MATH 5651 Midterm 1 Study Guide

Comprehensive Exam Prep (Updated)

Contents

Part I: Reference Tables

Discrete Probability Distributions

You must memorize the PMF, Range, and Expected Value for these distributions.

Distribution	**Notation**	**PMF** $P(X = k)$	**Range**	**Mean** $E[X]$
Bernoulli	$Ber(p)$	$p^k(1-p)^{1-k}$	$\{0, 1\}$	p
Binomial	$Bin(n, p)$	$\binom{n}{k}p^k(1-p)^{n-k}$	$\{0, \dots, n\}$	np
Geometric	$Geo(p)$	$(1-p)^{k-1}p$	$\{1, 2, \dots\}$	$1/p$
Poisson	$Poi(\lambda)$	$\frac{e^{-\lambda}\lambda^k}{k!}$	$\{0, 1, \dots\}$	λ
Discrete Uniform	$Uni[n]$	$1/n$	$\{1, \dots, n\}$	$(n+1)/2$

Continuous Probability Distributions

Distribution	**Notation**	**PDF** $f(x)$	**Range**	**Mean** $E[X]$
Uniform	$U[a, b]$	$\frac{1}{b-a}$	$[a, b]$	$\frac{a+b}{2}$
Exponential	$Exp(\lambda)$	$\lambda e^{-\lambda x}$	$[0, \infty)$	$1/\lambda$

Fundamental Counting Formulas

- **Permutations (Order Matters):** $P(n, k) = \frac{n!}{(n-k)!}$
- **Combinations (Order Doesn't Matter):** $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- **Stars and Bars (Multisets):** $\binom{n+k-1}{k}$ (Choosing k items from n categories with replacement)

Part II: Required Derivations and Proofs

The following pages contain the key derivations emphasized in Homework 1, Homework 2, and the Textbook.

Derivation 1: Geometric Series

Question:

Prove the summation formula for an infinite geometric series.

Show that for $|r| < 1$:

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

Context: This mathematical tool is required to prove the validity of the Geometric probability mass function.

Derivation 2: Geometric Distribution Validity

Question:

Let $X \sim Geo(p)$ with probability mass function $P(X = k) = (1 - p)^{k-1}p$ for $k = 1, 2, \ldots$.

Show that:

$$\sum_{k=1}^{\infty} P(X = k) = 1$$

Derivation 3: Expectation of Geometric Distribution

Question:

Let $X \sim Geo(p)$.

Derive the Expected Value:

$$E[X] = \frac{1}{p}$$

Hint: You may need to use the derivative of a geometric series.

Derivation 4: Taylor Series for e^x

Question:

Write down the Taylor Series expansion for e^{λ}.

Show that:

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}$$

Context: This is required to prove the validity of the Poisson distribution.

Derivation 5: Poisson Distribution Validity

Question:

Let $X \sim Poi(\lambda)$ with probability mass function $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$ for $k = 0, 1, \ldots$.

Show that:

$$\sum_{k=0}^{\infty} P(X = k) = 1$$

Derivation 6: Expectation of Poisson Distribution

Question:

Let $X \sim Poi(\lambda)$.

Derive the Expected Value:

$$E[X] = \lambda$$

Derivation 7: Linearity of Expectation

Question:

Let X and Y be discrete random variables defined on the same sample space.

Prove that:

$$E[X+Y] = E[X] + E[Y]$$

Hint: Use the definition of expectation involving a double summation over the joint probability mass function $P(X = x, Y = y)$.

Derivation 8: Exponential Distribution Validity

Question:

Let $T \sim Exp(\lambda)$ with probability density function $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$.

Show that:

$$\int_0^\infty \lambda e^{-\lambda t}\, dt = 1$$

Derivation 9: Expectation of Exponential Distribution

Question:

Let $T \sim Exp(\lambda)$.

Derive the Expected Value:

$$E[T] = \frac{1}{\lambda}$$

Hint: Use integration by parts.

Derivation 10: Poisson as a Limit of Binomial

Question:

Let $X_n \sim Bin(n, p_n)$. Assume that as $n \to \infty$, $p_n \to 0$ such that $np_n \to \lambda$.

Show that:

$$\lim_{n\to\infty} P(X_n = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Derivation 11: Exponential as a Limit of Geometric

Question:

Let T_{geo} be a geometric random variable representing the number of trials until success, where the probability of success is $p = \lambda/n$. Let time t be measured in steps of size $1/n$.

Show that:

$$\lim_{n\to\infty} P(T_{geo} > t) = e^{-\lambda t}$$

Derivation 12: One Poisson Arrival (Double Integral)

Question:

Let $T_1 \sim Exp(\lambda)$ and $T_2 \sim Exp(\lambda)$ be independent. T_1 is the time of the first arrival, and T_2 is the time between the first and second arrival.

Calculate the probability of exactly one arrival in time t:

$$P(T_1 \leq t \cap T_1 + T_2 > t)$$

Set this up as a double integral over the joint density of T_1 and T_2.

Part III: Counting & Combinatorics Problems

These questions cover material from Day 1, Day 2/3 (Activity 2a), Day 4, and Day 6.

