Math 122-A Winter 2022 HOMEWORK# 4 (due February. 10)

1. In each case find all the values of z such that:

   (a) $z = i^i$
   From previous proofs we know

$$i = e^{i(\frac{\pi}{2}+(2\pi)k)} \; for \; k \in \mathbb{Z}$$

   So,

$$\begin{aligned} z &= (e^{i(\frac{\pi}{2}+(2\pi)k)})^i \\ &= e^{i^2(\frac{\pi}{2}+(2\pi)k)} \\ &= e^{-\frac{\pi}{2}-(2\pi)k} \end{aligned}$$

   (b) $z = (1-i)^{1+i}$

$$\begin{aligned} z &= e^{ln(1-i)^{1+i}} \\ &= e^{1+i(ln(1-i))} \end{aligned}$$

   Now to find the polar form of the equation,

$$|1-i| = \sqrt{2}$$

$$arg(1-i) = \frac{-\pi}{4} + 2\pi(k)$$

   Inserting this into the equation we get

$$z = e^{(1+i)(ln(\sqrt{2})+i(\frac{-\pi}{4}+2\pi(k))}$$

$$z = e^{(1-i)(\frac{ln2}{2})+i(\frac{\pi}{4}+2\pi(k))}$$

$$= e^{\frac{ln(2)}{2}-\frac{\pi}{4}+2\pi(k)}e^{i\frac{ln(2)}{2}-\frac{\pi}{4}+2\pi(k)} \; k \in \mathbb{Z}$$

   (c) $e^{1/z} = 1 + i\sqrt{3}$.

$$\begin{aligned} e^{1/z} &= 1 + i\sqrt{3} \\ = ln(e^{\frac{1}{z}} &= ln(1+i\sqrt{3}) \\ = \frac{1}{z} &= ln(1+i\sqrt{3}) \end{aligned}$$

   To convert the equation into polar form we have,

$$\sqrt{(1)^2 + (\sqrt{3})^2} = 2$$

$$tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

   So,

$$\begin{aligned} \frac{1}{z} &= ln(2) + i(\frac{\pi}{2} + 2\pi(k) \\ z &= \frac{1}{ln(2) + i(\frac{\pi}{2} + 2\pi(k)} \\ &= \frac{ln(2) + i(\frac{\pi}{2} + 2\pi(k)}{ln^2(2) + \frac{\pi}{3} + 2\pi(k)} \end{aligned}$$

2. For any $z \in \mathbb{C}$ define:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \sinh(z) = \frac{e^{z} - e^{-z}}{2}, \cosh(z) = \frac{e^{z} + e^{-z}}{2}$$

Prove the following identities hold.

(a) $\sin(-z) = -\sin(z), \quad \cos(-z) = \cos(z), \sin(-z) = -\sin(z)$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\rightarrow \sin(-z) = \frac{e^{i(-z)} - e^{-i(-z)}}{2i}$$

$$= \frac{e^{-iz)} - e^{iz}}{2i}$$

$$== \frac{-e^{i(-z)} + e^{-i(-z)}}{2i}$$

$$= -sin(z)$$

$$\cos(-z) = \cos(z)$$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\cos(-z) = \frac{e^{i(-z)} + e^{-i(-z)}}{2}$$

$$= \frac{e^{-iz} + e^{iz}}{2}$$

= -cos(z)

(b) $\sin^2(z) + \cos^2(z) = 1, \quad \cosh^2(z) - \sinh^2(z) = 1,$

$$sin^2(z) + cos^2(z) = 1$$

$$= (\frac{e^{iz} - e^{-iz}}{2i})^2 + (\frac{e^{iz} + e^{-iz}}{2})^2 = 1$$

$$= \frac{(e^{iz})^2 - 2e^{iz}e^{-iz} + (e^{-iz})^2}{-4} + \frac{(e^{iz})^2 + 2e^{iz}e^{-iz} + (e^{-iz})^2}{4} = 1$$

