Lesson Notes

MODULE 2 | LESSON 2

VARIANCE SWAPS: SPANNING WITH OPTIONS

Reading Time	30 minutes
Prior Knowledge	Contingent claim, State price, Spanning, Replication, Log contract, Constant dollar gamma, Risk-neutral pricing
Keywords	Basic Probability Theory, Basic Calculus, Basic Stochastic Calculus, Black-Scholes, The Greeks, Partial differential equation (PDE), Option moneyness (in the money/at the money/out of the money)

In the last lesson, we discussed volatility swaps and variance swaps at a rather high level, focusing on how their mark-to-market values are calculated. We also discussed their growing popularity. In this lesson, we go deeper: We discuss the technical—but model-free and risk-neutral—replication of variance swaps, as well as some of the reasons investors and portfolio managers choose to trade them.

1. Probability Theory, Calculus, and Stochastic Calculus

Let's get this out in the open upfront: The main reading for this lesson is somewhat more mathematically technical than the last module, but it is math that you have seen before and you should feel comfortable with it, even if that means you need to do some review. You should do your best to understand the derivation in the "Spanning with Options" section of Hilpisch's "Model-Free Replication of Variance." It involves some basic probability theory in that the expected value or expectation of a payoff is defined by the integral of the payoffs over all possible outcomes. It also involves some basic calculus, specifically integration by parts, which is actually required twice. Do your best to convince yourself that this derivation holds true. There is even some stochastic calculus, where again you should be able to rely on your prior knowledge of stochastic differential equations in order to follow the mathematical argument. The overarching goal of this lesson is to better understand the nature of volatility in the service of better risk management. Another more specific goal for this lesson is familiarity with the variance swap itself, in order to put this product in your arsenal of risk and portfolio management techniques.

2. Hedging and Replication

You may have noticed that in the last lesson, we focused on the introduction of variance swaps as a risk management product, but the notion of swaps opened the door to our discussion of collateralization and mark-to-market margining—a key credit risk mitigation technique in over-the-counter trading. By the same token, we dig deeper into the mathematical mechanics of the variance swaps in our effort to better understand the nature of volatility and variance, but we will not pass up the opportunity to investigate the replication techniques. Certainly, understanding the replication of a product provides extremely valuable insights into the products themselves. Recall how your understanding of options deepened when you learned about put-call parity and the role of stock and the risk-free zero-coupon bond in that model. With that said, beyond replication as an excellent demonstration of a product's payout profile, replication is central to the concept of hedging. So the science of replication is worthwhile for its own sake.

In the last lesson, we hinted that variance swaps and options with the same underlying were actually deeply related, which was probably already obvious in some ways. For example, we mentioned that the strike for a volatility swap is set to the implied volatility, which means the implied volatility is used to define the variance swap strike as well (once it is squared). We'll see in this lesson that the options-variance swap relationship actually runs much deeper than this.

Have a look at the short required-reading video "States & Complete Markets" (Cochrane) in order to remind yourself that the state prices for elementary securities reflect the risk-neutral pricing for the contingent claim related to that state. (Hilpisch refers to these contingent claims as "elementary" or "*Arrow-Debreu*" securities; don't let this confuse you.) Since a contingent claim pays one unit in one state of the world and zero in every other state of the world, the market price of a contingent claim tells you the market's view of the probability of that state of the world. Simplifying the scenario only slightly, if the price of a contingent claim (known as the **state price**) is 0.2 units, then the market sees the probability of that state as 0.2 or 20%. This is risk-neutral pricing, and it makes perfect sense when you think about it. If you believed the probability of a 1-unit payout for a contingent claim was higher than 20%, say 22%, then statistically you should buy the contingent claim. Depending on the elasticity of the supply of contingent claims, you (possibly along with others who believed in the higher probability of payout) would drive the price up to 0.22. This is what Hilpisch means when he says, "every contingent claim can be priced by multiplying the contingent claim's state-dependent payoff by the respective state prices."

