Math 122-A Winter 2022 HOMEWORK# 6 (due February. 28)

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(1) Consider the function $f(z) = (z+1)^2$ and the region R bounded by the triangle with vertices $0, 2, i$ (its boundary and interior). Find the points where $|f(z)|$ reaches its maximum and minimum value of R.

$$\begin{aligned}
&\gamma_1 : [0,1] \to \mathbb{C} \\
&t \to 2t \\
|f(2t)| = &|(2t+1)^2| = |4t^2+4t+1| \\
&for\ t=0 : |4(0)^2+4(0)+1| = 1 \\
&for\ t=1 : |4+4+1| = 9
\end{aligned}$$

$$\begin{aligned}
&\gamma_2 : [0,1] \to \mathbb{C} \\
&t \to 2-2t+ti \\
&|f(2-2t+ti)| = |(2-2t+ti)^2+1| = |5t^2-12t+9| \\
&for\ t=0 : |5(0)-12(0)+9| = 9 \\
&for\ t=1 : |5-12+9| = 2
\end{aligned}$$

$$\begin{aligned}
&\gamma_3 : [0,1] \to \mathbb{C} \\
&t \to i-ti \\
&|f(i-ti)| = |(i-ti)^2+1| = |t^2+1| \\
&for\ t=0 : |(0)^2+1| = 1 \\
&for\ t=1 : |1+1| = 2
\end{aligned}$$

Therefore, our max is 9 and our min is 1.

(2) Find the maximum of $|\sin(z)|$ on $[0, 2\pi] \times [0, 2\pi]$.

First we must expand sin(z).

$$\begin{aligned}
sin(z) &= \frac{e^{iz}-e^{-iz}}{2i} \\
&= \frac{e^{i(x+iy)}-e^{-i(x+iy)}}{2i} \\
&= \frac{e^{ix}e^{-y}-e^{-ix}e^{y}}{2i} \\
&= \frac{(cosx+isinx)e^{-y}-(cosx-isinx)e^{y}}{2i} \\
&= \frac{1}{2i}(cosx(e^{-y}-e^{y})+isinx(e^{-y}-e^{y}))
\end{aligned}$$

$$= cosx(\frac{-1}{i})(\frac{e^y - e^{-y}}{2}) + sinx(\frac{e^y + e^{-y}}{2})$$

$$= icos(x)sinh(y) + sin(x)cosh(y)$$

Since x is an increasing function, the max of —sin(z)— and $|sin(z)|^2$ will be the same. So we can now use $|sin(z)|^2$ to find our max.

$$|sin(z)|^2 = cos^2(x)sinh^2(y) + sin^2(x)cosh^2(y)$$

$$= cos^2(x)sinh^2(y) + sin^2(x)(1 + sinh^2(y))$$

$$= (cos^2(x) + sin^2(x)) + (sin^2(x) + sinh^2(y))$$

$$|sin(z)|^2 = sin^2(x) + sinh^2(y)$$

Now, using this function, we can try to find out max.

$$sin^2(x) = 1$$

$$x = \pi/2, 3\pi/2$$

Since $sinh^2(y)$ is a hyperbolic function, it reaches its max at the bound. So, $y = 2\pi$

(3) Calculate:

(a) $\int_0^{2\pi} \frac{d\theta}{a+b\cos(\theta)} \quad 0 < b < a.$

HINT: Work backward using that $\cos(\theta) = \left(e^{i\theta} + e^{-i\theta}\right)/2$ to convert the integral into a complex integral along the curve $|z| = 1$

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos(\theta)} \quad 0 < b < a$$

$$= \int_0^{2\pi} \frac{d\theta}{a + b(\frac{e^{i\theta}+e^{-i\theta}}{2})}$$

Let $z = e^{i\theta}$ and $\frac{1}{z} = e^{-i\theta}$ $dz = ie^{i\theta}d\theta$ $\frac{1}{iz}dz = d\theta$

$$\frac{1}{i}\oint_{|z|=1} \frac{dz}{az + \frac{1}{2}bz(z + \frac{1}{z})}$$

$$= \frac{2}{i}\oint_{|z|=1} \frac{dz}{2az + bz^2 + 1}$$

Now, to find our singularities in $|z| = 1$ we isolate f(z) and solve.

