

# Advanced Digital Signal Processing

高等數位訊號處理

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課程網頁：<https://djj.ee.ntu.edu.tw/ADSP.htm>

歡迎大家來修課，也歡迎有問題時隨時聯絡！

## 上課方式

(1) 錄影，影片將藉由 NTU Cool 下載 <https://cool.ntu.edu.tw>

(2) 現場 (週三下午 15:30~18:20，明達館205室)

## 作業和報告繳交方式：

用 NTU Cool 來繳交作業與報告的電子檔 <https://cool.ntu.edu.tw>

注意，Tutorial 一定要交Word 或 Latex 原始碼

Wiki 要寄編輯條目的連結給老師

上課時間：14 週

2/21,

3/6, 出 HW1

3/13,

3/20, 交 HW1

3/27, 出 HW2

4/3,

4/10, 交 HW2

4/17, 出 HW3

4/24,

5/1, 交 HW3

5/8, 出 HW4

5/15,

5/22, 交 HW4, 張國章博士代課

5/29, 出 HW5

6/5 之前, Oral Presentation

6/12, 交 HW5 及 term paper

原則上: 3n-1 週出作業, 3n+1 週繳交

- 評分方式：

**Basic: 15 scores**

原則上每位同學都可以拿到 12 分以上，

另外，會有額外問答題，每位同學四次，每答對一次加0.8分

**Homework: 60 scores (5 times, 每 3 週一次)**

請自己寫，和同學內容極高度相同，將扣 70% 的分數

就算寫錯但好好寫也會給 40~95% 的分數，

遲交分數打 8 折，**不交不給分**。不知道如何寫，可用 E-mail 和我聯絡，或於上課時發問

**禁止 Ctrl-C Ctrl-V 的情形。**

**Term paper 25 scores**

## Term paper 25 scores

方式有五種

### (1) 書面報告

10頁以上(不含封面)，中英文皆可，11或12的字體，題目可選擇和信號處理(包括信號、通訊、影像、音訊、生醫訊號、經濟信號處理等等)有關的任何一個主題。

包括 abstract, conclusion, 及 references，並且要分 sections。儘量工整  
鼓勵多做實驗及模擬，

有做模擬的同學請將程式附上來，會有額外加分。

嚴禁 Ctrl-C Ctrl-V 的情形，否則扣 70% 的分數

### (2) Tutorial (對既有領域做淺顯易懂的整理) *full pw: give 1997*

限十八個名額，和書面報告格式相同，但頁數限制為18頁以上(若為加強前人的 tutorial，則頁數為  $(2/3)N + 13$  以上， $N$  為前人 tutorial 之頁數)，題目由老師指定，以清楚且有系統的介紹一個主題的基本概念和應用為要求，為上課內容的進一步探討和補充，交 Word 檔。

選擇這個項目的同學，學期成績加 4分

—

### (3) 口頭報告 full

限十個名額，每個人 15~40分鐘，題目可選擇和課程有關的任何一個主題。口頭報告的同學請在6月7日以前將影片錄好，並且把影片(或連結)寄給老師。有意願的同學，請儘早告知，以先登記的同學為優先。

選擇這個項目的同學，學期成績加2分

口頭報告時，希望同學們至少能參與線上觀看，並將做為第五次作業的其中一題。

### (4) 編輯 Wikipedia

中文或英文網頁皆可，至少 2 個條目，但不可同一個條目翻成中文和英文。總計 80行以上。限和課程相關者，自由發揮，越有條理、有系統的越好

選擇編輯 Wikipedia 的同學，請於 5月29日前，向我登記並告知我要編輯的條目(2 個以上)，若有和其他同學選擇相同條目的情形，則較晚向我登記的同學將更換要編輯的條目

書面報告和編輯 Wikipedia，期限是 6月12日

以上(1), (2), (3), (4) 不管選哪個題目，若有做實驗模擬，請附上程式碼，會有額外的加分 (鼓勵不強制)

## (5) 程式編寫，協助信號處理程式資料庫的建立

選擇 Page 10 當中其中一組題目，來編寫相關的程式，程式用 Matlab, Python, 或 C 編寫皆可 (共6組題目， $6 \times 3 = 18$ 個名額)

選擇這個題目的同學，期末要用 NTUCool 交程式，並另外寫一個 ReadMe 的檔案，說明程式該如何執行，並舉例子顯示程式執行結果。

## Tutorial 可供選擇的題目(可以略做修改)

- ✓(1) Automatic Music Evaluation
- ✓(2) Transformer in Natural Language Processing
- ✓(3) Speech Recognition in Multi-Speaker Scenario
- ✓(4) Color Coordinate Transform
- (5) Advanced Multimedia Security Techniques
- ✓(6) Semantic Segmentation
- ✓(7) Instance and Panoptic Segmentation
- ✓(8) Panorama Image Processing
- ✓(9) Reflection Removal in Image
- ✓(10) Alternating Direction Method of Multipliers (ADMM) for Optimization
- ✓(11) Optimization for  $L_0$  Norm Problems
- ✓(12) Beam Forming

## Tutorial 可供選擇的題目(可以略做修改)

✓(13) BM3D Image Denoising Method

• (14) Primitive Polynomial

✓(15) Galois field

✓(16) Biosignature Identification

• (17) Electroretinogram (ERG)

✓(18) Electrooculogram (EOG)

程式編寫可供選擇的題目

(有意願的同學可選擇其中一組，用 Matlab, Python，或 C++ 皆可，要加說明檔 ReadMe，不可 copy 網路程式)

- ✓(1) (i) Step Invariance IIR Filter Design  
(ii) MSE FIR Filter Design with Weights and Transition Bands
- Mat (2) (i) Minimax Filter for Types II  
Py (ii) Minimax Filter for Types III  
(iii) Minimax Filter for Types IV
- ✓(3) (i) Use the DCT to Compute the DFT for Even Inputs  
(ii) Use the DCT to Compute the DFT for Odd Inputs  
(iii) Discrete-Hartley Transform
- ✓(4) (i) Change the Time of a Signal without Varing the Frequency  
(ii) Change the Frequency of a Signal without Varing the Time
- ✓(5) (i) Sectioned Convolution  
(ii) Use the Recursive Method to Implement the Convolution with  $a^n u[n]$
- ✓(6) (i) Orthogonal Frequency-Division Multiplexing  
(ii) Modulation and Demodulation by CDMA Using Walsh Bases

## Matlab Program

Download: 請洽台大各系所

### 參考書目

洪維恩，Matlab 程式設計，旗標，台北市，2013 . (合適的入門書)

張智星，Matlab 程式設計入門篇，第四版，碁峰，2016.

預計看書學習所花時間： 3~5 天

## Python Program

Download: <https://www.python.org/>

### 參考書目

葉難， Python程式設計入門，博碩，2015

黃健庭， Python程式設計：從入門到進階應用，全華，2020

The Python Tutorial      <https://docs.python.org/3/tutorial/index.html>

研究所和大學以前追求知識的方法有什麼不同？

研究所：觀念的學習，整理與分析，創造

大學：記憶，熟練，速度

## Question:

$$\rightarrow \cos(2\pi ft) - j \sin(2\pi ft)$$

Fourier transform:  $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$

## Why should we use the Fourier transform?

(1) spectrum analysis

(2) convolution  $\rightarrow$  multiplication

$$y(t) = x(t) * h(t) \quad Y(f) = X(f)H(f)$$

$$= \int x(t-\tau)h(\tau) d\tau$$

Is the Fourier transform the best choice in any condition?

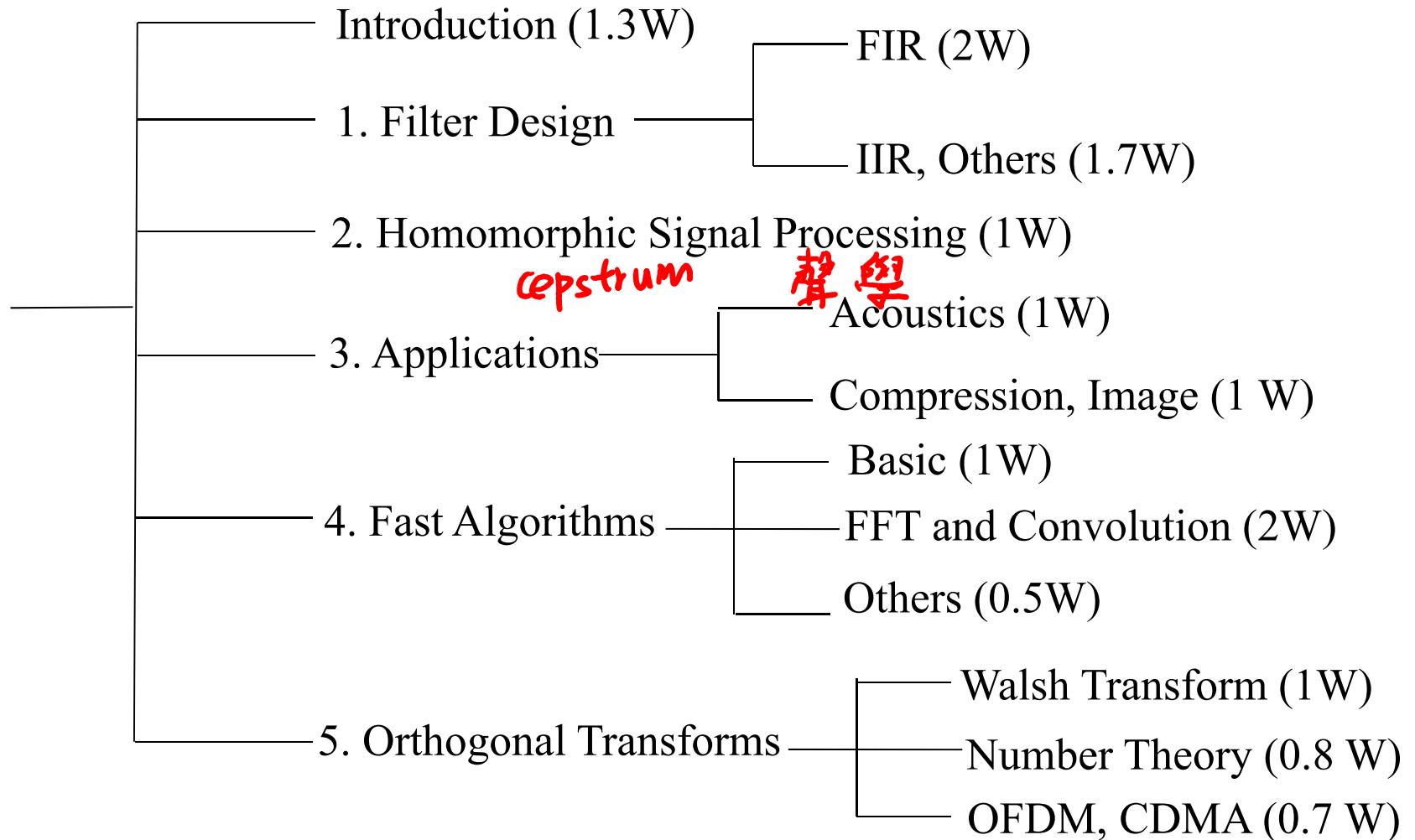
disadvantage  $(a+jb)(c+jd) = ac - bd + j(ad + bc)$

(1) not real operation

(2) irrational number multiplication

# I. Introduction

## Outline



目標：

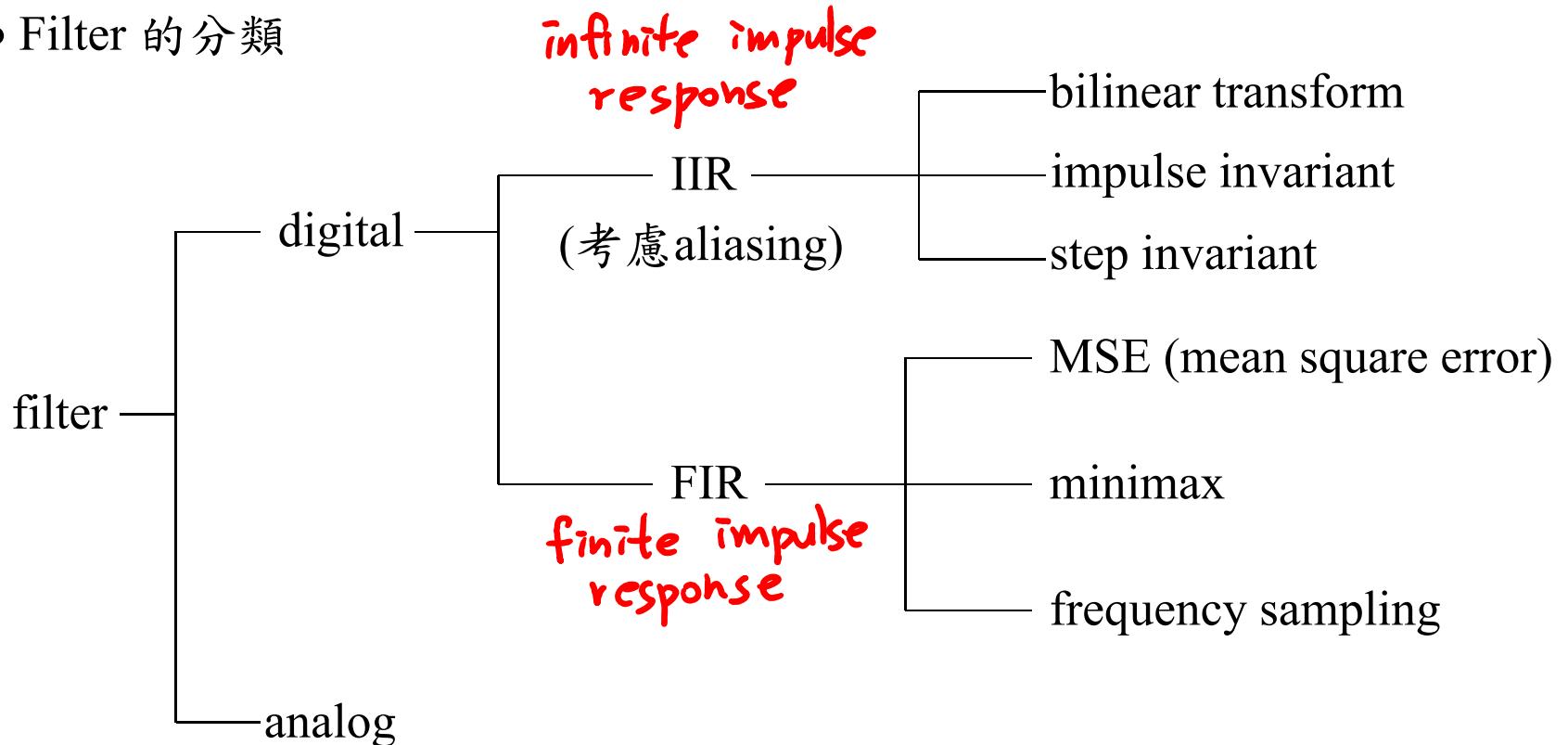
- (1) 對 Digital Signal Processing 作更有系統且深入的了解
- (2) 學習 Digital Signal Processing 幾個重要子領域的基礎知識

## Part 1: Filter

$$y(t) = x(t) * h(t) = \int x(t-\tau) h(\tau) d\tau$$

$x(t)$ : input,  $h(t)$ : impulse response

- Filter 的分類



(技術較早開發)

If  $\text{length}(x) = N$ ,  $\text{length}(h) = L$   
 $\text{length}(y) = N + L - 1$

$$y[n] = x[n] * h[n] = \sum_m x[n-m] h[m]$$

: IIR

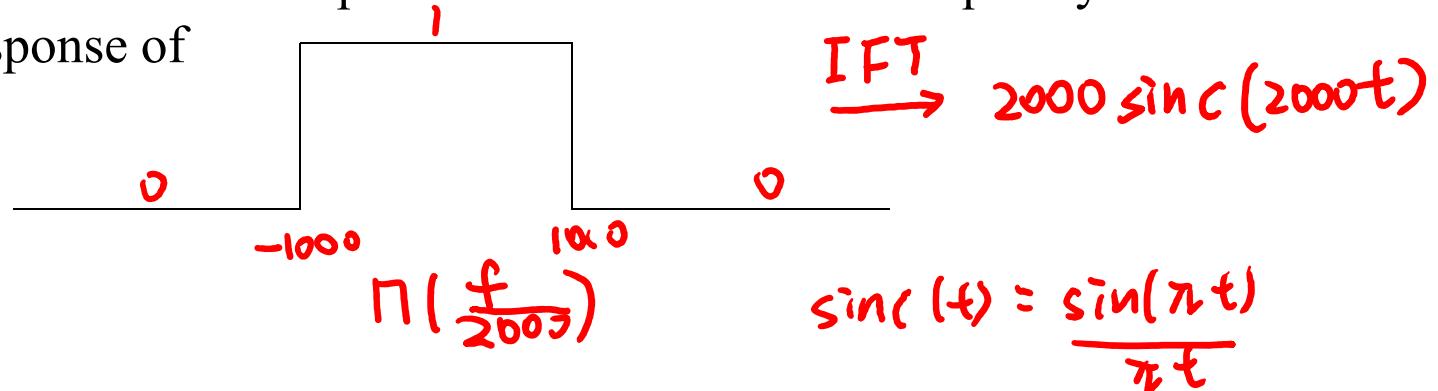
IIR filter 的優點：(1) easy to design

(2) (sometimes) easy to implement *page 3)*

缺點：  
 (1) usually more computation loading  
 (2) output has infinite length  
 (3) may not be stable

FIR filter 的優點：  
 (1) usually less computation  
 (2) output has finite length  
 (3) always stable

缺點：An FIR filter is impossible to have the ideal frequency response of



## Part 2: Homomorphic Signal Processing

- 概念：把 convolution 變成 addition

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &\downarrow F^T \\ Y(f) &= X(f) H(f) \\ &\downarrow \log \\ \log(Y(f)) &= \log(X(f)) + \log(H(f)) \end{aligned}$$

## Part 3: Applications of DSP

filter design, data compression (image, video, text), acoustics (speech, music), image analysis (structural similarity, sharpness), 3D accelerometer

- Part 4: Fast Algorithms
- Basic Implementation Techniques

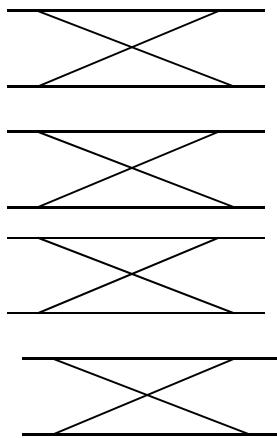
Example: one complex number multiplication

= ? Real number multiplication.

