Abstract

This paper will use the heuristics of a finite Boolean algebra with operators to examine a restricted family of setwise stabilizers of 2-modular representations of the symmetric group with which polynomial time algorithms might be possible. The key virtue of the approach will be to use single-voice musical passages (modeled as partial algebras over a free semigroup) to help the search.

1 Notation

Throughout this article, when it's clear that a field is being expected, I will denote, GF(k) by k, so that the binary matrix algebra will be denoted $2^{n\times n}$. In addition, $\mathbf{n}$ will denote $\{1,\cdots,n\}$. It will be used both as a set and list (tuple). When convention will call to start at 0, I'll use the notation $\mathbf{n}_0 := \{0,\cdots,n-1\}$. An n-bit-vector v of atoms in a BA 2^n will correspond to its Boolean sum where needed.

The existence of an a-ary operator $f : (2^n)^a \to 2^n$ uniquely implies the existence of two more fundamental forms of a function on the generators (atoms), ${}^Bf : \mathbf{n}^a \to 2^n$ ${}^Vf : \mathbf{n}^a \to 2^n$ where the first instance of 2^n refers to the Boolean algebra and the second instance of 2^n refers to the binary vector space. It will always be clear which is intended by the superscript of the function.

We will use *elementary atomaic Boolean operators* quite frequently, so will shorten it to *elementary operators.* and by this I mean, a map $e_{(x_i,\ldots,x_n),v_j}$: $\mathbf{n}^a \to \mathbf{n}$ which maps a single atom-tuple $(x_1,\ldots,x_a)$ in the Cartesian power to x, (where $x_i, x \in \mathbf{n}$), and map the rest of the atom-tuples to 0. We will denote such a function by $(x_1,\ldots,x_n) \mapsto_n x$, and drop the subscript if the width is clear. To account for the wide variety of Boolean operators, we will look at the Boolean algebra whose atoms are all the elementary opeartors of a given width and arity, and see that any Boolean operator of a given width corresponds to an element in this Boolean operator space algebra as a superposition of elementary operators in the natural way (using complete distributivity).

We will only be interested in finite collections of operators of finite arity, and will utilize the multiplicity function $\mu : \mathbb{N}_0 \to \mathbb{N}_0$, whereby $\mu(a) = r$ if the BAO has r operators of arity a. We will always assume that μ is zero on all but a finite number of elements. When we write $op^a_\mu(i)$ to denote the *ith* operator of arity a, we assume that $0 < i \le \mu(a)$.

2 Introduction

Let n be some natural number, and consider a finite BAO B of width n, with some multiplicity signature. It's easy to see that the automorphism group of the universe of B is isomorphic to S_n, and then that raises the question of which permutations on the atoms induce which changes in the operators. The case of seeing which automorphisms respect the operators can be determined by looking at the automorphism group of the full structure (not just the universe) and won't be treated of here, as it is probably treated of elsewhere. However, if we examine which permutations of the atoms map one operator to another (assuming a "large" set of operators), things get more interesting, as now we are in a position to examine this using 2-modular representations that act on finite subspaces of $\bigoplus_{a=0}^{\infty} \mathbf{Op}_a^n$, where $\mathbf{Op}_a^n := 2^{n^{a+1}}$, the binary vector space, associated with the Boolean algebra on elementary operators discussed above. This is so defined because there are n^{a+1} elementary Boolean operators of arity a, since each operator is determined by the single element domain a-vector and the image. We will verify that they obey the laws of a binary vector space, (or better, 2-algebra) below.

3 Boolean Operator Algebra

To verify that the elementary Boolean operators do indeed generate a 2-algebra, we proceed as follows : First, we recall that, in any binary field (a field of characteristic 2), addition corresponds to pointwise Boolean exclusive disjunction of the bit-vectors, multiplication corresponds to pointwise join, minus corresponds to pointwise complement, and 0 is the n-vector $(0, \ldots, 0)$, while 1 is the n-vector $(1, \ldots, 1)$. The case of the operator space is not too different. We will restrict ourselves, here, to binary operators, with the certainty that the generalization to an arbitrary arity is straightforward. We will define the additive inverse of an elemntary binary operator $(a, b) \mapsto p$ to be the map that maps all atom tuples not equal to (a, b) to 1, and (a, b) to the complement $-p$. That is:

$$-(a, b) \mapsto p := (x, y) \mapsto \begin{cases} -p \text{ if } (x, y) = (a, b) \\ 1 \text{ if } (x, y) \neq (a, b) \end{cases} \tag{1}$$

There will be a slight abuse of notation in that the variable p, when occuring in an elementary operator represents an integer from 1 to n, whereas when occuring in the image of an arbitrary Boolean operator, represents an n-vector vector $(0, \ldots, 1, \ldots, 0)$, with the 1 in the p-th position, so that, $-p = (1, \ldots, 0, \ldots 1)$, with 0 in the p-th position, corresponding to the usual additive inverse in a binary vector space.

We define addition to map identical domains to the sum (exclusive join), and different domains to the bit-vectors themselves :

$$(a, b) \mapsto p \ + \ (c, d) \mapsto q := \begin{cases} (a, b) \mapsto p, (c, d) \mapsto q, \\ \text{all else} \mapsto 0, \text{ if } (a, b) \neq (c, d) \\ (a, b) \mapsto p + q, \\ \text{all else} \mapsto 0, \text{ if } (a, b) = (c, d) \end{cases} \tag{2}$$

There is, once again, the same abuse of notation that we discussed above, with $p + q$.

We define multiplication, similarly, as mapping elementary operators with different domains to 0, while same domains to the product (Boolean meet).

$$(a, b) \mapsto p \cdot\ (c, d) \mapsto q := \begin{cases} 0 \text{ if } (a, b) \neq (c, d) \\ (a, b) \mapsto p \cdot q, \\ \text{all else} \mapsto 0,\ \text{ if } (a, b) = (c, d) \end{cases} \tag{3}$$

It's easy to see that any arbitrary binary Boolean operator $f : \mathbf{n}^2 \mapsto 2^n$ can be decomposed uniquely to a sum of elementary binary operators by first resolving the function to a sum of operators defined on single ordered pairs in $\mathbf{n}^2$, and then resolve each of those elementary operators according to the atom decomposition of the function value.

Once, we do that we can define multiplication and addition by using the distributive law on the decomposition, so that our space automatically satisfies the 2-algebra axioms (the field action being the trivial action that sends 0 to the 0 map, and 1 to the identity map).

Of course, it might be convenient to not have to decompose each time we apply a 2-algebra operation, and perhaps I will compute the operations directly, according to the domain sets, zero values and whatnot.

Thus we have defined $\mathbf{Op}_2^n \cong 2^{n^3}$ as a 2-algebra, and the direct sum across all arities gives us the infinite-dimensional space $\mathbf{Op}^n := \bigoplus_{a=0}^{\infty} \mathbf{Op}_a^n$, where $\mathbf{Op}_a^n$ is defined similarly, and is isomorphic to $2^{n^{a+1}}$.

In practice, though we will have a fixed finite set of operators, and we will restrict ourselves to the finite space generated by them, and when we look at the representation of the symmetric group on the atoms, we will examine the setwise stabilizers of the given set of operators, within this generated subspace.

4 A New Transducer Approach

Rather than using partial algebras to model the musical process, we will try transducers. The input words will correspond to the passive internal genetic music, and more generally to an abstract genetic state, while the output words will correspond to the "transcription", the typed music. The formal states of the transducer, will be defined as subsets of the setwise stabilizer of the operators $\{\mu_\Xi\}$ in $A := (2^m, \mu_\Xi)$, That is, $Q := \mathscr{P}(S^{\{\mu_\Xi\}})$. These will also be the initial and final states of our transducer.

The input alphabet will be arbitrary subsets of permutations of m elements (not just from the setwise stabilizer), and will be denoted $P := \mathscr{P}(S_m)$. The output alphabet will be $T := \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, /, *, -\}$, the musical alphabet. Therefore we define the *musical output transducer* as $\mathcal{T} := (Q, E)$, where $E \subseteq Q \times P^* \times T^* \times Q$.

We will also, tentatively, impose the requirement that a transition's origin be included in each of the input alphabetic states.

Essential to our argument will be the countable atomless Boolean algebra, and we will use as isomorphism type the interval algebra on rational endpoints. I still have to work out what the free generators are explicitly. I'm also learning about ordered permutation groups, so as to more closely characterize the types of actions of A and B on our interval algebra that are "allowed." Since A and B are finite, I'm optimistic.

With that in mind, we will also define two maps that associate characters in T to subsets of operators of A and B. This will be essential in exploring the connections between P and T, which don't have to be explicitly computed, but should obey certain rules.

So, we'll assume the existence of $\alpha : T \to \mathscr{P}(\mu_\Xi)$ and $\beta : T \to \mathscr{P}(\eta_Z)$.

Somewhere along the line we will posit that input and output words in $\mathcal{T}$ be of the same length.

We will also define at some point, a *musical input transducer*, which will be a "reversal" of the above transducer in that it will assume a priori a musical passage $p \in T^*$, some or all of whose factors form the input words, and whose output words are the "Boolean states."

To return to our musical output transducer, we will explore utilizing the fact that its specification will imply a direct pattern between states in P and Q, quite independent of the output alphabet, and to achieve this, we will correleate with it context-free grammar, $\mathscr{G} := (P\backslash Q, Q, R)$, where $R \subseteq (P\backslash Q) \times P^*$. This way, we can formaalize the fact that the purification of spiritual state is "algebraic", and only dependent on the finite states available, (and perhaps on local properties of the Boolean operators).

