5651 HW 1 - Matthew Pipes

1 (a) $\sum_{K=1}^{\infty} \frac{1}{2^K} = 1$

Utilizing the sum of an infinite geometric series formula $S = \frac{a_1}{1-r}$

with $a_1 = \frac{1}{2}$, $r = \frac{1}{2}$ yields $S = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$, as desired.

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1 (b) $\sum_{K=1}^{\infty} q^{K-1} p = 1$

Begin by re-writing q^{K-1} as $\frac{q^K}{q}$, and moving terms without K out of the summation $\rightarrow \frac{p}{q} \sum_{K=1}^{\infty} q^K$

Expand the first few terms to seek a pattern

$\rightarrow \frac{p}{q}(q + q^2 + q^3 + \ldots) \rightarrow p(1 + q + q^2 + \ldots)$

Note this is a geometric series with $a_1 = p$ and $r = q = 1-p$.

So using the infinite geometric series formula once more yields

$\frac{p}{1-(1-p)} = \frac{p}{p} = 1.$ Thus we have shown

$$\sum_{K=1}^{\infty} q^{K-1} p = 1.$$

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2. Show that

$$\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^{k}}{k!} = 1.$$

Move $e^{-\lambda}$ out of summation.

$$\rightarrow e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^{k}}{k!}$$

Note, the taylor series expansion for

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!}.$$

Thus, $\sum_{k=0}^{\infty} \frac{\lambda^{k}}{k!} = e^{\lambda}$. Finally,

$$e^{-\lambda} \cdot e^{\lambda} = e^{0} = 1,$$ as desired.

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(3) If $X \sim Geo(p)$, find $E(x)$ in as many ways possible.

$$E(x) = \sum_{X=1}^{\infty} x\, P(X=x).$$ $$P(X=x) = p(1-p)^{x-1}$$ Thus,

$$E(x) = \sum_{x=1}^{\infty} x p(1-p)^{x-1}.$$

List some of the first elements:

$$p + 2p(1-p) + 3p(1-p)^2 + 4p(1-p)^3 + \dots$$ since $q = 1-p$

$$E(x) = p + 2qp + 3q^2p + 4q^3p + \dots \quad (1)$$

Multiply by q, form new equation

$$q\,E(x) = qp + 2q^2p + 3q^3p + \dots \quad (2)$$

Perform (1) - (2): $E(x) - qE(x) = p + qp + q^2p + q^3p + \dots$

Factor $E(x)$ on LHS. Factor p on RHS. RHS becomes infinite geometric series with Sum = 1.

$$E(x)[\underbrace{1-q}_{p}] = 1$$

$pE(x) = 1$, thus $E(x) = \frac{1}{p}$

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Problem 4: If $X \sim Poi(\lambda)$, find $E(x)$

$$E(x) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

- Proceed by factoring the constant term $e^{-\lambda}$ out. and see that the first term is zero, so we can begin at $x=1$.

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{x \cdot \lambda^x}{x!}$$

- Can clear x in numerator by rewriting $x!$ as $x(x-1)!$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}$$

- Strip off a λ term, rewrite λ^x as $\lambda^{x-1} \cdot \lambda$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1} \cdot \lambda}{(x-1)!}$$

- Move constant λ term out. Let $\ell = x-1$

$$= \lambda \cdot e^{-\lambda} \cdot \sum_{x=1}^{\infty} \frac{\lambda^{\ell}}{\ell!}$$

- Notice that this is the taylor series expansion for e^{λ}, like earlier. Thus,

$$E(x) = \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda e^0 = \lambda.$$

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Problem 5: Prove $E(x) + E(Y) = E(X+Y)$

Given $E(X+Y) = \sum_i \sum_j (x_i + y_j) \cdot P(X = x_i, Y = y_j)$

We Distribute $(x_i + y_j)$ with $P(X = x_i, Y = y_j)$:

$$\sum_i \sum_j x_i \cdot P(X = x_i, Y = y_j) + y_j \cdot P(X = x_i, Y = y_j)$$

We split the summations:

$$\sum_i \sum_j x_i \cdot P(X = x_i, Y = y_j) + \sum_i \sum_j y_j \cdot P(X = x_i, Y = y_j)$$

x_i does not depend on j, move out by one summation.