$$= \frac{e^{2iz} - 2e^0 + e^{-2iz}}{-4} + \frac{e^{2iz} + 2e^0 + e^{-2iz}}{4} = 1$$

$$= \frac{-2}{-4} + \frac{2}{4} = 1$$

$$1 = 1$$

$$cosh^2(z) - sinh^2(z) = 1$$

$$= (\frac{e^z + e^{-z}}{2})^2 - (\frac{e^z - e^{-z}}{2})^2 = 1$$

$$\frac{(e^{2z}) + 2e^z e^{-z} + (e^{-2z})}{4} - \frac{(e^{2z})^2 + 2e^z e^{-z} + (e^{-2z})}{4} = 1$$

$$\frac{1}{2} + \frac{1}{2} = 1$$

(c) $\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$,

$$= (\frac{e^{iz_1} + e^{-iz_1}}{2})(\frac{e^{iz_2} + e^{-iz_2}}{2}) - (\frac{e^{iz_1} - e^{-iz_1}}{2i})(\frac{e^{iz_2} - e^{-iz_2}}{2i})$$

$$= \frac{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)} + e^{i(z_1-z_2)} + e^{i(z_2-z_1)}}{4} + \frac{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)} - e^{i(z_1-z_2)} - ^{i(z_2-z_1)}}{4}$$

$$= \frac{2e^{i(z_1+z_2)} + 2e^{-i(z_1+z_2)}}{4}$$

$$= \frac{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}}{2} = cos(z_1 + z_2)$$

(d) $\cosh(z_1 + z_2) = \cosh(z_1)\cosh(z_2) + \sinh(z_1)\sinh(z_2)$,

$$(\frac{e^{z_1} + e^{-z_1}}{2})(\frac{e^{z_2} + e^{-z_2}}{2}) - (\frac{e^{z_1} - e^{-z_1}}{2})(\frac{e^{z_2} - e^{-z_2}}{2})$$

$$= \frac{e^{(z_1+z_2)} + e^{-(z_1+z_2)} + e^{(z_1-z_2)} + e^{(z_2-z_1)}}{4} + \frac{e^{(z_1+z_2)} + e^{-(z_1+z_2)} - e^{(z_1-z_2)} - ^{(z_2-z_1)}}{4}$$

$$= \frac{2e^{(z_1+z_2)} + 2e^{-(z_1+z_2)}}{4}$$

$$= \frac{e^{(z_1+z_2)} + e^{-(z_1+z_2)}}{2} = cos(z_1 + z_2)$$

(e) $(\sin(z))' = \cos(z)$, $\quad (\cos(z))' = \sin(z)$,

$$(\sin(z))' = (\frac{e^{iz} - e^{-iz}}{2i})'$$

$$= (\frac{e^{iz}}{2i}(\frac{d}{dz}) - (\frac{e^{-iz}}{2i})(\frac{d}{dz})$$

$$= \frac{ie^{iz}}{2i} - \frac{-ie^{-iz}}{2i}$$

$$= \frac{e^{iz} + e^{-iz}}{2} = cos(z)$$

$$(cos(z))' = (\frac{e^{iz} + e^{-iz}}{2})'$$

$$= \frac{ie^{iz} - ie^{-iz}}{2}$$

$$= \frac{i^2(e^{iz} - e^{-iz}}{2i}$$

$$= \frac{e^{iz} + e^{-iz}}{2i} = -sin(z)$$

(f) $(\sinh(z))' = \cosh(z), \quad (\cosh(z))' = \sinh(z),$

$$(\sinh(z))' = (\frac{e^z - e^{-z}}{2})(\frac{d}{dx})$$

$$= \frac{e^z + e^{-z}}{2} = \cosh(z)$$

3. Evaluate the following integrals:

(a)

$$\int_1^2 \left(\frac{1}{t} + i\right)^2 dt,$$

$$= \int_1^2 \left(-(\frac{1}{t})^2 + 2i(\frac{1}{t}) - 1\right) \, dt$$

$$= (-\frac{1}{t} + 2iln(\frac{1}{t}) - t)\Big|_1^2$$

$$= (\frac{1}{2} + 2iln(\frac{1}{2}) - 2) - (-1 + 0 + -1)$$

$$= 2iln(\frac{1}{2}) - \frac{1}{2}$$

(b)

