ECE 4550 Lecture 10

Background on Discrete-Time Systems

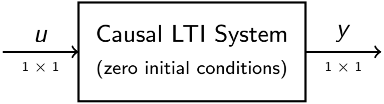
Topics Covered

Similarity of continuous time and discrete time systems

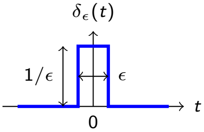
Continuous Time	Topic	Discrete Time
?	convolutions	?
?	transforms	?
?	partial fraction expansions	?
?	exponential functions	?
?	state-space representations	?
?	stability theory	?

Convolutions

Measurement based modeling



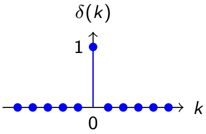
$$\delta(t) = \lim_{\epsilon \to 0} \delta_\epsilon(t)$$



$$u(t) = \delta(t) \quad \Rightarrow \quad y(t) = h(t)$$

$$y(t) = h(t) * u(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau$$

$$\delta(k) = \begin{cases} 1, & k = 0 \\ 0, & \text{else} \end{cases}$$



$$u(k) = \delta(k) \quad \Rightarrow \quad y(k) = h(k)$$

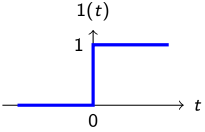
$$y(k) = h(k) * u(k) = \sum_{i=0}^{k} h(k-i)u(i)$$

Transforms

Definitions and basic pairs

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

$$1(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{else} \end{cases}$$



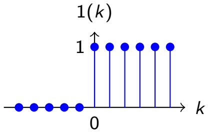
$$\delta(t) \;\leftrightarrow\; 1$$

$$1(t) \;\leftrightarrow\; \frac{1}{s}$$

$$e^{at}\cdot 1(t) \;\leftrightarrow\; \frac{1}{s-a}$$

$$\mathcal{Z}\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k)z^{-k}$$

$$1(k) = \begin{cases} 1, & k \geq 0 \\ 0, & \text{else} \end{cases}$$



$$\delta(k) \;\leftrightarrow\; 1$$

$$1(k) \;\leftrightarrow\; \frac{z}{z-1}$$

$$a^k\cdot 1(k) \;\leftrightarrow\; \frac{z}{z-a}$$

Transforms

Basic properties

$$a_1 f_1(t) + a_2 f_2(t) \;\leftrightarrow\; a_1 F_1(s) + a_2 F_2(s)$$

$$\dot{f}(t) \;\leftrightarrow\; sF(s) - f(0)$$

$$f(t - T) \cdot 1(t - T) \;\leftrightarrow\; e^{-sT} F(s), \quad T > 0$$

$$h(t) * u(t) \;\leftrightarrow\; H(s)U(s)$$

$$U(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s) = H(s)U(s)$$

$$a_1 f_1(k) + a_2 f_2(k) \;\leftrightarrow\; a_1 F_1(z) + a_2 F_2(z)$$

$$f(k + 1) \;\leftrightarrow\; zF(z) - zf(0)$$

$$f(k - K) \cdot 1(k - K) \;\leftrightarrow\; z^{-K} F(z), \quad K > 0$$

$$h(k) * u(k) \;\leftrightarrow\; H(z)U(z)$$

$$U(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z) = H(z)U(z)$$

Partial Fraction Expansions

Distinction between two fundamental transform pairs

$$F(\star) = \frac{-2\star^2 - 3\star + 2}{(\star + 1)(\star - 2)}, \quad \star = \{s, z\}$$

