Problem 1.

Using both relativistic and non-relativistic kinematics, calculate the kinetic energy of a proton with β=0.001, 0.01, 0.1, 0.2 and 0.5. Estimate where you start seeing a significant (>5%) difference between the relativistic and non-relativistic energies.

Solution

When approaching this relativistically, we know that $E = \gamma mc^2$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, and $E = T + mc^2$.

$$\begin{aligned} E &= T + mc^2 \\ T &= E - mc^2 \\ &= \gamma mc^2 - mc^2 \\ &= (\gamma - 1)mc^2 \\ &= \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) mc^2 \end{aligned} \tag{1}$$

When considering this from a classical perspective, we know that:

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}mc^2\frac{v^2}{c^2} \\ &= \frac{1}{2}\beta^2 mc^2 \end{aligned} \tag{2}$$

Since the mass of a proton is $m_{\mathrm{p}^+} = 938.272MeV$, the nonrelativistic and relativistic kinetic energies are captured in Table 1.

β	$T_{classical}$ (2)	$T_{relativistic}$ (1)
0.001	$4.69 * 10^-4$	$4.69 * 10^-4$
0.01	$4.69 * 10^-2$	$4.69 * 10^-2$
0.1	4.69	4.73
0.2	18.8	19.3
0.5	117	145

Table 1: Classical and Relativistic Kinetic Energies of p^+ as a function of β

To determine at what energy the error exceeds 5%,

$$
\begin{aligned}
0.05 &> \frac{T_{relativistic} - T_{classical}}{T_{relativistic}} \\
&> \frac{\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)\cancel{mc^2} - \frac{1}{2}\beta^2\cancel{mc^2}}{\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)\cancel{mc^2}} \\
&> 1 - \frac{\beta^2}{2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)} \\
\frac{\beta^2}{2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)} &> 0.95 \\
\beta^2 &> 1.90 * \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)
\end{aligned}
$$

Solving this for β gives:

$$\beta > 0.257$$

Problem 2.

Using relativistic kinematics, calculate the neutron threshold energy for: $n + {}^{12}\mathrm{C} \longrightarrow n + 3\,\alpha, (\alpha = {}^4\mathrm{He})$

Solution

Specify a 4-momentum vector $(\vec{p}, i\,(E_n + E_{^{12}\mathrm{C}}))$.

Initial state:

$$(\vec{p}, i\,(E_n + E_{^{12}\mathrm{C}}))$$

Assuming the ^{12}C is at rest, $E_{^{12}\mathrm{C}} = m\cancel{c^2}$ (for c=1)

$$\begin{aligned}(\vec{p_n}, i\,(E_n + m_{^{12}\mathrm{C}}))^2 &= \left({p_n}^2 - \left({E_n}^2 + {m_{^{12}\mathrm{C}}}^2 + 2m_{^{12}\mathrm{C}}E_n\right)\right) \\ &= \cancel{p_n^2} - \left(\cancel{p_n^2} + m_n^2 + m_{^{12}\mathrm{C}}^2 + 2E_n m_{^{12}\mathrm{C}}\right)\end{aligned}$$

Final state (all particles at rest in CM frame):

$$(\cancel{\vec{p}}, i\,(m_n + 3m_\alpha))^2 = -\left(m_n^2 + 9m_\alpha^2 + 6m_n M_\alpha\right)$$

Since 4-momentum is conserved, we can set these as equal to each other:

$$\begin{aligned}\cancel{m_n^2} + m_{^{12}\mathrm{C}}^2 + 2E_n m_{^{12}\mathrm{C}} &= \cancel{m_n^2} + 9m_\alpha^2 + 6m_n M_\alpha \\ m_{^{12}\mathrm{C}}^2 + 2E_n m_{^{12}\mathrm{C}} &= 9m_\alpha^2 + 6m_n M_\alpha \\ E_n &= \frac{9m_\alpha^2 + 6m_n m_\alpha - m_{^{12}\mathrm{C}}^2}{2m_{^{12}\mathrm{C}}^2} \\ &= 944.15MeV\end{aligned}$$