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THE ROBUSTNESS OF EQUILIBRIA TO INCOMPLETE INFORMATION

BY ATSUSHI KAJII AND STEPHEN MORRIS[1]

A number of papers have shown that a strict Nash equilibrium action profile of a game may never be played if there is a small amount of incomplete information (see, for example, Carlsson and van Damme (1993a)). We present a *general approach* to analyzing the robustness of equilibria to a small amount of incomplete information. A Nash equilibrium of a complete information game is said to be *robust to incomplete information* if every incomplete information game with payoffs almost always given by the complete information game has an equilibrium which generates behavior close to the Nash equilibrium. We show that many games with strict equilibria have no robust equilibrium and examine why we get such different results from existing refinements. If a game has a unique correlated equilibrium, it is robust. A natural many-player many-action generalization of risk dominance is a sufficient condition for robustness.

KEYWORDS: Higher order beliefs, incomplete information, refinements, robustness.

1. INTRODUCTION

BEFORE HARSANYI'S SEMINAL CONTRIBUTION (Harsanyi (1967)), game theory was subject to the apparently compelling criticism that all conclusions relied on the assumption that payoffs were common knowledge. Harsanyi's formulation of games of incomplete information allowed analysis of situations where payoffs are not common knowledge. In this paper, we return to the classic question of how sensitive the conclusions of game theory are to the common knowledge of payoffs assumption.

Consider an analyst who plans to model some strategic situation by a particular complete information game. He believes that the game describes the environment correctly with high probability. But he is also aware that the players may be facing some uncertainty about others' payoffs which they will take into account in choosing their strategies. If it is guaranteed that the analyst's prediction based on the complete information game is not qualitatively different from some equilibrium of the real incomplete information game being played, then the analyst will be justified in ignoring subtle informational complications.

To be specific, suppose we fix a Nash equilibrium of a complete information game. Say that it is *robust to incomplete information* if behavior close to it is an equilibrium of *every* nearby incomplete information game. By "nearby" incomplete information game, we mean that the sets of players and actions are the same as in the complete information game and, with high probability, each player knows that his payoffs are the same. We do not make any common

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knowledge or approximate common knowledge assumptions about payoffs; and we allow for the possibility that payoffs are very different with very low probability. By "behavior close to it," we mean that the distribution over actions generated by the Nash equilibrium is close to the distribution over actions generated by an equilibrium of the nearby incomplete information game.

Although this definition has the flavor of earlier "refinements," we show that it has very different properties. In particular, we show that there exists an open set of games which have a unique (strict) Nash equilibrium that is not robust. The argument is based on an "infection argument" similar to that of Rubinstein (1989). Fix a complete information game with a unique (strict) Nash equilibrium. Consider an incomplete information game where payoffs are always the same as the complete information game, except that a single, low probability, "crazy" type of player 1 has a dominant strategy to play an action which is not part of the unique Nash profile. Suppose that some type of player 2 puts high conditional probability on the crazy type of player 1. Even though this "infected" type's payoffs are given by the complete information game, her best response need not be part of the unique Nash profile, since she puts high probability on player 1 not playing according to the unique Nash profile. Thus the crazy type of player 1 determines the equilibrium behavior of the infected type of player 2. Since the crazy type of player 1 has low ex ante probability, the laws of *conditional* probability ensure that the infected type of player 2 must also have low ex ante probability. But if there is a type of player 3 who puts high probability on the infected type of player 2, and another type of player 1 who puts high probability on that type of player 3, and so on, then the argument can be iterated to ensure that a large number of types of all players are infected. Because we place no restrictions on the structure of players' information, behavior at an arbitrarily small event can rule out *even* strict Nash equilibria.

Our robustness test is thus very stringent. Nonetheless, we are able to present two different kinds of sufficient conditions for robustness, although we emphasize that both are very strong. The first comes from the observation that if an incomplete information game is near a complete information game, then any equilibrium of the incomplete information game generates a distribution over actions which is an approximate correlated equilibrium of the complete information game (the payoff uncertainty allows correlation of actions). Thus *if* a complete information game has a unique correlated equilibrium, then that equilibrium—which must be also a Nash equilibrium—is robust, by the upper hemicontinuity of the correlated equilibrium correspondence. This is the first sufficient condition for robustness.

Our second, more interesting, sufficient condition works by showing a necessary connection between the ex ante probability of an event and the probability that players have certain higher order beliefs about that event. In order to develop this result, we first review some earlier work.

Following Morris, Rob, and Shin (1995), say that an action profile $a \equiv (a_1, \dots, a_I)$ in an I player game is a $(p_1, \dots, p_I)$-dominant equilibrium if each action a_i is a best response to any conjecture putting probability at least p_i on other players choosing a_{-i}. Write $\mathbf{p} \equiv (p_1, \ldots, p_I)$ for such a profile of probabilities. Following Monderer and Samet (1989), say that an event is **p**-evident if each individual i attaches probability at least p_i to the event whenever it is true. These definitions can be related together to prove a result about the equilibria of incomplete information games. Suppose that a complete information game has a **p**-dominant equilibrium a; suppose also that an incomplete information game contains an event E which both is **p**-evident and has the property that payoffs are given by the complete information game at all states in E. Then the incomplete information game has an equilibrium where a is played at all states in E: if all players follow a on E, then each player i assigns conditional probability at least p to others playing a_{-i}, so a_i is a best response no matter what actions are taken outside E. Hence we can let the players choose optimal actions outside E, and we still have an equilibrium.

This result suggests one way of identifying robust equilibria. Suppose a complete information game has a **p**-dominant equilibrium a. If we could show that every nearby incomplete information game contains a high probability event E which satisfies the two properties cited above, we would be done. In fact, Monderer and Samet (1989) have provided an algorithm which—for any given information structure—finds the largest probability event E satisfying those two properties. Say that an event is **p**-believed if each individual i believes it with probability at least p_i. Start with the event where the payoffs are given by the complete information game and eliminate those states where that event is not **p**-believed. Now eliminate states where the remaining event is not **p**-believed, and iterate this procedure. Formally, say that an event is common **p**-belief if it is **p**-believed, it is **p**-believed that it is **p**-believed, etc. ... Then the largest event E satisfying the two properties above will be the set of states where it is common **p**-belief that payoffs are given by the complete information game.

Because we don't make any assumptions about players' higher order beliefs about payoffs, there is no guarantee that such an event E has high probability. We only know that the original game is played with high probability. When is it the case that if the probability of an event is high then the probability of the set of states where that event is common **p**-belief is high? Surprisingly, we are able to show that if $\sum_{i=1}^{I} p_i < 1$, the probability that a high probability event is common **p**-belief is high, independent of the fine details of the information structure. Conversely, if $\sum_{i=1}^{I} p_i \geq 1$, then it is possible to construct an information structure which has an event with probability arbitrarily close to 1, that is *never* common **p**-belief. This result shows a surprising necessary connection between the ex ante probability of an event and the ex ante probability of individuals having certain higher order beliefs. As an immediate corollary, we have that if an action profile a is a **p**-dominant equilibrium with $\sum_{i=1}^{I} p_i < 1$, then a is robust.

Although both our sufficient conditions are very strong, they can be used to give a complete characterization of robustness in one special but much studied class of games. In generic two player, two action games, there is always exactly one robust equilibrium: if the game has a unique pure strategy equilibrium, it is robust; if the game has two pure strategy equilibria, then the *risk dominant* equilibrium (Harsanyi and Selten (1988)) is robust; if the game has no pure strategy equilibrium, then the unique mixed strategy equilibrium is robust.

These are the main results of the paper. We will now relate them back to the extensive related literature. Our work follows a classical viewpoint in the refinements literature. Given that an analyst does not know the exact game being played, is there a strategy profile which is close to an equilibrium strategy profile in all nearby games (see, for example, Kohlberg and Mertens (1986))? In the refinements literature, perturbations have typically been either directly on players' action choices (the "trembles" approach) or on the payoffs of a complete information game (as in Kohlberg and Mertens (1986)). We, on the other hand, perturb payoffs indirectly, via the information structure. In this, we follow the view that "perturbations" should be explicitly modelled as arising from player types with different payoffs (see Kreps (1990)). In particular, our technique of "embedding" a complete information game in "nearby" incomplete information games closely follows Fudenberg, Kreps, and Levine (1988) and Dekel and Fudenberg (1990).[2] However, this difference in modelling perturbations does not account for why we get such different results—in particular, the result that even unique strict Nash equilibria need not be robust. In Section 5, we observe that if we restricted attention to nearby incomplete information games with *either* bounded state spaces *or* independent signals about payoffs, then all *strict* equilibria would be robust. Thus the differences arise because of the richness of the information structures that we consider.

Our work builds most strongly on a literature relating higher order beliefs to the equilibria of incomplete information games, especially Monderer and Samet (1989) and Morris, Rob, and Shin (1995), and we use extensively techniques and results from those papers. However, our approach differs from the higher order beliefs literature in a fundamental way. Instead of making *assumptions* about players' higher order beliefs about payoffs, we make assumptions about the ex ante probabilities of payoffs and *deduce* the required properties of players' higher order beliefs, which can then be used to characterize equilibria. Thus in this paper, we ask *which* Nash equilibria of a complete information game can always be played in equilibria of nearby incomplete information games, without allowing almost common knowledge assumptions in the definition of nearby games. In contrast, the higher order beliefs literature has asked how must the notion of a "nearby" incomplete information games be strengthened in order to ensure that, for *every* Nash equilibrium of a complete information game, similar behavior is generated by an equilibrium of *every* nearby incomplete information

[2] But those papers tested whether an outcome can be justified by *some* sequence of perturbed games, while we require robustness to *all* such sequences. Thus our work relates to their work as stability type refinements relate to perfection type refinements.

game.[3] Monderer and Samet (1989) showed that if payoffs of a complete information game are common p-belief, with high probability for some p close to 1, then all Nash equilibria are robust.[4]

Finally, Carlsson and van Damme (1993a) have considered a closely related robustness question. They suppose that each player of a two player, two action game observes a noisy signal of the payoffs. They show that as the noise goes to zero, the unique (Bayesian Nash) equilibrium has the risk dominant (Nash) equilibrium of the complete information game being played.[5] There are many reasons why our results are not directly comparable; but our work should be seen as replicating some of their results in a different setting, as well as providing new results. The key difference is that while Carlsson and van Damme consider a certain critical class of payoff perturbations (with continuous signals), we characterize robustness to *all* perturbations (with countable state spaces).

The paper is organized as follows. In Section 2, we present our approach to embedding complete information games in nearby incomplete information games and define the notion of robustness. In Section 3, we show that a unique correlated equilibrium must be robust but that a unique (strict) Nash equilibrium need not be. In Section 4, we review results on belief operators and common p-belief and present new results on ex ante probabilities and higher order beliefs. In Section 5, these results are applied to give further positive robustness results. We also provide a complete characterization of robustness for two player, two action games. Section 6 concludes.

