Implied Volatility

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This literature review summarizes the academic research on option-implied volatility. It describes algorithms for calculating implied volatility and various weighting schemes used to derive a single volatility estimate from the prices of multiple options, summarizes evidence in the debate on whether to use historical data or implied volatility in forecasting, and reviews several other papers on the various uses of implied volatility, including market efficiency studies and event studies. This review also suggests that implied volatility is being widely misused in practice, describes the schizophrenic notion of the volatility smile, and reviews various methods for calculating a risk-neutral density function consistent with option prices and the new generation of option pricing models (such as Rubenstein's implied binomial tree method) based on these implied volatilities.

Option pricing formulas, such as the Black–Scholes formula, relate the price of an option to the underlying asset price, the volatility of the underlying asset, and other parameters that influence option prices. The underlying stock price and the other parameters, including the strike price of the option, time to expiration, interest rate, and dividend yield of the underlying asset, are relatively easy to observe. Given that these values are known, the pricing formula relates the option price to the volatility of the underlying asset. Historical stock price data may be used to estimate the volatility parameter, which then can be plugged into the option pricing formula to derive option values. As an alternative, one may observe the market price of the option, then invert the option pricing formula to determine the volatility implied by the market price. The market's assessment of the underlying asset's volatility, as reflected in the option price, is known as the *implied volatility* of the option.

Traditionally, implied volatility has been calculated using either the Black–Scholes formula or the Cox–Ross–Rubinstein binomial model. Under the strict assumptions of the Black–Scholes model, implied volatility is interpreted as the market's estimate of the constant volatility parameter. If the underlying asset's volatility is allowed to vary deterministically over time, implied volatility is interpreted to be the market's assessment of the average volatility over the remaining life of the option.[1] Option pricing formulas other than the Black–Scholes or binomial also may be used to calculate implied volatilities. If the volatility of the underlying asset is itself a random process, as is assumed in "stochastic volatility" models, the market prices of options can still be used to estimate the parameters of the underlying asset process.[2]

Although the concept of implied volatility is commonly associated with standard stock options or stock index options, it is also quite useful in other contexts. Implied volatility may be calculated from the prices of exotic options, as demonstrated by Ball, Torous, and Tschoegl (1985). Several authors have examined implied volatility using options on commodity or currency futures,[3] and the prices of bond options can be used to estimate the parameters of an underlying term structure model.[4]

Option pricing formulas often cannot be inverted analytically, so implied volatility must be calculated numerically. In general, this calculation is accomplished by feeding the value–price difference,

$$C(\sigma) - C_M, \tag{1}$$

into a root–finding program, where $C(\,)$ is an option pricing formula, σ is the volatility parame-

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ter,[5] and C_M is the observed market price of the option. Various algorithms can be used to find the value of σ that makes this expression equal to zero. Choosing among them involves a tradeoff between robustness and speed of convergence. A simple approach that is very slow but reliable is to try a series of different values for σ and choose the one that comes closest to satisfying condition (1). Sometimes known as the "shotgun method," this approach is easy to implement but inefficient compared with other techniques such as the bisection method that for all practical purposes are just as robust.[6] Faster convergence can be achieved if an analytic expression is known for the option's "vega"—the derivative of the option price with respect to the volatility parameter. Such is the case for the Black–Scholes formula, for which a Newton–Raphson algorithm can usually achieve reasonably accurate estimates within two or three iterations.[7] Resorting to numerical procedures is not always necessary; for the special case of at-the-money options, Brenner and Subrahmanyam (1988) showed that the Black–Scholes formula can be inverted to derive a simple formula for implied volatility.

WEIGHTED-AVERAGE IMPLIED VOLATILITIES

Often, many options, which vary in strike price and time to expiration, are written on the same underlying asset. If the Black–Scholes model held exactly, these options would be priced so that they all have exactly the same implied volatility, which of course, is not the case. Systematic deviations from the predictions of the Black–Scholes model, often called the "volatility smile," are discussed in a later section of this review. Even if market participants were to price options according to Black–Scholes, price discreteness, transactions costs, and nonsynchronous trading would cause observed implied volatilities to differ across options.

In response to this problem, an early branch of literature suggested calculating implied volatilities for each option and then using a weighted average of these implied volatilities as a point estimate of future volatility. The idea behind this approach is simple: If the model is correct, then deviations from the predicted prices represent noise, and noise can be reduced by using more observations. The simplest weighting scheme, used by Trippi (1977) and by Schmalensee and Trippi (1978), places equal weights on all N implied volatilities:

$$\hat{\sigma} = \frac{1}{N}\sum_{i=1}^{N}\sigma_i. \tag{2}$$

One concern with using equal weights is that the model is not, in fact, correct. The Black–Scholes model prices some options more accurately than others, and placing more weight on observations for which the model performs better is reasonable. Trippi and Schmalensee and Trippi dealt with this problem by simply throwing out options that are near expiration or far from the money.

