is based on the continuous time representation. The dynamics of the system is governed by ordinary differential equations (ODE) describing the balance of the inventory. In this framework all operations are performed in the time domain. Forrester [10] had a widely recognized pioneering contribution to the investigation of order amplification caused by non-zero lead time and demand signal processing. He also derived a second-order ordinary differential equation with a special term approximating the effect of lead time. Another path followed by Warburton [16] leads to the field of delay differential equations (DDE), requiring more elaborate solution strategies. The difficulty lies in the fact that the resulting characteristic equation is transcendent, with an infinite number of roots. Asl *et al.* [1] proposed a solution strategy by exploiting the advances in the theory of the Lambert W function due to Corless *et al.* [6]. The basic idea is that the solution of the inventory balance equation is assumed in the form of complex exponentials, and the integration constants are obtained in terms of the Lambert W function. Warburton [16] applied these ideas to present approximate analytic solutions for the inventory balance equation. He also explored the response of this continuous model to a variety of demand patterns. In [16] the basis of the approximate exponential formalism are laid and the startup of a new distribution facility facing a constant demand is modelled. The response of the inventory to a step function, a ramp and an impulse function is studied in [17–19], respectively.

This brief summary demonstrates, that the Lambert W function based exponential approximation has become a popular tool for the investigation of different replenishment policies applied to a variety of demand patterns in the continuous framework. The main reasons are:

- the tractability of the analysis,
- easy to determine stability properties,
- clear dependence on model parameters,
- a number of convincing test cases.

However, a careful study of the model equations justifies, that subtle, yet important components of the technique are not formulated by adequate precision in the literature. Thus, the aim of this paper fourfold. First, we present the complete description of the solution strategy emphasising certain underexposed components. This part of the paper also serves for future reference. Second, we dive into the details of one particular part of the solution strategy, *i.e.* the determination of certain integration constants. A part of the published theory is valid only in special cases, while other parts are in error. We give the derivation of the proper formulae and support the study with the presentation of test cases. The third aim is to demonstrate the limitations of the approach. In [16] the author claims, that the exponential approximation performs well for the whole range of parameters. We present test cases contradicting to this statement. The fourth aim is to give a detailed analysis of non-oscillatory exponential decaying solutions, that are not investigated in the related works [16–19].

In the following paragraphs we revisit the basic problem of continuous inventory control. The focus is on the pioneering paper of Warburton [16] that describes the foundations of the exponential approximation strategy applied to the startup of a distribution facility. For simplicity, WIP terms and demand smoothing are omitted just like in [16]. The structure of this paper is the following. In section 2 we introduce the labelling conventions and formulate the differential equation governing the dynamics of the inventory. In section 3 the exact analytic solution for vanishing lead time is described. The exponential approximation for non-zero lead time is analysed in section 4. Supporting test cases are presented in section 5. The implications of the analytic derivations are discussed in section 6 while some concluding remarks are given in the last section.

## 2. The inventory balance equation

We consider the problem of linear inventory control. The objective is to continuously satisfy the observed demand while maintaining an inventory to guard against the stochastic nature of real life processes. The demand rate of  $D(t)$  items per time unit is completely fulfilled if possible. Unfulfilled demands are backlogged. The aim of the adopted inventory replenishment policy is to drive the actual inventory level  $I(t)$  towards its desired target value  $\tilde{I}$ . At time  $t = 0$  the inventory is at  $I(0) = I_0$ . In the works of Warburton [17–19] it is assumed that

$$I_0 = \tilde{I}. \quad (1)$$

Here we release this assumption and study the more general case by letting the initial and the target inventory levels to be different. In order to manage the inventory, orders are placed with a rate of  $O(t)$ , and as an effect, items are received at the rate of  $R(t)$ . The temporal variation of  $I(t)$  is governed by the inventory balance equation

$$\frac{dI}{dt} = R(t) - D(t). \quad (2)$$

It is assumed, that the lead time  $\tau$  is a non-negative constant number, *i.e.* the receiving rate follows the delayed order rate according to

$$R(t) = O(t - \tau). \quad (3)$$

Inserting the last equation into (2), we obtain another form of the inventory balance equation, clearly reflecting its special delay character

$$\frac{dI}{dt} = O(t - \tau) - D(t). \quad (4)$$

The dynamic response of the inventory to the observed demand is characterized by the adopted inventory replenishment policy. In this paper we employ a fractional order-up-to policy that is a simplified variant of the *automatic pipeline, inventory and order-based production control system* (APIOBPCS) of John *et al.* [11]

$$O(t) = \frac{\tilde{I} - I(t)}{T}, \quad (5)$$

where  $T$  is a relaxation factor, also known as a controller, adjusting the responsiveness of the policy. The higher the value of  $T$ , the slower the response of the policy is. The system is closed by the demand rate that is chosen to be a step function through this paper

$$D(t) = D_0 \text{ for } t < 0 \text{ and } D(t) = D_1 = D_0 + \hat{D} \text{ for } t \geq 0, \quad (6)$$

where  $\hat{D} = D_1 - D_0$  is the jump in the demand.

### 2.1. Initial conditions

In the case of  $\tau = 0$  the inventory balance equation (4) becomes a simple first-order ODE that is subject to an initial condition at  $t = 0$

$$I(0) = I_0. \quad (7)$$

However, the choice of  $\tau > 0$  changes the entire character of the problem, since now we deal with a DDE and the initial condition turns to be a functional equation [1]. The standard procedure to treat this setup is as follows. For  $t < 0$  we assume that the inventory is in equilibrium with  $R(t) = D(t) = O(t) = D_0$  and  $I(t) = I_0$ . At  $t = 0$  the demand changes, and the solution is readily calculated for  $0 \leq t \leq \tau$ . By following established nomenclature [1], interval  $0 \leq t \leq \tau$  is referred to as the *pre-interval*, while the corresponding exact solution  $\phi(t)$  is called the *pre-shape function*. The functional initial condition for the inventory balance equation is

$$I(t) = \phi(t) \text{ if } 0 \leq t \leq \tau. \quad (8)$$

Now the task is to find a particular solution of equation (4) that satisfies condition (8). As we discuss later in this paper, in practical applications the exact satisfaction of (8) is relaxed due to the mathematical difficulties characterizing DDEs.

