Economics on the Edge of Chaos:

How does economics deal with complexity and the implications for systems management

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We discuss some issues and challenges facing economic modellers when confronted with data generated within a non-linear world. The pitfalls associated with the linearization of inherently non-linear models are raised and the concept of *robustness* defined and proposed as a property of scientifically valid models. The existence of chaos in economic time series is discussed and an example, using financial data, presented.

Keywords: *linearization; robustness, structural stability, chaos, attractor.*

> ***"……. since our premises are necessarily false, good theorising consists to a large extent in avoiding assumptions....(with the property that)....a small change in what is posited will seriously affect the conclusions."*** **Baumol (1958)**

1. INTRODUCTION

Clearly the economic world is nonlinear, so it would appear that focusing on linear dynamics is of limited interest. However economists have typically found nonlinear models to be so difficult and intractable that they have adopted the technique of linearisation to deal with them. Thus linear dynamics have retained a role in economic analysis.

The dynamics which are the focus of this paper cannot arise in a linear dynamical system. Let the vector **x** represent the deviations (possibly in logs) of economic variable from their equilibrium values and *A* an $n \times n$ matrix. Then the equation:

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1}$$

represents a linear dynamical system. Any solution path of (1) either:

- converges to the origin (corresponding to equilibrium of the underlying model) or
- diverges to infinity or
- is a cyclical path surrounding the origin (the phase portrait is a centre).

This last possibility is *structurally unstable* in the sense that arbitrarily small perturbations of the matrix *A* will cause the solution paths to transmute into either divergent or convergent paths. Thus regular and persistent cycles cannot occur in a linear model in a structurally stable way. We return to structural instability below.

When nonlinear models are admitted the dynamic possibilities become much more interesting. Suppose (2) is replaced by:

$$\dot{\mathbf{x}} = F(\mathbf{x}) \tag{2}$$

where F is any (reasonable but possibly nonlinear) function. The dynamic behaviour of (2) includes the following possibilities:

- convergence to points
- divergence to infinity
- convergence to limit cycles
- convergence to strange (fractal) attractors.

The last of these occurs in chaotic systems, and plays an important role in section XX below, which is concerned with empirical chaos.

At this stage it is worthwhile drawing the important distinction between *structural stability* and *robustness.* The former is a property of *whole phase portraits*: we define a structurally stable system as one whose phase portrait transmutes into a qualitatively equivalent (homeomorphic) phase portrait in response to arbitrarily small parameter perturbations. The latter is a property of *individual solution paths*: a robust path is one which transmutes into another path with the same limiting behaviour (as $t \to \infty$) in response to arbitrarily small parameter perturbations (where parameters now include boundary conditions). The closed orbit (centre) phase portrait of a linear model is *structurally unstable*. Saddlepoint models of the type discussed in sections 2 and 3 below are *structurally stable*, and in such a model divergent paths are robust, while convergent paths are not. It will be argued in section 3 that non-robust paths are unobservable and thus, for scientific purposes, can be ignored.

Section 2 below deals with the technique of linearisation and examines its shortcomings. Section 3 develops the notion of robustness and uses it to show that many conventional macroeconomic models can be rejected on methodological grounds alone. Both sections use a simple 'money and growth' model to illustrate the argument. This model has certain implications for hyperinflation which are examined empirically in section 4. Section 5 turns to the question of chaos and reports the results of research on the detection of chaos in financial data. Section 6 concludes.

2. LOCAL V. GLOBAL ANALYSIS

Most economic models are inherently nonlinear but, as mentioned in the Introduction, economists have typically found nonlinear dynamics to be intractable and have, accordingly, resorted to the method of local linearisation. This approach relies on a theorem due to Hartman which provides a local linear approximation to a dynamical system such as equation 2 above. The theorem tells us that:

- provided the phase portrait of equation 2 is not a centre, it will be *locally* homeomorphic to the phase portrait of its linearisation (which can be located by evaluating the matrix of first partial derivatives of $F(\mathbf{x})$).