$$f(z) = \frac{1}{bz^2 + b + 2az}$$

$$z = \frac{-2a \pm 2\sqrt{a^2 - b^2}}{2b}$$

Case 1: both points are in the circle

$$\oint_{|z|=1} \frac{dz}{(z-(\frac{-a+\sqrt{a^2+b^2}}{b})(z-(\frac{-a-\sqrt{a^2+b^2}}{b})}$$

Now let $z_0 = \frac{-a}{b} + \frac{\sqrt{a^2+b^2}}{b}$ and $f(z) = \frac{1}{(z-(\frac{-a-\sqrt{a^2+b^2}}{b})}$

$$= \frac{2\pi i}{(\frac{-a+\sqrt{a^2+b^2}}{b}) - (\frac{-a+\sqrt{a^2-b^2}}{b})}$$

$$= \frac{4\pi i\sqrt{a^2+b^2}}{b}$$

If we let

$$z_0 = \frac{-a}{b} + \frac{\sqrt{a^2+b^2}}{b}$$

Then we get

$$= -\frac{4\pi i\sqrt{a^2+b^2}}{b}$$

$$= \frac{4\pi i\sqrt{a^2+b^2}}{b} + -\frac{4\pi i\sqrt{a^2+b^2}}{b} = 0$$

Case 2: Both points are outside of —z—=1 By Cauchy 1, $\oint f(z)dz = 0$

Case 3: One point out, one point in. Since $0 \le b \le a$, the only point we have to worry about is $z = \frac{-a}{b} + \frac{\sqrt{a^2+b^2}}{b}$

$$\oint_{|z|=1} f(z) = \frac{f(z)}{z-z_0}$$

$$= 2\pi i(f(z_0))$$

$$= 0$$

(b) $\int_0^{2\pi} \frac{d\theta}{(a+b\cos(\theta))^2}$

Let

$$u = a + bcos(\theta) \qquad \frac{d}{du}(\frac{1}{u}) = \frac{-1}{u^2}$$

$$\int_0^{2\pi} \frac{1}{u^2} du = -\int_0^{2\pi} \frac{d}{du}(\frac{1}{u})$$

$$= \frac{1}{u}\Big|_0^{2\pi}$$

$$= \frac{-1}{a+bcos(\theta)}\Big|_0^{2\pi}$$

$$= 0$$

(c) $\int_0^{2\pi} \frac{\sin(\theta)d\theta}{(a+b\cos(\theta))^2}, \quad 0 < b < a.$

$$u = a + bcos(\theta) \qquad du = -bsin(\theta)d\theta \qquad \frac{du}{-b} = sin(\theta)d\theta$$

$$\frac{-1}{b}\int_0^{2\pi} \frac{1}{u^{-2}}du$$
$$= \frac{1}{b}(\frac{1}{u}\Big|_0^{2\pi})$$
$$= \frac{1}{b}(\frac{1}{a+bcos(\theta)}\Big|_0^{2\pi})$$
$$= 0$$

(4) Prove that if $f : \mathbb{C} \to \mathbb{C}$ is entire such that for some $n \in \mathbb{N}$

$$\lim_{|z|\to\infty} \frac{|f(z)|}{|z|^n} = M < \infty$$

then f is a polynomial of degree at most n.

PROOF: First we must consider Cauchy's inequality,

$$|f^{n+1}(z_0)| \leq (n+1)(|z_0 + Re^{it}|) \leq (n++1)(|z_0 + R|)$$

Now we must first prove that $f^{n+1}(z_0) = 0, \forall z \in \mathbb{C}$

$$f^{n+1}(z_0) = \frac{(n+1)!}{2\pi i}\int_{|z-z_0|=R|} \frac{f(z)}{(z-z_0)^{n+2}}dz$$
$$= \frac{1}{\pi i}\int_0^{2\pi} \frac{f(z_0 + Re^{it}iRe^{it}}{R^{n+2}e^{i(n+2)t}}dt$$

Since $z(t) = z_0 + Re^{it}, t \in [0, 2\pi]$

$$|f^{n+1}(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(z_0 + Re^{it})|}{R^{n+1}}dt$$
$$\lim_{R\to\infty} \frac{(n+1)(|z_0| + R)}{R^{(n+1)}} = 0$$
$$\to |f^{n+1}(z_0)| \leq 0$$

Since the derivative of $f^{n+1} = 0$, f^n must be a constant. Our hypothesis is true, therefore f is a polynomial of degree at most n.