In this paper we investigate the initial setting considered by Warburton [16], where the following conditions are adopted:

$$D_0 = 0 \text{ for } t < 0, \quad I(0) = I_0 \text{ and } D(t) = D_1 = \hat{D} \text{ for } t \geq 0, \quad (9)$$

describing the startup of a distribution facility facing a constant demand.

### 3. The exact solution for $\tau = 0$

In the case of vanishing lead time ( $\tau = 0$ ) the inventory equation takes the following simple form

$$\frac{dI(t)}{dt} = \frac{\tilde{I} - I(t)}{T} - D_1. \quad (10)$$

The corresponding homogeneous equation

$$\frac{dI(t)}{dt} + \frac{I(t)}{T} = 0 \quad (11)$$

has a general solution as a simple exponential

$$I(t) = ce^{-\frac{t}{T}}. \quad (12)$$

The particular solution of inhomogeneous equation (10) satisfying initial condition (7) is:

$$I(t) = \left( I_0 - \tilde{I} + D_1 T \right) e^{-\frac{t}{T}} + \tilde{I} - D_1 T. \quad (13)$$

The corresponding order rate is

$$O(t) = D_1 - \left( \frac{I_0 - \tilde{I}}{T} + D_1 \right) e^{-\frac{t}{T}}. \quad (14)$$

Equations (13) and (14) imply that for zero lead time the system is always stable, *i.e.* it exponentially relaxes to a steady state. The asymptotic behaviour of the inventory

$$\lim_{t \rightarrow \infty} I(t) = \tilde{I} - D_1 T \quad (15)$$

reflects the presence of the permanent inventory deficit with the magnitude of  $D_1 T$ . This feature is a well known deficiency of ordering policy (5) focusing only on the replenishment of the inventory. On the other hand, the order rate asymptotically approaches the demand rate

$$\lim_{t \rightarrow \infty} O(t) = D_1. \quad (16)$$

#### 4. The approximate analytic solution for $0 < \tau$

In this section we present the approximate solution of the inventory balance equation in the form of complex exponentials. First we give a short introduction into the Lambert W function, since it plays a fundamental role in the solution of the emerging characteristic equation. Next, we calculate the pre-shape function, that is followed by the derivation of the particular approximate analytic solution. Since the description of the matching procedure between the preshape function and the approximation is not consistently covered in the literature, we give a detailed discussion of the subject.

##### 4.1. The Lambert W function

The Lambert W function is defined by  $W : \mathbb{C} \rightarrow \mathbb{C}$  satisfying equation

$$W(z) e^{W(z)} = z, \quad (17)$$

where  $z$  is a complex variable. An excellent overview of its history, applications, and related numerical analysis can be found in the work of Corless *et al.* [6]. It is highly relevant to our work that the Lambert W function is multivalued with an infinite number of branches. The  $m$ -th branch labelled by  $W_m$  satisfies equation (17) for all integer values of  $m$ . Amongst many fields of science it has its application in the solution of linear first-order delay differential equations [1], since the corresponding characteristic equation can be cast into form (17). In the field of linear inventory control the potential of the Lambert W function has been exploited by Warburton [16]. The 0-th branch ( $W_0$ ) is referred to as the principal branch of the Lambert W function. Since it plays a fundamental role in our study, its basic properties are summarized below when its domain is reduced to  $\mathbb{R}^-$ . The real ( $Re(W_0)$ ) and the imaginary ( $Im(W_0)$ ) components of  $W_0$  are plotted on figure 1. There are two significant values of  $z \in \mathbb{R}^-$ :  $-\pi/2$  and  $-1/e$ . When  $-\pi/2 < z < 0$ ,  $Re(W_0)$  is negative. The zeros of  $Re(W_0)$  are at  $z = -\pi/2$  and  $z = 0$ . The minimum value of  $Re(W_0)$  is taken at  $z = -1/e$

