Bonus

in order to ensure that weights are always positive, set the integral of η along the latitude of μ_1 to be $2\sqrt{1-\mu_i^2}/\pi$. this gives the relationship

$$2\sqrt{1-\mu_i^2}/\pi = \frac{\sum_i \mu_i p_i}{\sum_i p_i}$$

and for $\mu_1 = 2\sqrt{1-\mu_{n/2}^2}/\pi$

this gives the result

$$\mu_1 = \frac{(n-2)(1-\sqrt{1-\alpha}) - (n-5)\alpha}{(n-5)^2\alpha - (n-2)(n-8)}$$

where $\alpha = \left[\left(\frac{4}{\pi}\right)^2 - 1\right]^2$

that said, I couldn't make this come out correctly, and the values in the table in the reference I found (pg 12 & 15 for equations, page 17 for quadrature sets) don't match the ones in the book, so I figure there's some wiggle room that is, if the area of the octant is measured in units of $\pi/2$, then

$$\sum_{j=1}^{n/2} w_j \mu_j^2 = \frac{1}{3} \tag{7}$$

so that one moment condition is satisfied by any completely symmetric set. Hence, one can choose μ_1 and the w_j to satisfy the $n/2 + 1$ even-moment conditions

$$\int_{-1}^{1} \frac{\mu^{2i} d\mu}{2} = \int_{0}^{1} \mu^{2i} d\mu = \frac{1}{2i + 1} = \sum_{j=p}^{n/2} w_j \mu_j^{2i} \tag{8}$$

for $i = 0, 1, \ldots, n/2$. Completely symmetric quadrature sets (which automatically satisfy the odd moments over the entire range of μ because of symmetry) obtained by satisfying (8) are given in Table I. However, for $n > 22$, such sets lead to negative w_j which are undesirable because of numerical truncation errors.

As an alternative to matching even moments, all half-range moments

$$\int_{0}^{1} \mu^{i} d\mu = \frac{1}{i + 1} = \sum_{j=1}^{n/2} w_j \mu_j^{i} \tag{9}$$

$i = 0, 1, \ldots, n/2$, can be matched, but this procedure leads to negative weights for $n \geq 12$. Table II displays sets obtained by satisfying equation (9).

A method of moment matching which does not lead to negative weights is obtained by matching half-range level moments. Instead of satisfying successively higher moments by choice by level weight, sequences of lower order moments are matched by choosing point weights. For example, in

TABLE I

Completely Symmetric Quadrature Sets Satisfying Even Moment Conditions.[a]

		μ_i	μ_i^2	w_i	p_i
n = 4	1	0.3500212	0.1225148	0.3333333	0.3333333
	2	0.8688903	0.7549704	0.1666667	
n = 6	1	0.2666355	0.0710945	0.2547297	0.1761263
	2	0.6815076	0.4644527	0.1572071	0.1572071
	3	0.9261808	0.8578110	0.0880631	
n = 8	1	0.2182179	0.0476191	0.2117283	0.1209877
	2	0.5773503	0.3333333	0.1370370	0.0907407
	3	0.7867958	0.6190476	0.0907407	0.0925926
	4	0.9511897	0.9047619	0.0604938	
n = 12	1	0.1672126	0.0279601	0.1639814	0.0707626
	2	0.4595476	0.2111840	0.1190886	0.0558811
	3	0.6280191	0.3944080	0.0631890	0.0373377
	4	0.7600210	0.5776319	0.0624786	0.0502819
	5	0.8722706	0.7608559	0.0558811	0.0258513
	6	0.9716377	0.9440799	0.0353813	
n = 16	1	0.1389568	0.0193090	0.1371702	0.0489872
	2	0.3922893	0.1538909	0.1090850	0.0413296
	3	0.5370966	0.2884727	0.0442097	0.0212326
	4	0.6504264	0.4230545	0.0643754	0.0256207
	5	0.7467506	0.5576364	0.0400796	0.0360486
	6	0.8319966	0.6922183	0.0392569	0.0144589
	7	0.9092855	0.8268001	0.0413296	0.0344958
	8	0.9805009	0.9613820	0.0244936	0.0085179
n = 20	1	0.1206033	0.0145452	0.1195893	
	2	0.3475743	0.1208079	0.1026829	
	3	0.4765193	0.2270706	0.0282212	
	4	0.5773503	0.3333333	0.0739389	
	5	0.6630204	0.4395960	0.0181985	
	6	0.7388226	0.5458588	0.0471265	
	7	0.8075404	0.6521215	0.0313726	
	8	0.8708526	0.7583842	0.0270754	
	9	0.9298639	0.8646469	0.0332842	
	10	0.9853475	0.9709096	0.0185105	

[a]The weights given sum to 0.5. The point weights are those of Fig. 3.