$$e^{at} \cdot 1(t) \;\leftrightarrow\; \frac{1}{s - a}$$

$$\frac{F(s)}{1} = \frac{-2s^2 - 3s + 2}{(s + 1)(s - 2)}$$

$$= c_0 + \frac{c_1}{s + 1} + \frac{c_2}{s - 2}$$

$$\Downarrow$$

$$f(t) = c_0 \delta(t) + c_1 e^{-t} + c_2 e^{2t}$$

$$a^k \cdot 1(k) \;\leftrightarrow\; \frac{z}{z - a}$$

$$\frac{F(z)}{z} = \frac{-2z^2 - 3z + 2}{z(z + 1)(z - 2)}$$

$$= \frac{c_0}{z} + \frac{c_1}{z + 1} + \frac{c_2}{z - 2}$$

$$\Downarrow$$

$$f(k) = c_0 \delta(k) + c_1 (-1)^k + c_2 (2)^k$$

Partial Fraction Expansions

Application of cover-up rule for distinct roots

$$\frac{-2s^2 - 3s + 2}{(s+1)(s-2)} = c_0 + \frac{c_1}{s+1} + \frac{c_2}{s-2}$$

$$c_0 = \lim_{s\to\infty} \frac{-2s^2 - 3s + 2}{(s+1)(s-2)} = -2$$

$$c_1 = \left.\frac{-2s^2 - 3s + 2}{s-2}\right|_{s=-1} = -1$$

$$c_2 = \left.\frac{-2s^2 - 3s + 2}{s+1}\right|_{s=2} = -4$$

$$\frac{-2z^2 - 3z + 2}{z(z+1)(z-2)} = \frac{c_0}{z} + \frac{c_1}{z+1} + \frac{c_2}{z-2}$$

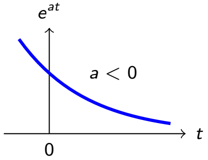
$$c_0 = \left.\frac{-2z^2 - 3z + 2}{(z+1)(z-2)}\right|_{z=0} = -1$$

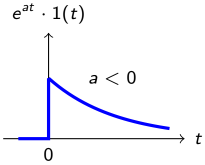
$$c_1 = \left.\frac{-2z^2 - 3z + 2}{z(z-2)}\right|_{z=-1} = 1$$

$$c_2 = \left.\frac{-2z^2 - 3z + 2}{z(z+1)}\right|_{z=2} = -2$$

Exponential Functions

Two cases: scalars and matrices



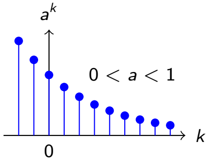


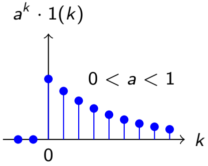
scalar a (1×1)

$$e^{at} = \sum_{i=0}^{\infty} a^i \frac{t^i}{i!}$$

matrix A $(n \times n)$

$$e^{At} = \sum_{i=0}^{\infty} A^i \frac{t^i}{i!}$$





scalar a (1×1)

$$a^k = \underbrace{(a)(a)\cdots(a)}_{k \text{ times}}$$

matrix A $(n \times n)$

$$A^k = \underbrace{(A)(A)\cdots(A)}_{k \text{ times}}$$

Exponential Functions

Distinction between scalar and matrix cases

if a is a scalar

$$e^{at} \cdot 1(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

if A is a matrix

$$e^{At} \cdot 1(t) = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \stackrel{?}{\Rightarrow} \quad e^{At} = \begin{bmatrix} e^{1t} & e^{2t} \\ e^{3t} & e^{4t} \end{bmatrix}$$

no

if a is a scalar

$$a^k \cdot 1(k) = \mathcal{Z}^{-1}\left\{\frac{z}{z-a}\right\}$$

if A is a matrix

$$A^k \cdot 1(k) = \mathcal{Z}^{-1}\left\{z\,(zI - A)^{-1}\right\}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \stackrel{?}{\Rightarrow} \quad A^k = \begin{bmatrix} 1^k & 2^k \\ 3^k & 4^k \end{bmatrix}$$

no

Exponential Functions

Calculating matrix exponential functions

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} s & 0 \\ 0 & s-1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/s & 0 \\ 0 & 1/(s-1) \end{bmatrix}$$