The two vital aspects of this result are:

- The equivalence of phase portraits is *topological* (homeomorphic) only. This equivalence preserves only topological properties, not, for example, distances, angles, curvature or rates of change.
- The equivalence is *local*. It holds only in an arbitrarily small neighbourhood of the point at which the linearisation is calculated. (This could be any point, but in economic models is usually taken to be an equilibrium, i.e. a point $\mathbf{x}^*$ such that $F(\mathbf{x}^*) = \mathbf{0}$.) Outside such an arbitrarily small neighbourhood the equivalence might easily break down.

Both these features of the linearisation process can be misleading when it is applied to economic models, which in many cases, require global rather than local analysis. In this section we develop a simple perfect foresight, market clearing "money and growth" model due originally to Burmeister and Dobell (1971). We compare the local and global dynamics of this model. Local linearisation generates the standard saddlepoint dynamics with a unique long run equilibrium possessing familiar macroeconomic properties. Global analysis reveals that the model has two long run equilibria, a saddlepoint and a stable equilibrium. Convergence to the latter is robust, while convergence to the former is non robust.

Assume a constant returns to scale production function:

$$Y = F(K, L) \tag{3}$$

where Y = output, K = capital stock, L = labour. This may be written in intensive form:

$$y = f(k) \tag{4}$$

where $y = Y/L$ and $k = K/L$. A simple proportionate consumption function is assumed:

$$C = cY \qquad (0 < c < 1) \tag{5}$$

giving a goods market equilibrium condition:

$$Y = cY + \dot{K} + \delta K \qquad (0 < \delta < 1) \tag{6}$$

where δ = depreciation rate. The goods market is assumed to clear continually and thus equation (6) is permanently satisfied. From equations (3) to (6) it is easy to derive the following differential equation:

$$\dot{k} = sf(k) - (\delta + n)k \equiv B(k) \tag{7}$$

where $s = 1 - c$ and n = (exponential) growth of labour force.

Turning now to the money market, a constant exogenous growth rate θ is assumed for the nominal money supply (M):

$$\dot{M} = \theta M \tag{8}$$

A standard demand for money function is assumed:

$$m \equiv M/L = P.G(y, r) \tag{9}$$

where P = price level, r = nominal interest rate and we assume:

$$G_y > 0 \text{ and } G_r < 0 \tag{10}$$

Equation (9) can be rewritten as:

$$x \equiv m/P = G(y, r) \tag{11}$$

where x = per capita real money balances. The money market is assumed to clear continually and the conditions of the implicit function theorem are assumed to hold for equation (11). We may then write:

$$r = H(y, x) \tag{12}$$

for some function H. There are in effect only two assets in the model, money and physical capital. The nominal interest rate may therefore be identified with the expected money yield on physical capital. Assuming the rate of inflation is perfectly foreseen, we may write:

$$r = f'(k) - \delta + \dot{P}/P \tag{13}$$

Equation (13) effectively identifies the real rate of interest with the marginal product of capital, net of depreciation.

From equations (9) to (14) it is easy to derive the following differential equation in x:

$$\dot{x} = x(f'(k) + \theta - \delta - n - H(f(k), x)) \equiv D(k, x) \tag{14}$$

Equations (7) and (14) together constitute a dynamical system in k and x, though note that (7) can be solved independently of (11) since the former equation does not involve x. Local linearisation of this model reveals the existence of an equilibrium (k^*, x^*) such that $k^*, x^* > 0$ and:

$$B(k^*) = 0, \qquad D(k^*, x^*) = 0 \tag{15}$$

In the neighbourhood of this equilibrium dynamics take the saddlepoint form, as illustrated in Figure 1. Paths starting on the stable branch converge to equilibrium. All other paths are divergent and thus treated as economically meaningless (we return to this point in section 3 below, where the "transversality condition" is discussed). Note, however, that *all* solution paths satisfy the perfect foresight and market clearing conditions. In long run equilibrium the level of per capita real money balances is constant and hence the steady state rate of inflation is equal to the (exogenous) rate of money growth minus the (exogenous) rate of population growth.