TABLE II

Completely Symmetric Quadrature Sets Satisfying Odd Moment Conditions.[a]

	μ_i	μ_i^2	w_i	p_i
n = 4	0.2958759	0.0875425	0.3333333	0.3333333
	0.9082483	0.8249149	0.1666667	
n = 6	0.1838670	0.0338071	0.2178992	0.1024651
	0.6950514	0.4830964	0.2308682	0.2308682
	0.9656013	0.9323858	0.0512325	
n = 8	0.1422555	0.0202366	0.1721829	0.1090122
	0.5773503	0.3333333	0.2101402	0.0631708
	0.8040087	0.6464300	0.0631708	0.2939388
	0.9795543	0.9595267	0.0545061	
n = 12	0.0935899	0.0087591	0.1168911	
	0.4511138	0.2035036	0.2531215	
	0.6310691	0.3982482	-0.1410287	
	0.7700602	0.5929927	0.2658355	
	0.8875457	0.7877373	-0.0388597	
	0.9912022	0.9824819	0.0440403	

[a]The weights given sum to 0.5. The point weights are those of Fig. 3.

Figure 2, the normalized integral of η on the unit sphere along the latitude of μ_1 is $2\sqrt{1 - \mu_1^2}/\pi$. Defining this quantity as

$$2\sqrt{1 - \mu_1^2}/\pi = \Sigma p_i \mu_i / \Sigma p_i \tag{10}$$

where the point weights are those belonging to the μ_i, gives a sequence of low-order moment conditions, one for each μ level. These moment equations for Figure 2 are then

$$\begin{aligned} p_1\mu_1 + p_2\mu_3 + p_1\mu_3 &= (p_1 + p_2 + p_1)2\sqrt{1 - \mu_1^2}/\pi \\ p_2\mu_1 + p_2\mu_2 &= (p_2 + p_2)2\sqrt{1 - \mu_2^2}/\pi \\ p_1\mu_1 &= 2p_1\sqrt{1 - \mu_3^2}/\pi \end{aligned} \tag{11}$$

For general even n, the relations analogous to (11) give n/2 relations for the n/2 quantities p_i and μ_1. However, the last two relations

$$\mu_1 + \mu_2 = 2\sqrt{1 - \mu_{n/2-1}^2}/\pi \tag{12a}$$

$$\mu_1 = 2\sqrt{1 - \mu_{n/2}^2}/\pi \tag{12b}$$

cannot both be satisfied. To obtain a consistent set of equations, Eq. (12b), representing the smallest latitudinal area, is deleted and, instead, (6) is satisfied so that (7) is also satisfied. Thus the zeroth, second, and a sequence of first-order moments are matched. Then (12a) with (4) serves to define μ_1

$$\mu_1 = \frac{(n - 2)(1 - \sqrt{1 - \alpha}) - (n - 5)\alpha}{(n - 5)^2\alpha - (n - 2)(n - 8)} \tag{13}$$

and hence all μ_i. Above, $\alpha = [(4/\pi)^2 - 1]^2$. The remaining $n/2 - 1$ p_j are found from equations analogous to (11). Sets obtained in this manner are displayed in Table III. Weights obtained in this manner are apparently always positive.

BIASED SYMMETRIC QUADRATURE SETS

Complete symmetry is required only in three-dimensional geometries. In lower dimensional geometries a relaxation of symmetry requirements allows additional degrees of freedom. A simple such relaxation is to keep the point and level arrangement of complete symmetry while allowing the points on each axis to be chosen from an independent set. In this case the requirement that points be on the unit sphere

$$\mu_i^2 + \eta_j^2 + \xi_k^2 = 1.0 \tag{14}$$

is solved by

$$\begin{aligned} \mu_m^2 &= \mu_1^2 + (m - 1)\Delta \\ \eta_m^2 &= \eta_1^2 + (m - 1)\Delta \\ \xi_m^2 &= \xi_1^2 + (m - 1)\Delta \\ \Delta &= 2(1 - \mu_1^2 - \eta_1^2 - \xi_1^2)/(n - 2) \end{aligned} \tag{15}$$

where $m = 1,2,\ldots,n/2$.

TABLE III

Completely Symmetric Quadrature Sets Satisfying Level Moment Conditions.[a]

	μ_i	μ_i^2	w_i	p_i
n = 4	0.3120418	0.0975949	0.3333333	0.3333333
	0.8971121	0.8048102	0.1666667	
n = 6	0.2390944	0.0571661	0.2582459	0.1831585
	0.6865981	0.4714169	0.1501748	0.1501748
	0.9410992	0.885668	0.0915792	
n = 8	0.2010510	0.0404215	0.2174330	0.1264098
	0.5773503	0.3333333	0.1283389	0.0910232
	0.7913565	0.6262452	0.0910220	0.0746315
	0.9587268	0.9191570	0.0632048	
n = 12	0.1596536	0.0254893	0.1726823	0.0781264
	0.4584710	0.2101957	0.1022793	0.0516366
	0.6284124	0.3949021	0.0738241	0.0429194
	0.7613203	0.5796086	0.0605145	0.0351903
	0.8742511	0.7643150	0.0516366	0.0309047
	0.9741773	0.9490214	0.0390632	
n = 16	0.1364305	0.0186133	0.1475402	0.0565552
	0.3917822	0.1534933	0.0874396	0.0361572
	0.5370040	0.2883733	0.0631648	0.0285758
	0.6505792	0.4232533	0.0519818	0.0262421
	0.7470832	0.5581334	0.0451381	0.0234298
	0.8324742	0.6930134	0.0402906	0.0188960
	0.9098865	0.8278934	0.0361672	0.0178932
	0.9812102	0.9627734	0.0282776	0.0156931

[a]The weights given sum to 0.5. The point weights are those of Fig. 3.