MATH 5651 Midterm 1 Study Guide

Updated with Exam 1 Topics & Common Distributions

Contents

Part I: Common Distributions to Memorize

Source: "Common Distributions.pdf" and "Exam 1 Topics.pdf"

Discrete Distributions

You must know the PMF $P(X = k)$ and Mean $E[X]$.

Name	**Notation**	**Range**	**PMF** $P(X = k)$	**Mean**
Discrete Uniform	$Uni[n]$	$\{1, 2, \ldots, n\}$	$\frac{1}{n}$	$\frac{n+1}{2}$
Bernoulli	$Ber(p)$	$\{0, 1\}$	$p^k(1-p)^{1-k}$	p
Binomial	$Bin(n, p)$	$\{0, \ldots, n\}$	$\binom{n}{k}p^k(1-p)^{n-k}$	np
Geometric	$Geo(p)$	$\{1, 2, \ldots\}$	$(1-p)^{k-1}p$	$\frac{1}{p}$
Poisson	$Poi(\lambda)$	$\{0, 1, \ldots\}$	$\frac{e^{-\lambda}\lambda^k}{k!}$	λ

Note on Geometric: The sheet confirms X counts *trials* until success (range starts at 1).

Continuous Distributions

You must know the PDF $f(x)$, CDF $F(x) = P(X \leq x)$, and Mean.

Name	**Notation**	**Range**	**PDF** $f(x)$	**CDF** $F(x)$	**Mean**
Uniform	$Uni(0, b)$	$(0, b)$	$\frac{1}{b}$	$\frac{x}{b}$	$\frac{b}{2}$
Exponential	$Exp(\lambda)$	$(0, \infty)$	$\lambda e^{-\lambda t}$	$1 - e^{-\lambda t}$	$\frac{1}{\lambda}$

Part II: Key Derivations

Source: Homework 1 & 2

1. **Geometric Series Sum:** Show $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$.
2. **Geometric PMF Sum:** Show $\sum_{k=1}^{\infty} (1-p)^{k-1}p = 1$.
3. **Poisson PMF Sum:** Show $\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = 1$ (requires Taylor Series $e^x = \sum x^k/k!$).
4. **Expectation of Geometric:** Derive $E[X] = 1/p$ using derivative of geometric series.
5. **Expectation of Poisson:** Derive $E[X] = \lambda$.
6. **Expectation of Exponential:** Derive $E[T] = 1/\lambda$ using integration by parts.
7. **Memoryless Property:** Show $P(T > s + t|T > s) = P(T > t)$ for Exponential.

Part III: Counting & Combinatorics

Source: Day 1, Day 4, Exam 1 Topics

Problem 1: Multinomial Coefficients (Arrangements)

Topic: Exam 1 Topics list explicitly mentions "Multinomial Coefficients".

Question:

(a) How many distinct ways can you rearrange the letters in the word **MINNESOTA**?

(b) How many ways can 10 distinct students be assigned to three different dorm rooms, if Room A holds 4 students, Room B holds 3 students, and Room C holds 3 students?

Problem 2: Stars and Bars (Multisets)

Topic: Day 4 Problems (Produce Stand)

Question:

A grocery store has 5 types of soda. You want to buy 12 bottles.

(a) How many different combinations of soda can you buy? (Order does not matter, repetition is allowed).

(b) How many combinations are possible if you must buy at least one of each type?

Problem 3: Distinguishable vs. Indistinguishable Partitioning

Topic: Day 4 Problems (Basketball Teams)

Question:

(a) **Distinguishable Groups:** 10 people want to play basketball. How many ways can you split them into a "Blue Team" of 5 and a "Red Team" of 5?

(b) **Indistinguishable Groups:** 10 people want to play basketball. How many ways can you split them into *two teams* of 5? (Where "Team A vs Team B" is the same matchup as "Team B vs Team A").

Problem 4: Poker Hands

Topic: Day 1 Problems

Question:

From a standard 52-card deck:

(a) **Full House:** How many hands contain 3 cards of one rank and 2 cards of another?

