MATH 5651 Midterm 1 Study Guide

Comprehensive Exam Prep

Contents

Part I: Reference Tables

Discrete Probability Distributions

You must memorize the PMF, Range, and Expected Value for these distributions.

Distribution	**Notation**	**PMF** $P(X = k)$	**Range**	**Mean** $E[X]$
Bernoulli	$Ber(p)$	$p^k(1-p)^{1-k}$	$\{0, 1\}$	p
Binomial	$Bin(n, p)$	$\binom{n}{k}p^k(1-p)^{n-k}$	$\{0, \dots, n\}$	np
Geometric	$Geo(p)$	$(1-p)^{k-1}p$	$\{1, 2, \dots\}$	$1/p$
Poisson	$Poi(\lambda)$	$\frac{e^{-\lambda}\lambda^k}{k!}$	$\{0, 1, \dots\}$	λ
Discrete Uniform	$Uni[n]$	$1/n$	$\{1, \dots, n\}$	$(n+1)/2$

Continuous Probability Distributions

Distribution	**Notation**	**PDF** $f(x)$	**Range**	**Mean** $E[X]$
Uniform	$U[a, b]$	$\frac{1}{b-a}$	$[a, b]$	$\frac{a+b}{2}$
Exponential	$Exp(\lambda)$	$\lambda e^{-\lambda x}$	$[0, \infty)$	$1/\lambda$

Fundamental Counting Formulas

- **Permutations (Order Matters):** $P(n, k) = \frac{n!}{(n-k)!}$
- **Combinations (Order Doesn't Matter):** $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- **Stars and Bars (Multisets):** $\binom{n+k-1}{k}$ (Choosing k items from n categories with replacement)

Part II: Required Derivations and Proofs

The following pages contain the key derivations emphasized in Homework 1, Homework 2, and the Textbook. You should be able to reproduce these from scratch.

Derivation 1: Geometric Series

Question:

Prove the summation formula for an infinite geometric series.

Show that for $|r| < 1$:

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

Context: This mathematical tool is required to prove the validity of the Geometric probability mass function.

Derivation 2: Geometric Distribution Validity

Question:

Let $X \sim Geo(p)$ with probability mass function $P(X = k) = (1 - p)^{k-1}p$ for $k = 1, 2, \ldots$.

Show that:

$$\sum_{k=1}^{\infty} P(X = k) = 1$$

Derivation 3: Expectation of Geometric Distribution

Question:

Let $X \sim Geo(p)$.

Derive the Expected Value:

$$E[X] = \frac{1}{p}$$

Hint: You may need to use the derivative of a geometric series.

Derivation 4: Taylor Series for e^x

Question:

Write down the Taylor Series expansion for e^λ.

Show that:

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^\lambda$$

Context: This is required to prove the validity of the Poisson distribution.

Derivation 5: Poisson Distribution Validity

Question:

Let $X \sim Poi(\lambda)$ with probability mass function $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$ for $k = 0, 1, \ldots$.

Show that:

$$\sum_{k=0}^{\infty} P(X = k) = 1$$

Derivation 6: Expectation of Poisson Distribution

Question:

Let $X \sim Poi(\lambda)$.

Derive the Expected Value:

$$E[X] = \lambda$$

Derivation 7: Linearity of Expectation

Question:

Let X and Y be discrete random variables defined on the same sample space.

Prove that:

$$E[X+Y] = E[X] + E[Y]$$

Hint: Use the definition of expectation involving a double summation over the joint probability mass function $P(X = x, Y = y)$.

Derivation 8: Exponential Distribution Validity

Question:

Let $T \sim Exp(\lambda)$ with probability density function $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$.

Show that:

$$\int_0^\infty \lambda e^{-\lambda t}\, dt = 1$$

Derivation 9: Expectation of Exponential Distribution

Question:

Let $T \sim Exp(\lambda)$.

Derive the Expected Value:

$$E[T] = \frac{1}{\lambda}$$

Hint: Use integration by parts.

Derivation 10: Poisson as a Limit of Binomial

Question:

Let $X_n \sim Bin(n, p_n)$. Assume that as $n \to \infty$, $p_n \to 0$ such that $np_n \to \lambda$.

Show that:

$$\lim_{n\to\infty} P(X_n = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Derivation 11: Exponential as a Limit of Geometric

Question:

Let T_{geo} be a geometric random variable representing the number of trials until success, where the probability of success is $p = \lambda/n$. Let time t be measured in steps of size $1/n$.

