Some Results on "An Income Fluctuation Problem"*

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A consumer at each period, given the income available, y, has to decide how much to consume and save. If he consumes $c \geqslant 0$ units he gets $u(c)$ units of satisfaction or utility, and if $x = y - c \geqslant 0$ is the amount saved then the available income in the next period is $rx + \omega^k$, where ω^k is a random variable, and r is an interest factor that is assumed to be known with certainty. Infinite time horizon problems are considered, and it is shown that if $0 < \delta r < 1$, where $0 < \delta < 1$ is a discount factor, then the limiting policy is optimal. Questions about the behavior of the stock level, such as boundness, are considered, and an example is given that shows that the stock level might converge almost surely to infinity. Finally an economic explanation is given.

INTRODUCTION

In this paper we consider the following situation. A consumer has a t period planning horizon, and at each period he must decide how much to consume and save, in order to maximize the total expected utility accumulated in the t periods. If y is the available income at the kth period, and $0 \leqslant x \leqslant y$ is the amount saved at that period, then the available income in the next period is $rx + \omega^k$, where ω^k is a random variable and r is an interest factor

that is assumed to be known with certainty. If $c = y - k$ units are consumed in period k, this produces $\delta^k u(c)$ units of satisfaction or utility to the consumer, $0 < \delta < 1$ is a discount factor. It is assumed that $u(c)$ is strictly concave, increasing, and differentiable and that the random variables ω^k are i.i.d.

In Section 1 we describe the model and state the general results that are going to be used; they are basically the concepts of competitive prices and policies for the stochastic case.

In Section 2 we consider the deterministic case, i.e., $\omega \equiv a$. It is shown that the limiting policy is optimal when $0 < \delta r < 1$, and the behavior of the consumption, except for the infinite time horizon problem, is studied. In Section 3 we consider the stochastic case; i.e., ω is a nondegenerated random variable. For the case $0 < \delta < 1$, $r = 1$, it is shown that the limiting policy is optimal and also that the level of income is bounded above. For $0 < \delta < 1$, $\delta r < 1$, and $r > 1$, it is also shown that the limiting policy is optimal, but it is not necessarily true that the stock level will always stay in an interval $[0, \bar{y}]$. An example is given for which the stock level converges; almost surely, to infinity. Sufficient conditions are obtained in order to assure the existence of a bound for the stock level.

We should also mention that some of the results of this paper were presented in the First Brazilian Symposium in Probability held at Rio de Janeiro in 1974.

1. THE MODEL AND SOME GENERAL RESULTS

Finite Time Horizon Problem

A consumer is faced with the following situation. He has a t period planning horizon, and at each period he must decide how much to consume and save, in order to maximize the total expected utility accumulated in t periods. If y is the available income at the kth period, and $0 \leqslant x \leqslant y$ is the amount saved at period k, then the available income in the next period is $rx + \omega^k$, where ω^k is a random variable and r is an interest factor that is assumed to be known with certainty. If $c = y - x$ units are consumed in period k, then this produces $\delta^k u(c)$ units of satisfaction or utility to the consumer, and $0 < \delta < 1$ is a discount factor. We will assume that $u(c)$ is strictly concave, increasing, and differentiable and that the random variables ω^k are i.i.d. The range of ω is an interval $[a, A]$, $A < +\infty$ and $\text{Prob}[\omega = a] > 0$. More formally our problem is maximize

$$E\left[\sum_{k=0}^{T} \delta^k u(c^k)\right]$$

subject to the stochastic constraints

$$c^k + x^k = y^k,$$
$$rx^{k-1} + \omega^{k-1} = y^k,$$
$$c^0 + x^0 = y^0,$$
$$c^T = y^T, \qquad \text{for } 1 < k < T.$$

$c^k \geqslant 0$, $x^k \geqslant 0$ for $1 \geqslant k \leqslant T$, and y^0 is the initial income available.

If $V_t(y)$ is the maximum expected utility that we can get when y is the income available and we have t periods to go, then the usual dynamic programming formulation leads us to the functional equations:

$$V_t(y) = \max_{\substack{c+x=y \\ c \geqslant 0, x \geqslant 0}} u(c) + \delta E V_{t-1}(rx + \omega) \qquad \text{for } t > 1,$$

$$V_1(y) = \max_{\substack{c+x=y \\ c \geqslant 0, x \geqslant 0}} u(c). \tag{1.1}$$

It can be shown that $V_t(y)$ inherits the basic properties of $u(c)$, i.e., $V_t(y)$ is increasing, strictly concave, and differentiable.

The derivatives of $V_t(y)$ will be denoted by $p_t(y)$ and the solution of (1.1) will be denoted by $\{c_t(y), x_t(y)\}$.

The following properties of $\{c_t(y), x_t(y)\}$ and $p_t(y)$ will be stated without proof (for proofs see [3]):

(1.2) $p_t(y)$ is a positive decreasing function, $p_t(y) \geqslant p_{t-1}(y)$ for all y and all t, and if $u'(c)$ is a convex function then $p_t(y)$ is also a convex function for all t.