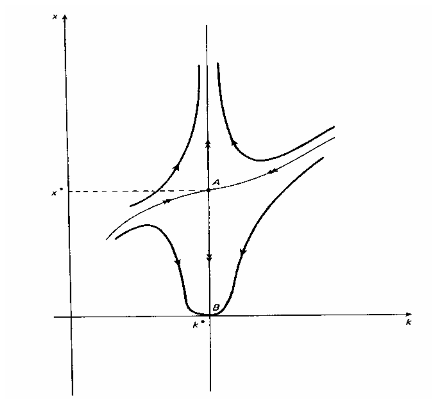

Figure 1

Thus controlling money growth is necessary and sufficient for the (long run) control of inflation.

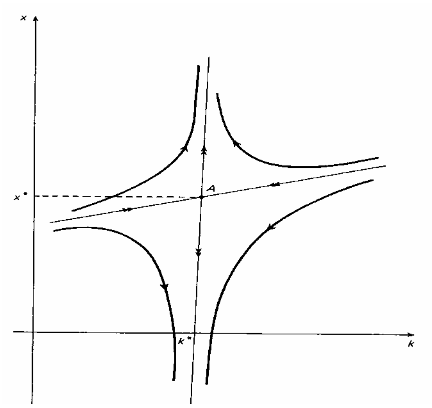

Figure 2

The global dynamics of this model are quite different from the local dynamics (see figure 2). In addition to the saddlepoint discussed above (point A in figure 2), there is a globally stable equilibrium, (k^*,0) (point B in figure 2). Note that Hartman's Theorem does indeed hold in the neighbourhood of point A. Paths with initial conditions lying above the stable branch entail real per capita money diverging to infinity and might therefore be rejected. However, paths with initial conditions lying below the stable branch entail real per capita money tending to zero. There is no reason to reject these paths: they all satisfy the market clearing and perfect foresight conditions and (as will be discussed below) most transversality conditions. Since real per capita money tends to zero along such a path, the price level must eventually grow faster than the rate of growth of nominal money minus the population growth rate. Using standard functional forms for the production function and demand for money function, it is easy to show that the inflation rate diverges to infinity along such a path, and hence the control of the nominal money supply is *not* sufficient for the control of inflation. These paths represent hyperinflationary bubbles, but they are wholly rational, equilibrium bubbles, with no inbuilt tendency to burst. In section 3 below we argue that they represent the major scientifically valid implication of the model.

The model developed above illustrates clearly the dangers of local linearisation. Hartman's theorem only holds locally, so that treating the linearisation as globally valid can lead to to the neglect of dynamics which may have important economic implications. Thus a global approach to dynamics is essential in most economic applications.

3. ROBUSTNESS

It is widely accepted that the demarcation between scientific and non-scientific assertions should be based on the requirement that a scientific assertion be susceptible, at least in principle, to refutation by empirical data. An observation which refutes an implication of a theory also refutes at least one of that theory's underlying assumptions. It follows that the underlying assumptions of a scientific theory need not be open to empirical test, provided that testable implications can be drawn from them. In practice it is generally the case that the assumptions underlying a theory cannot be *exactly* true, but rather, it is to be hoped, are close approximations to reality. Under these circumstances it is essential that the implications of a theory are robust to small variations in its underlying assumptions. Without this property, empirical testing of a theory is impossible.