(b) **Two Pair:** How many hands contain 2 cards of one rank, 2 cards of a different rank, and 1 card of a third rank?

Part IV: Discrete Probability & Logic

Source: Activity 2a, Homework 1

Problem 5: Unions and Intersections (Yahtzee)

Topic: Activity 2a (Dice)

Question:

You roll 5 six-sided dice.

(a) What is the size of the sample space?

(b) What is the probability of rolling a "Large Straight" (either 1-2-3-4-5 OR 2-3-4-5-6)? *Note: Order of dice matters in sample space, but not for the hand type.*

(c) What is the probability of rolling five-of-a-kind?

Problem 6: Logic and Complements

Topic: Exam 1 Topics (Set Theory)

Question:

Let A and B be two events with $P(A) = 0.4$ and $P(B) = 0.5$.

(a) If A and B are independent, find $P(A \cup B)$.

(b) If A and B are mutually exclusive (disjoint), find $P(A \cup B)$.

(c) If $A \subset B$, find $P(A|B)$.

Part V: "Day 6" Problems (Wait Times & Sampling)

Source: Day 6 Problems (The Sock Drawer & Ping Pong)

Problem 7: The "Waiting for the Second" Distribution

Topic: Day 6 Problem 1

Question:

A jar contains 6 Red balls and 2 Blue balls. Balls are drawn **one at a time without replacement**.

(a) Let X be the draw number on which the **first** Blue ball is found. Find the probability distribution (PMF) of X. (i.e., find $P(X = k)$ for valid k).

(b) Let Y be the draw number on which the **second** Blue ball is found. Find the probability distribution of Y.

(c) Explain why the probabilities for X and Y are symmetric (hint: consider the positions of the balls in a line).

Problem 8: The Sock Drawer (Matching Pairs)

Topic: Day 6 Problem 2

Question:

My sock drawer has 4 distinct pairs of socks (8 socks total: 2 Red, 2 Blue, 2 Green, 2 Black).

I pull socks out one by one without replacement. Let X be the number of socks drawn until I get a **matching pair**.

(a) What is the minimum possible value of X?

(b) What is the maximum possible value of X?

(c) Find $P(X = 3)$.

Part VI: Calculus-Based Probability

Source: Homework 2, Exam 1 Topics (CDFs)

Problem 9: CDFs and PDFs

Topic: Exam 1 Topics / Homework 2

Question:

Let X be a random variable with PDF $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ (Exponential).

(a) Derive the Cumulative Distribution Function (CDF), $F(x) = P(X \leq x)$, by integrating the PDF.

(b) Use the CDF to find $P(1 < X \leq 3)$.

(c) Use the CDF to find the Median of X (the value m such that $F(m) = 0.5$).

Problem 10: Bayes' Theorem with Continuous Variables

Topic: Homework 2 Q1 (The Bayes/Urn Problem)

Question:

Let X be a random variable representing the number of coins we pick. $P(X = 1) = 0.2$, $P(X = 2) = 0.8$. If we pick X coins, we flip them. Let Y be the number of Heads obtained (assume fair coins).

(a) Find the conditional probability $P(Y = 1|X = 2)$.

(b) Find the joint probability $P(X = 2 \cap Y = 1)$.

(c) Find the marginal probability $P(Y = 1)$.

(d) **Bayes:** Find $P(X = 2|Y = 1)$.

Problem 11: Joint Distributions (Max/Min)

Topic: Homework 2 Q2

Question:

Roll two fair 6-sided dice, X_1 and X_2. Let $Y = \max(X_1, X_2)$.

(a) Find $P(Y \leq 4)$. (Hint: This requires $X_1 \leq 4$ AND $X_2 \leq 4$).

(b) Find the CDF of Y, $F_Y(k)$.

(c) Find the PMF of Y, $P(Y = k)$, by using the logic $P(Y = k) = F_Y(k) - F_Y(k-1)$.