Trade-off: “Multiplication” takes longer than “addition”

- FFT and Convolution

Due to the Cooley-Tukey algorithm (butterflies),  
the complexity of the FFT is:



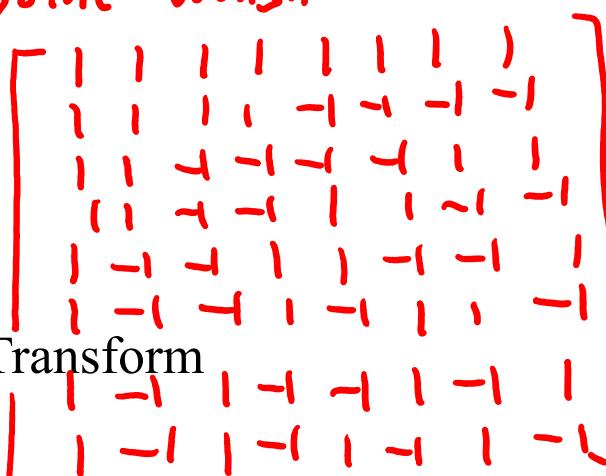
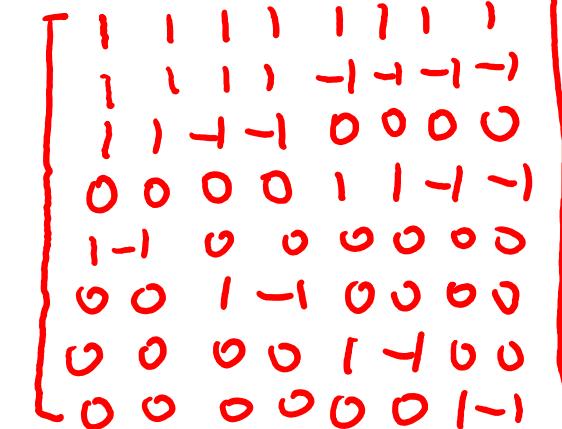
The complexity of the convolution is: 3個 DFTs,  $O(N \log_2 N)$

$\downarrow$  reduced  
 $O(N)$

- Part 5: Orthogonal Transforms

DFT 的兩個主要用途：

Question: DFT 的缺點是什麼？  $DFT(x[n]) = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi mn}{N}}$

<ul style="list-style-type: none"> <li>• Walsh Transform (CDMA)</li> <li>• Number Theoretic Transform</li> </ul>	<p><i>8-point Walsh</i></p>  <p>The 8-point Walsh transform matrix is an 8x8 binary matrix. It consists of two 4x4 Hadamard matrices stacked vertically. The first 4x4 matrix has columns: (1, 1, 1, 1), (1, -1, 1, -1), (1, 1, -1, -1), (1, -1, -1, 1). The second 4x4 matrix has columns: (1, 1, 1, 1), (-1, -1, 1, 1), (-1, 1, -1, 1), (1, 1, -1, -1).</p>	<p><i>Haar</i></p>  <p>The 8-point Haar transform matrix is an 8x8 binary matrix. It consists of two 4x4 Haar matrices stacked vertically. The first 4x4 matrix has columns: (1, 1, 1, 1), (1, 1, -1, -1), (1, -1, 1, -1), (1, -1, -1, 1). The second 4x4 matrix has columns: (1, 1, 1, 1), (0, 0, 0, 0), (1, -1, 0, 0), (0, 0, 1, -1).</p>
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- Orthogonal Frequency-Division Multiplexing (OFDM)
- Code Division Multiple Access (CDMA)

# Review 1: Four Types of the Fourier Transform

- 四種 Fourier transforms 的比較

	time domain	frequency domain
(1) Fourier transform	continuous, aperiodic	continuous, aperiodic
(2) Fourier series	continuous, <u>periodic</u> (or continuous, only the value in a finite duration is known)	<u>discrete</u> , aperiodic 只要是週期性的訊號頻譜就會是離散的
(3) discrete-time Fourier transform <u>(DTFT)</u>	<u>discrete</u> , aperiodic	continuous, <u>periodic</u> 如果 input 在 time domain 是離散的，在 frequency domain 就一定是週期的。
(4) discrete Fourier transform <u>(DFT)</u> implemented by FFT (fast Fourier transform)	discrete, <u>periodic</u> (or discrete, only the value in a finite duration is known)	discrete, <u>periodic</u> 如果 input 在 time domain 是週期的，在 frequency domain 就一定是離散的。 離散 → 週期

$$x(t+T) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f(t+T)} df \quad 23$$

## (1) Fourier Transform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \quad ,$$

Alternative definitions

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad ,$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

$$f = \frac{m}{T} = m \underline{df}$$

$$\sum_m e^{j2\pi m t} = \delta(t)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$\sum_m e^{j\frac{2\pi}{T} m(t-t_1)} = 0 \quad \text{if } t \neq t_1 \\ = \delta\left(\frac{t-t_1}{T}\right) = T\delta(t-t_1)$$

## (2) Fourier series (suitable for period function)

periodic 時 積分從  
負無限大到正無限  
大變成0到T

$$X[m] = \int_0^T x(t) e^{-j\frac{2\pi m}{T} t} dt$$

$$T: \text{週期} \quad x(t) = x(t+T)$$

$$x(t) = T^{-1} \sum_{m=-\infty}^{\infty} X[m] e^{j\frac{2\pi m}{T} t}$$

' $m \in \mathbb{Z}$ : 從後方看Σ'

$$e^{j2\pi m} = (\cos(2\pi m) + j \sin(2\pi m)) = 1$$

Possible frequencies are to satisfy:

$$e^{j2\pi f t} = e^{j2\pi f(t+T)} = e^{j2\pi f t} e^{j2\pi f T}$$

①某 phase 是 π 的整數倍

$$\frac{1}{T} \text{ 整數倍 (m倍)} \quad e^{j2\pi f T} = 1$$

頻率和  $m$  之間的關係 :  $f = \frac{m}{T}$

$$x(t) = C \sum_m X[m] e^{j\frac{2\pi m}{T} t}$$

$$= C \sum_m \int_0^T x(t_1) e^{-j\frac{2\pi m}{T} t_1} e^{j\frac{2\pi m}{T} t} dt$$

$$C = \frac{1}{T}$$

$$= C \int_0^T x(t_1) \left( \sum_m e^{j\frac{2\pi}{T} m(t-t_1)} \right) dt$$

$$= C \int_0^T x(t_1) T \delta(t-t_1) dt = CT x(t)$$

假設  $x(t) = \int_{-\infty}^{\infty} x_a(f) e^{-j2\pi f t} df$   
 sample by time 等於 analytic

$$x[n] = x_a(n\Delta_t)$$

### (3) Discrete-time Fourier transform (DSP 常用) (DTFT) from dt

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n \Delta_t}, \quad x[n] = \Delta_t \int_0^{1/\Delta_t} X(f) e^{j2\pi f n \Delta_t} df$$

$$X(f + \frac{1}{\Delta_t}) = \sum_n x[n] e^{-j2\pi(f + \frac{1}{\Delta_t})n \Delta_t} = \sum_n x[n] e^{-j2\pi f n \Delta_t} e^{-j2\pi n}$$

$$X(F) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi n f_s} = \sum_{n=-\infty}^{\infty} e^{-j2\pi F_n} x[n] \quad x[n] = \int_0^{1/F} X(F) e^{j2\pi F n} dF = X(f)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n \Delta_t} \quad x[n] = \frac{\Delta_t}{2\pi} \int_0^{2\pi/\Delta_t} X(\omega) e^{j\omega n \Delta_t} d\omega \quad (\because F = \frac{f}{f_s} = \frac{f}{\Delta_t})$$

$$X(F) = X(F+1)$$

Set  $f_s = \frac{1}{\Delta_t}$  (sampling frequency)

$$\underline{X(f + f_s) = X(f)}$$

$$dF = \Delta_t df$$

### (4) Discrete Fourier transform (DFT) (DSP 常用)

$$x[n] = x_c(n\Delta_t) \quad x[n] = x[n+N] \quad (x_c(n\Delta_t) = x_c(n\Delta_t + N\Delta_t))$$

$$X[m] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi m n}{N}}, \quad x[n] = \frac{1}{N} \sum_{m=0}^{N-1} X[m] e^{j\frac{2\pi m n}{N}}$$

$$t = n\Delta_t \quad T = N\Delta_t \quad f = \frac{m}{T} = \frac{m}{N\Delta_t} \quad e^{-j2\pi f n \Delta_t} = e^{-j2\pi \frac{m}{N\Delta_t} n \Delta_t} = e^{-j\frac{2\pi}{N} m n}$$

$$\text{頻率和 } m \text{ 之間的關係: } f = \frac{m}{N\Delta_t} = \frac{m}{N} f_s \quad \Delta_t df = \Delta_t \frac{f_s}{N} = \frac{1}{N}$$

$$X_c(f) = X_c(f + \frac{1}{\Delta_t})$$

$$X[m] = X_c(m\Delta_f) = X_c\left(\frac{m}{N\Delta_t}\right) \quad \text{where } f_s = 1/\Delta_t \text{ (sampling frequency)}$$

$$X[m+N] = X_c\left(\frac{m+N}{N\Delta_t}\right) = X_c\left(\frac{m}{N\Delta_t} + \frac{1}{\Delta_t}\right)$$

Modified :

$$(1) X(f) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi f t} dt$$

该式为连续信号的傅立叶变换，但此非离散化形式。

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### (3) Discrete-time Fourier transform (DTFT)

即  $x[n]$  discrete  
范围为  $n \sim -\infty \sim +\infty$

用 ZC 来 implement  
不容易  
且用 ZC 未实现的  
 $\rightarrow$  (4) DFT

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n \Delta_t}, \quad x[n] = \frac{1}{\Delta_t} \int_0^{1/\Delta_t} X(f) e^{j2\pi f n \Delta_t} df$$

$\Delta_t$ : sampling interval

$$X(f + f_s) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi n \Delta_t (f + f_s)} = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi n \Delta_t f} e^{-j2\pi n \Delta_t f_s} = X(f)$$

$f_s = \frac{1}{\Delta_t}$

Note:  $X(f) = X(f + f_s)$

where  $f_s = 1/\Delta_t$  (sampling frequency)

It can be rewritten as:

$$X(F) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi n F}$$

$F$  为 normalized frequency

$$x[n] = \int_0^1 X(F) e^{j2\pi n F} dF$$

$$\Delta_t dF = dF$$

$X(F) = X(F + 1)$ ;  $F = \frac{f}{f_s}$  normalized 之后  $f_s \rightarrow 1$

$$F = \frac{f}{f_s} = f \Delta_t \quad (\text{normalized frequency, see page 26})$$

频率  
频率

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n \Delta_t} \quad x[n] = \frac{\Delta_t}{2\pi} \int_0^{2\pi/\Delta_t} X(\omega) e^{j\omega n \Delta_t} d\omega$$

$$(1) X(f) = \int_{-\infty}^{\infty} x_c(t) e^{-j2\pi f t} dt$$

#### (4) Discrete Fourier transform (DFT)

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(DSP 常用)

$$X[m] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi m n}{N}}, \quad x[n] = \frac{1}{N} \sum_{m=0}^{N-1} X[m] e^{j\frac{2\pi m n}{N}}$$

$t = n\Delta_t$

$f = m\Delta_f = \frac{m}{N\Delta_t}$

$e^{-j2\pi f t} = e^{-j2\pi n\Delta_t + j\frac{m}{N\Delta_t} t} = e^{-j\frac{2\pi nm}{N}}$

$$dt df = \Delta_t \Delta f = \frac{1}{N}$$

$$X[m+N] = \sum_n x[n] e^{-j\frac{2\pi}{N}(m+N)n} = \sum_n x[n] e^{-j\frac{2\pi mn}{N}} e^{-j\frac{2\pi Nn}{N}} = X[m]$$

Note:  $x[n] = x[n+N]$ ,  $X[m] = X[m+N]$

If  $x[n]$  was sampled from  $x_c(t)$ ,  $x[n] = x_c(n\Delta_t)$ , then

$$x_c(n\Delta_t) = x_c(n\Delta_t + N\Delta_t)$$

continuous

頻率和  $m$  之間的關係 :  $f = \frac{m}{N\Delta_t} = \frac{m}{N} f_s$  非整數值.

i.e.,  $f = m\Delta_f$  where  $\Delta_f = \frac{1}{N\Delta_t}$

↑ 整數倍.      ↑ 週期      ↑ 週期

$f = m\Delta_f = m \cdot \frac{1}{N\Delta_t}$

$\Rightarrow \Delta f = \frac{1}{N\Delta_t}$

$\therefore \Delta t \cdot \Delta f = \frac{\Delta t}{N\Delta_t} = \frac{1}{N}$

大部分現實條件 continuous signal  
取樣

## Review 2: Normalized Frequency

Why use normalized frequency? A: 在很多情況下離散訊號的取樣範圍而未知。

(時間/空間)

$\Delta_t = \text{sampling interval}$

### (1) Definition of normalized frequency $F$ :

$$F = \frac{f}{f_s} = f \Delta_t = \frac{\omega \Delta_t}{2\pi} \quad \text{where } f_s = 1/\Delta_t \text{ (sampling frequency)}$$

$$W = 2\pi F \quad X(w) = X(w+2\pi) \quad \Delta_t : \text{sampling interval}$$

ex: vocal signal  
20Hz ~ 20000Hz

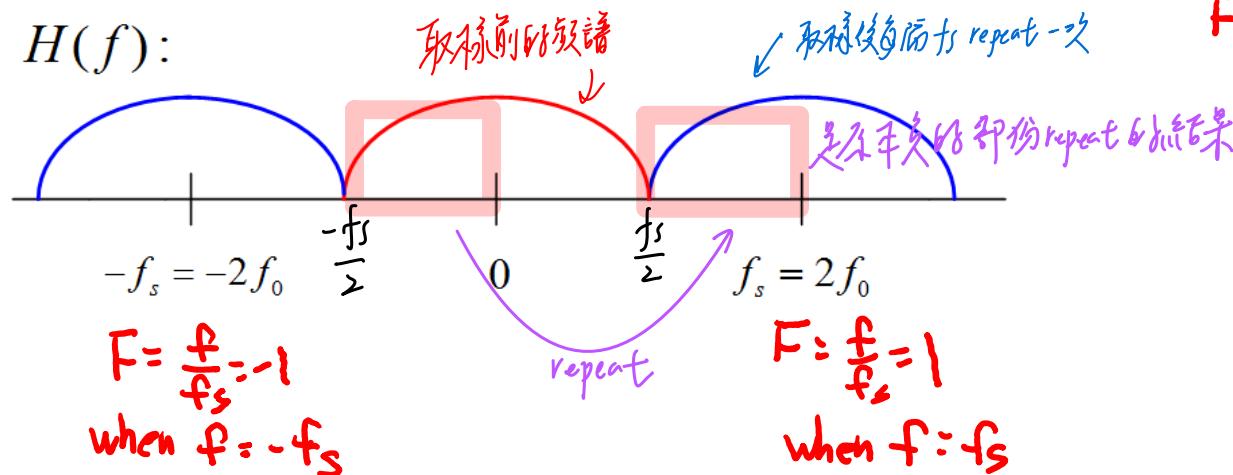
$$f_s > 40000\text{Hz}$$

### (2) folding frequency $f_0$

$$f_0 = \frac{f_s}{2}$$

若以 normalized frequency 來表示，  
folding frequency = 1/2

$$\begin{aligned} \text{If } f_s = 40000 \\ F=1 \rightarrow f=40000 \\ F=0.5 \rightarrow f=20000 \end{aligned}$$



For the discrete time Fourier transform

$$(1) G(f) = G(f + f_s) \longrightarrow \text{i.e., } \underline{G(F)} = \underline{G(F + 1)}. \quad \text{Normalized}$$