$$\min_{z < 0} Re(W_0(z)) = Re(W_0(-1/e)) = -1. \quad (18)$$

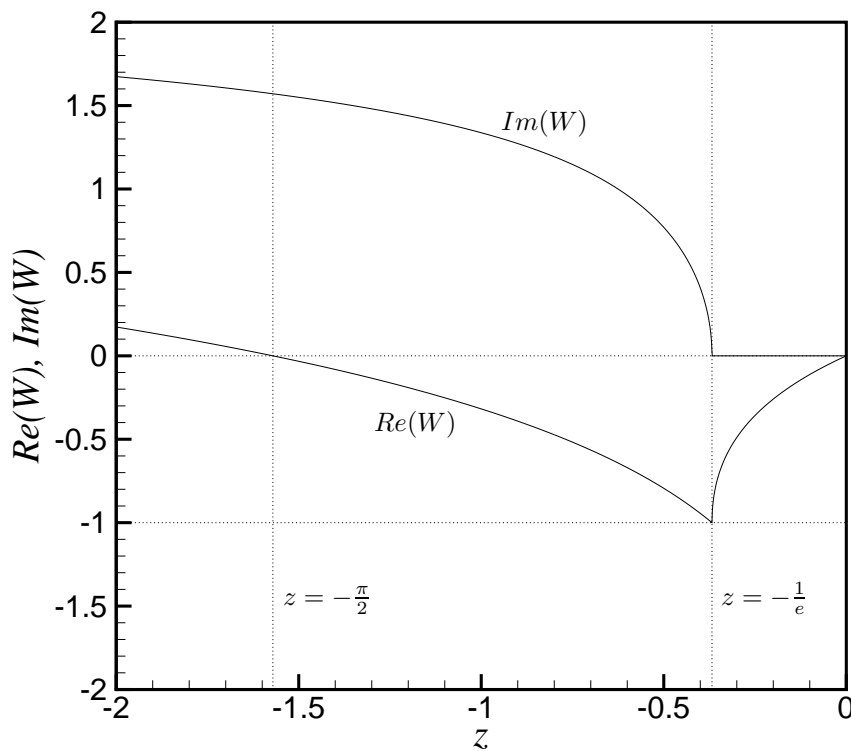


Figure 1: The real and the imaginary components of the principal branch of the Lambert W Function  $W_0(z)$  in the case of purely real  $z$ . The important values of  $z = -\pi/2$  and  $z = -1/e$  are labelled by dotted lines.

When  $z < -\pi/2$ ,  $Re(W_0)$  is positive. For  $-1/e \leq z \leq 0$   $W_0$  is purely real, i.e.  $Im(W_0) = 0$ . Finally,  $Im(W_0)$  is positive for  $z < -1/e$ . This information will be exploited in the following paragraphs.

#### 4.2. Computation of the pre-shape function

Condition (9) implies, that in the case of non-zero lead time no orders are delivered until  $t = \tau$ , thus, for  $0 \leq t < \tau$  the receiving rate vanishes:  $R(t) = D_0 = 0$ . The corresponding form of the inventory equation is



$$\frac{dI(t)}{dt} = -\hat{D}. \quad (19)$$

The solution matching initial condition (7) provides the pre-shape function as

$$\phi(t) = I_0 - \hat{D}t, \quad (20)$$

while the corresponding order rate is:

$$O(t) = \frac{\tilde{I} - I_0}{T} + \frac{\hat{D}}{T}t. \quad (21)$$

### 4.3. The approximate exponential solution for $\tau \leq t$

For  $\tau \leq t$  the inventory equation becomes an inhomogeneous delay differential equation

$$\frac{dI(t)}{dt} = \frac{\tilde{I} - I(t - \tau)}{T} - D_1, \quad (22)$$

that can be solved by the method described in [1, 16]. First, the general solution to the homogeneous equation is obtained, next a particular solution to the inhomogeneous equation is calculated.

#### 4.3.1 The homogeneous term

The homogeneous part of equation (22) is

$$\frac{dI(t)}{dt} + \frac{I(t - \tau)}{T} = 0. \quad (23)$$

Following the basic concepts of [1, 16], we look for the solution in the form of

$$I(t) = Ae^{qt}, \quad (24)$$

where  $A$  and  $q$  are constant complex numbers. The actual level of the inventory is considered to be the real part of (24). Substituting expansion (24) into equation (23) we arrive to

$$Ae^{qt} \left( q + \frac{e^{-q\tau}}{T} \right) = 0. \quad (25)$$

The corresponding characteristics equation is

$$q + \frac{e^{-q\tau}}{T} = 0. \quad (26)$$

By straightforward algebraic manipulations the above equation can be transformed into

$$q\tau e^{q\tau} = -\frac{\tau}{T}, \quad (27)$$

that is formally identical to equation (17) serving the basis for the introduction of the Lambert W function. Comparison of equations (27) and (17) immediately yields, that there are an infinite number of values exist for  $q$ . The  $m$ -th value of  $q$  labelled by  $q_m$  is associated with the  $m$ -th branch of the Lambert W Function

$$W_m(z) = q_m\tau \quad \text{and} \quad z = -\frac{\tau}{T}, \quad (28)$$

leading to

$$q_m = \frac{W_m(-\tau/T)}{\tau}. \quad (29)$$

The  $m$ -th mode of the solution of the inventory equation depending on the  $m$ -th branch is

$$I_m(t) = A_m e^{q_m t}, \quad (30)$$

where  $q_m$  represents known complex coefficients, while  $A_m$  labels the unknown complex integration constants depending on  $m$ . Due to the linearity of the inventory equation, any linear combination of the modes is a solution to (23)

$$I(t) = \sum_{m=-\infty}^{\infty} A_m e^{q_m t}. \quad (31)$$

#### 4.3.2 The inhomogeneous term

Since the system is excited by a constant term, we look for a particular solution of the inhomogeneous equation (22) in the following form

$$I(t) = B, \quad (32)$$

where  $B$  is a constant coefficient. Substitution of expansion (32) into equation (22) yields

$$B = \tilde{I} - D_1 T. \quad (33)$$

#### 4.3.3 The general solution

The general solution of equation (22) containing the yet unknown constant coefficients  $A_m$  is the sum of the general solution of the homogeneous equation and a particular solution of the inhomogeneous equation

$$I(t) = \tilde{I} - D_1 T + \sum_{m=-\infty}^{\infty} A_m e^{q_m t}. \quad (34)$$

The corresponding order rate is

$$O(t) = D_1 - \sum_{m=-\infty}^{\infty} \frac{A_m}{T} e^{q_m t}. \quad (35)$$

#### 4.3.4 Notes on the exponential expansion

The particular solution of equation (22) corresponding to the investigated problem has to satisfy the related initial condition given by (8):

$$\tilde{I} - D_1 T + \sum_{m=-\infty}^{\infty} A_m e^{q_m t} = \phi(t) \quad \text{if } 0 \leq t \leq \tau, \quad (36)$$

where, in principle,  $\phi$  could be any *reasonable* functions. The last equation assumes, that any functions that are differentiable over  $[0, \tau]$ , except at a finite number of points, can be expanded as the infinite sum of complex exponentials, with the strong restriction, that the exponents are determined by the branches of the Lambert W function evaluated at  $z = -\tau/T$  (see equation (29)). According to our best knowledge, such a proof does not exist. Nevertheless, in a practical calculation only a finite number of  $2M + 1$  (usually symmetric) modes are taken into consideration, leading to the following approximate solution of the inventory equation (4) subject to condition (8)