(1.3) $c_t(y)$, $x_t(y)$ are nondecreasing continuous functions and $c_t(y) \leqslant c_{t-1}(y)$ $(x_t(y) \geqslant x_{t-1}(y))$ for all t.

(1.4) The functions $c_t(y)$, $x_t(y)$ satisfy the following relations:

(1.5) $p_t(y) \geqslant \delta r E p_{t-1}(rx_t(y) + \omega)$ with equality if $x_t(y) > 0$;

(1.6) $p_t(y) \geqslant u'(c_t(y))$ with equality if $c_t(y) > 0$.

Notation. If $\omega \equiv v$ (a constant), then the corresponding prices and policies will be denoted by $p_t^v(y)$ and $(c_t^v(y), x_t^v(y))$.

Now we will state a theorem that permits us to compare $p_t(y)$ and consequently $c_t(y)$ with $p_t^a(y)$ with $p_t^a(y))$, $p_t^A(y)(c_t^A(y))$, and $p_t^{\bar{\omega}}(y)(c_t^{\bar{\omega}}(y))$ where $\bar{\omega}$ is the $E\omega$. A proof of it can be seen in [3].

THEOREM 1.1. *The following relations are true:*

(i) $p_t(y) \leqslant p_t^a(y)$ *for all t, and consequently $c_t(y) \geqslant c_t^a(y)$;*

(ii) $p_t(y) \geqslant p_t^A(y)$ *for all t and consequently $c_t(y) \leqslant c_t^A(y)$;*

(iii) *if $u'(c)$ is a convex function then $p_t(y) \geqslant p_t^{\bar{\omega}}(y)$ and consequently $c_t(y) \leqslant c_t^{\bar{\omega}}(y)$, where $\bar{\omega} = E\omega$.*

Infinite Time Horizon Problems

Now the consumer objective is to maximize

$$E\left[\sum_{t=0}^{\infty} \delta^t u(c^t)\right]^1$$

subject to the stochastic constraints

$$c^t + x^t = rx^{t-1} + \omega^{t-1} \quad \text{for } t > 1,$$
$$c^0 + x^0 = y^0,$$
$$c^t \geqslant 0, \quad x^t \geqslant 0 \quad \text{for } t \geqslant 0.$$

DEFINITION. A policy is a pair of functions $(c(y), x(y))$ such that $c(y) \geqslant 0$, $x(y) \geqslant 0$, $c(y) + x(y) = y$. If y is the stock of capital available then $c(y) \geqslant 0$ tells us how much to consume and $x(y) \geqslant 0$ how much to save.

Given a policy $(c(y), x(y))$ the stock of capital available at period t will be denoted by y^t (observe the superscript).

For some cases the series $E[\sum_{t=0}^{\infty} \delta^t u(c^t)]$ might diverge and in order to compare the return of policies we will introduce the following definition; for a more detailed discussion see [1].

DEFINITION. A policy $(c(y), x(y))$ is said to overtake another policy $(\hat{c}(y), \hat{x}(y))$ if starting from the same initial stock y^1, the policies $(c(y), x(y))$, $(\hat{c}(y), \hat{x}(y))$ yield (stochastic) consumptions paths c^t and $\hat{c}^t$ that satisfy

$$E\left[\sum_{t=0}^{T} u(c^t) - u(\hat{c}^t)\right] > 0 \quad \text{for all } T > \text{some } T_0.$$

DEFINITION. A policy is said to be optimal if it overtakes all other policies.

Now we will state without a proof a theorem that gives sufficient conditions for the optimality of a policy $(c(y), x(y))$. A proof of it for a similar problem can be seen in [2].

THEOREM 1.2. *If there exists a positive and nonincreasing function $p(y)$, and a policy $(c(y), x(y))$ that satisfies the following conditions, then the policy $(c(y), x(y))$ is an optimal policy:*

(1.7) $p(y) \geqslant u'(c(y))$ *with equality if $c(y) > 0$;*

(1.8) $p(y) \geqslant \delta r E p(rx(y) + \omega)$ *with equality if $x(y) > 0$;*

(1.9) $\lim_{t \to \infty} E\delta^t p(y^t) y^t = 0.$

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[1] Observe the superscript c^t indicates the consumption at the tth period, now counting from the first period.

In general it is not easy to find an optimal policy to infinite time horizon problems. Natural candidates are of course the limiting policies $c(y) = \lim_{t \to \infty} c_t(y)$ and $x(y) = \lim_{t \to \infty} x_t(y)$ which are continuous nondecreasing functions. If bounds for $p_t(y)$ can be found, limits in (1.5) and (1.6) can be taken and they yield

$$p(y) \geqslant u(c(y))$$

and

$$p(y) \geqslant \delta r E p(rx(y) + \omega),$$

where $p(y) = \lim_{t \to \infty} p_t(y)$. Hence whenever a bound for $p_t(y)$ can be found, to show that the limiting policy is optimal it will suffice to show that $\lim_{t \to \infty} E\delta^t p(y^t) y^t = 0$, where y^t is the stock available at the tth period when we use the limiting policy.