$$\int_0^{\pi/3} e^{it} dt,$$

$$= ie^{it}\Big|_0^3$$

$$= i(e^{i\frac{\pi}{3}} - 1)$$

(c)

$$\int_0^{2\pi} e^{ikt} dt, \quad k \in \mathbb{Z}.$$

$$= ike^{ikt}\Big|_0^2 \pi$$

$$= ik(e^{ik2\pi} - 1) \ \ k \in \mathbb{Z}$$

4. In each case write the equation of the curve representing:

(a) the segment joining i and i,

$$\gamma : [0, 1] \to \mathbb{C}$$

$$\to \gamma(t) = (1 - t)z_0 + tz_1$$

$$= (1 - t) + ti$$

(b) the circumference of center $1 - i$ and radius 2, in the counter clockwise,

$$\gamma : [0, 2\pi] \to \mathbb{C}$$

$$t \to (cos(t) + isin(t))$$

$$\gamma(t) = 2cos(t) + 1 + i(sint - 1)$$

(c) the triangle with vertices $1, i, -2$

$$\gamma_1 : [0, 1] \to \mathbb{C}$$

$$= 3t + 2$$

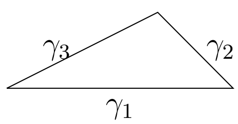
$$\gamma_2 : 0, 1] \to \mathbb{C}$$

$$= (i - 1)t + 1$$

$$\gamma_3 : [0, 1] \to \mathbb{C}$$

$$= -2t - it + 1$$

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3$$



5. Evaluate the following integrals:

(a) $\int_\gamma xdz$, γ the boundary on the unit square
First, we parameterize each line of the square.

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$$

$$\begin{array}{ll} \gamma_1 : [0, 1] \to \mathbb{C} & \gamma_1(t) = (1 - t) + (1 + i)t = 1 + it \\ \gamma_2 : [0, 1] \to \mathbb{C} & \gamma_2(t) = 1 - t + i \\ \gamma_3 : [0, 1] \to \mathbb{C} & \gamma_3(t) = (1 - t)i = i - it \\ \gamma_4 : [0, 1] \to \mathbb{C} & \gamma_4(t) = t \end{array}$$

$$\int_\gamma xdz, \quad \gamma = \int_0^1 f(\gamma_n(t))\gamma_n' dt$$
$$\int_0^1 f(\gamma_1(t))\gamma_1' dt = \int_0^1 idt = i$$
$$\int_0^1 f(\gamma_2(t))\gamma_2' dt = \int_0^1 -(1-t)dt = -\frac{1}{2}$$
$$\int_0^1 f(\gamma_3(t))\gamma_3' dt = \int_0^1 0(-i)dt = 0$$
$$\int_0^1 f(\gamma_4(t))\gamma_4' dt = \int_0^1 tdt = \frac{1}{2}$$
$$\to \int_\gamma xdz = i$$

(b) $\int_\gamma e^z dz, \quad \gamma$ the portion of the unit circle joining 1 and i in the counter clockwise direction.

By a theorem gone over in class, we know that $\int_\gamma e^z dz$ can be written as F(b)-F(a). So,

$$\int_\gamma e^z dz = e^i - e$$

(c) $\int_\gamma xdy, \gamma$ is the boundary of a bounded region $A \subset \mathbb{R}^2$ (without holes) in the counter clockwise direction. HINT : use Green's Theorem.

By Green's Theorem, we know that

$$\oint, P_{(x,y)}dx + Q_{(x,y)}dy = \iint_\Omega (\delta_x Q - \delta_y P)dxdy$$

Let $\Omega = A$, P= 0, and Q= x. Then,

$$\oint x, dy = \iint (\delta_x)dA = A$$