FYS4150 - Project 5
The Rossby waves equation

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1 Abstract

This project is aimed to solve the Rossby wave equation in two difference domains, periodic and basin boundaries. Both analytical and numerical solutions were used in this study. Numerical experiments were conducted under two initial conditions which are sine and Gaussian functions. Truncation errors and stability were studied by implementing two time-stepping methods, forward-Euler and leapfrog. Results showed that the leapfrog time stepping approach is more stable than the Euler algorithm in 1D simulation. Therefore, the leapfrog was applied to solve time-stepping for two dimensions Rossby wave equation.But, the leapfrog seems to has instability in 2D simulation. Hovmuller diagram was introduced in order to study phase velocity of the wave by extracting from the slope. Both analytical and numerical solution agree with phase speed at magnitude of $6.3x10^{-3}$ with the westward propagation (relative error is about 10^{-7}). Our program may have some missing parameters which lead to unexpected streamfunction field in 2D-basin domain simulation at the starting time step $t = 0$. However, wave reasonable adjusted to the boundary condition at the next time step $t = 1$.

All resources which are associated with this project were uploaded on the GitHub (`https://github.com/pipatths/FYS4150-Computational-physics/tree/master/Project5`). All fortran codes, results and this report are provided there.

2 Introduction

Rossby or planetary waves are occur within both ocean and atmosphere. These waves are generated by the Earth's rotation. The Rossby waves are long waves which have scales of hundreds to thousands of kilometers They can cause the changes of climate conditions, sea level rising, and also El Nino [1]. Horizontal wave speed varies with the latitudes as known as coriolis effect. Coriolis acceleration is almost zero at the equator and largest at the pole (positive in Northern Hemisphere and negative in Southern Hemisphere). This results in changes of vorticity if the fluid parcel move to different latitudes and can express as the vorticity groverning equation as

$$\partial_t \zeta + \beta \partial_x \psi = 0. \tag{1}$$

where ζ is the vorticity and ψ is the streamfunction, which determines the horizontal velocities

$$u = -\partial_y \psi \quad , \quad v = \partial_x \psi, \tag{2}$$

So the vorticity is defined as

$$\zeta = \partial_x v - \partial_y u = \nabla_H^2 \psi. \tag{3}$$

where the coriolis parameter defined as

$$f = 2\Omega \sin(\theta) \tag{4}$$

where θ is the latitude and Ω is the rotation rate of the Earth, $\Omega = 2\pi/day$. We often approximate f as a linear function, centered on a latitude θ_0:

$$f \approx f_0 + \beta y \tag{5}$$

where $f_0 = 2\Omega \sin(\theta_0)$, $\beta = 2\Omega \cos(\theta_0)/R_e$ and $y = R_e(\theta - \theta_0)$, if R_e is the Earth's radius. This linear representation is called the β-plane approximation, and accounts for the β term in (1).

Thus the vorticity equation can be written in terms of the streamfunction called the barotropic Rossby wave equation as

$$\partial_t \nabla_H^2 \psi + \beta \partial_x \psi = 0. \tag{6}$$

In this project, analytical and numerical solution of the barotropic Rossby wave will be examine in two different domain. One is periodic domain which wraps around (represents the atmosphere), while another one is basin or closed domain which has solid walls like continents. Two time stepping approaches, forward-Euler and leapfrog, will be implemented to study truncation errors and stability of simulation. We also apply sine and Gaussian functions as initial conditions and then calculate phase speed of the waves. Finally, two dimentional simulations will be introduced with Jacabi's iteration.

3 Method

3.1 Analytical solution in periodic domain

First, we consider the vorticity equation 6 in 1-dimension. For periodic domain, we defined $x \in [0, L]$, the assumed suitable streamfunction as

$$\psi = A \cos\left(\frac{2n\pi x}{L} - \omega t\right) \tag{7}$$

where n is an integer ($n = 1, 2, \dots$), A is the amplitude of the wave and ω is the wave angular frequency. This streamfunction has satisfy the boundary condition in periodic domain where $\psi(0, t)$ has to be equal to $\psi(L, t)$ as follow

$$\begin{aligned}
\psi(0,t) &= A\cos(0 - \omega t) = A\cos(-\omega t) \\
\psi(L,t) &= A\cos\left(\frac{2n\pi L}{L} - \omega t\right) \\
&= A\left[\cos(2n\pi)\cos(-\omega t) - \sin(2n\pi)\sin(-wt)\right] \\
&= A\cos(-\omega t)
\end{aligned}$$

We applied the streamfunction 7 to the barotropic Rossby wave equation 6, in order to find wave frequency of dispersion relation (ω). All derivatives term were calculated as follow

$$v = \partial_x \psi = -\frac{2n\pi A}{L} \sin\left(\frac{2n\pi x}{L} - \omega t\right) \tag{8}$$

$$\partial_x v = -\frac{(2n\pi)^2 A}{L^2} \cos\left(\frac{2n\pi x}{L} - \omega t\right) \tag{9}$$

$$\partial_t \partial_x v = -\frac{(2n\pi)^2 A\omega}{L^2} \sin\left(\frac{2n\pi x}{L} - \omega t\right) \tag{10}$$

then substituting these derivatives into 6. Next, we got wave dispersion relation

$$\omega = -\frac{\beta L}{2n\pi}. \tag{11}$$

Here, phase velocity of the wave can be calculated from

$$v_p = \frac{\omega}{k}. \tag{12}$$

where $k = 2\frac{\pi}{\lambda}$ is wavenumber in x (east-west) direction, and λ is the wavelenght.