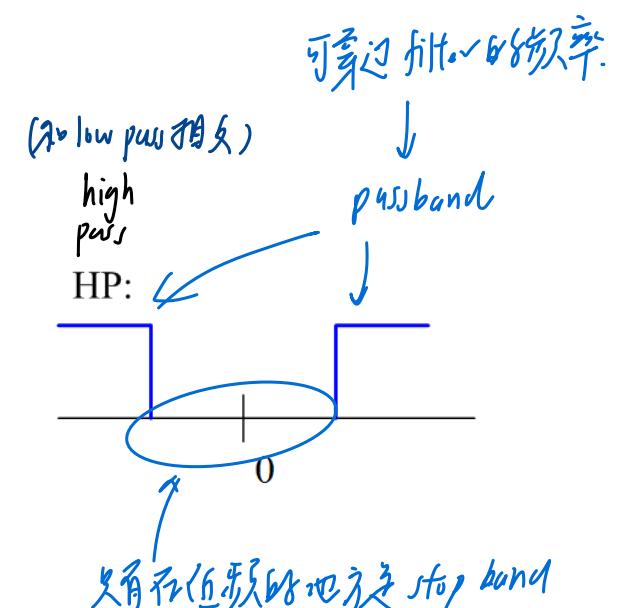
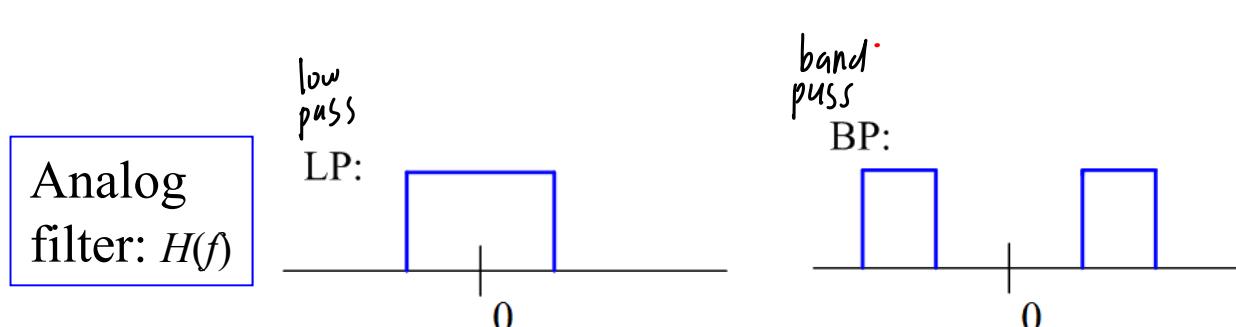
$$(2) \text{ If } g[n] \text{ is real} \longrightarrow G(F) = G^*(-F) \quad (* \text{ means conjugation})$$

只需知道  $G(F)$  for  $0 \leq F \leq \frac{1}{2}$  (即  $0 < f < f_0$ )

就可以知道全部的  $G(F)$

$$(3) \text{ If } g[n] = g[-n] \text{ (even)} \longrightarrow G(F) = G(-F),$$

$$g[n] = -g[-n] \text{ (odd)} \longrightarrow G(F) = -G(-F)$$



只步走離散的  $\rightarrow$  頻譜具有週期性，且週期  $f_s$

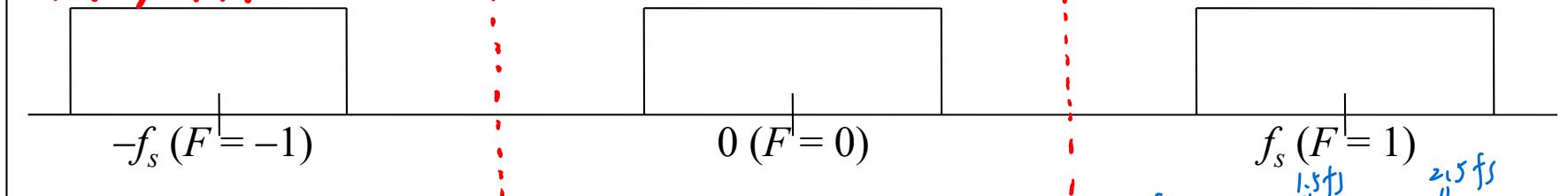
(或 normalized 稱  $\omega$ )

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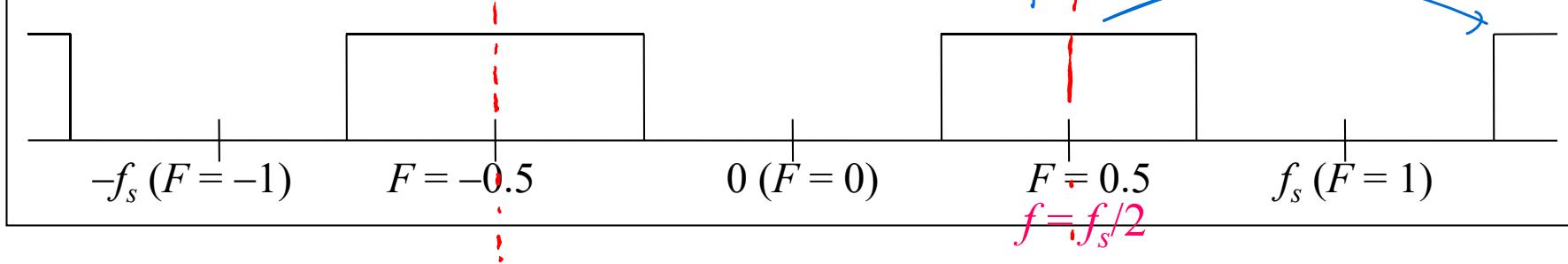
- Discrete time Fourier transform of the lowpass, highpass, and band pass filters

low pass filter ( pass band 在  $f_s$  的整數倍附近 )

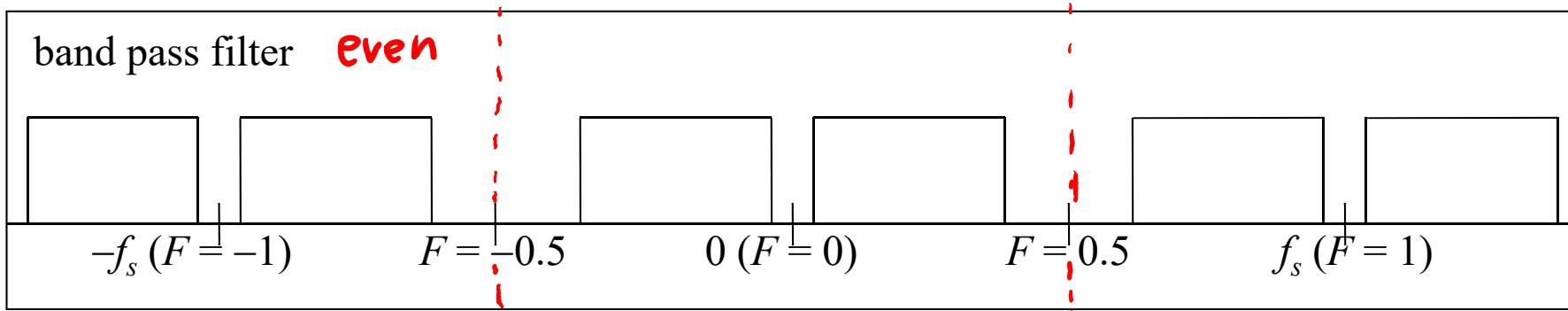
$$H(f) = H(-f) \text{ even}$$



high pass filter even



band pass filter even



$$\omega = -\pi$$

$$-\frac{f_s}{2} \text{ (normalized: } -0.5\text{)}$$

$$\omega = \pi$$

$$\frac{f_s}{2} \text{ (normalized: } 0.5\text{)}$$

## Review 3: Z Transform and Laplace Transform

- **Z-Transform**

suitable for **discrete** signals

$$G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n}$$

Compared with the discrete time Fourier transform:

$$G(f) = \sum_{n=-\infty}^{\infty} g[n]e^{-j2\pi f n \Delta_t}$$

↓  
 (discrete)

离散傅里叶  
 $\tilde{F}(f) = \sum_{n=-\infty}^{\infty} f[n] e^{-j2\pi f n t}$ ,  $|z|=1$   
 ↓

$$z = e^{j2\pi f \Delta_t} = e^{j2\pi F}$$

∵  $z^{-n} = (e^{j2\pi f \Delta t})^{-n}$   
 $= e^{-j2\pi f n \Delta t}$

## • Laplace Transform

suitable for **continuous** signals

↗ 為解決之齊條件

One-sided form  $G(s) = \int_0^\infty g(t)e^{-st} dt$

Two-sided form  $G(s) = \int_{-\infty}^\infty g(t)e^{-st} dt$

当  $s$  是純虛數時,  
Laplace  $\Rightarrow$  Fourier

Compared with the Fourier transform:

$$G(f) = \int_{-\infty}^\infty g(t)e^{-j2\pi f t} dt$$

$\hookrightarrow s = j2\pi f$   
Laplace transform 等於 Fourier transform

$s = j2\pi f$  (pure imaginary)

when  $s = \sigma + j2\pi f$

$$G(s) = \int_{-\infty}^\infty g(t)e^{-\sigma t} e^{-j2\pi f t} dt$$

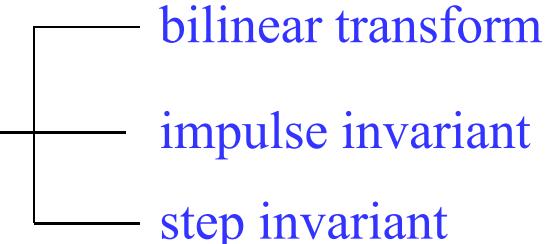
## Review 4: IIR Filter Design

Two types of digital filter:

(1) IIR filter (infinite impulse response filter)

(2) FIR ~~filter~~ (finite impulse response filer)

There are 3 popular methods to design the IIR filter



Advantage:

Disadvantage:

An IIR Filter May Not be Hard to Implement

$\rightarrow$  IIR

$$\text{Ex : } h[n] = (0.9)^n \quad , \text{ for } n \geq 0 , \quad h[n] = 0 , \text{ otherwise}$$

$$y[n] = x[n] * h[n]$$

Z transform (page 28)

$$\begin{aligned} Y(z) &= X(z) H(z) \\ &= \frac{X(z)}{1 - 0.9z^{-1}} \end{aligned}$$

$$(1 - 0.9z^{-1}) Y(z) = X(z)$$

inverse  $\Downarrow z$

$$Y[n] - 0.9 Y[n-1] = X[n]$$

$$Y[n] = 0.9 Y[n-1] + X[n]$$

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h[n] z^{-n} \\ &= \sum_{n=0}^{\infty} 0.9^n z^{-n} = \sum_{n=0}^{\infty} (0.9z^{-1})^n \\ &= \frac{1}{1 - 0.9z^{-1}} \quad \begin{matrix} \text{pole : } 0.9 \\ \text{stable} \end{matrix} \\ z^{-1} Y(z) &\xrightarrow{z^{-1}} Y[n-1] \end{aligned}$$

Oppenheim, Signals and Systems

## Method 1: Impulse Invariance

白話一點，就是直接做 sampling

analog filter  $h_a(t)$

digital filter  $h[n]$

*sampling directly* 
$$h[n] = h_a(n\Delta_t)$$

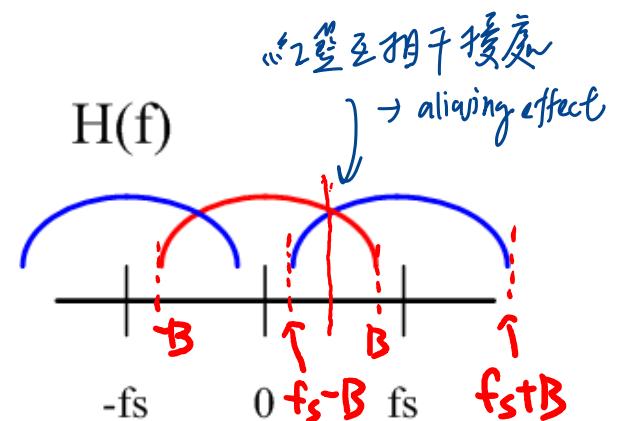
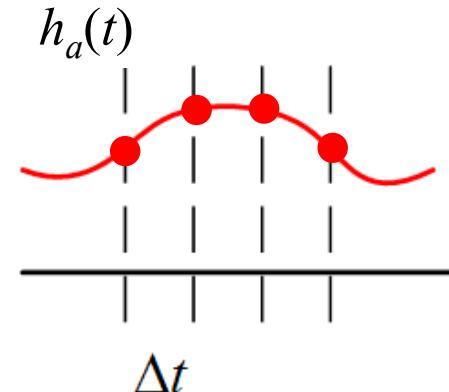
Advantage : Simple

Disadvantage : (1) infinite

(2) aliasing effect

取樣的頻率不夠高  
→干擾。

→對提高  $f_s$  → 錯覺量增加。



If  $f_s - B < B$        $f_s < 2B$

$f_s < 2B$ , aliasing effect occurs

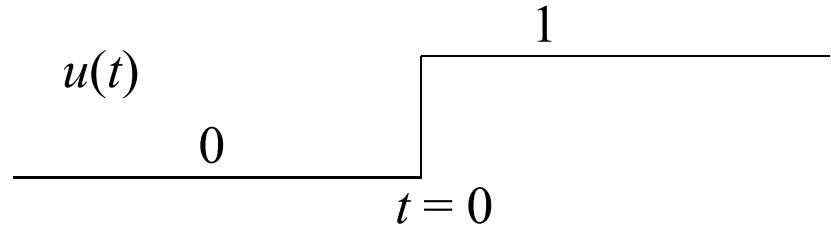
## Method 2: Step Invariance

→ 對 step function 的 response 作 sampling

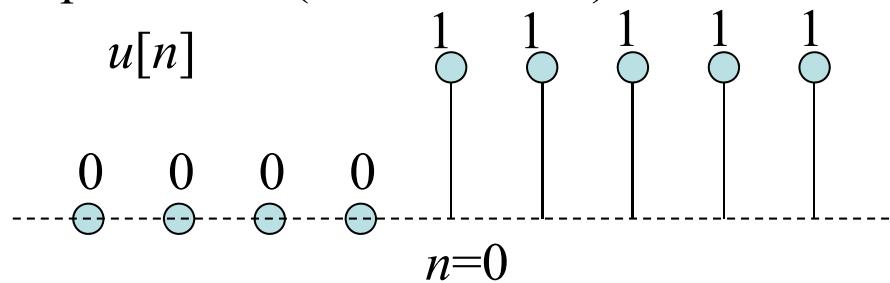
analog filter  $h_a(t)$

digital filter  $h[n]$

step function (continuous form)



step function (discrete form)



Laplace transform of  $u(t)$ :

$$\frac{1}{s}$$

Fourier transform of  $u(t)$ :

$$\frac{1}{j2\pi f}$$

Z transform of  $u[n]$ :

$$\frac{1}{1 - z^{-1}}$$

用 method 1  
 → impulse variance ≠ 高頻段有 aliasing effect  
 Step 1 Calculate the convolution of  $h_a(t)$  and  $u(t)$   
 → 高頻度 (t大)  
 → 分母大  
 → 降下來高頻的  
 地方被拉下來  
 (見右圖)

在 time domain (B 依舊有  
 = Frequency domain 除 J2πf (FT property)  
 $\begin{cases} \tau \leq t \\ \tau > t \end{cases}$   
 $u(t-\tau)=1 \text{ for } t-\tau > 0$   
 $u(t-\tau)=0 \text{ for } \tau > t$

$$h_{a,u}(t) = h_a(t) * u(t) = \int_{-\infty}^{\infty} h_a(\tau) u(t-\tau) d\tau = \int_{-\infty}^t h_a(\tau) d\tau$$

$H_{a,u}(f) = \frac{H_a(f)}{j2\pi f}$  (其實就是對  $h_a(t)$  做積分)  
 ↓  
 依樣用 → 除  $j2\pi f$   
 模分 乘

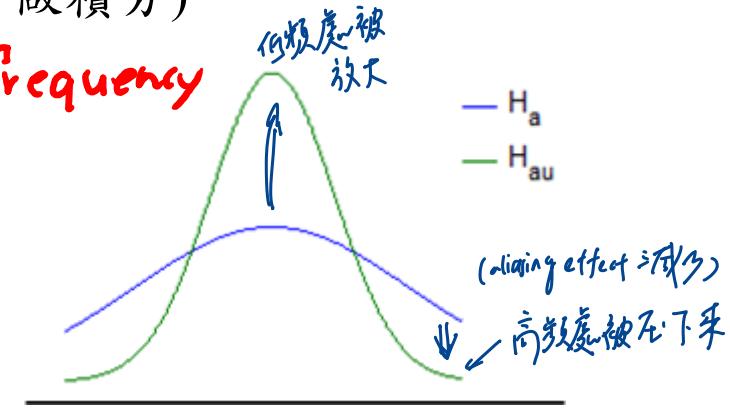
Decrease the high-frequency part.

Step 2 Perform sampling for  $h_{a,u}(t)$

$$h_u[n] = h_{a,u}(n\Delta_t)$$

↓  
做高分來還原.

Step 3 Calculate  $h[n]$  from  $h[n] = h_u[n] - h_u[n-1]$



Note: Since  $h_u[n] = h[n] * u[n]$        $H_u(z) = \frac{1}{1-z^{-1}} H(z)$

$$H(z) = (1 - z^{-1}) H_u(z)$$

$$\begin{aligned} &= H_a(f) + \\ &H_a(f - f_s) \\ &= H_a(f) + \\ &H_a^*(f_s - f) \end{aligned}$$

so       $h[n] = h_u[n] - h_u[n-1]$

Advantage of the step invariance method:

\* 主要 Advantage:

Disadvantage of the step invariance method:

較為間接，設計上稍微複雜

避免混叠  
aliasing effect  
↓

### Method 3: Bilinear Transform

Suppose that we have known an analog filter  $h_a(t)$  whose frequency response is  $H_a(f)$ .

