Fox derivatives, group cohomology, and Kazhdan's property (T)

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Introduction

Outline

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- We focus on finitely presented groups

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- Results

Relations between main concepts

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- Fox derivatives define group cohomology

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\boxed{H^*} \\
\boxed{\text{property (T)}} \arrow[u]
\end{tikzcd}$$

Relations between main concepts

- Fox derivatives define group cohomology
- Kazhdan's property (T) can be viewed cohomologically

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- Elements of interest: Laplacians
- We work with matrices over $\mathbb{R}G$

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- We decide SOS property with convex optimization

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- Defined by Ralph. H. Fox in 1950s

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- $\mathbb{Z}^2 = \langle a, b | aba^{-1}b^{-1} \rangle$:

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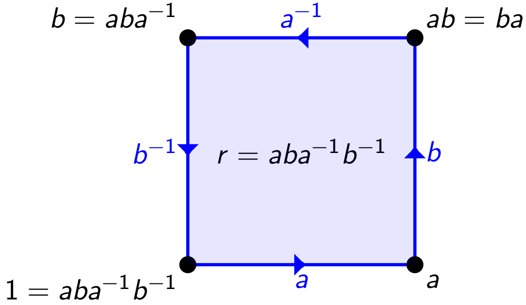

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- $0 \to V \xrightarrow{d_0} V^n \xrightarrow{d_1} V^m \to \cdots$ (Lyndon, 1950s)

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- Unlike d_0 and d_1, no explicit formula

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- (T) iff $\Delta^2 - \lambda\Delta = \mathrm{SOS}$ for $\lambda > 0$ (Ozawa, 2016)
- $\Delta = d_0^* d_0 = \sum_{i=1}^{n}(1 - s_i)^*(1 - s_i)$

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- Caution: $\overline{H}^i \neq \operatorname{Ker}\left(H^i(x) \to H^i(\{x_0\})\right)$

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Proposition (Dymara-Januszkiewicz)

For any $i \geq 2$ there exists a group G_i with reduced H^i and $H^i(G, \rho_0) \neq 0$ for some unitary representation ρ_0.

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- *H^i vanish and H^{i+1} are reduced.*

Vanishing condition

- Suppose we compute cohomology of G from

$$\cdots \to (\mathbb{R}G)^{k_{i-1}} \xrightarrow{d_{i-1}} (\mathbb{R}G)^{k_i} \xrightarrow{d_i} (\mathbb{R}G)^{k_{i+1}} \to \cdots$$

- $\Delta_i = d_i^* d_i + d_{i-1} d_{i-1}^* \in M_{k_i}(\mathbb{R}G)$

Theorem (Bader and Nowak, 2020)

TFAE for G and $i \geq 1$:

- *H^i vanish and H^{i+1} are reduced.*
- *$\Delta_i - \lambda I = \mathrm{SOS}$ for some $\lambda > 0$.*

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Proposition (Bader and Nowak, 2020)

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- Translation to the group ring setting:

Proposition (Bader and Nowak, 2020)

$\Delta_i(\rho)-\lambda I \succeq 0$ for each unitary ρ iff $\Delta_i-\lambda I = \mathrm{SOS}$ for some $\lambda>0$.

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- The converse implications remain open

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- $M = \mathrm{SOS}$ implies $M(\rho) \succeq 0$ for every unitary ρ
- Does the converse hold for $M = (\Delta_j^{\pm})^2 - \lambda\Delta_j^{\pm}$?

Results

(joint work with M. Kaluba and P. Nowak)

SDP problem for matrix SOS

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$M = \mathrm{SOS}$ *iff there exists* $P \succeq 0$ *such that* $M = \mathrm{y}^* P \mathrm{y}$.

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Lemma

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- Convex optimization for $M = \Delta_1 - \lambda I$:

$$\begin{aligned} \text{maximize:} \quad & \lambda \\ \text{subject to:} \quad & M_{i,j}(g) = \langle \delta_{i,j} \otimes \delta_g, P \rangle, \\ & P \succeq 0. \end{aligned}$$

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Proposition

If $(\Delta_2^-)^2 - \lambda \Delta_2^-$ is an SOS for G_1 and G_2, then it is an SOS for $G = G_1 * G_2$.

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- We use the following presentation of $\mathrm{SL}_3(\mathbb{Z})$:

$$\mathrm{SL}_3(\mathbb{Z}) = \langle \{E_{i,j}\} | \cdots, (E_{1,2}E_{2,1}^{-1}E_{1,2})^4 \rangle$$

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Theorem (Kaluba, M., Nowak)

For G, $\Delta_1 - \lambda I = \mathrm{SOS}$ for any $\lambda \leq 0.32$.

Corollary

The first cohomology of $\mathrm{SL}_3(\mathbb{Z})$ vanishes, and the second is reduced.

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- $(\Delta_2^-)^2 - n^2 \Delta_2^- = n^2 d_1^2 - n^2 d_1^2 = 0 = \mathrm{SOS}$
- Corollary: $*_{k=1}^m C_{n_k}$ has reduced second cohomologies

Thank you for attention!