$$I_M(t) = \tilde{I} - D_1 T + \sum_{m=-M}^M A_m e^{q_m t}. \quad (37)$$

The lack of sound analytic investigation of the convergence of expansion (36) has two implications. First, it is not clear how large the value of  $M$  should be in order to get an acceptable solution. Second, there is no unique method for the calculation of the corresponding  $A_m$  coefficients. Due to the delay nature of the problem, we not only have to match  $I(t)$  at  $t = 0$ , but at all points within interval  $[0, \tau]$ . Clearly, in a general case an exact fit does not exist (*e.g.* consider a linear pre-shape function). Thus, approximate fitting mechanisms have to be applied, that are not unique.

There are three conceptually different principles can be found in the literature to approach the problem. Asl *et al.* [1] propose a method to fit approximation (37) at  $2M + 1$  equidistant points over the pre-interval, and to obtain coefficients  $A_m$  from a set of linear algebraic equations, provided that the corresponding matrix is invertible. Warburton and Disney [19] argue that the matrix is badly conditioned and that a large number of modes (20-30) are needed to be taken into consideration to get near the pre-shape function. They choose to follow the idea of Corless [5] by minimizing the following integral

$$\int_0^{\tau} [\phi(t) - I_M(t)]^2 dt. \quad (38)$$

Unfortunately, the mathematical details of this minimization procedure are omitted in [19]. A careful examination of the presented material implies that the results should be

treated with caution, warranting for further inspection. In this paper we do not aim the clarification of the issue.

The third option is to set  $M = 0$  and approximate the exact solution  $\mathcal{I}(t)$  of inventory equation (4) subject to condition (8) via  $I_0(t)$ , depending on the principal mode of the Lambert W function:

$$\mathcal{I}(t) \approx I_0(t). \quad (39)$$

In all related studies of Warburton [16–19] it is concluded, that this choice provides a sufficiently accurate representation of the inventory in practical applications. The author claims, that the highest deviation of  $I_0(t)$  from the reference solution is less than 3% at the first peak. However, as we point out later, this statement does not accurately reflect reality.

In this paper our interest goes exclusively into the investigation of approximation (39). For the sake of simplicity, from this point on subscript 0 is omitted, and we adopt the following labelling conventions

$$W = W_0(-\tau/T), \quad A = A_0 \quad \text{and} \quad I(t) \equiv I_0(t) = \tilde{I} - D_1 T + A e^{\frac{Wt}{\tau}}, \quad (40)$$

where  $\tau$  and  $T$  are real constants,  $W$  is a known complex constant defined in terms of  $\tau$  and  $T$ , and  $A$  is a complex integration constant specified below.

#### 4.3.5 Computation of the integration constants

Let us start by studying the behaviour of the *exact* solution  $\mathcal{I}(t)$  and its derivative at the left and at the right vicinities of  $t = \tau$ . The known pre-shape function (20) provides the left limit

$$I_L = \lim_{t \rightarrow \tau-} \phi(t) = \lim_{t \rightarrow \tau-} (I_0 - \hat{D}t) = I_0 - \hat{D}\tau, \quad (41)$$

$$\partial I_L = \lim_{t \rightarrow \tau-} \frac{d\phi(t)}{dt} = \lim_{t \rightarrow \tau-} \frac{d}{dt} (I_0 - \hat{D}t) = -\hat{D}. \quad (42)$$

Approximation (34) yields the right limit

$$I_R = \lim_{t \rightarrow \tau+} I(t) = \lim_{t \rightarrow \tau+} \left( \tilde{I} - D_1 T + A e^{\frac{Wt}{\tau}} \right) = \tilde{I} - D_1 T + A e^W, \quad (43)$$

$$\partial I_R = \lim_{t \rightarrow \tau+} \frac{dI(t)}{dt} = \lim_{t \rightarrow \tau+} \frac{d}{dt} \left( \tilde{I} - D_1 T + A e^{\frac{Wt}{\tau}} \right) = \frac{AW}{\tau} e^W. \quad (44)$$

Note, that inventory equation (4) itself defines the known exact value of  $dI(t)/dt$  also at the left vicinity of  $\tau$  (with  $O(t) = 0$ ) and at the right vicinity of  $\tau$  (with  $O(t)$  computed by equation (5)):

$$\lim_{t \rightarrow \tau-} \frac{dI(t)}{dt} = -\hat{D}, \quad (45)$$

$$\lim_{t \rightarrow \tau+} \frac{dI(t)}{dt} = \lim_{t \rightarrow \tau+} \left( \frac{\tilde{I} - I_0 + \hat{D}(t - \tau)}{T} - D_1 \right) = \frac{dI(t)}{dt} \Big|_{t=\tau} = \frac{\tilde{I} - I_0}{T} - D_1. \quad (46)$$

The last two equations imply that in general, the first-derivative of the exact solution is right continuous and left discontinuous at  $t = \tau$ . The slope is continuous at  $t = \tau$  if equation (1) holds and  $D_0 = 0$ .

Now we are ready to formulate appropriate matching conditions. Recall, that we are looking for a single complex exponential function approximating the unknown solution of the inventory balance equation. In order to complete our derivations, we have yet to determine the value of the remaining integration constant  $A$ , that is a complex number. There are two scalar unknowns, so we can require the satisfaction of two independent scalar conditions at most. One option would be to minimize the following integral

$$\int_0^{\tau} [\phi(t) - Re(I(t))]^2 dt. \quad (47)$$

Here we follow another approach proposed by Warburton [16]. The basic principle is to ensure, that at  $t = \tau$  the exponential approximation is launched from its proper level with the proper slope. In other words, we have to match approximation (34) and its first-derivative with their respective exact values.