The discussion so far has been very abstract, in relation to unspecified "states," so let's be more concrete now and consider that the state with which we are concerned is the state of the stock market. In the reading for the last lesson, Hilpisch illustrated the workings of the VSTOXX using the underlying Euro Stoxx 50 as the relevant index. However, the mechanics of variance swaps are independent of the underlying; in fact, the underlying need not even be an index. The underlying may just as well be a sector or even a single-name stock. Really, as we will see, the only requirement is that the underlying have a wide enough range of call and put strikes for the maturity of interest. That implies that variance swaps can be traded on interest rates and foreign exchange—and this is in fact the case. Such variance swaps are traded every day.

3. From Option Prices into State Probabilities

The "Spanning with Options" section of the main reading glides over a pretty astounding claim that deserves a moment of recognition: Let us assume that there is a strike for every possible level of some broad stock market index, and for each of these strikes, there are European put or call options for a given maturity (that is, the maturity of the variance swap we are interested in). Then *if we know the prices for these options, we know the risk-neutral probability assigned to each of these possible states or price levels*. Just as the state prices told us the risk-neutral probabilities of states in the example of generic contingent claims above, put and option prices can tell us the risk neutral probabilities of index price levels. How can we transform these option prices into the risk-neutral probability density? Simply by taking the second derivative of the call or put prices with respect to the strike associated with the price level of interest. Be sure you appreciate and accept this bold assertion before moving past this section, as it is fundamental to what follows. If there were such a thing as a continuous strike surface, we could build a risk-neutral probability density function (pdf) using the relevant options. An obvious caveat is that, in general, there are *not* options with strikes for every single possible price level, so this pdf is but an illustrative pipe dream.

$$p(S_T, T; S_t, t) = \left.\frac{\partial^2 P(S_t, K, T)}{\partial K^2}\right|_{S_T=K} = \left.\frac{\partial^2 C(S_t, K, T)}{\partial K^2}\right|_{S_T=K}$$

Here, of course, P and C are the put and call option prices, which are functions of the strike K, the time to maturity T, and the state (or underlying price) at time T, ST. Since we can think of p as a probability density, what is the integral of p with respect to K over the interval zero to infinity? (Your answer should be "one" unless you were considering the possibility of products with negative prices, like, for example, certain spreads. If that was the case, what is the integral of p with respect to K over the interval negative infinity to infinity? Now we are finally on the same page.)

With this understanding of p as a probability density, the following equation from Hilpisch makes sense:

$$\mathbf{E}_t(g(S_T)|S_t) = \int_0^\infty g(K)p(K, T; S_t, t)dK$$

And from here, we split the integral into two integrals, one each for where the calls and puts are most liquid.

$$\mathbf{E}_t(g(S_T)|S_t) = = \int_0^F g(K)\frac{\partial^2 P(S_t, K, T)}{\partial K^2}dK + \int_F^\infty g(K)\frac{\partial^2 C(S_t, K, T)}{\partial K^2}dK$$

At this point, we have not said anything specific about the payout function *g*, only that its expected value is the probability-weighted payout for each state covered by the integral. Indeed, the rest of the section "Spanning with Options" leaves this payout function almost completely unspecified in order to conclude with the fascinating and broad claim that "any twice continuously differentiable payoff g due at T can be replicated by infinite strips of European put and call options maturing at T" (Hilpisch). That is, the only requirement for g is that it is twice continuously differentiable.

4. Defining the Payout: Why the Log Contract?

The log contract is not actually traded, but it has very helpful theoretical properties. For one, the log contract captures the realized volatility of an underlying, namely in the second of the terms in the last row below on the right.

$$\begin{aligned}\log\frac{S_T}{F} &= \log\frac{S_T}{S_0} \\ &= \int_0^T d\log(S_t) \\ &= \int_0^T \frac{dS_t}{S_t} - \int_0^T \frac{\sigma^2(S_t)}{2}dt\end{aligned}$$

Furthermore, since we are assuming a risk-neutral world where stock prices are already equal to their discounted expectations, we can set the first time on the right to zero. This leaves us with a relatively simple equality between the expected value of the log contract and the realized volatility of the underlying. (Sure, the latter is scaled by two, but we still have some work to do around scaling, so we won't concern ourselves with the two at this point.)

$$\mathbf{E}\left(\int_0^T \sigma^2(S_t)dt\right) = -2\mathbf{E}\left(\log\frac{S_T}{F}\right)$$

Knowing what we know now, the log contract seems like an excellent candidate for our payout function. What should we do to test its viability for our split integral above? We know that, to use the log contract as our payout function, it has to be a “twice continuously differentiable.” Let’s see.