2. FRAMEWORK

2.1. *Complete Information Games*

Throughout our analysis, we fix a complete information game $\mathscr{G}$ consisting of a finite collection of players $\mathscr{I}=\{1,\dots,I\}$ and, for each player i, a finite action set A_i and payoff function $g_i: A \to \Re$, where $A = A_1 \times \cdots \times A_I$. Thus $\mathscr{G} = \{\mathscr{I}, \{A_i\}_{i\in\mathscr{I}}, \{g_i\}_{i\in\mathscr{I}}\}$. We shall denote $\prod_{j\neq i} A_j$ by A_{-i} and a generic element of A_{-i} by a_{-i}. Similar conventions will be used whenever they are clear from the context.

For any finite set S, denote by $\Delta(S)$ the set of all probability measures on S.

DEFINITION 2.1: An *action distribution*, $\mu \in \Delta(A)$, is a *correlated equilibrium of* $\mathscr{G}$ if, for all $i \in \mathscr{I}$ and $a_i, a_i' \in A_i$,

$$\sum_{a_{-i}\in A_{-i}} g_i(a_i, a_{-i})\mu(a_i, a_{-i}) \geq \sum_{a_{-i}\in A_{-i}} g_i(a_i', a_{-i})\mu(a_i, a_{-i}).$$

[3] For nonstrict Nash equilibria, it is necessary to weaken the solution concept in the incomplete information game to *interim ε-equilibrium*; that is, each player's payoff conditional on each information set must be within ε of the best response.

[4] Monderer and Samet (1996) and Kajii and Morris (1994) showed that this notion of closeness is also *necessary* for this conclusion.

[5] Similar techniques also work in some many player settings—see Carlsson and van Damme (1993b) and Kim (1996).

An action distribution μ is a *Nash equilibrium* if it is a correlated equilibrium and, for all $a \in A$,

$$\mu(a) = \prod_{i \in \mathscr{I}} \mu_i(a_i),$$

where $\mu_i \in \Delta(A_i)$ is the marginal distribution of μ on A_i.

This indirect way of defining (mixed strategy) Nash equilibrium is equivalent to the standard one.

2.2. *Embedding Complete Information Games in Incomplete Information Games*

We will require a way of comparing complete and incomplete information games. For our purposes, an incomplete information game $\mathscr{U}$ consists of (1) the collection of players, $\mathscr{I} = \{1, \ldots, I\}$; (2) their action sets, $A_1, \ldots, A_I$; (3) a countable state space, Ω; (4) a probability measure on the state space, P; (5), for each player i, a partition of the state space, $\mathscr{Q}_i$; and (6), for each player i, a bounded state dependent payoff function, $u_i : A \times \Omega \to \mathfrak{R}$. Thus $\mathscr{U} = \{\mathscr{I}, \{A_i\}_{i \in \mathscr{I}}, \Omega, P, \{\mathscr{Q}_i\}_{i \in \mathscr{I}}, \{u_i\}_{i \in \mathscr{I}}\}$. We write $P(\omega)$ for the probability of the singleton event $\{\omega\}$ and $Q_i(\omega)$ for the (unique) element of $\mathscr{Q}_i$ containing ω. Throughout the paper we will restrict attention to incomplete information games where every information set of every player is possible, that is $P[Q_i(\omega)] > 0$ for all $i \in \mathscr{I}$ and $\omega \in \Omega$. Under this assumption the conditional probability of state ω given information set $Q_i(\omega)$, written $P[\omega | Q_i(\omega)]$, is well-defined by the rule $P[\omega | Q_i(\omega)] = P(\omega)/P[Q_i(\omega)]$. If $\mathscr{U}$ satisfies all the above properties, we say that $\mathscr{U}$ *embeds* $\mathscr{G} = \{\mathscr{I}, \{A_i\}_{i \in \mathscr{I}}, \{g_i\}_{i \in \mathscr{I}}\}$, we write $E(\mathscr{G})$ for the set of incomplete information games which embed $\mathscr{G}$.

A (*mixed*) *strategy* for player i is a $\mathscr{Q}_i$-measurable function $\sigma_i : \Omega \to \Delta(A_i)$. We denote by $\sigma_i(a_i | \omega)$ the probability that action a_i is chosen given ω under σ_i. A *strategy profile* is a function $\sigma = (\sigma_i)_{i \in \mathscr{I}}$ where σ_i is a strategy for player i. We write Σ for the collection of such strategy profiles. We denote by $\sigma(a | \omega)$ the probability that action profile a is chosen given ω under σ; we write σ_{-i} for $(\sigma_j)_{j \neq i}$; when no confusion arises, we extend the domain of each u_i to mixed strategies and thus write $u_i(\sigma(\omega), \omega)$ for $\sum_{a \in A} u_i(a, \omega)\sigma(a | \omega)$. Now the payoff of strategy profile σ to player i is given by the expected utility $\sum_{\omega \in \Omega} \sum_{a \in A} u_i(a, \omega)\sigma(a | \omega)P(\omega)$ which can also be written as $\sum_{\omega \in \Omega} u_i(\sigma(\omega), \omega)P(\omega)$.

DEFINITION 2.2: A strategy profile σ is a *Bayesian Nash equilibrium* of $\mathscr{U}$ if, for all $a_i \in A_i$ and $\omega \in \Omega$,

$$\sum_{\omega' \in Q_i(\omega)} u_i(\sigma(\omega'), \omega')P[\omega' | Q_i(\omega)] \geq \sum_{\omega' \in Q_i(\omega)} u_i(\{a_i, \sigma_{-i}(\omega')\}, \omega')P[\omega' | Q_i(\omega)].$$

A strategy profile σ specifies the probability of a given action profile being played at a given state. However, an outside observer who is unsure of the exact information structure would observe only actions, not states. So we will be interested in a reduced form representation of the strategy profile, where we only report the ex ante probability of certain action profiles being played.

DEFINITION 2.3: An action distribution, $\mu \in \Delta(A)$, is an *equilibrium action distribution* of $\mathscr{U}$ if there exists a Bayesian Nash equilibrium σ of $\mathscr{U}$ such that $\mu(a) = \sum_{\omega \in \Omega} \sigma(a|\omega)P(\omega)$.

2.3. *Robustness*

We want to formalize the idea that an incomplete information game $\mathscr{U}$ is close to a complete information game $\mathscr{G}$ if the payoff structure under $\mathscr{U}$ is equal to that under $\mathscr{G}$ with high probability. Thus, for each incomplete information game $\mathscr{U} \in E(\mathscr{G})$, write $\Omega_{\mathscr{U}}$ for the set of states where payoffs are given by $\mathscr{G}$, and every player knows his payoffs:

$$\Omega_{\mathscr{U}} \equiv \{\omega : u_i(a, \omega') = g_i(a) \text{ for all } a \in A,\ \omega' \in Q_i(\omega), \text{and } i \in \mathscr{I}\}.$$

DEFINITION. 2.4: The incomplete information game $\mathscr{U}$ is an *ε-elaboration* of $\mathscr{G}$ if $\mathscr{U} \in E(\mathscr{G})$ and $P[\Omega_{\mathscr{U}}] = 1 - \varepsilon$. Let $E(\mathscr{G}, \varepsilon)$ be the set of all ε-elaborations of $\mathscr{G}$.

To clarify the idea of ε-elaborations, consider 0-*elaborations*. The *degenerate* 0-elaboration is the game where Ω is a singleton set, and the original complete information game is played with probability 1. The set of equilibrium action distributions of the degenerate incomplete information game is just the set of Nash equilibria of the complete information game. But 0-elaborations can entail more complicated information structures that allow players to correlate their actions. Indeed, an action distribution is an equilibrium action distribution of *some* 0-elaboration of $\mathscr{G}$ if and only if it is a correlated equilibrium of $\mathscr{G}$ (see Aumann (1987)). Thus while two 0-elaborations may appear close to an outside observer, they are not necessarily close from the point of view of the players in the game.

We will measure the distance between action distributions by the max norm:

$$\|\mu - \nu\| = \max_{a \in A} |\mu(a) - \nu(a)|.$$

DEFINITION 2.5: An action distribution μ is *robust to incomplete information* in $\mathscr{G}$ if, for every $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \leq \bar{\varepsilon}$, every $\mathscr{U} \in E(\mathscr{G}, \varepsilon)$ has an equilibrium action distribution ν with $\|\mu - \nu\| \leq \delta$.

Notice that if μ is robust to incomplete information, it is a Nash equilibrium of $\mathscr{G}$ since the degenerate 0-elaboration belongs to $E(\mathscr{G},0)$.

Thus suppose that an analyst thinks that $\mathscr{G}$ is the true game with high probability. Suppose μ is robust to incomplete information in $\mathscr{G}$. Then the analyst can be confident that the actual game has an equilibrium which is arbitrarily close to μ, as long as the actual game is sufficiently close (in the ε-elaboration sense) to $\mathscr{G}$. Conversely, suppose μ is a Nash equilibrium of $\mathscr{G}$ which is not robust. Then the analyst knows that there exists $\delta > 0$ and a sequence of incomplete information games $\mathscr{U}^k \in E(\mathscr{G}, \varepsilon^k)$ with $\varepsilon^k \to 0$ such that every equilibrium action distribution ν^k of $\mathscr{U}^k$ has $|\mu - \nu^k| > \delta$. In that sense, equilibrium μ is not a robust prediction. If $\mathscr{G}$ has no robust equilibrium, then the analyst knows that equilibrium play must be extremely sensitive to the information structure.

REMARK 1: The definition of robustness concerns the behavior of the equilibrium action distribution, i.e., the *average behavior across states*. Thus there is no guarantee that equilibrium behavior in the ε-elaboration is close to the equilibrium behavior of $\mathscr{G}$ *at any given state*. We shouldn't expect behavior at any given state to be close to any Nash equilibrium: consider, for example, information structures which "purify" mixed Nash equilibria of the complete information game.

REMARK 2: It is important that we allow payoffs under an ε-elaboration $\mathscr{U}$ to be very different from $\mathscr{G}$ outside the event $\Omega_{\mathscr{U}}$. If we required, in addition, payoffs outside $\Omega_{\mathscr{U}}$ to be uniformly within ε of $\mathscr{G}$, quite different results would follow. In particular, strict Nash equilibria would be robust.

REMARK 3: An alternative formulation would require every $\mathscr{U} \in E(\mathscr{G}, \varepsilon)$ to have an equilibrium σ with ex ante payoffs within δ of payoffs under μ. This would give the same results, as long as we altered the definition of robustness to allow $\bar{\varepsilon}$ to be a function of the bound on payoffs.

REMARK 4: For simplicity, our definition focuses on elaborations where, with high probability, each player *knows* that his payoffs are given by $\mathscr{G}$. Essentially the same results would follow if we replaced "knows" by "believes with high probability" (assuming we again allowed $\bar{\varepsilon}$ to be a function of the bound on payoffs). Players' uncertainty about their own payoff function is orthogonal to the issues considered in this paper.

