1.3.E. I must verify that the union of a co-dense and a nowhere dense set is a co-dense set.

Proof. Let C be co-dense and N nowhere dense. Let U be a non-empty open set. We must show that $U \setminus (C \cup N) = U \cap (X \setminus (C \cup N))$ is non-empty. But this equals $(U \setminus C) \cap (U \setminus N) \supset (U \setminus C) \cap (U \setminus \overline{N})$. If $U \setminus \overline{N} = \varnothing$, then $X = \overline{U \setminus \overline{N}} = \varnothing$, a contradiction. So $U \setminus \overline{N}$ is a non-empty open set, which therefore must intersect non-trivially with $U \setminus C$, completing the proof.

1.3.F. I must show that any open subset of a dense in itself space is dense in itself.

Proof. Let U be an open subset of the dense in itself topological space X. Let $u \in U$. By assumption, $u \in \overline{X \setminus \{u\}} = \overline{(U \setminus \{u\}) \cup (X \setminus U)} = \overline{U \setminus \{u\}} \cup \overline{X \setminus U}$. If $u \notin \overline{U \setminus \{u\}}$, then $u \in \overline{X \setminus U}$. But since, $u \notin X \setminus U$, then $X \setminus U$ is not closed, which means U is not open, a contradiction. Therefore u is an accumulation point, and U is dense in itself, completing the proof.

1.3.G. I must show that the family of all Borel sets in the real line can be reprsented as the union $\Phi = \bigcup_{\alpha < \omega_1} \mathcal{F}_\alpha$, where $\mathcal{F}_0$ is the family of all closed sets, and $\mathcal{F}_\alpha$ consists of all countable unions of sets from $\Phi_\alpha = \bigcup_{\xi < \alpha} \mathcal{F}_\xi$ for an odd ordinal number α, and of all countable intersections of sets from Φ_α for an even ordinal number α. Note that all families $\mathcal{F}_\alpha$ are of cardinality $\mathbf{c}$ and deduce that the real line has subsets which are non-Borel sets.

Proof. I will be using axioms BS1', BS2, BS3' from the text for all Borel sets. The proof will proceed in two steps. The first step is to show that Φ satisfies the axioms for Borel sets; the second step is to show that Φ is the minimal family satisfying those axioms.

Step 1: BS1' and BS3' follow trivially.

Now for BS2: I must show that if $A \in \Phi$, then $\mathbb{R} \setminus A \in \Phi$. In other words, I must show that for all ordinals $\kappa < \omega_1$ and $A \in \mathcal{F}_\kappa$, that $\mathbb{R} \setminus A \in \Phi$. I'll proceed by transfinite induction.

The base case follows from realizing that a closed set, i.e. a member of $\mathcal{F}_0$ is an open set, which, as stated in the introductory note to the problem, is a Borel set and obeys BS2.

Now suppose that κ is a limit ordinal, i.e. ω_0, and let $A \in \mathcal{F}_{\omega_0}$. Then A is the countable intersection of elements from $\mathcal{F}_r$ for various finite r. Thus, by DeMorgan's Law, $\mathbb{R} \setminus A$ is the countable union of complements of various elements from the $\mathcal{F}_r$s, each complement an element of Φ by the inductive hypothesis, implying that their union, $\mathbb{R} \setminus A$ is by definition contained in some $\mathcal{F}_\alpha$, where α is the first odd ordinal greater than the union of each index corresponding to the $\mathcal{F}_?$ containing the complement. Thus $\mathbb{R} \setminus A \in \Phi$.

Now suppose that κ is a successor ordinal, and let $A \in \mathcal{F}_k$. Then A is the countable intersection or union of elements from $\mathcal{F}_{k-1}$, so $\mathbb{R} \setminus A$ is the countable union or intersection of complements of elements in $\mathcal{F}_{k-1}$, each complement, by the inductive hypothesis, an element of Φ, so essentially using the same argument as in the limit ordinal case, adjusting for intersection versus union and odd versus even, $\mathbb{R} \setminus A \in \Phi$.

Step 2: Let $\mathcal{B}$ be the smallest family obeying the Borel axioms in $\mathbb{R}$. I must show that $\Phi \subset \mathcal{B}$, or equivalently, that for all $\alpha < \omega_1, \mathcal{F}_\alpha \subset \mathcal{B}$. We'll proceed by transfinite induction.

The base case follows, since all closed sets of $\mathbb{R}$ are contained in $\mathcal{B}$.

Now suppose that for all $\alpha < \omega_0$, the only limit ordinal less than ω_1, that $\mathcal{F}_\alpha \subset B$. Then $\mathcal{F}_{\omega_0}$, the union of all such sets is also contained in $\mathcal{B}$.

For κ a successor ordinal, suppose that $\mathcal{F}_{\kappa-1} \subset \mathcal{B}$. Then since $\mathcal{F}_\kappa$ is the set of countable unions (intersections) of elements from $\mathcal{F}_{\kappa-1}$, for odd (even) κ, then we can just apply BS3 (BS 3'), since once again $\mathcal{F}_{\kappa-1} \subset \mathcal{B}$, to complete the induction.

For the last remark, note that the union of $\mathbf{c}$ sets of cardinality $\mathbf{c}$, has cardinality $\mathbf{c}$; hence, so does the set of Borel sets, whereas the power set of the real line has cardinality $2^{\mathbf{c}}$, and by Cantor's Theorem the power set properly contains the family of Borel sets.