First we focus on the computation of  $I_R$  that is obtained from the continuity of the inventory. In the time continuous framework it is natural to assume that the inventory is continuous at  $t = \tau$ , unless the demand is represented by an impulse function at that instance; a theoretically possible case that we do not cover. Thus, matching the inventory (43) to its exact value (41) yields

$$\tilde{I} - D_1 T + A e^W = I_0 - \hat{D}\tau, \quad (48)$$

Next, the derivative of the exponential approximation (44) is matched with the exact value of the slope (46)

$$\frac{AW}{\tau} e^W = \frac{\tilde{I} - I_0}{T} - D_1. \quad (49)$$

By introducing the following labelling conventions

$$J_0 = I_0 - \tilde{I} + D_1 T - \hat{D}\tau, \quad (50)$$

$$J_1 = \left( \frac{\tilde{I} - I_0}{T} - D_1 \right) \tau, \quad (51)$$

equations (48) and (49) can be transformed into the following system of algebraic equations containing complex coefficients to be solved for  $A$ :

$$A e^W = J_0, \quad (52)$$

$$AW e^W = J_1. \quad (53)$$

At this point it is useful to decompose  $A$  and  $W$  to their real and imaginary components

$$A = a + \alpha i, \quad (54)$$

$$W = w + \Omega i. \quad (55)$$

### The $\Omega \neq 0$ case

First we assume, that  $W$  has a non-zero imaginary part, *i.e.*  $\Omega \neq 0$ . This scenario happens only when  $\tau/T > 1/e$ . Since we use complex quantities for the representation of the inventory, equation (52) represents two scalar conditions for the two components of  $A$ . Thus, no more degrees of freedom are left for setting the value of the slope. Since in the given framework the satisfaction of the imaginary part means no practical benefit, we can relax the full satisfaction of equations (52) and (53) and focus only to the real parts

$$e^w (a \cos \Omega - \alpha \sin \Omega) = J_0, \quad (56)$$

$$e^w [a(w \cos \Omega - \Omega \sin \Omega) - \alpha(w \sin \Omega + \Omega \cos \Omega)] = J_1, \quad (57)$$

respectively. The solution of equations (56) and (57) for the components of  $A$  is

$$a = \frac{J_0(\Omega \cos \Omega + w \sin \Omega) - J_1 \sin \Omega}{e^w \Omega}, \quad (58)$$

$$\alpha = \frac{J_0(w \cos \Omega - \Omega \sin \Omega) - J_1 \cos \Omega}{e^w \Omega}. \quad (59)$$

These solutions were properly obtained in [16]. However, equations (58) and (59) loose their validity when  $\Omega = 0$ . This case was not investigated by Warburton, claiming, that it leads to permanent inventory deficit. While it is most certainly the case under the present circumstances, in the following paragraph we give the details of this particular scenario because it opens up an unexplored class of solutions to the inventory equation.

### The $\Omega = 0$ case

If  $0 < \tau/T \leq 1/e$ , then  $\Omega = 0$ , *i.e.*  $W$  becomes purely real. Now the real parts of equations (52) and (53) take the following, particularly simple form

$$a e^w = J_0, \quad (60)$$

$$aw e^w = J_1. \quad (61)$$



It turns out, that the real and the imaginary components of  $A$  are completely decoupled, thus, both equations (60) and (61) contain only  $a$  as an unknown,  $\alpha$  does not appear. The solutions for  $a$  by equations (60) and (61) are, respectively

$$a = \frac{J_0}{e^w}, \quad (62)$$

$$a = \frac{J_1}{w e^w}. \quad (63)$$

Equations (62) and (63) can only be simultaneously satisfied if

$$J_1 = J_0 w. \quad (64)$$

Theoretically, for any meaningful combination of  $T$  and  $\tau$  one can set  $I_0$  and/or  $\tilde{I}$  such, that equation (64) is satisfied. However, if  $I_0$  and  $\tilde{I}$  are given in an application, as it is usual, the satisfaction of equation (64) can not be guaranteed. In general, whenever  $\tau/T \leq 1/e$  holds, the slope of the inventory can not be matched with its exact value at  $t = \tau$ , only the value of the inventory can be set. This clear deficiency of the present framework based on the principal branch of the Lambert W function is not given in the related references [16–19]. Indeed, the complex exponential approximation starts with the correct value at  $t = \tau$ , but its slope is incorrect. This feature predicts errors in the solution whenever  $\Omega = 0$  holds.

#### 4.4. Notes on the matching

The matching procedure described above is based on the practically relevant principle, *i.e.* setting the level of inventory and its derivative at  $t = \tau$ . We noted, that in general, the derivative of the inventory is discontinuous at this point. It is only continuous if the initial and the target inventory levels are equal, *i.e.* equation (1) holds.

In order to analyse some models presented in the literature, for  $I_0 \neq \tilde{I}$  we shall investigate the consequence of improper matching based on the requirement of  $C_0$  and  $C_1$  continuity of the inventory at  $t = \tau$ . Even though condition (1) does not hold,  $C_1$  continuity can be enforced by matching the approximate solution (44) with (42). This choice can be conveniently implemented by replacing equation (51) with

$$J_1 = -\hat{D}\tau, \quad (65)$$

while all the following formulas stay unchanged.

## 5. Results

In this section we present the solution of test problems designed for highlighting the most significant properties of the complex exponential approximation. All cases derive from a single scenario considered by Warburton for demonstrating the accuracy of the model (see figure 7 of ref. [16]). The setup describes the startup of a distribution facility. Accordingly, for  $t < 0$  orders are not issued, *i.e.*  $O(t) = D_0 = 0$ . The lead time is  $\tau = 10$ . Unfortunately, in the given reference the numerical value of the constant demand rate is not given, nevertheless, it can be guessed from the figure to be  $D_1 = 20$ . At  $t = 0$  the facility starts its operation by replenishment policy (5) targeting  $\tilde{I} = 1000$ . The remaining parameters  $I_0$  and  $T$  are case dependent and given below. Note, that Warburton plotted the results from  $t = 0$  until  $t = 50$ , while we take a 20% larger interval with  $t = 60$ .

