Math 122-A Winter 2022 HOMEWORK# 5 (due February. 17)

1. Let $z_0 \in \mathbb{C}$ be any interior point to any positive oriented simple closed curve C. Prove

$$\oint_C \frac{dz}{z-z_0} = 2\pi i, \quad \oint_C \frac{dz}{(z-z_0)^{n+1}} = 0, n = 1, 2, 3, \dots$$

$$\oint_C \frac{dz}{z-z_0} = 2\pi i$$
$$C : [0, 2\pi] \to \mathbb{C}$$
$$t \to z_0 + Re^{it}$$

Note that $C(t) = z_0 + Re^{it}$ and $C'(t) = iRe^{it}$

$$\oint_C \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{C'(t)}{C(t) - z_0} dt$$
$$= \int_0^{2\pi} \frac{iRe^{it}}{Re^{it}} dt$$
$$= i \int_0^{2\pi} dt \ = 2\pi i$$

$$\oint_C \frac{dz}{(z-z_0)^{n+1}} = 0, n = 1, 2, 3, \dots$$
$$C : [0, 2\pi] \to \mathbb{C}$$
$$t \to z_0 + Re^{it}$$
$$\oint_{C \frac{dz}{(z-z_0)^{n+1}}}$$
$$= \int_0^{2\pi i} \frac{iRe^{it}}{(Re^{it})^{n+1}}$$
$$= \frac{i}{r^n} \int_0^{2\pi} e^{-it(n+1)} dt$$
$$= \frac{i}{r^n} \frac{e^{it(n+1)}}{i(n+1)} \Big|_0^{2\pi}$$
$$= 0$$

2. Let C be the contour of the circle $|z - i| = 2$ in the positive sense. Find

(a) $\oint_C \frac{dz}{z^2+4}$,

$$= \oint_C \frac{dz}{(z+2i)(z-2i)}$$

$$= \oint_C \frac{\frac{1}{z+2i}}{z-2i} dt, \quad f(z) = \frac{1}{z+2i} \quad z_0 = 2i$$

$$= 2\pi i (\frac{1}{2i+2i}) = \frac{\pi}{2}$$

(b) $\oint_C \frac{e^z dz}{z - \pi i/2}$,

$$f(z) = e^z \quad z_0 = \frac{\pi i}{2}$$

$$= (e^{\pi i/2}) 2\pi i$$

$$= i(s\pi i) = -2\pi$$

(c) $\oint_C \frac{\cos(z) dz}{(z^2+16)\, z}$

$$= \oint_C \frac{\frac{cos(z)}{x^2+16}}{z} dz, \quad f(z) = \frac{cos(z)}{x^2+16} \quad z_0 = 0$$

$$= 2\pi i (\frac{cos(0)}{0^2+16}) = \frac{\pi i}{8}$$

(d) $\oint_C \frac{dz}{2z+1}$.

$$= \oint_C \frac{1/2}{z - (-\frac{1}{2}} dz$$

$$f(z) = \frac{1}{2} \quad z_0 = -\frac{1}{2}$$

$$= \frac{1}{2}(2\pi i) = \pi i$$

3. For $z \in \mathbb{C}$ with $|z| \neq 3$, denote C the contour of the circle $|z| = 3$ in the positive sense and define

$$g(z) = \oint_C \frac{2w^2 - w - 2}{w - z} dw.$$

Find the values of $g(2)$ and $g(3+2i)$.

$$g(2) = 2(2)^2 - 2 - 2(2\pi i)$$

$$= 4(2\pi i) = 8\pi i$$

3+2i is outside of C, therefore g(3+2i) = 0

4. Assuming that the given contour ids positive oriented, compute

(a) $\displaystyle\oint_{|z|=3} \frac{(e^z+z)\,dz}{z-2}$,

$$z_0 = 2, \quad f(z) = e^z + z$$

$$f(2) = (e^2+2)(2\pi i)$$

(b) $\displaystyle\oint_{|z|=1} \frac{e^z dz}{z^2}$,

$$f(z) = e^z \quad z_0 = 0$$

$$(2\pi i)f(0) = 2\pi i$$

(c) $\displaystyle\oint_{|z|=2} \frac{dz}{z^2+z+1}$,

$$= \oint_{|z|=2} \frac{dz}{z-(\frac{-1+\sqrt{3}i}{2})(z-(\frac{-1-\sqrt{3}i)}{2})}$$

$$f(z) = \frac{1}{z-(-\frac{-1-\sqrt{3}i}{2}} \quad z_0 = \frac{-1}{2} + \frac{\sqrt{3}i}{2}$$

