Discussion of

A Non-Linear Market Model

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Main Findings of the Paper

- Many market data-based anomalies exhibit a particular pattern of **underperforming** the market in **both good and bad times**.
  - This pattern is absent from accounting data-based anomalies.
- Their returns are **non-linearly** related to the market.
- Their CAPM αs are mostly explained by a **non-linear** transformation of the market factor.
  - It **does not** help price the accounting data-based anomalies.
- This is an **underperformance** premium rather than a **downside** premium:
  - Roughly **half of the αs** are accounted for by the **increasing** component of the projection of the pricing kernel on the market factor: "upside" risk premium.

The Message of the Paper

Main message:

- The (projection of) the pricing kernel on the market **must be increasing** beyond some threshold to explain the underperformance premium.
- This non-linearity is driven by **time variation in systematic variance**.
  - Hence all is well with economic theory!
  - But I'd like Tobias to **dig deeper**.

Salient point:

- Conditioning is crucial: without **time variation** in risk premiums, a non-linear model does not explain CAPM αs.

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Revisiting pricing kernel shape restrictions (1)

The paper states:

> *[...] for non-linear pricing to be relevant, one has to find a systematic relationship $g(R^m)$ between the CAPM pricing errors ε and R^m, that is **in addition inversely related to** $M(R^m)$*

Can we sharpen this description?

Revisiting pricing kernel shape restrictions (2)

Start with linear SDF: $M_{t+1} = 1 - \delta R^m_{t+1}$.

Same g for every j?

Price return on asset j: $R^j = \beta^j R^m + \varepsilon^j$. **Assume** $\mathbb{E}\left(\varepsilon^j | R^m\right) = \boxed{g(R^m)}$.

$$\mathbb{E}_t\left(\underbrace{\left(1 - \delta R^m_{t+1}\right)}_{M_{t+1}} R^j_{i,t+1}\right) = \mathbb{E}_t\left(\underbrace{\left(1 - \delta R^m_{t+1}\right)}_{M_{t+1}} \underbrace{\left(\beta^j R^m_{t+1} + \varepsilon^j_{t+1}\right)}_{\text{Return on asset } j}\right)$$

$$= \underbrace{\boxed{0}}_{\text{Linear pricing}} + \mathbb{E}_t\left(\left(1 - \delta R^m_{t+1}\right) \underbrace{\mathbb{E}\left(\varepsilon_{i,t+1} \mid R^m_{t+1}\right)}_{\text{Apply tower property}}\right) = \mathbb{E}_t\left(\left(1 - \delta R^m_{t+1}\right) g(R^m_{t+1})\right) =: \underbrace{\boxed{\alpha^j_t}}_{\text{Lin. SDF's pricing error}}$$

Revisiting pricing kernel shape restrictions (3)

Consider a non-linear SDF: $M_{t+1} = 1 - \delta R^m_{t+1} + \boxed{f(R^m_{t+1})}$.

Assume $\mathbb{E}_t\left(f(R^m_{t+1})\right) = 0$ and $\boxed{\mathbb{E}_t\left(R^m_{t+1} f(R^m_{t+1})\right) = 0}$.

NL component **does not price** linear payoffs

Price return on asset j: $R^j = \beta^j R^m + \varepsilon^j$.

$$\mathbb{E}_t\left(\overbrace{\left(1 - \delta R^m_{t+1} + f(R^m_{t+1})\right)}^{M_{t+1}} \overbrace{\left(\beta^j R^m_{t+1} + \varepsilon^j_{t+1}\right)}^{R^j}\right)$$

$$= \boxed{\alpha^j_t} + \mathbb{E}_t\left(f(R^m_{t+1})\varepsilon^j_{t+1}\right) = \boxed{\alpha^j_t} + \boxed{\mathbb{E}_t\left(f(R^m_{t+1}) g(R^m_{t+1})\right)}$$

Lin. SDF's pricing error

From tower property

Revisiting pricing kernel shape restrictions (4)

Correct pricing: $\mathbb{E}_t\left(M_{t+1}R^j_{t+1}\right) = 0$ requires

$$\alpha^j_t = -\mathbb{E}_t\left(f(R^m_{t+1})g(R^m_{t+1})\right)$$

$$\alpha^j_t = \mathrm{Cov}_t\Big(\underbrace{-f(R^m_{t+1})}_{\text{Non-linear SDF component}}, g(R^m_{t+1})\Big)$$

- Precise statement of "inversely related,"
- Another test?