What we can say, for sure is that when the transducer reaches input words of the form Q^*, then the computations involved in producing the notes should be directly consequent from some automorphisnm group of (a subgroup of) $S^{\{\mu_\Xi\}}$.

Much of this is speculation, and before I continue with the exact nature of the computations involved, I think it will help to first work on the implications of the context-free grammar $\mathscr{G}$, coupled with certain "intuitive" facts concerning our metaphysical motivation.

Also, I am quite absorbed in learning about ordered permutation groups, so that once again, it will be clear which bijections from the atoms of A and B to free generators in our rational interval algebra will be compatible with permutations of those atoms.

That's all for now! Exciting stuff!

5 Parameterizing partial operations

Let $\alpha : \mathbb{N}_0 \to \mathbb{N}_0$ and $\beta : \mathbb{N}_0 \to \mathbb{N}_0$ be the multiplicity signatures of BAs 2^m and 2^n respectively, and let $\Xi := \bigcup_{a \in \mathbb{N}_0, \alpha(a) > 0} \{a\} \times \alpha(a)^{\{\}}$ and $\Xi' := \bigcup_{a \in \mathbb{N}_0, \beta(a) > 0} \{a\} \times \beta(a)^{\{\}}$ parameterize their operators, so that $A := ((2^m), f_\Xi)$ and $B := ((2^n), g_{\Xi'})$, $m > n$. Let $S := \Psi(f_\Xi)$ and $S' := \Psi(g_{\Xi'})$ be matrix subsets of $\mathcal{TM}_n$ corresdponding to the operators of A and B.

Let $\underline{P} := (T^+, (\mu_\sigma)_{\sigma \in S_m}), (\eta_g)_{g \in g_{\Xi'}})$ be a partial algebra on the free semigroup on $T := \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, /, *, -\}$ corresponding to a 13-letter musical alphabet that is convenient for 10-key musical typing. Guitarists can note that these are fret numbers of the A-string, a good place to start, while pianists have to do more work! My preferred interperetation of the symbols is 0-9 = A-F#, / = G, * = G#, - = rest/sustain. When dealing with the free monoid T^*, we will denote the neutral element by ε.

Next, let F be the the interval algebra on rational endpoints (which is isomorphic to the the unique countable atomless BA or alternatively, the free countable algebra). Let $F := \langle E \rangle$ where $E := \{[0, r) | r \in \mathbb{Q}\}$. Any automorphism of F would have to presreve the linear order of every rational endpoint of E, and vice versa, so we see, $\mathbf{Aut}(F) \cong \mathbf{Aut}_<(\mathbb{Q})$.

We will be interested in the *countable free kernels* K and L, defined so that $2^m \cong F/K$ and $2^n \cong F/L$.

Also, we will equip F with the signature $(F, f_\Xi, g_{\Xi'})$, where the analogous operators are chosen so that the free projections mentioned above are indeed homomorphisms. This and the fact that A and B are finite, means that the automorphisms of the universes of A, B, and F will coincide with their automorphisms as BAOs.

Let $\Delta : S_m \to GL(\mathcal{TM}_m)$ and $\Gamma : S_n \to GL(\mathcal{TM}_n)$ be the infinite dimensional representations of S_m and S_n, respectively, that permute columns and block columns and block columns of block columns in the natural, uniform, defining way.

Although this is useful, for our purposes we will have to focus ourselves on restrictions, so let $S_\Delta := \mathbf{Stab}_\Delta S$ and $S_\Gamma := \mathbf{Stab}_\Gamma S'$ be setwise stabilizers and $\delta : S_\Delta \to GL(\langle S \rangle)$ and $\gamma : S_\Gamma \to GL(\langle S' \rangle)$ be natural restrictions of Δ and Γ.

One important thing to note is that the signature of $\underline{P}$ induces an action of a symmetric group on the partial algebra by permuting the partial operations according to the indices of μ_σ and η_f. We will use the free quotient of the intersection of the free kernels $C := (F/(K \cap L), f_\Xi, g_{\Xi'})$, where the operators are chosen so that a member of a coset gets mapped to the same value as the finitary representative. The advantage of this is that the symmetric group S_r, where r is the width of C, paramaterizes both $(\mu_\sigma)_{\sigma \in S_m}$ (through the composition of the given permutation's automorphism of C with the canonical projection of C on A) – and also parameterizes $(\eta_g)_{g \in g_{\Xi'}}$ (through the *action* of the composition of the given permutation's automorphism of C with the canonical projection of C on B).

The next step of course will be to see how the defined representations get extended on S_r, how the setwise stabilizers of $f_\Xi \cup g_{\Xi'} \subset \mathbf{Op}_r$ under those extensions are related to their component counterparts S_Δ and S_Γ.

Also, we will use our partial algebra to compare the paramaters which yield intersecting graphs for partial operations μ_σ and η_g (i.e. overlapping musical word values on those partial operations) and compare the permutations from S_r that determines them, and use these permutations to construct an element (preferably a generator!) in the setwise stabilizer S_Γ.

More to come.

6 Representations

Now, to return to the basic model of a finite Boolean algebra 2^n with normal, completely distributive operators $f_\xi, \xi \in \Xi$, in light of the 2-algebra isomorphism $\psi_1 : \mathbf{Op}_{1,n} \to 2^{n\times n}$ above, we can examine the linear action of S_n on the vector space $2^{n\times n}$, and we see that this gives rise to the *defining Boolean operator representation*, $\delta : S_n \to GL(2^{n\times n})$ (afforded by the $2S_n$-module $2^{n\times n}$). It's clear that δ merely permutes the columns of a binary matrix. And of course we have the *defining Boolean representation*, similarly defined, corresponding to the action of a permutation of atoms on its Boolean sum: $\rho : S_n \to GL(2^n)$ (afforded by the $2S_n$-module 2^n). That is, it's a permutation representation on the binary vector space.

It turns out that the two homomorphisms are inherently connected by the formula $\delta(\sigma)(R)v = R\rho(\sigma)(v)$, for $R \in 2^{n\times n}$, $v \in 2^n$, and $\sigma \in S_n$. (We can write these more clearly as actions, $(\sigma\cdot R)v = R(\sigma\cdot v)$. Indeed this follows from the fact that multiplying a Boolean vector by a matrix whose columns have been permuted by σ, is the same as multiplying the non-permuted matrix by the vector whose elements have been permuted by σ. Note, the above equality is **not** the same as $\sigma\cdot(Rv) = \Sigma_{i\leq Rv}\sigma(i)$, the sum of the atoms obtained by permuting those below the Boolean vector Rv with σ.

7 Stabilizers

Given our defining unary Boolean operator representation, $\delta : S_n \to GL(2^{n\times n})$, a natural question to ask in its own right (which becomes more manageable in light of the operator algebra isomorphism) is which permutations stabilize (permute) sets of operators. In the matrix algebra formulation, this becomes: given a set $S \subset 2^{n\times n}$ of matrices, compute the stabilizer $\mathbf{Stab}_\delta(S) := \{\sigma \in S_n | \sigma S \subseteq S\}$.

So we lose nothing in merely restricting ourselves to the matrix representations in our quest for stabilizers. As one might expect, sets as important as these stabilizers don't come easy, in light of the current non-existence of polynomial time algorithsm to compute setwise stabilizers.

I'm in the process of trying to see if there's an elegant calculation of the automorphism group of $(2^n, S)$, in terms of perhaps the stabilizer $A := \mathbf{Stab}_\delta(S)$. It seems that they should be connected, and so far I reasoned that $\mathbf{Aut}(2^n, S) = \{\sigma \in S_n | \exists \eta \in A, \forall M \in S, x \in 2^n, M(\sigma\cdot x) = \sigma((\eta M)\cdot x)\}$ I'll let you know how the calculation turns out! Before, I'll get to the "music theory," I'll leave you with a

Conjecture 1. *For all $n \in \mathbb{N}$, there exists an $S \subset 2^{n\times n}$, such that the defining unary Boolean operator representation representation $\delta : S_n \to GL(2^{n\times n})$, when restricted to $S' := \boldsymbol{Stab}_{2^{n\times n}}(S)$, the setwise stabilizer of S, is equivalent to the restriction of the defining unary Boolean permutation representation $\rho : S_n \to GL(2^n)$ to S'.*

The reason I believe this might be true has to do with the fact that digital Boolean algebras have their "brains" embedded in their Boolean existence, and therefore they both register and manifest automorphisms in the same digital space.

8 Music Theory

We are not interested in a completist account of music theory which explains all of its mysterious combinatorics or neurological phenomena, or tries to create music algorithmically, because that is foreign to our purposes. We need the bare minimum of music theory required to aid in computation.

8.1 New Partial Approach

It's clear that since we're interested in the connections between a group (the setwise stabilizers in the defining representations) and certain strings of notes, that we should be interested in the free semigroup on 13 generators (12 for each note and one signifying the rests/sustains that provide the rhythm). Octave ambiguities wouldn't matter in the sense that all musical passges would still be musical with notes of the same value but different octave. It might strike one as less catchy than another, but it would undeniably be music, and the mere existence of a certain aesthetic best among equivalents is enough to earmark the equivalence class, even if not define a canonical representative.

So, the idea is that we will employ a partial algebra whose universe is the free semigroup on 13 generators, which we'll denote by F_{13}, and whose function symbols are any labels of the "build-up / resolution" pattern that is central to musical sensation.

The "built-up passages" (which of course can be nested), correspond to word arguments in a partial function, whose resolution is denoted by the function value word. By earmarking the parts through an audio device and a human listener, we can assemble a set of partial functions corresponding to the sensory flavor of the passage. Of course, a passage grows, and peformers usually like to add to songs, as do composers and improvisers, so this just corresponds to new partial functions on the words (subwords of the musical passage).

