

Problem 1: Ball Selection and Coin Flips

$X = 1$	$X = 2$	$X = 3$
$Y = 0$	$Y = 2$	$Y = 3$
	$Y = 1$	$Y = 2$
$Y = 1$		$Y = 1$
	$Y = 0$	$Y = 0$

Figure 1: Sample Space Partition for Problem 1

Consider the following features of the sample space partition that we can utilize to solve the given problems:

1. $X \sim \text{Uni}[3]$, so $P(x = 1) = P(x = 2) = P(x = 3) = \frac{1}{3}$.
2. Since X and Y are dependent, we can find the probability of a particular cell with the **Multiplication Rule** given by $P(\text{cell}) = P(X = x) \cdot P(Y = y | X = x)$
3. To find the total probability for specific number of heads $P(Y = y)$, we sum the Multiplication Rule across all values of x , yielding the **Law of Total Probability**:

$$P(Y = y) = \sum_{x=1}^3 P(X = x) \cdot P(Y = y | X = x)$$

4. To find the probability we rolled a certain number X , given we saw a certain number of heads Y , we can use **Bayes Rule**:

$$P(X = x | Y = y) = \frac{P(X = x) \cdot P(Y = y | X = x)}{P(Y = y)}$$

Conceptually, we are highlighting only the cells where $Y = 2$. Of the cells where $Y = y$, how much of these cells are $X = x$?

With that being said, we can now solve the problems given. (on the next page)

(a) $P(X = 2 \text{ and } Y = 2)$

From 2, we recognize we must solve $P(X = 2) \cdot P(Y = 2 | X = 2)$. Thus, we solve

$$P(X = 2 \text{ and } Y = 2) = \left(\frac{1}{3}\right) \cdot \left(\binom{2}{2} \cdot (0.4)^2\right) = \boxed{\frac{4}{75}}$$

(b) $P(X = 3 \text{ and } Y = 2)$

From 2, this is nearly the same as the last problem, except we are restricted to the $x = 3$ column:

$$P(X = 3 \text{ and } Y = 2) = \left(\frac{1}{3}\right) \cdot \left(\binom{3}{2} \cdot (0.4)^2 \cdot (0.6)\right) = \boxed{\frac{12}{125}}$$

(c) $P(Y = 2 | X = 3)$

$$P(Y = 2 | X = 3) = \binom{3}{2} \cdot (0.4)^2 (0.6) = \boxed{\frac{36}{125}}$$

(d) $P(Y = 2)$

Here, we move onto using the law of total probability. For every column x , we sum up the probability $P(Y = y | X = x)$. We first note that Y will never be 2 for $X = 1$. Thus, we solve:

$$P(X = 2) \cdot P(Y = 2 | X = 2) + P(X = 3) \cdot P(Y = 2 | X = 3)$$

Since we have already solved for these quantities in (a) and (b), we add them:

$$P(Y = 2) = \frac{4}{75} + \frac{12}{125} = \boxed{\frac{56}{375}}$$

(e) $P(X = 3 | Y = 2)$

Notice that this is simply Bayes theorem for values that we have already solved. That is,

$$P(X = 3 | Y = 2) = \frac{P(X = 3) \cdot P(Y = 2 | X = 3)}{P(Y = 2)} = \frac{\frac{36}{125}}{\frac{56}{375}} = \boxed{\frac{9}{14}}$$

Problem 2: Two Dice Problem

$Y = 6$	6	6	6	6	6	6
$Y = 5$	5	5	5	5	5	6
$Y = 4$	4	4	4	4	5	6
$Y = 3$	3	3	3	4	5	6
$Y = 2$	2	2	3	4	5	6
$Y = 1$	1	2	3	4	5	6
	$X = 1$	$X = 2$	$X = 3$	$X = 4$	$X = 5$	$X = 6$

Figure 2: Sample Space Partition for Problem 2

(a) $P(M=6)$

Visual Method: Count the number of squares with $M = 6$, divide by the total number of squares. The result is $\boxed{\frac{11}{36}}$.

Unions and Intersections Method: Let A, B be sets such that $A = \{X = 6\}$ and $B = \{Y = 6\}$. Note, that A, B are uniform, thus $P(A) = P(B) = \frac{1}{6}$, and $P(A \cap B) = P(A)P(B) = \frac{1}{36}$. We calculate the union of A and B :

$$P(M = 6) = P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \boxed{\frac{11}{36}}$$

Compliment Method: Note that $P(A^c) = P(B^c) = \frac{5}{6}$. Then, we can write:

$$P(M = 6) = P(A \cup B)^c = 1 - [P(A^c) \cap P(B^c)] = 1 - \left[\frac{5}{6} \cdot \frac{5}{6}\right] = \boxed{\frac{11}{36}}$$

(b) $P(M=1)$

Visual Method: Count the number of squares with $M = 1$, divide by the total number of squares. The result is $\boxed{\frac{1}{36}}$.