Consider, for example, a chemical theory which predicts the outcome of a particular chemical reaction under conditions of constant ambient temperature. Whatever care the experimental chemist takes, he will not be able to hold the ambient temperature *exactly* constant: it is bound to fluctuate slightly during the course of the experiment. Suppose now that the outcome of the experiment is substantially different from what the theory predicted. Is the theory refuted? The theoretical chemist can always reply that the ambient temperature was not *exactly* constant as her theory required, and that the experiment does not, therefore, constitute a refutation of her theory. This defence would not be possible if the robustness property were required *ab initio*.

We go further than this, and claim that non-robust implications of a theory are scientifically meaningless and should be ignored. It is a necessary (though possibly not sufficient) condition for scientific validity that a theory make robust predictions.

We may define robustness more precisely as follows:

> ***Definition.*** *Any property of a model will be called robust if the set of parameter values for which it occurs is of strictly positive Lebesgue measure.*

A non-robust property is one which occurs for a parameter set of zero Lebesgue measure and can thus be thought of as having a zero probability of occurring. Of course it is a well-known conundrum of probability theory that although a non-occurring event has probability zero, the converse does not hold. An event with zero probability could occur, but we consider it appropriate to label such events as "unobservable". Note that it is not enough for robustness that the relevant parameter set is dense, because dense sets can easily have zero measure. For example, the set of rational numbers is dense in the set of real numbers, but it is countable, and therefore certainly of measure zero. For the case of dynamic models, the definition of robustness given in the *Introduction* is a special case of the definition above, where the property at issue is the limiting behaviour of a solution path as $t \to \infty$. In a saddlepoint model, such as the linearised model of section 2, convergence is non-robust while divergence is robust. Since the "monetary control of inflation" result depends upon convergence, it too is non-robust, and must therefore be considered an unobservable event. In the global version of the model convergence to the saddlepoint equilibrium (point A) is non-robust, while convergence to the stable equilibrium (point B) is robust and hence so are the hyperinflationary bubbles that this convergence entails, According to the global dynamics of the model then, the "monetary control of inflation" property is unobservable, while hyperinflationary bubbles are observable. In general we conclude that *any* assertion which depends upon convergence in a saddlepoint model can be safely dismissed as non-scientific. Thus a large part of macrodynamics can be dismissed on methodological grounds alone.

A favourite technique for avoiding the non-robustness problem in saddlepoint models is to add auxiliary assumptions to the model in an attempt to rule out divergent paths. Suppose, for example, that the economy is controlled by an agent choosing per capita consumption (z) to maximise a utility function such as:

$$\int_0^\infty U(z,x)e^{(n-\rho)t}dt \qquad (16)$$

subject to the constraints of equations (7) and (14). To ensure convergence of the integral, take $n < \rho$. Applying Pontryagin's Maximum Principle to this problem yields the transversality condition:

$$\lambda k + \mu x \to 0 \quad \text{as} \quad t \to \infty \qquad (17)$$

where λ and μ are the costate variables for equations (7) and (14) respectively. In the linearised saddlepoint model of section 2, only convergent paths satisfy this transversality condition. Hence the addition of a utility maximising assumption shrinks the set of solution paths down to the set of convergent paths, thus avoiding the non-robustness problem.

There are a number of difficulties with this argument:

1. Short of a centrally planned economy it is hard to see exactly who the controlling agent is supposed to be. The utility maximising approach seems to involve teleology.
2. The transversality condition is *not*, in general, a necessary condition for the solution of the Pontryagin problem. (e.g. see Halkin, 1971)
3. In the global version of the model in section 2, the transversality condition (17) is satisfied on paths converging to the stable equilibrium (point B), for a wide class of utility functions *U*.
4. In the linearised version of the model, it is not clear how the economy is supposed to get on to the stable branch if it should become displaced from it (for example by a shift in policy parameters).

Problem no.4 is "solved" by appeal to the notion of a "jump variable". If the price level jumps by exactly the right amount, real per capita balances can jump by just the amount required to place the economy on the stable branch, and thus ensure convergence. Again, short of a centrally planned economy, it is hard to see what mechanism is supposed to bring this jump about.