$$= \frac{2\pi i}{(\frac{-1+\sqrt{3}i}{2}) - (\frac{-1-\sqrt{3}i}{2})}$$

$$= \frac{2\pi}{\sqrt{3}}$$

Now, let

$$f(z) = \frac{1}{z-(-\frac{-1+\sqrt{3}i}{2}} \quad z_0 = \frac{-1}{2} - \frac{\sqrt{3}i}{2}$$

$$= \frac{2\pi i}{(\frac{-1-\sqrt{3}i}{2}) - (\frac{-1+\sqrt{3}i}{2})}$$

$$= -\frac{2\pi}{\sqrt{3}}$$

Now we must add them together

$$\frac{2\pi}{\sqrt{3}} + (-\frac{2\pi}{\sqrt{3}}) = 0$$

(d) $\displaystyle\oint_{|z|=1} \frac{dz}{z^2-1}$.

Professor Ponce said we don't have the skills to do this problem yet.

DEFINITION: A $f:\mathbb{C}\to\mathbb{C}$ is an ENTIRE function if f is analytic in all $\mathbb{C}$.

5. Prove that if f is entire and there exist $z_0 \in \mathbb{C}$ and $r > 0$ such that

$$f(\mathbb{C}) \cap \{z \in \mathbb{C} : |z - z_0| < r\} = \emptyset$$

then f is a constant function.

Let $g = \frac{r}{f(z)-z_0}$ We know g is entire $\frac{r}{z}$ is analytic except when $f(z) - z_0 = 0$ which can't happen. We can also say that g is bounded because $|g| = \left|\frac{r}{f(z)-z_0}\right| = \frac{r}{f(z)-z_0}$ By our hypothesis, $|f(z) - z_0 \geq r \to 1 \geq \frac{r}{|f(z)-z_0}$ So, since g is entire and bounded, g is a constant by Liouvilles theorem. So,

$$c = g = \frac{r}{f(z) - z_0}$$

$$r = c(f(z) - z_0)$$

$$f(z) = \frac{r}{c} + z_0$$

6. Identify all entire functions f such that $\forall z \in \mathbb{C} \quad |f(z)| \leq 2|z|$.

$$f^n(z_0) = |\frac{n!}{2\pi i}| \int |\frac{dw}{(w - z_0)^{n+1}} dw|$$

Now,

$$|\frac{dw}{(w - z_0)^{n+1}} dw| \leq \frac{M}{R^{n+1}}$$

Since, $|w - z_0 = R$

$$\to |f^n(z_0) \leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} l(r)$$

Therefore, every derivative is bounded $|f| \leq |2z|$ We claim: $f = az + b$ So its enough to show that f"(z)= 0, f'(z)=a and f(z)= az+b.
To show f"(z)=0:

$$f'(z_0) = \frac{2!}{2\pi i)} \oint \frac{f(z)}{(z - z_0)^3} dz$$

$$\gamma(t) = z_0 + Re^{it} \gamma : [0, 2\pi] \quad \gamma'(t) = iRe^{it}$$

$$f''(z_0) = \frac{1}{\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it} iRe^{it}}{R^3 e^{3it}} dt$$

$$|f''(z_0)| = |\frac{1}{\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it} iRe^{it}}{R^3 e^{3it}} dt| \leq \frac{1}{\pi} \int_0^{2\pi} \frac{f(z_0 + Re^{it} iRe^{it}}{R^2} dt$$

$$\leq \frac{1}{\pi} \int_0^{2\pi} \frac{2|z_0| + 2|Re^{it}|}{R^2} dt$$

By the triangle inequality

$$= \frac{1}{\pi} \frac{2|z_0| + 2|R|}{R^2} t \Big|_0^{2\pi}$$

$$|f''(z_0) = \leq \frac{4|z_0 + 4R}{R^2}$$

$$\frac{|f''(z_0)| - \frac{4}{R}}{|z_0|} \leq \frac{4}{R^2}$$

As R goes to infinity,

$$\frac{|f''(z_0)|}{|z_0} = 0$$

$$f''(z_0) = 0$$

By the Louiville theorem, since f"(z)=0, f'(z) must be a constant.

$$\int f'(z) = az + b = f(z)$$

$$|f(z)| \leq 2|z|$$

$$|b| \leq 0 \quad f(z) = az + b \quad f(0) = b$$

This cannot happen, so f(z)= az

$$|az| \leq 2|z|$$

$$|a||z| \leq 2|z|$$

$$|a| le 2$$