2. DETERMINISTIC CASE $\omega \equiv a$, $0 < \delta < 1$, $r \geqslant 1$, $\delta r \leqslant 1$

In this section we will consider the case $\omega = a > 0$, $0 < \delta < 1$, $r \geqslant 1$, and $\delta r < 1$. We will obtain bounds for the prices $p_t(y)$, study the behavior of the income level when we follow the limiting policy, and finally it will be shown that the limiting policy is optimal.

LEMMA 2.1. *If* $0 < \delta < 1$, $r \geqslant 1$, $\delta r < 1$, *then* $c_t(y) \geqslant a$ *for* $y \geqslant a$ *and* $c_t(y) = y$ *for* $y \leqslant a$.

Proof. Suppose not, i.e., $x_t(a) > 0$. Hence from (15) and (1.6) we get

$$
\begin{aligned}
p_t(y) &= \delta r p_{t-1}(rx_t(y) + a) \\
&< p_{t-1}(x_t(y) + a) \\
&\leqslant p_t(x_t(y) + a) \qquad \text{by (1.2).}
\end{aligned}
$$

So, $a \geqslant rx_t(y) + a$, by the monotonicity of $p_t(y)$, which is a contradiction. ∎

LEMMA 2.2. *If* $0 < \delta < 1$, $r > 1$, $\delta r < 1$ *then* $c_t(y) \geqslant [(r-1)/r] y$ *for all* y.

Proof. The proof depends on the value that $x_t(y)$ assumes:

(a) $x_t(y) = 0$. Then $c_t(y) = y$ and consequently $c_t(y) \geqslant [(r-1)/r] y$.

(b) $x_t(y) > 0$. From the optimality condition we get $p_t(y) = \delta r p_{t-1}(rx_t(y) + a)$. Now, from (1.2), the monotonicity of $p_t(y)$ and the fact that $\delta r < 1$ we get $p_t(y) < p_t(rx_t(y))$ and consequently we get $y/r > x_t(y)$ and $c_t(y) \geqslant [(r-1)/r] y$. ∎

From Lemmas 2.1 and 2.2 it follows that $p_t(y) \leqq u'[(r-1)/r] a\}$. So, the limiting policy satisfies the relations

$$p(y) \geqslant \delta rp(rx(y) + a) \qquad \text{with equality if } x(y) > 0, \qquad (2.1)$$
$$p(y) = u'(c(y)), \qquad c(y) > 0. \qquad (2.2)$$

We should mention at this point that if we assume that $u'(0) < \infty$, we could drop the assumption that $a > 0$, and bounds for the prices would then be $u'(0)$.

The next theorem tells us how the stock level behaves when we follow the limiting policy.

THEOREM 2.3. *For the limiting policy $y^{t+1} \leqslant y^t (c(y^{t+1})) \leqslant c(y^t)$, and if $y^{0\,2}$ is the initial stock, then there exist a $T(y^0)$ such that $y^t = a$ for all $t > T(y^0)$.*

Proof. First we will show that $y^{t+1} \leqslant y^t$. If $y^t > a$ and $x(y^t) = 0$, then $y^{t+1} = a$ and so $y^{t+1} < y^t$. If $y^t > a$ and $x(y^t) > 0$ then from (2.1) we get

$$p(y^t) = \delta rp(y^{t+1})$$
$$< p(y^{t+1}).$$

So, $y^t > y^{t+1}$ by the monotonicity of $p(y)$. If $y^t = a$ from Lemma 2.1 we get that $x(y^t) = 0$ and hence that $y^{t+1} = a$, so $y^{t+1} \leqslant y^t$.

To show the existence of $T(y^0)$ we will argue by contradiction, i.e., suppose that $y^t > a$ for all t, hence $x(y^t) > 0$ for all t. So, from (2.1) we get

$$p(y^0) = (\delta r)^t p(y^t) \qquad \text{for all } t, \qquad (2.3)$$

but $0 < \delta r < 1$ and $p(y) \leqslant u'(a)$ and consequently (2.3) cannot hold for all t. ∎

From the previous theorem it follows that the sequences y^t converge to a in a finite number of periods.

The behavior of the sequence of capital $\{y^t\}$, consumption $\{c^t\}$, and saving $\{x^t\}$ can be represented by the following graphs:
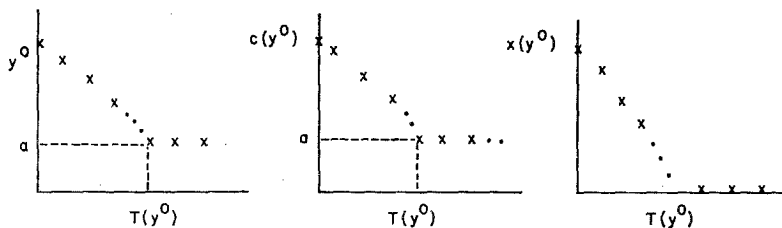
THEOREM 2.4. *The limiting policy is optimal.*

Proof. It suffices to show that $\lim_{t \to \infty} \delta^t p(y^t) y^t = 0$. From Lemma 2.2 and Theorem 2,3 we have that $p(y^t) \leqslant u'\{[(r-1)/r] a\}$ and $y^t \leqslant y^0$ for all t. Hence $0 \leqslant \delta^t p(y^t) y^t \leqslant \delta^t u'\{[r-1)/r] a\} y^0$, for all t. So, $\lim_{t \to \infty} \delta^t p(y^t) y^t = 0$, since $0 < \delta < 1$.