In summary then, the non-robustness problem which necessarily arises with saddlepoint dynamics, can be avoided by (a) imposing a transversality condition and (b) appealing to the notion of a "jump variable". As discussed above both are unsatisfactory assumptions, and both take the model a long way from its (relatively) innocuous origins in perfect foresight and market clearing. We propose that the robustness property be required of any model claiming scientific status, and that *ad hoc* auxilliary assumptions, designed to ensure robustness, be disallowed. On this basis, virtually all the macroeconomics based on saddlepoint dynamics can be rejected on methodological grounds alone.

In the next section we will pursue the issue of empirical observability and support for nonlinear models via an example (other examples can be found in George and Oxley 2000). In section 4 we investigate the existence of chaos using 16127 daily observations from the Standard & Poor's Composite Price Index (this example is taken from Harrison et al. 1999).

4. CHAOS IN THE S&P 500

Chaos is widely found in the fields of physics and other natural sciences, however, the existence of chaos in economic data is still an open question. Various contributions have been made to this economic literature, including Barnett et al. (1994), Barnett and Chen (1988), Chen (1996), Brock and Sayers (1989), and Ramsey, Sayers and Rothman (1990). In addition, a new international journal, which is exclusively devoted to, and entitled, *Studies in Non-linear Dynamics and Econometrics,* has recently been founded which is testimony to the interest in this area.

Some of the main problems, which pervade the area of economic time series evidence on chaos, are the effects of noise, trend, and more general structural change. Of these noise and time evolution appear to be the most problematic with the latter often modelled via ARCH/GARCH processes that allow changing means and variances. These problems are frequently compounded by the paucity of available data. In attempting to answer questions relating to the existence of nonlinearities and chaos in economic data, researchers have normally used either the Hinich bispectrum test, the BDS test of Brock, Dechert and Scheinkman (1996), White's (1989), test or more recently Kaplan's test, to identify nonlinearities and, when considering chaos, have used tools based on phase space reconstruction developed and used successfully in the physical sciences. The most commonly used of these chaos tests are the Lyapunov exponents test and the Grassberger-Procaccia (GP) correlation dimension test. While these tests have revealed an abundance of previously unexplained non-linear structure and yielded a deeper understanding of the dynamics of many different economic time series, the case of deterministic chaos in these types of series is yet to be clarified. The main problem we identify in this study is that of noise which degrades these measurement techniques. The use of conventional filtering methods such as low pass filtering using Fourier transforms, moving averages etc., and also singular spectrum analysis based on singular value decomposition commonly used in economics, can lead to distortion of the dynamics.

4.1 The Testing Approach

Before economic data can be analysed for the existence of deterministic chaos, the twin problems of growing time trends and noise require consideration. The main contribution of this paper will be to the latter where new non-linear noise reduction (NNR), techniques will be applied to the data. However, the following general methodology will be followed. Firstly, the (log) data will be adjusted to remove systematic calendar effects and trend effects by differencing. Secondly, in order to reconstruct a chaotic attractor in phase space, two basic parameters, the embedding dimension *m,* and delay time *h,* must be correctly determined. The embedding theorem (*m=2d+1*, where *d* is correlation dimension) provides a sufficient condition for reconstructing an attractor from a scalar time series. An efficient method to determine an acceptable minimum *m*, from experimental time series is the so-called *false nearest neighbour* (FNN), recently developed using a geometrical construction. It monitors the behaviour of near neighbours under changes in the embedding dimension from $m \to m+1$. When the number of the false nearest neighbours arising through projection is zero in dimension *m*, the attractor has unfolded in this embedding dimension *m*. This technique is robust to the noise and a correct region of the embedding dimension can be determined in the presence of noise, which is important for the type of data used here. An estimate of the value of the delay time *h,* is provided by the autocorrelation function (ACF).