Finally, phase speed of our wave in the periodic domain is

$$v_p = \frac{\omega L}{2n\pi} = -\frac{\beta L^2}{(2n\pi)^2}. \tag{13}$$

The negative phase speed shows that the wave propagates to the west (westward propagation).

3.2 Analytical solution in solid boundaries

In case of Rossby wave in solid boundaries or basin, wave can not propagate trough the walls or solid boundaries, therefore, boundary conditions have to be $\psi(0) = \psi(L) = 0$. For the Rossby wave in basin, we assumed streamfunction as

$$\psi = A(x)\cos(kx - \omega t) \tag{14}$$

where $A(x)$ is wave amplitude which depends on position (x) and k is wavenumber. Again, substituting derivative terms of streamfunction to the barotropic Rossby wave equation 6, then figure out wave dispersion relation and phase speed. After substituting the derivatives the equation 6 can be represented as

$$(A''(x)\omega - \beta A(x)k - A(x)\omega k^2)\sin(kx - \omega t) + (2A'(x)\omega k + \beta A'(x)\cos(kx - \omega t) = 0 \tag{15}$$

To satisfy the equation 15, sine and cosine terms can be neglect, and left with coefficient terms

$$2A'(x)\omega k + \beta A'(x) = 0 \tag{16}$$

$$A''(x)\omega - \beta A(x)k - A(x)\omega k^2 = 0. \tag{17}$$

After solving these two equations, wave dispersion relation is

$$\omega = -\frac{\beta}{2k} \tag{18}$$

Next step, we have to calculate the wave's amplitude by substituting dispersion relation into 17, then got

$$A''(x) + k^2 A(x) = 0 \tag{19}$$

This second-order differential equation has solution in form of sine and cosine

$$A(x) = B\cos(kx) + C\sin(kx) \tag{20}$$

At this step, wave's amplitude have to satisfy the boundary condition. Thus

$$A(0) = B = 0 \tag{21}$$

$$A(L) = C\sin(kL) = 0 \tag{22}$$

$$sin(kL) = 0 \tag{23}$$

$$k = \frac{n\pi}{L} \tag{24}$$

So,

$$A(x) = C\sin(kx) \tag{25}$$

Finally, the streamfunction of the Rossby wave in basin is

$$\psi = C\sin(kx)\cos(kx - \omega t) \tag{26}$$

and phase speed is

$$v_p = \frac{\omega}{k} = -\frac{\beta}{2k^2} \tag{27}$$

This wave also propagates to the west as the wave in periodic domain. From the streamfunction, wave harmonizes from the cosine term and moves up and down from the sine term.

3.3 Numerical solution

In order to solve the Rossby wave equation 6 numerically, we may rewrite the equation into two non-dimensional Partial Differential Equations (PDEs) as

$$\partial_t \zeta + \partial_x \psi = 0. \tag{28}$$

$$\partial_{xx} \psi + \partial_{xx} \psi = \zeta. \tag{29}$$

which describes time evolution of vorticity and is a Poisson equation of the vorticity. These allow us to calculate vorticity forward in time, at every time step, the Poisson equation will be solved and new state streamfunctuion will be updated. These algorithms will be discussed in detail later, after section of discretization and finite differences approximation.

3.3.1 Discretization and finite difference approximation

First step in numerical simulations is discretizing independent variables. In this project, space ($xandy$) and time (t) will have to specify the discrete approximation. We define $x \in [0, 1]$, $y \in [0, 1]$ and $t \in [0, T]$ where T is upper time limit. The discrete point in both space and time are given by

$$\begin{aligned} x_j &= j\Delta x \ , \qquad & j &= 0, 1, \ldots, N_x - 1 \\ y_k &= k\Delta y \ , \qquad & k &= 0, 1, \ldots, N_y - 1 \\ t^n &= n\Delta t \ , \qquad & n &= 0, 1, \ldots \end{aligned} \tag{30}$$

where N_x and N_y are number of discrete grid points in x and y direction, respectively, and defined by

$$N_x = \frac{1}{\Delta x} + 1 \ , \qquad N_y = \frac{1}{\Delta y} + 1 \tag{31}$$