Another problem with equal weights is that some options are more sensitive to volatility than others; estimation errors (such as those induced by price discreteness or asynchronous data) are likely to be higher for options whose prices are insensitive to volatility. Therefore, placing more weight on options with higher vegas (higher sensitivities to volatility) appears to be preferable to equal weighting. Latané and Rendleman (1976) suggested this weighting scheme:

$$\hat{\sigma} = \frac{1}{\sum_{i=1}^{N} w_i}\sqrt{\sum_{i=1}^{N} w^2{}_i\sigma^2{}_i}\,, \tag{3}$$

where the weights, w_i, are the Black–Scholes vegas of the options. This forecast has the advantage of weighting options according to their sensitivities, but it is subject to the criticism that it is biased because the weights do not sum to 1. Chiras and Manaster (1978) suggested weighting not by vegas but by volatility elasticities:

$$\hat{\sigma} = \frac{\sum_{i=1}^{N}\sigma_i\frac{\delta C_i}{\delta\sigma_i}\frac{\sigma_i}{C_i}}{\sum_{i=1}^{N}\frac{\delta C_i}{\delta\sigma_i}\frac{\sigma_i}{C_i}}. \tag{4}$$

Beckers (1981) and Whaley (1982) suggested choosing to minimize

$$\sum_{i=1}^{N} w_i[C_i - BS_i(\hat{\sigma})]^2, \tag{5}$$

where C_i is the market price and BS_i the Black–Scholes price of option i. The weights, w_i , may be chosen in many ways, the most obvious choices being equal weights or Black–Scholes vegas.

Which of these many methods is best at predicting volatility? Beckers (1981) addressed this question empirically using daily prices of equity options from October 13, 1975, to January 23, 1976. He compared the forecasting ability of three measures of implied volatility. The first was the measure suggested by Latané and Rendleman and given in equation (3). The second was the quadratic loss function given by equation (5), and the third was simply the implied volatility of the option with the highest vega.[8] Beckers found that the squared-error-minimizing technique led to better forecasts than the Latané and Rendleman measure. He also found, however, that using the implied volatility of the option with the highest vega outperforms either of the other two techniques.

Beckers's results seem to make the weighted–average approach to estimating implied volatilities obsolete, because using the implied volatility of the nearest-in-the-money call option appears to do as well at forecasting future volatility as does any weighted average; it also has the advantage of being easier to calculate. Nevertheless, a fair amount of subsequent research has been devoted to this field. Brenner and Galai (1981) found that additional forecasting power can be achieved by calculating the weighted-average implied volatility several times during the day and averaging the results instead of using closing prices. This finding suggests that intraday frictions may significantly affect implied volatility estimates.[9]

Whaley (1982) used a minimized-squared-pricing-error implied volatility to investigate empirically the various models for dividend-paying American call options. Using weekly closing prices for Chicago Board Options Exchange (CBOE) options on 91 dividend-paying stocks from January 1975 to February 1978, he found that this measure of implied volatility is more accurate than equally weighted, vega-weighted, or elasticity-weighted average implied volatility. He also found that the different dividend models give approximately the same implied volatilities.

Gemmill (1986) looked at 13 equity options traded on the London Traded Options Market. Using monthly closing prices from May 1978 to July 1983, he compared six different implied volatility weighting schemes—equal weights, elasticity weights, minimized squared pricing errors, nearest the money only, farthest out of the money only, and farthest in the money only—to see which best predicts future volatility. Using a regression test, he found (consistent with Beckers) that the nearest-the-money measure contains the most information about future volatility, followed by the minimum-squared-pricing-error measure.

Scott and Tucker (1989) examined the relative performance of various weighting schemes for calculating implied volatility from currency options. Using transactions data for American-style FX calls on the British pound, Canadian dollar, deutschemark, yen, and Swiss franc from March 14, 1983, to March 13, 1987, these authors inverted the Garman–Kohlhagen (1983) currency option formula to calculate implied volatilities.[10] They examined three weighting schemes: vega weights, minimized squared pricing error, and the method that places all the weight on the option nearest the money. They found that all three weights perform equally well and that adding historical volatility does not improve predictive accuracy. Additional research on the forecasting power of the implied volatility of currency options has been reported by Fung, Lie, and Moreno (1990) and by Edey and Eliot (1992). Turvey (1990) tested alternative weighting schemes for calculating implied volatilities for options on soybean and live cattle futures.