If $g(S_T) = log(S_T)$ and we take the first continuous derivative, we find $g'(S_T) = 1/(S_T)$.

Then, we take the second continuous derivative to find that $g''(S_T)$ is $-1/(S_T)^2$.

The form of these equations implies smooth differentiability under the assumption that S is itself smooth. And this in turn implies that we can indeed use the above split integral to find the expectation for this payout function. More specifically, we can replicate this payout function using the call and put option prices given a wide array of strikes.

5. Aggregating the Option Strips: Scaling

The rest of the Hilpisch reading describes how best to scale the strip of options so that the dollar gamma for each strike is

1. constant across strikes;
2. as narrow as possible so that they better approximate the contingent claims discussed above (with payout at one and only one state).

Furthermore, in order for the variance swap to have constant exposure to variance regardless of the underlying price level, the contributions for low-strike and high-strike options must be equalized. The code provided helps you visualize the reasoning for each approach, so be sure to run it.

6. Speculating and Hedging with Variance Swaps

Now that you know a bit more about the payout profile of variance swaps and how they are structured, you might be curious who trades variance swaps and why. As is mentioned in the reading, volatility has become an asset class in its own right, so many portfolio managers, especially those pursuing a macro strategy, speculate on the outright realized variance of various indices, sectors, or single-name equities. They might also speculate on the relative levels of these in relation to each other. Their views might lead to taking the long sides, the short sides, or a combination of long and short sides of the variance swap.

In addition to these “risk as asset class” investors, there are many types of market participants that have naturally or implicitly “short” volatility exposure due to their other trading strategy, as described by Demeterfi et al. in the following list:

• Risk arbitrageurs assume the spread between stocks of companies involved in a merger will narrow. However, if overall market volatility increases, the merger plans may fall apart and the spreads will actually widen.

• Investors beholden to any types of fixed allocations (for example, benchmarking strategies) will need to rebalance more frequently in volatile markets, which means higher transaction costs during these periods.

• Similarly, portfolio managers with benchmarking strategies will likely incur higher tracking error in periods of higher volatility. When this tracking error results in worse performance than the benchmark, there will likely be costs, for example, related to redemptions.

• Equity funds are, in a sense, indirectly short volatility given the negative correlation between index level and volatility. Especially in light of the increasing correlation across global equity indices, geographic diversification is a less effective portfolio as a risk-reduction technique. Since volatility tends to increase when global equity declines, a long volatility position (e.g., via a variance swap) may indeed hedge some of this risk. Though, in this case, a more direct hedge would likely be a combination of put options. (Demeterfi et al. 3).

Conversely, a portfolio manager that uses long call and put options as a relatively cheap way to get positive and negative delta exposure to underlying securities—for example, instead of using up capital to buy a share of stock outright, one might simply buy a call option that is, for example, at the money or out of the money—is implicitly long volatility. To hedge this exposure so that the portfolio reflects changes in underlying prices in isolation from the volatility, such a manager might enter a variance swap on the short side.

7. Conclusion

In this lesson, we learned more about realized variance and its difference from option-implied variance--and how this difference or spread can be replicated and traded. The discussion so far has assumed the continuous price movements. In the next lesson, we look at the efficacy of models that includes jumps, in addition to the continuous Brownian motion dynamics.

References

- Cochrane, John. "4a.1 States & Complete Markets." *YouTube*, https://www.youtube.com/watch?v=BRPkSh8ET6A.
- Hilpisch, Yves. "Model-Free Replication of Variance." *GitHub*, https://github.com/yhilpisch/lvvd/blob/master/code/03_model_free_replication.ipynb.
- Demeterfi, Kresimir et al. "More Than You Ever Wanted To Know About Volatility Swaps." *Quantitative Strategies Research Notes*, GoldmanSachs, March 1999.