$$e^{At} \cdot 1(t) = \mathcal{L}^{-1}\left\{ \begin{bmatrix} 1/s & 0 \\ 0 & 1/(s-1) \end{bmatrix} \right\} = \begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix} \cdot 1(t)$$

$$z\,(zI - A)^{-1} = z \begin{bmatrix} z & 0 \\ 0 & z-1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & z/(z-1) \end{bmatrix}$$

$$A^k \cdot 1(k) = \mathcal{Z}^{-1}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & z/(z-1) \end{bmatrix} \right\} = \begin{bmatrix} \delta(k) & 0 \\ 0 & 1 \end{bmatrix} \cdot 1(k)$$

State-Space Representations

Application of transforms for response evaluation

- standard form

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

- transform property

$$\dot{x}(t) \;\leftrightarrow\; sX(s) - x(0)$$

- s-domain response

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

$$X(s) = (sI - A)^{-1}\left(x(0) + BU(s)\right)$$

- standard form

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

- transform property

$$x(k+1) \;\leftrightarrow\; zX(z) - zx(0)$$

- z-domain response

$$zX(z) - zx(0) = AX(z) + BU(z)$$
$$Y(z) = CX(z) + DU(z)$$

$$X(z) = (zI - A)^{-1}\left(zx(0) + BU(z)\right)$$

State-Space Representations

Time domain responses involve matrix exponential functions

- in the s-domain

$$X_{zi}(s) = (sI - A)^{-1}x(0)$$
$$X_{zs}(s) = (sI - A)^{-1}BU(s)$$

- in the t-domain $(t > 0)$

$$x_{zi}(t) = e^{At}x(0)$$
$$x_{zs}(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- in the z-domain

$$X_{zi}(z) = (zI - A)^{-1}zx(0)$$
$$X_{zs}(z) = (zI - A)^{-1}BU(z)$$

- in the k-domain $(k > 0)$

$$x_{zi}(k) = A^k x(0)$$
$$x_{zs}(k) = \sum_{i=0}^{k-1} A^{k-1-i}Bu(i)$$

State-Space Representations

Eigenvalues and transfer functions

- standard form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

- the eigenvalues of the system are the roots of the characteristic polynomial

$$\det(sI - A)$$

- the transfer function of the system is

$$H(s) = C\left(sI - A\right)^{-1}B + D$$

- standard form

$$\begin{aligned}x(k+1) &= Ax(k) + Bu(k)\\ y(k) &= Cx(k) + Du(k)\end{aligned}$$

- the eigenvalues of the system are the roots of the characteristic polynomial

$$\det(zI - A)$$

- the transfer function of the system is

$$H(z) = C\left(zI - A\right)^{-1}B + D$$

State-Space Representations

Definition of poles and zeros for siso systems

- the transfer function is

$$H(s) = \frac{\det \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix}}{\det \left[sI - A \right]} = \frac{\mathcal{N}(s)}{\mathcal{D}(s)}$$

- the zeros of the system are the roots of $\mathcal{N}(s)$ after canceling common factors
- the poles of the system are the roots of $\mathcal{D}(s)$ after canceling common factors
- the hidden modes of the system are the roots of the common factors

- the transfer function is

$$H(z) = \frac{\det \begin{bmatrix} zI - A & B \\ -C & D \end{bmatrix}}{\det \left[zI - A \right]} = \frac{\mathcal{N}(z)}{\mathcal{D}(z)}$$

- the zeros of the system are the roots of $\mathcal{N}(z)$ after canceling common factors
- the poles of the system are the roots of $\mathcal{D}(z)$ after canceling common factors
- the hidden modes of the system are the roots of the common factors

Stability Theory

Internal stability definition and analysis

definition: a system is internally stable if the zero-input response of all its state variables asymptotically approaches zero as time evolves