Maloney and Rogalski (1989) found that option prices reflect predictable seasonal patterns in volatility. Morse (1991) also looked at the seasonality of implied volatility, finding that the difference between call and put implied volatility tends to drop on Fridays and rise on Mondays. Resnick, Sheikh, and Song (1993) described an "expiration-specific" weighted-average implied volatility, taking into account monthly patterns that differ systematically with market capitalization. Franks and Schwartz (1991), using options on the Financial Times Stock Exchange Index from May 1984 to December 1989, examined the time-series properties and macroeconomic determinants of Chiras–Manaster weighted-average implied volatility. They found that shocks to implied volatility do not persist for long and that leverage, inflation, and long-term nominal interest rates help explain implied volatility.

Weighted-average implied volatilities have also been used to construct volatility indexes. Before the advent of index options, Gastineau (1977) suggested a volatility index constructed from the implied volatilities of individual stock options.[11] Recently, the CBOE introduced its Market Volatility Index (VIX). This index was designed to represent the implied volatility of an at-the-money option on the S&P 100 (OEX) Index with 22 trading days to maturity. Whaley (1993) described the construction of this index and discussed hedging strategies based on futures and options written on a volatility index.

The VIX is a weighted average of implied volatilities of eight OEX option contracts, four calls and four puts, using the two strike prices nearest the money and the nearest two expiration dates. Implied volatility is calculated for each option using a Cox–Ross–Rubinstein binomial framework that accounts for early exercise and discrete dividends. The weighted average of the eight is constructed by (1) averaging the implied volatilities of the put and the call, holding strike and time to expiration constant; (2) averaging across strike prices, weighting in proportion to the distance from the strike price to the current index value; and (3) averaging across times to expiration with weights proportional to the difference between time to expiration of the option and 22 days. The second step is to standardize the index to represent an at-the-money option, and the third is to standardize the index to represent an option with 22 trading days to expiration. Fleming, Ostdiek, and Whaley (1995) examined the time-series properties of this index.

In summary, various weighted-average techniques for calculating implied volatility have been suggested and have received quite a bit of attention despite widespread admission that the underlying model used to calculate the volatilities is incorrect and despite empirical evidence suggesting that the near-the-money option is as good as a weighted average at predicting volatility.

IMPLIED VOLATILITY VERSUS HISTORICAL DATA

Perhaps the most important issue in volatility forecasting is whether the forecast should be based on historical price data, implied volatility, or some combination of the two. Latané and Rendleman (1976), Schmalensee and Trippi (1978), Chiras and Manaster (1978), Beckers (1981), and several others all found that implied volatility is better than historical standard deviation at forecasting future realized volatility. Latané and Rendleman found this result using 39 weekly observations for options on 24 stocks from October 1973 to June 1974. Schmalensee and Trippi used 56 weekly observations for options on six stocks from April 1974 to May 1975. Besides finding no significant relationship between historical volatility and implied volatility, they also found that implied volatility seems to decline following price increases and that implied volatilities are positively correlated across stocks. Trippi (1977) described a trading strategy based on implied volatilities that appears to have been capable of earning abnormal returns.

Unfortunately, the papers of Latané and Rendleman, Schmalensee and Trippi, and Trippi failed to account for dividends. Chiras and Manaster, using a sample of 23 monthly observations from June 1973 to April 1975, accounted for dividends by converting realized dividends to a continuous rate. They found that during the first nine months of their sample, implied volatility was not significantly better than historical standard deviation at forecasting volatility. That result, however, is reversed in the remainder of the sample, leading the authors to conclude that the market took some time after the opening of the CBOE in 1973 before beginning to incorporate volatility forecasts into option prices. Melnick and Yannacopoulos compared a volume-weighted implied standard deviation to a number of alternative measures of volatility for IBM call options over the period November 1, 1976, to July 3, 1980, and concluded that implied volatility incorporates all the relevant information in past prices. Using data from 1973 to 1981, Heaton (1986) came to a similar conclusion.

In short, the early literature found implied volatility to be better at forecasting future volatility than estimators based on historical data. Subsequent research has generally supported this conclusion, but results have been mixed. For one, these early papers generally used the historical standard deviation of returns based on a time series of closing prices. Since then, other, more powerful methods have been developed for predicting future volatility from historical data. One such approach takes advantage of the information in daily high, low, opening, and closing prices.[12] Marsh and Rosenfeld (1986) argued that these extreme-value estimators are quite sensitive to microstructural frictions such as infrequent trading and bid–ask bounce. Beckers (1983) suggested incorporating implied volatility to increase the power of these tests and found that doing so leads to relatively small incremental forecasting power.