3. STOCHASTIC CASE, I. E., ω IS A NONDEGENERATED RANDOM VARIABLE

In this section we will study the stochastic case in which $\omega \in [a, A]$, $A < \infty$ is a nondegenerated random variable, and $\mathrm{Prob}[\omega = a] = \alpha > 0$.

For the deterministic case we have shown that in a finite number of periods, when we follow the limiting policy, the stock level reaches the level a. The analog for the stochastic case would be a renewal process. This is in fact true whenever $0 < \delta < 1$ and $r = 1$, the stock $\{y^t\}$ evolves in an interval $\{a, \bar{y}]$ and a is a renewal point. But, for $0 < \delta < 1$, $\delta r < 1$, $r > 1$ this result is not necessarily true, we will give an example in which $\{y^t\} \to_{\text{a.s.}} \infty$.

(a) *The Case* $0 < \delta < 1, r = 1$

Will show that the limiting policy is optimal, but first it is necessary to find bounds for the prices $p_t(y)$.

LEMMA 3.1.
$$p_t(y) \leqslant u'(a).$$

Proof. From Theorem 1.1 and Lemma 2.1 we get

$$p_t(y) \leqslant p_t{}^a(y) \leqslant u'(a) \qquad \text{for all } y \geqslant a \tag{3.1}$$

and all t, consequently $c_t(y) \geqslant c_t{}^a(y) \geqslant a$.

THEOREM 3.2. *The limiting policy is optimal.*

Proof. First the limiting policy satisfies the conditions

$$p(y) \geqslant \delta E p(x(y) + \omega), \tag{3.2}$$
$$p(y) = u'(c(y)), \tag{3.3}$$

which is obtained from the optimality conditions (1.5) and (1.6) after taking the limits and using Lemma 3.1. To prove that the limiting policy is optimal it suffices to show that $\lim_{t \to \infty} E \delta^t p(y^t) y^t = 0$. From Lemma 3.1 we have that $p(y) \leqslant u'(a)$. Now as $\omega \leqslant A$ we have that $y^t \leqslant tA + y^0$. So,

$$\lim_{t \to \infty} \delta^t p(y^t) y^t \leqslant \lim_{t \to \infty} (\delta^t(tA + y^0)) u'(a) = 0.$$

THEOREM 3.3. *There exists a $\bar{y}$ such that for all $y \geqslant \bar{y}$ $x(y) + A \leqslant y$.*

Proof. First we will show that there is a $\bar{y}$ such that $x(\bar{y}) + A \leqslant \bar{y}$. By contradiction suppose not, i.e., for all y, $x(y) + A > y$; hence, $c(y) < A$ for all y. So, $\lim_{y \to \infty} c(y) = M$ and

$$\lim_{y \to \infty} p(y) = u'(M) > 0, \qquad \text{since } M \geqslant c(y) > 0 \text{ for } y > 0. \qquad (3.4)$$

From (3.2) and (3.3) we have

$$p(y) = \delta E p(x(y) + \omega) \leqslant \delta p(x(y) + a). \qquad (3.5)$$

Now, taking the limit in (3.5), using (3.4) and the fact that $\lim_{y \to \infty} x(y) = \infty$ we get that $M \leqslant \delta M$ which is a contradiction since $0 < \delta < 1$. The theorem now follows from the fact that $c(y)$ is a nondecreasing function which implies that $c(y) \geqslant A$ for all $y \geqslant \bar{y}$ and consequently that $x(y) + A \leqslant y$ for all $y \geqslant \bar{y}$. ∎

From the previous theorem it follows that the process evolves in an interval $[a, \bar{y}]$. For the deterministic case we were able to show that $\{y^t\}$ converges in a finite number of periods to a (whenever the process reaches the stock level a we will say that the process is in state a; for the stochastic case we will show that the process $\{y^t\}$ is a delayed renewal process, in which state a is a renewal point, i.e., each time the process reaches state a it starts all over again.

Let X_0 be the number of periods that it takes to reach for the first time state a. And let X_j be the number of periods that elapses between the $(j - 1)$th time we reach state a and the jth time we reach it again. The random variables $X_0, X_1, ..., X_j, ...$ are independent and $X_1, X_2, ..., X_j$, are identically distributed. Hence the process X_j is a delayed renewal process. To show that it is nonterminating we need to show that $EX_j < \infty$. In order to show that X_j has finite expectation we will consider an artificial process that is described below.

