Problem 1.

Repeat the Lagrangian derivation using time, t, as the parameter instead of distance, s, along the direction of travel.

Solution

$$
\begin{aligned}
\Psi(t) &= \Psi(x_0, y_0, z_0, E_0, \hat{\Omega}_0, t_0, t) \\
x(t) &= x_0 + \int_0^t \frac{dx}{dt} dt = x_0 + \hat{\Omega} \cdot \hat{i} v t \\
y(t) &= y_0 + \int_0^t \frac{dy}{dt} dt = y_0 + \hat{\Omega} \cdot \hat{j} v t \\
z(t) &= z_0 + \int_0^t \frac{dz}{dt} dt = z_0 + \hat{\Omega} \cdot \hat{k} v t \\
E(t) &= E_0 + \int_0^t \frac{dE}{dt} dt = E_0 \\
\hat{\Omega}(t) &= \hat{\Omega}_0 + \int_0^t \frac{d\hat{\Omega}}{dt} dt = \hat{\Omega}_0 \\
t(t) &= t_0 + t \\
\Psi(t + dt) &= \Psi(t) - \Psi(t)\sigma_t(t) dt + q(t) dt \\
\frac{d\Psi}{dt} &= q(t) - \Psi(t)\sigma_t(t) \\
\frac{\mathrm{d}\Psi}{\mathrm{d}t} &= \left[\frac{\partial \Psi}{\partial t}\frac{\partial t}{\partial t} + \frac{\partial \Psi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \Psi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \Psi}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial \Psi}{\partial E}\frac{\partial E}{\partial t} + \frac{\partial \Psi}{\partial \hat{\Omega}}\frac{\partial \hat{\Omega}}{\partial t}\right] \\
&= \left[\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial x}\hat{\Omega} \cdot \hat{i} v + \frac{\partial \Psi}{\partial y}\hat{\Omega} \cdot \hat{j} v + \frac{\partial \Psi}{\partial z}\hat{\Omega} \cdot \hat{k} v\right] \\
q(t) &= \frac{1}{v}\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial x}\hat{\Omega} \cdot \hat{i} + \frac{\partial \Psi}{\partial y}\hat{\Omega} \cdot \hat{j} + \frac{\partial \Psi}{\partial z}\hat{\Omega} \cdot \hat{k} + \Psi(t)\sigma_t \\
q(\vec{r}, E, \hat{\Omega}, t) &= \frac{1}{v}\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial x}\hat{\Omega} \cdot \hat{i} + \frac{\partial \Psi}{\partial y}\hat{\Omega} \cdot \hat{j} + \frac{\partial \Psi}{\partial z}\hat{\Omega} \cdot \hat{k} + \Psi(\vec{r}, E, \hat{\Omega}, t)\sigma_t(\vec{r}, E, \hat{\Omega}, t)
\end{aligned}
$$

Problem 2.

How would the equation look for a charged particle with stopping power (i.e., energy loss per unit distance) of $S(E)$?

Solution

$$
\begin{aligned}
\Psi(t) &= \Psi(x_0, y_0, z_0, E_0, \hat{\Omega}_0, t_0, t) \\
x(s) &= x_0 + \int_0^s \frac{dx}{ds} ds = x_0 + \hat{\Omega} \cdot \hat{i} s \\
y(s) &= y_0 + \int_0^s \frac{dy}{ds} ds = y_0 + \hat{\Omega} \cdot \hat{j} s \\
z(s) &= z_0 + \int_0^s \frac{dz}{ds} ds = z_0 + \hat{\Omega} \cdot \hat{k} s \\
E(s) &= E_0 + \int_0^s \frac{dE}{ds} ds = E_0 - S(E) s \\
\hat{\Omega}(t) &= \hat{\Omega}_0 + \int_0^t \frac{d\hat{\Omega}}{dt} dt = \hat{\Omega}_0 \\
t(s) &= t_0 + \frac{s}{v}
\end{aligned}
$$

Combine these and take derivatives as before:

$$
q(\vec{r}, E, \hat{\Omega}, t) = \frac{1}{v} \frac{\partial \Psi(\vec{r}, E, \hat{\Omega}, t)}{\partial t} - \frac{\partial \Psi(\vec{r}, E, \hat{\Omega}, t)}{\partial E} S(E) + \vec{\nabla} \Psi(\vec{r}, E, \hat{\Omega}, t) \cdot \hat{\Omega} + \Psi(\vec{r}, E, \hat{\Omega}, t) \sigma_t(\vec{r}, E, \hat{\Omega}, t)
$$

Problem 3.

Fermi developed his age theory by assuming that neutron scattering was a continuous process (instead of happening instantaneously at each collision). Using ξ (average lethargy gain per collision), show that $S(E) = E\xi\sigma_s$.

Solution

On average, every time a neutron travels one mean free path, $\frac{1}{\sigma}$, it has a collision and its lethargy, u increases by ξ. Energy is related to lethargy by:

$$\begin{aligned}
u &= \ln \frac{E_0}{E} \\
&= \ln E_0 - \ln E \\
\frac{\mathrm{d}u}{\mathrm{d}E} &= \frac{-1}{E}
\end{aligned}$$

Therefore:

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}s} &= \frac{\mathrm{d}E}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}s} \\
&= \left(\frac{\mathrm{d}u}{\mathrm{d}E}\right)^{-1}\frac{\mathrm{d}u}{\mathrm{d}s} \\
&= -E\xi\sigma
\end{aligned}$$