The advantages of partial algebras is that they can construct "relatively free algebras over partial relative substructures" and we can already see some proof of this in connection with partial subalgebras. These come in three flavors – closed, relative, and weak relative. The closed subalgebra would correspond to some notion of "algorithmic melody construction" which as some algorithmic composers have found, can be used to undeniably create music. The relative subalgebra would correspond to restricting against certain examples of music that always sound bad to you. And the weak relative would correspond to filtering by removing notes and perhpas preserving rhythm.

This is more intuitive, and there are subtle points that have to be worked out, to both solidify the ideas, and so that we dont' lose sight of the forest for the trees.

More to come.

8.2 Old Partial Approach

As an alternative to the above beginnings, let's try using the technique of partial algebras in conjunction with Boolean algebras with operators, to utilize an interaction between the possible members of 13^L, where L is the length of our musical passage. We denote the set of all elements $\{1, \ldots, L\}$ by L_1. To reflect the structure in an improvised (or composed) passage, it's nice to use partial functions to express the possibilities and constraints that go through a musicians head as he improvises. For instance, a musician might know the last two notes he wants to play, and the first two notes, he wants to play, but not the middle two notes. This can be represented informally as, $t_1 \mapsto t_2 \to 13 \to 13 \mapsto t_3 \mapsto t_4$. In general, for any subset $S \subset L_1$, we can imagine an $L-$ bit-vector of the mask of the set, as it shifts throughout all the $L - |b_S|$ possibilities, where b_S is the number of spaces between the left-most 1 and the right-most 1. So, each partial function in the algebra that we will soon formally define represents, a musical choice, within the context of preceding notes, with or without gaps. So, the first 4 notes in the musical passage p, and if we know thisnote is t_5, we notate this by $t_1 \mapsto t_2 \mapsto t_3 \mapsto t_4 \mapsto t_5$. So, when we wish to examine all choices corresponding to this pattern of 4 consecutive notes, then we simply use an operation of arity 4, which is in $13^4 \to 13$. But, we are implicitly assuming that this corresponds to 4 consecutive notes. As we noted above, there can be very real musical choices made with gaps in the middle, so to remedy that we can using the power set algebra 2^L to index partial dictive functions So, the value $\{t_i\}_4$,

With this basic idea, we can examine the partial algebra $\mathcal{D}(\surd) := \{f_S | S \subset L_1\} where L is the length of p$

A brief outline :

The letter $\mathcal{D}$ was chosen to denote diction.

More to come!

8.3 Old Flawed Approach

So, we start by looking at the Cartesian product of 13^M, where M is the (presumably large) number of notes in the musical passage, and 0-11 signify A-F#, while 12 signifies a rest or sustain. The idea is to get the bare minimal notion of music that could drive our computations. We don't distinguish between notes an octave apart, and we use the blank note 12 to define rhythm, which of course is quite complicated for complicated rhythms, but assuming nothing too avant-garde, we can capture most hummable melodies (in an equivalence class whose octave ambiguities are probably still musical).

We define the projection function $i_p : \{M\} \to 13$ which maps the index of the passage p to its note value. For instance, if the 3rd note in passage p is a

C#, then $i_p(3) = 4$.

To create the BA with ops, we will use the standard procedure of starting with a relational structure, and then defining its uniquely determined complex algebra.

We'll let R_p denote a collection of relations of unrestricted arities determined by the passage p (and perhaps determined by music-theoretic principles). The primary relations that are uniquely determined by p are the unary relations $\rho_t \subset 13^M$, where $\rho_t a \leftrightarrow i(a) = t$. One useful class of operators is the k-ary relations $\sigma_{k,t} \subset 13^K$ where $(t_1, \ldots, t_k) \in \sigma_{k,t}$ iff there is a successive sequence of k components in p which contain t. We can also, instead of specifying only one tone to search for, search for a potential sub-passage. In any event, we will restrict our attention to the complex algebra B_p of R_p, which is the power set algebra on $\{M\}$, with an operator $f_\rho : \{M\}^k \to M$, for some relation $\rho \subseteq \{M\}^{k+1}$, sticking to the convention that

$$(t_1, \cdots, t_{k+1}) \in \rho \leftrightarrow f_\rho(t_1, \cdots, t_k) = t_{k+1}$$

.

So, the hope is that since Boolean homomorphisms are tantamount to ideals, which in finite BAs are simple enough, the main work will be generating enough relevant operators in B_p that are compatible with the defining Boolean operator matrix representation of S_N (which it'll be helpful to calculate explicitly soon!!) and a given set of operators S whose setwise stabilizer you want to find in the given BA $B := 2^N$, which imply the existence hopefully of a homomorphism of BAs with ops between B_p and B, whose stabilizers under the actions of the symmetric groups are comparable.

The hope is that it'll be much easier to compute setwise stabilizers in B_p than in B, since we get to choose our operators that are generated by the primary ones, and by whatever aesthetic constraints or persistent musical patterns we can describe. The variety is much more manageable in a music theoretic BA than in a BA with arbitrary operators.

9 More elaboration of philosophical conjecture

FIX THIS, BUT JUST IGNORE THE TENSOR LEFTOVERS. To elaborate more on the topic of setwise stabilizers, here's another conjecture :

Conjecture 1. *The setwise stabilizer* $\{G_\Xi := \boldsymbol{Stab}(f(\Xi))$ *of the Boolean Operator Representation acting on the operators' generated subspace of the tensor matrix algebra is always a symmetric group.*

Pretty bold claim, and one that I can only describe in terms of a hypothetical digital system primarily determined by the variables and opeators of one specific very large finite Boolean algebra, corresponding to the gates. Now, it's not my claim that the Boolean properties of the gates determine most of the functionality, but the types of universal disfluencies that might threaten fault tolerance are all dependent and dare I say manipulable by the symmetries in the gates. So, the reason I believe this superstitious and bold statement is true is the following : The only automorphisms possible in exact digital circuitry abiding by the laws of physics are those which map an operator to another, thereby using existing functionality in new symmetric ways. And those are the flukes and some men's only hope. Enough of those inevitably signal the Physics Police, because then the broken automorphisms ensue (they're **not** in the setwise stabilizer!). My point is that the full automorphism group of a subalgebra (with restricted operators) of the entire Boolean algebra would have magical and neutral behavior. Which is precisely the setwise stabilizer of given operators of our given representation, provided such things exist, and perhaps a crystallographer can make a better and more sane case than that, but now it's time to prove this supposedly obvious statement.

There's also

Conjecture 1. *The quotient of the setwise stabilizer* $G_\Xi := \boldsymbol{Stab}(f(\Xi))$ *by the pointwise stabilizer* $H_\Xi := \mathbf{Stab_1}(f(\Xi))$ *of the Boolean Operator Representation acting on the operators' generated subspace of the direct sums of powers of binary matrix algebras is always a symmetric group. Or put another a way, any representation of the symmetric group into the binary vectory space generated by a relation structure whose universe is of the same cardinality as the symmetric group's degree (and the action on relations being via points) always induces symmetric groups.*

Similar reasoning – factoring out the netural elements doesn't alter behavior. More to come.

10 Loose Outline of Proof of Conjecture

Utilizing the setting and structures already established, we have A, B, and the BORs $\Gamma : S_A :\to \langle S\rangle$, $\Delta : S_B \to \langle S'\rangle$. $S_A := \mathbf{Stab}_A(S)$ $S_B := \mathbf{Stab}_B(S')$. And we are interested in the quotients of the repsective setwise stabilizers and pointwise stabilizers Q_A and Q_B. We will try to use the parameterization of S_A and S_B into (μ_σ) and (η_σ), which is a bit of a change from above. I seem to believe that utilizing the setwise stabilizer to paramaterize, rather than some composition of operator and permutation/automorphism, is better and more natural. So, we use the permutations in the setweise setabilizers to parmaaterize, and we examine the quotients, to see if we can prove that an arbitrary coset corresponds to a transposition in the left regular action. And if we can embed an arbitrary transposition in the stabilizer quotient, then we're fine. So, to being the proof, we will assume that we have two Boolean algebras A, and B, mentioned above, and described above in prewvious sections. And $m < n$, where m and n are the repsective widths, and by assuming that in both A and B there is an element in the symmetric group, whose coset of the pointwise stabilizer corresponds to a transposition in the left regular action not contained in Q_A or Q_B, we can derive the contradiction that perhaps $m = n$, or some such, by presupposing and "exhausting" the finitary paramterization mentioned above, and using free monoids on the musical alphabet to derive a correpsondence between overlapping partial operations that classify growing musical scores, and the permutations that determine these operatorions, in the not-yet described paramaterization. So, it's just a question of exhausting patterns, because everything is finitary. The infinite words, and whatnot, will only be used up to a point, there will be some convergence of words, and blam, a contradiction. Now for learning more free semigroup theory and combinatorics! And trying to translate overlapping words in a free monoid with automorphisms of the direct product $A \times B$, and decompos directly perhaps, but more learning!