Consider a process with $T'(\bar{y}) = T(\bar{y}) + 1$ states, where $T(\bar{y})$ is the number of periods that it takes to reach the state a when we assume that $w \equiv a$ and we follow the optimal policy (see Theorem 2.3); $\bar{y}$ is an upper level for which $y^t \leqslant \bar{y}$ for all t, which existence we have shown before. Now returning to the stochastic case, at each period we observe the value of w. If $w = a$ and we are at state $1 < j \leqslant T'(\bar{y})$ then we move to state $j - 1$ and if we are at state 1 we stay at that state. If $w \neq a$ and we are at state $1 \leqslant j \leqslant T'(\bar{y})$ we move to state $T'(\bar{y})$. The following picture describes the process, where the numbers in the arrows are the transition probabilities. If z is the random variable that represents the number of periods that the process takes to reach the state 1 having started from state $T'(\bar{y})$, it can be shown that

$$EZ = \frac{1 + \alpha + \cdots + \alpha^{T(\bar{y})-1}}{\alpha^{T(\bar{y})}},$$
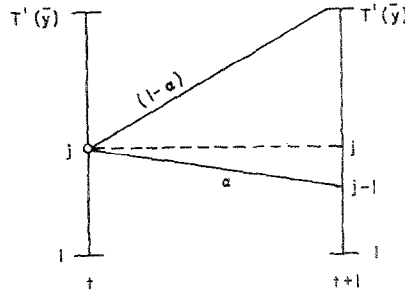
and it is easy to see that $X_j \leqslant Z$ and so consequently we will have that $EX_j < \infty$.

Using a well-known result from the renewal theory (for instance, see [4, p. 365], we can state the following theorem.

THEOREM 3.4. *If ω is a denumerable random variable, then the process $\{y^t\}$ has a limiting distribution.*

(b) *Case* $0 < \delta < 1, r > 1, \delta r < 1$

First we have from Theorem 1.1 and Lemma 2.2 that $p_t(y) \leqslant p_t{}^a(y) \leqslant u'\{[(r-1)/r] \, y\} \leqslant u'\{[(r-1)/r]a$ for all $y \geqslant a$ and all t. Hence, the limiting policy satisfies

$$p(y) \geqslant \delta r E p(rx(y) + \omega), \tag{3.6}$$

$$p(y) = u'(c(y)). \tag{3.7}$$

THEOREM 3.5. *The limiting policy is optimal.*

Proof. It suffices to show that $\lim_{t \to \infty} E \delta^t p(y^t) \, y^t = 0$. Now, $y^t \leqslant r^t \{y^0 + [(r-1)/r] \, A\}$ and $p(y^t) \leqslant u'\{[(r-1)/r] \, a\}$, so

$$E \delta^t p(y^t) \, y^t \leqslant (\delta r)^t \left(y^0 + \frac{r-1}{r} A \right) u' \left(\frac{r-1}{r} a \right) \qquad \text{for all } t. \tag{3.8}$$

Taking the limit and using the fact that $\delta r < 1$, we get the desired result. ∎

When $0 < \delta < 1, r = 1$, we were able to show that there exists a $\bar{y}$ such that $x(y) + A \leqslant y$ for all $y \geqslant \bar{y}$. We could expect this result holds for $r > 1$, $\delta r < 1$. The following example will show that in some cases we could get $\{y^t\} \to_{\text{a.s.}} \infty$.

Consider the following problem expressed in its dynamic programming formulation.

PROBLEM I. $V_t(y) = \text{Max}\{-e^{-c} + \delta E V_{t-1}(rx + \omega)\}$,

$$c + x = y,$$

$$c \geqslant 0, \qquad x \geqslant 0,$$

$$V_1(y) = -e^{-y}.$$

To find an explicit solution of Problem I is "very difficult," and as a metter of fact, we are not going to solve it. We will find the solution of another problem, and then show that for the limiting policy the stock level $\{y^t\}$ of Problem I converges almost sure to $+\infty$.

Consider the following problem, in its dynamic programming formulation.

PROBLEM II. $V_t(y) = \text{Max}\{-e^{-c} + \delta E V_{t-1}(rx + \omega)\},$

$$c + x = y,$$

$$V_1(y) = -e^{-y}.$$

Observe that now we do not impose the condition of nonnegativeness on c and x. This fact will allow us to find an explicit solution of Problem II. This solution is given in Appendix A, and the limiting policy is

$$\bar{c}(y) = \frac{r-1}{r} y - \frac{1}{r-1} \log \delta r E[e^{-\omega(r-1)/r}], \tag{3.9}$$

$$\bar{x}(y) = \frac{y}{r} + \frac{1}{r-1} \log \delta r E[e^{-\omega(r-1)/r}]. \tag{3.10}$$

The solution of Problem II, its prices, and the stock level will be denoted by $\{\bar{c}(y), \bar{x}(y)\}, \bar{p}(y)$, and $\{\bar{y}^t\}$ and the corresponding ones for Problem I by $\{c(y), x(y)\}, \{p(y)\}$, and $\{y^t\}$.

