1.1.A. I must verify that, for all subsets A and B of a topological space, we have **(a)** $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ and **(b)** $\overline{A} \setminus \overline{B} \subset \overline{A \setminus B}$. I must also verify **(c)**, that the inclusions can't be replaced by equalities.

(a) $A \subset \overline{A}$ and $B \subset \overline{B}$, so $A \cap B \subset \overline{A} \cap \overline{B}$, which is closed, therefore contains the minimal closed set containing $A \cap B$, so $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

(b) $A = (A \setminus B) \cup (A \cap B)$, so $\overline{A} = \overline{A \setminus B} \cup \overline{A \cap B} \subset \overline{A \setminus B} \cup (\overline{A} \cap \overline{B})$, from **(CO3)** and part **(a)**. Therefore $\overline{A} \setminus \overline{B} \subset (\overline{A \setminus B} \cup (\overline{A} \cap \overline{B})) \setminus \overline{B} = \overline{A \setminus B} \setminus \overline{B} \subset \overline{A \setminus B}$.

1.1.B. I must show **(a)** that for any sequence $A_1, A_2, \ldots$ of subsets of a topological space, we have

$$\overline{\bigcup_{i=1}^{\infty} A_i} = \bigcup_{i=1}^{\infty} \overline{A_i} \cup \bigcap_{i=1}^{\infty} \overline{\bigcup_{j=0}^{\infty} A_{i+j}}$$

. I must also **(b)**, show by an example that the above equality does not hold when the second term on the right side is omitted. (Which I'll do later.)

(a) Proof. Let $x \in \overline{\bigcup_{i=1}^{\infty} A_i} = \overline{\bigcup_{i=1}^{r} A_i} \cup \overline{\bigcup_{i=r+1}^{\infty} A_i}$, for all $r \geq 0$. If $x \notin \bigcup_{i=1}^{\infty} \overline{A_i}$, then $x \notin \bigcup_{i=1}^{r} \overline{A_i} = \overline{\bigcup_{i=1}^{r} A_i}$, so $x \in \overline{\bigcup_{i=r+1}^{\infty} A_i}$, for all $r \geq 0$. Thus $x \in \bigcap_{r=0}^{\infty} \overline{\bigcup_{i=r+1}^{\infty} A_i} = \bigcap_{i=1}^{\infty} \overline{\bigcup_{j=0}^{\infty} A_{i+j}}$, completing one direction of inclusion.

For the reverse direction, suppose $x \in \bigcup_{i=1}^{\infty} \overline{A_i}$. Then, there exists a k, such that $x \in \overline{A_k} \subset \overline{A_k \cup \bigcup_{i \neq k} A_i} = \overline{\bigcup_{i=1}^{\infty} A_i}$. On the other hand, if $x \in \bigcap_{i=1}^{\infty} \overline{\bigcup_{j=0}^{\infty} A_{i+j}}$, then, for all $i \geq 1, x \in \overline{\bigcup_{j=0}^{\infty} A_{i+j}}$. So, this holds in particular for $i = 1$, which completes the proof.