Another approach has been to describe the time series of the underlying stock using generalized autoregressive conditional heteroscedasticity (GARCH) models.[13] For example, Day and Lewis (1992) studied the relative forecasting power of implied volatility versus historical data by adding implied volatility as an explanatory variable in a GARCH model. They found that for OEX options, both implied volatility and historical data contain incremental information about future volatility. Xu and Taylor (1993) extended this approach to account for the term structure of volatility and found that for three out of four foreign exchange options, implied volatility is the best one-period predictor of volatility and that historical data add no additional explanatory power. Choi and Wohar (1993) found that returns forecasted from a GARCH model are consistent with the implied volatilities observed in option prices. Fung and Hsieh (1991) addressed the forecasting issue in the context of a GARCH model, using transactions-level data from options on S&P futures, U.S. Treasury bond futures, and deutschemark futures. Of the three, implied volatility helped forecast the volatility only of deutschemark futures. Noh, Engle, and Kane (1994) compared the forecasting ability of implied volatility with that of a GARCH model by comparing the returns to delta-neutral straddles of S&P 500 options based on the two forecasts. They found that a strategy based on the GARCH forecast generated higher returns than a similar strategy based on implied volatility. Strong and Dickinson (1994) also discussed how to use both historical and implied volatility to forecast implied volatility and hedge ratios.

Lamoureux and Lastrapes (1993) used the GARCH formulation and implied volatilities to test the Hull–White class of stochastic-volatility option pricing models. These models assume that volatility risk is unpriced (see Hull and White 1987). If a model in this class is correct, then all available information will be incorporated into the market's prediction of future volatility.[14] The data for the test include transactions data for CBOE options trading on ten actively traded stocks from April 19, 1982, to March 31, 1984. None of the stocks paid dividends during this period, and the option prices used to calculate implied volatilities were constructed by taking the midpoint of the inside bid–ask quote for at-the-money, intermediate-term options. Out of these data, a single, representative daily implied volatility observation was constructed for each stock, and the resulting time series was subjected to analysis. For seven of the ten stocks, the authors rejected the Hull–White class of models in favor of a more general GARCH model.[15] Diz and Finucane (1994) examined the relationship between implied volatility and the average volatility expected over the remaining life of options. Using data from OEX options, they concluded that, consistent with the Hull–White class of option pricing models, implied volatility is not significantly different from expected average volatility for a variety of volatility process specifications.

Stein (1989) used implied volatility to test the "overreaction" hypothesis in the options market. If volatility follows a mean-reverting process, then shocks to the implied volatility of near-expiration options should be accompanied by shocks in the same direction (but smaller in magnitude) to the implied volatility of long-term options. Stein hypothesized an AR(1) process for the volatility of the OEX and estimated the mean-reversion parameter using historical data. He found that the implied volatility of the long-term OEX options seems to overreact to changes in the implied volatility of short-term OEX options, given the observed level of mean reversion in volatility. This result conflicts with the earlier work of Poterba and Summers (1986), who examined properties of historical volatility and of the volatilities implied by the CBOE call option index and by the Value Line three-month and six-month options index. They concluded that volatility shocks do not persist very long and that the implied volatility of long-run options does not move very much in response to shocks in that of short-run options. The results of Diz and Finucane (1993) supported the findings of Poterba and Summers and contradicted those of Stein. Using an alternative empirical specification, Diz and Finucane (1993) found strong evidence against the over-reaction hypothesis and limited evidence in favor of under-reaction in some periods.

Using a statistical technique that explicitly accounts for overlapping observations, Fleming (1994) studied the forecasting power of implied volatility for OEX options.[16] He calculated implied volatility very carefully using a binomial tree that incorporated dividends, early exercise, and the embedded "wildcard option."[17] Fleming concluded that implied volatility is an upwardly biased estimator of future volatility but that the magnitude of the bias is not economically significant. He also concluded that implied volatility dominates historical volatility as a forecast of future volatility.

Canina and Figlewski (1993) also investigated the ability of implied volatility to forecast actual volatility, but they came to the opposite conclusion. They claimed that implied volatility has virtually no explanatory power but that estimates of historical volatility can explain some of the variation in realized volatility. The authors iterated a 500-step binomial tree to obtain daily observations of implied volatility for OEX call options from March 1983 to March 1987, excluding options with fewer than 7 or more than 127 days to maturity. They used realized dividend values to adjust for dividends, allowed for early exercise, and threw away option prices that violated the static-arbitrage boundary. They calculated implied volatilities for four "time-to-maturity" groups and eight "intrinsic value" groups so as to reduce the effects of systematic volatility-smile effects. They then estimated the realized volatility over the remaining life of the option and (for each group) regressed that estimate on implied volatility. They found the regression coefficients to be statistically insignificant and so concluded that implied volatility poorly forecasts actual volatility. A regression of realized volatility on historical volatility, however, appeared to have some explanatory power. Additional research on this topic by Geske and Kim (1994) and by Christensen and Prabhala (1994) casts serious doubt on these results.