From (3.10) we get the following relation for the stock level

$$\bar{y}^{t+1} = \bar{y}^t + [r/(r-1)] \log \delta r E[e^{-[(r-1)/r]\omega}] + \omega;$$

i.e., $\{\bar{y}^t\}$ is a random walk in which the random variable is

$$\omega' = \omega + [r/(r-1)] \log \delta r E[e^{-[(r-1)/r]\omega}].$$

Now, if we take $\text{Prob}[\omega = 0] = 1 - \theta$ and $\text{Prob}[\omega = A > 0] = \theta$ we will have that

$$E\omega' = \theta A + \frac{r}{r-1} \log \delta r + \log[1 - \theta + \theta e^{-A(r-1)/r}] \frac{r}{r-1}$$

for a given θ if we choose A sufficiently large, and we will have that $E\omega' > 0$ and so $\{\bar{y}^t\} \to_{\text{a.s.}} +\infty$. Now, we will show that $\bar{y}^t < y^t$ for all t, but first we need a lemma, whose proof is given in Appendix B.

LEMMA 3.6.

$$\bar{x}(y) \leqslant x(y) \quad \text{for } y \geqslant 0.$$

THEOREM 3.7.

$$\bar{y}^t \leqslant y^t \quad \text{for all } t, \text{ if } y^0 = \bar{y}^0.$$

Proof. The proof will be done by induction. First, it is true for $t = 0$ since $y^0 = \bar{y}^0$. Now assume that it is true for t, i.e., $\bar{y}^t \leqslant y^t$. If $\bar{y}^{t+1} < 0$ then $\bar{y}^{j+1} < y^{t+1}$ since $y^t \geqslant 0$ for all t. If $\bar{y}^{t+1} > 0$, we have

$$\bar{y}^{t+1} = r\bar{x}(\bar{y}^t) + \omega^t \leqslant rx(\bar{y}^t) + \omega^t$$

by monotonicity of $\bar{x}(\cdot)$ and the induction hypothesis. Now, using Lemma 3.6 we get

$$\bar{y}^{t+1} = r\bar{x}(y^t) + \omega^t$$
$$\leqslant r\bar{x}(y^t) + \omega^t$$
$$\leqslant rx(y^t) + \omega^t = y^{t+1}. \quad \blacksquare$$

From the previous results we could state that if $u(c) = -e^{-c}$ and for a given θ A is sufficiently large, then $\{y^t\} \to_{\text{a.s.}} +\infty$.

Since in general it is not necessarily true that there exists a $\bar{y}$ such that $y^t \in [0, \bar{y}]$ for all t, it would be interesting to find conditions under which $y^t \in [0, \bar{y}]$ for some $\bar{y}$.

THEOREM 3.8. *A sufficient condition for the existence of a $\bar{y}$ such that* $rx(y) + A \leqslant y$ *for all* $y \geqslant \bar{y}$ *is that*

$$\lim_{y \to \infty} \frac{Ep(rx(y) + \omega)}{p(rx(y) + A)} \leqslant 1. \tag{3.11}$$

Proof. From (3.11) we have that given an $\epsilon > 0$ there exists a $\bar{y}$ such that

$$\frac{Ep(rx(y) + \omega)}{p(rx(y) + A)} \leqslant 1 + \epsilon \qquad \text{for all } y \geqslant \bar{y}.$$

Hence,

$$\frac{p(y)}{\delta rp(rx(y) + A)} \leqslant 1 + \epsilon$$

and

$$\frac{p(y)}{p(rx(y) + A)} \leqslant \delta r + \epsilon \delta r.$$

If, $\epsilon < (1 - \delta r)/\delta r$, then $p(y) \leqslant p(rx(y) + A)$ for all $y \geqslant \bar{y}$ and consequently $y \geqslant rx(y) + A$ for all $y \geqslant \bar{y}$. $\quad \blacksquare$

One case that immediately satisfies the last theorem is the case $u(c) = Kc + g(c)$, where $K > 0$ is a constant and $g(c)$ is a strictly concave function, increasing and differentiable.

The last result would be of limited application if we were restricted to the last example. But as we will see, using the concept of exponent of a function (used by Brock and Gale in [1]) we will be able to apply it to a very general class of problems.

DEFINITION. The exponent of the function f at the point x is

$$e_f(x) = \log_x f(x),$$

and the asymptotic exponent e_f of f is $\lim_{x \to \infty} e_f(x)$.

If $f(x)$ is a positive function, then the following fact can be shown (see [1] for a proof).

