Hicksian aggregation season 2

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1. Where have I seen this before?

Recall that in this problem, given $u : \mathbb{R}^{n+1} \to \mathbb{R}$, one wishes to find $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ and $U : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\begin{aligned} \mathbf{x}(\mathbf{p}, q, y) := \arg\max_{\mathbf{x},z} \quad & u(\mathbf{x}, z) \\ \text{s.a.} \quad & \mathbf{px} + qz = y. \end{aligned} , \tag{1}$$

and

$$\begin{aligned} X(f(\mathbf{p}), q, y) := \arg\max_{X,z} \quad & U(X, z) \\ \text{s.a.} \quad & PX + qz = y \end{aligned} \tag{2}$$

satisfy

$$X(f(\mathbf{p}), q, y) = g\big(\mathbf{x}(\mathbf{p}, q, y)\big).$$

Typically, these functions do not exist except in very restrictive cases. Thus, in the general case the equality will not hold and consequently the approximation error could be directly measured by

$$\mathcal{D}(\mathbf{p}, q, y) := \Big| X(f(\mathbf{p}), q, y) - g\big(\mathbf{x}(\mathbf{p}, q, y)\big) \Big|$$

For general functions u and U there is no algebraic expression of the demands X and $\mathbf{x}$. Hence, in our last meeting we discussed a method to compute this difference based on a Taylor approximation of both demand functions. In order to obtain the derivatives one could implicitly differentiate the lagrangians of problems (1) and (2). To put in practice this plan in a particular example one wishes to choose some "nice" utility functions u and U and aggregators f and g to see how these calculations work out. Sadly, the common Cobb-Douglas utility function is of no interest in this setting. Indeed, if $\alpha \in (0, 1)^n$ with $|\alpha|_1 < 1$, then

$$\hat{u}_\alpha(\mathbf{x}, z) = \underbrace{\left(\prod_{j=1}^{n} x_j^{\alpha_j} \right)}_{:=v(\mathbf{x})} z^{1-|\alpha|_1},$$

and thus

$$\hat{u}_\alpha(\mathbf{x}, z) = \tilde{u}(v(\mathbf{x}), z),$$

with

$$\tilde{u}(v, z) = vz^{1-|\alpha|_1}.$$

Since $v : \mathbb{R}^n \to \mathbb{R}$ is also an homothetic function, then functional separability holds and good aggregation is possible by choosing the proper aggregators f and g. Consequently, in order to find an interesting example we must choose some function u that does not allow z to be "independent" of the goos in x. A simple tweak of the function u_α does this trick and gives some interesting results. Let α be as above and define

$$u_\alpha(\mathbf{x}, z) = \sum_{j=1}^{n} x_j^{\alpha_j} z^{1-\alpha_j}. \tag{3}$$

Observe that if all α_j are different, then it is not possible to factor z in the above expression, solving the problem. Indeed, as Varian (1992) points out, functional separability is present only if the preference relation $\succ_u$ represented by the utility function satisfies:

$$(\mathbf{x}, z) \succ (\mathbf{x}', z) \Longleftrightarrow (\mathbf{x}, z') \succ (\mathbf{x}', z') \qquad \forall\ z, z', \mathbf{x}, \mathbf{x}' \tag{4}$$

It is straightforward to see that for the preference relation represented by the function u_α defined in (3), property (4) does not hold unless all α_j are equal.

Observe that when $\alpha_j = \alpha_0$ for every j, then

$$u_{\alpha_0}(\mathbf{x}, z) = \left(\sum_{j=1}^{n} x_j^{\alpha_0} \right) z^{1-\alpha_0}.$$

The parenthesis in this last expression suggest a particular form of aggregation

$$g(\mathbf{x}) := \left(\sum_{j=1}^{n} x_j^{\alpha_0} \right)^{1/\alpha_0}.$$

This function g is also present in the context of international trade. In Helpman et al. (2008) the authors develop a model where g is precisely the utility function of a country. They also suggest a certain price index that could also be useful in this setting but I have not investigated it yet. Given this function g, the aggregate function U should be defined as follows:

$$U_{\alpha_0}(X, z) = X^{\alpha_0} z^{1-\alpha_0},$$

which is the usual Cobb-Douglas utility function. In the general case, when the α_j are not equal, we could define U as follows:

$$U_\alpha(X, z) = X^{\overline{\alpha}} z^{1-\overline{\alpha}},$$

where $\overline{\alpha} = \frac{1}{n}\sum_{j=1}^{n} \alpha_j$. This definition has the property that if $\alpha_j = \alpha_0$ for every j, then $\overline{\alpha} = \alpha_0$.

The only problem with the function u_α is the following. The first order conditions of problem (1) are

$$\alpha_j x_j^{\alpha_j} z^{1-\alpha_j} = \lambda_d p_j x_j, \qquad j = 1, \ldots, n \tag{5}$$

$$\sum_{j=1}^{n} x_j^{\alpha_j} z^{1-\alpha_j} - \sum_{j=1}^{n} \alpha_j x_j^{\alpha_j} z^{1-\alpha_j} = \lambda_d q z \tag{6}$$

Summing (5) over j and replacing in (6) we have

$$\sum_{j=1}^{n} x_j^{\alpha_j} z^{1-\alpha_j} - \lambda_d \sum_{j=1}^{n} p_j x_j = \lambda_d q z,$$

and thus

$$\lambda_d = \frac{u(x, z)}{y}.$$

This means we do not have an analytic expression for the demand $\mathbf{x}(\mathbf{p}, q, y)$ but computing the derivatives should be possible by implicit differentiation of the lagrangian. The derivatives of X can be easily computed since

$$X(f(\mathbf{p}), q, y) = \overline{\alpha}\frac{y}{f(\mathbf{p})}$$

References

E. Helpman, M. Melitz, and Y. Rubinstein. Estimating Trade Flows: Trading Partners and Trading Volumes*. *The Quarterly Journal of Economics*, 123(2):441–487, 05 2008. ISSN 0033-5533. doi: 10.1162/qjec.2008.123.2.441. 2

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