Although the debate is still open, the general conclusion to be drawn from this large body of research is that for forecasting volatility, implied volatility tends to be more useful than historical data. Time-series models that incorporate both, however, show a great deal of promise. A number of recent papers shed more light on the intertemporal relationship between implied volatility and underlying price dynamics. Sheikh (1993), for example, examined the time series of implied volatility and its relationship to returns in the underlying stock for a number of equity and index options trading on the CBOE. He found positive autocorrelation in the time series of implied volatilities and a positive relationship between returns and lagged implied volatility.

Kawaller, Koch, and Peterson (1994) used intraday data to examine the lead–lag relationship between implied and historical volatility for options on futures. Included in their study were options on S&P 500 futures, deutschemark futures, Eurodollar futures, and live cattle futures. Dividing the day into ten intervals, they investigated the intraday relationship between implied and historical volatility. They found that implied volatility never leads historical volatility, suggesting that option traders cannot anticipate impending changes in volatility. Using daily data, this result reverses in some markets. Their study also found a strong link between trading volume and historical volatility but no stable relationship between volume and implied volatility.[18] Boyle and Park (1994) showed that a lead–lag relationship between stock and option markets can lead to a bias when implied volatility is calculated using contemporaneous observations. In particular, the size of the bias is positively related to the volatility of the underlying asset when the option market leads the stock market and is negatively related when the stock market leads the options market.

Lo and Wang (1995) pointed out that because prices are sampled discretely, an estimate of historical volatility will depend on the analyst's belief about the level of mean reversion in stock prices.[19] This result has subtle implications for the debate on whether to use implied volatility or historical data. On the one hand, it means that measuring an asset's instantaneous volatility using historical data is difficult; on the other hand, it also means that using implied volatility to estimate the variance of the asset's price at a future date is difficult. Perhaps the proper choice of whether to use implied volatility or historical data depends on the forecasting horizon.

IMPLIED VOLATILITY EVENT STUDIES

Because implied volatility is widely interpreted as the market's forecast of future volatility, movements in implied volatility have been interpreted as reflecting the market's response to new information about the future volatility of the underlying stock. This interpretation has led several authors to conduct event studies examining the impact of new information on the implied volatility of options.

Patell and Wolfson (1981) examined the properties of the implied volatility of equity options at the time of quarterly earnings announcements. Other authors have found historical volatility to be high near earnings announcements. If this volatility is reflected in option prices, then implied volatility should fall after earnings announcements. Patell and Wolfson verified this prediction.

A number of authors have examined the behavior of implied volatility in response to stock splits. Historical volatility tends to be high following stock splits. French and Dubofsky (1986) found a small increase in implied volatility in response to stock splits. This finding is contradicted by the results of Klein and Peterson (1988) and Sheikh (1989), who found that implied volatility does not seem to respond to stock splits.

Day and Lewis (1988) found that implied volatility is higher around the expiration dates of stock index futures and stock index options. Bailey (1988) examined the response of implied volatility to the release of (M1) money supply information. Gemmill (1992) examined the pattern of implied volatility in British markets immediately prior to the election of 1987. Madura and Tucker (1992) considered the effect of U.S. balance-of-trade deficit announcements on the implied volatility of currency options. Levy and Yoder (1993) investigated the behavior of implied volatility around merger and acquisition announcements, and Barone-Adesi, Brown, and Harlow (1994) used the implied volatility of options on target firms to estimate the probability of a successful takeover. Jayaraman and Shastri (1993) examined the relationship between implied volatility and announcements of dividend increases.

IMPLIED VOLATILITY SMILES

Over the years, it has become quite clear that the market does not price all options according to the Black–Scholes formula. The consensus opinion is that the model performs reasonably well for at-the-money options with one or two months to expiration, and this experience has motivated the choice of such options for calculating implied volatility. For other options, however, the discrepancies between market and Black–Scholes prices are large and systematic. If the market were to price options according to the Black–Scholes model, all options would have the same implied volatility. Because the Black–Scholes model holds reasonably well for some options and not for others, different options on the same underlying security must have different Black–Scholes implied volatilities. It is now well known that the implied volatilities of options differ systematically across strike prices and across time to expiration. The pattern of implied volatilities across times to expiration is known as the "term structure of implied volatilities," and the pattern across strike prices is known as the "volatility skew" or the "volatility smile," a term that is sometimes used generally to refer to the pattern across both time to expiration and strike.