REMARK 5: The definition requires $\bar{\varepsilon}$ to be chosen as a function of δ alone. Thus we require a property to hold uniformly over ε-elaborations, as we vary *all* characteristics of the incomplete information games. Allowing $\bar{\varepsilon}$ to depend also on the state space and information partitions would in principle weaken the definition of robustness. However, it will be clear from the arguments which follow that our results would be unchanged if we allowed $\bar{\varepsilon}$ to depend on the state space and information partitions, as long as we still required uniformity with respect to probability distributions on that state space.

REMARK 6: Allowing players to choose actions that are close to a best response (conditional on their information sets) would not significantly weaken our notion of robustness. Formally, suppose that instead of requiring (Bayesian Nash) equilibria in the elaborations, we required only *interim ε-equilibria* (Fudenberg and Tirole (1991, page 563)). We will verify when we give our nonexistence example that this would not change our results.

REMARK 7: We allow players to correlate their actions via information in the elaborations. Yet we require (in our definition of robustness) that an action distribution be (nearly) played in an equilibrium of every nearby elaboration, *including* degenerate ones where no correlation is possible. A more reasonable definition might allow players access to payoff-irrelevant randomizing devices with private signals, uncorrelated with the states of the elaborations (see Cotter (1991)). Thus say that $\mu \in \Delta(A)$ is a *correlated equilibrium action distribution* of $\mathscr{U}$ if there exists a correlated equilibrium $\xi \in \Delta(\Sigma)$ of $\mathscr{U}$ with

$$\mu(a) = \sum_{\sigma \in \Sigma} \sum_{\omega \in \Omega} \xi(\sigma)\sigma(a|\omega)P(\omega).$$

and say that $\mu \in \Delta(A)$ is *robust with correlation to incomplete information* if it satisfies Definition 2.5 with "equilibrium action distribution" replaced by "correlated equilibrium action distribution." By definition, any robust μ is also robust with correlation; thus our positive results would continue to hold with this definition. We will verify that our negative result also holds with this definition.

3. THE ROBUSTNESS OF UNIQUE NASH EQUILIBRIA

In this section, we consider complete information games which have a unique Nash equilibrium and examine when it is robust. Intuitively, this should be the easiest setting in which to demonstrate robustness. However, we first provide an open class of games with a unique (strict) Nash equilibrium which is not robust.

3.1. *Nonexistence*

EXAMPLE 3.1—*The Cyclic Matching Pennies Game*: Consider the following game $\mathscr{G}$. There are 3 players and each player has three possible actions: Heads (H), Tails (T), and Safe (S). Each player's payoff depends only on his own action and the action of his "adversary." Player 3's adversary is player 2, player 2's adversary is player 1, and player 1's adversary is player 3. Thus, for example, 1's payoff is completely independent of 2's action. Each player has a safe action under which he gets 1 (independent of his adversary's action). If he does not play his safe action, then he is playing a cyclic matching pennies game, where he tries to choose the face of the coin different from his adversary's. Each player gets -4 if he matches his adversary's choice, 4 otherwise. Each player gets 0 if his adversary chooses the safe action and he does not. Thus, for example, $g_1(H, H, H) = g_1(H, T, H) = g_1(H, S, H) = -4$, while $g_1(H, H, T) = 4$ and $g_1(H, H, S) = 0$.

Note for future reference that if any player puts probability strictly greater than $5/8$ on his adversary choosing H or T, he has a strict best response to do the opposite (and not play S). Thus if 1 thinks that player 3 will play H with probability $q > 5/8$, his payoff to T is $q(4) + (1-q)(-4) = 8q - 4 > 1$, while the payoff to H is $q(-4) + (1-q)(4) = 4 - 8q < -1$, while the payoff to S is 1.

This game has a *unique* Nash equilibrium where all players choose S. To see why, first suppose that 1 plays H in equilibrium with positive probability, but not T. Then 2 never plays H. So 3 never plays T. So 1 never plays H, a contradiction. Suppose now that 1 plays both H and T in equilibrium. Then 3 must play H and T with equal probability. So the payoff to 1 of playing H and T is 0 which is strictly less than 1, the payoff to playing S. So we have another contradiction.

Now consider the following ε-elaboration of $\mathscr{G}$. Let Ω be the set of nonnegative integers and $P(\omega) = (1 - \sqrt{1-\varepsilon})(\sqrt{1-\varepsilon})^{\omega}$. Partitions are given by $\mathscr{Q}_1 = (\{0,1,2\},\{3,4,5\},\dots)$; $\mathscr{Q}_2 = (\{0\},\{1,2,3\},\{4,5,6\},\dots)$, and $\mathscr{Q}_3 = (\{0,1\}, \{2,3,4\}, \{5,6,7\},\dots)$. This information structure can be generated by a communication protocol similar to that in Rubinstein (1989). Suppose that player 1 sends a message to player 2 which gets lost with probability $1 - \sqrt{1-\varepsilon}$. If the message is lost, no more messages are sent. If the message is received, player 2 sends the message on to player 3; again, it gets lost with (independent) probability $1 - \sqrt{1-\varepsilon}$. If player 3 receives the message, he sends a confirmation back to player 1. Messages continue to be sent along the cycle $1 \to 2 \to 3 \to 1 \to \dots$ until one is lost. If the state ω is the number of messages successfully received, this communication protocol generates exactly the information structure outlined above.

To complete the description of the ε-elaboration, we must specify payoffs. Suppose player 3 has a dominant strategy to play H at information set $\{0,1\}$, player 2 has a dominant strategy to play T at information set $\{0\}$, and payoffs are given by $\mathscr{G}$ everywhere else. Thus $u_i(a,\omega) = g_i(a)$ unless *either* $i = 3$ and $\omega \in \{0,1\}$ *or* $i = 2$ and $\omega = 0$; $u_3((a_1,a_2,H),\omega) = 1$, $u_3((a_1,a_2,T),\omega) = 0$, and $u_3((a_1,a_2,S),\omega) = 0$ for all a_1, a_2, and $\omega \in \{0,1\}$; and $u_2((a_1,T,a_3),0) = 1$, $u_2((a_1,H,a_3),0) = 0$, and $u_3((a_1,S,a_3),0) = 0$ for all a_1, a_3. Note that $\Omega_{\mathscr{U}} = \{2,3,4,\dots\}$, so $P[\Omega_{\mathscr{U}}] = 1 - (1-\sqrt{1-\varepsilon}) - (1-\sqrt{1-\varepsilon})\sqrt{1-\varepsilon} = 1 - \varepsilon$.

In any equilibrium, player 3 must play H at states 1 and 2. But at information set $\{0,1,2\}$, 1 assigns probability at least $2/3$ to 3 playing H, so 1 must play T. But by a similar argument, 2 must then play H at $\{1,2,3\}$, 3 must play T at $\{2,3,4\}$, 1 must play H at $\{3,4,5\}$, and so on. Safe is played nowhere. Thus for any $\varepsilon > 0$, we can construct an ε-elaboration where the unique Nash equilibrium is never played in the unique Bayesian Nash equilibrium. Table I describes the actions in that unique Bayesian Nash equilibrium.

This negative example does, however, suggest a strategy for identifying robust equilibria in other games. The unique Bayesian Nash equilibrium of the ε-elaboration described above involves players alternating between the six action profiles, $(T,T,H),(T,H,H),(T,H,T),(H,H,T),(H,T,T),(H,T,H)$. The

TABLE I

ω	0	1	2	3	4	5	6	...
1's action	T	T	T	H	H	H	T	...
2's action	T	H	H	H	T	T	T	...
3's action	H	H	T	T	T	H	H	...

Pareto-dominated outcomes (H, H, H) and (T, T, T) are thus avoided. Thus as $\varepsilon \to 0$, the equilibrium action distribution generated by the unique Bayesian Nash equilibrium will converge to a uniform distribution on those six profiles. This action distribution is a correlated equilibrium of $\mathscr{G}$. We will show that convergence to a correlated equilibrium is a general property, which we will be able to exploit in providing our first positive robustness result in the next section. But first we will verify that the selection of this equilibrium is robust to a number of features of the construction.

(i) In the above argument, each optimal action gives a payoff of at least $4/3$, and thus gives a payoff at least $1/3$ higher than the next best action. Thus allowing players to take actions which are approximate but not exact best responses would not help in this example.

(ii) The above argument also shows that the strategies identified are the unique (incomplete information game) strategies which survive iterated deletion of strictly interim dominated strategies. This ensures that the unique correlated equilibrium of the incomplete information game entails each player choosing those strategies with probability one. Thus if any action distribution is going to be robust with correlation (see definition in Remark 7), it must be the correlated equilibrium identified above. But notice that we could easily alter the above sequence of ε-elaborations to ensure a *different* limit. Suppose that (for each ε) we increased the probability of states 0, 6, 12, etc. ... by a small amount while decreasing the probability of all other states. If the change was not too large, the same strategy profile would remain the unique one surviving iterated deletion. But the limit equilibrium action distribution would put strictly higher probability on action profile (T, T, H) and strictly lower probability on the others. Thus the cyclic matching pennies game *also* has no correlated equilibrium which is robust *with correlation* to incomplete information.

(iii) If we perturbed payoffs slightly, the argument (relying as it does on iterated deletion of *strictly* dominated strategies) would go through unchanged. Thus the set of complete information games where there is no robust equilibrium is open in the set of payoff matrices.

3.2. *A Sufficient Condition for Robustness*

The previous discussion suggests that the limit of equilibrium action distributions of a sequence of ε-elaborations of a complete information game must be a correlated equilibrium. This gives the following proposition.

PROPOSITION 3.2—Unique Correlated Equilibrium: *If $\mathscr{G}$ has a unique correlated equilibrium μ^*, then μ^* is the unique robust equilibrium of the game.*

The condition that there is a unique correlated equilibrium is very strong but far from vacuous. If a two player, two action game has no pure strategy Nash equilibrium, then the unique mixed strategy equilibrium is the unique correlated equilibrium. Dominance solvable games (where a unique action profile survives iterated deletion of strictly dominated strategies) have unique correlated equilibria; so do two player zero sum games with a unique Nash equilibrium (Aumann (1987) and Forges (1990)). Neyman (1995) and Cripps (1994) give sufficient conditions for all correlated equilibria to be convex combinations of Nash equilibria, and thus for uniqueness of Nash equilibrium to imply uniqueness of correlated equilibrium.

The proof of Proposition 3.2 requires some preliminary definitions and lemmas.

DEFINITION 3.3: Action distribution μ is an η-correlated equilibrium of $\mathscr{G}$ if for all $i \in \mathscr{I}$ and $f: A_i \to A_i$,

$$\sum_{a \in A} \{(g_i(a) - g_i(f(a_i), a_{-i}))\mu(a)\} \geq -\eta.$$

Note that 0-correlated equilibrium is equivalent to the definition of correlated equilibrium (Definition 2.1).

LEMMA 3.4: *For any complete information game $\mathscr{G}$ and $\eta > 0$, there exists $\bar{\varepsilon} > 0$ such that every equilibrium action distribution of every ε-elaboration of $\mathscr{G}$ with $\varepsilon \leq \bar{\varepsilon}$ is an η-correlated equilibrium.*

The straightforward proof is omitted. Since $\Delta(A)$ is compact, the following corollary is an immediate implication of Lemma 3.4.

