Math 122-A Winter 2022 HOMEWORK# 7-8 (due March. 9th)

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(1) Let $D_1(z_0) = \{z \in \mathbb{C} : |z - z_0| < 1\}$. Let $f, g : D_1(z_0) \to \mathbb{C}$ be two analytic functions on $D_1(z_0)$. Prove that if

$$f^{(n)}(z_0) = g^{(n)}(z_0), \quad n = 0, 1, 2, 3, \ldots\ldots$$

then $f(z) = g(z), \forall z \in D_1(z_0)$.

Let f(z)= $\sum_{n=0}^{\infty}(z-z_0)^n \frac{f^n(z)}{n!}$ and g(z)= $\sum_{n=0}^{\infty}(z-z_0)^n \frac{g^n(z)}{n!}$
Then, f(z) = g(z) implies

$$\sum_{n=0}^{\infty}(z-z_0)^n \frac{f^n(z)}{n!} = \sum_{n=0}^{\infty}(z-z_0)^n \frac{g^n(z)}{n!}$$

$(z-z_0)^n$ and n! cancel leaving $f^n = g^n$.

(2) Let $D_1(z_0) = \{z \in \mathbb{C} : |z - z_0| < 1\}$. Let $f : D_1(z_0) \to \mathbb{C}$ be an analytic function on $D_1(z_0)$ such that it has a zero of order $N \in \mathbb{N}$ at z_0, i.e.

$$f(z_0) = f'(z_0) = \ldots. = f^{N-1}(z_0) = 0, \quad f^n(z_0) \neq 0.$$

(i) Prove that exists $g : D_1(z_0) \to \mathbb{C}$ analytic on $D_1(z_0)$ with $g(z_0) \neq 0$ and $f(z) = (z - z_0)^N g(z)$. f(z) = $\sum_{k=N}^{\infty} \frac{f^k(0)}{k!} z^k$

$$= z^n(\frac{f^n(0)}{N!} + a_1 z + a_2 z^2 + \ldots) = g(z)$$

$$for \frac{f^n(0)}{N!} \neq 0$$

$$\to f(z) = z^N g(z)$$

(ii) There exists $\delta > 0$ such that if $0 < |z - z_0| < \delta$ such that $f(z) \neq 0$. (The zeros of a non-trivial analytic function are isolated)

Since g(z) is continuous at 0 and g(z) $\neq 0$, $\exists \delta$ such that $|z| \leq \delta \Rightarrow |g(z)| \neq 0$

(3) Let $f(z) = \sin(\pi/z)$. Thus, $f(1/n) = 0$. Does this contradict the result in (2)?

No, this does not contradict the result in (2) because the domain of f(z) is not open connected which doesn't satisfy a requirement of the theorem. Therefore, the theorem does not apply here.

(4) Find the order of each of the zeros of the given functions:

(a) $(z^2 - 4z + 4)^2$,

$$= (z-2)^4$$

So z=2, order 4

(b) $z^2(1-\cos(z))$,

z= 0, order 3 and since $f'(0) = -sin(0) = 0$ and $f''(0) = cos(0) = 1 \neq 0$ z= $s\pi k$ order 2 for $k \in \mathbb{N}$.

(c) $e^{2z} - 3e^z - 4$

$$= (e^z - 4)(e^2 - 1)$$

$e^z = 4$ has roots $z = e^{ln(4)+2\pi ki}$ and $e^z = 1$ has roots $z = e^{\pi i(2l+1)}$ for $k, l \in \mathbb{Z}$ order 1 for each of these.

(5) Locate the isolated singularity of the given function and tell whether is removable singularity, a pole or an essential singularity.

