Abstract

This paper will use the heuristics of a finite Boolean algebra with operators to examine (using representation theory and graded Grothendieck groups) a restricted family of setwise stabilizers of 2-modular representations with which polynomial time algorithms might be possible.

1 Introduction

Throughout this article, when it's clear that a field is being expected, I will denote GF(k) by k. Also, I will use $\{n\}$ in matrix and Kronecker-delta indices to denote a list of elements, so that $a_{i\{n\}} = (a_{i1}, a_{i2}, \ldots, a_{in})$. (Singletons will be clear by context). An n-bit-vector v of atoms in a BA 2^n will correspond to the finite set of atoms it represents, which I'll call $S : 2^n - > \mathscr{P}(I_n)$, where $I_n = (1, \ldots, n)$ is the ordered list of the n atoms, which we will also treat as a set.

Let n be some natural number, and consider a finite Boolean algebra 2^n with normal, completely distributive operators $f_\xi, \xi \in \Xi$. It's easy to see that the automorphism group of the universe of 2^n is isomorphic to S_n, and then that raises the question of which permutations on the atoms induce which changes on the operators. The case of seeing which automorphisms respect the operators can be determined by looking at the automorphism group of the full structure (not just the universe) and won't be treated of here, as it is probably treated of elsewhere. However, if we examine which permutations of the atoms map one operator to another (assuming a "large" set of operators), things get more interesting, as now we are in a position to examine this using group representations over a finite binary field, or what turns out to be the same thing, yet more manageable, a group algebra representation into the binary matrix algebra (and also, not necessarily equivalently as yet, but a representation into the endomorphism ring of a boolean module, i.e. a boolean group with a matrix algebra action to be determined below. Work that out.)

In the case of a single unary operator, our algebra is quite simple, and for a set of k-ary operators, it seems easy to believe that we have nice semisimplicity of some sort. So, for a unary operator $f : 2^n \to 2^n$, f's values are determined by its values on the atoms, by complete distributivity and atomicity, so we are interested in the function space $(2^n)^{I_n}$,

So, we can easily bijectively associate the binary matrix algebra $2^{n \times n}$ with the set of unary operators on 2^n: Indeed, if an arbitrary matrix entry b_{ij} in a matrix in the algebra is 1, we can interpret that as expressing the fact that atom i is in the atomic decomposition (join of all atoms below it) of the Boolean element that atom j maps to under the matrix's corresponding given operator. A 0, on the other hand, would signify that the atom i is not in atom j's image's decomposition. So, $a_{ij} := \chi_{\{k \in I_n : k <= f(j)\}}(i)$, where χ is the characteristic function. Note that the ith row of a binary matrix $A = (a_{ij})$ is the set of all atoms whose image under $f : 2^n \to 2^n$ dominate i. More formally, $a_{i\{n\}} = (\chi_{\{k \in I_n : i <= f(k)\}}(I_n)$ and $a_{\{n\}j} = (\chi_{\{k \in I_n : k <= f(j)\}}(I_n)$.

And for the other direction, we can define a specific operator corresponding to a binary matrix to be mere matrix multiplication by a Boolean element's decomposition vector, which we see corresponds to a completely distributive, normal Boolean operator by construction, needing only to define the operator on the "standard basis" $e_i := \delta_{i\{n\}}$, where i is an atom. and mapping 0 to 0.

It remains to determine a meaningful structural correspondence of the matrix algebra with $\mathbf{Op}_{1,n}$ (the set of all normal completely distributive unary operators on 2^n), to accompany the bijective correspondence $\psi : \mathbf{Op}_{1,n} \to 2^{n\times n}$ discussed above. But it is clear that ψ is a 2-module homomorphism (and might be a 2-algebra homomorphism) when we define $\mathbf{Op}_{1,n}$ to be the 2-module whose action takes 0 to the zero-map and 1 to the identity-map, and whose abelian addition corresponds to pointwise exclusive-join. Let $f \sim A = (a_{ij})$ and $g \sim B = (b_{ij})$ This of course determines some $h \sim A \times B = C(c_{ij})$, which to compute, we must remember h maps atom j to the Boolean sum of the set represented by the j-th column of C, and that by the definition of matrix multiplication, the j-th column of C is the list $(a_{1\{n\}} \cdot b_{\{n\}j},\ a_{2\{n\}} \cdot b_{\{n\}j},\ \cdots, a_{n\{n\}} \cdot b_{\{n\}j})$. The i-th element in this list represents N mod 2, where N is the number of atoms below the image of j under g whose image under f is also above i. More formally $a_{i\{n\}} \cdot b_{\{n\}j} = |\{k \in I_n : i <= f(k) \text{ and } k <= g(j)\}| \bmod 2$, so that $h(j) = \bigvee_{i=1}^{n} S(k \in I_n : i <= f(k) \text{ and } k <= g(j))$. Now to figure out how to simplify!!

More later!

TODO!!!!

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This naturally leads to B, the $2^{n\times n}$-module whose (abelian) group operation is the join of 2^n, and whose action from $2^{n\times n}$ is the matrix multiplication by the column-vector decomposition of an element in the Boolean group. So, for each representation $\phi : 2S_n \to \mathbf{End}_2 B$, we can associate each corresponding matrix representation $\Phi : 2S_n \to 2^{n\times n}$ with $\Phi' : 2S_n \to 2\mathbf{Op}_{1,n}$, its composition with ψ. And we can also compare and contrast it with a natural consideration, $\rho : 2S_n \to 2\mathbf{Op}_{1,n}$, which maps σ to the unary operator resulting from applying σ to the atoms below it. That is to say $\rho(\sigma) = x \mapsto \bigvee_{i<=x} \sigma i$.

2 Stabilizers

Now, to return to the basic model of a finite Boolean algebra 2^n with normal, completely distributive operators $f_\xi, \xi \in \Xi$, in light of the bijective (and hopefully 2-algebra isomorphism) $\psi : \mathbf{Op}_{1,n} \to 2^{n\times n}$ above. Once more, the setwise stabilizer of a given set of operators lets you know which permutations permute them, i.e. under which permutations is your given set $f\Xi'$ closed, for $\Xi' \subset \Xi$. We see that the stabilizer of a given subset of operators, under the "operator space representation" $\Phi' : 2S_n \to 2\mathbf{Op}_{1,n}$ corresponds directly to the standard module-theoretic representation $\Phi : 2S_n \to 2^{n\times n}$, and its setwise setwise stabilizer is $\mathbf{Stab}(\Xi') := S_{n\psi(f(\Xi'))}$. So we lose nothing in merely restricting ourselves to the representations and finding, instead, stabilizers of sets of binary matrices. But, now we see the power of setwise stabilizers, which we already knew from the fact that polynomial time algorithms don't exist for computing its generators in general. But, I will suggest a small iota of an approach based on iterating through a lattice of subgroups in between the pointwise stabilizer $S_{n\Delta}$, $\Delta \subset 2^{n\times n}$ and the setwise stabilizer $S_{n\Delta}$. The heuristics of this approach invovles then being able to project key groups (like in the music-theoretic portions, the Grothendieck group of a subset of a free monoid) on a key member of the lattice, which is plausible, if the stabilizers are determined by the "way they are assembled." So for a Boolean partition π_n of 2^n, we look at stabilizers $S_{n,\Delta}$ where Δ is a union of elements in π_n. We want to decompose the group algebra $2\mathbf{Stab}(\Delta)$ into a direct sum, so we can use grading theory. The partition π_n might be able to be determined so that $2\mathbf{Stab}(\Delta) = \bigoplus_i P \in \$pi_n$.

More to come! Happy Hacking!