The Lyapunov exponents test and the Grassberger-Procassia correlation dimension method are well-documented methods used in the quantitative analysis of time series data as tests for chaos; see for example, Abarbanel et al. (1993). Here we concentrate on the latter.

The geometrical features of an attractor can be specified using the Grassberger-Procaccia correlation *dimension.* Suppose we have a scalar time series x_i $(i = 1,2,\ldots N)$ of a dynamical variable sampled at an equal time interval Δt from which the *K* vectors $\mathbf{Y}_j$ *(j=1, 2,...K)* in the *m*-dimensional phase space can be reconstructed using the time delay technique. Then the *correlation dimension* D_2 is defined and calculated as:

$$D_2 = \lim_{\varepsilon \to 0} \frac{\log_2 C_m(\varepsilon)}{\log_2 \varepsilon}, \tag{18}$$

where $C_m(\varepsilon)$ is known as the correlation integral and can be computed as

$$C_m(\varepsilon) = \lim_{K \to \infty} \frac{1}{K(K-1)} \sum_{ij}^{K} \theta(\varepsilon - \|\mathbf{Y}_i - \mathbf{Y}_j\|), \tag{19}$$

Where $\theta(x)$ is the Heaviside step function and $\|\mathbf{Y}_i - \mathbf{Y}_j\|$ is the distance between the vectors $\mathbf{Y}_i$ and $\mathbf{Y_j}$. Thus, the sum $\sum_{ij}^{K} \theta(\varepsilon - \|\mathbf{Y}_i - \mathbf{Y}_j\|)$ is equal to the number of pairs *(i,j)* whose distance $\|\mathbf{Y}_i - \mathbf{Y}_j\|$ in the reconstructed phase space is less than the distance ε. For a chaotic attractor, D_2 is a non-integer, the value of which determines whether the system is low- or high dimensional.

The use of this approach must, however, be applied with caution since it describes a kind of scaling of behaviour in the limit as the distance between points on the attractor approaches zero and therefore is sensitive to the presence of noise. Indeed our numerical experiments have shown that a noise level as small as 2~5% of the time series content can make these measurements inaccurate and inconclusive. Moreover, noise can also prevent precise prediction. Here we use NNR algorithms based on finding and extracting the approximate trajectory, which is close to the original clean dynamics in reconstructed phase space from the observed time series. The implementation of the algorithms involves three basic steps: i) to reconstruct the underlying attractor from the observed series, ii) to estimate the local dynamical behaviour choosing a class of models and fitting the parameters statistically, and iii) to adjust the observations to make them consistent with the clean dynamics. The technique can reduce noise by about one order of magnitude. If some standard techniques are employed to pre-process the data, such as band-pass filtering, filtered embedding and singular value decomposition, significantly larger amounts of noise can be reduced since the local dynamics are enhanced.

These non-linear noise reduction algorithms have been developed under the assumption that the noise is additive rather than dynamic. In practice, the two may not be distinguishable, being based on data only and both of them can be reduced, as long as the exact dynamics can be reconstructed.

While the correlation dimension measurement is often accepted as ‘proof’ of chaos, it is not a definitive test against time series data with certain types of coloured noise. This issue will be resolved using *surrogate* techniques. The correlation dimension method must be applied together with a surrogate technique to reliably discriminate between chaos and noise from a time series, so as to avoid claims of chaos when simpler explanations (such as linearly correlated noise) are adequate. The consideration of surrogate data is based on the following. 1) Statement of a null hypothesis that shall be tested for consistency with the recorded original data. 2) Generation of a number of *surrogate* data sets; an algorithm is to randomise the phase of the raw data so that the surrogated set has the same Fourier spectra as the original. 3) Calculation of the value of interest, e.g., correlation dimension, Lyapunov exponents etc., for the original and all the surrogates. 4) Calculate mean and spread of the results obtained from the surrogates to determine whether the difference to the original, if any, is statistically significant.