### 5.1. Oscillatory solution, $I_0 = \tilde{I}$

In the original test case detailed above  $I_0 = \tilde{I}$ , therefore the inventory is  $C_1$  continuous at  $t = \tau$ . Here we consider only a single value of the adjustment controller,  $T = 4$ . The solution is given on the left of figure 2. The thick solid line represents the reference solution, computed by a simple Runge-Kutta method. Since the level of the numerical error is not visible on the scale of the plot, we can well consider it as the exact solution  $\mathcal{I}(t)$ . The thin solid line corresponds to the known preshape function for  $0 \leq t < \tau$ , continued by the approximate exponential solution starting at  $t = \tau$ . On the right of the figure the signed relative error is plotted:

$$\epsilon(t) = \frac{\mathcal{I}(t) - I(t)}{\mathcal{I}(t)}. \quad (66)$$

Observe, that the exponential approximation seems to closely follow the exact solution. Warburton even concluded that the error is less than 3%. However, figure 2 implies that the error grows to 15% if we extend our investigation beyond  $t = 50$ .

### 5.2. Oscillatory solution, $I_0 \neq \tilde{I}$

Now we slightly perturb the setup of the previous test case, and reduce the starting inventory by 10%, *i.e.*  $I_0 = 900$ . Since  $I_0 \neq \tilde{I}$ , the exact solution will be  $C_1$  discontinuous at  $t = \tau$ , with a slope that is precisely captured by the approximate solution. The results are shown on figure 3. The thick solid line corresponds to the exact solution in the sense discussed above. The thin solid line represents the  $C_1$  discontinuous solution, with the exact values of  $I(\tau) = \mathcal{I}(\tau)$  and  $dI/dt|_{t=\tau} = d\mathcal{I}/dt|_{t=\tau}$  imposed. The thin dashed line

corresponds to the incorrect  $C_1$  continuous solution at  $t = \tau$ . Observe the significant errors on the right figure. For the theoretically correct case the deviation is 32%, while the incorrect enforcement of  $C_1$  continuity increases the error to 71%.

### 5.3. Non-oscillatory stable solution, $I_0 = \tilde{I}$

Now we turn our attention towards non-oscillatory solutions, when  $0 \leq \tau/T \leq 1/e$ . We take the limiting value of  $T = e\tau$ , resulting in  $W(-1/e) = -1$ . As in the original case,  $I_0 = 1000$ . Although the exact solution is  $C_1$  continuous at  $t = \tau$ , the approximate inventory does not reflect this property as predicted in section 4.3.5. This feature is well demonstrated by figure 4. The highest deviation from the reference solution is 9%. Observe, that the asymptotic solution suffers from permanent inventory deficit with a magnitude of  $D_1T$ .

### 5.4. Non-oscillatory stable solution, $I_0 \neq \tilde{I}$

Finally, we stay in the non-oscillatory domain with  $T = e\tau$  and decrease the starting inventory to  $I_0 = 500$ . The solution is presented on figure 5. The discontinuity of the exact solution is well pronounced in this case. The highest deviation from the reference solution is 25%. The effect of permanent inventory deficit is clearly captured.

## 6. Discussion

The scope of the presented study is the analytic investigation of linear inventory control in the framework of DDEs. The corresponding theory has been developed in a series of papers by Warburton [16–19]. Although these studies are definitely progressive, certain avenues have not yet been explored, and some misconceptions have led the author to suboptimal conclusions. In order to improve the present status of the available material, in this paper we revisited the subject, focusing on [16] that introduced the formalism. For completeness we gave a detailed description of the basic solution procedure, that we extended by some additional derivations. In this section we discuss the main implications of the theory and the test calculations.

### 6.1. The approximate nature of the exponential solution

In references [16–18] it is emphasised, that the presented theory provides exact solutions of the inventory balance equation, without the need of approximations. This statement is true for the general solution. However, any particular solution has to satisfy the corresponding initial condition *exactly*, that is in fact a functional condition given by (8). In

a practical example of [16], for  $0 \leq t \leq \tau$  the particular approximate solution, that is a single complex *exponential* function, has to match *exactly* the *linear* pre-shape function, which is not possible. The only theoretical possibility to achieve an exact particular solution in the given framework is related to the specific case, when the pre-shape function is purely exponential. Thus, the complex exponential solution based on the Lambert W function is in fact an exact analytic *approximation* to the particular solution defined by inventory equation (4) and initial condition (8).

## 6.2. Accuracy of the exponential approximation

Reference [16] provides the following conclusion: “In practical situations where there is likely to be noise in the data, the one-term Lambert W function provides an easy-to-compute, accurate representation of the inventory response.” As a supporting example, figure 7 of [16] displays plots containing both the reference solutions and the exponential approximations. In those particular cases the error is claimed to be less than 3%. However, in section 5 we extended the integration time of the very same test case by 20%, and found, that the relative error rapidly grows to 15%. If we let the starting inventory to differ from the target inventory by only 10%, the error climbs to 30%, that is not considered to be small anymore. In conclusion, we contradict to references [16–18] by stating, that the error is very much dependent on the parameters and on the integration time, rendering the model highly inaccurate at certain occasions.