According to non-dimensional equation of barotropic Rossby wave, approximation of derivative terms is needed. First, finite differences arrive from Taylor expansion, so we were implementing the Taylor expansion for the derivatives of streamfunction and vorticity around the point x, y, t for the function $\psi(x \pm \Delta x, y, t)$, $\psi(x, y \pm \Delta y, t)$ and $\zeta(x, y, t \pm \Delta t)$. Thus

$$\partial_t \zeta^n_{j,k} = \frac{\zeta^{n+1}_{j,k} - \zeta^n_{j,k}}{\Delta t} \tag{32}$$

$$\partial_t \zeta^n_{j,k} = \frac{\zeta^{n+1}_{j,k} - \zeta^{n-1}_{j,k}}{2\Delta t} \tag{33}$$

$$\partial_x \psi^n_{j,k} = \frac{\psi^n_{j+1,k} - \psi^n_{j-1,k}}{2\Delta x} \tag{34}$$

$$\partial_{xx} \psi^n_{j,k} = \frac{\psi^n_{j+1,k} - 2\psi^n_{j,k} + \psi^n_{j-1,k}}{\Delta x^2} \tag{35}$$

$$\partial_{yy} \psi^n_{j,k} = \frac{\psi^n_{j,k+1} - 2\psi^n_{j,k} + \psi^n_{j,k-1}}{\Delta y^2} \tag{36}$$

These approximation shows the truncation errors by Taylor expansion are second order in both time and space.

3.2.2 One spatial dimension and time

Consider one spatial dimension, y-dimension terms were neglected from the vorticity equation 6, so

$$\partial_t \zeta + \partial_x \psi = 0 \tag{37}$$

$$\partial_{xx}\psi = \zeta \tag{38}$$

To solve the equation 37, forward-Euler and leapfrog time-stepping schemes were implemented. Therefore, vorticity with forward-Euler time-stepping can express as

$$\zeta_j^{n+1} = \zeta_j^{n-1} - \frac{\Delta t}{2\Delta x}(\psi_{j+1}^n) - \psi_{j-1}^n)) \tag{39}$$

while, vorticity with leapfrog time-stepping can be written as

$$\zeta_j^{n+1} = \zeta_j^{n-1} - \frac{\Delta t}{\Delta x}(\psi_{j+1}^n) - \psi_{j-1}^n)) \tag{40}$$

Both two time-stepping schemes will be discussed in term of stability in result. Next, two boundary conditions were implemented to numerical simulations, then solving 1 dimension Poisson equation 38 in every time steps. We apply standard discretization 35 to Poisson equation and got

$$-\psi_{j+1}^{n+1} + 2\psi_j^{n+1} - \psi_{j-1}^{n-1} = -\Delta x^2 \zeta_j^{n+1}\ , \qquad j = 0, \dots N_x - 1 \tag{41}$$

This linear system can be solved using tridiagonal solver, and can be rewritten as

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & \cdots \\ 0 & -1 & 2 & -1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \psi_1^{n+1} \\ \psi_2^{n+1} \\ \vdots \\ \psi_{N_x-3}^{n+1} \\ \psi_{N_x-2}^{n+1} \end{bmatrix} = \begin{bmatrix} -\Delta x^2 \zeta_1^{n+1} \\ -\Delta x^2 \zeta_2^{n+1} \\ \vdots \\ -\Delta x^2 \zeta_{N_x-3}^{n+1} \\ -\Delta x^2 \zeta_{N_x-2}^{n+1} \end{bmatrix} \tag{42}$$

for basin domain where $\psi_0^{n+1} = \psi_{N_x-1}^{n+1} = 0$. Note that, Poisson equation represents the interior wave, therefore, tridiagonal system has $N_x - 2$ dimensions. Here, Thomas algorithm is introduced to solve the system. For the periodic domain where $\psi_0^{n+1} = \psi_{N_x-1}^{n+1}$, the system can be written as

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & \cdots \\ 0 & -1 & 2 & -1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & -1 & 2 & -1 \\ -1 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \psi_0^{n+1} \\ \psi_1^{n+1} \\ \vdots \\ \psi_{N_x-3}^{n+1} \\ \psi_{N_x-2}^{n+1} \end{bmatrix} = \begin{bmatrix} -\Delta x^2 \zeta_0^{n+1} \\ -\Delta x^2 \zeta_1^{n+1} \\ \vdots \\ -\Delta x^2 \zeta_{N_x-3}^{n+1} \\ -\Delta x^2 \zeta_{N_x-2}^{n+1} \end{bmatrix} \tag{43}$$

This is not the tridiagonal system, so, we introduced least square approximation instead. To sum up, algorithm of the Rossby wave equation in one dimension is provided in algorithm 1

3.2.4 Two spatial dimensions and time

For two spatial dimensions case, we let x and y direction are in the $[0, 1]$. This lead to streamfunction are zero at all boundaries, in basin case. Thus

$$\psi(0, y, t) = \psi(1, y, t) = \psi(x, 0, t) = \psi(x, 1, t) = 0 \tag{44}$$

While, boundary conditions for periodic domain are

$$\psi(0, y, t) = \psi(1, y, t) \qquad \text{and} \qquad \psi(x, 0, t) = \psi(x, 1, t) \tag{45}$$