To design the digital filter  $h[n]$  with the frequency response  $H(f)$ ,

$$H(f_{new}) = H_a(f_{old}) \quad f_{old} \in (-\infty, \infty)$$

$$f_{new} \in (-f_s/2, f_s/2)$$

$$f_s = 1/\Delta_t \text{ (sampling frequency)}$$

- The relation between  $f_{new}$  and  $f_{old}$  is determined by the mapping function

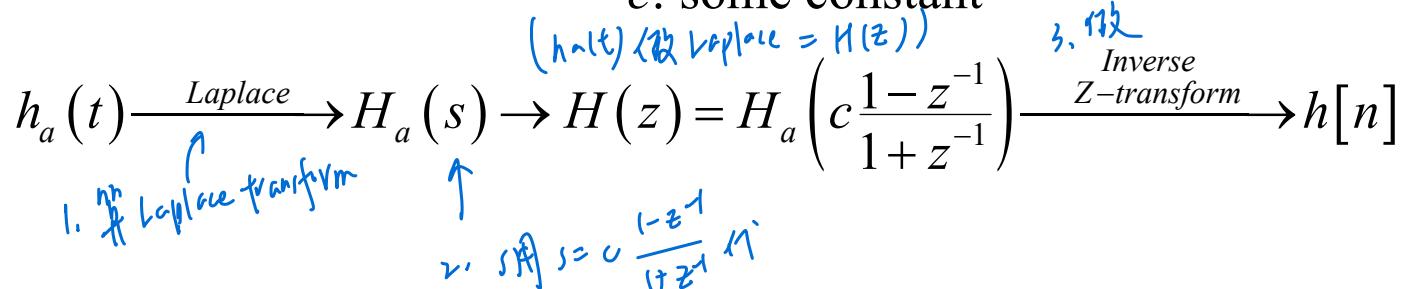
$$s = c \frac{1 - z^{-1}}{1 + z^{-1}}$$

$s$ : index of the Laplace transform

$z$ : index of the Z transform

$c$ : some constant

$H(z)$ : Z transform  
of  $h[n]$



$$s = c \frac{1 - z^{-1}}{1 + z^{-1}}$$

p. 79  
s =  $j2\pi f_{old}$

p. 28  
 $z = e^{j2\pi f_{new} \Delta_t}$

代入

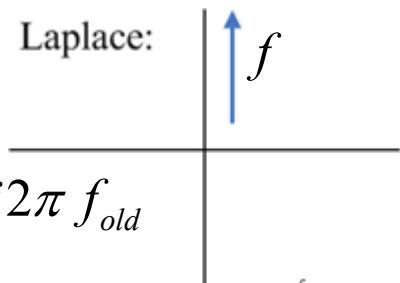
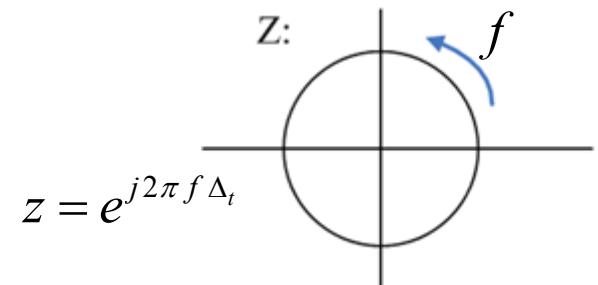
参考page 28、page29

$$\begin{aligned} j2\pi f_{old} &= c \frac{1 - e^{-j2\pi f_{new} \Delta_t}}{1 + e^{-j2\pi f_{new} \Delta_t}} = c \frac{e^{j\pi f_{new} \Delta_t} - e^{-j\pi f_{new} \Delta_t}}{e^{j\pi f_{new} \Delta_t} + e^{-j\pi f_{new} \Delta_t}} \\ &= c \frac{j \sin(\pi f_{new} \Delta_t)}{\cos(\pi f_{new} \Delta_t)} \end{aligned}$$

$$2\pi f_{old} = c \tan(\pi f_{new} \Delta_t)$$

$$f_{new} = \frac{1}{\pi \Delta_t} \operatorname{atan} \left( \frac{2\pi}{c} f_{old} \right) = \frac{f_s}{\pi} \operatorname{atan} \left( \frac{2\pi}{c} f_{old} \right)$$

$$f_s = \frac{1}{\Delta t}$$



$$s = j2\pi f_{old}$$

$$\begin{aligned} e^{j2\pi f_{new} \Delta_t} &= \cos(2\pi f_{new} \Delta_t) + j \sin(2\pi f_{new} \Delta_t) \\ e^{-j2\pi f_{new} \Delta_t} &= \cos(-2\pi f_{new} \Delta_t) + j \sin(-2\pi f_{new} \Delta_t) \end{aligned}$$

$$\cos(2\pi f_{new} \Delta_t)$$

- Suppose that the Laplace transform of the analog filter  $h_a(t)$  is  $H_{a,L}(s)$

The Z transform of the digital filter  $h[n]$  is  $H_z(z)$

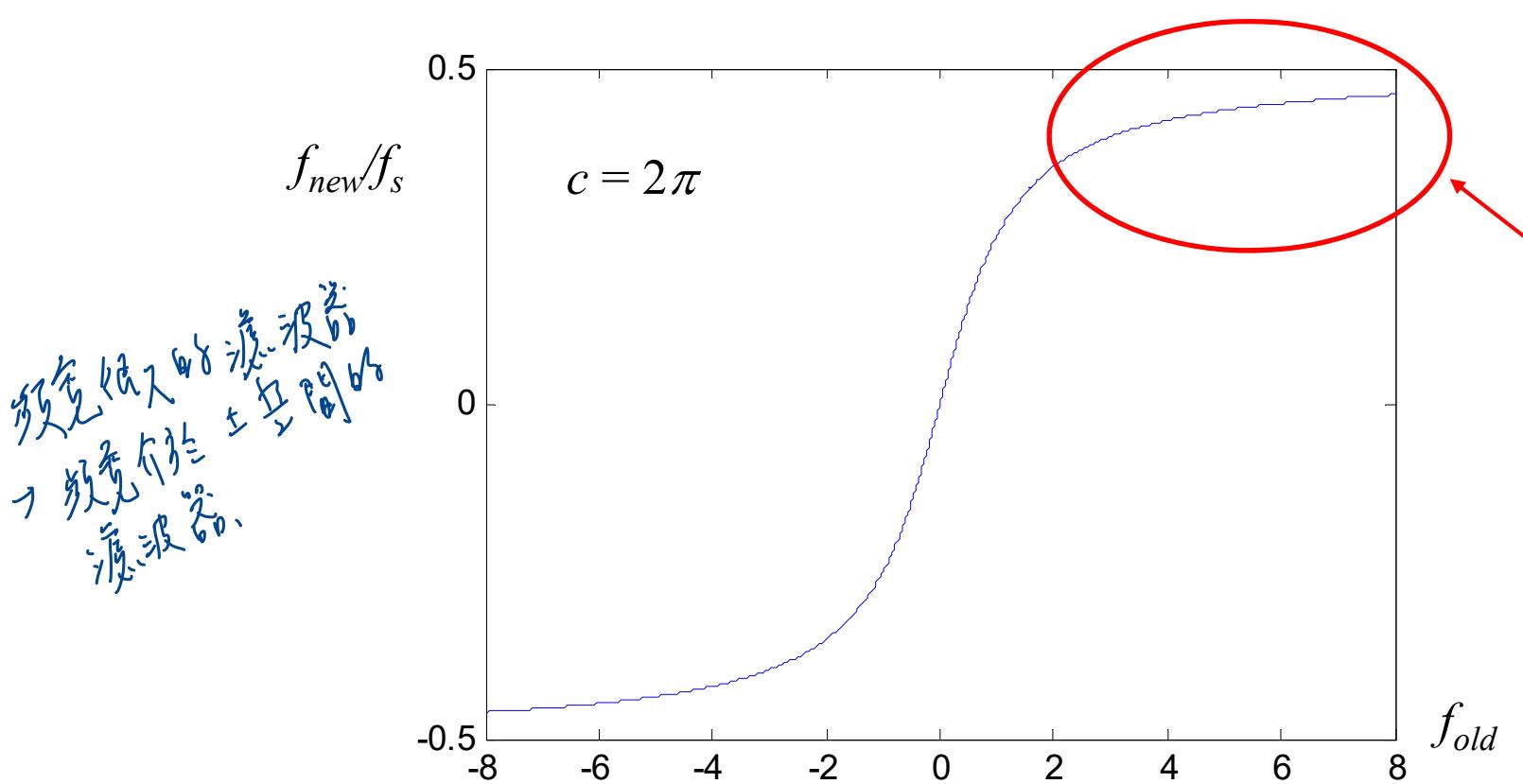
$$H_z(z) = H_{a,L} \left( c \frac{1 - z^{-1}}{1 + z^{-1}} \right)$$

$$\frac{f_s}{\pi} \left( -\frac{\pi}{2} \right) < f_{new} < \frac{f_s}{\pi} \left( \frac{\pi}{2} \right)$$

$$-\frac{f_s}{2} < f_{new} < \frac{f_s}{2}$$

$$f_{new} = \frac{f_s}{\pi} \operatorname{atan} \left( \frac{2\pi}{c} f_{old} \right)$$

$f_{old}$	$-\infty$	0	$\infty$	1
$f_{new}$	$-\frac{f_s}{2}$	0	$\frac{f_s}{2}$	$\frac{f_s}{\pi} \operatorname{atan} \left( \frac{2\pi}{c} \right)$

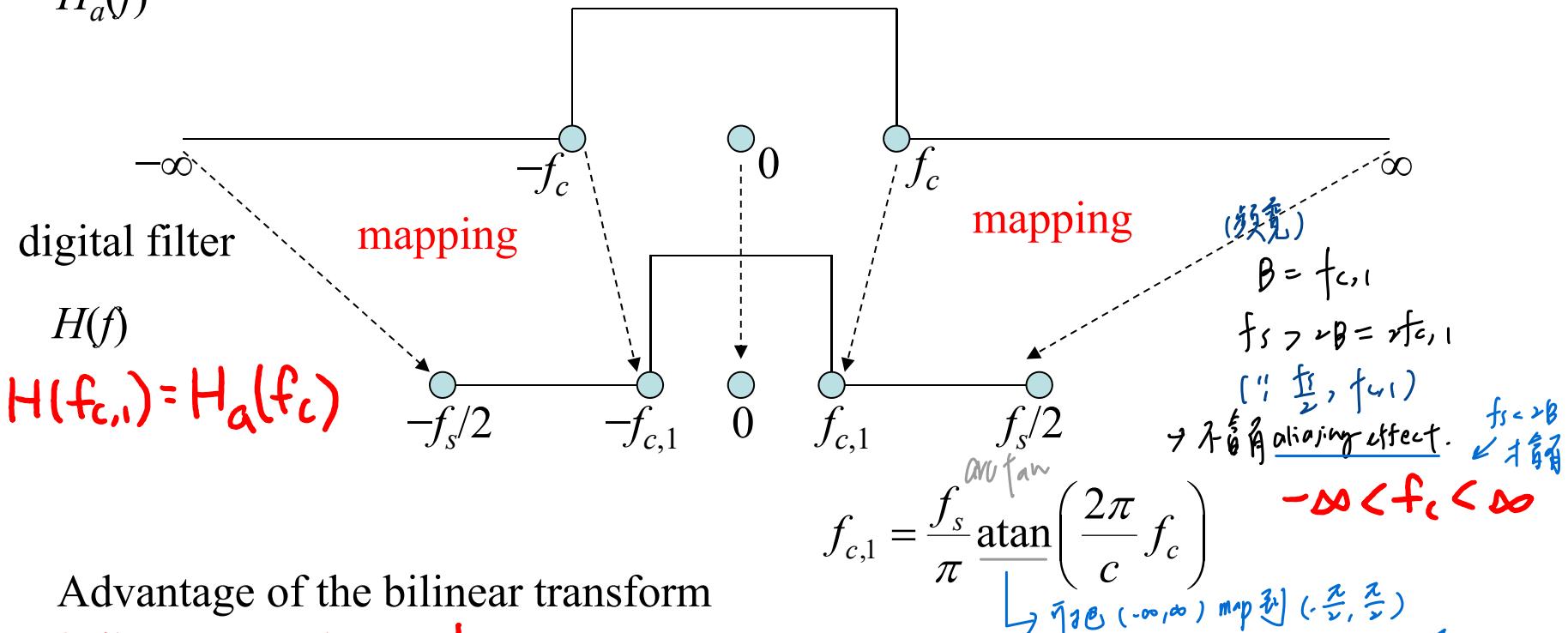


原本的範圍是  $(-\infty, \infty)$ , 如果我們用適當的 mapping, 把範圍變成  $(-\frac{f_s}{2}, \frac{f_s}{2})$   
 → 取樣就不會有 aliasing effect 產生。

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analog filter

$$H_a(f)$$



Advantage of the bilinear transform

fully avoid the aliasing effect

Disadvantage of the bilinear transform

1. 設計技術 (英文字母 approach)
  1. impulse invariance → 直接取樣
  2. step invariance → 積分後取樣

運算量較大

2. 高頻失真 distortion  
 $(\text{只取在 } \pm \frac{\pi}{2})$

$$-\frac{\pi}{2} < \text{atan}(x) < \frac{\pi}{2}$$

$$-\frac{f_s}{2} < f_{c,1} < \frac{f_s}{2}$$

$$\begin{aligned} \tan(\frac{\pi}{2}) &= \infty \\ \tan(-\frac{\pi}{2}) &= -\infty \end{aligned}$$

## 附錄一：學習 DSP 知識把握的要點

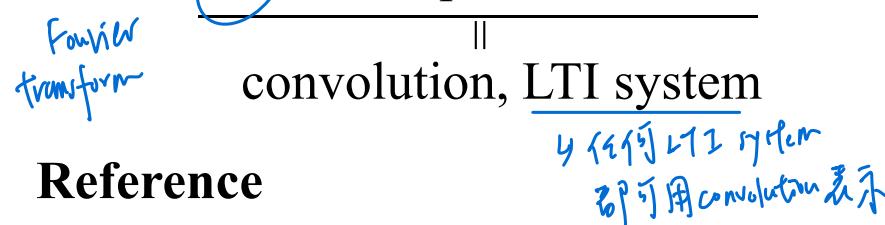
- (1) Concepts: 這個方法的核心概念、基本精神是什麼
- (2) Comparison: 這方法和其他方法之間，有什麼相同的地方？  
有什麼相異的地方
- (3) Advantages: 這方法的優點是什麼
  - (3-1) Why? 造成這些優點的原因是什麼
- (4) Applications: 這個方法要用來處理什麼問題，有什麼應用
- (5) Disadvantages: 這方法的缺點是什麼
  - (5-1) Why? 造成這些缺點的原因是什麼
- (6) Innovations: 這方法有什麼可以改進的地方  
或是可以推廣到什麼地方

## 2. Digital Filter Design (A)

任何可以用來去除 noise 作用的 operation，皆被稱為 filter

↳ 廣義：可表示 convolution 的 operation

甚至有部分的 operation，雖然主要功用不是用來去除 noise，但是可以用 FT + multiplication + IFT 來表示，也被稱作是 filter

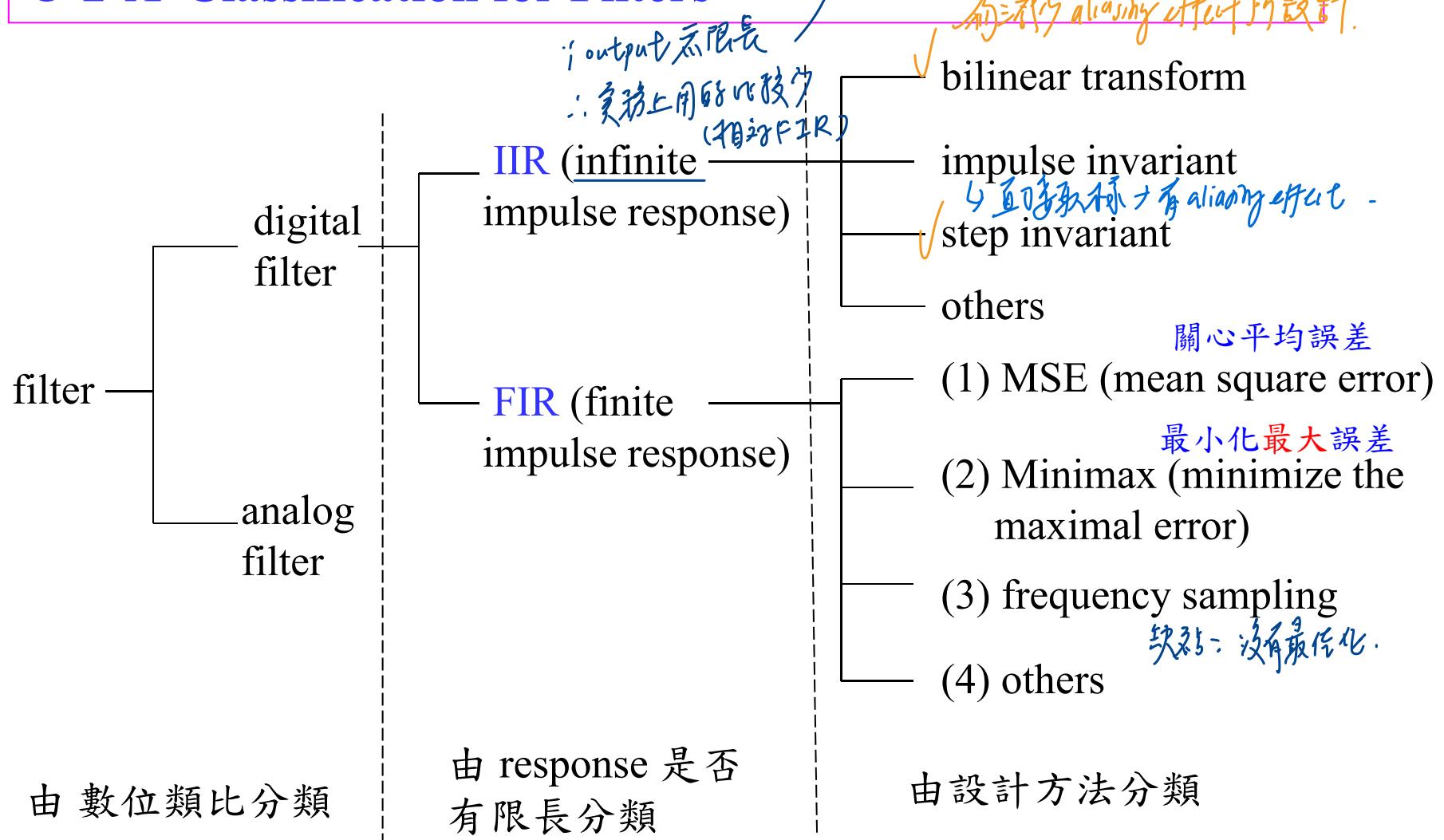


Reference

→ : 甚至可說所有的 LTI system  $\Rightarrow$  不是只有做雜訊去除  
都是一種 filter  
(filter)  
也可做 edge detection.  
移動均等.