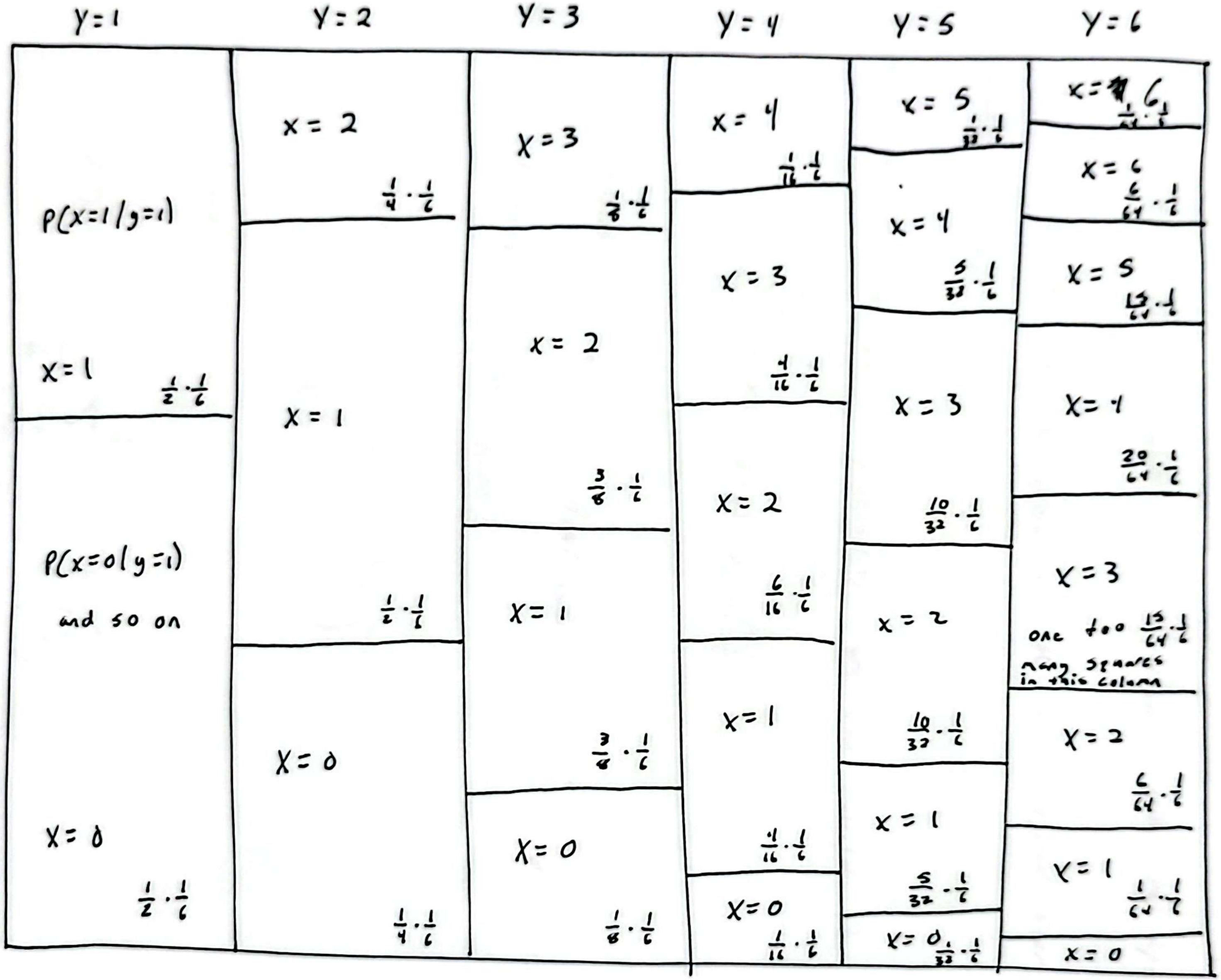
Flip $\sum_i \sum_j$ in the double summation with y_j. Move y_j out one summation.

$$\sum_i x_i \underbrace{\sum_j P(X = x_i, Y = y_j)}_{P(X = x_i)} + \sum_j y_j \underbrace{\sum P(X = x_i, Y = y_j)}_{P(Y = y_j)}, \quad \text{thus}$$

$\sum_i x_i \cdot P(X = x_i) + \sum_j y_j \cdot P(Y = y_j)$. By definition of Expected Value:

$E(x) + E(Y)$. Thus, $E(X+Y) = E(x) + E(Y)$.

$\square$



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Thus, from the previous drawing, we can begin to build a solution.

We note that Y follows a uniform distribution, thus $P(Y=y) = \frac{1}{6}$ in all cases.

$P(X=x \mid Y=y)$ yields the height of a segment within the specific column

$P(X=x \mid Y=y) \cdot P(Y=y)$ yields the area of the prior segment from the whole solution space.

And, using the law of total probability, we can calculate that

$P(X=x)$ using $\sum_{y=1}^{6} P(X=x \mid Y=y) \cdot P(Y=y)$

Since X describes a binomial distribution, and Y describes a uniform distribution, we have,

$$P(X=x) = \sum_{y=\max(1,x)}^{6} \binom{y}{x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{n-x} \cdot \frac{1}{6}$$

or more cleanly

$$P(X=x) = \frac{1}{6} \sum_{y=\max(1,x)}^{6} \binom{y}{x} \left(\frac{1}{2}\right)^{y}$$