We apply only leapfrog scheme due to the stability study which will be discussed in results (section 4.1.1). Applying leapfrog scheme, the equation becomes

$$\zeta_{j,k}^{n+1} = \zeta_{j,k}^{n-1} - \frac{\Delta t}{\Delta x}(\psi_{j+1,k}^n - \psi_{j-1,k}^n) \tag{46}$$

Algorithm 1 1D Rossby wave equation

- Initialize Poisson equation in form of matrix.
- Set initial conditions for ζ and ψ.
if Domain is solid boundaries **then**
  - Apply boundary conditions
end if
while loop over time **do**
  - Advance ζ to the next time step for both forward-Euler and leapfrog
  if Domain is solid boundaries **then**
    - Update ψ by Thomas algorithm
  end if
  if periodic BC's **then**
    - Update ψ by least squares approximation
    - Apply periodic boundary conditions
  end if
end while

Then, applying centered differences to update ψ at $n+1$ time step

$$\frac{\psi_{j+1,k}^{n+1} - 2\psi_{j,k}^{n+1} + \psi_{j-1,k}^{n+1}}{\Delta x^2} + \frac{\psi_{j,k+1}^{n+1} - 2\psi_{j,k}^{n+1} + \psi_{j,k-1}^{n+1}}{\Delta y^2} = \zeta_{j,k}^{n+1} \tag{47}$$

Next, solving for $\psi_{j,k}^{n+1}$ using Jacobi's iteration yields

$$\psi_{j,k}^{n+1} = \frac{\Delta y^2(\psi_{j+1,k}^{n+1} + \psi_{j-1,k}^{n+1}) + \Delta x^2(\psi_{j,k+1}^{n+1} + \psi_{j,k-1}^{n+1}) - \Delta x^2 \Delta y^2 \zeta_{j,k}^{n+1}}{2(\Delta x + \Delta y)} \tag{48}$$

In conclusion of 2D Rossby wave equation, numerical processes are provided in algorithm 2.

Algorithm 2 2D Rossby wave equation

- Initialize conditions for ζ and ψ
if Domain is solid boundaries **then**
  - Apply boundary conditions
end if
while loop over time **do**
  - Advance ζ to the next time step for leapfrog
  - Update ψ by Jacobi's method (apply for both basin and periodic domain)
end while

4 Results and discussion

In this section all simulations will be discuss in term of stability (1D simulation), and comparison between two initial conditions, sine and Gaussian conditions (both 1D and 2D simulations).

4.1 One spatial dimension and time

To test the stability of time-stepping schemes, various Δt were employed with same Δx. We varied four difference time steps, $\Delta t = 0.5$, 0.25, 0.05, and 0.005. We applied sine function as initial condition of $\psi(x, 0) = \sin(4\pi x)$. For the Gaussian condition, we apply $\psi = exp - (\frac{x-x0}{\sigma})^2$ where σ is width of the Gaussian and equal 0.1. Only simulations at time of 150 were discuss. From figure 1, forward-Euler scheme converge to leapfrog scheme with small time step, and those schemes have same streamfunction with $\Delta t = 0.005$. It is could save computational resources obviously shown that the leapfrog scheme has more stability than the forward-Euler. The leapfrog simulations provide stabled streamfuction with the large time step ($\Delta t = 0.5$) which could save computational resources. In contrast, the forward-Euler scheme need small time step (at least $\Delta t = 0.005$ from the simulations) in order to getting close to steady state. This stability conclusion is agree with both boundary conditions but only solid boundaries condition is presented in this report. From this result, we implement the leapfrog time-stepping to the further study. We also tried to animate the wave propagation, and the results show that waves are westward propagation. This agrees with the analytical solution, as mentioned above.