To even talk about volatility smiles is schizophrenic: First, a constant volatility is assumed to derive the model; then, many different volatilities are calculated for the same underlying asset. Once the Black–Scholes model is rejected, Black–Scholes implied volatility has no real meaning and, of course, should no longer be interpreted as the market's assessment of the underlying asset's volatility. The real phenomenon underlying volatility smiles is that either (1) market imperfections systematically prevent prices from taking their true Black–Scholes values or (2) the underlying asset price process differs from the (lognormal diffusion) process assumed by the Black–Scholes model.[20] The volatility smile is just a convenient way of illustrating this observation that probably developed by historical accident—motivated by the fact that options traders have grown accustomed to thinking of trading in terms of Black–Scholes implied volatility.

The earliest papers that found evidence for volatility smiles did not formulate the results in such terms but instead described how Black–Scholes pricing errors vary systematically with strike price or with time to expiration. MacBeth and Merville (1979), for example, reported that the Black–Scholes model undervalues in-the-money and overvalues out-of-the-money call options. Subsequent authors found the contrary result—that the Black–Scholes model undervalues out-of-the-money calls. These and other relevant results are summarized by Galai (1983).

Of the studies documenting volatility smiles, the most systematic and complete is that of Rubinstein (1985). Rubinstein examined matched pairs of call option transactions from the Berkeley Options Data Base to conduct nonparametric tests of the Black–Scholes null hypothesis that implied volatilities exhibit no systematic differences across strike prices or across time to maturity for otherwise identical options.[21] If deviations from the Black–Scholes model are white noise, the option with the lower strike price should have a higher implied volatility for about half the observations. A similar argument applies for the option with the shorter time to maturity.

Rubinstein's most robust result is that for out-of-the-money calls implied volatility is systematically higher for options with shorter times to expiration. His other results were statistically significant but changed across subperiods. He divided the sample into two subperiods: Period I from August 23, 1976, to October 21, 1977, and Period II from October 24, 1977, to August 31, 1978. For at-the-money calls, Rubinstein found that in Period I, implied volatility for options with short times to expiration was higher than for those with longer times to expiration but that the result was the opposite in Period II. Moreover, in Period I, implied volatility was higher for options with lower striking prices, but again, the result was reversed in Period II. Thus, systematic deviations from the Black–Scholes model appear to exist, but the pattern of deviations varies over time.[22]

Subsequent studies by Sheikh (1991) and by Heynen (1994) used Rubinstein's nonparametric tests to examine implied volatility patterns in index options. Sheikh found smile effects using transactions data for OEX call options from March 1983 to March 1987. He observed that the smiles constitute evidence against the Black–Scholes model and in favor of an option pricing model that incorporates stochastic volatility. Heynen examined the implied volatility of European Options Exchange (EOE) stock-index options, which are European style options on an index of 25 active stocks on the Amsterdam Stock Exchange. Using Rubinstein's approach and transactions data from January 23 to October 31, 1989, he found systematic smile effects, including a U-shaped term structure of implied volatility. He reviewed the predictions of various stochastic volatility models, found the observed smile pattern to be inconsistent with them, and suggested an alternative explanation, based on market imperfections, for the volatility smile.

Many other authors found evidence for volatility smiles and nonflat term structures of implied volatilities in various markets. Shastri and Tandon (1986), for example, used the Geske–Johnson approach (see Geske and Johnson 1984) to price American options on futures and found volatility-smile and term-structure effects in the markets for options on S&P 500 futures and on deutschemark futures. They also suggested using yesterday's estimate of implied volatility as an input for today's option pricing model and examined the performance of this approach relative to using historical volatility. Xu and Taylor (1994) examined the term structure of volatility implied by options on four Philadelphia Stock Exchange currency options using data from 1985 to 1989. Heynen, Kemna, and Vorst (1994) examined the ability of various GARCH models to explain the observed term structure of implied volatilities.

In short, the discrepancy between market prices and the predictions of the Black–Scholes model has attracted a great deal of interest. But again, looking at Black–Scholes volatility smiles is awkward—in essence, we must assume that they do not exist in order to derive them. The next section of this article describes another, more adequate method for dealing with the observed failures of the Black–Scholes model.

THE PROBABILITY DISTRIBUTION IMPLIED BY OPTION PRICES

Although volatility smiles have only recently come to the attention of the academic community, their existence and even their shape have long been familiar to savvy individuals and institutions familiar with options markets. The reality of volatility smiles has led to general dissatisfaction with the Black–Scholes and other models that are inconsistent with them. Recent developments in the academic literature (and certainly within the realm of proprietary research) have considered models that either allow for volatility smiles or explicitly incorporate them into the option pricing process.