11 Automorphisms of First-Order Structure : Know Best Focus

So, we look at an ordered permutation group $G \subseteq \mathrm{Aut}(\mathbb{Q}, <)$, which we associate with the countable atomless free algebra and a chain of free generators $\{I_q\}_{q\in\mathbb{Q}}$ order-isomorphic to $\mathbb{Q}$. By a well-known theorem [??], It turns out that G is the automorphism group of a model characterizesd by a theory if and only if it's oligomorphic. So, the idea is that the model of this theory would have to have certain properties, namely some type of "maximum of 'available' automorphisms," by which I mean with respect to certain setwise stabilizers. Another property it would be nice to have is some type of maximal universality, if not pure freedom. So, whether or no I'm right that G and not just $\mathrm{Aut}(\mathbb{Q}, <)$ is guaranteed to be oligomorphic, we can at least explore that condition. So, first suppose G is oligomorphic, and thereefore that it is the automorphism group of a countable model determined by an $\mathscr{L}$-sturcture $\mathfrak{M}$ that is the unique model of a theory T in a first-order language $\mathscr{L}$. We wish to explore the relationship between $\mathfrak{M}$ and the subojects, in light of Fraisse's theorem. (Which I have to pore through.) We know the number of orbits of $\mathrm{Aut}(\mathbb{Q}, <)$ on a tuple of distinct elements of $\mathbb{Q}$ is $n!$. This I got from Cameron's "Oligomorphic Permutation Groups" One thing that might be critical is as is discussed, characterizing the sequences of orbit frequencies over lengths of tuples, which I will now read more about. Good to be naturally curious over something that's not conventionally behavioral!

12 Tristone Systems

One can create a ternary alphabet $C = \{+, ?, -\}$, and informally view the symbols as, repectively, $\{$*yes*, *freely generate*, *no*$\}$, corresponding to a sort of philosophical Stone's Representation Theorem Trichotomy system of decisions, whereby a binary decision can be reached either by conventional binary means, corresponding to + or -, or it can be achieved by manipulating the exhaustive and fundamental two-valued decisions (2-valued homomorphisms) guaranteed by Stone's Representation Theorem, corresponding to ?. For this last case, there's a not fully determined interface with free algebras, all of which might generalize to infinite BAs too one day, we'll see.

We'll start things off by defining actions. First, there's the induced action of S_n on the BA 2^n, which acts merely as automorphisms, or in other words, permutes the atoms below the given element.

Then there are the following *tristone actions*

1. (a) The left action S_n on S_{2^n}, by $(\sigma\cdot f)x := \sigma(fx)$, where $\sigma \in S_n$, $f \in S_{2^n}$, and $x \in 2^n$, and the action is the induced action above.
   (b) The right action of S_n on S_{2^n}, by $(f\cdot\sigma)x := f(\sigma x)$, again under the induced action.
2. The left action of S_n on $\mathbf{Op_n}$, by permuting the elementary restrictions, as discussed above, i.e. $\sigma(i \mapsto x) := (\sigma i) \mapsto x$, for each atom-value pair.
3. The left action of S_{2^n} on $\mathbf{Op}_n$, where this time the permutation acts on the value. $f(i \mapsto x) = i \mapsto (fx)$, for each atom-value pair.

Having defined these actions, a *tristone* is merely a triple of atom permutations, Boolean permutations, and Boolean operators, whose tristone actions between them are compatible as setwise stabilizers. We refer to Q, F, and O as *stones*.

Definition 1. *An n-tristone is a triple (Q, F, O), with $Q \subseteq S_n$, $F \subseteq S_{2^n}$, and $O \subseteq \mathbf{Op}_n$ such that*

3#1 $Q \subseteq Stab_{S_n}F$*, for each (left, right) action.*

3#2 $Q \subseteq Stab_{S_n}O$.

3#3 $F \subseteq Stab_{S_{2^n}}O$.

Eventually we hopefully will realize systems of tristones that are loosely stable (closed) with respect to changes in Q, F, or O. It's unclear which changes are admissible, and which further constraints they uniquely determine. To be determined. We can have atoms permutations act on Q (or get added to Q), Boolean permutations act on or get added to F, and operators get added to O.

The rest of what follows is tentative.

Definition 2. *An n-tristone system (or n-3# system) is a collection of n-tristones each member (Q, F, O) of which is contained in the set of tristones determined by any of the following changes in a single stone.*

Q *(a)* $RQ := \{\gamma\sigma | \gamma \in R, \sigma \in Q\}$
(b) $QR := \{\sigma\gamma | \gamma \in R, \sigma \in Q\}$.
(c) $R \cup Q$.

F *(a)* $GF := \{\gamma\sigma | \gamma \in G, \sigma \in F7\}$.
(b) $FG := \{\sigma\gamma | \gamma \in G, \sigma \in F\}$.

   (c) $F \cup G$

O *(a)* $P \cup O$

The connection with this and our "ternary" alphabet C is that, for a tristone system $\mathcal{T}$, we label, $(Q', F', O') = (Q, F, O) \oplus (R, G, P)$ if $Q' = Q \cup R$. (Perhaps, G or P can be left out if the result is already determined.) Likewise, $(Q', F', O') = (Q, F, O) \oslash (R, G, P)$ if $F' = F \cup G$, and $(Q', F', O') = (Q, F, O) \ominus (R, G, P)$ if $O' = O \cup P$.

So, $C := \{+, ?, -\}$ corresponds to respectively to ©$:= \{\oplus, \oslash, \ominus\}$, with the necessary paramaters as an extra specification, and given a tristone $\mathcal{T}$, we can determine how a word over C can "instruct" some sequence of commands on $\mathcal{T}$.

The question of generation is one that will be postponed, since it might be as pluralistic as in partial algebras.

The next natural question is how many or which tristones can exist for a given stone. If (Q, F, O) is a tristone, we write $3\#(Q, F, O)$. And to denote the set of all tristones with one or more component given, we write, $3\#(S) := \{(Q, F, O) | S \in \{Q, F, O\}$ and $3\#(Q, F, O)$ and $3\#(S, T) := \{(Q, F, O) | \{S, T\} \subset \{Q, F, O\}$ and $3\#(Q, F, O)$.

So, it would be nice if $3\#(S, T)$ were small, maybe even a singleton, for specific values of S and T, and soon we will go through consequences of non-singleton values of $3\#(S, T)$. We will also do basic inclusion lemmas for this, to express intuitive patterns like that, an increase in the number of operators in O, decreases the size of Q for a fixed F.

And in general other characterizations of $3\#(Q, F, O)$ will be sought.

0One big goal in all of this is to see if we can determine "new" permutations in Q or elements in the other stones, which could have applications to computational group theory in situations where you are given only a specification of a group, and need to determine concrete generators. So, we will find ways to describe $3\#(F, O)$ given $3\#(Q, F, O')$, and $O \subset O'$. So, let's start with some easy lemmas.

Lemma 1. *$3\#(Q', F, O')$ and $O \subsetneq O'$ implies, there exists a Q such that $3\#(Q, F, O)$ and $Q \supseteq Q'$.*

Proof. Choose $Q := \mathrm{Stab}_{S_n} F \cap \mathrm{Stab}_{S_n} O$. By hypothesis, $Q' \subseteq \mathrm{Stab}_{S_n} F \cap \mathrm{Stab}_{S_n} O' \subseteq \mathrm{Stab}_{S_n} F \cap \mathrm{Stab}_{S_n} O = Q$, and by construction, (3#1) and (3#2) are satisfied. For (3#3), we see $F \subseteq \mathrm{Stab}_{S_{2^n}} O' \subseteq \mathrm{Stab}_{S_{2^n}} O$. □

But the real excitement about something like this, would be if the containment were proper, and I'm investigating deciding if/when, proper containment is preserved by full setwise stabilizers. It might be. In which case I'll strengthen the above.

In relation to music, can we associate with any musical passage, or, any formal word, a word in $\{+, ?, -\}*$ that can compute parameters in the given passage? Perhaps too vaguely stated.

Building some type of holistic non-determinism in the very core of a ternary computational alphabet, should be interesting, if only to just bounce back and forth between all the dependent notions!

Of course we will work with automata theory, in conjunction with music, and arbitrary semigroups, and hopefully even formal languages that can be turned into a compiler! Lots of semi-routine fodder to evaluate, like the compatibility constraints in ternary words under given tristone systems.

Also interfaces with 2-modular representation theory in relation to the generated Boolean operator spaces of a tristone.

13 Sieves

Intuitively a useful pattern to observe is when you have a "kind of free" algebra. For instance, maybe an algebra always has a canoncical map onto each member of a class of algebras. In fact, I remember reading something about this in my Universal Algebra book, which I will review, but for now, I will settle upon this definition.

Definition 3. *A sieve is a finite group G, such that there exists (possibly infinite) family $\mathscr{S}_G$ of finite BAOs (not necessarily of the same width) such that every quotient of the setwise stabilizer by pointwise stabilizer of the operators, under permutation of the atoms, is a homomorphic image of G.*

Formally,

$$\begin{aligned} &G \textit{ is a sieve} \leftrightarrow \exists \mathscr{S}_G, \\ &\qquad \forall B := (2^n, \{f_\Xi\}) \in \mathscr{S}_G, Stab_{S_n}\{f_\Xi\}/Stab_{S_n}(f_\Xi) \\ &\qquad \textit{is a homomorphic image of } G \end{aligned}$$

One challenge will be to construct minimal subgroups of $\mathrm{Aut}(Q, <)$ that project onto a sieve. Construct them with the operators, and perhaps using tristones! More to come!