- [1] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*, London: Prentice-Hall, 3<sup>rd</sup> ed., 2010.
- [2] D. G. Manolakis and V. K. Ingle, *Applied Digital Signal Processing*, Cambridge University Press, Cambridge, UK. 2011.
- [3] B. A. Shenoi, *Introduction to Digital Signal Processing and Filter Design*, Wiley-Interscience, N. J., 2006.
- [4] A. A. Khan, *Digital Signal Processing Fundamentals*, Da Vinci Engineering Press, Massachusetts, 2005.
- [5] S. Winder, *Analog and Digital Filter Design*, 2<sup>nd</sup> Ed., Amsterdam, 2002.

## ◎ 2-A Classification for Filters



even:  $h[n] = h[-n]$   
odd:  $h[-n] = -h[n]$

[除] all pass 外，都是在濾除雜訊。

(delay)

Classification for filters (依型態分)

pass-stop filter

→ 1 0 even  
impulse response

$h[n]$ : the pattern  
to be found  
 $h^*[n-h]$

matched filter

↳ 物件辨識  
(ML起碼以前常用)

Wiener filter

↳ noise removal

equalizer filter  
multipath

others

ex: 1) 以  $j\omega$  分

乘  $(j\omega)^3$

做 FT

↓  
乘  $j\omega$  除  $j\omega$

做 inverse FT

high pass filter  
ex: 心電圖

low pass filter

ex: edge detection  
band pass filter

band stop filter

all pass filter  
delay  $\times[n-T]$

Notch filter

= band stop filter  
with a narrow stop band

是一種 band stop filter (但 stop band

$\rightarrow -fs/2$

$F = -\frac{1}{2}$

$\rightarrow \frac{1}{2}$

$F = \frac{1}{2}$

$\rightarrow \text{filter range}$

完整 page 27

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|H(f)|



0

$F = \frac{1}{2}$

$F = -\frac{1}{2}$

$\rightarrow \text{filter range}$

$|e^{-j2\pi f}| = 1$   
delay  $\rightarrow$  延遲但保留  
是延遲

"even symmetric"  
(之後  $\rightarrow$  repeat)

← 異頻的範圍，

濾除特定頻率的聲音  
以: fm Hz 的利少聲

$H(f) = \frac{1}{1 + j\frac{f}{f_0}}$   
 $f < 0$   $f > 0$

$\rightarrow$  做 FT, 一边上  $-j$ , 一边上  $+j$

又或是 pass-stop filter  
 $H(f) = H(-f)$

$\rightarrow$  prediction  
ui: 預測明天的氣溫, 股價 ...

$$\begin{aligned} \xrightarrow{\text{Fourier Transform}} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned} \quad \downarrow \text{inverse} \quad \Rightarrow X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

↳ 未 j2πf 印象分. 44

## ◎ 2-B FIR Filter Design

FIR filter: impulse response is nonzero at finite number of points

impulse response

$$h[n] = 0 \text{ for } n < 0 \text{ and } n \geq N$$

在線上

( $h[n]$  has  $N$  points,  $N$  is a finite number)

通常假設  $\curvearrowleft$   
 $h[n]$  is causal

$$(h[n]=0 \text{ for } n<0)$$

$$y[n] = \sum_m x[n-m] h[m]$$

ex:  $y = 20$  被打一下  
 $y = 19$  欲有反應  
 $\rightarrow$  不太可能.



when  $m < 0, n-m > n$   
future info

- FIR is more popular because its impulse response is finite.

金山区海事处

if  $\mathcal{M}$  filter  $\rightarrow$  even symmetric

(假設一)

If  $h[n] = h[-n]$ , then  $H(f) = H(-f)$

time domain  
even symmetric

45

Specially, when  $h[n]$  is even symmetric  $h[n] = h[N-1-n]$

and  $N$  is an odd number

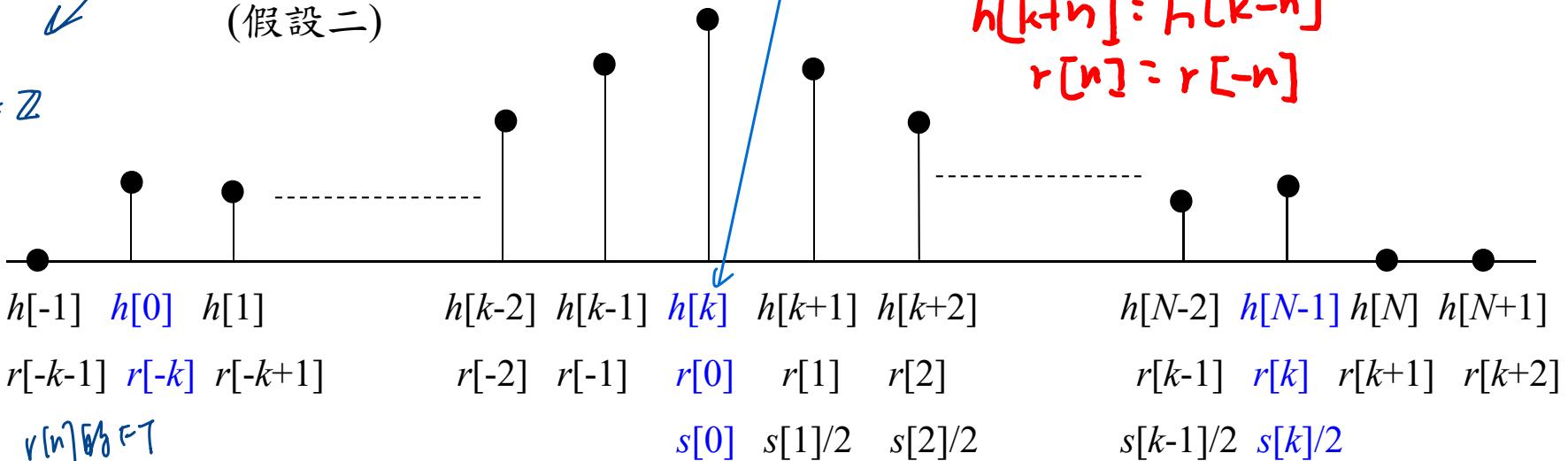
之稱 中心平移 ( causal )

$$h[k+n] = h[k-n]$$

$$r[n] = r[-n]$$

不二譜

$$k = \frac{N-1}{x} \in \mathbb{Z}$$



$$R(F) = e^{j2\pi F k} H(F), \quad R(F) \text{ is real}$$

$$(a) \quad r[n] = h[n + k], \quad \text{where } k = (N-1)/2.$$

$$(b) s[0] = r[0], \quad s[n] = 2r[n] \text{ for } 0 < n \leq k.$$

② 底層有N-1個條件數字

→ 可利用擇設成  $r_{13}$  來求  $k_1$  的係數。

Impulse Response of the FIR Filter:

$$h[n] \quad (h[n] \neq 0 \text{ for } 0 \leq n \leq N-1)$$

$$r[n] = h[n+k], \quad k = (N-1)/2 \quad (r[n] \neq 0 \text{ for } -k \leq n \leq k, \text{ see page 45})$$

上圖可視為  $r[n]$ .

Suppose that the filter is **even**,  $r[n] = r[-n]$ .

用 DTF1  
之 FIR 機構  
discrete, aperiodic

Set  $s[0] = r[0]$ ,  $s[n] = 2r[n]$  for  $n \neq 0$ .

Then, the discrete-time Fourier transform of the filter is

$$H(F) = \sum_{n=-\infty}^{\infty} h[n] e^{-j2\pi F n} \quad (F = f\Delta_t \text{ is the } \mathbf{normalized frequency})$$

See page 25

$$H(F) = e^{-j2\pi F k} R(F)$$

$$R(F) = \sum_{n=-k}^k r[n] e^{-j2\pi F n}$$

$$= \sum_{n=-k}^{-1} r[n] e^{-j2\pi F n} + r[0] + \sum_{n=1}^k r[n] e^{-j2\pi F n}$$

$$= \sum_{n=1}^k r[-n] e^{j2\pi F n} + r[0] + \sum_{n=1}^k r[n] e^{-j2\pi F n}$$

$$= \sum_{n=1}^k r[n] e^{j2\pi F n} + r[0]$$

**k+1 unknowns**  $s[0], s[1], \dots, s[k]$

$$\cancel{R(F) = \sum_{n=0}^k s[n] \cos(2\pi n F)}$$

$\because s[n] = r[n]$ ,  $s[0] = r[0]$

誤差越小 → 越有可能得到正確濾波器。

## ◎ 2-C Least MSE Form and Minimax Form FIR Filters

偏微分 + 一  
→ 可用矩阵運算  
做 optimization 技術

(1) least MSE (mean square error) form

↓

(關心 平均 誤差)

$$\text{L}_2\text{-norm problem} \rightarrow \text{MSE} = f_s^{-1} \int_{-f_s/2}^{f_s/2} |H(f) - H_d(f)|^2 df, \quad f_s: \text{sampling frequency}$$

$H(f)$ : the spectrum of the filter we obtain

$H_d(f)$ : the spectrum of the desired filter

desired 正確濾波器。

$$\text{MSE} = \int_{-V_2}^{V_2} |R(F) - H_d(F)|^2 dF$$

$H(F)$  用  $R(F)$  取代  
(為了解方便)

(2) mini-max (minimize the maximal error) form

(關心 最大 誤差)

$$\text{L}_\infty\text{-norm problem} \rightarrow \text{maximal error: } \max_f |H(f) - H_d(f)|$$

The transition band is always ignored

$$\max_F |R(F) - H_d(F)|$$

設計出來的 filter  
→ symmetric

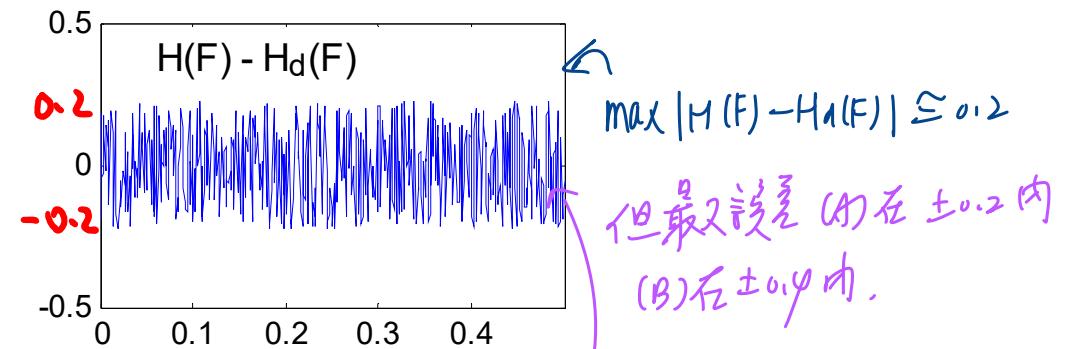
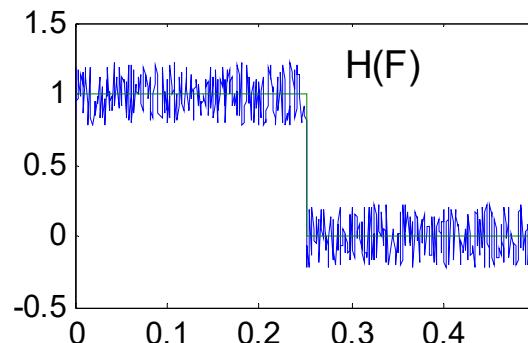
Example:

desired output  $H_d(F)$

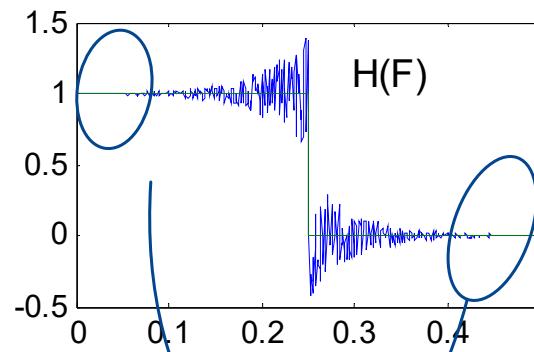
① 采用兩種方法

未設計 filter :

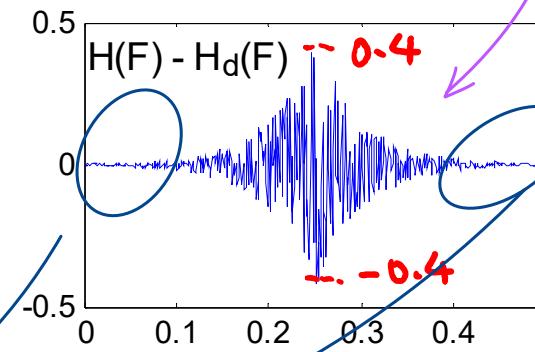
- (A) larger MSE, but smaller maximal error



- (B) smaller MSE, but larger maximal error



左右兩邊的  
誤差都趨近於 0



## ● 2-D Review: FIR Filter Design in the MSE Sense

$$R(F) = \sum_{n=0}^k s[n] \cos(2\pi n F)$$

$$\begin{aligned}
 MSE &= f_s^{-1} \int_{-f_s/2}^{f_s/2} |R(f) - H_d(f)|^2 df = \underbrace{\int_{-1/2}^{1/2} |R(F) - H_d(F)|^2 dF}_{F=f/f_s} \\
 &= \int_{-1/2}^{1/2} \left| \sum_{n=0}^k s[n] \cos(2\pi n F) - H_d(F) \right|^2 dF \quad |X(F)|^2 = X(F)X^*(F) \\
 &= \int_{-1/2}^{1/2} \left( \sum_{v=0}^k s[v] \cos(2\pi v F) - H_d(F) \right) \left( \sum_{\tau=0}^k s[\tau] \cos(2\pi \tau F) - H_d(F) \right) dF \\
 \xrightarrow{\text{optimize}} \xrightarrow{\text{partial derivative} \Rightarrow} \frac{\partial MSE}{\partial s[n]} &= \int_{-1/2}^{1/2} \cos(2\pi n F) \left( \sum_{\tau=0}^k s[\tau] \cos(2\pi \tau F) - H_d(F) \right) dF \\
 &\quad + \int_{-1/2}^{1/2} \left( \sum_{v=0}^k s[v] \cos(2\pi v F) - H_d(F) \right) \cos(2\pi n F) dF = 0
 \end{aligned}$$

$$\boxed{\frac{\partial MSE}{\partial s[n]} = 2 \sum_{\tau=0}^k s[\tau] \int_{-1/2}^{1/2} \cos(2\pi n F) \cos(2\pi \tau F) dF - 2 \int_{-1/2}^{1/2} H_d(F) \cos(2\pi n F) dF = 0}$$

Σ, ∫ 顺序待定

$$\frac{\partial MSE}{\partial s[n]} = 2 \sum_{\tau=0}^k s[\tau] \int_{-1/2}^{1/2} \cos(2\pi n F) \cos(2\pi \tau F) dF - 2 \int_{-1/2}^{1/2} H_d(F) \cos(2\pi n F) dF = 0$$

From the facts that

$$\cos \alpha \cos \beta = \frac{1}{2} \cos(\alpha + \beta) + \frac{1}{2} \cos(\alpha - \beta)$$

$$\int_{-\nu_2}^{\nu_2} \cos(2\pi(n+\tau)F) dF + \int_{-\nu_2}^{\nu_2} \cos(2\pi(n-\tau)F) dF$$

orthogonal

$$\int_{-1/2}^{1/2} \cos(2\pi n F) \cos(2\pi \tau F) dF = 0 \quad \text{when } n \neq \tau,$$

$$\int_{-1/2}^{1/2} \cos(2\pi n F) \cos(2\pi \tau F) dF = 1/2 \quad \text{when } n = \tau, \quad n \neq 0, \quad \left. \frac{-1}{2\pi(n-\tau)} \sin(2\pi(n-\tau)F) \right|_{-1/2}^{1/2}$$

$$\int_{-1/2}^{1/2} \cos(2\pi n F) \cos(2\pi \tau F) dF = 1 \quad \text{when } n = \tau, \quad n = 0. \quad = 0$$

since

$$\sin(\pi k) = 0$$

if  $k$  is an

when  $n = \tau = 0$  integer

$$\frac{1}{2} \int_{-1/2}^{1/2} 1 dF + \frac{1}{2} \int_{-1/2}^{1/2} 1 dF = 1$$

Therefore,

$$\frac{\partial MSE}{\partial s[0]} = 2s[0] - 2 \int_{-1/2}^{1/2} H_d(F) dF = 0$$

$$\frac{\partial MSE}{\partial s[n]} = s[n] - 2 \int_{-1/2}^{1/2} \cos(2\pi n F) H_d(F) dF = 0 \quad \text{for } n \neq 0.$$

$$n \neq 0 \quad \int_{-1/2}^{1/2} \cos(2\pi n F) \cos(2\pi \tau F) dF = \frac{1}{2}$$

$$\rightarrow 2 \sum_{\tau=0}^k s[\tau] \int_{-1/2}^{1/2} \cos(2\pi n F) \cos(2\pi \tau F) dF = 2 \sum_{\tau=0}^k s[\tau] \cdot \frac{1}{2} = \sum_{\tau=0}^k s[\tau]$$

Minimize MSE  $\rightarrow$  Make  $\frac{\partial MSE}{\partial s[n]} = 0$  for all  $n$ 's



$$\therefore \boxed{s[0] = \int_{-1/2}^{1/2} H_d(F) dF}, \quad \boxed{s[n] = 2 \int_{-1/2}^{1/2} \cos(2\pi n F) H_d(F) dF}.$$

$s[n]$  は設計の目的で  $h[n]$  に  $\rightarrow$  ある

Finally, set  $h[k] = s[0]$ ,

$$h[k+n] = s[n]/2, \quad h[k-n] = s[n]/2 \quad \text{for } n = 1, 2, 3, \dots, k,$$

$$h[n] = 0 \text{ for } n < 0 \text{ and } n \geq N.$$

Then,  $h[n]$  is the impulse response of the designed filter.