If $x > x'$ and $\underline{\alpha} < e_f < \bar{\alpha}$ then for x large

$$(x'/x)^{\underline{\alpha}} < f(x)/f(x') < (x/x')^{\bar{\alpha}}. \tag{3.12}$$

THEOREM 3.9. *If $u'(c)$ has an asymptotic exponent e_f, then there exists a $\bar{y}$ such that $rx(y) + A \leqslant y$ for all $y \geqslant \bar{y}$.*

Proof. If $\lim x(y) = K$, then for $y \geqslant tK + A$, $rx(y) + A \leqslant y$. If $\lim_{y \to \infty} x(y) = \infty$ then the proof is as follows.

$$\frac{Ep(rx(y) + \omega)}{p(rx(y) + A)} \leqslant \frac{p(rx(y) + a)}{p(rx(y) + A)}$$

$$= \frac{u'(c(rx(y) + a))}{u'(c(rx(y) + A))} \leqslant \left(\frac{c(rx(y) + A)}{c(rx(y) + a)} \right)^{\alpha} \quad \text{for } y \text{ large,} \tag{3.13}$$

by (3.12), and using the fact that $\lim_{y \to \infty} c(y) = \infty$ and $c(tx(y) + A) \geqslant c(rx(y) + a)$.

Now, $c(rx(y) + A) = c(rx(y) + a) + h(y)(A - a)$, were $0 \leqslant h(y) \leqslant 1$. Hence, from (3.13) we get

$$\frac{Ep(rx(y) + \omega)}{p(rx(y) + A)} \leqslant \left(\frac{c(rx(y) + a) + h(y)(A - a)}{c(rx(y) + a)} \right)^{\alpha}$$

$$= \left(\frac{1 + \dfrac{h(y)(A - a)}{c(rx(y) + a)}}{1} \right)^{\alpha} \quad \text{for } y \text{ large.}$$

Now, taking the limit as $y \to \infty$ we get $Ep(rx(y) + \omega)/[p(rx(y) + A)] = 1$. The theorem now follows from Theorem 3.8.

Now, whenever there exists a $\bar{y}$ such that $rx(y) + A \leqslant y$ for all $y \geqslant \bar{y}$, we will have that the process evolves in an interval $[a, \bar{y}]$. As in the case $0 < \delta < 1, r = 1$ it follows also that the process $\{y^t\}$ is a delayed renewal process, as are its consequences.

In order to give some economic flavor to our result we should first mention that there is a close relationship between the limiting exponent of the marginal utility $u'(y)$ and the Arrow–Pratt relative risk avertion measure $R_R(y) = -u''(y)/u'(y)$. In [7, p. 110], it is shown that if $R_R(y) \leqslant R$ for all $y \geqslant y_0$ for some y_0 then $u'(y)/u'(y_0) \geqslant (y_0/y)^R$ for all $y \geqslant y_0$, that is inequality (3.12) obtained from the existence of the limiting exponent for the marginal utility also holds. A possible economic explanation of our result could be argued in

the following way: At each period the consumer is guided in his choice between consumption and saving by the values of an extra unit in face of his actual wealth, y, and his next $rx(y) + w$, a random variable, that depends on his next income w. Those values are expressed by $p(y)$ and $p(rx + w)$, the latter being also a random variable. His uncertainty is about the changes in the value $p(rx + w)$. Now, if $p(rx + A)/[Ep(rx + A)]$ is "close" to 1, we could say that the consumer forsees small variation of $p(rx + w)$ with respect to $p(rx + A)$. If we assume that the consumer's behavior could be expressed by a utility function whose marginal utility has a limiting exponent or a uniformly bounded relative risk avertion for large values of y we will have that $\lim_{y \to \infty} \{p(rx(y) + A)/[Ep(rx(y) + w)]\} \to 1$ (see theorem 3.9) and if we also take into account the fact that $\delta r < 1$ we will have that for sufficiently large value of y the consumer will behave as in the deterministic case, i.e., his wealth is going to decrease no matter what happens with his future income.[3] The utility function e^{-y} does not have a uniformly bounded relative risk avertion, nor does its marginal utility have a limiting exponent, that is the variability of $p(rx + w)$ with respect to $p(rx + A)$ might not decrease to an extent that makes him behave as in the "deterministic case." This is in fact what our previous example shows, namely, given a binomial distribution for w if we choose A sufficiently large the sequence of wealths $\{y^t\}$ that we get when we follow the optimal policy converges almost surely to infinity. On the other hand if A is sufficiently small, for instance, $A \leqslant -\log \delta r$, it can be shown that $y^t \geqslant A$ implies $y^t \geqslant y^{t-1}$ for all t. A reason for this result could be given in a similar way as before. First, for different values of the wealth $rx(y) + w$, the greatest difference in consumption is A, and if A is "small" this will imply from the continuity of $u'(y)$ that $p(rx + w)$ do not vary relatively to $p(rx + A)$ to a great extent.

We also should point out a fact observed early by Pratt [8] that the function $u'(y) = e^{-y}$ in spite of its nice graph has great relative variations when we compare values that differ in a relatively small proportion. For instance, if $y = 10^6$ and $y' = y + 10^{-5}y$ or $y' = 10^6 + 10$ we have that $e^{-y}/e^{-y'} = e^{10}$, which is a very large number!