$$(a)\frac{e^z-1}{z}, \quad (b)\frac{z^2}{\sin(z)}, (c)\frac{e^z-1}{e^{2z}-1}, (d)\frac{z^4-2z^2+1}{(z-1)^2}$$

a)

$$\lim_{z\to 0} \frac{e^z - 1}{z}$$

$$= \frac{1-1}{0} = 0$$

So, z=0, removable singularity

b)

$$\lim_{z\to 0} \frac{z^2}{sin(z)}$$

$$= \frac{0}{0} = 0$$

So, z=0, removable singularity and

$$\lim_{z\to 2\pi k} \frac{z^2}{sin(z)}$$

$$= \frac{\pi k}{0}$$

So, a pole at z= πk for $k \in \mathbb{Z}$

c)

$$\lim_{z\to 0} \frac{e^z - 1}{e^{2z} - 1}$$

$$= \frac{1-1}{1-1} = 0$$

So, z= $0 + 2\pi ki$, removable singularity and

$$\lim_{z\to \pi i(2\pi k+1} \frac{e^z - 1}{e^{2z} - 1}$$

pole at z= $\pi i + 2\pi ki$ for $k \in \mathbb{Z}$

d)

$$\lim_{z\to 1} \frac{z^4 - 2z^2 + 1}{(z-1)^2}$$

$$= \frac{1}{1} = 1$$

z=1 is a removable singularity

(6) Find the Laurent series for a given function about the point $z = 0$ and find the residue at that point (a) $\frac{e^z-1}{z}$, $(b)\frac{z}{(\sin(z))^2}$, $(c)\frac{1}{e^z-1}$, $(d)\frac{1}{1-\cos(z)}$. In (c) and (d) compute only three terms of the Laurent series.

a)

$$e^z - 1 = \frac{z}{1!} + \frac{z^2}{2!} + ...$$

So,

$$\frac{e^z-1}{z} = 1 + \frac{z}{2!} + ...$$

and since this is a removable singularity, Res(f(z),0) = 0

b)

$$\sin(z) = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - ...$$

$$(sin(z))^2 = z^2 - \frac{2z^4}{3!} + ...$$

Since this function has a simple pole, we only need to concern ourselves with the coefficient of $\frac{1}{z}$. The first term of our function is

$$\frac{z}{((sin(z))^2} = \frac{z}{z^2} = \frac{1}{z}$$

So, our residue at (f(z), 0) = 1

c)

$$\frac{1}{e^z-1} = \frac{1}{\frac{z}{1} + \frac{z^2}{2!} + \frac{z^3}{3!} + ...}$$

$$\frac{1}{z(1 + \frac{z}{2!} + ... + \frac{z^{n-1}}{n!}}$$

$$1 = (1 + \frac{z}{2!} + ...)(a_0 + a_1 z + ...)$$

$$1 = a_0 1 =\rightarrow a_0 = 1$$

$$0 = a_0\frac{1}{2!} + a_1 1 \rightarrow a_1 = \frac{1}{2}$$

$$0 = a_2 1 + a_1\frac{1}{2!} + a_0\frac{1}{3!} \rightarrow a_2 = \frac{1}{12}$$

$$\frac{1}{z}(1 - \frac{z}{2} + \frac{z^2}{12}$$

our residue of (f(z),0) = 1

d)

$$\frac{1}{1-cos(z)} = \frac{z^2}{2! + \frac{z^4}{4!} - \frac{z^6}{6!} + ...}$$

$$\frac{2}{z^2}(\frac{1}{\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...}$$

$$= \frac{2}{z^2}(1 + (\frac{2z^2}{4!} - \frac{2!z^4}{6!} + ...))$$

Res(f(z),0) = 0.

(7) Find the residue of $f(z) = 1/\left(1+z^n\right)$ at the point $z_0 = e^{i\pi/n}$.

f(z) has a simple pole so to find our residue we find

$$\lim_{z \to z_0} \frac{z - z_0}{1 + z^n}$$

Applying L'Hopitals rule we get,

$$\lim_{z \to z_0} \frac{1}{nz^{n-1}}$$

$$= \frac{1}{n(e^{\pi i/(n-1)}}$$

(8) Calculate : (a) $\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)} dx$

$$\oint f(z) = \int_{-R}^{R} \frac{x^2}{\left(1+x^2\right)\left(4+x^2\right)} + \int_{C_r} \frac{z^2}{\left(1+z^2\right)\left(4+z^2\right)}$$

$$\int_{-R}^{R} \frac{x^2}{\left(1+x^2\right)\left(4+x^2\right)} = 2\pi i(Res_{z=i}f(z) + Res_{z=2i}f(z)$$