## ⑤ 2-E FIR Filter Design in the Mini-Max Sense

It is also called “Remez-exchange algorithm”

or “Parks-McClellan algorithm”

### References

- [1] T. W. Parks and J. H. McClellan, “Chebychev approximation for nonrecursive digital filter with linear phase”, *IEEE Trans. Circuit Theory*, vol. 19, no. 2, pp. 189-194, March 1972.
- [2] J. H. McClellan, T. W. Parks, and L. R. Rabiner “A computer program for designing optimum FIR linear phase digital filter”, *IEEE Trans. Audio-Electroacoustics*, vol. 21, no. 6, Dec. 1973.
- [3] F. Mintz and B. Liu, “Practical design rules for optimum FIR bandpass digital filter”, *IEEE Trans. ASSP*, vol. 27, no. 2, Apr. 1979.
- [4] E. Y. Remez, “General computational methods of Chebyshev approximation: The problems with linear real parameters,” AEC-TR-4491. ERDA Div. Phys. Res., 1962.

Suppose that:

- ① Filter length =  $N$ ,  $N$  is odd,  $N = 2k+1$ .
- ② Frequency response of the **desired filter**:  $H_d(F)$  is an even function  
( $F$  is the normalized frequency)
- ③ The **weighting function** is  $W(F)$

Two constraints

从: 希望 pass band 誤差小  $\rightarrow$   $W(F)$  在 pass band 为 1.

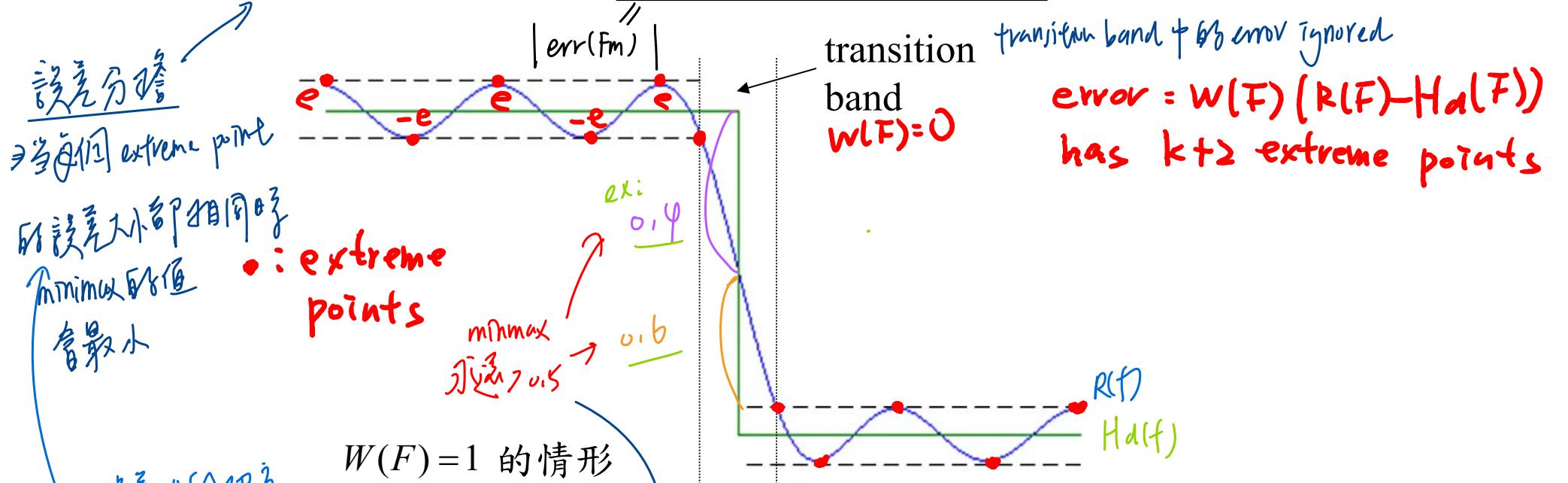
$\downarrow$   
和 signal 有界

(误差不离被调 signal, 可有些 noise 没滤除干净)

討論中

用 Mini-Max 方法所設計出的 filters，一定會滿足以下二個條件

- (1) 有  $k+2$  個以上的 extreme points Error 的 local maximal  
(see page 63) local minimum
- (2) 在 extreme points 上， $\frac{W(F_m)|R(F_m)-H_d(F_m)|}{\text{transition band}} = \text{定值}$



T. W. Parks and J. H. McClellan, "Chebychev approximation for nonrecursive digital filter with linear phase", *IEEE Trans. Circuit Theory*, vol. 19, no. 2, pp. 189-194, March 1972.

$\therefore W(F_m) |R(F_m)-H_d(F_m)|$  之定值

- Generalization for Mini-Max Sense by weight function

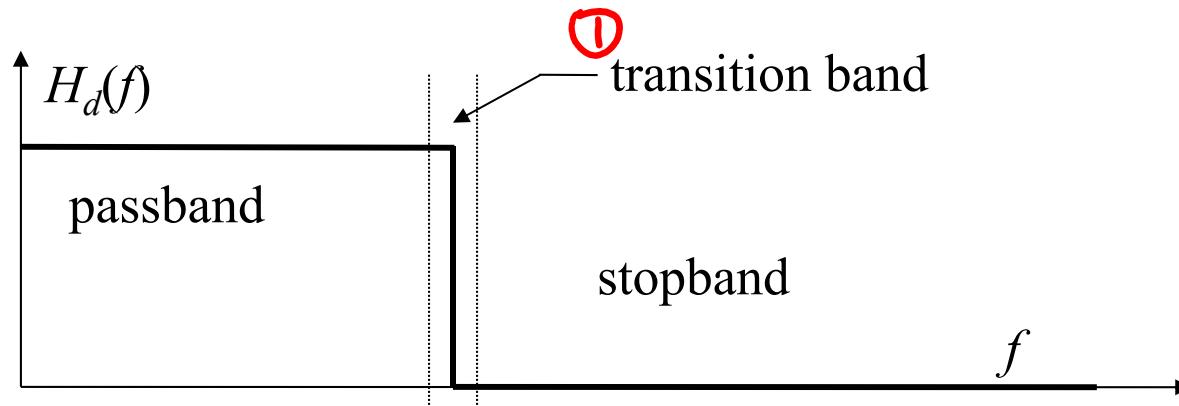
maximal error:  $\underset{f, f \notin \text{transition band}}{\text{Max}} |R(f) - H_d(f)|$

weighted maximal error:  $\underset{f, f \notin \text{transition band}}{\text{Max}} |W(f)[R(f) - H_d(f)]|$

where  $W(f)$  is the <sup>②</sup> weight function.

$$\underset{F, F \notin \text{transition band}}{\text{Max}} |W(F)[R(F) - H_d(F)]|$$

The weight function is designed according to which band is more important.



Q: How do we choose  $W(f)$  When SNR ↑ ?

**Example:** If we treat the passband the same important as the stopband.

$W(f) = 1$  in the passband,  $W(f) = 1$  in the stopband

Q1:  $W(f) = 1$  in the passband,  $W(f) < 1$  in the stopband 代表什麼？

Q2:  $W(f) < 1$  in the passband,  $W(f) = 1$  in the stopband 代表什麼？

Q3: 如何用來壓縮特定區域(如 transition band 附近)的 error？

Q4: Weighting function 的概念可否用在 MSE sense？

Minimax 法的步驟：

→ recursive

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-  $\hat{s}[n]$  的 extreme pursuit 算法  $\rightarrow$  用來猜  $F_m$

$$\hat{s} = A^{-1} H$$

$$s[n]$$

Step1

↓  
Step2

$$err(F)$$

↓  
Step3

$$\text{new } F_m$$

↓  
Step4

$$0 \leq E_1 - E_0 \leq \Delta \rightarrow \text{收斂}$$

↓  
Step5

$$s[n] \xrightarrow{\text{最後結果}} h[n] \quad (p.45)$$

↓  
Step6

why? '1 normalized frequency  $\Rightarrow$  週期 = 1 ( $X(F) = X(F+1)$ )

p. 28 由單點知道  $\omega_0 \sim \omega_5$  的設置  $\Rightarrow$  其它的也知道

## ◎ 2-F Mini-Max Design Process

但我們只需求  $\omega_0 \sim \omega_5$  ! PX 設定假設  $H(f) = H(-f)$   
設計出來的 filter  $\Rightarrow$  even (derived filter  $\rightarrow$  our function)  
p.53

(Step 1): Choose arbitrary  $k+2$  extreme frequencies in the range of

- 頂點是 extreme  
pm, 位置  $\rightarrow$  不能完全隨  
便設  $\rightarrow$  有限制. ① ~ ③

$0 \leq F \leq 0.5$ , (denoted by  $F_0, F_1, F_2, \dots, F_{k+1}$ )

Note: (1) Exclude the transition band.

②

(2) The extreme points cannot be all in the stop band.

③

Set  $E_1$  (error)  $\rightarrow \infty$

Extreme frequencies:

The locations where the error is maximal.

第一個點的設置  $\rightarrow$  [  $R(F_0) - H_d(F_0)$  ]  $W(F_0) = -e$   
filter 的 filter weight function

[  $R(F_2) - H_d(F_2)$  ]  $W(F_2) = -e$

$\vdots$

$\vdots$

[  $R(F_{k+1}) - H_d(F_{k+1})$  ]  $W(F_{k+1}) = (-1)^{k+2} e$

$F = [0:0.0001:0.5]$   $\rightarrow$  matlab

(5001 points)

離散化求  $|R(\omega)|$  maximal / minimal  
 $\rightarrow$  不用微微分。

(某些地方也沒辦法微分)

Why 每個點的誤差都是土  $e$  ?

A: (p.54) extreme points 上,  $|Err(\omega)|$

$[R(\omega) - H_d(\omega)] W(\omega) = e$

$= |W(\omega)(R(\omega) - H_d(\omega))|$  为誤差

$[R(\omega_3) - H_d(\omega_3)] W(\omega_3) = e$

(參考 page 54)

$$W(F_m) : [R(F_m) - H_d(F_m)] = W^{-1}(F_m)(-1)^{m+1}e$$

$$\Rightarrow H_d(F_m) = R(F_m) + W^{-1}(F_m)(-1)^m e = \sum_{n=0}^k s[n] \cos(2\pi n F_m) + W^{-1}(F_m)(-1)^m e$$

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**(Step 2):** From page 46,  $[R(F_m) - H_d(F_m)]W(F_m) = (-1)^{m+1}e$  (where  $m = 0, 1, 2, \dots, k+1$ ) can be written as

$$\sum_{n=0}^k s[n] \cos(2\pi F_m n) + (-1)^m W^{-1}(F_m) e = H_d(F_m)$$

where  $m = 0, 1, 2, \dots, k+1$ .  $(m=0, 1, \dots, k+1)$

Expressed by the matrix form:

	$s[0]$	$s[1]$	$\downarrow$	$s[2]$		$\cdots$	$s[k]$		$e$		
{	$m=0$	$1 \quad \cos(2\pi F_0)$	$\cos(4\pi F_0)$	$\cdots$	$\cos(2\pi k F_0)$		$1/W(F_0)$		$s[0]$		$H_d[F_0]$
	$m=1$	$1 \quad \cos(2\pi F_1)$	$\cos(4\pi F_1)$	$\cdots$	$\cos(2\pi k F_1)$		$-1/W(F_1)$		$s[1]$		$H_d[F_1]$
	$m=2$	$1 \quad \cos(2\pi F_2)$	$\cos(4\pi F_2)$	$\cdots$	$\cos(2\pi k F_2)$		$1/W(F_2)$		$s[2]$		$H_d[F_2]$
	$\vdots$	$\vdots$	$\ddots$		$\vdots$				$\vdots$		$\vdots$
	$m=k$	$1 \quad \cos(2\pi F_k)$	$\cos(4\pi F_k)$	$\cdots$	$\cos(2\pi k F_k)$		$(-1)^k / W(F_k)$		$s[k]$		$H_d[F_k]$
	$m=k+1$	$1 \quad \cos(2\pi F_{k+1})$	$\cos(4\pi F_{k+1})$	$\cdots$	$\cos(2\pi k F_{k+1})$		$(-1)^{k+1} / W(F_{k+1})$		$e$		$H_d[F_{k+1}]$

$(k+2) \times (k+2)$  matrix

$A$

可取最值 it's change while change  
↓ (4 extreme points) 由點到線  
stop band < pass band  
↓ p. 74 explain

Solve  $s[0], s[1], s[2], \dots, s[k]$  from the above matrix

(performing the matrix inversion).

解 S :

$$AS = H$$

$$S = A^{-1}H$$

假設的未知數 (unknown)

Square matrix

前面的 extreme points 及其後的 :: 計算  $\text{err}(F)$  來找更多的 extreme points  $F$  與  $P$ .

(Step 3): Compute  $\text{err}(F)$  for  $0 \leq F \leq 0.5$ , exclude the transition band.

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$$\text{err}(F) = [R(F) - H_d(F)]W(F) = \left\{ \sum_{n=0}^k s[n] \cos(2\pi n F) - H_d(F) \right\} W(F)$$

Set  $W(F) = 0$  at the transition band.  $\rightarrow$  為了方便

(Step 4): Find  $k+2$  local maximal (or minimal) points of  $\text{err}(F)$

①  $q(\tau)$  比左右的值都大  $\Rightarrow \tau$  : local maximal  
local maximal point: (if  $q(\tau) > q(\tau + \Delta_F)$  and  $q(\tau) > q(\tau - \Delta_F)$ ,

then  $\tau$  is a local maximal of  $q(x)$ .

local minimal point: if  $q(\tau) < q(\tau + \Delta_F)$  and  $q(\tau) < q(\tau - \Delta_F)$ ,

then  $\tau$  is a local minimal of  $q(x)$ .

Other rules: Page 63

Denote the local maximal (or minimal) points by  $P_0, P_1, \dots, P_k, P_{k+1}$

$F_m = P_m$  new extreme points

These  $k+2$  extreme points could include the boundary points of the transition band

## (Step 5):

Set  $E_0 = \text{Max}(|\text{err}(F)|)$ .

$$\begin{cases} E_0 : \text{現在的Max} |\text{err}(F)| \\ E_1 : \text{前一次iteration的Max} |\text{err}(F)| \end{cases}$$

(Case a) If  $E_1 - E_0 > \Delta$ , or  $E_1 - E_0 < 0$  (or the first iteration) →  
set  $F_n = P_n$  and  $E_1 = E_0$ , return to Step 2.

(Case b) If  $0 \leq E_1 - E_0 \leq \Delta$  → continue to Step 6.  
*代表收斂*

## (Step 6):

Set  $h[k] = s[0]$ ,

*optimization*



$h[k+n] = s[n]/2, h[k-n] = s[n]/2$  for  $n = 1, 2, 3, \dots, k$

(referred to page 45)

Then  $h[n]$  is the impulse response of the designed filter.

## ◎ 2-G Mini-Max FIR Filter 設計時需注意的地方

(1) Extreme points 不要選在 transition band

Initial guess的extreme points只要注意別取在transition band裡，即能保證 converge，不同的guess會影響converge的速度但不影響結果

(2)  $E_1$  (error of the previous iteration)  $< E_0$  (present error) 時，亦不為收斂

(3) Remember to update  $W(F_m)$  and  $H_d(F_m)$  according to the locations of  $F_m$ .

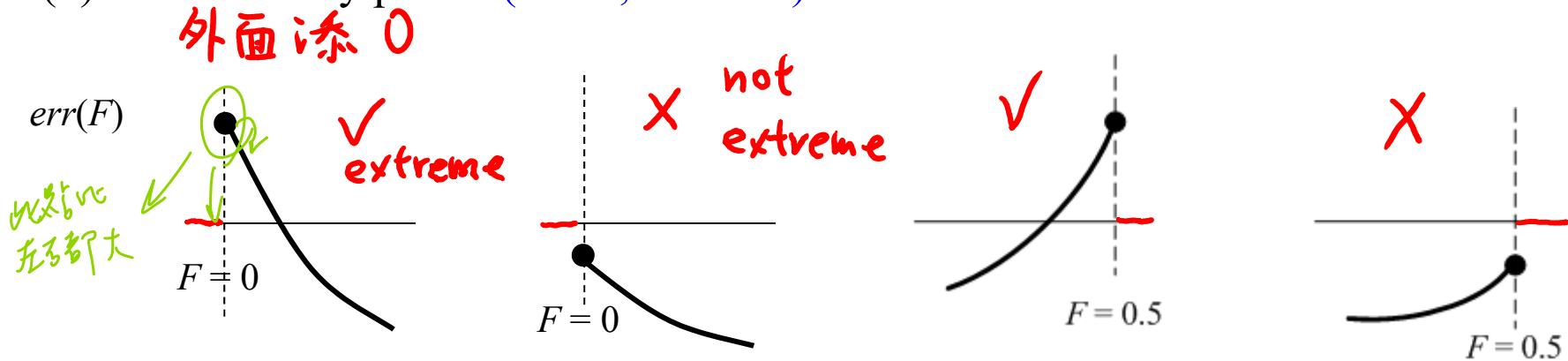
(4) Extreme points 判斷的規則：

(a) The local peaks or local dips that are not at boundaries must be extreme points.