COROLLARY 3.5: *Suppose $\varepsilon^k \to 0$ as $k \to \infty$, each $\mathscr{U}^k$ is an ε^k-elaboration of $\mathscr{G}$, and each μ^k is an equilibrium action distribution of $\mathscr{U}^k$. Then a subsequence of μ^k converges to some correlated equilibrium of $\mathscr{G}$.*

A related result appears in Shin and Williamson (1996). Now the proof of Proposition 3.2 is completed as follows. Suppose a unique correlated equilibrium μ^* is not robust. Then there exists $\delta > 0$ and a sequence of ε^k-elaborations $\mathscr{U}^k$ where $\varepsilon^k \to 0$ such that for every equilibrium σ^k of $\mathscr{U}^k$ with induced action distribution μ^k, $\| \mu^k - \mu^* \| > \delta$ for all k. By Corollary 3.5, μ^k must have a convergent subsequence whose limit is a correlated equilibrium, which must be different from μ^* since $\| \mu^k - \mu^* \| > \delta$; this contradicts uniqueness.

4. **p**-BELIEF

In this section, we introduce belief operators and the notion of common p-belief. We report and discuss our *critical path result*, showing the connection between ex ante probability and statements about higher order beliefs. This result is used in the next section to prove a positive robustness result.

Fix the information system part of an incomplete information game, i.e., $\{\mathscr{I}, \Omega, P, \{\mathscr{Q}_i\}_{i\in\mathscr{I}}\}$. We maintain the assumption that $P[Q_i(\omega)] > 0$, for all $i \in \mathscr{I}$ and $\omega \in \Omega$. For any number $p_i \in (0,1]$, define

$$B_i^{p_i}(E) \equiv \{\omega : P[E|Q_i(\omega)] \geq p_i\}.$$

That is, $B_i^{p_i}(E)$ is the set of states where player i believes E with probability at least p_i. For any row vector $\mathbf{p} = (p_1, \ldots, p_I) \in (0, 1]^I$, $B_*^{\mathbf{p}}(E) \equiv \bigcap_{i\in\mathscr{I}} B_i^{p_i}(E)$; $B^{\mathbf{p}}{}_*(E)$ is the set of states where E is **p**-*believed*, i.e., each player i believes E with probability at least p_i. An event is **p**-*evident* if it is **p**-believed whenever it is true, i.e., $E \subseteq B_*^{\mathbf{p}}(E)$. An event is *common* **p**-*belief* if it is **p**-believed, it is **p**-believed that it is **p**-believed, etc. ...; thus E is common **p**-belief at state ω if $\omega \in C^{\mathbf{p}}(E) \equiv \bigcap_{n\geq 1} [B_*^{\mathbf{p}}]^n(E)$. Monderer and Samet (1989) introduced such belief operators and characterized common **p**-belief for symmetric **p**, i.e., $C^{(p,\ldots,p)}(E)$. Their common p-belief result remains true essentially as stated in the case of asymmetric **p**, so we omit the proof of the following straightforward extension.

THEOREM 4.1—The Common **p**-Belief Theorem: *E is common* **p**-*belief at ω (i.e., $\omega \in C^{\mathbf{p}}(E)$) if and only if there is a* **p**-*evident event F with $\omega \in F \subseteq B_*^{\mathbf{p}}(E)$.*

What is the connection between the (ex ante) probability of an event E and the ex ante probability of the event $C^{\mathbf{p}}(E)$? The following proposition shows that if $\sum_{i\in\mathscr{I}} p_i < 1$, then $P[C^{\mathbf{p}}(E)]$ is close to 1 whenever $P[E]$ is close to 1, regardless of the state space. Write $\mathscr{F}_i$ for the σ-field generated by $\mathscr{Q}_i$ and say that event E is *simple* if $E = \bigcap_{i\in\mathscr{I}} E_i$, each $E_i \in \mathscr{F}_i$.

PROPOSITION 4.2—The Critical Path Result: *If $\sum_{i\in\mathscr{I}} p_i < 1$, then in any information system $\{\mathscr{I}, \Omega, P, \{\mathscr{Q}_i\}_{i\in\mathscr{I}}\}$, all simple events E satisfy*:

$$P[C^{\mathbf{p}}(E)] \geq 1 - (1 - P[E])\left(\frac{1 - \min\limits_{i\in\mathscr{I}}(p_i)}{1 - \sum\limits_{i\in\mathscr{I}} p_i}\right).$$

What is the significance of the condition $\sum_{i\in\mathscr{I}} p_i < 1$? Consider the case where $I = 2$ and $p_1 = p_2 = p$, so the condition reduces to $p < 1/2$. Suppose we had $\Omega = \{1, 2, \ldots, \infty\}$, $\mathscr{Q}_1 = (\{1\}, \{2,3\}, \ldots, \{\infty\})$, $\mathscr{Q}_2 = (\{1,2\}, \{3,4\}, \ldots, \{\infty\})$, and $P(\omega) = \pi_\omega$. Let $E = \Omega \setminus \{1\} = \{2, 3, \ldots, \infty\}$ and suppose that each time we apply the "everyone believes" operator we knock out one more state, i.e., $B_*^{(p,p)}(E) =$

$\{3,4,\ldots,\infty\}$, $[B_*^{(p,p)}]^2(E)=\{4,5,\ldots,\infty\}$, and so on, so that $C^{(p,p)}(E)=\{\infty\}$. Since $2\notin B_*^{(p,p)}(E)$, we must have $\pi_2/(\pi_1+\pi_2)<p$, i.e., $\pi_2<(p/(1-p))\pi_1$. Since $3\notin[B_*^{(p,p)}]^2(E)$, we must have $\pi_3/(\pi_2+\pi_3)<p$, i.e., $\pi_3<(p/(1-p))\pi_2<(p/(1-p))^2\pi_1$. Thus a lower bound on the probability $C^{(p,p)}(E)$ is $1-\pi_1(1+(p/(1-p))+\cdots+(p/(1-p))^K)$. If $p\geq\frac{1}{2}$, this expression drops below zero as K becomes large, and thus is not a useful bound. But if $p<\frac{1}{2}$, this expression tends to $1-\pi_1((1-p)/(1-2p))$ as K tends to infinity. But since $\pi_1=1-P[E]$, this equals $1-(1-P[E])((1-p)/(1-2p))$, which is exactly the lower bound on $P[C^{(p,p)}(E)]$ given by Proposition 4.2 in the special case where $I=2$ and $p_1=p_2=p$.

The argument of the above paragraph explains where the bound comes from, *if* the information structure has the special form outlined. It turns out that a generalized version of the information system described above is always the "worst case," in the sense of maximizing the probability knocked out by successive applications of the $B_*^{\mathbf{p}}$ operator. But to prove this, we must formalize the idea of maximizing over information systems. The trick is to observe that we are not interested in the whole structure of the information system, but only in the ex ante probability of certain events. Thus we may, without loss of generality, focus on the probability of events which concern us. But the definitions of those events impose certain linear inequalities on their ex ante probabilities. Our maximization problem "over all information systems" can thus be reduced to a linear programming problem and a duality argument. The proof is presented in the Appendix.

Before we use this result to prove robustness results, we make a few further observations.

(i) Proposition 4.2 is tight. For instance, it can be shown by construction that if $\sum_{i\in\mathscr{I}}p_i\geq 1$ and $q<1$, then there exists an information system $\{\mathscr{I},\Omega,P,\{\mathscr{Q}_i\}_{i\in\mathscr{I}}\}$ and a simple event E, with $P[E]=q$ and $C^{\mathbf{p}}(E)=\varnothing$.

(ii) If we did not restrict attention to simple events, a version of Proposition 4.2 still holds with a slightly weaker bound:

COROLLARY 4.3: *In any information system $\{\mathscr{I},\Omega,P,\{\mathscr{Q}_i\}_{i\in\mathscr{I}}\}$, if $\sum_{i\in\mathscr{I}}p_i<1$, then for all events E,*

$$P[C^{\mathbf{p}}(E)]\geq 1-(1-P[E])\left(1+\sum_{i\in\mathscr{I}}\frac{p_i}{1-p_i}\right)\left(\frac{1-\min_{i\in\mathscr{I}}(p_i)}{1-\sum_{i\in\mathscr{I}}p_i}\right).$$

(iii) As Monderer and Samet (1989) observed, the characterization of common **p**-belief described in Theorem 4.1 was independent of the common prior assumption. That is, if we endowed each individual with a different prior measure P_i, and defined belief operators by $B_i^{p_i}(E)\equiv\{\omega:P_i[E|Q_i(\omega)]\geq p_i\}$, Theorem 4.1 and its proof would remain unchanged. Indeed, the game theoretic results which Monderer and Samet prove using common p-belief would also be unchanged. By contrast, Proposition 4.2 relies on the common prior assumption. Suppose $\mathscr{I}=\{1,2\}$; $\Omega=\{1,2,\dots,2N\}$;

$$P_1(\omega)=\begin{cases}\dfrac{1-\varepsilon}{N}, \text{if } \omega \text{ is odd,}\\ \dfrac{\varepsilon}{N}, \text{if } \omega \text{ is even;}\end{cases} \qquad P_2(\omega)=\begin{cases}\dfrac{\varepsilon}{N}, \text{if } \omega \text{ is odd,}\\ \dfrac{1-\varepsilon}{N}, \text{if } \omega \text{ is even;}\end{cases}$$

$\mathscr{Q}_1=(\{1,2\},\{3,4\},\dots,\{2N-1,2N\})$ and $\mathscr{Q}_2=(\{1\},\{2,3\},\dots,\{2N\})$. Consider the event $E=\{3,4,\dots,2N\}$; $P_1[E]=P_2[E]=(N-1)/N$, so we can make both ex ante probabilities of event E arbitrarily close to 1 by choosing N sufficiently large. But $C^{\mathbf{p}}(E)=\varnothing$, for all $\mathbf{p}$ with $p_1>\varepsilon$ and $p_2>\varepsilon$.

5. ROBUSTNESS AND p-DOMINANCE

It is an implication of the work of Monderer and Samet (1989) that any *strict* Nash equilibrium action profile will be played with high probability in some equilibrium action distribution if there is common **p**-belief of payoffs, for **p** close to **1**, with high ex ante probability. But what if we have common **p**-belief of payoffs for some **p** which is not close to **1**?

DEFINITION 5.1: Action profile a^* is a **p**-*dominant equilibrium*[6] of $\mathscr{G}$ if for all $i\in\mathscr{I}$, $a_i\in A_i$, and all $\lambda\in\Delta(A_{-i})$ with $\lambda(a^*_{-i})\geq p_i$,

$$\sum_{a_{-i}\in A_{-i}}\lambda(a_{-i})g_i(a_i^*,a_{-i})\geq\sum_{a_{-i}\in A_{-i}}\lambda(a_{-i})g_i(a_i,a_{-i}).$$

Note that a dominant strategy equilibrium is a $(0,\dots,0)$-dominant equilibrium, and that any pure strategy Nash equilibrium is a **1**-dominant equilibrium. It is easy to see if a^* is a **p**-dominant equilibrium and $\mathbf{p}'\leq\mathbf{p}$ (with the usual vector ordering), then a^* is a $\mathbf{p}'$-dominant equilibrium. So for any pure strategy equilibrium a^*, we are interested in the smallest **p** for which a^* is a **p**-dominant equilibrium.