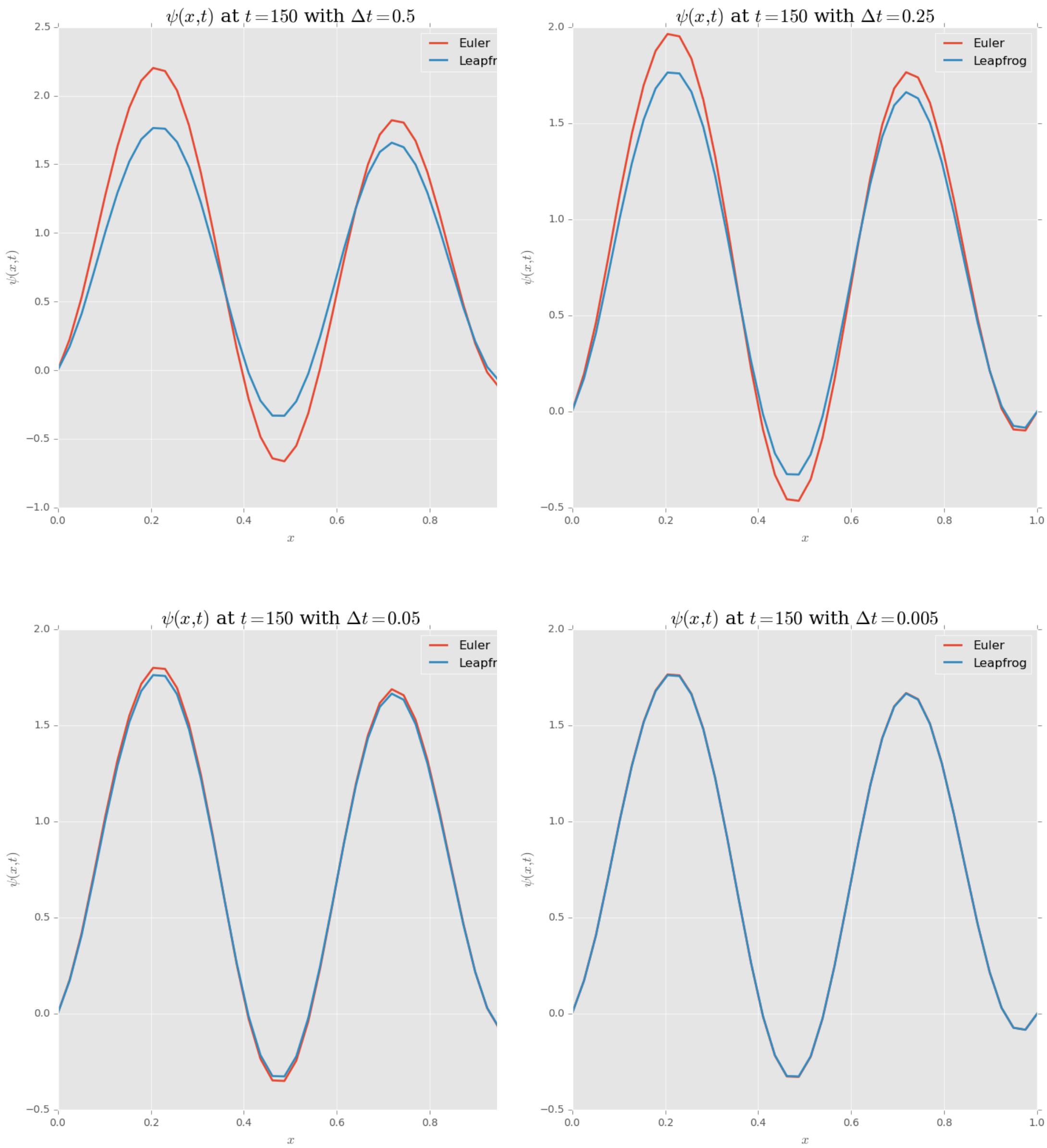

Figure 1: Streamfunction in basin with initial condition of sine function at time of 150. The graphs show both forward-Euler and Leapfrog schemes. Each plot represents different Δt with fixed $\Delta x = 0.025$.

Hovmuller diagram is introduced in order to study wave propagation. Streamfunctions were plot as function of time. Again, the results show that waves propagate to the west (Figure 2 and 3). Wave phase speed can be calculated from the slope of the graphs. As seen in Figure 2 and 3, it is difficult to calculate phase speed from the Gaussian initial condition. So, we extract the phase speed from the sine wave in periodic domain as an example. Analytical and numerical phase speed are agreed well at magnitude of $-6.3x10^{-3}$, roughly, with the relative error about 10^{-5}.