Local peaks:  $err(F) > err(F + \Delta_F)$  and  $err(F) > err(F - \Delta_F)$

Local dips:  $err(F) < err(F + \Delta_F)$  and  $err(F) < err(F - \Delta_F)$

(b) For boundary points ( $F = 0, F = 0.5$ )



Add a zero to the outside and conclude whether the point is a local maximum or a local minimum.

(5) 有時，會找到多於  $k+2$  個 extreme points, 該如何選  $P_0, P_1, \dots, P_k, P_{k+1}$

*boundary: 0, 0.5*

(a) 優先選擇 不在 boundaries 的 extreme points

(b) 其次選擇 boundary extreme points 當中  $|\text{err}(F)|$  較大的，

直到湊足  $k+2$  個 extreme points 為止

(c) 找好 extreme points 之後，要記得重新依  $F$  值大小排序

$$F_m < F_{m+1} \quad \begin{array}{l} \text{「假設前一階的設差和下階不同} \\ \text{例如子中由 } l \rightarrow -l + l \rightarrow l \dots \end{array}$$

## ◎ 2-H Examples for Mini-Max FIR Filter Design

$\therefore \text{过渡带半宽 } k = \frac{N-1}{2} = 4 \Rightarrow \text{有 } 4+2=6 \text{ 个极点} \Rightarrow \text{形成 } 6 \times 6 \text{ 矩阵}$

- Example 1: Design a 9-length highpass filter in the mini-max sense

ideal filter:  $H_d(F) = 0$  for  $0 \leq F < 0.25$ ,

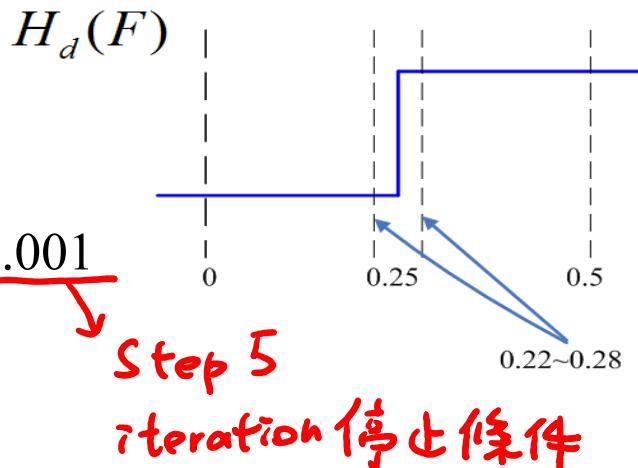
$H_d(F) = 1$  for  $0.25 < F \leq 0.5$ ,

transition band:  $0.22 < F < 0.28$

weighting function:  $W(F) = 0.25$  for  $0 \leq F \leq 0.22$ ,

$W(F) = 1$  for  $0.28 \leq F \leq 0.5$ ,

$$\Delta = 0.001$$



(Step 1) Since  $N = 9$ ,  $k = (N-1)/2 = 4$ ,  $k+2 = 6$ ,

→ Choose 6 extreme frequencies

角频率的选取: (e.g.,  $F_0 = 0, F_1 = 0.1, F_2 = 0.2, F_3 = 0.3, F_4 = 0.4, F_5 = 0.5$ ) → 不包括在  $0.22 \sim 0.28$  (transition band)

$$[R(F_n) - H_d(F_n)]W(F_n) = (-1)^{n+1}e, \quad n = 0, 1, 2, 3, 4, 5.$$

page 59

(stop band)  
 $w(F) = 0 \sqrt{3} \quad 0 \leq F \leq 0.22$

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**(Step 2)**  $h:0$   $n=1$   $\cos(2\pi F_m n)$   $n=2$   $n=3$   $n=4$   $\frac{(-1)^m}{w(F_m)}$   $H_d(F_m)$

$F_m=0$	$m=0$	1	1	1	1	1	$s[0]$	0
0.1	1	1	0.809	0.309	-0.309	-0.809	$s[1]$	0
0.2	2	1	0.309	-0.809	-0.809	0.309	$s[2]$	0
0.3	3	1	-0.309	-0.809	0.809	0.309	$s[3]$	1
0.4	4	1	-0.809	0.309	0.309	-0.809	$s[4]$	1
0.5	5	1	-1	1	-1	1	$e$	1

$\downarrow$  當  $F_m$  在 stopband 時， $w(F_m) = 0$

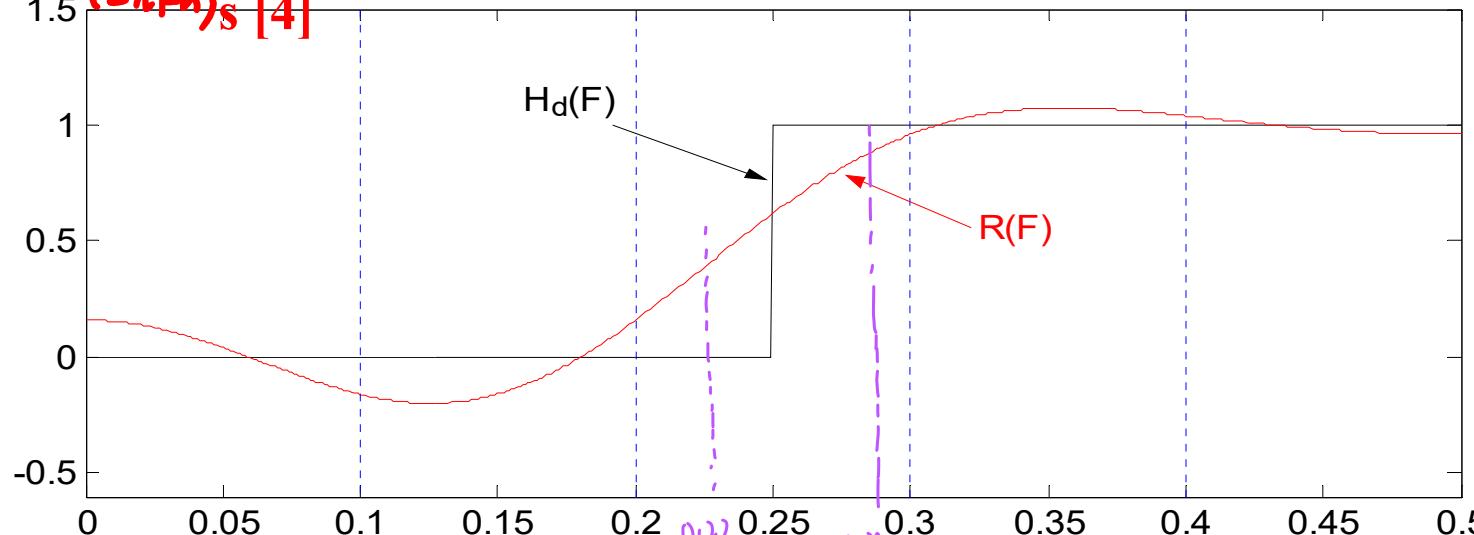
$\downarrow$  當  $F_m$  在 passband 時， $w(F_m) = 1$

$w(F) = 1$  for  $0 \leq F \leq 0.5$  (pass band)

$$\begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ s[3] \\ s[4] \\ e \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 4 \\ 1 & 0.809 & 0.309 & -0.309 & -0.809 & -4 \\ 1 & 0.309 & -0.809 & -0.809 & 0.309 & 4 \\ 1 & -0.309 & -0.809 & 0.809 & 0.309 & -1 \\ 1 & -0.809 & 0.309 & 0.309 & -0.809 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5120 \\ -0.6472 \\ -0.0297 \\ 0.2472 \\ 0.0777 \\ -0.040 \end{bmatrix}$$

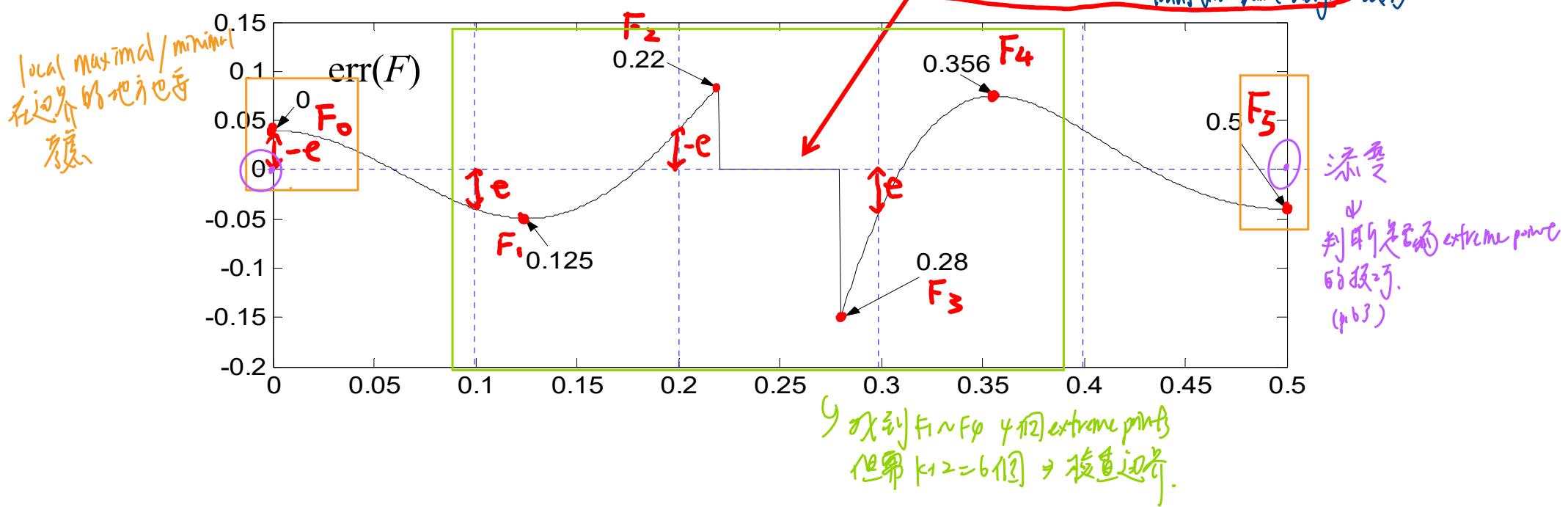
$$R(F) = 0.5120 - 0.6472\cos(2\pi F) - 0.0297\cos(4\pi F) + 0.2472\cos(6\pi F)$$

$\sum_{n=0}^{k+1} S[n] \cos(2\pi Fn) + 0.0777\cos(8\pi F)$



$$(\text{Step 3}) \text{ err}(F) = [R(F) - H_d(F)]W(F)$$

$W(F) = 0$  for  $0.22 < F < 0.28$   
transition band weight



(Step 4) Extreme points: *由 error function 算出的 extreme points*  
 $F_0 = 0, F_1 = 0.125, F_2 = 0.22, F_3 = 0.28, F_4 = 0.356, F_5 = 0.5$

*if  $\text{err} = 2 \times 10^{-3}$*   
*回到 step 2*  
*重新取些*  
*点 EPF.*

(Step 5)  $E_0 = \text{Max}[|\text{err}(F)|] = 0.1501$ , return to Step 2.

$$E_1 = \infty \quad E_1 - E_0 > \Delta$$

*距离第一端点，前一端的误差可设为  $\infty$*

(Step 2) Using  $F_0 = 0, F_1 = 0.125, F_2 = 0.22, F_3 = 0.28, F_4 = 0.356, F_5 = 0.5$   
 $\rightarrow s[0] = 0.5018, s[1] = -0.6341, s[2] = -0.0194, s[3] = 0.3355,$   
 $s[4] = 0.1385$

(Step 3)  $\text{err}(F) = [R(F) - H_d(F)]W(F)$ ,

(Step 4) extreme points : 0, 0.132, 0.22, 0.28, 0.336, 0.5

(Step 5)  $E_0 = \text{Max}[|\text{err}(F)|] = 0.0951$ , return to Step 2.

$$E_1 - E_0 = 0.1501 - 0.0951 = 0.055 > \Delta$$

### Third iteration

(Step 2), (Step 3), (Step 4), peaks : 0, 0.132, 0.22, 0.28, 0.334, 0.5

(Step 5)  $E_0 = \underline{0.0821}$ , return to Step 2.

### Fourth iteration

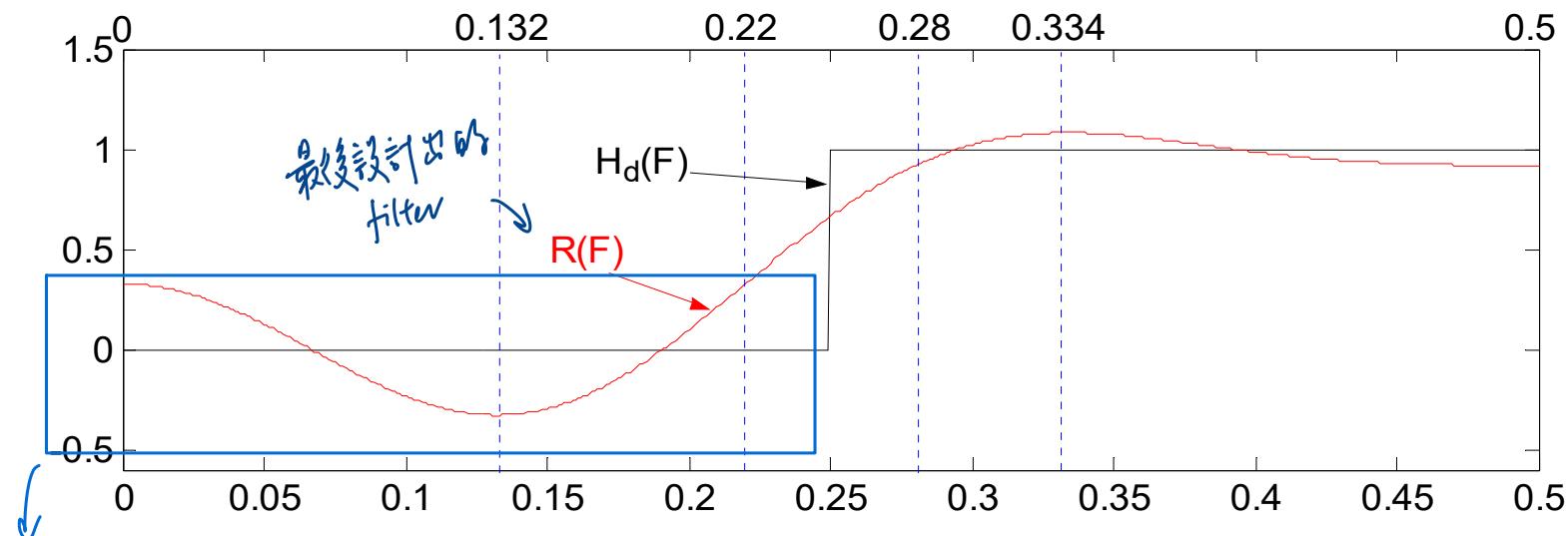
(Step 2), (Step 3), (Step 4), peaks : 0, 0.132, 0.22, 0.28, 0.334, 0.5

(Step 5)  $E_0 = \underline{0.0820}$ ,  $E_1 - E_0 = 0.0001 \leq \Delta$ , continues to Step 6.