APPENDIX A

In this Appendix we will find the solution of the problem:

Maximum

$$E \sum_{t=0}^{T} \delta^t u(c^t)$$

---

[3] From the fact that $\lim[p(tx + A)/Ep(rx + A)] \to 1$, the optimality condition $p(y) = \delta r Ep(rx(y) + w)$ could be replaced by $p(y) \simeq (\delta r + \epsilon)p(rx(y) + A)$, where ϵ can be made arbitrarly small by choosing y arbitrarly large.

subject to $c^t + x^t = y^t,$ $0 \leqslant t < T$

$$y^{t+1} = rx^t + \omega^t$$

and

$$x^T = 0, \qquad c^T = y^T,$$

where

$$u(c) = -e^{-c}, \qquad \omega^t \in [a, A] \; \omega^t \text{ are i.i.d.}$$

The dynamic programming formulation leads to the functional equations:

$$V_t(y) = \max\{-e^{-c} + \delta EV_{t-1}(rx + \omega^t)\},$$

$$x + c = y,$$ (A.1)

and

$$V_1(y) = -e^{-y}, \qquad c_1(y) = y, \qquad \text{and} \qquad x_1(y) = 0.$$

We claim that the solution of (A.1) for any $t \geqslant 1$ has the form

$$c_t(y) = \frac{r^{t-1}}{r^{t-1} + \cdots + 1} \, y - A_t \log \delta r - B_t,{}^4$$

$$x_t(y) = \frac{r^{t-2} + \cdots + 1}{r^{t-1} + \cdots + 1} \, y + A_t \log \delta r + B_t.$$

It is true for $t = 1$. Now assume that is true for t, then

$$B_{t+1} = \left(B_t + \log E \exp \left(-\frac{r^{t-1}}{r^{t-1} + \cdots + 1} \, \omega \right) \right) \times \frac{r^{t-1} + \cdots + 1}{r^t + \cdots + 1}.$$

From the recurrence relation, we will have that the limiting policy is

$$c(y) = \frac{r-1}{r} \, y - \frac{1}{r-1} \log \delta r - \frac{1}{r-1} \log E[e^{-((r-1)/r)\omega}],$$

$$x(y) = \frac{y}{r} + \frac{1}{r-1} \log \delta r + \frac{1}{r-1} \log E[e^{-((r-1)/r)\omega}].$$

APPENDIX B

In this appendix we will obtain a result that permits us to compare the optimal policies for the problems:

PROBLEM I.

$$\text{Max} \sum_{t=0}^{T} \delta^t u(c^t)$$

---

[4] Where the sum in the denominator ranges over L's such that $t - i \geqslant 0$.

Subject to

$$y^{t+1} = rx^t + \omega^t,$$
$$x^t + c^t = y^t,$$
$$x^t \geqslant 0, \qquad c^t \geqslant 0.$$

PROBLEM II.

$$\text{Max} \sum_{t=0}^{T} \delta^t u(c^t)$$

Subject to

$$y^{t+1} = rx^t + \omega t,$$
$$x^t + c^t = y^t,$$
$$c^T = y^t, \qquad x^T = 0.$$

The prices and policies for Problem I will be denoted by $p_t(y)$, $\{c_t(y), x_t(y)\}$ and the corresponding ones to Problems II by $\bar{p}_t(y)$, $\{\bar{c}_t(y), \bar{x}_t(y)\}$.

THEOREM B1. *For $y \geqslant 0$, $\bar{x}_t(y) < x_t(y)$ and consequently $\bar{x}(y) \leqslant x(y)$.*

Proof. The proof will be done by induction. It is true for $t = 1$ since $x_1(y) = 0$ and $\bar{x}_1(y) = 0$. Now, assume that it is true for $t - 1$, i.e., $x_{t-1}(y) \geqslant \bar{x}_{t-1}(y)$ for $y \geqslant 0$ and by contradiction that $x_t(y) < \bar{x}_t(y)$. Then $y - c_t(y) < y - \bar{c}_t(y)$ and consequently $c_{t-1}(y) > \bar{c}_{t-1}(y)$ and $p_{t-1}(y) < \bar{p}_{t-1}(y)$. From the optimalities conditions we get

$$p_t(y) = Ep_{t-1}(rx_t(y) + \omega)$$
$$\geqslant Ep_{t-1}(r\bar{x}_{t-1}(y) + \omega) \qquad \text{since } p_{t-1}(\cdot) \tag{B.1}$$

is nonincreasing and $r\bar{x}_{t-1}(y) > x_t(y) \geqslant 0$. But from the induction hypothesis we have that $c_{t-1}(y) \leqslant \bar{c}_{t-1}(y)$ and so, $\bar{p}_{t-1}(y) \geqslant \bar{p}_{t+1}(y)$. Then (B.1) implies

$$p_t(y) \geqslant Ep_{t-1}(r\bar{x}_{t-1}(y) + \omega)$$
$$= \bar{p}_{t-1}(y),$$

which is a contradiction. ∎

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