14 New Epiphany

To sum up the previous discussions, it would behoove someone (me!!), to code a musical automaton that works under the process of elimnation, via two directions. But in order to not lose sight of the forest for the trees, let's announce the intention. The intention of the musical hermeneutics engine, which I will slowly start to assemble soon – using GAP, Haskell, Finite Model Theory, some free automaton engine, etc. The goal will be to compute setwise stabilizers. Eventually, hopefully arbitrary ones, but, if this isn't possible, we'll at least settle for stabilizers of given relations under permutation of atoms. It will use a large musical database and algorithms that associate "notewise" ternary comfort levels of "approve / tolerate / dislike" (and perhaps also "listen actively / listen passively / try to not register"), with instructions as to multiply class of possible setwise stabilizers by elements in one of the three union operands in the decomposition below :

$$\begin{aligned} \mathrm{Aut}(\mathfrak{M}) = \\ \mathrm{Stab}\{\mathscr{R}\}\backslash Stab(\mathscr{R}) \cup \\ Stab(\mathscr{R}) \cup \\ \mathrm{Aut}(\mathfrak{M})\backslash \mathrm{Stab}\{\mathscr{R}\} \end{aligned}$$

,

where $\mathscr{R}$ is a set of relations on the to be determined first order structure $\mathfrak{M}$.

Henceforward we will refer to these three operands as the *plus constitutient, slash constitutient and minus constitutiend.*

In the process of determining the stabilizer we will need to establish it as automorphism group of some relational model to be determined. The fact that finite model theory really only needs relations, means that, in theory, we can transfer the relational structure to a Boolean algebra with operators – which latter would be the more convenient setting for the representation theory, whereas the model-theoretic setting is more appropriate for, well, models.

So, the idea is that when you like an instant of a musical passage, then you are in the non-pointwise setwise stabilizer, when you are neutral or in a passive state, then you are in the pointwise stabilizer, and otherwise you're in a broken state, i.e. a state that it's in the setwise stabilizer's complement.

The good news is that there are only a finite number of judgments of a given musical piece of length n, 3^n. Each one of these determines a stabilizer pattern, which can be used to construct the stabilizer itself, and eventually, hopefully, the relation structure.

The starting point in applications of finite model theory to formal semantics is of course (the indices) of a given word itself.

I have more to learn as to whether or not we can use this approach.

Another variation will be to use automorphisms of the Fraisse limit to some exhaustive family of structures during the neutral phases, and perhaps even the focus phases, so that all magical moods factor through the Fraisse limit, i.e. the universal homogeneous structure of given age.

Futhermore, the second way we can more uniquely determine a musical struture is by narrowing the possibilities of our domain set and relation set with informal musical semantics – i.e. saying, if I don't like easy major key, diatonic melodies, then that further weeds out a lot of patterns that the musical software can filter. The idea is to use standard paradigms of music in the consistency sense – if you like this progresssion, then you will like this similar progression.

I've only begun to learn about finite model theory, so more to come!

14.1 Algorithms

Let's cut to the chase, given a permutation group G on Ω, and $\Delta \subseteq \Omega$, we know that it's not too expensive to compute the pointwise stabilizer. But the setwise stabilizer is non-polynomial. In general. But, here's what we can try – harness some of music's mysterious combinatorics, as follows: Examine the canonical relational structure (the orbits of G on Δ as relations) and I seaem to recall that computing orbits is not terribly expensive.

Now, here's the part that I have to work out before I annoucne it as a step in the computation, but we somehow have to generate the stabilizer of G on Δ from the action of G on the orbits of G's action on Ω. The idea is that we want to find a set M of the same cardinality as Δ, only inhabited by relations on Δ, such that the automorphism group of the relation structure of M is isomorphic to $\mathrm{Stab}_G\{\Delta\}$.

We will somehow use a finite model on M for this, and use music to construct it. The overall transformatoin is still in the air, but we will mention the following "computation."

Given a musical passage $p \in T*$ and a critique of it $j \in +, /, -*$, with $|p| = |j|$, I haven't fully figured out how I want to handle p, but I know how I want to handle j. We will assume that the critique is annoyance-free. Then we will perhaps express in first-order language that each automorphism in the three ternary

It seems that the key step in the transitionining between the plus constituent and an adjacent slash constitutent is that the latter is compulsory. I.e. If after a certain amount of build-up, you are required to repose, and any other notes than what the passage indicates would be a broken autmorphism. This is pretty close to reality, and if it's a little off, it's still a viable pattern.

Let's see if we can express this in a first-order language. other than

We will repeatedly make use of the fact that there are many musical passages that sound "universally" good, provided you're in the mood for it – i.e., they don't annoy. These precious passages each determine a set of n permutations whose action on the closure system, corresponds to the $+$, We look at j as a "form." Given the set of closed sets containing the pointwise stabilizesr of Δ, we can view each note judgment in our p

We will have two complementary tendencies throughout this work – first, determining the automorpism group and structure of a given musical word, (usually given a positive critique of it). Second, we will also start with an arbitrary permutation group and subset of points, and compose music or consult a database to determine the setwise stabilizer.

We'll start with the first:

14.2 Narrowing the closed set

Let $w \in T^*$ be a musical passage and $j \in 3^*$ be a critique of it, and $|w| = |j| = n$. We are interested in determining both the isomorphism type of a model, and then its automorphism group. We will use as our domain of the model, $\mathbf{n}$, and the relations we'll add, after each transition from a period of focus to a period of repose. The reason this is philosophically valid, is as mentioned above, the idea that there is a constraint imposed after the completion of a build-up, such that the next notes have to be precisely what they are to maintain the quality of the piece. This is, from experience, the feeling at least, whereby one feels like improvisational music is composed of pushing freely during build-up, and letting go under constraint, during release.

First, we will use synonyms for {-, /, +} that behave better with semigroup notation. $\{n, s, p\} :=$ {-, /, +} respectively.

We will use the notation $\mathscr{S}_G\mathfrak{M}$ to denote the familiy of all subgroups of G that contain the pointwise stabilizer of $\mathfrak{M}$, and call it the *stabilizer closure system of* $\mathfrak{M}$ *with respect to* G.

We'll let s_i denote a word of i s's, and p_i denoate word of i p's. So, to begin, we will assume that $j = p_a s_b j'$.

We will build up our model M letter by letter through j, and transition by transition through $\mathfrak{M}$'s relations. So, initially $\mathfrak{M} = (\varnothing, \varnothing)$. After the first note, it becomes $(\mathbf{1}, \{w_1\})$ where $\{w_1\}$ is the set of all indices whose value matches the note w_1, which is of course just $\{1\}$. $\mathscr{S}_{S_1}(\mathbf{1}, \{w_1\}) = \{1_{S_1}, S_1\}$. Assuming that $a \geq 2$, the second note is where the atomic structure begins to show. Now the minimal structure is only dependent on whether $w_1 = w_2$, and on a simple stabilizer relation, and we will choose an additional relation R_2, such that for all $H \in \mathscr{S}_{S_2}$ $(\mathbf{2}, \{w_1\}, \{w_2\}, R_2)$, and for all $\sigma \in S_2$, $H\sigma$

15 GAP Code

The good news is that while I'm obviously not a capable research mathematician, I can do a million variations in code on key properties that are obviously important to the mission of best focus : This file is more like a journal, part philosophy, part math, but the theme still stands true. The only thing is that the ultimate goal of managing setwise stabilizers with music, will never be attained if it is aimed for directly. Rather to retain terra firma, I will have to resort to key mathematical properties that I can code that will motivate better directions. The music component will probably involve finite model theory, in addition to group theory and BAOs, so I'm not too optimistic of reaching the holy grail. My hope rather is I can create tools in GAP that will help others along this path, and help myself. So, anyway, the key property is the following:

Remark 1. *Given a BAO* $B := (2^n, f_\Xi)$*, and a subset of operators* $\{f_\Delta\}$*,* $\Delta \subseteq \Xi$*, if* $\mathrm{Stab}_{S_n}\{f_\Delta\}$ *properly contains* $\mathrm{Stab}_{S_n}(f_\Delta)$*, determine which additional operators* $\{g_\Omega\}$ *we have to add to* $\{f_\Delta\}$*, so that the new setwise stabilizer is equal to the pointwise stabilizer, and the new pointwise stabilizer is isomorphic to the old pointwise stabilizer.*

I have already begun learning GAP, and hopefully will code simple instances of this, and generalize approparitely.

16 Finite Models

Let n be a positive integer and let Ξ be a finite set of symbols which index a family of relations on $\mathbf{n}$, where the arity of $\xi \in \Xi$ is $\lambda(\xi)$, and the corresponding relation is $\rho(\xi)$. Let $\mathcal{R} := \{\rho(\xi)|\xi \in \Xi\}$ so that we define a relational structure $B := (\mathbf{n}, \mathcal{R})$. Now let $G \leq S_n$ be the subgroup of all permutations which also permute the relations in $\mathcal{R}$, in the manner discussed in the previous sections. That is to say G permutes relations by points. Of course the odds that this will happen non-trivially are against us, but not completely. So, we then arrive at an action of G on $\mathcal{R}$. The induced subgroup of this action G_0 will be utilized as a structure $\mathfrak{G} := (G_0, \gamma)$, where γ is the usual set of relations of inverses, identities, and product that a group possesses. Now, let T be an alphabet (in our case, presumably the 13-character musical alphabet discussed above). For each $w \in T^*$, let w_t be the unary relation on $|w|$ that consists of all indices of value t in w. With this notation, define the word signature over $\sigma = \{I_t|t \in T\}$ (the universe once again $|w|$). My conjecture is :

Conjecture 2. *For every word over the musical alphabet, there exists an interpretation of the word signature in the group signature such that all inducing groups are permutation groups on relations of non-empty-intersection via points, and whose induced relations are the same as the word's. Formally, for every $w \in T^*$, there is a k-dimensional $\mathcal{L}$-interpretation $\Pi := (\pi_{uni}(x_1, \ldots, x_k)$, $(\pi_t(x_1, \ldots, x_k)_{t\in T}))$ of $\sigma(T)$ in γ such that, for each inducing group, each $\mathcal{L}[\gamma]$ formula in Π, $(\pi_t(x_1, \ldots, x_k))$, has a domain of $k-tuples$, whose positions (in some ordering of $(\pi_{uni}(\_, \ldots, \_))$, are the same as I_t's in w. Furthermore, any such inducing group K acts on some relational structure B as described above of relations of non-empty intersection.*

Proof. It seems intuitively clear that if this statement were true, it could only be proven non-constructively. So, we'll proceed by contraposition. Assume that for all k-dimensional interpretations $\Pi := (\pi_{\mathrm{uni}}(\bar{x}), (\pi_i(\bar{x}_1, \ldots, \bar{x}_{\rho_i})_{i\in\Xi}))$ of σ_T in γ, there exists an inducing group G that acts degenerately on every set of relations $\mathcal{R} := \{R_i\}_{i\in I}$ of possibly varying arities, over every fixed initial segment domain $\mathbf{n}$ via points. That is to say that each permutation in $G \leq S_n$ fails to permute the R_is of every relational structure – this is almost always the case! But, not necessarily always. Which leads itself well to the 0-1 laws of finite model theory. More on that later.

