Spectral gaps for cohomological Laplacians of $SL_n(Z)$

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$$G = \langle 5 \rangle$$

Theorem (Ozawa 14)

$$H^{L} \equiv 0 \iff \Delta^{2} - \lambda \Delta \geqslant 0$$
 for some $\lambda > 0$, where $\Delta = |S| - \sum_{s \in S} s \in RG$.

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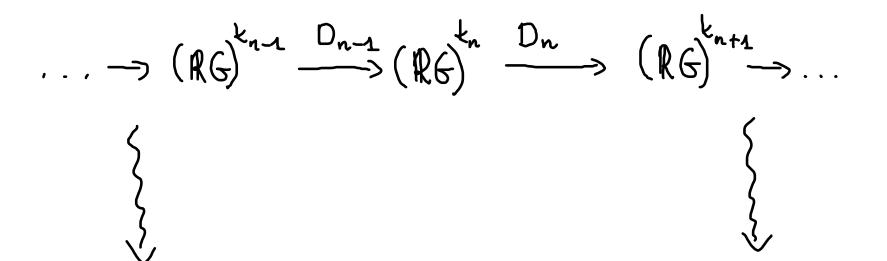
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• How to show $H^n \equiv 0$, ... not only for n=1?

 $(RG)^{k_{n-1}} \xrightarrow{D_{n-1}} (RG)^{k_n} \xrightarrow{D_n} (RG)^{k_{n+1}} \longrightarrow \dots$



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Thm (Bader, Nowak '20) $H^{n} \equiv 0, \text{ provided } \Delta_{n} - \lambda I \geqslant 0 \text{ for some } \lambda > 0,$ where $\Delta_{n} = D_{n}^{*} D_{n} + D_{n-1} D_{n-1}^{*} \in M_{k_{n}}(RG).$

• $G_n = SL_n(Z)$, Aut(F_n)

• Gn = $SL_n(Z)$, Aut (Fn) => expanders, random elts in finite groups • Gn = SLn(Z), Aut(Fn)

=> expanders, random elts in finite groups

Thm (Kaluba, Kielak, Nowak, Ozawa '19, '21)

$$\bigvee_{\substack{n > 3 \\ > 5}} \exists_{\lambda_n > 0} \Delta^2 - \lambda \Delta \geqslant 0 \quad (\Rightarrow) H^1(G_n, \Pi) = 0).$$

· G_n = $SL_n(Z)$, Aut(F_n) => expanders, random elts in finite groups Thm (Kaluba, Kielak, Nowak, Ozawa '19,121)

 $\bigvee_{n \geqslant 3} \exists_{\lambda_n > 0} \Delta^2 - \lambda \Delta \geqslant 0 \quad (\Rightarrow) H^1(G_n, \pi) = 0.$

Idea:

prove for low degrees and induce for higher using the elementary matrix presentation $G_n = \langle E_{ij} \rangle$ and the decomposition $\Delta^2 = Sq + Adj + Op$.

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$$\Delta_{\Lambda} = J^*J^* + dd^*, \quad J = \begin{bmatrix} \frac{\partial r_i}{\partial s_j} \end{bmatrix}, \quad d = [\Lambda - s_i]$$

• Reprove the thing using Δ_{\perp} instead of Δ $SL_{n}(Z) = \langle E_{ij} | [E_{ij}, E_{kl}], [E_{ij}, E_{jk}] = \frac{1}{ik} \rangle$

 $\Delta_1 = J^*J^+dd^*, \quad J = \begin{bmatrix} \frac{\partial r_i}{\partial s_j} \end{bmatrix}, \quad d = \begin{bmatrix} 1 - s_i \end{bmatrix}$ Thm (Kaluba, H., Nowak 22)

 $\Delta_1 - 0.32 I > 0$ for n=3.

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- $\Delta_1 = J^*J + dd^*, J = \begin{bmatrix} \frac{\partial r_i}{\partial s_j} \end{bmatrix}, d = [1 s_i]$ Thm (Kaluba, H., Nowak '22)

$$\Delta_1 - 0.32T \geqslant 0$$
 for $n=3$.

· What about $SL_n(2)$, n > 4?

$$\Delta_1 = Sq + Adj + Op$$

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Condusions:

$$\Delta_1 = Sq + Adj + Op$$

Thm (Kaluba, M. 123)

 $Adj - 0.2 (n-2) \underline{\Gamma} \geqslant 0$. Hence $\Delta_1 - 0.2 (n-2) \underline{\Gamma} \geqslant 0$.

Condusions:

- alternative proof of (T) for SLn(Z)

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Thm (Kaluba, M. '23)

 $Adj - 0.2 (n-2) \underline{\Gamma} \geqslant 0$. Hence $\Delta_1 - 0.2 (n-2) \underline{\Gamma} \geqslant 0$.

Condusions:

- alternative proof of (T) for SLn(Z)
- induction method can be applied to SAut(Fn) as well.