A3P4

1. The Centered Finite Difference Scheme is defined as follows:

$$f_h'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2(x_{i+1} - x_{i-1})} \tag{1}$$

2. The error e_i is defined as follows:

$$e_i = |f'(x_i) - f_h'(x_i)| \tag{2}$$

3. The Talor Series is defined as follows:

$$f(x) = \frac{f(a)}{0!} + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + ... = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)(x-a)^n}{n!} \tag{3}$$

where a is a fixed known value and $f^{(n)}(x)$ refers to the nth derivative of $f(x)$.

4. Using the Taylor Series and a known value of $a = x_i$, $f(x_{i+1})$ can be written as:

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)(x_{i+1} - x_i)^2}{2!} + ... = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_i)(x_{i+1} - x_i)^n}{n!} \tag{4}$$

5. Define h as the step size, meaning $x_{n+1} - x_n = h$. $f(x_{i+1})$ becomes:

$$f(x_{i+1}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_i)h^n}{n!} \tag{5}$$

6. Similary, use the Taylor Series to write $f(x_{i-1})$:

$$f(x_{i-1}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_i)(x_{i-1} - x_i)^n}{n!} = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_i)(-h)^n}{n!} \tag{6}$$

7. Substituting Equation (1) into Equation (2):

$$e_i = \left| f'(x_i) - \frac{f(x_{i+1}) - f(x_{i-1})}{2(x_{i+1} - x_{i-1})} \right| \tag{7}$$

8. Substituting Equations (5) and (6) into Equation (7), and again using the fact that $x_{n+1} - x_n = h$:

$$e_i = \left| f'(x_i) - \frac{\sum_{n=0}^{\infty} \frac{f^{(n)}(x_i)h^n}{n!} - \sum_{n=0}^{\infty} \frac{f^{(n)}(x_i)(-h)^n}{n!}}{2h} \right| \tag{8}$$

9. Note that the summations cancel out for all even values of n (when $(-h)^n = h^n$). For odd values, the terms become doubled. Thus, Equation (8) becomes:

$$e_i = \left| f'(x_i) - \frac{2\sum_{n=0,odd}^{\infty} \frac{f^{(n)}(x_i)h^n}{n!}}{2h} \right| = \left| f'(x_i) - \frac{f'(x_i)h + f^{(3)}(x_i)h^3/6 + ...}{h} \right| \tag{9}$$

10. Dividing each term of the sum by h:

$$e_i = \left| f'(x_i) - f'(x_i) - f^{(3)}(x_i)h^2/6 - f^{(5)}(x_i)h^4/120 - ... \right| \tag{10}$$

11. Finally, canceling out the $f'(x_i)$ terms yields:

$$e_i = \left| -f^{(3)}(x_i)h^2/6 - f^{(5)}(x_i)h^4/120 - ... \right| = \left| \sum_{n=3,odd}^{\infty} \frac{f^{(n)}(x_i)h^{n-1}}{n!} \right| \tag{11}$$

12. The leading term contains h^2, meaning this is a second order approximation.