So, my first attempt in proving will be to let $\mathfrak{G}$ be the family of all finite groups that induce under some interpretation of σ_T in γ. Now, we let $\mathfrak{H}$ be the family of all finite groups that act on some set of relations via points (the relations not necessarily of the same arity, but over the same set).

Clearly both sets are countable, so we have a bijection $\Psi : \mathfrak{G} \to \mathfrak{H}$ and we will proceed by a back-and-forth argument to derive a fixed point of the

Everything below is a kind of superstitious intuitive wishy washy. We will proceed to construct relations based on the implied combinatorics of the "induced word" w over which G^{Π} is a σ_T-structure. Our assumptions are that music exists and that of course Π exists.

G of course is finite, yet the set of relation structures that resist an action via points is infinite (by hypothesis). To generate a counterexample, we have no recourse but the defining formulas of Π (and of course musical intuition).

The first piece of intuition is that there would be some homomorphisms $\psi : T^* \to T^*$ which respect the induced index relations. Intuitively this means that a good musical passage retains its good quality after being subjected to some semigroup homomorphism.

The idea is to construct a well-thought of relational structure based on the connection between the induced word and its corresponding automorphisms in such a way as to force a relation-stable (sub)group.

The easiest "musical" relation homomorphism that preserves the automorphism quality is transposition by a musical interval.

That begs that question as to which other homomorphisms or perhaps even general maps can achieve this same homomorphic transformation.

At our disposal is the fact that $G^{\Pi} \models \pi_{\text{uni}}(\_)$.

We will generate these relations by describing "music" informally, not by the choice of notes, like the algorithmic music theorists, but by their behavior with respect to alterations. The simplest alteration shifts 0-* (A - G#) modularly, while keeping '-' (the rest/sustain) fixed. This corresponds to transposing by an interval and ignoring rests and sustains.

□

17 Weighted Automata

I've abandoned the previous model-theoretic approach to musical words, and will instead focus on weighted automata. I am preparing to digest a book on Power Algebras over Semirings. The challenge of course will be to somehow parameterize the symmetric group into a semiring. I'm tentatively constructing a semiring appropariate for musical words as follows. Let $n \geq 1$ denote the length of a given word. Let $P_n := \{(i,j) | i + j \leq n, i, j \in \mathbb{N}\}$. Think of this as partitioning an initial segment of positive integers into three parts – the middle part corresponds to the factor of the given word – while the left and right are the prefix and postfix. P_n just represents i-translations of a segment of length j, not exceeding n's initial segment. i is an offset and j is a length. Both run from 0 to n. Next we will examine the set $W_n := P_n^{\infty}$, i.e. the set of all finite or infinite words generated by P_n. Let $1_S := (0,n)^{\omega}$ and $0_S := (n,0)$ be the one and zero of our imminent semiring S. The addition operation will correspond to "pointwise intersection of factor indices," whereby the sum of two generalized words (finite or infinite) $f_n, g_n : \mathbb{N} \to P_n$ will be mapped to (dropping the n-indices) $f + g : k \mapsto (\mathbf{max}((fk)_1, (gk)_1), \mathbf{max}(fk, gk)_2 - ||fk| - |gk||)$, where, for $p, q \in P_n$, $\mathbf{max}(p,q) = (\mathbf{max}(p_1,q_1), {}_{\text{-}2})$, i.e. is determined by only the first component, while the second component corresponds to the first. We also define $|p| := p_1 + p_2$. The 1s and 2s are of course projections.

The multiplicative operation will be appending, with an "appending" of an infinite word from the right being identical to the left operand. So,

$$f \cdot g := \begin{cases} fg \text{ if } f \text{ is finite.} \\ f \text{ if } f \text{ is infinite.} \end{cases} \tag{4}$$

A finite generalized word is a function $f : \mathbb{N} \to P_n$, where there exists some ℓ (a length), such that $fk \neq (0, n), \forall k < \ell$ and $fk = (0, n), \forall k \geq \ell$. That is the support of f_n is a finite initial segment of the natural numbers. And an appending is, of course, defined by

$$(fg)k = \begin{cases} fk \text{ if } k < \ell \\ g(k - \ell) \text{ if } k \geq \ell \end{cases} \tag{5}$$

. Now to prove that $S_n := (P_n, +, \cdot, 0_S, 1_S)$ is a semiring!!

18 Clones

Given the discussion above of elementary relations of a finite relational structure in the usual sense, it extends easily to BAOs which makes it natural to adapt as the projections of a clone. The further decomposition of an arbitrary operator on the atoms of a BA into compositions in fact makes it ideal for clone theory, and seems to completely determine the exotic "birelational" representation of S_n (more later, and I'm not sure of this claim).

It seems that the mysterious complexity of these relation-permuting permutations on a finite set (akin to simple groups), is that the clone behavior explodes computationally at n = 3 (see Algebras and Orders, [Rosenberg and Sabidussi v.389 in NATO ASI series, article "Essentially Minimal Groupoids", by Machida and Rosenberg]) or perhaps more faithfully, becomes unmanageable at that point, whereby little is known. More reason to invest in clone theory, i.e. it's ideal for finance, cryptography, and metaphysics.

Now the connection between clones and birelational represenations of the symmetric seems to hinge on the simplicity of both projections and elementary maps, and the conceptual notion of a composition as a sort of set-theoretic representation of a subset of given operators.

An elementary Boolean operator is (presumably) merely an atom in a field of sets (atomic of course), and it seems these "operator atoms" would determine, not only an operator, under the Boolean order, but with equal precision, our representation.

Now that I have made a tentative claim, I have to formalize a bit to work out at least three key issues –

- The relation between Boolean operators to elementary operators that paramaterize some aspect of our given ones.
- The relation of each projection of the Boolean operators with both of the above.

- The precise relation between a composition and a representation, perhaps using some type of Stone duality, in ways, I will elaborate on later . . . .

Let $B := (2^n, \{f_i\}_z)$ be a finite BAO with a finite number of operators. Let $\{e_{i,j}\}_{i,j}$ be their decompositions into elementary operators, i.e. for a fixed i, let $f_{i,j} := \Sigma_{j=1}^n e_{i,j}$, (plus will be defined perhaps later) – where once again $e_{i,j} := i \mapsto j, i \neq k \mapsto 0$. And as is standard in model theory, let $\bar{i}$ be an arbitrary tuple of unspecificied arity. When we write $f\bar{x}$, we can then assume a well-defined

It seems that decomposition is the key to connecting our elementary relations and projections. First we'll examine operations on atoms, then operations on arbitrary Boolean elements. Even though an operator is determined by its values on atoms, the values are expressed as sums of atoms, so there's that additional decomposition.

Rather than arguing over chicken and egg – let's stick to the fundamental need to examine what determines a given set of Boolean operators' capability of admitting representations of its birelational space by atomic permutations.

It's clear that (at least a specific class of) the birelational representation behavior will be determined by which elementary operators are below the set of all given operators, and that compositions will determine another layer. Now, let's see what this looks like formally, i.e. we will try to express a given operator not just in terms of a sum, but in terms of compositions – this will determine how operators of different arities will interact. First let's see if any arbitrary set of operators (of varying arities) can be reduced to compositions of elementary operators. Let $\Omega := \{f_i\}_{i<z}$ and $f_i : n^{a(i)} \to n, f_j : n^{a(j)} \to n$ be arbitrary Boolean operators, $a(i) < a(j)$. Let O^n signify the set of all operations of all arities on n, the universe.

We will attempt to examine the set of all decompositions, probably parameterized by (a subset of) the power set algebra 2^z, whereby we will associate z with the set of all given operators Ω, and a subset of z with a subset of Ω. So, let $E \subseteq z$ be the following set of integers between 1 and the number z: Let E be defined inductively as follows :

$$\begin{cases} e_{\bar{i},j} \leq_O \Sigma_{k=1}^z f_k \text{ for some } \bar{i}, j \to e_{\bar{i},j} \in E \\ f\bar{g} \in E \\ \text{ for some } f, g_i \in E \text{ and } \bar{g} := (g_i, \cdots, g_n) \end{cases} \tag{6}$$

So, $f\bar{g}\bar{x} := f(g_1\bar{x}, \ldots, g_c\bar{x})$, and g_iś arity is handled by choosing the new operator's parameter as the largest arity from the g_is, and the conventions from model theory about not substituting variables beyond the arity, etc.