4.2 Some results based upon the S&P500

Although several economic data sets have been considered, we will report only one based upon the Standard and Poor’s Composite Price Index, which comprises 16127 daily observations on the logarithmic price change $x_t = 100[\log(p_t) - \log(p_{t-1})]$. For details of the data set see Gallant et al. (1993). The time series x_t has been adjusted to remove systematic calendar and trend effects and is taken to be jointly stationary. A representative window of the raw series taken after 1947 is shown as Figure 3(a).

Figure 3

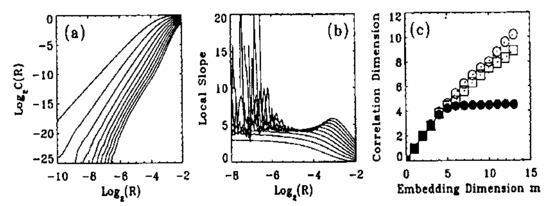

Figure 2. (a) Logarithmic plot of correlation function $C_m(R)$ versus correlation distance R for embedding dimension $m = 3, 4, \ldots, 13$, (b) Local slope as a function of $\text{Log}_2(R)$ derived from (a), and (c) Correlation dimension D_2 versus embedding dimension m: □ – data before NNR, ● – data after NNR, and ○ – surrogate set from the data after NNR.

Our analysis, based upon the GP correlation dimension measurement, in conjunction with the non-linear noise reduction filtering and surrogate technique provides strong evidence in favour of chaos in this data. Some results are detailed below. As shown in Figure 4 (c) (open squares), the raw data gave no saturated correlation dimension on increasing embedding dimension, suggesting that the data may be noise-dominated. Figure 4(b) presents the data after NNR of the raw data.

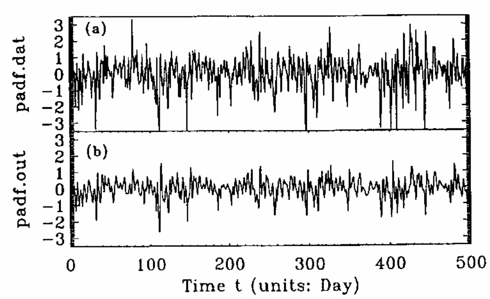

Figure 1. Two time series of $x(t)$ (a) raw data and (b) data after NNR.

Figure 4

Comparison of this with the raw data shows that the effect of noise is manifested as relatively large amplitude random fluctuations, masking the overall deterministic patterns in which the noise level is estimated to be ~90% of the clean signal. Correlation dimension analysis of this noise-filtered data revealed a clear (scaling) saturation region on increasing the embedding dimension as shown in Figure 4 (solid circles in (c)), indicative of deterministic chaos. In confirming that the convergent correlation dimension is a result of chaotic dynamics, both the raw and noise-filtered series were surrogated to randomise the phase and so destroy the deterministic structure. The results show the dimension to diverge as shown in Figure 4(c) (open circles), consistent with stochastic behaviour, from which we confirm that the saturated correlation dimension of the noise-filtered data in Figure 4(c) arises from an endogenous deterministic mechanism.

5. CONCLUSIONS

The paper had several aims. Firstly, we wish to emphasise the need to consider nonlinearities in economic models and to this end highlight some of the dangers of linearisation and using 'tricks' to force behaviours on models which, in general cases, do not arise. Secondly, we argue in favour of "robustness" being considered a necessary property of economic models and make the important distinction between structural stability and robustness. Finally, we use an example to consider the role of nonlinearities in economic behaviour. The example considers whether evidence of deterministic chaos exists in the Standard and Poor's Composite Price Index. When the noise is removed via nonlinear noise reduction techniques, evidence of deterministic chaos is, in fact, found. Nonlinear models and nonlinear methods have a major role to play in economics and econometrics and although they may involve more sophisticated modelling and estimation techniques, the rewards are enormous.

6. ACKNOWLEDGEMENTS

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