$$Res_{z=1}f(z) = \lim_{z \to 1} \frac{(z-1)z^2}{(z-1)(z+1)(4+z^2)}$$

$$= \frac{i^2}{(2i)(3)} = \frac{-1}{6i}$$

$$Res_{z=2i}f(z) = \lim_{z \to 2i} \frac{(z-2i)z^2}{(z^2+1)(z+2i)(z-2i)}$$

$$= \frac{(2i)^2}{(2i)^2+1)(4i)} = \frac{1}{3i}$$

$$2\pi i(-\frac{1}{6i} + \frac{2}{6i}) = \frac{\pi}{3}$$

(b) $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} (= \pi/2)$, Following the same formula as above, all we really need to do is find the residue of f(z) and multiply it by $2\pi i$. Letting

$$f(z) = \frac{dz}{(1+z^2)^2}$$

$$Res_{z=i} f(z) = (\lim_{z \to i} \frac{d}{dz} (\frac{(z-i)^2}{(z+i)^2(z-i)^2})$$

$$= 2\pi i (\lim_{z \to i} \frac{-2}{(z+i)^3})$$

$$= 2\pi i (\frac{-2}{4i}) = \frac{\pi}{2}$$

(c) $\int_{-\infty}^{\infty} \frac{x \sin(ax)}{x^2+b^2} dx \ (= \pi e^{-ab})$,

$$f(z) = \frac{ze^{iaz}}{z^2+b^2}$$

$$Res_{z=bi} f(z) = \lim_{z \to bi} \frac{(z-bi)(ze^{iaz}}{(z+bi)(z-bi)}$$

$$= \frac{e^{-ab}}{2i}$$

So,

$$2\pi i (\frac{e^{-ab}}{2i}) = \pi e^{-ab}$$

(d) $\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx (= \pi)$,

$$f(z) = \frac{sin(z)}{z}$$

$$\int_C \frac{sin(z)}{z} = \frac{1}{2i} \Big(\int_c \frac{e^{ix}}{x}$$

$$Res_{z=0} f(z) = \lim_{z \to 0} \frac{ze^{iz}}{z} = 1$$

$$\frac{1}{2i}(2\pi i) = \pi$$

(e) $\int_0^{2\pi} \frac{dt}{2+\cos^2(t)}$ Let t= θ

$$\int_0^{2\pi} \frac{d\theta}{2 + cos^2(\theta)}$$

Let $z = e^{i\theta}$ $d\theta = \frac{dz}{iz}$

$$= \oint_{|z|=1} \frac{1}{2 + (\frac{z+\frac{1}{z}}{2})^2} (\frac{dz}{iz})$$

$$= \frac{4}{i} \oint_{|z|=1} \frac{dz}{z^3 + 10z + \frac{1}{z}}$$

$$= \frac{4}{i} \oint_{|z|=1} \frac{zdz}{z^4 + 10z^2 + 1}$$

$$= \frac{4}{i} \oint_{|z|=1} \frac{zdz}{(z + i\sqrt{5 - 2\sqrt{6}})((z + i\sqrt{5 - 2\sqrt{6}})(z - i\sqrt{5 - 2\sqrt{6}})(z - i\sqrt{5 + 2\sqrt{6}})}$$

$$Res_{z=i\sqrt{5-2\sqrt{6}}} = \lim_{z \to i\sqrt{5-2\sqrt{6}}} \frac{z(z + i\sqrt{5 - 2\sqrt{6}})}{(z + i\sqrt{5 - 2\sqrt{6}})((z + i\sqrt{5 + 2\sqrt{6}})(z - i\sqrt{5 - 2\sqrt{6}})(z - i\sqrt{5 + 2\sqrt{6}})}$$

$$= \frac{i\sqrt{5 - 2\sqrt{6}}}{(i\sqrt{5 - 2\sqrt{6}} + i\sqrt{5 + 2\sqrt{6}})(i\sqrt{5 - 2\sqrt{6}} - i\sqrt{5 - 2\sqrt{6}})(i\sqrt{5 - 2\sqrt{6}} - i\sqrt{5 + 2\sqrt{6}})}$$

$$= \frac{1}{4\sqrt{6}}$$

So,

$$\oint_{|z|=1} \frac{1}{2 + (\frac{z+\frac{1}{z}}{2})^2} (\frac{dz}{iz}) = 2\pi i(\frac{4}{i}(\frac{1}{4\sqrt{6}})) = \frac{2\pi}{\sqrt{6}}$$