The following result uses the core idea of Monderer and Samet's (1989) main result.

LEMMA 5.2: *Suppose action profile a^* is a **p**-dominant equilibrium of $\mathscr{G}$. Consider any $\mathscr{U}\in E(\mathscr{G})$ and let F be a **p**-evident event such that $F\subseteq\Omega_{\mathscr{U}}$. Then $\mathscr{U}$ has a Bayesian Nash equilibrium where $\sigma_i(a_i^*\mid\omega)=1$ for all $i\in\mathscr{I}$ and $\omega\in F$.*

PROOF: Let $F_i=B_i^{p_i}(F)$, so $F\subseteq\bigcap_{i\in\mathscr{I}}F_i$ by assumption. Consider the modified incomplete game $\mathscr{U}'$ where each player's strategy must satisfy $\sigma_i(a_i^*\mid\omega')=1$,

[6]This definition extends the definition in Morris, Rob, and Shin (1995) to the many player case with asymmetric **p**; we have also replaced strict inequalities by weak inequalities.

for all $\omega' \in F_i$ and $i \in \mathcal{I}$. There exists an equilibrium σ of the modified game. We shall show that σ is in fact an equilibrium of $\mathcal{U}$. By construction, for every i, at any $\omega \notin F_i, \sigma_i$ is a best response to σ_{-i}. Let $\omega \in F_i$. Then by definition, $P[F|Q_i(\omega)] \geq p_i$; thus by the construction of σ, the conditional probability of a^*_{-i} being played is at least p_i, so a^*_i is a best response against σ_{-i}. Thus σ is also an equilibrium of the original game. *Q.E.D.*

It is an immediate corollary that if a^* is a strict Nash equilibrium of $\mathcal{G}$, then there exists $p<1$ such that if $P[C^{(p,\dots,p)}(\Omega_{\mathcal{U}})] \geq p$, $\mathcal{U}$ has an equilibrium action distribution where a^* is played with probability at least p.

This does not suffice to prove a robustness result because there is no guarantee that a high value of $P[\Omega_{\mathcal{U}})]$ implies a high value of $P[C^{(p,\dots,p)}(\Omega_{\mathcal{U}})]$. This would be true if we restricted attention to elaborations $\mathcal{U}$ with either independent types or a bound on the number of states of the world. But assuming independent types or a bounded state space are just indirect ways of restricting higher order beliefs.

The cyclic matching pennies game (Example 3.1) showed why there is (in general) a difference between assuming only that $\Omega_{\mathcal{U}}$ has high probability, and that $C^{(p,\dots,p)}(\Omega_{\mathcal{U}})$ has high probability. The unique (strict) Nash equilibrium in that example is in fact $(4/5, 4/5, 4/5)$-dominant, but is not played in any equilibrium of the nearby elaborations we constructed. This is because there was no high probability $(4/5, 4/5, 4/5)$-evident event contained in $\Omega_{\mathcal{U}}$.

To establish robustness, we need to show that for any $\delta > 0$, $[C^{\mathbf{p}}(\Omega_{\mathcal{U}})] \geq 1-\delta$ for *any* ε-elaboration $\mathcal{U}$ where ε is small enough. But since $C^{\mathbf{p}}(\Omega_{\mathcal{U}})$ is the largest **p**-evident set contained in $\Omega_{\mathcal{U}}$, this is exactly what Proposition 4.2 provides. In particular, we have the following proposition:

PROPOSITION 5.3: *Suppose action profile a^* is a* **p***-dominant equilibrium of $\mathcal{G}$ with $\sum_{i\in\mathcal{I}} p_i < 1$. Then a^* is robust to incomplete information.*

PROOF: Write μ^* for the distribution putting probability 1 on a^*. Fix any $\delta > 0$. By Proposition 4.2, we can choose $\varepsilon > 0$, such that $P[E] > 1-\varepsilon$ implies $P[C^{\mathbf{p}}(E)] > 1-\delta$. Thus by construction of ε, for any $\mathcal{U} \in E(\mathcal{G}, \varepsilon)$, we have $P[C^{\mathbf{p}}(\Omega_{\mathcal{U}})] > 1-\delta$. By Lemma 5.2, there exists a Bayesian Nash equilibrium of $\mathcal{U}$ with $\sigma_i(a^*_i|\omega) = 1$, for all $\omega \in C^{\mathbf{p}}(\Omega_{\mathcal{U}})$. Thus there exists an equilibrium action distribution of $\mathcal{U}$ with $\mu(a^*) \geq P[C^{\mathbf{p}}(\Omega_{\mathcal{U}})] > 1-\delta$. Therefore, $|\mu(a) - \mu^*(a)| < \delta$ for all $a \in A$, so μ^* is robust. *Q.E.D.*

When can we show that there is a unique robust equilibrium? There exist nongeneric games where many action profiles satisfy the sufficient condition of Proposition 5.3. For example, consider a degenerate game where payoffs are constant for every player, regardless of actions. Every action profile is a $(0,\dots,0)$-dominant equilibrium of such a game and thus, by Proposition 5.3, every action profile is robust. But consider the following slight refinement of **p**-dominant equilibrium.

DEFINITION 5.4: Action profile a^* is a *strict* **p**-*dominant equilibrium* of $\mathscr{G}$ if for all $i \in \mathscr{I}, a_i \in A_i$, and all $\lambda \in \Delta(A_{-i})$ with $\lambda(a^*_{-i}) > p_i$,

$$\sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i(a_i^*, a_{-i}) > \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i(a_i, a_{-i}).$$

That is, playing a_i^* is a unique best response if i believes that a^*_{-i} is played with probability strictly greater than p_i. By continuity, a strict **p**-dominant equilibrium is *always* a **p**-dominant equilibrium. In the degenerate game above, no action profile is a strict **p**-dominant equilibrium, for *any* **p**. So strict **p**-dominance is a strict refinement of **p**-dominance. But for a generic choice of payoffs, any **p**-dominant equilibrium is a strict **p**-dominant equilibrium.

If a game $\mathscr{G}$ has a *strict* **p**-dominant equilibrium a^* for some **p** with $\sum_{i \in \mathscr{I}} p_i \leq 1$, then there exists a nearby incomplete information game with a unique equilibrium where a^* is played at every state.

LEMMA 5.5: *Suppose action profile a^* is a strict* **p**-*dominant equilibrium of $\mathscr{G}$ with $\sum_{i \in \mathscr{I}} p_i \leq 1$. Then, for all $\varepsilon > 0$, there exists $\mathscr{U} \in E(\mathscr{G}, \varepsilon)$ such that the unique equilibrium action distribution μ of $\mathscr{U}$ has $\mu(a^*) = 1$.*

PROOF: The proof is by construction of $\mathscr{U}$.[7] Suppose a^* is strict **p**-dominant with $\sum_{i \in \mathscr{I}} p_i \leq 1$. Let $q_i = (p_i / \sum_{j \in \mathscr{I}} p_j) \geq p_i$ for each $i \in \mathscr{I}$. It follows that $\sum_{i \in \mathscr{I}} q_i = 1$. Now let $\Omega = \mathscr{I} \times \mathscr{Z}_+$ and $P(i,k) = \varepsilon(1-\varepsilon)^k q_i$. Let each $\mathscr{Q}_i$ consist of (i) the event $E_i^0 = \{(j,0)_{j \neq i}\}$; and (ii) all events of the form $E_i^k = \{(i, k-1), (j,k)_{j \neq i}\}$, for each integer $k \geq 1$. Let

$$u_i(a, \omega) = \begin{Bmatrix} g_i(a), & \text{if } \omega \notin E_i^0 \\ 1, & \text{if } \omega = E_i^0 \text{ and } a_i = a_i^* \\ 0, & \text{if } \omega = E_i^0 \text{ and } a_i \neq a_i^* \end{Bmatrix},$$

and let σ be any equilibrium of $\mathscr{U}$. Write $\mu(a_{-i}|\omega)$ for the probability i attaches to action profile a_{-i} given strategy profile σ_{-i}, i.e.,

$$\mu(a_{-i}|\omega) = \sum_{\omega' \in Q_i(\omega)} \left\{ \prod_{j \neq i} \sigma_j\left(a_j | \omega'\right) \right\} P[\omega' | Q_i(\omega)].$$

We show that $\sigma_i(a_i^*|\omega) = 1$ for all $\omega \in \Omega$ and $i \in \mathscr{I}$, by induction. By construction, $\sigma_i(a_i^*|\omega) = 1$ for all $\omega \in E_i^0$ and $i \in \mathscr{I}$, since a_i^* is a dominant action at such states. Now our inductive hypothesis is that $\sigma_i(a_i^*|\omega) = 1$ for all

[7] The proof is a many player extension of an argument in Morris, Rob, and Shin (1995).

$\omega \in E_i^k$ and $i \in \mathscr{I}$. Consider any $\omega \in E_j^{k+1}$. By construction of the state space, we have

$$P\left[\{(j,k)\} | Q_j(\omega)\right] = \frac{\varepsilon(1-\varepsilon)^k q_j}{\varepsilon(1-\varepsilon)^k q_j + \left(\sum_{i \neq j} \varepsilon(1-\varepsilon)^{k+1} q_i\right)}$$

$$= \frac{q_j}{q_j + (1-\varepsilon)\left(\sum_{i \neq j} q_i\right)} > q_j \geq p_j.$$

Thus by the inductive hypothesis, we have $\mu(a_{-j}^* | \omega) > p_j$. Since a^* is a strict **p**-dominant equilibrium, this shows that $\sigma_j(a_j^* | \omega) = 1$. Thus our inductive hypothesis holds for $k+1$. *Q.E.D.*

Now combining Proposition 5.3 with Lemma 5.5 gives the following corollary.

COROLLARY 5.6—Uniqueness: *Suppose action profile a^* is a strict **p**-dominant equilibrium in $\mathscr{G}$ with $\sum_{i \in \mathscr{I}} p_i < 1$. Then a^* is the unique robust equilibrium of the game.*

PROOF: Action profile a^* is a **p**-dominant equilibrium in $\mathscr{G}$ with $\sum_{i \in \mathscr{I}} p_i < 1$, and so is robust by Lemma 5.3. But by Lemma 5.5, no action profile other than a^* is played in any robust equilibrium. *Q.E.D.*

EXAMPLE 5.7—Pure Co-ordination Game: Let $A_i = \{a^1, \ldots, a^N\}$ for each $i \in \mathscr{I}$. Let

$$g_i(a) = \begin{cases} x_i^n, & \text{if } a_j = a^n, \text{for all } j \in \mathscr{I}, \\ 0, & \text{otherwise,} \end{cases}$$

where $x_i^n > 0$ for all $i \in \mathscr{I}$ and $n = 1, \ldots, N$.