Polynomial pricing kernel

Specification:

- $M_{t+1} = c_{0,t} + \sum_{i=1}^{N} c_{i,t} (R_{t+1}^{m})^{i} \rightarrow f(R^m) = c_{2,t}(R^m)^2 + \cdots + c_{N,t}(R^m)^N$
- **Does not** meet orthogonality condition $\mathbb{E}_t \left(R^m f(R^m)\right) \neq 0$.
- Consequence: **unclear** role of non-linear component,
- Could be significant due to linear pricing.

Suggestion: orthogonalize the polynomial.

- Use conditional $\mathbb{P}$ and $\mathbb{Q}$ measures,
- Schneider (2015) for details.

Are these really state variables? (1)

The paper states:

> *This result is encouraging as it offers a plausible economic interpretation for the upside risk premium, namely compensation for systematic variance risk.*

- I believe this statement is based on the CAPM + variance swap model;
- The other two models do not contain state variable dynamics,
  - That they only work **conditionally** is not sufficient to make that statement.

Let's have a closer look at this SDF...

Are these really state variables? (2)

SDF:

Long variance swap payoff

$$M^{VS}_{t+\boxed{\tau}} = a_{0,t} + b_1 R^m_{t+\tau} + b_2 \frac{\boxed{RV_{t+\tau} - VIX^2_{t,\tau}}}{\sigma_t^{d_1}}$$

Consider a flexible horizon

State dynamics are hidden in $RV_{t+\tau}$ (*skipping* jumps...):

Measurement noise

$$RV_{t+1} = \boxed{\int_t^{t+\tau} \sigma_s^2 ds} + \boxed{\nu_{t+1}}$$

State evolution

- How much state evolution? Depends on τ!
- How much noise? Depends on RV **data frequency**!

Are these really state variables? (3)

An experiment:

- Consider $\tau = 1$ day and a daily data frequency:
  - $\int_t^{t+\tau} \sigma_s^2 ds \approx \boxed{\sigma_t^2 \times \tau}$ ← Approximate expected variance

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- At 1-day horizon **and** frequency, RV is **little information on state dynamics**.

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An experiment:

- Consider $\tau = 1$ day and a daily data frequency:
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  - $v_{t+1} = \left(R_{t+1}^m\right)^2 - \sigma_t^2 \times \tau$
- At 1-day horizon **and** frequency, RV is **little information on state dynamics**.
- Take VIX_t^2 as a proxy for σ_t^2:

Variable	$\rho(\text{Variable}, RV_{t+1})$
R_{t+1}^m	≈ 0
VIX_t^2	$\geq 50\%$
$VIX_{t+1}^2 - VIX_t^2$	≈ 0

- At $\tau = 21$ (this paper), it *could be* much better, but **we aren't sure**.

Are these really state variables? (4)

Pure state-variable tests:

- Consider two estimates of state of economy σ_t^2:
  - $VIX^2_{t,\tau}$,
  - $\widehat{\sigma}_t^2$, **spot variance** estimate from high-frequency data.
- Specify SDFs:
  - $M^{VIX}_{t+\tau} = a_{0,t} + b_1 R^m_{t+\tau} + \sum_{i=2}^{N} c_{i,t} (R^m_{t+1})^i + b_{N+1,t} \Delta VIX^2_{t+1,\tau}$,
  - $M^{spot}_{t+\tau} = a_{0,t} + b_1 R^m_{t+\tau} + \sum_{i=2}^{N} c_{i,t} (R^m_{t+1})^i + b_{N+1,t} \Delta \widehat{\sigma}^2_{t+1}$,
  - Account for **measurement error** in $\widehat{\sigma}_t^2$ (instrument with estimate from **preceding day**; see Jegadeesh et al., 2019).
- Horse race between higher moments and state variable dynamics.

Are these really state variables? (5)

Horizon-dependent tests:

- Return to $M^{VS}_{t+\tau}$;
- Exploit SPX dailies and weeklies to perform tests at different horizons,
  - $VIX^2_{t,\tau}$ for 1 day or week ahead...
- If variance swap payoff is **significant** for **very small horizons**, I would interpret this as evidence **against** the state variable interpretation.