One approach is to use stochastic volatility models, the focus of one of the most productive research areas in option pricing during the past few years. If the price of the underlying asset is assumed to follow a sufficiently complicated stochastic process, nearly any type of volatility smile can be generated.

Another approach is to use the information contained in option prices to estimate the risk-neutral density of the terminal stock price. This method relies on a general result from modern option pricing theory: Under certain conditions, a contingent claim that depends on the terminal stock price (and that cannot be exercised early) can be priced by describing the contract as a bundle of state-contingent claims, multiplying the payoff in each state by the corresponding "Arrow–Debreu" state price, and summing across states.[23] Thus, given N different states, the time t price of a contingent claim expiring at time T would be calculated by the equation

$$C(t) = \sum_{s=1}^{N} V(s)p(s), \tag{6}$$

where $V(s)$ describes the payout at time T and $p(s)$ the Arrow–Debreu price of state s. For simplicity, suppose that the risk-free rate, denoted by r, is constant. Because the owner of a complete set of state-contingent claims is guaranteed one dollar at expiration, the sum of the state prices must then be $e^{-r(T-t)}$, the value of one dollar discounted to the risk-free rate. If equation (6) is rewritten as

$$\begin{aligned} C(t) &= \sum_{s=1}^{N} e^{-r(T-t)}V(s)\frac{p(s)}{e^{-r(T-t)}} \qquad (7) \\ &\equiv \sum_{s=1}^{N} e^{-r(T-t)}V(s)\pi(s), \end{aligned}$$

then the $\pi(s)$ sum to 1. Because neither state prices nor discount rates can be negative, each $\pi(s)$ is non–negative, and the set of $\pi(s)$ terms has the two essential properties of a probability density.

In fact, if market participants were risk neutral, then for each state, these $\pi(s)$ terms would be the same as the objective probability of that state. In other words, the set of state-contingent prices divided by the discount factor would simply be the underlying probability density. This relation is why the $\pi(s)$ terms are often called "risk-neutral probabilities," which together form the "risk-neutral density."

When the state space is continuous, the price of a contingent claim is derived by integrating the payoff over a density function (the risk-neutral density) of the underlying asset and then discounting at the risk-free rate. The price of a contingent claim at time t is given by the equation

$$C(t) = e^{-r(T-t)} \int_0^\infty V(s)f(s)ds, \tag{8}$$

where $f(s)$ is the risk-neutral density. For a call option, $V(s) = \max(0, s - K)$, and for a put option, $V(s) = \max(0, K - s)$, where K represents the strike price.

The traditional way to implement this result is to impose a restriction such as the Black–Scholes (constant-variance) assumption on the underlying asset process and to use the resulting density $f(s)$ to price options. The new approach, suggested in recent papers by Rubinstein (1994) and other authors cited therein, is to start with the market prices of options, then find some density $f(s)$ that is consistent with those prices, given the pricing equation (8). The variance of this market-implied distribution may then be used as a measure of future volatility over the remaining life of the option.[24] Potentially, the implied risk-neutral distribution contains even richer information about the market's expectations for future movements in the underlying asset. This distribution can then be used, for example, to calibrate a binomial or trinomial tree that is consistent with the observed prices of all options. Methods of incorporating the volatility smile into tree-based option pricing models have been suggested by Rubinstein (1994), Derman and Kani (1994), and Dupire (1994).

Kuwahara and Marsh (1994) used Rubinstein's method in their investigation of Japanese equity warrant pricing. They found that the probability density implied by Nikkei index options is negatively skewed, as is the case for S&P index options, but that the density implied by a cross-section of warrants is positively skewed. Equivalently, the volatility smile seems to be downward sloping for index options and upward sloping for the warrants. A good deal of theoretical and empirical research remains to be done on this topic.

Rubinstein (1994) discussed three methods for estimating the risk-neutral density implied by option prices. The first, attributed to Francis Longstaff, is simply to derive a step-function approximation to the risk-neutral density, where the step function is as coarse as the interval between successive strike prices of traded options (five dollars for most options traded in the United States). Rubinstein showed that for some parameter values, this method yields poor results.[25]

A second method, introduced by Shimko (1991), relies on the fact that the risk-neutral density is equal to the second derivative of the call price with respect to the strike price.[26] Thus, if a continuum of option prices were observable, the risk-neutral density would also be observable. In fact, option prices are only observable for a few, discretely spaced strike prices. Shimko suggested estimating the option price as a continuous function of the strike price by interpolating the prices of market-traded options, then deriving the risk-neutral density from the interpolated values. Of course, call option prices can be interpolated in many ways. Shimko's approach is to calculate the Black–Scholes implied volatility for each option, then use least squares to fit a quadratic function to the volatility smile. The fit values from the least squares procedure give implied volatility as a continuous function of the strike price, which may then be mapped back through the Black–Scholes equation to obtain a continuum of call option prices, which in turn yields the risk-neutral density.

