GEORGIA INSTITUTE OF TECHNOLOGY

SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 4550 — Control System Design — Summer 2020

Problem Set #9: Comparison of Two Implementations of Integral Control

The objective of this problem set is to help you understand the distinctions between two design philosophies when developing integral controllers. Submit your solution online by 7/21.

Consider a continuous-time plant

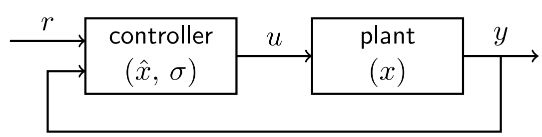
$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{aligned}
$$

and a corresponding discrete-time controller

$$
\begin{bmatrix} \hat{x}[k+1] \\ \sigma[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} \star & \star \\ \star & \star \end{bmatrix}}_{F} \begin{bmatrix} \hat{x}[k] \\ \sigma[k] \end{bmatrix} + \underbrace{\begin{bmatrix} \star & \star \\ \star & \star \end{bmatrix}}_{G} \begin{bmatrix} y[k] \\ r[k] \end{bmatrix}
$$

$$
u[k] = \underbrace{\begin{bmatrix} \star & \star \end{bmatrix}}_{H} \begin{bmatrix} \hat{x}[k] \\ \sigma[k] \end{bmatrix}
$$

where $\star$ entries are chosen by the designer. The block diagram



visually illustrates how the plant and controller are connected where

$$
r[k] = r(kT), \quad y[k] = y(kT), \quad u(t) = u[k] \ \forall\ t \in [kT, kT+T)
$$

defines the action of the interfacing data converters with update period T.

1. This problem focuses on <u>indirect</u> digital design (i.e. <u>continuous-time</u> philosophy). In this approach, there are two steps: (i) the controller is designed as a continuous-time system; (ii) an approximating discrete-time controller is determined. The first step is based on

$$
\begin{aligned}
u(t) &= -K_1\hat{x}(t) - K_2\sigma(t) \\
\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) - L\left(C\hat{x}(t) - y(t)\right) \\
\dot{\sigma}(t) &= y(t) - r(t)
\end{aligned}
$$

where L satisfies

$$
\det\left(sI - (A - LC)\right) = (s + \lambda_e)^n
$$

and $\mathcal{K}$ satisfies

$$
\det\left(sI - (\mathcal{A} - \mathcal{B}\mathcal{K})\right) = (s + \lambda_r)^{n+1}
$$

with

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}.$$

If using the forward Euler approximation method, the second step is based on

$$\begin{aligned} u[k] &= -K_1\hat{x}[k] - K_2\sigma[k] \\ \hat{x}[k+1] &= \hat{x}[k] + T\left(A\hat{x}[k] + Bu[k] - L\left(C\hat{x}[k] - y[k]\right)\right) \\ \sigma[k+1] &= \sigma[k] + T\left(y[k] - r[k]\right). \end{aligned}$$

(a) Derive general expressions for controller matrices F, G and H.

(b) Compute F, G and H assuming that

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \lambda_e = 12, \quad \lambda_r = 3, \quad T = 0.096.$$

(c) Simulate the control system associated with (b) over time interval $0 \leq t \leq 4$, assuming that $r(t) = 1$ and zero initial conditions for the plant and the controller. Use the attached Matlab simulation template with $h = T/100$, and provide the response plots.

2. This problem focuses on <u>direct</u> digital design (i.e. <u>discrete-time</u> philosophy). In this approach, there are two steps: (i) the plant is modeled as a discrete-time system; (ii) the controller is designed as a discrete-time system. The first step is based on

$$\begin{aligned} x[k+1] &= A_d x[k] + B_d u[k], \quad A_d = e^{AT}, \quad B_d = \int_0^T e^{At}B\,dt \\ y[k] &= Cx[k] \end{aligned}$$

and the second step is based on

$$\begin{aligned} u[k] &= -K_1\hat{x}[k] - K_2\sigma[k] \\ \hat{x}[k+1] &= A_d\hat{x}[k] + B_d u[k] - L\left(C\hat{x}[k] - y[k]\right) \\ \sigma[k+1] &= \sigma[k] + T\left(y[k] - r[k]\right) \end{aligned}$$

where L satisfies

$$\det\left(zI - (A_d - LC)\right) = (z - \gamma_e)^n, \quad \gamma_e = e^{-\lambda_e T}$$

and $\mathcal{K}$ satisfies

$$\det\left(zI - (\mathcal{A}_d - \mathcal{B}_d\mathcal{K})\right) = (z - \gamma_r)^{n+1}, \quad \gamma_r = e^{-\lambda_r T}$$

with

$$\mathcal{A}_d = \begin{bmatrix} A_d & 0 \\ TC & 1 \end{bmatrix}, \quad \mathcal{B}_d = \begin{bmatrix} B_d \\ 0 \end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}.$$

(a) Derive general expressions for controller matrices F, G and H.

(b) Compute F, G and H assuming that

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \lambda_e = 12, \quad \lambda_r = 3, \quad T = 0.096.$$

(c) Simulate the control system associated with (b) over time interval $0 \leq t \leq 4$, assuming that $r(t) = 1$ and zero initial conditions for the plant and the controller. Use the attached Matlab simulation template with $h = T/100$, and provide the response plots.

Matlab Template for Control System Simulations
Continuous-Time Plant and Discrete-Time Controller
(T = Timer ISR Period)

```
for n = 0:length(t)-1

    %% discrete-time update

    if mod(n*h,T) == 0

        % sensor measurements
        y = C*x;
        r = ?;

        % controller output
        u = H*v;

        % controller dynamics
        v = F*v+G*[y;r];

    end

    %% continuous-time update

    % data logger
    Y(n+1) = C*x;
    U(n+1) = u;

    % plant dynamics
    x = x+h*(A*x+B*u);

end

subplot(211), plot(t,Y), xlabel('t'), ylabel('y')
subplot(212), stairs(t,U), xlabel('t'), ylabel('u')
```