Description of polynomial pricing kernel methodology

The current write-up is confusing:

- Non-standard use of "price of risk": $\underbrace{\mathbb{E}_t^{\mathbb{Q}}\left((R_{t+1}^m)^N\right)}_{\text{Risk-neutral moment}}$ vs in $\mathbb{E}(R^j) = \beta^j \times \underbrace{\lambda}_{\text{Price of risk (Cochrane, 2001)}}$.
  - In continuous-time literature: "market price of risk:" $(\mu - r)/\sigma$.
- Paper claims SDF only estimated using option data (plus R^m), but **test asset returns** used to pin down time-variation in risk premiums (GMM).
- Estimation?
  - $M_{t+1} = \boxed{c_{0,t}} + \sum_{i=1}^{N} \boxed{c_{i,t}} (R_{t+1}^m)^i$
  - Estimation: determination of **coefficients** $\boxed{c_{i,t}}$ in M_{t+1}.
  - This paper: calculation of **risk-neutral moments** from option prices.

Streamlining the description

Main idea in the paper:

- If SDF components (“factors”) are **tradable**, coefficients can be estimated **from pricing the factors**.
- The SDF: $M_{t+1} = c_{0,t} + \boxed{\overset{\text{Traded!}}{R^m_{t+1}}} + \boxed{\overset{\text{Not so much...}}{\sum_{i=2}^{N} c_{i,t} (R^m_{t+1})^i}}$
- Big 🧠 moment: use **option prices** to estimate **prices** of the **higher moments**.
  - Factors become $F_{i,t+1}(R^m_{t+1})^i - \mathbb{E}^{\mathbb{Q}}\left((R^m_{t+1})^i\right)$
- Then standard asset pricing theory gives is the SDF:

$$M = a_t + \mathbf{b}_t'\mathbf{F}, \quad a_t = \frac{1}{R_{f,t}} - \mathbf{b}_t'\mathbb{E}_t(\mathbf{F}), \quad \mathbf{b}_t = \frac{1}{R_{f,t}}\mathbb{E}_t(\mathbf{F}\mathbf{F}')^{-1}\mathbb{E}_t(\mathbf{F})$$

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- Big 🧠 moment: use **option prices** to estimate **prices** of the **higher moments**.
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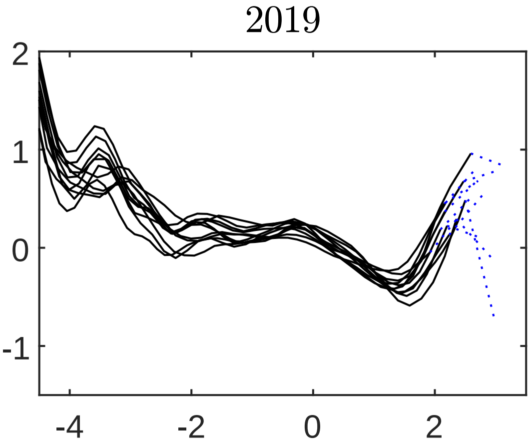
$$M = a_t + \mathbf{b}_t'\mathbf{F}, \quad a_t = \frac{1}{R_{f,t}} - \mathbf{b}_t'\mathbb{E}_t(\mathbf{F}), \quad \mathbf{b}_t = \frac{1}{R_{f,t}}\mathbb{E}_t(\mathbf{F}\mathbf{F}')^{-1}\mathbb{E}_t(\mathbf{F})$$

- Focus on the 🧠 innovation, all the rest is standard.

Role of Jump Risk

The quadratic pricing kernel is not enough:

- The quadratic pricing kernel **does not capture** the anomaly.
- The U-shape only kicks in for **high return values**.
- Is **jump risk** the missing piece?
  - State variable analysis would be **more difficult** if jump intensity does not depend on the volatility state.



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Conclusion

- Great empirical evidence on the **common feature** in multiple anomalies.
- Their alphas can be resolved with:
  - an SDF that is **non-linear** in the market factor**...**
  - and has **time-varying** risk premiums.
  - **Both components** are crucial.
- I see the "state variable" vs. "we have a problem" issue as **not resolved**.

References

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