## 6.3. Dependence on $\tau/T$

Reference [16] concludes: “Treating  $A$  as a complex constant results in a solution that turns out to provide an excellent representation of the inventory over the entire range of  $\tau/T$ .” As we pointed out in paragraph 4.3.5, this procedure works well if the imaginary part of  $W$  is not vanishing. However, in domain  $0 \leq \tau/T \leq 1/e$   $W$  is purely real, thus we can only match the level of the inventory at  $t = \tau$ , and we have no control over the derivative. This fundamental difference can lead to the appearance of considerable errors as justified by the results of section 5, especially if equation (1) does not hold.

## 6.4. The $C_1$ discontinuity of the inventory at $t = \tau$

It is somewhat surprising, that in [16–18] the matching conditions are derived by targeting both  $C_0$  and  $C_1$  continuity of the inventory at  $t = \tau$  for the complex exponential approximation. Indeed, in [18] we find: “To determine  $A$ , we recognize that the inventory and its derivative must be continuous at  $t = \tau$ .” Equations (45) and (46) of the present paper imply that this statement is incorrect. The derivative of the inventory is continuous at  $t = \tau$  only if

$$D_0 = \frac{\tilde{I} - I_0}{T}, \quad (67)$$

otherwise it is necessarily discontinuous. Thus, a more appropriate philosophy for obtaining constant  $A$  is to set both the level of the approximate exponential and its derivative to their respective exact values at  $t = \tau$ . In figures 3 and 5 we present solutions with  $I_0 \neq \tilde{I}$ . The non-continuity of the exact solution is apparent. Although equation (1) does not hold, the slope of the exponential approximation may still be forced to be continuous, regardless of the fact, that theoretically it is incorrect. This choice leads to an undesired undershoot followed by a phase shift, increasing the relative error above 70% in one particular example. On the other hand, if the inventory and its derivative are matched with their exact values at  $t = \tau$ , the approximate exponential starts at the proper level with the right slope. In this case figure 3 presents exaggerated overshoots around the maxima and an error function increasing above 30% at certain instances. Another interesting approach to determine constant  $A$  could be to analytically minimize integral (47).

## 7. Concluding remarks

The analytic investigation supported by computational evidence presented in this paper contradicts to the literature. In particular, references [16–18] conclude that the complex exponential approximation based on the principal branch of the Lambert W function provides an easy-to-compute, accurate representation of the inventory response. By applying slight perturbations on the setup of a single test case in [16] we found, that the accuracy of the analytic approximation is sensitive to the choice of the model parameters and the integration time. The relative error varies over a wide range, reaching even 30%, which is not acceptable in most applications. Clearly, a careful parameter study could reveal even much higher errors. It is not a surprising conclusion though, considering the fact that the solution of a DDE is approximated by one single complex exponential. In all the test cases of [16–18] it is assumed, that initially the inventory is at its target value. We released this assumption, since in real life it can not always be accommodated. Numerical examples imply considerable deviations from the unknown exact solution in this case, revealing a definite limitation of the model.

The accuracy is expected to increase by including more modes in expansion (37), as proposed by Warburton and Disney [19]. Due to the lack of corresponding convergence analysis the approximation has to be verified by a fairly simple numerical integration procedure representing the exact solution with a very high accuracy. If so, the question naturally arises: what is the point of the analytic efforts in increasing the accuracy, if the numerical solution is extremely fast, reliable and accurate? Perhaps the most benefit can

be gained from the investigation of the single exponential approximation based on the principal branch of the Lambert W function. Indeed, it is relatively simple to manipulate, and most results can be obtained in closed analytic form. It seems, that the scope of the presented methodology has its most value in the stability analysis of the inventory balance equation. Indeed the principal branch has the strictest stability condition that is embraced by the stability condition of the other branches [1]. The corresponding formula limits the ratio of the lead time over the adjustment time to  $0 \leq \tau/T \leq \pi/2$  for getting a stable response. It also provides hints to what parameter values to choose in order to avoid the generation of oscillatory orders by a steady demand.

Starting from the concepts discussed above we explored the behaviour of the exponential approximation in the parameter domain corresponding to stable non-oscillatory solutions. As an example, we studied the inventory response to a sudden increase in the demand followed by a constant state. From the point of view of inventory management non-oscillatory decaying solutions could be preferable over oscillatory ones. Surprisingly, the *optimal* ordering policy proposed by Warburton [16–18] positions the system in the stable *oscillatory* domain by claiming, that non-oscillatory solutions are prone to permanent inventory deficit. This defect is induced by incomplete ordering policies neglecting the so-called demand term. However, inclusion of this term removes the deficit and opens up the path to stable non-oscillatory solutions of the inventory balance equation. This will be the subject of our upcoming publication.

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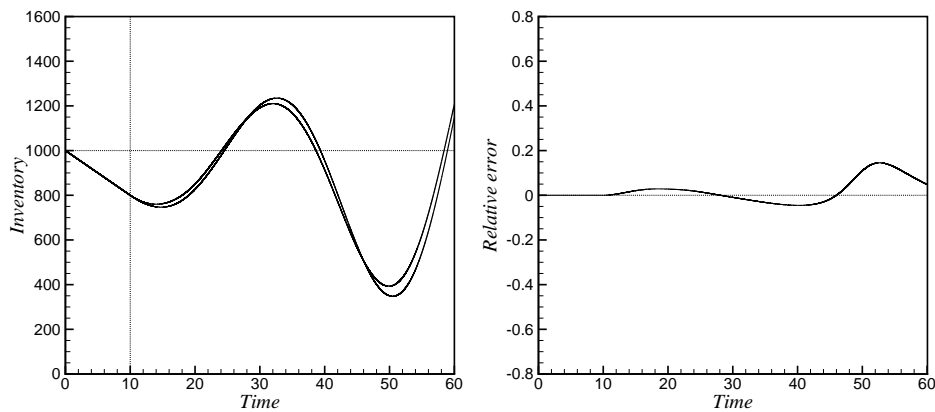


Figure 2: Inventory response in the oscillatory unstable domain. The parameters are:  $\tilde{I} = 1000$ ,  $I_0 = 1000$ ,  $D_0 = 0$ ,  $D_1 = 20$ ,  $\tau = 10$ ,  $T = 4$ . Thick solid line: reference solution. Thin solid line: exponential approximation. Left side: inventory response. Right side: relative error of the approximate solution ( $|\max \epsilon(t)| = 15\%$ ).

