GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 4550 — Control System Design — Summer 2020

Problem Set #8: Discrete-Time Systems Concepts

The objective of this problem set is to gain proficiency with some discrete-time systems concepts frequently encountered in control system design. Submit your solution online by 7/15.

1. Consider the process of financing retirement. During working years you deposit funds monthly from income, and during retirement years you withdraw funds monthly from savings. Your account balance $y(k)$ for month k is obtained by adding together the previous month's balance $y(k-1)$, the interest earned on that previous month's balance $ry(k-1)$, and any positive or negative contributions $u(k)$, so that

$$y(k) = ay(k-1) + u(k), \quad k \geq 0$$

where r denotes the monthly interest rate and $a = 1 + r$. Your monthly sequence of deposits b and withdrawals c is assumed to satisfy

$$u(k) = \begin{cases} b & , \ k = 0, \ldots, m-1 \\ -c & , \ \text{otherwise} \end{cases}$$

where m is the number of working months. Note that the parameters $\{a, b, c\}$ are not allowed to vary with time, in order to focus on the simplest version of this problem.

   (a) Defining the state variable of this discrete-time system as

$$x(k) = y(k) - u(k),$$

   express its model in standard state-space form and provide the $\{A, B, C, D\}$ coefficients.

   (b) For future reference, find c_1 and c_2 such that

$$\frac{1}{(z-a)(z-1)} = \frac{c_1}{z-a} + \frac{c_2}{z-1}.$$

   (c) Find $x(k)$ as an explicit function of $k \in \{0, \ldots, m\}$ for the working era.

   (d) Find $x(i)$ as an explicit function of $i \in \{0, \ldots, ?\}$ for the retirement era, where $i = k - m$.

   (e) Find the duration of the retirement era, i.e. the value of i such that $x(i) = 0$, as an explicit symbolic equation involving a logarithm. For the special case of $a = 1.005$ and $m = 360$, plot the duration of the retirement era versus ratio c/b, taking care to limit this ratio to the range $6 \leq c/b \leq 10$.

2. Find conditions on real coefficients α and β guaranteeing that all roots of $z^2 + \alpha z + \beta$ are located in the discrete-time stability region defined by $|z| < 1$. Provide your answer in two forms: (i) a set of inequalities; and (ii) a sketch showing a shaded region in the (α, β) plane.

3. An ideal integrator shown below is described by

$$y(t) = \int_0^t u(\tau)\, d\tau, \quad t \geq 0.$$

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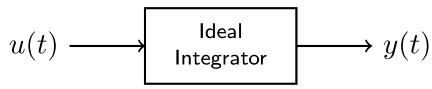
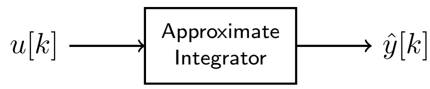


A discrete-time system to approximate integration is depicted by



where $u[k] = u(kT)$, $y[k] = y(kT)$, $\hat{y}[k] \approx y[k]$ and T is the sampling period.

(a) Determine the transfer function $G(s)$ of the ideal integrator.

(b) Using the forward approximation to numerical integration:

  i. Determine the difference equation relating $u[k]$ and $\hat{y}[k]$.

  ii. Determine the transfer function $H(z)$, from $U(z)$ to $\hat{Y}(z)$.

(c) Using the backward approximation to numerical integration:

  i. Determine the difference equation relating $u[k]$ and $\hat{y}[k]$.

  ii. Determine the transfer function $H(z)$, from $U(z)$ to $\hat{Y}(z)$.

(d) Using the trapezoidal approximation to numerical integration:

  i. Determine the difference equation relating $u[k]$ and $\hat{y}[k]$.

  ii. Determine the transfer function $H(z)$, from $U(z)$ to $\hat{Y}(z)$.

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