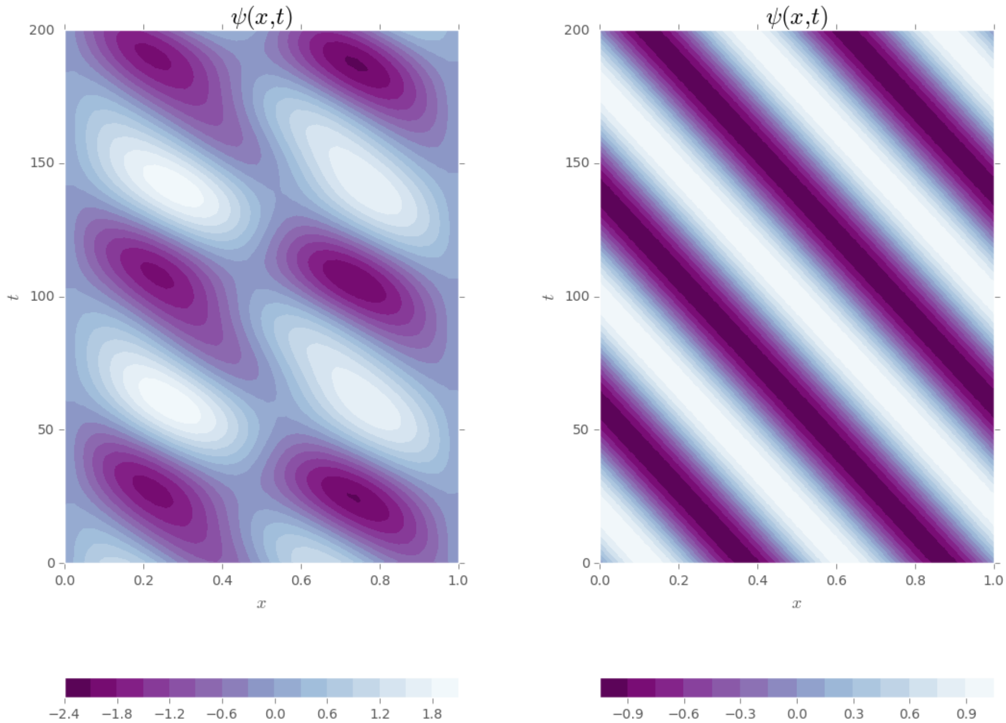

Figure 2: Streamfunction with initial condition of sine function, leapfrog time-stepping, $\Delta x = 0.025 and \Delta t = 0.001$ in basin (left) and periodic domain (right).

In term of wave structure, the Gaussian wave has more complex structure. The wave may be a combination of sine and cosine functions, thus, wave has different phase speeds. This also illustrates in the Hovmuller diagram of Gaussian wave in the complexity of streamfunction

We also tried to vary the Gaussian width σ to observe the wave structure. Results is shown in Figure 4. Small Gaussian width results in more homogeneous wave structure. This could imply that there is one dominated wave function. With this small Gaussian width, we can calculate the phase speed from the Hovmuller diagram.

4.2 Two spatial dimension and time

We studied sine wave in both basin and periodic domain. Implementation of leapfrog time-stepping is also involved in 2D simulations. In this section, wave structure (streamfunction) at starting time will be focused. The parameter Δx is set equal to Δy at 0.025 and $\Delta t = 0.001$ was used. Consider wave in periodic domain, 2D simulation is corresponding with 1D simulation. Wave propagates to the west. The wave stability also is examined but it was difficult to determine the stable criterion. However, leapfrog scheme seem unstable for large time step size. This means that the leapfrog scheme is definitely stable for the simulation, in particular, for 2D simulation. In basin domain, we have to satisfy stremfunction at all boundaries by adding y dependence term as

$$\psi(x, y, 0) = \sin(\pi y)\sin(4\pi x) \tag{49}$$

Results obviously shown adjustment of waves to the boundaries. In basin domain, wave vanishes at the walls (Figure 6 and 7), where as, continued streamfunction appear at edges

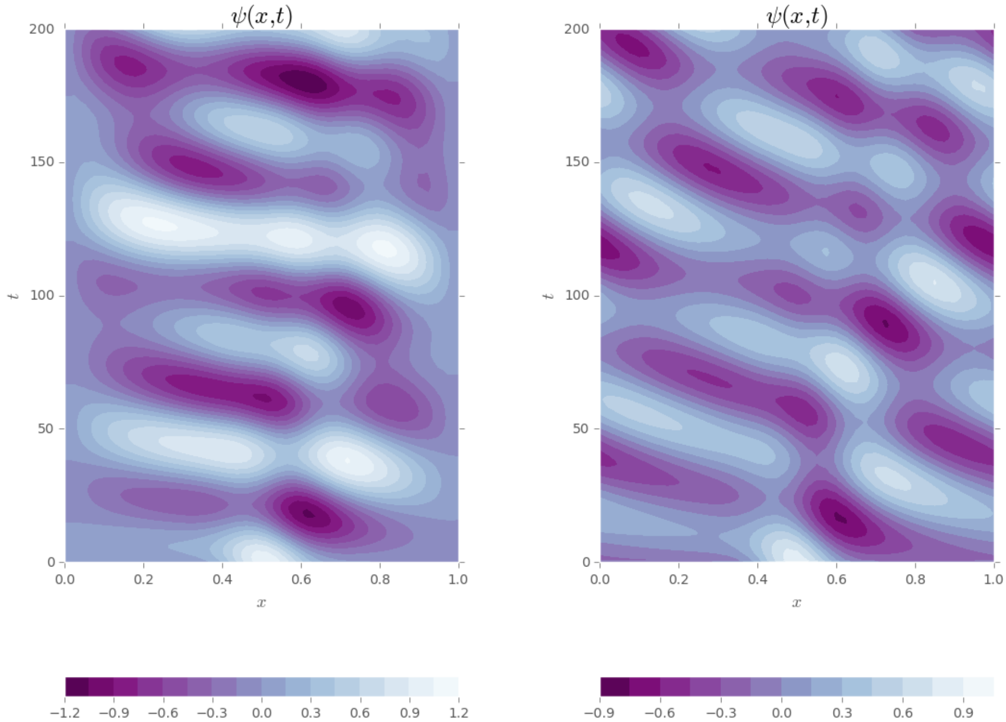

Figure 3: Streamfunction with initial condition of Gaussian function, leapfrog time-stepping, $\Delta x = 0.025 and \Delta t = 0.001$ in basin (left) and periodic domain (right).

of periodic domain (Figure **??**). Considering Figure 6, streamfunction at $t = 0$ in basin domain, wave almost like the wave in the periodic domain at the same time, but paying more attention at the boundary, wave still vanishes at the boundary (no smooth). Moreover, wave reasonably adjust to the boundaries at next time step $t = 1$ in Figure 7. This may be result of missing some parameters or term in the program.