This game has N symmetric pure strategy strict Nash equilibria. Equilibrium $a = (a^n, \ldots, a^n)$ is strict **p**-dominant if $p_i x_i^n > (1-p_i) x_i^m$ for all $m \neq n$ and $i \in \mathscr{I}$, i.e., if $p_i > \max_{m \neq n} x_i^m / (x_i^m + x_i^n)$ for all $i \in \mathscr{I}$. Thus, by Corollary 5.6, $a = (a^n, \ldots, a^n)$ is the unique robust equilibrium if $\sum_{i \in \mathscr{I}} (\max_{m \neq n} x_i^m / (x_i^m + x_i^n)) < 1$.

Let us consider two special cases:

(i) *Two players*. If the game has a *strictly Pareto dominant* equilibrium ((a^n, a^n) is strictly Pareto dominant if $x_i^n > x_i^m$ for $i = 1,2$ and all $m \neq n$), then the strictly Pareto dominant equilibrium is (p, p)-dominant for some $p < 1/2$, and thus the unique robust equilibrium.

(ii) *Symmetric payoffs*. Suppose $x_i^n = x^n$ for all $i \in \mathscr{I}$ and $x^1 > x^2 \ldots > x^n$. Then x^1 is the unique robust equilibrium if $x^2 < x^1/(I-1)$. Thus there always exists a unique robust equilibrium if there are two players. But, for fixed $(x^1, \ldots, x^n)$, no equilibrium satisfies our sufficient condition for robustness if I is sufficiently large. The sufficient condition of Corollary 5.6 becomes close to requiring that the equilibrium be a dominant strategy equilibrium as the number of players becomes large.

The sufficient conditions of Propositions 3.2 and 5.3, and Corollary 5.6, provide only a limited characterization of robustness, in general. But they are enough to give a complete characterization of robustness in generic two player, two action games. Except for some nongeneric cases, a two player, two action game has (i) a unique (strict) pure strategy equilibrium; (ii) two (strict) pure strategy equilibria; or (iii) no pure strategy equilibrium and a unique mixed strategy equilibrium.

(i) If there is a unique pure strategy equilibrium, then at least one player has a dominant strategy to play his action in that equilibrium. Say that it is player 1. Then the unique equilibrium is $(0, p)$-dominant for some $p < 1$. Thus the unique Nash equilibrium is robust by Proposition 5.3.

(ii) If there are two pure strategy equilibria, then (generically) exactly one of them is *risk dominant* in the sense of Harsanyi and Selten (1988). In a two player, two action game, an equilibrium is risk dominant exactly if it is a strict (p_1, p_2)-dominant equilibrium for some (p_1, p_2) with $p_1 + p_2 < 1$. Now by Corollary 5.6, the risk dominant equilibrium is the unique robust equilibrium.

(iii) If a generic two player two action game has no pure strategy Nash equilibrium, then the (unique) mixed strategy equilibrium is the unique correlated equilibrium. So that unique equilibrium must be robust by Proposition 3.2.

6. CONCLUSION

Why should one strict Nash equilibrium be more likely to be played than any other? The literature provides different approaches to answering this question. Many approaches, though, agree in predicting that in a two player, two action game with two strict Nash equilibria, the *risk dominant* equilibrium is more likely to be played. What is driving the selection of the risk dominant equilibrium in the different approaches? Existing approaches rely on symmetry assumptions.[8] Thus Carlsson and van Damme (1993a) showed that a lack of common knowledge of payoffs might lead to the selection of the risk dominant equilibrium. However, they considered one particular (if natural) class of symmetric perturbations. They did not address the question of whether another set of perturbations might lead to another equilibrium being selected. Morris, Rob, and Shin (1995) clarified the relation of Carlsson and van Damme's argument to higher order beliefs, and showed that a strong version of their argument could

[8] Harsanyi and Selten's (1988) axiomatic justification relies on a symmetry assumption combined with a best response invariance property. Deterministic dynamic justifications (e.g., Harsanyi and Selten's tracing procedure) rely on a symmetric prior in the initial condition, combined with the larger basin of attraction property. Stochastic stability arguments (e.g., Kandori, Mailath, and Rob (1993)) are driven by symmetry assumptions about mutations (see Bergin and Lipman (1996)).

not work for equilibria of two player, two action games which were not risk dominant. This paper, on the other hand, provides a general notion of robustness of equilibria to incomplete information. The risk dominant equilibrium of a complete information game is shown to be robust to *every* perturbation where payoffs are given by the original game with high probability and players know their own types with high probability. This result is driven not by some ad hoc symmetry restriction from outside the game but on a new and surprising property of higher order beliefs under the common prior assumption (Proposition 4.2).

However the analysis of the paper is more general than the two player, two action case. We formulate a general notion of robustness of equilibria to incomplete information. We show that even games with strict Nash equilibria may have no robust equilibrium. There is an open set of such games. We have provided two different kinds of positive results. The correlated equilibrium result is straightforward. The common **p**-belief results utilized new work showing the existence of a surprising amount of structure in arbitrary belief hierarchies with common priors. We believe that this "hidden content" of the common prior assumption is an important area of future work.

Beyond these contributions, this paper makes a substantive connection between the higher order beliefs literature and the refinements literature.[9] Why is this important? We believe that economists modelling strategic situations may have good information about the players' payoffs. They are unlikely to be well informed about players' beliefs about others' beliefs about payoffs, and other such higher order beliefs. Thus we considered a notion of "closeness" of games that makes *no assumptions* about players' higher order beliefs in the perturbed game. We said that an incomplete information game is a small perturbation of a complete information game (from the analyst's point of view) if the payoffs are almost always the same. This view of perturbed games follows the existing refinements literature. We differ from the existing refinements literature in having a rich structure of correlated types, which allows a rich structure of higher order beliefs; thus the connection to higher order beliefs.

Given that some games have no robust equilibrium, and in the light of the refinements literature, it is natural to ask if a *set* of equilibria is robust. Say that a set of action distributions is robust to incomplete information if any nearby incomplete information game has an equilibrium that generates an action distribution that is close to *some* element of the set. By Corollary 3.5, it is immediate that the set of all correlated equilibria is robust in this sense. But which robust sets of action distributions are minimal under set theoretic inclusion? Our results can be interpreted as identifying settings where the minimal set is a singleton.[10]

[9] See Börgers (1994) for one earlier paper that linked higher order beliefs to refinements.
[10] A referee emphasized this interpretation of our results to us.

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APPENDIX

We shall prove Proposition 4.2 and Corollary 4.3. We first provide two lemmas that relate together the ex ante probability of an event and the ex ante probability of first order beliefs about that event (Lemmas A and B). Then (in Lemma C), we provide an inequality relating the ex ante probability of an event (i.e., $P[E]$) and the ex ante probability that that event is Kth order $\mathbf{p}$-belief (i.e., $P[[B^{\mathbf{p}}]^K(E)]$). The proof of this lemma constitutes the bulk of the proof. Finally (in Lemma F) we examine what happens to that inequality as K becomes large. We show that if $\sum_{i\in\mathcal{I}} p_i < 1$, the limit of the K-inequality exists and is the desired bound.

LEMMA A: *For all events E, if $F \in \mathcal{F}_i$ and $F \subseteq \Omega \backslash B_i^{p_i}(E)$, then $P[E\cap F] \leq (p_i/(1-p_i))P[F\backslash E]$.*

PROOF: If $P[F]=0$, the result holds trivially. If $P[F]>0$, $F\in\mathcal{F}_i$ and $F\subseteq \Omega\backslash B_i^{p_i}(E)$, then $P[E|F]<p_i$, so $P[E\cap F]=P[E|F]P[F]<p_iP[F]$; then $(1-p_i)P[E\cap F]<p_iP[F\backslash E]$. *Q.E.D.*

LEMMA B: *For all events E, $1-P[B_*^{\mathbf{p}}(E)] \leq (1+\sum_{i\in\mathcal{I}}(p_i/(1-p_i)))(1-P[E])$.*

PROOF: First note that

$$P[E\backslash B_*^{\mathbf{p}}(E)] \leq \sum_{i\in\mathcal{I}} P[E\backslash B_i^{p_i}(E)] \leq \sum_{i\in\mathcal{I}}\left(\frac{p_i}{1-p_i}\right)P[\Omega\backslash(E\cup B_i^{p_i}(E))].$$

So

$$\begin{aligned}
1-P[B_*^{\mathbf{p}}(E)] &= P[E\backslash B_*^{\mathbf{p}}(E)] + P[\Omega\backslash(E\cup B_*^{\mathbf{p}}(E))],\\
&\leq \sum_{i\in\mathcal{I}}\left(\frac{p_i}{1-p_i}\right)P[\Omega\backslash(E\cup B_i^{p_i}(E))] + P[\Omega\backslash(E\cup B_*^{\mathbf{p}}(E))],\\
&\leq \left(1+\sum_{i\in\mathcal{I}}\left(\frac{p_i}{1-p_i}\right)\right)P[\Omega\backslash(E\cup B_i^{p_i}(E))],\\
&\leq \left(1+\sum_{i\in\mathcal{I}}\left(\frac{p_i}{1-p_i}\right)\right)(1-P[E]). \qquad \textit{Q.E.D.}
\end{aligned}$$

Since $P[C^{\mathbf{p}}(E)]=\lim_{K\to\infty}P[[B_*^{\mathbf{p}}]^K(E)]$, we would like to provide an upper bound for $1-P[[B_*^{\mathbf{p}}]^K(E)]$ as a function of $1-P[E]$. Iterated application of Lemma B gives us that

$$1-P[[B_*^{\mathbf{p}}]^K(E)] \leq \left(1+\sum_{i\in\mathcal{I}}\frac{p_i}{1-p_i}\right)^K(1-P[E]).$$

This implies that for any given $\mathbf{p}\in(0,1)^I$ and K, we can guarantee (uniformly across information systems) that $P[[B_*{}^{\mathbf{p}}]^K(E)]$ is large by setting $P[E]$ sufficiently close to 1. But for any $\mathbf{p}\in(0,1)^I$,

$(1+\sum_{i\in\mathcal{I}} p_i/(1-p_i))^K \to \infty$ as $K\to\infty$, so this inequality will be of no use in bounding $P[C^{\mathbf{p}}(E)]$. We need a tighter bound. The bound is constructed from the following $I\times I$ matrix $\mathbf{R}$:

$$\mathbf{R}=\begin{pmatrix} 0 & \frac{p_2}{1-p_2} & \frac{p_3}{1-p_3} & & \frac{p_I}{1-p_I} \\ \frac{p_1}{1-p_1} & 0 & \frac{p_3}{1-p_3} & \cdots & \frac{p_I}{1-p_I} \\ \frac{p_1}{1-p_1} & \frac{p_2}{1-p_2} & 0 & & \frac{p_I}{1-p_I} \\ & \vdots & & \ddots & \vdots \\ \frac{p_1}{1-p_1} & \frac{p_2}{1-p_2} & \frac{p_3}{1-p_3} & \cdots & 0 \end{pmatrix}.$$

Note that $\mathbf{R}$ depends on $\mathbf{p}$. Writing $[\mathbf{x}']_i$ for the ith element of column vector $\mathbf{x}'$, let $\xi^*(\mathbf{p},K)=\max_{i\in\mathcal{I}}([(\mathbf{I}+\mathbf{R}+\cdots+\mathbf{R}^K)\mathbf{1}']_i)$.