Once again, $\leq_B$ is the Boolean order, and we define $f \leq_O g$ inductively as follows :

$$\begin{cases} e_{\bar{i},j} \leq_O e_{\bar{i},q} \text{ if } j \leq_B q \\ e_{\bar{i},j} \leq_O f \text{ if } e_{\bar{i},j} \leq_O e_{\bar{i},k} \text{ for some } k \leq_B f\bar{i} \end{cases} \tag{7}$$

To return to the forest from the trees, let's formalize a bit our previously mentioned task to create a composition chain between to Boolean operators. First we'll start with atomic operators.

seeming semi-routine task of deriving projections in terms of elementary functions, we'll do first projections of atomic operations (then Boolean operations). Let $f : n^s \to n$ be an

18.1 Having read Lau

Having found the essential reference, Dietlinde Lau's "Function Algebras" on Springer, I will postpone further development of the previous main section's ideas, for deference to standard notation, and of course existing theory. The key that I want to empahsize is that the algebra of atomic operations is no different than any other function algebra, and these determine by a few additional operations the Booelan operators. Having made that claim, I will now formalize it:

The key (not for formalizing, but applying) will be isolating the exact point at which the birelational represenation of the symmetric group interfaces with superposition operations. Let's try to formalize both. From [Lau], we know that the set P_n of atomic operations on $\mathbf{n}$ (the atoms of the BA 2^n) – can be superposed, and this process constitutes P_n's operators. Suppose we have a subset of Boolean operators F that are above the "elementary" operators E_F. And furthermore that the birelational vector space $V_F := \langle E_F \rangle$ admits a representation of S_n. The idea is to assocate with this representation, combinations of Mal'Tsev operations, which hopefully would be seen to have nice properties. So now let's . . . first see the image of a transposition under this representation. The first instinct is to associate it with the Mal'Tsev ζ, so let's elaborate : Let $\tau := (i\ j)$ be a transposition in S_n. To not confuse nomenclature with the standard function algebra nomenclature, let's refer to atomic operations as $\alpha_p^{\bar{a}}$ $(\bar{a} \mapsto p)$.

So we know there exists a $\beta_q^{\bar{b}}$ such that the given representation maps appropriately : $\tau \cdot \alpha_p^{\bar{a}} = \beta_q^{\bar{b}}$. It's clear first of all that $\bar{a}$ and $\bar{b}$ have the same arity. We know this of course since representations always induce automorphisms. Now let's elucidate the case as to whether or not the action of these two elements is within the same operator, and which, or different. [FORMALIZE!]

[Another thing to formalize – examine the connection between a birelational representation and the representation acting on all atomic operations below the given operators.]

regarding each atomic operation as "elementary" (non-standard)) – anyway we will try to establish some correlation between a Mal-Tsev operation and a representation of S_n that acts on some subset of P_n. We will choose the subset carefully, as most subsets will not yield representations, but, let's try to first suppose a representation of S_n into some binary vector space $V := \langle E_F \rangle$, where the generators E_F are merely the atomic operations which constitute each element in $F \subseteq P_n$, where according to standard nomenclature, P_n is the function algebra on $\mathbf{n}$.

each "elementary constituent" (atomic operation) contained in the "constituent union" of all atomic operations. And let E be the set of generators

The 5 Mal'Tsev operations, as mentioned in [Lau],

So, for now, my superstitious belief is that the subspace generated by the given Boolean operators has the same dimension as the number of atomic operations below them, and let's see if this bears out: Let E_F by the set of atomic operations below $F \subseteq P_n$. My claim is that $V := \langle E_F \rangle \cong W := \langle F \rangle$. And once again we will use the relational notation when speaking of representations, since finite BAs are determined by their atoms. We will use the same notation for atomic operations as for elementary relations. The next question as to whether or not the two representations are equivalent, will be postponed, until we can identify a 2-isomorphism. So, we'll tentatively define $\psi : V \to W$ . . . Better. We'll start with a transposition $\tau \in S_n$, and $f \in F$. Let α_f be the set of atomic operations below f. So, in birelational space language (2-vector spaces), $f := \Sigma\alpha_f$. The first question is whether or not τ permutes α_f. In general, there's no reason why it should, so perhaps we can find a maximal subset of α_f that gets permuted, in which case this maximal subset $\beta_f \subset \alpha_f$, would not be fixed by τ, and would only be valid as a component of the representation if there were another operator g above β_f. Additionally, the image of τ on α_f (am not sure of this!!!)

The idea is that the determination of these other analogues, might be possible if the dimension of the subspace generated by our given Boolean operators is the same as the number of atomic operations below that same given set.

So perhaps this indicates a way to construct birelational relationships, i.e. by choosing relations that certain permutations fix or break, and the ones that break require its image to be another relation.

But for now, let's start with a single relation $R = \Sigma_{\bar{a},p}\alpha_p^{\bar{a}}$. Let S^R be the setwise stabilizer of the action of S_n on R, and let T^R be its complement. For each permutation in T^R the goal will be to add a relation, so that the vector space generated by R and the new relation U contains the image of R under the action of an element in T^R.

On a different note, let's formalize the remark I'm trying to prove (and remember I switch transparently between relational algebraic language and BAO language):

Remark 1. *For a given set $F \subseteq P_n$, let $V := \langle E_F \rangle$ be a binary vector space, where E_F is the set of elementary relations below each relation in F, and let $W := \langle F \rangle$ (generated by the formal sums of tuples over the field 2). Suppose further that W admits a representation under the action on the universe* $\mathbf{n}$. *I must show that* $\dim_2 W = |E_F| = \dim_2 V$.

Proof. First, I have no reason to believe this, other than that, it would make my life easier. So, let's try a naive approach, and try to associate a basis vector of V with a basis of vector of W in some compatible way (even though full compatibility of representation might be postponed). One reason why this remark is naive is that I see no dependency between the representation and the vector space.

□

19 New Directions with Inner Automorphisms of Function Algebras

The key test to see if anything like my conjecture (that birelational representations of the symmetric group only induce symemtric groups) is true is to work first with the most natural way representations acting on a birelational space can be non-trivial – as mentioned before – with superposition operations. The good news is that Mal'Tsev's theorem from his celebrated 1966 paper [Lau], giving conditions for all automorphisms of a (finite subset of) a (sublcass of) a given function algebra to be inner to be that the subclass contains certain unary functions, the "characteristic step functions" and the setwise stabilizer of a distinguished subset of the initial segment $E \subseteq E_k$. See Theorem 9.12.4 in [Lau]. Anyway, at the expense of misusing the expression "Mal'Tsev Condition," I'll ask whether inner automorpghism groups of function algebras satisfying Mal'Tsev conditions are necessarily symmetric. It will be helpful to utilize some of the notions discussed previously in this "paper." Namely, the induced group of the representation. The kernel of the action (the pointwise stabilizer) would seem to be the symmetric group on the complement of E. The setwise stabilizer would be the Young Subgroup composed of the symmetric groups on E and $E \backslash E_k$. So the quotient (the induced group) would be S_E.

Now to convince myself that any of this is really really true, I will have to formalize more. Namely, i will have to see if the extra "compatibility conditions" of a subclass alter the characterizations of automorphism groups of function algebras satisfying these Mal'Tsev conditions by a mere symmetric group on its distinguished subset of the univere's segment. More specifically, there will be issues regarding finitude, i.e. automorphisms on subclasses, versus invariant bijections on finite subsets of automorphism groups, which doesn't seem to pose too much of a conceptual hurdle, but which is not mentioned in the text. Also, I will further require the automorphism groups in question to not have extraneous automorphisms that are not the result of applying superposition operations to finite subsets of S and T – by which the finite subset . . . presumably somehow characteristic step functions (parameterized by elements in E and E_k), and the setwise stabilizer of E.

19.1 Special Case of Conjecture

So, now we'll see if something like the previous conjecture applies to special cases, as discussed. Many of the constructs here are derived from Mal'Tsev's work, as treated of in [Lau].

Conjecture 3. *Let* $\varnothing \neq E \subseteq E_k$ *and* $S := P_k^1[k] \cap Pol_k E$ *and* $T := \{t_{a,b} | a \in$

$E \wedge b \in E_k\}$ where

$$t_{a,b}(x) := \begin{cases} a \text{ if } x \in E \\ b \text{ otherwise} \end{cases} \tag{8}$$

. Let $A \subseteq P_k$ be a finite set of functions, each generated by subsets of $S \cup T$. The inner automorphism group of A is isomorphic to the symmetric group S_E.

Proof. Let $E := \{e_1 \dots e_r\}$, $r < n$ and $A := \{\alpha_1 \dots \alpha_a\}$ and G be the inner automorphism group of A. S of course is the setwise stabilizer of E in S_k, so occurs as a Young subgroup $S \cong S_E \times S_{E \setminus E_k}$. In the sequel we will freely pass between isomorphism and refer to an element in S as a product of the two corresponding elements. We must show that the symmetric group S_E induces the same automorphisms on A as G. Let $\mathscr{D} := \{D \subseteq S \cup T \mid \forall \alpha \in A$, D generates $\alpha\}$. Let $D \in \mathscr{D}$ and $T_1 := T \cap D$ and $S_1 := S \cap D$. Let $e \in E$ and $\alpha \in A$ and $\sigma \in S_E$, $t \in T$.

Clearly S_E stabilizes S and T under the action via inner automorphisms, but S_E doesn't necessarily stabilize S_1 or T_1, nor does it necessarily act on them. This should come as no surprise since they're not unique, by virtue of D's non-uniqueness. There are many ways to assemble functions. But the hope is that the totality of possibilities will be neatly categorized.