5 Conclusion

This project studied Rossby wave equation in two different domain. One is the basin or solid boundaries domain which is represented the oceanic Rossby wave. Another one is periodic domain where wave can continuously propagate and representing the atmospheric Rossby wave. The barotropic Rossby wave equation has derivative in space and time. This leads to implementing time-stepping approaches and finite differences. Two time-stepping schemes which are leapfrog and forward-Euler were introduced in this project. The 1D simulations result in instability of forward-Euler. Therefore, we recomment to implement the leapfrog scheme for time-stepping. In contrast, the leapfrog scheme seems not satisfy the stability in 2D simulations. Using time-stepping schemes could be discuss more in the 2D simulation. Moreover, we also studied phase speed of the waves. Numerical results were illustrated in Hovmuller diagram, here, phase speed was extracted from the slope of the diagram. Numerical phase speed agrees well with analytical solution. From the 2D simulations, abnormal streamfunction was observed at the starting time, but reasonable adjustment was simulated at the next time step. This may be a consequence of missing parameters in the program.

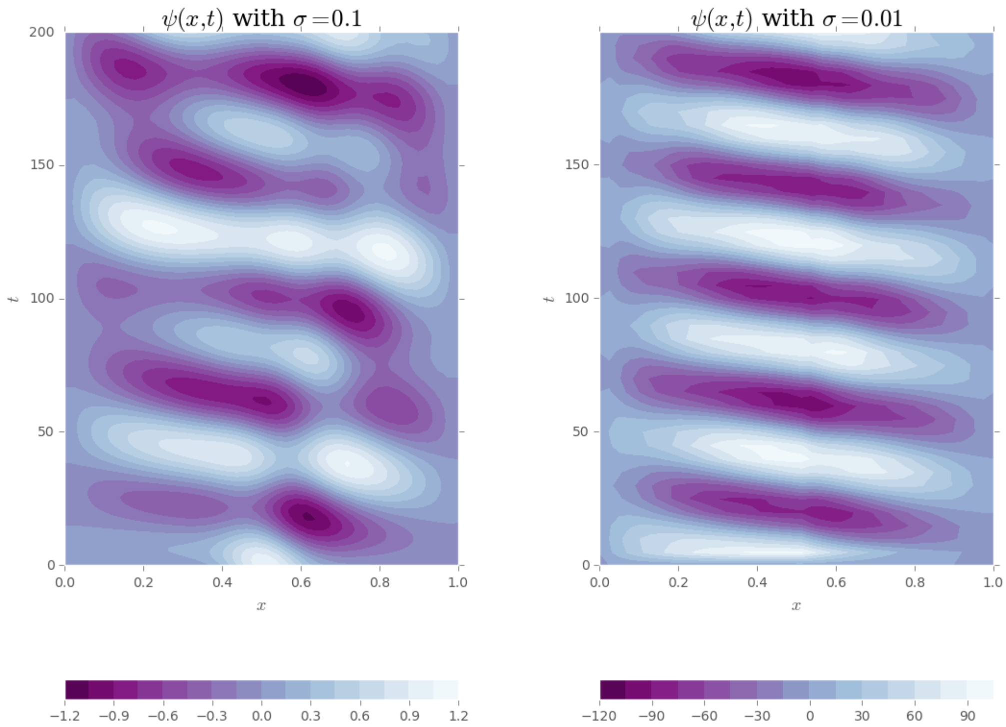

Figure 4: Hovmuller diagram of streamfunction with Gaussian initial condition, leapfrog time-stepping, $\Delta x = 0.025 and \Delta t = 0.001$ in basin domain. The simulation of Gaussian width $\sigma = 0.1$ is on the left and $\sigma = 0.01$ is on the right

6 References

1 National Ocean Service, National Oceanic and Atmospheric Adminsistration (NOAA), 2019. What is a Rossby wave? `https://oceanservice.noaa.gov/facts/rossby-wave.html`

2 Joseph LaCasce. Geophysical fluid dynamics. `http://folk.uio.no/josepl/papers/dynbook7.pdf`, 2017.

3 Morten Hjorth-Jensen (2019) Computational Physics Lectures. `http://compphysics.github.io/ComputationalPhysics/doc/pub/mcint/html/mcint.html`

