Math 122 A Homework 2

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1 Problem 1

Prove $f : C\ R^2 \to C\ R^2$
a) Maps the line $x = a_0$ onto $u = a_0^2 - \frac{v^2}{4a_0^2}$

$$u = a_0^2 - y^2 \quad v = 2a_0$$

$$u = a_0^2 - (\frac{v}{2a_0})^2$$

where

$$y = \frac{v}{2a_0}$$

$$u = a_0^2 - (\frac{v^2}{4a_0^2})$$

b) Maps the line $y = -b_0$ onto $u = -b_0^2 + \frac{v^2}{4b_0^2}$

$$u = x^2 - b_0^2 \quad v = 2xb_0$$

$$u = (\frac{v}{2b_0})^2 - b_0^2 \quad x = \frac{v}{2b_0}$$

$$u = \frac{v^2}{4b_0^2} - b_0^2$$

c) Maps the hyperbola $x^2 - y^2 = c_0$ onto the line $u = c_0$

$$u = x^2 - y^2$$

Plug in c_0 for u

$$x^2 - y^2 = c_0$$

d) Maps the hyperbola $xy = d_0$ onto the line $v = 2d_0$

$$v = 2xy$$

$$\frac{v}{2} = xy$$

Plug in d_0 for xy

$$\frac{v}{2} = d_0$$

$$v = 2d_0$$

2 Problem 2

Prove that the function $f : C\text{-}0 \to C$ defined as $\frac{1}{z}$ maps any line and any circle onto a line or a circle

Proof:

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

$$u(x, y) = \frac{x}{x^2 + y^2}$$

$$v(x, y) = -\frac{y}{x^2 + y^2}$$

To show $\frac{1}{z}$ maps a circle on to a line:

$$\frac{1}{z} = \frac{1}{r}e^{-i\theta} = |z - 1| = 1$$

$$|z - 1| = |(x - 1) + iy|^2 = 1$$

$$(x - 1)^2 + y^2 = 1$$

$$x^2 - 2x + 1 + y^2 = 1$$

$$= x^2 + y^2 = 2x$$

So, $u = \frac{x}{2x} = \frac{1}{2}$ and $y = -\frac{y}{2x} = 0$

Which maps the line $x = \frac{1}{2}$

Now for mapping a line to a circle: Show $f(z) = \frac{1}{z}$ maps $x = c_0$ say, x=1, onto a circle, $(u - \frac{1}{2})^2 + v^2 = \frac{1}{4}$

$$u = \frac{1}{1 + y^2} \quad v = -\frac{y}{1 + y^2}$$

$$(u - \frac{1}{2})^2 = (\frac{1}{1 + y^2})^2$$

$$= (\frac{1 - y^2}{2 + 2y^2})^2$$

$$= \frac{1 - 2y^2 + 4y^2}{4(1 + 2y^2 + y^4}$$

$$v^2 = \frac{y^2}{1 + 2y^2 + y^4} = \frac{4y^2}{1 + 2y^2 + y^4}$$

$$(u - \frac{1}{2})^2 + v^2 = \frac{1 + 2y^2 + y^2}{4(1 + 2y^2 + y^4} = \frac{1}{4}$$

Therefore, the line x=1 maps onto the circle $(u - \frac{1}{2})^2 + v^2 = \frac{1}{4}$

3 Problem 3

a) Prove $\lim_{z\to 0}\frac{\overline{z}}{z}$ does not exist.

$$\lim_{z=x+iy\to 0}\frac{x-iy}{x+iy}$$

$$\lim_{y\to 0}\frac{x-i0}{x+i0}=1$$

$$\lim_{x\to 0}\frac{0-iy}{0+iy}=-1$$

$-1\neq 1$ So the limit does not exist.

b) Prove $\lim_{z\to 0}\frac{\overline{z}\overline{z}}{z}=0$

$$\lim_{z\to 0}\frac{\overline{z}\overline{z}}{z}=\lim_{z\to 0}\frac{(x-iy)^2}{x+iy}$$

$$\lim_{x\to 0}\frac{(x-iy)^2}{x+iy}=\frac{(0-iy)^2}{iy+0}=0$$

$$\lim_{y\to 0}\frac{(x-iy)^2}{x+iy}=\frac{(x-0)^2}{x+0}=0$$

Therefore, $\lim_{z\to 0}\frac{\overline{z}\overline{z}}{z}=0$

4 Problem 4

Prove

$$\lim_{z\to w} z^n = w^n \;\; \forall n\in N$$

Proof: For the base case, let n=1. Then,

$$\lim_{z\to w} z = w \;\; \forall n\in N$$

Let $\epsilon>0$. Choose some δ such that $\delta=\epsilon$. So,

$$0<|z-w|<\delta$$

$$\to |w-z|<\delta| \leftrightarrow |w-z|<\epsilon$$

Therefore, the base case holds.

Now, for proof by induction, assume $\lim_{z\to w} z^n = w^n$ holds for n. We can use this to prove it also holds for n+1.

$$\lim_{z\to w} z^{n+1} = \lim_{z\to w} z^n\cdot z$$

$$= \lim_{z \to w} z^n \cdot \lim_{z \to w} z$$
$$= w^n \cdot w$$
$$= w^{n+1}$$

5 Problem 5

For any a,b,c,d $\in C$ with ad-bc $\neq 0$, define $T_A : C - \frac{d}{c} \to C$ if $c \neq 0, T_A : C \to C$ if c=0 with

$$T_A = \frac{az+b}{cz+b}$$

a) Prove if $c \neq 0$ then

$$\lim_{z \to \infty} T_A(z) = \frac{a}{c}, \lim_{z \to \frac{-d}{c}} T_A(z) = \infty$$

$$T_A(\frac{-d}{c}) = \infty \quad T_A(\infty) = \frac{a}{c}$$

b) Prove $T_A : C \bigcup \infty \to C \bigcup \infty$ is one to one.

$$\forall z, z' \in C - \frac{-d}{c}$$

, z does not equal z' $T_A(z)$ does not equal $T_A(z')$

$$\leftrightarrow \forall z, z' \in C - \frac{-d}{c}, \ \ T_A(z) = T_A(z') \to z = z'$$

$$T_A(z) = T_A(z') \to \frac{az+b}{cz+d} = \frac{az'+b}{cz'+b}$$
$$\leftrightarrow (az+b)(cz'+b) = (az'+b)(cz+d)$$
$$...$$
$$\leftrightarrow z = z'$$

c) $T_A^{-1} = T_{A^{-1}}$ I have no idea ahh!! //d)