Show that:

$$\lim_{n\to\infty} P(T_{geo} > t) = e^{-\lambda t}$$

Derivation 12: One Poisson Arrival (Double Integral)

Question:

Let $T_1 \sim Exp(\lambda)$ and $T_2 \sim Exp(\lambda)$ be independent. T_1 is the time of the first arrival, and T_2 is the time between the first and second arrival.

Calculate the probability of exactly one arrival in time t:

$$P(T_1 \leq t \cap T_1 + T_2 > t)$$

Set this up as a double integral over the joint density of T_1 and T_2.

Part III: Counting & Combinatorics Problems

These questions cover material from "Day 1 Problems" and "Day 4 Problems".

Problem 1: Poker Hands

Question:

(a) In how many ways can you select five cards from a standard deck of 52?

(b) How many ways can you select a **Full House** (three of one value and two of another)?

(c) How many ways can you select a **Flush** (all five cards of the same suit)?

Problem 2: Coin Flip Sequences

Question:

(a) How many sequences of eight coin flips contain exactly 3 Heads and 5 Tails?

(b) How many sequences of eight coin flips contain 5 Heads and 3 Tails?

Problem 3: Stars and Bars (The Produce Stand)

Question:

A produce stand has n kinds of fruit. You wish to buy k pieces of fruit.

(a) If you buy 12 pieces of fruit from 2 kinds (A and B), how many selections can you make?

(b) Derive the general formula: How many selections are possible if you choose k items from n distinct categories, allowing replacement (duplicates)?

Problem 4: Partitioning (Basketball Teams)

Question:

In order to play a game of basketball, 10 children at a playground divide into two teams of five players.

How many different games are possible?

Note: Team A vs Team B is the same game as Team B vs Team A.

Problem 5: Strategic Expectation (The Biathlon)

Question:

You have two shooting strategies for a biathlon (5 shots). 1. Shoot fast: 10 seconds per shot, 85% hit rate ($p = 0.85$). 2. Shoot controlled: 12 seconds per shot, 90% hit rate ($p = 0.90$).

A miss adds a 35-second penalty loop. Which strategy has the lower **Expected Value** for total time?

Part IV: Probability Calculations

These questions cover material from Homework 1 and Homework 2.

Problem 6: Conditional Probability (Urns/Jars)

Question (HW 2 Q1):

A jar contains three ping pong balls, numbered 1, 2, 3. Let X be the number on the ball selected ($X \sim Uni[1]$). After selecting the ball, flip X coins that land heads with probability 0.4. Let Y be the number of heads.

Find $P(X = 3|Y = 2)$**.**

Problem 7: Joint Random Variables (Dice)

Question (HW 2 Q2):

Roll two six-sided dice. Let X be the value of the first die and Y be the value of the second die. Let $M = \max(X, Y)$.

(a) Find $P(M = 6)$.

(b) Find $P(M = 1)$.

(c) Find $E[M]$.

Problem 8: The Radioactive Decay (Calculus)

Question (HW 2 Q3):

Let $T_1 \sim Exp(\lambda)$. T_1 models the decay of a radioactive atom. If the atom decays before time $t = 1$, immediately replace it with a second radioactive atom $T_2 \sim Exp(\lambda)$.

(a) Write an integral which expresses $P(T_1 \leq 1)$.

(b) Evaluate the integral to find the probability.

(c) Find $P(T_2 \leq s | T_1 < 1)$.

Problem 9: Custom Discrete Distribution

Question (HW 1 Q6):

Let Y be the outcome of rolling a six-sided die ($Y \sim Uni[2]$). Then flip Y coins. Let X be the number of heads.

Find the distribution (PMF) of X. *(Write the formula for $P(X = k)$).*

Problem 10: Bayes' Theorem (Medical Testing)

Question (General Example):

A disease affects 1% of the population. A test for the disease is 99% accurate (if you have the disease, it says positive 99% of the time; if you do not have the disease, it says negative 99% of the time).

If you test positive, what is the probability you actually have the disease?

Problem 11: Set Theory and Logic

Question (Activity 2a):

You draw two balls from a jar of red and blue balls without replacement.

(a) What is the probability that the first is red AND the second is blue?

(b) What is the probability that at least one ball is red? (Solve this using the complement).

Problem 12: Independent Events

Question:

Let A and B be independent events.

(a) Prove that A^c and B are independent.

(b) If $P(A) = 0.3$ and $P(B) = 0.4$, find $P(A \cup B)$.