Unions and Intersections Method: Let A, B be sets such that $A = \{X = 1\}$ and $B = \{Y = 1\}$. Note, that for $P(M = 1)$, both dice must be 1 simultaneously. Thus, We calculate the intersection of A and B :

$$P(M = 1) = P(A \cap B) = P(A)P(B) = \frac{1}{6} \cdot \frac{1}{6} = \boxed{\frac{1}{36}}$$

Compliment Method: Note that $P(A^c) = P(B^c) = \frac{5}{6}$. Then, we can write:

$$P(M = 1) = P(A \cap B) = 1 - P(A \cap B)^c = 1 - (P(A)^c \cup P(B)^c) = 1 - (P(A^c) + P(B^c) - P(A^c)P(B^c)) = 1 - \left(\frac{5}{6} + \frac{5}{6} - \frac{25}{36}\right) = \boxed{\frac{1}{36}}$$

(c) $P(M=5)$

Visual Method: Count the number of squares with $M = 5$, divide by the total number of squares. The result is $\boxed{\frac{9}{36}}$.

Unions and Intersections Method: Let A, B be sets such that $A = \{X = 5 \cap Y \leq 5\}$ and $B = \{Y = 5 \cap X \leq 5\}$. Thus:

$$P(M = 5) = P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{5}{36} + \frac{5}{36} - \frac{1}{36} = \boxed{\frac{9}{36}}$$

Compliment Method: It is helpful to think about what the sets A^c and B^c contain in the context of this problem. Without loss of generality, note that:

$$A = \{X = 5 \cap Y \leq 5\}, \text{ therefore } A^c = \{X \neq 5 \cup Y > 5\}$$

Notably, in reading the definition for A , we read: “ A is the set of all ordered pairs where $X = 5$ **AND** $Y \leq 5$. When we take the complement, we must apply De Morgan’s law’s to the statement. That is, A^c reads” A^c is the set of all ordered pairs where $X \neq 5$ **OR** $Y = 6$.” If we did not emphasize this point, it may be easy to recognize that $(5, 6) \in A^c$, despite our definition for A^c reading $X \neq 5$. There is an **or** in the definition we need to be careful to account for. Thus:

$$P(M = 5) = P(A \cup B) = 1 - P(A \cup B)^c = 1 - \frac{27}{36} = \boxed{\frac{9}{36}}$$

(d) $E(M)$

We know that $E(M)$ is defined as

$$E(M) = \sum_{k=1}^6 k \cdot P(M = k)$$

But how do we figure out $P(M = k)$? A feature of note in our partitioned sample space is that values of $M \in [1, 6]$ occur at their corresponding equivalents for $X, Y \in [1, 6]$. That is, when $M = 6$, we have both $X, Y = 6$. These patterns form an L-shape, and contain a smaller square inside of a larger square. Since we know that the sum of all probabilities in our sample space sum to one, we can find values for $P(M \leq k)$ by plugging $k = 1, 2, \dots, 6$ into $\frac{k^2}{36}$, yielding:

k	$P(M \leq k)$ Formula	$P(M \leq k)$ Value
1	$1^2/36$	$1/36$
2	$2^2/36$	$4/36$
3	$3^2/36$	$9/36$
4	$4^2/36$	$16/36$
5	$5^2/36$	$25/36$
6	$6^2/36$	$36/36$

From our Cumulative Distribution Function, we can derive our Probability Mass Function in the usual way, yielding:

k	0	1	2	3	4	5	6
$P(M = k)$	0	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

We now have everything we need to calculate $E(M)$:

$$E(M) = \sum_{k=1}^6 k \cdot P(M = k) = 1 \left(\frac{1}{36} \right) + 2 \left(\frac{3}{36} \right) + 3 \left(\frac{5}{36} \right) + 4 \left(\frac{7}{36} \right) + 5 \left(\frac{9}{36} \right) + 6 \left(\frac{11}{36} \right) = \boxed{\frac{161}{36}}$$

Problem 3: Radioactive Decay - Exponential Model

(a) Integral expressing $P(T_1 \leq 1)$

Since $T_1 \sim \text{Exp}(\lambda)$, we know it has PDF $f(t) = \lambda e^{-\lambda t}$ where $t \geq 0$. Thus, we can set up an integral:

$$P(T_1 \leq 1) = \int_0^1 \lambda e^{-\lambda t} dt$$

(b) Evaluating the integral

$$\int_0^1 \lambda e^{-\lambda t} dt = [-e^{-\lambda t}]_0^1 = (-e^{-\lambda \cdot 1}) - (-e^{-\lambda \cdot 0}) = 1 - e^{-\lambda}$$

(c) Complement of $P(T_1 \leq 1)$

$$P(T_1 > 1) = 1 - P(T_1 \leq 1) = 1 - (1 - e^{-\lambda}) = e^{-\lambda}$$

(d) Conditional Probability T_1, T_2

Since T_1 and T_2 are independent:

$$P(T_2 \leq s | T_1 < 1) = P(T_2 \leq s)$$

$$P(T_2 \leq s) = \int_0^s \lambda e^{-\lambda t} dt = [-e^{-\lambda t}]_0^s = 1 - e^{-\lambda s}$$