Let $\gamma \in S_1$, $\gamma = \epsilon\upsilon$ for some $\epsilon \in E$ and $\upsilon \in E \setminus E_k$. So $\sigma \cdot (\epsilon\upsilon)$ maps $i \in E_k$ to $\sigma(\epsilon\upsilon(\sigma^{-1} i))$.

Another structure we will study is the symmetric group on each subset of E. $\cdots$

The idea is that reality doesn't know which D to pick, nor which automorphism types are "canonical," so we will have to deal with a lot of families.

So the challenge now will be to choose an action of S_E on some function-algebraic structure. The first guess is to just take the full image $S_E\mathscr{D} := \{S_E D \mid D \in \mathscr{D}\}$, where $S_E D := \{\sigma\delta \mid \sigma \in S_E, \delta \in D\}$. Now let $D_0 := \bigcap S_E \mathscr{D}$.

First, let's verify that all these things are well-defined. Since each function in A is generated by Let G be the inner automorphism group of A, and let g in G.

————————————— First let's formally define an inner automorphism for a finite subset of a subclass of a function algebra, to make sure it's the same.

$\square$

19.2 Finitary Aspects

The fact that an automorphism can be defined on a finite subset of a function algebra, without the given set being closed, opens itself to all manner of possibilities. I think the following definition has the advantage of perhaps the more conventional one using partial automorphisms. Perhaps I can even prove the equivalence of the two.

The definition of *finite automorphism* we will consdier as a bijection on a finite that respects relations based on Mal'Tsev operations between the elements. [Note to self: Prove equivalence] All the more so if there are relevant generators (not of the set but of each element).)

Definition 4. *We define the binary relation* pointwise strictly generates $\rightleftharpoons$ *inductively as follows*

$$\begin{cases} \varnothing \rightleftharpoons \varnothing \\ \forall E, F \subseteq P_k, \\ \quad E \rightleftharpoons F \Rightarrow \\ \quad \forall e \in P_k, \circ \in \{\zeta, \tau, \Delta, \nabla\} \\ \quad E \cup \{e\} \rightleftharpoons F \cup \{\circ e\} \\ \forall E, F \subseteq P_k, \\ E \rightleftharpoons F \Rightarrow \\ \quad \forall e, f \in P_k \\ \quad E \cup \{e, f\} \rightleftharpoons F \cup \{e \star f\} \end{cases} \tag{9}$$

This is the strongest variant. A weaker variant that allows redundant generators is *pointwise generates* ($\rightharpoonup$).

Definition 5. *For each $E \subseteq P_k$,*

$$\begin{cases} E \rightharpoonup \varnothing \\ \forall F \subset P_k, \\ \quad E \rightharpoonup F \Rightarrow \\ \quad \forall e \in E, \circ \in \{\zeta, \tau, \Delta, \nabla\} \\ \quad E \rightharpoonup F \cup \{\circ e\} \\ \forall F \subset P_k, \\ \quad E \rightharpoonup F \Rightarrow \\ \quad \forall e, e' \in E \\ \quad E \rightharpoonup F \cup \{e \star e'\} \end{cases} \tag{10}$$

Of course an easier way to express pointwise generators is

$$E \rightharpoonup F \Leftrightarrow E' \rightleftharpoons F \text{ for some } E' \subseteq E \tag{11}$$

. Another variant that allows the ommission of intermediate generated terms is *subpointwise generates* ($\rightharpoondown$).

Definition 6. *$E \rightharpoondown F \Leftrightarrow E \rightleftharpoons F'$ for some $F' \supseteq F$.*

The goal of these definitions is to determine the "automorphisms" of F in terms of those of E. It could be that one is easier to compute than the other. In order to talk about automorphisms of finite and finitely generated sets of functions, we will need to introduce partial algebras. Indeed, an "automorphism" of a subset of a universal algebra is a bijection that respects the operations. Therefore the image (under the bijection) of any application of elements (under the operations) is the application of the images, and since the given domain is closed under the bijection, saying you have a bijection on a set that respects the operations of an algebra implies that the set is closed. We're not interested in closed sets of a function algebra here, since they are not necessarily finite, and can be unmanageable (as can their "automorphism" groups). Therefore, we resort to partial function algebras, i.e. algebras of Mal'Tsev operations defined only on certain points. Before reviewing the various types of homomorphisms that can be defined on partial algebras, we will define the three finitary generation relations above in terms of partial algebras.

For $E, F \subseteq P_k$, $E \rightleftharpoons F$, $\circ \in \{\zeta, \tau, \Delta, \nabla\}$, we will define the domain $D_E^\circ := \{e \in E | \circ e \in E\}$, and correspondingly $D_E^\star := \{e \in E | \exists e' \in E, \star ee' \in E\}$. These domains turn $(E, \zeta, \tau, \Delta, \nabla, \star)$ into a partial function algebra. Likewise we turn F into a partial algebra, and to examine the compatibility of Mal'Tsev operations (and their domains), we will resort to the standard definitions and properties of partial algebras.

????? Perhaps using a binary field would be relevant to Boolean-specific operators, but first: preliminaries, like the precise definition. Think of this as a precursor to code, from hereon. All my math definitions and this file in general is a first-draft of code. The only reason to figure this out is so I can then code it!

So, let $F \subseteq P_k$, be a finite subset ??????

Being ever skeptical, I will have to code to see whether the finitary definition of automorphism – a bijection that resepcts operations – is relevant. A reference to an automorphism group on a finite set is made use of in [Lau], but a full discussion was ommitted, no doubt due to space. Coding it shouldn't be too difficult, after I finish some more work on permutations.

Let's err on the side of heuristics, and try to reduce an inner automorphism of a function algebra to a permutation of generators.

There's still some uncertainty, which hopefully will be cleaned up with code, as to whether or not a bijection on a finite set of functions that respects the Mal'Tsev operations can be considered an automorphism even if the given domain is not a closed set.

Let α be an inner automorphism on a class. ——————— TO BE CONTINUED . . .

19.3 Partial Function Algebras

Let $[n] := 1 \ldots n$. Let $\Omega := \tau, \zeta, \delta, \nabla, \star$ be the 5 symbols for Mal'Tsev Operations and $S := (S_\tau, S_\zeta, S_\delta, S_\nabla, S_\star)$ be the 5 sorts correspoding to each operation. Let $\Sigma := (S, \alpha : \Omega \to S^\star \times S)$ For the computations we will assume a Σ-algebra $(A := (A_s)_{s \in S}, (\sigma^A)_{\sigma \in \Omega})$, where $A_s \subseteq P_k$. We will oftentimes postulate that A is a relatively free partial $\mathscr{A}$-algebra generated by some set of functions (of unrestricted arity) $v : [r] \to S$, where $\mathscr{A}$ is a set of elementary implications. All this terminology comes from [Reichel][footnote].

We are choosing to split the sorts (subsets of P_n) into 5, rather than one single sort, for bookkeeping reasons, to be able to specify a universe for each operation for three reasons :

- Restrict the iterative free partial generation from turning into a bona fide total function algebra.
- Classifications.
- To apply partial algebraic machinery.

.

The above considerations will facilitate the current brass ring – computing automorphism groups of relatively free partial function algebras. Associated with that goal, will of course be the need to prove properties. The former will hopefully be handled by GAP, the latter by Coq.

19.4 Computations

At this point, to get more concrete, I'm reading through Jeffrey S. Leon's, "Computing Automorphism Groups of Combinatorial Objects," [footnote]. If we can parameterize appropriately to define precisely what I mean First, we'll establish some notation.

20 Function Algebras and Representations

Up until now I have only intuited that there's a connection between inner automorphism groups of function algebras and 2-modular representations of the symmetric group, but I'm still a bit hesitant regarding some details, so we'll get formal. Groups act from the right.

Let $F = \langle S \rangle \subseteq P_k$ be a finitely generated function algebra, and let G be its inner automorphism group. Eventually we will try to associate G with submodules of the induced subgroup of a representation of S_k (with the standard 0-counting caveat) over the binary vector space $V := \langle A_k \rangle$, where $A_k := \{s{\cdot}\sigma | s \in S, \sigma \in S_k\}$, where the action $f(x_1, \cdots, x_n){\cdot}\sigma := f(x_1\sigma^{-1}, \cdots, x_n\sigma^{-1})\sigma$ and the product is the defining action.

One mystery will be the cardinality of A_k, since we would need a solution of the word problem to compute it. (FORMALIZE THIS!) Our semantic correspondence is not immediate for two reasons: the binary vector space would allow you to delete certain generators, yielding different function algebras (and a "portmanteau represenatation") but it's useful to examine the possibility of each permutation acting on a generator (n-fold function) via the universe of the function. We have to permute this basis obviously, which implies a symmetric group, but we still have to examine the induced subgroup (the quotient of the setwise and pointwise stabilizer of the action on a suitable basis for a $2S_k$-submodule), which quotient we have no reason to believe is symmetric.

We will clarify that the basis A_k contains elements that correspond to not arbitrary functions themselves in a function algebra, but elements induced by generators and permutations (and not Mal'Tsev operator applications) which determine the inner-automorphic properties of the whole algebra. (FORMALIZE THIS!) What a formal sum of basis vectors (an element in the 2-vector space V) corresponds to semantically is a set of generators, which are acted on by each permutation on the universe of the function algebra, in the standard conjugate way. (Forget my flawed oversight earlier in the file regarding just using the "natural" action!)

Probably the easiest next step is to formalize some type of direct sum of $2S_k$-modules that "fit" into V, and check for compatibility.

A lot of questions remain, and we will just try to nit-pick ad infinitum until we get something. Or at least I will!

. . . . . .