- consider eigenvalue of system at

$$s = \alpha + j\beta$$

- eigenvalue response will be

$$\begin{aligned} e^{st} &= e^{(\alpha+j\beta)t} \\ &= e^{\alpha t} e^{j\beta t} \\ &= e^{\alpha t}\left(\cos(\beta t) + j\sin(\beta t)\right) \end{aligned}$$

- internal stability requires $\mathrm{Re}\{s\} < 0$

$$e^{\alpha t} \to 0 \quad \Rightarrow \quad \alpha < 0$$

- consider eigenvalue of system at

$$z = \alpha e^{j\beta}$$

- eigenvalue response will be

$$\begin{aligned} z^k &= \left(\alpha e^{j\beta}\right)^k \\ &= \alpha^k e^{j\beta k} \\ &= \alpha^k \left(\cos(\beta k) + j\sin(\beta k)\right) \end{aligned}$$

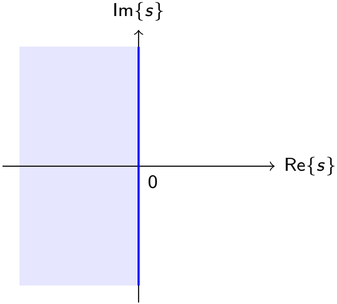
- internal stability requires $|z| < 1$

$$\alpha^k \to 0 \quad \Rightarrow \quad |\alpha| < 1$$

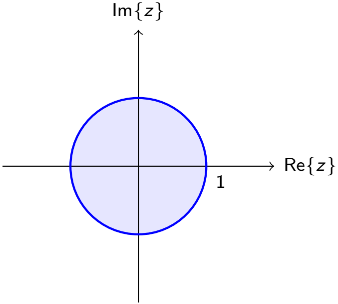
Stability Theory

Internal stability theorems

- theorem: a continuous-time system is internally stable if and only if all its eigenvalues have real parts less than 0
- the continuous-time stability region is the open left-half plane



- theorem: a discrete-time system is internally stable if and only if all its eigenvalues have magnitudes less than 1
- the discrete-time stability region is the open unit disk

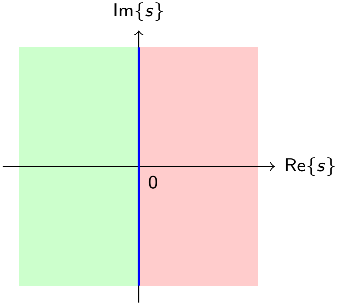


Stability Theory

Mapping between stability regions

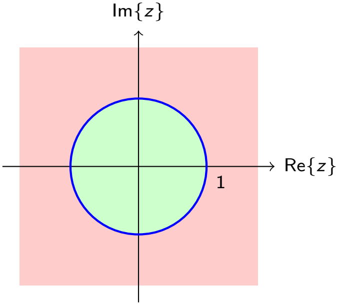
- given a z-domain polynomial, a corresponding s-domain polynomial may be derived by substituting

$$z = (1+s)/(1-s)$$



- given a s-domain polynomial, a corresponding z-domain polynomial may be derived by substituting

$$s = (z-1)/(z+1)$$



Stability Theory

Routh-Hurwitz analysis example

$$\star^2 - \star + a, \quad \star = \{s, z\}$$

- s-domain polynomial

$$s^2 - s + a$$

- Routh-Hurwitz analysis

$$\begin{array}{c|cc} s^2 & 1 & a \\ s^1 & -1 & \\ s^0 & a & \end{array}$$

- there is no value of a for which all roots of $s^2 - s + a$ satisfy $\mathrm{Re}\{s\} < 0$

- z-domain polynomial

$$z^2 - z + a$$
$$(2 + a)s^2 + 2(1 - a)s + a$$

- Routh-Hurwitz analysis

$$\begin{array}{c|cc} s^2 & 2 + a & a \\ s^1 & 2(1 - a) & \\ s^0 & a & \end{array}$$

- all roots of $z^2 - z + a$ satisfy $|z| < 1$ if and only if $0 < a < 1$