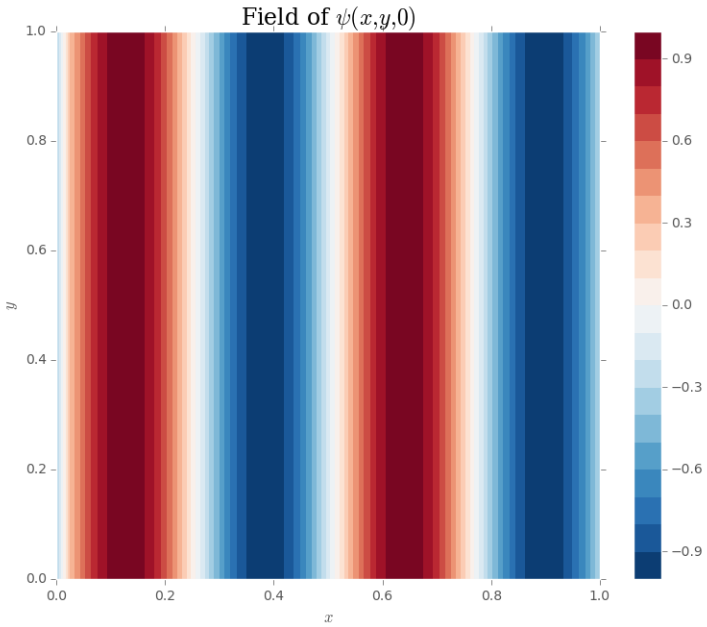

Figure 5: Contour plot of streamfunction with sine initial condition, leapfrog time-stepping, $\Delta x = 0.025 and \Delta t = 0.001$ in periodic domain at starting time $t = 0$

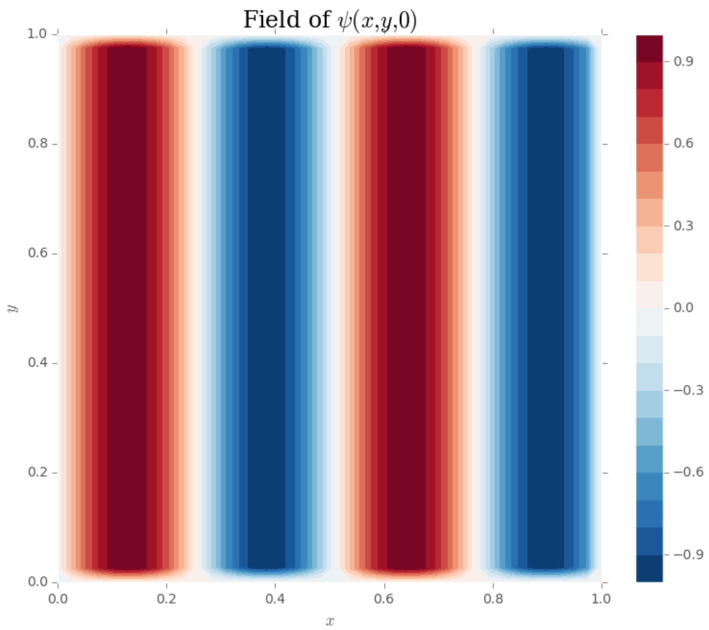

Figure 6: Contour plot of streamfunction with sine initial condition, leapfrog time-stepping, $\Delta x = 0.025 and \Delta t = 0.001$ in basin domain at starting time $t = 0$

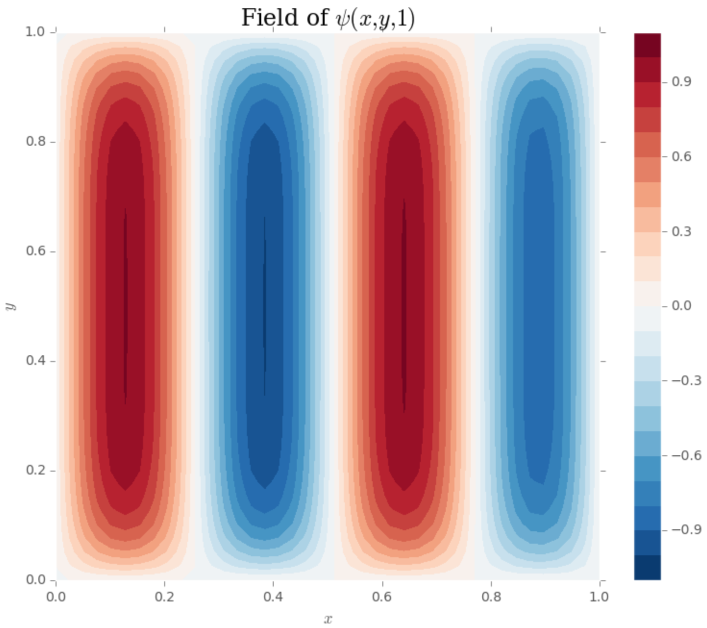

Figure 7: Contour plot of streamfunction with sine initial condition, leapfrog time-stepping, $\Delta x = 0.025 and \Delta t = 0.001$ in basin domain at $t = 1$