Lemma C: *In any information system* $\{\mathcal{I},\Omega,P,\{\mathcal{Q}_i\}_{i\in\mathcal{I}}\}$, *for any simple event* E,

$$1-P[[B_*^{\mathbf{p}}]^K(E)]\le(1-P[E])\xi^*(\mathbf{p},K).$$

Intuitively, we want to maximize (over *all* information systems and *all* simple events) the value of $1-P[[B_*^{\mathbf{p}}]^K(E)]$, subject to the constraint that $P[E]\ge 1-\varepsilon$, and show that the maximand of this problem is $\varepsilon\xi^*(\mathbf{p},K)$. Our proof of Lemma C involves solving this maximization problem, which is (implicitly) parameterized by $\mathbf{p}$, K, and ε. Our proof utilizes some notation providing a labelling of different regions of the state space, which is outlined below.

Fix $\{\mathcal{I},\Omega,P,\{\mathcal{Q}_i\}_{i\in\mathcal{I}}\}$ and, for each i, $E_i^1\in\mathcal{F}_i$. Thus $E^1=\bigcap_{i\in\mathcal{I}}E_i^1$ is a simple event. Define inductively $\{E_1^n,\ldots,E_I^n,E^n\}_{n=1}^{K+1}$ as follows: $E^n=\bigcap_{i\in\mathcal{I}}E_i^n$ and $E_i^{n+1}=B_i^{p_i}(E^n)$. Thus $E_i^2=B_i^{p_i}(E^1)$ and $E_i^n=B_i^{p_i}([B_*^{\mathbf{p}}]^{n-2}(E^1))$ for all $n\ge 2$. By convention, let $E_i^0=\Omega$, for all i. Let $D_i^n=E_i^n/E_i^{n+1}$ for all $n=0,K$, and $D_i^{K+1}=E_i^{K+1}$. Since $B_*^{\mathbf{p}}(E)\subseteq E$ for all simple events E, the sets $\{D_i^n\}_{n=0}^{K+1}$ partition Ω for each i. In particular, $\{D_i^n\}_{n=0}^{K+1}$ is a coarser partition than $\mathcal{Q}_i$. Let $\mathbf{n}=(n_1,\ldots,n_I)$ be a typical element of $\{0,1,\ldots,K+1\}^I$. We shall denote by $\min(\mathbf{n})$ the smallest number in $\{n_1,\ldots,n_I\}$. Define $L(\mathbf{n})\equiv\bigcap_{i\in\mathcal{I}}D_i^{n_i}$ and $\pi(\mathbf{n})\equiv P[L(\mathbf{n})]$ for all $\mathbf{n}\in\{0,\ldots,K+1\}^I$. This notation allows us to characterize every relevant region of the state space by an "address" $\mathbf{n}\in\{0,\ldots,K+1\}^I$; note that the $L(\mathbf{n})$ are disjoint. With two individuals, we can represent the state space by the following box:

$L(0,0)$	$L(0,1)$	$\cdots$	$L(0,n_2)$	$\cdots$	$L(0,K+1)$	$\leftarrow D_1^0$
$L(1,0)$	$L(1,1)$		$L(1,n_2)$		$L(1,K+1)$	$\leftarrow D_1^1$
$\vdots$		$\ddots$				
$L(n_1,0)$	$L(n_1,1)$		$L(n_1,n_2)$		$L(n_2,K+1)$	$\leftarrow D_1^{n_1}$
$\vdots$				$\ddots$		
$L(K+1,0)$	$L(K+1,1)$		$L(K+1,n_2)$		$L(K+1,K+1)$	$\leftarrow D_1^{K+1}$
$\uparrow$ D_2^0	$\uparrow$ D_2^1		$\uparrow$ $D_2^{n_2}$		$\uparrow$ D_2^{K+1}	

Thus for all $n = 0, K+1$, and $i \in \mathscr{I}$,

(6.2) $$D_i^n = \bigcup_{\{\mathbf{n} \in \{0, \ldots, K+1\}^I : n_i = n\}} L(\mathbf{n}),$$

(6.3) $$E^n = \bigcup_{\{\mathbf{n} \in \{0, \ldots, K+1\}^I : \min(\mathbf{n}) \geq n\}} L(\mathbf{n}).$$

Now for all $n = 1, K$, and $i \in \mathscr{I}$, $D_i^n \subseteq \Omega \backslash E_i^{n+1} = \Omega \backslash B_i^{p_i}(E^n)$; so we have by Lemma A,

(6.4) $$P[D_i^n \cap E^n] \leq \frac{p_i}{1-p_i} P[D_i^n \backslash E^n].$$

Combining (6.2), (6.3), and (6.4), gives, for all $n = 1, K$, and $i \in \mathscr{I}$,

(6.5) $$\sum_{\{\mathbf{n} \in \{0, \ldots, K+1\}^I : n_i = n \text{ and } \min(\mathbf{n}) = n\}} \pi(\mathbf{n}) \leq \frac{p_i}{1-p_i} \left(\sum_{\{\mathbf{n} \in \{0, \ldots, K+1\}^I : n_i = n \text{ and } \min(\mathbf{n}) < n\}} \pi(\mathbf{n}) \right).$$

These inequalities follow from the construction of the events. Now our maintained hypothesis that $P[E] \geq 1 - \varepsilon$ implies

(6.6) $$\sum_{\{\mathbf{n} \in \{1, \ldots, K+1\}^I : \min(\mathbf{n}) = 0\}} \pi(\mathbf{n}) \leq \varepsilon.$$

Observe that $[B_*^{\mathbf{p}}]^K(E^1) = E^{K+1}$, so

$$1 - P[[B_*^{\mathbf{p}}]^K(E^1)] = \Sigma_{\{\mathbf{n} \in \{0, \ldots, K+1\}^I : \min(\mathbf{n}) \leq K\}} \pi(\mathbf{n}).$$

Thus we are interested in the following linear programming problem (P):

(6.7) $$\max \sum_{\{\mathbf{n} \in \{0, \ldots, K+1\}^I : \min(\mathbf{n}) \leq K\}} \pi(\mathbf{n})$$

subject to $\pi(\mathbf{n}) \geq 0$, (6.5), and (6.6).

Thus the statement of Lemma C has been reinterpreted as a linear programming problem. It remains only to show that the maximand is $\varepsilon \xi^*(\mathbf{p}, K)$.

The solution to (P) has the property that only certain critical locations have positive probability. Write $c(i,n)$ for the location where all components are equal to $n+1$ except the ith which is n, i.e., $c(i,n) = (n+1, \ldots, n+1, n, n+1, \ldots, n+1)$, and say that $\mathbf{n}$ is a *critical location* if it can be written in this form. The critical path constraint requires:

(6.8) $$\pi(\mathbf{n}) = 0, \quad \text{if } \mathbf{n} \text{ is not a critical location.}$$

The critical path intuition suggests that the following problem (P′) has the same value as problem (P).

(6.9) $$\max \sum_{\{\mathbf{n} \in \{0, \ldots, K+1\}^I : \min(\mathbf{n}) \leq K\}} \pi(\mathbf{n})$$

subject to $\pi(\mathbf{n}) \geq 0$, (6.5), (6.6), and (6.8).

We will prove this by looking at the dual problems of (P) and (P′). First, the dual problem (D) of (P) is as follows. Call the constraint corresponding to individual i and n in (6.5) (i,n), and denote by $\lambda(i,n)$ the shadow price of the constraint. Similarly, denote by ξ the shadow price of the constraint in (6.6). Consider any $\mathbf{n} = (n_1, \ldots, n_I)$ with $0 < \min(\mathbf{n}) \leq K$. Then $\pi(\mathbf{n})$ appears in the left-hand side of (6.5) if and only if $n_i = \min(\mathbf{n})$, i.e., $\pi(\mathbf{n})$ appears in all inequalities (i,n) with $n = \min(\mathbf{n})$; $\pi(\mathbf{n})$ appears in the right-hand side of every inequality (i,n) with $n_i = n > \min(\mathbf{n})$. If $k = 0$, then $\pi(\mathbf{n})$ also appears in the inequality in (6.6). Thus the dual problem has the following constraints:

For each $\mathbf{n} = (n_1, \ldots, n_I)$ with $\min(\mathbf{n}) = 0$,

$$\xi - \sum_{\{i : n_i > 0\}} \frac{p_i}{1 - p_i} \lambda(i, n_i) \geq 1; \tag{6.10}$$

for $\mathbf{n}$ with $0 < \min(\mathbf{n}) < K$,

$$\sum_{\{i : n_i = \min(\mathbf{n})\}} \lambda(i, n_i) - \sum_{\{i : n_i > \min(\mathbf{n})\}} \frac{p_i}{1 - p_i} \lambda(i, n_i) \geq 1; \tag{6.11}$$

and for $\mathbf{n}$ with $\min(\mathbf{n}) = K$,

$$\sum_{\{i : n_i = K\}} \lambda(i, K) \geq 1. \tag{6.12}$$

Thus the dual problem (D) of (P) is

$$\min \varepsilon \cdot \xi \tag{6.13}$$

subject to $\lambda(i, n) \geq 0$, $\xi \geq 0$, (6.10), (6.11), and (6.12).

Next, let us construct the dual (D′) of the constrained primal problem (P′). Look at the constraints' corresponding critical addresses, which will give us the dual form of the critical path:

For $c(i, 0)$, from (6.10), we have

$$\xi - \sum_{j \neq i} \frac{p_j}{1 - p_j} \lambda(j, 1) \geq 1 \quad \text{for each } i; \tag{6.14}$$

for $c(i, n)$, $1 \leq n < K$, from (6.11), we have

$$\lambda(i, n) - \sum_{j \neq i} \frac{p_j}{1 - p_j} \lambda(j, n + 1) \geq 1 \quad \text{for each } i \text{ and } n; \tag{6.15}$$

and for $c(i, n)$, $n = K$, from (6.12), we have

$$\lambda(i, K) \geq 1. \tag{6.16}$$

Thus the dual problem (D′) is:

$$\min \varepsilon \cdot \xi \tag{6.17}$$

subject to $\lambda(i, n) \geq 0$, $\xi \geq 0$, and (6.14), (6.15), and (6.16).

By construction, the minimum of (D′) is no larger than (D) since (D′) has less constraints.

It will turn out that (6.14), (6.15), and (6.16) are binding: set $\lambda^*(i, K) = 1$ for all $i \in \mathcal{I}$, $\lambda^*(i, k) = 1 + \sum_{j \neq i} (p_j/(1 - p_j)) \lambda^*(j, k + 1)$ and for all $i \in \mathcal{I}$ and $1 \leq k < K$; and $\xi^* = 1 + \max_{i \in \mathcal{I}} \{\sum_{j \neq n} (p_j/(1 - p_j)) \lambda^*(j, 1)\}$. In matrix notation, $\lambda^*(i, k) = [(\mathbf{I} + \mathbf{R} + \cdots + \mathbf{R}^{K-k})\mathbf{1}']_i$ and $\xi^* = \max_{i \in \mathcal{I}} \lambda^*(i, 0)$.