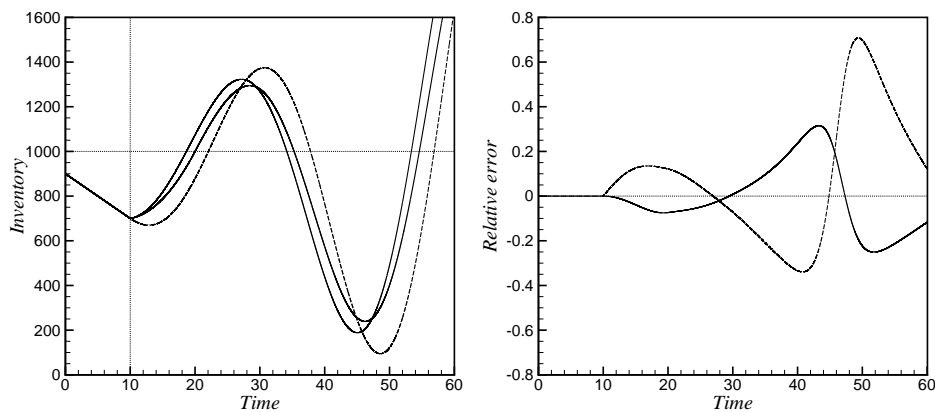


Figure 3: Inventory response in the oscillatory unstable domain. The parameters are:  $\tilde{I} = 1000$ ,  $I_0 = 900$ ,  $D_0 = 0$ ,  $D_1 = 20$ ,  $\tau = 10$ ,  $T = 4$ . Thick solid line: reference solution. Thin solid line: exponential approximation with  $C_1$  discontinuity ( $|\max \epsilon(t)| = 32\%$ ). Thin dashed line: exponential approximation with  $C_1$  continuity ( $|\max \epsilon(t)| = 71\%$ ). Left side: inventory response. Right side: relative error of the approximate solutions.

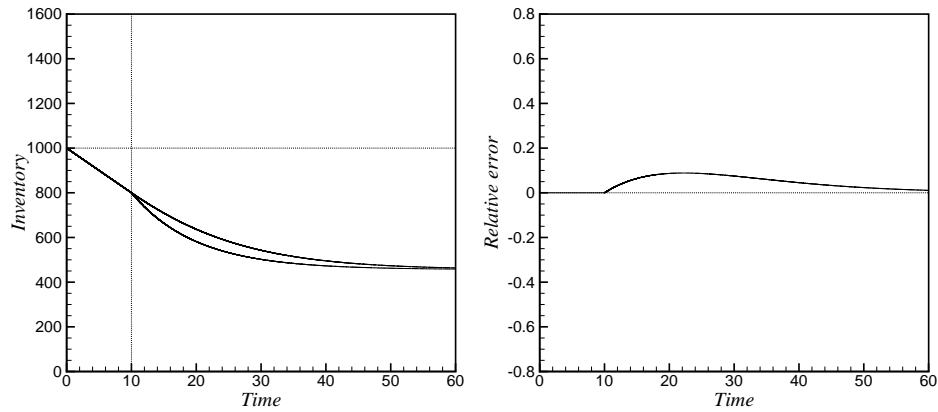


Figure 4: Inventory response in the non-oscillatory stable domain. Thick solid line: reference solution. Thin solid line: exponential approximation with  $C_1$  discontinuous inventory. The parameters are:  $\tilde{I} = 1000$ ,  $I_0 = 1000$ ,  $D_0 = 0$ ,  $D_1 = 20$ ,  $\tau = 10$ ,  $T = e\tau$ . Left side: inventory response. Right side: relative error of the approximate solution ( $|\max \epsilon(t)| = 9\%$ ).

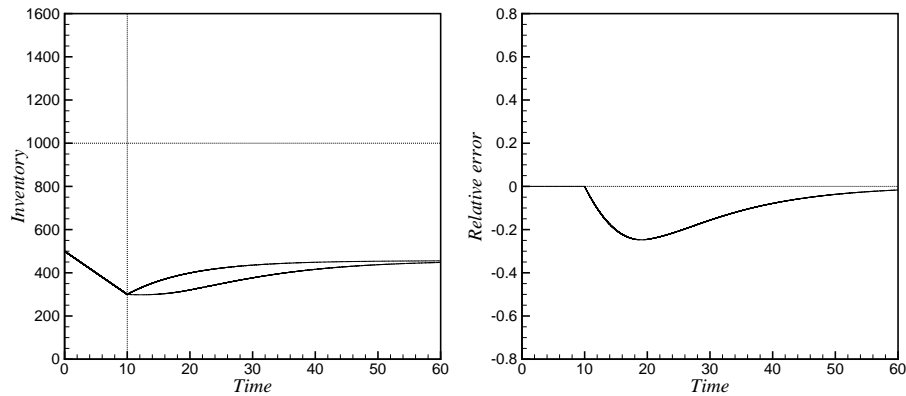


Figure 5: Inventory response in the non-oscillatory stable domain. Thick solid line: reference solution. Thin solid line: exponential approximation with  $C_1$  discontinuous inventory. The parameters are:  $\tilde{I} = 1000$ ,  $I_0 = 500$ ,  $D_0 = 0$ ,  $D_1 = 20$ ,  $\tau = 10$ ,  $T = e\tau$ . Left side: inventory response. Right side: relative error of the approximate solution ( $|\max \epsilon(t)| = 25\%$ ).



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