Problem 1: Poker Hands (Two Pair)

Concept: Day 1 (Combinations)

Question:

From a standard deck of 52 cards:

(a) How many ways can you select a **Two Pair** hand? (Two cards of one rank, two cards of a different rank, and one card of a third rank).

(b) How many ways can you select a **Three of a Kind** hand? (Three cards of one rank, and two other cards that do not match the first rank and do not match each other).

Problem 2: Coin Flip Sequences

Concept: Day 1 (Permutations with repetition)

Question:

(a) How many sequences of 10 coin flips contain exactly 4 Heads and 6 Tails?

(b) How many sequences of 10 coin flips contain at least 9 Heads?

Problem 3: Stars and Bars (The Bakery)

Concept: Day 4 (Multisets/Stars and Bars)

Question:

A bakery sells 6 varieties of cookies. You wish to buy a box of 12 cookies.

(a) How many different selections of 12 cookies are possible? (Assume the bakery has at least 12 of every variety).

(b) How many selections are possible if you must buy at least one of each variety?

Problem 4: Partitioning (Study Groups)

Concept: Day 4 (Partitions/Unordered Groups)

Question:

A class of 12 students needs to divide into 3 study groups of 4 students each.

How many different ways can these groups be formed?

Note: Group A vs Group B is the same set of groups as Group B vs Group A. The groups are indistinguishable.

Problem 5: Dice Poker (Yahtzee)

Concept: Day 2/3 (Ordered vs Unordered Dice outcomes)

Question:

You roll five six-sided dice.

(a) How many total ordered outcomes are there?

(b) How many outcomes result in a **Large Straight**? (The dice show $\{1, 2, 3, 4, 5\}$ or $\{2, 3, 4, 5, 6\}$ in any order).

(c) What is the probability of rolling a "Five of a Kind" (e.g., 1-1-1-1-1)?

Problem 6: The Sock Drawer

Concept: Day 6 (Sampling without replacement / Matching)

Question:

My sock drawer has 5 distinct pairs of socks (10 socks total: 2 Red, 2 Blue, 2 Green, 2 White, 2 Black).

I pull socks out one by one without replacement. Let X be the number of socks I must pull out until I have a matching pair.

(a) What is the minimum and maximum value for X?

(b) Find $P(X = 3)$. (Hint: This means the first two were different, and the third matched one of the first two).

Problem 7: Distinct vs Indistinguishable Balls

Concept: Day 2/3 (Activity 2a)

Question:

A jar contains 8 ping pong balls. 3 are Red (numbered 1, 2, 3). 5 are Blue (numbered 4, 5, 6, 7, 8).

(a) You draw 2 balls without replacement. What is the probability that the first is Red and the second is Blue?

(b) You draw 2 balls without replacement. What is the probability that you get exactly one Red and one Blue (in any order)?

(c) What is the probability that the numbers on the two balls drawn are both even?

Problem 8: Order of Draws

Concept: Day 6 (Waiting for a specific outcome)

Question:

A jar contains 8 Red balls and 4 Blue balls. Balls are drawn one at a time without replacement.

(a) Let X be the number of the draw when the **first** Blue ball is drawn. Find $P(X = 1)$ and $P(X = 2)$.

(b) Let Y be the number of the draw when the **second** Blue ball is drawn. Find $P(Y = 3)$.

Problem 9: Strategic Expectation

Concept: Day 4 (Biathlon/Expected Value)

Question:

You have two strategies for a quiz with 5 questions. 1. **Rush:** You spend less time checking work. You have a 0.8 probability of getting each question right. 2. **Double-Check:** You spend more time. You have a 0.95 probability of getting each question right.

Each wrong answer costs you a 2-point penalty. (Score = Number Correct $\times$ 1 - Number Incorrect $\times$ 2).

Which strategy gives the higher **Expected Score**?

Part IV: Probability Calculations

These questions cover material from Homework 1 and Homework 2, slightly altered.

Problem 10: Conditional Probability (Urns/Jars)

Question (Altered HW 2 Q1):

A jar contains four balls, numbered 1, 2, 3, 4. Let X be the number on the ball selected. After selecting the ball, flip X coins that land heads with probability 0.5. Let Y be the number of heads.

Find $P(X = 4|Y = 3)$**.**

(Hint: Use Bayes' Theorem).

Problem 11: Joint Random Variables (Dice)

Question (Altered HW 2 Q2):

Roll two six-sided dice. Let X be the value of the first die and Y be the value of the second die. Let $m = \min(X, Y)$.

(a) Find $P(m = 6)$.

(b) Find $P(m = 1)$.

(c) Find $E[m]$.

Problem 12: Continuous Probability (Decay)

Question (Altered HW 2 Q3):

Let $T \sim Exp(2)$ (Exponential distribution with rate $\lambda = 2$).

(a) Write an integral which expresses $P(T \leq 2)$.

(b) Evaluate the integral to find the probability.

(c) Find $P(T > 4 | T > 2)$ using the memoryless property (or calculation).

Problem 13: Custom Discrete Distribution

Question (Altered HW 1 Q6):

Let Y be the outcome of rolling a 4-sided die (outcomes 1, 2, 3, 4). Then flip Y coins. Let X be the number of heads.

Find the probability that $X = 2$. *(Use the Law of Total Probability).*