Lemma D: (λ^*, ξ^*) *is a solution to* (D').

Proof: Note that given any λ, $\xi = 1 + \max_{i \in \mathcal{I}} \{\sum_{j = i} (p_j/(1 - p_j)) \lambda(j, 1)\}$ is clearly optimal since $\xi \geq 1 + \sum_{j = i} (p_j/(1 - p_j)) \lambda(j, 1)$ for all i. Since $\lambda(i, 1) \geq 1 + \sum_{j = i} (p_j/(1 - p_j)) \lambda(j, 2)$ from (6.15) and $\max_{i \in \mathcal{I}} \{\sum_{i \in \mathcal{I}} (p_j/(1 - p_j)) \lambda(j, 1)\}$ is a weakly increasing function of vector $(\lambda(i, 1))_{i \in \mathcal{I}}$, the minimum is attained at $\lambda(i, 1) = 1 + \sum_{j = i} (p_j/(1 - p_j)) \lambda(j, 2)$ for any given $\lambda(j, 2)$. The iterative argument shows $\lambda(i, k) = 1 + \sum_{j \neq n} (p_j/(1 - p_j)) \lambda(j, k + 1)$ and thus the statement of the lemma. *Q.E.D.*

LEMMA E: *(λ^*, ξ^*) is a solution to (D).*

PROOF: Since problem (D) includes all constraints of (D′), by Lemma D it suffices to show that (λ^*, ξ^*) satisfy (6.10), (6.11), and (6.12). Relation (6.10) is clear by construction. Note that for any k, $\lambda^*(i,k) \geq \lambda^*(i,k+1)$ holds for every i: it is straightforward to check that this is true for $K-1$, and if it is true for k, then

$$\lambda^*(i,k-1) = 1 + \Sigma_{j \neq n} \frac{p_j}{1-p_j} \lambda^*(j,k) \geq 1 + \Sigma_{j \neq n} \frac{p_j}{1-p_j} \lambda^*(j,k+1) = \lambda^*(i,k);$$

thus it is true for $k-1$.

Now for any $\mathbf{n}$ with $\min(\mathbf{n}) = 0$,

$$\xi^* - \sum_{\{i:n_i>0\}} \frac{p_i}{1-p_i} \lambda^*(i,n_i) \geq \xi^* - \sum_{\{i:n_i \geq 0\}} \lambda^*(i,1) \geq \xi^* - \sum_{j \neq i} \frac{p_j}{1-p_j} \lambda^*(j,1) \geq 1,$$

where the first inequality holds due to the monotonic property of λ^* shown above. Thus (6.10) is satisfied. Similarly, for $\mathbf{n}$ with $0 < \min(\mathbf{n}) = n < K$,

$$\begin{aligned}
&\sum_{\{i:n_i=n\}} \lambda^*(i,n_i) - \sum_{\{i:n_i>n\}} \frac{p_i}{1-p_i} \lambda^*(i,n_i)dn \\
&\quad \geq \sum_{\{i:n_i=n\}} \lambda^*(i,n) - \sum_{\{i:n_i>n\}} \frac{p_i}{1-p_i} \lambda^*(i,n+1)dn \\
&\quad \geq \lambda^*(i^*,n) - \sum_{j \neq i} \frac{p_j}{1-p_j} \lambda^*(j,n+1) = 0,
\end{aligned}$$

where i^* can be any i with $n_i = n$. Therefore (6.12) holds. *Q.E.D.*

By the duality theorem of linear programming, the value of problem (D) is the same as the value of problem (P), completing the proof of Lemma C, since $\xi^*(\mathbf{p}, K) = \xi^*$ (making the dependence on $\mathbf{p}$ and K explicit).

It remains only to consider what happens to $\xi^*(\mathbf{p}, K)$ as $K \to \infty$.

LEMMA F: *If $\Sigma_{i \in \mathcal{J}} p_i < 1$, then $\xi^*(\mathbf{p}, K) \to ((1 - \min_{i \in \mathcal{J}}(p_i))/(1 - \Sigma_{i \in \mathcal{J}} p_i))$ as $K \to \infty$.*

PROOF: First we establish some properties of the matrix $\mathbf{R}$. We will use the following decomposition of $\mathbf{R}$. Write $\mathbf{I}$ for the identity matrix. Write $\mathbf{D}$ for the diagonal matrix with ith diagonal element p_i. Write $\mathbf{X}$ for the matrix of 1's.

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} p_1 & 0 & 0 & & 0 \\ 0 & p_2 & 0 & \cdots & 0 \\ 0 & 0 & p_3 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p_I \end{pmatrix},$$

$$\mathbf{X} = \begin{pmatrix} 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & & 1 \\ & \vdots & & \ddots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Observe that $\mathbf{R} = (\mathbf{X} - \mathbf{I})\mathbf{D}(\mathbf{I} - \mathbf{D})^{-1}$.

Claim 1: $\mathbf{I} + \mathbf{R} + \mathbf{R}^2 + \cdots$ is bounded if $\Sigma_{i \in \mathcal{J}} p_i < 1$. To see this, $\Sigma_{j \in \mathcal{J}} p_j < 1 \Rightarrow (\Sigma_{j \in \mathcal{J}} p_j) - p_i < 1 - p_i$ for all $i \Rightarrow \alpha = \max_{i \in \mathcal{J}}(((\Sigma_{j \in \mathcal{J}} p_j) - p_i)/(1-p_i)) < 1$. Write $\mathbf{1}$ for the row vector of 1's. So $\mathbf{R}(\mathbf{1}' - \mathbf{p}') = (\Sigma_{i \in \mathcal{J}} p_i)\mathbf{1}' - \mathbf{p}' \leq \alpha(\mathbf{1}' - \mathbf{p}')$ and thus $(\mathbf{I} + \mathbf{R} + \mathbf{R}^2 + \cdots)(\mathbf{1}' - \mathbf{p}') \leq (1 + \alpha + \alpha^2 + \cdots)(\mathbf{1}' - \mathbf{p}') = (1/(1-\alpha))(\mathbf{1}' - \mathbf{p}')$. *Q.E.D.*

Claim 2: $[\mathbf{I}-\mathbf{R}]^{-1}=[\mathbf{I}-(\mathbf{X}-\mathbf{I})\mathbf{D}(\mathbf{I}-\mathbf{D})^{-1}]^{-1}=(\mathbf{I}-\mathbf{D})(\mathbf{I}+(1/\delta)\mathbf{XD})$ where $\delta=1-\sum_{i\in\mathcal{J}}p_i$. To see this, first observe that

$$\mathbf{XD}=\begin{pmatrix} p_1 & p_2 & p_3 & & p_I \\ p_1 & p_2 & p_3 & \cdots & p_I \\ p_1 & p_2 & p_3 & & p_I \\ & \vdots & & \ddots & \vdots \\ p_1 & p_2 & p_3 & \cdots & p_I \end{pmatrix} \quad \text{and so}$$

(6.18) $$\mathbf{XDXD}=(1-\delta)\mathbf{XD}.$$

Now:

$$\begin{aligned} &[\mathbf{I}-(\mathbf{X}-\mathbf{I})\mathbf{D}(\mathbf{I}-\mathbf{D})^{-1}](\mathbf{I}-\mathbf{D})\left(\mathbf{I}+\frac{1}{\delta}\mathbf{XD}\right) \\ &=\left\{\begin{matrix} (\mathbf{I}-\mathbf{D})\left(\mathbf{I}+\frac{1}{\delta}\mathbf{XD}\right) \\ -(\mathbf{X}-\mathbf{I})\mathbf{D}\left(\mathbf{I}+\frac{1}{\delta}\mathbf{XD}\right)\end{matrix}\right\} \\ &=\left\{\begin{matrix} \mathbf{I}+\frac{1}{\delta}\mathbf{XD}-\mathbf{D}-\frac{1}{\delta}\mathbf{DXD} \\ -\mathbf{XD}-\frac{1}{\delta}\mathbf{XDXD}+\mathbf{D}+\frac{1}{\delta}\mathbf{DXD}\end{matrix}\right\} \\ &=\frac{1}{\delta}\mathbf{XD}+\mathbf{I}-\frac{1}{\delta}(1-\delta)\mathbf{XD}-\mathbf{XD}, \quad \text{by (6.18)} \\ &=\mathbf{I}. \end{aligned}$$

Q.E.D.

Recall that $\xi^*(\mathbf{p},K)\equiv\max_{i\in\mathcal{J}}([(\mathbf{I}+\mathbf{R}+\cdots+\mathbf{R}^K)\mathbf{1}']_i)$. Claim 1 implies that $\xi^*(\mathbf{p},K)\to\max_{i\in\mathcal{J}}([[\mathbf{I}-\mathbf{R}]^{-1}\mathbf{1}']_i)$ as $K\to\infty$. Claim 2 shows that the (i,i)th element of $[\mathbf{I}-\mathbf{R}]^{-1}$ is $((1-p_i)/(1-\sum_{k\in\mathcal{J}}p_k))(1-\sum_{j\neq i}p_j)$ and the (i,j)th element is $((1-p_i)/(1-\sum_{k\in\mathcal{J}}p_k))p_j$ if $i\neq j$. Thus $[\mathbf{I}-\mathbf{R}]^{-1}\mathbf{1}'$ is a column vector with ith element $((1-p_i)/(1-\sum_{k\in\mathcal{J}}p_k))$. Thus

$$\begin{aligned}\lim_{K\to\infty}\xi^*(\mathbf{p},K)&=\max_{i\in\mathcal{J}}((1-p_i/1-\textstyle\sum_{k\in\mathcal{J}}p_k)) \\ &=((1-\min_{i\in\mathcal{J}}(p_i)/1-\textstyle\sum_{i\in\mathcal{J}}p_i)).\end{aligned}$$

Q.E.D.

Now Proposition 4.2 follows from Lemma F and Lemma C. Corollary 4.3 also follows since, for each event E, $B_*^{\mathbf{p}}(E)$ is a simple event and $C^{\mathbf{p}}(B_*^{\mathbf{p}}(E))=C^{\mathbf{p}}(E)$, so:

$$\begin{aligned}1-P[C^{\mathbf{p}}(E)]&=1-P[C^{\mathbf{p}}(B_*^{\mathbf{p}}(E))] \\ &\leq(1-P[B_*^{\mathbf{p}}(E)])\left(\frac{1-\min_{i\in\mathcal{J}}(p_i)}{1-\sum_{i\in\mathcal{J}}p_i}\right), \quad \text{by Proposition 4.2} \\ &\leq(1-P[E])\left(1+\sum_{i\in\mathcal{J}}\frac{p_i}{1-p_i}\right)\left(\frac{1-\min_{i\in\mathcal{J}}(p_i)}{1-\sum_{i\in\mathcal{J}}p_i}\right), \quad \text{by Lemma B.}\end{aligned}$$

Q.E.D.

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