Rubinstein (1994) suggested a third method, which is to choose the distribution that is closest (in a least-squares sense) to some "prior" distribution, subject to the constraint that the rational option prices derived from the chosen distribution must fall within the observed bid–ask spread for each traded option. A comprehensive investigation of the merits of alternative priors or objective functions constitutes a worthy agenda for further research.[27]

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FOOTNOTES

1. It is important to understand, however, that average *instantaneous* volatility is not the same thing as total variance over the remaining life of the option.
2. For example, Engle and Mustafa (1992) showed how to use option prices to estimate the volatility parameters when the underlying asset follows a GARCH process.
3. See the paper by Bodurtha and Shen (1994), who used the prices of currency options not only to calculate implied volatilities for two exchange rates but also to estimate the implied correlation between them.
4. Amin and Morton (1994), for example, used Eurodollar futures and options data to back out the implied volatility of interest rates in the Heath–Jarrow–Morton term-structure model.
5. In general, σ may be a vector of parameters describing the underlying stock process.
6. The bisection method is to bracket the root, then repeatedly cut the bracket in half to converge on the root.
7. The Newton–Raphson root-finding method speeds up convergence by taking advantage of information in the function's first derivative. Manaster and Koehler (1982) described how to choose a starting value for the first iteration to ensure that the Newton–Raphson algorithm will converge whenever a solution actually exists. For a review of how to apply both bisection and Newton–Raphson methods, see Kritzman (1991).
8. The option with the highest vega is usually very close to the money.
9. Brenner and Galai (1982) also examined the time series properties of these implied volatility estimates. Harvey and Whaley (1991) demonstrated the potential for microstructural data imperfections such as bid–ask spreads and asynchronous observation of option prices and the underlying stock price to bias implied volatility estimates and to induce spurious negative autocorrelation in the time series of implied volatility.
10. The Garman–Kohlhagen formula is similar to the Black–Scholes formula with the foreign interest rate acting as a continuous dividend yield.
11. For additional discussion on this issue, see the papers by Gastineau and Madansky (1979, 1984) and Eckardt and Williams (1984).
12. This branch of literature includes the papers of Parkinson (1980), Garman and Klass (1980), and Kunitomo (1992).
13. The GARCH class of time-series models allows the variance of the innovation to depend on lagged innovations and lagged levels of the process and to revert to a long-run mean. See Bollerslev (1986).
14. Technically, the market's variance forecast error at time t should be orthogonal to any information available at time t.
15. One potential criticism of this type of test is that biases can arise from the facts that (1) the implied variance from the Hull–White type models is only an approximation to the true subjective implied variance, and (2) the Brownian motions corresponding to the stock process and the volatility process may be correlated. To address these potential biases, the authors presented a simulation that investigated the effect of such misspecification on the results. They found that the bias is never more than 1.3 percent of the true variance.

16. Specifically, he used generalized method of moments estimation.
17. The wildcard option is the option to exercise an option after the settlement price is fixed. For example, OEX options may be exercised until 3:15 (CST), but the settlement price is fixed at 3:00.
18. The authors also found a strong U-shaped pattern in intraday historical volatility but no significant pattern in intraday implied volatility.
19. More generally, with discretely sampled data, the estimate of the diffusion term depends on the drift term.
20. Equivalently, the "risk-neutral" density function of the terminal stock price is not lognormal.
21. For a description of the Berkeley Options Data Base, see Rubinstein and Vijh (1987).
22. For a similar study using more recent data, see Culumovic and Welch (1994).
23. The Arrow–Debreu price is the price of a state-contingent claim that pays one dollar if the state occurs and zero otherwise. In this case, the "states" correspond to different terminal values of the underlying stock.
24. Remember, however, that this market-implied distribution is not the variance of the true probability distribution but rather of the distribution in a risk-neutral world. The theoretical relationship between the two, however, depends on the equilibrium price process in the economy, which in turn depends on investor preferences.
25. Mayhew (1995) generalized this approach by introducing a class of spline estimators.
26. This fact was first demonstrated by Breeden and Litzenberger (1978). See also Shimko (1993).
27. I would like to thank David Modest for guidance and support. I would also like to thank Mark Rubinstein, Terry Marsh, Bill Keirstead, and Van Harlow for valuable comments.