(5) Let $A \subset \mathbb{C}$ be an open set and $f : A \to \mathbb{C}$ be an analytic function on A. Assuming that $z_0 \in A$ such that at z_0

$$\{z \in \mathbb{C} : |z - z_0| \leq R\}, \quad R > 0$$

$f(z_0) = \frac{1}{\pi R^2} \iint_{|z-z_0|\leq R} f(x+iy)dxdy.$

First, we know that

$$f(z) = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(w)}{w-z}$$

If we parameterize the Cauchy integral from above, we get:

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})d\theta$$

$$\rightarrow \int_0^R f(z)dr = \int_0^R \int_0^{2\pi} f(z+re^{ie^{i\theta}})d\theta dr$$

Multiplying both sides by r we get

$$\int_0^R f(z)rdr = \frac{1}{2\pi}\int_0^R \int_0^{2\pi} f(z+re^{i\theta})r\ drd\theta$$

$$\rightarrow f(z)\int_0^R rdr = \frac{1}{2\pi}\int_0^R \int_0^{2\pi} f(z+re^{i\theta})r\ drd\theta$$

$$\rightarrow \frac{1}{2}R^2 f(z) = \frac{1}{2\pi}\int_0^R \int_0^{2\pi} f(z+re^{i\theta})r\ drd\theta$$

$$\rightarrow \frac{1}{2}R^2 f(z) = \frac{1}{2\pi}\int_{|(x+iy)|-z\leq R} f(x+iy)\ dxd\theta$$

$$\rightarrow f(z) = \frac{1}{\pi R^2}\int_{|(x+iy)|-z\leq R} f(x+iy)\ dxd\theta$$

(6) Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be defined as

$$f(x) = e^{-1/x^2} \quad \text{if} \quad x\neq 0, \quad f(0)=0$$

Show that f is infinitely differentiable and $\forall n\in\mathbb{N} \quad f^{(n)}(0)=0$. Verify that the power serie of f at $x=0$ does not agree with f in any neighborhood of 0 .

First we test f'(0)= 0.

$$f'(0) = \lim_{h\rightarrow 0}\frac{f(h)-f(0)}{h}$$

$$= \lim_{h\rightarrow 0}\frac{e^{\frac{-1}{h^2}}-0}{h}$$

$$= \lim_{h\rightarrow 0}\frac{e^{\frac{-1}{h^2}}}{h}$$

Let $x=\frac{1}{h}$

$$\rightarrow \lim_{x\rightarrow\infty} e^{-x^2}x$$

$$= \lim_{x\to\infty} \frac{x}{e^{x^2}} = 0$$

Now we test f"(0)=0.

$$f''(0) = \lim_{h\to 0} \frac{f'(h) - f'(0)}{h}$$

$$= \lim_{h\to 0} \frac{f'(h)}{h}$$

$$\lim_{h\to 0} \frac{(e^{\frac{1}{h^2}})'}{h}$$

$$= \lim_{h\to 0} \frac{(e^{\frac{1}{h^2}})\frac{2}{h^3}}{h}$$

$$= \lim_{h\to 0} \frac{2e^{\frac{1}{h^2}}}{h^4}$$

$$= \lim_{h\to 0} \frac{\frac{2}{h^4}}{e^{\frac{1}{h^2}}}$$

Let $t = \frac{1}{h}$

$$\lim_{t\to\infty} \frac{2t^4}{e^{t^2}} = 0$$

No matter how many times we take the derivative, e^{h^2} will always be in the denominator, telling us that the limit of $f^n(0)$ will always equal 0.