(Step 6) From  $s[0] = 0.4990$ ,  $s[1] = -0.6267$ ,  $s[2] = -0.0203$ ,  $s[3] = 0.3316$ ,  
 $s[4] = 0.1442$

$$N=9, k=4$$

$$\begin{aligned} h[4] &= s[0] = 0.4990, & h[3] &= h[5] = s[1]/2 = -0.3134, \\ h[2] &= h[6] = s[2]/2 = -0.0101, & h[1] &= h[7] = s[3]/2 = 0.1658, \\ h[0] &= h[8] = s[4]/2 = 0.0721. \end{aligned}$$



看起來  $R(P)$  跟 desired  $H_d(F)$   
 差很大，但  $R(F)$  還是 optimal of  
 → stop band看起來還很大；說明了 stop band weight 設得較低 (0.15) ↘ 差4倍  
 ↓ pass band (1)

$$k = \frac{N-1}{2} = 3 \quad \text{1. extreme points 例題 5(1)}$$

- **Example 2:** Design a 7-length digital filter in the mini-max sense  
 $N=7$

ideal filter:  $H_d(F) = 1$  for  $0 \leq F < 0.24$ ,

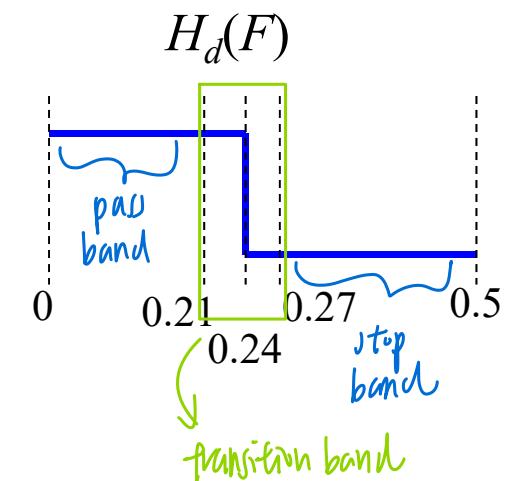
$H_d(F) = 0$  for  $0.24 < F \leq 0.5$ ,

transition band:  $0.21 < F < 0.27$

weighting function:  $W(F) = 1$  for  $0 \leq F \leq 0.21$ ,

$W(F) = 0.5$  for  $0.27 \leq F \leq 0.5$ ,

$\Delta = 0.001$



(Step 1) Since  $N = 7$ ,  $k = (N-1)/2 = 3$ ,  $k+2 = 5$ ,

→ Choose 5 extreme frequencies

(e.g.,  $F_0 = 0$ ,  $F_1 = 0.2$ ,  $F_2 = 0.3$ ,  $F_3 = 0.4$ ,  $F_4 = 0.5$ )

$\cos(\omega \pi n F_m)$

**(Step 2)**

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0.309 & -0.809 & -0.809 \\ 1 & -0.309 & -0.809 & 0.809 \\ 1 & -0.809 & 0.309 & 0.309 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

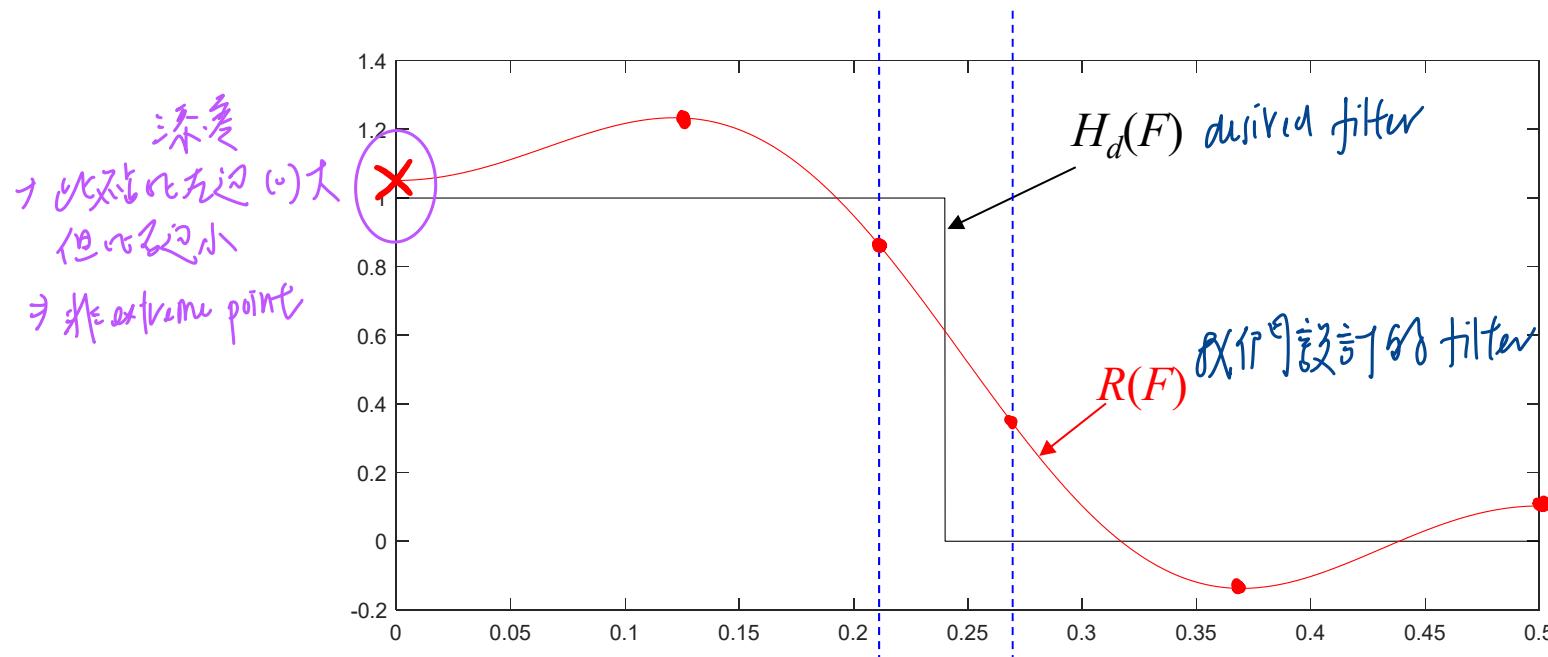
$\xrightarrow{\frac{(-1)^m}{w(F_m)}}$

$\begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ s[3] \\ e \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \\ 2 \end{bmatrix}$

passband

$F_0 = 0$   
 $F_1 = 0.2$   
 $F_2 = 0.3$   
 $F_3 = 0.4$   
 $F_4 = 0.3$

$\rightarrow s[0] = 0.5486, s[1] = 0.7215, s[2] = 0.0284, s[3] = -0.2472, e = -0.0514$



After Step 2,

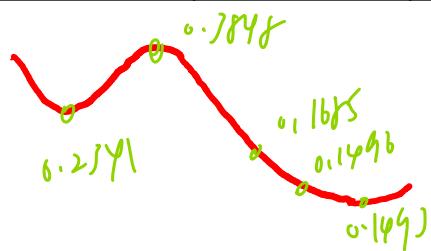
$$\begin{aligned} \text{(Step 3)} \quad \text{err}(F) = & [0.5486 + 0.7215 \cos(2\pi F) + 0.0284 \cos(4\pi F) \\ & - 0.2472 \cos(6\pi F) - H_d(F)] W(F) \end{aligned}$$

(Step 4) extreme points: 0.1217, 0.21, 0.27, 0.3698, 0.5.

(Step 5)  $E_0 = \text{Max}[|\text{err}(F)|] = 0.2341$ , return to Step 2.

第三次迭代，  
 $E_1 < E_0$   
 $> 341$

Iteration	1	2	3	4	5	6
$\text{Max}[ \text{err}(F) ]$	0.2341	0.3848	0.1685	0.1496	0.1493	0.1493

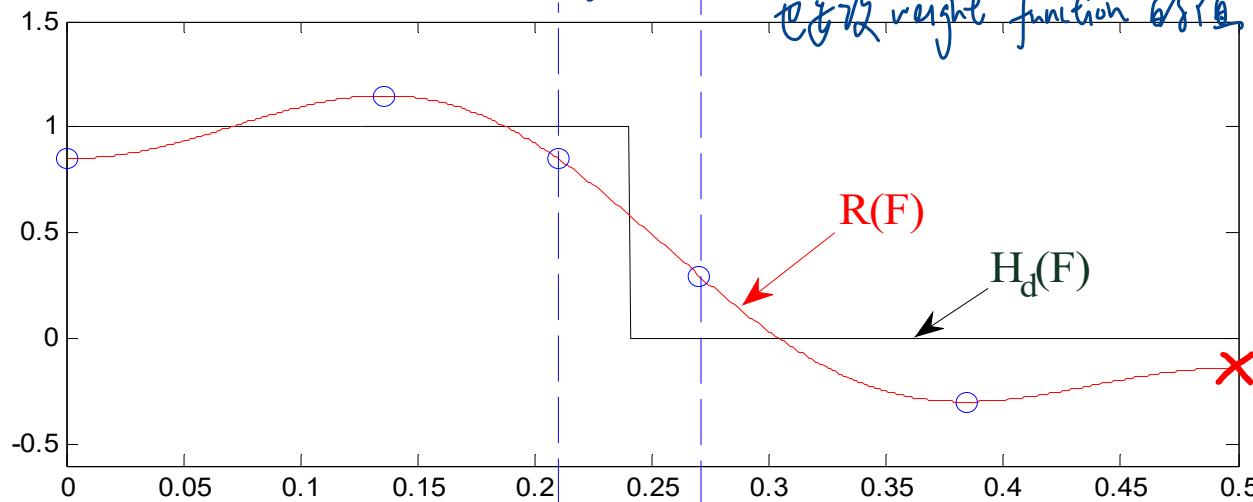


6

After 6 times of iteration

原本取 pass band : 3, stop band: > 112 extreme points  
 → 但最後變成 112, 113  
 → 是有 112 extreme points 過多, 當 pass band ≥ stop band  
 ∴ 此時須 = 1 改  $H_d(F)$  的值, (pass = 1, stop = 0)  
 也修改 weight function 的值 (p.59)

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$$s[0] = 0.4243, \ s[1] = 0.7559, \ s[2] = -0.0676, \ s[3] = -0.2619, \ e = 0.1493$$

(Step 6):

$$h[3] = 0.4243, \ h[2] = h[4] = s[1]/2 = 0.3780,$$

$$h[1] = h[5] = s[2]/2 = -0.0338,$$

$$h[0] = h[6] = s[3]/2 = -0.1309, \ h[n] = 0 \text{ for } n < 0 \text{ and } n > 6$$

## 附錄二：Spectrum Analysis for Sampled Signals

(學信號處理的人一定要會的基本常識)

已知  $x[n]$  是由一個 continuous signal  $y(t)$  取樣而得

$$x[n] = y(n\Delta_t)$$

DFT:  $X[m] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nm/N}$

$$\begin{aligned} n &= 0, 1, \dots, N-1 \\ m &= 0, 1, \dots, N-1 \end{aligned}$$

FT:  $Y(f) = \int_{-\infty}^{\infty} e^{-j2\pi f t} y(t) dt$

$$\begin{aligned} m &= 20000 \quad (m > \frac{N}{2}) \\ f &= 20000 \times \frac{6000}{30000} - 6000 \end{aligned}$$

$$\begin{aligned} \text{Ex: } x[n] &= y\left(\frac{n}{6000}\right) \\ n &= 0 \sim 29999 \\ (N &= 30000) \end{aligned}$$

$$X[m] \quad m = 600$$

$$f = 600 \times \frac{6000}{30000}$$

$$= 120 \text{ Hz}$$

Q:  $x[n]$  的 DFT 和  $y(t)$  的 Fourier transform 之間有什麼關係？

Basic rule : 把間隔由 1 換成  $f_s/N$  where  $f_s = 1/\Delta_t$



$$f = m \frac{f_s}{N}$$

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$$\text{D}(262 \text{ Hz}) \quad m = 262 \times \frac{N}{f_s} = 1310$$

for  $f = 150 \text{ Hz}$ ,  $m = ?$

$$f = m \Delta_f = \frac{m}{N \Delta_t} = \frac{m f_s}{N}$$

週期  $\frac{1}{\Delta t}$

(Very important)

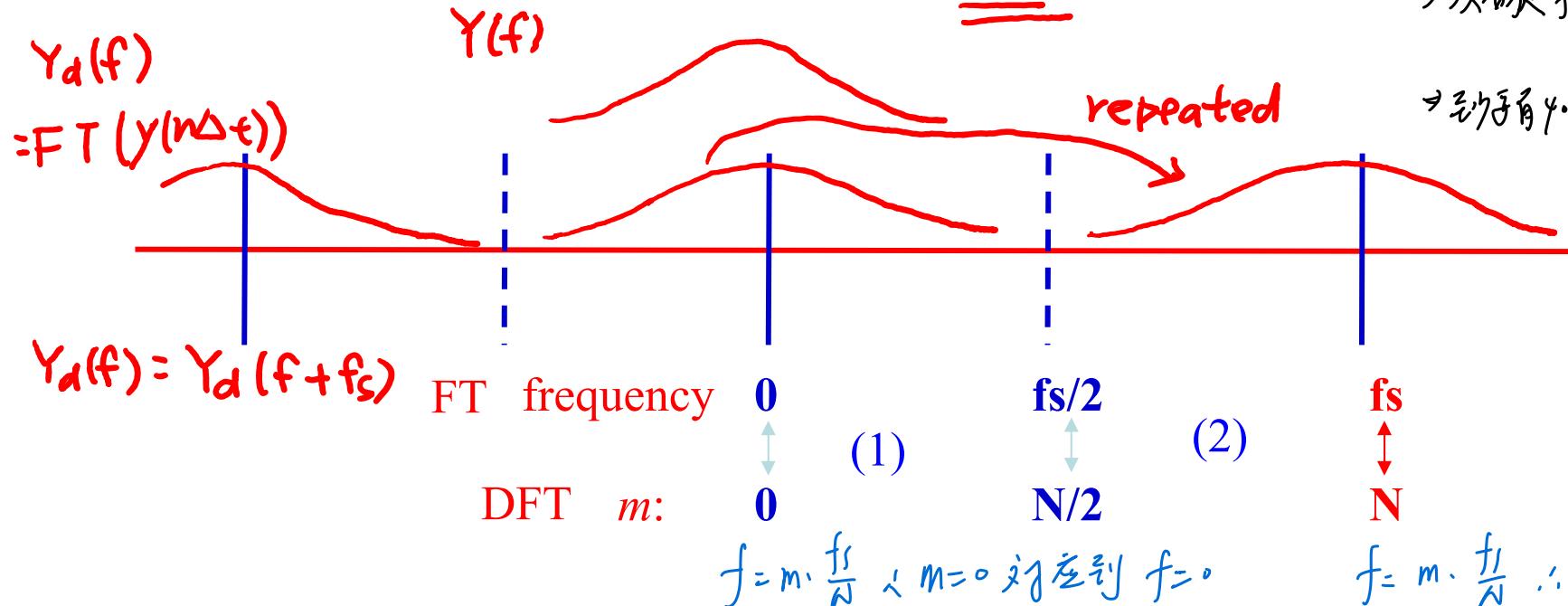
$$\text{when } m = \frac{N}{2}, f = \frac{f_s}{2}, m = N, f = f_s$$

從  $\Delta t \rightarrow$  物理意義：時間的變化量

from  $\Delta t = f$  其已取樣： $t = n\Delta t$

$$(1) X[m]\underline{\Delta_t} \cong Y\left(m \frac{f_s}{N}\right) \quad f_s = 1/\Delta_t \quad \text{for } m \leq N/2$$

$$(2) X[m]\underline{\Delta_t} \cong Y\left((m-N)\frac{f_s}{N}\right) = Y\left(m\frac{f_s}{N} - f_s\right) \quad \text{for } m > N/2$$



If the sampling frequency is  $f_s$ , the FT output has the period of  $f_s$  文字到  $f = N \cdot \frac{f_s}{N} = f_s$

The DFT output has the period of  $N$

Ex1:  $f_s = 40000$ ,  $N = 20000$  (2.5 sec)

if  $f = 162$  (D. of 頻率)  $m = ?$

$\therefore m \cdot \frac{f_s}{N} = m \cdot \frac{40000}{20000} = m \cdot \frac{1}{5} = 162 \therefore m = 162 \times 5 = 1310$  將最後 1310 的地方 = 162 亂寫文字的那個地方

ex: voice

$20 \sim 20000$  Hz

→ 人耳可以聽到的聲音

完全取樣 → 可還原出原來的信號

→ 球場是  $f_s > 2B = 40000$

頻寬

→ 球子有  $4000 \sim 10000$  以上的取樣速率 / sec

$$\text{Proof : } Y(f) = \int_{-\infty}^{\infty} e^{-j2\pi f t} y(t) dt$$

用  $t = n\Delta_t$ ,  $f = m\Delta_f$  代入

$$Y(m\Delta_f) \cong \sum_n e^{-j2\pi m\Delta_f n\Delta_t} y(n\Delta_t) \Delta_t = \Delta_t \sum_n e^{-j2\pi m\Delta_f n\Delta_t} x[n]$$

當  $\Delta_t \Delta_f = \frac{1}{N}$  i.e.,  $\Delta_f = \frac{1}{N\Delta_t} = \frac{f_s}{N}$

$$\begin{aligned} Y\left(m \frac{f_s}{N}\right) &\cong \Delta_t \sum_n e^{-j2\pi \frac{mn}{N}} x[n] \\ &= \Delta_t DFT\{x[n]\} \end{aligned}$$

Ex2:  $f_s = 40000$ ,  $N = 200000$

$X(10000)$  corresponds to  $f = ?$

其  $200000$  位的

第  $10000$  位的數字 = ?

$$\rightarrow f = m \cdot \frac{f_s}{N} = 10000 \times \frac{40000}{200000} = 10000 \times \frac{1}{5} = 2000 \text{ (Hz)}$$

$X(160000)$  corresponds to  $f = ?$

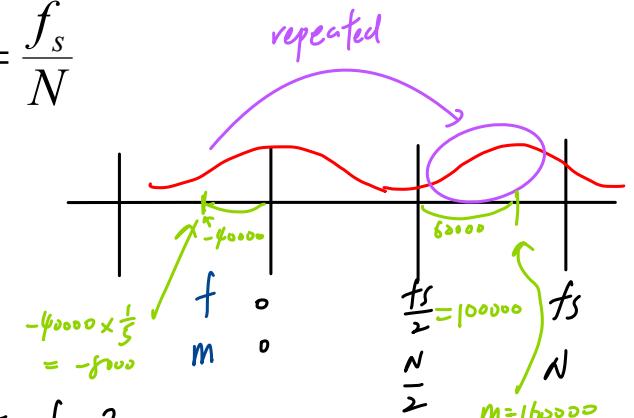
$$\rightarrow f = 160000 \times \frac{40000}{200000} = 160000 \times \frac{1}{5} = 32000 \text{ (Hz)}$$

有可能嗎？  
超過取樣率

但可看到  $m = \frac{N}{2}$  相應的已是  $f = \frac{f_s}{2}$

$$\text{由式知 } \frac{N}{2} = \frac{200000}{2} = 100000$$

$$\Rightarrow 32000 - f_s \Rightarrow 32000 - 40000 = -8000 \text{ 不正確。}$$



Ex3:  $f_s = 40000$ ,  $N = 600000$

$f = 500 \text{ Hz}$ ,  $m = ?$

$\rightarrow f = 500 \text{ Hz}$  是 20 位到  $\left( f = m \cdot \frac{f_s}{N} \Rightarrow 500 = m \cdot \frac{40000}{600000} \Rightarrow 500 = m \cdot \frac{1}{15} \Rightarrow m = 500 \times 15 = 7500 \right)$

②  $f = 6000$ ,  $f = ? \text{ Hz}$  考慮尾數 1, 6

$\rightarrow f = m \cdot \frac{f_s}{N} = 6000 \times \frac{40000}{600000} = 6000 \times \frac{1}{15} = 400 \quad \square$

Example : 已知

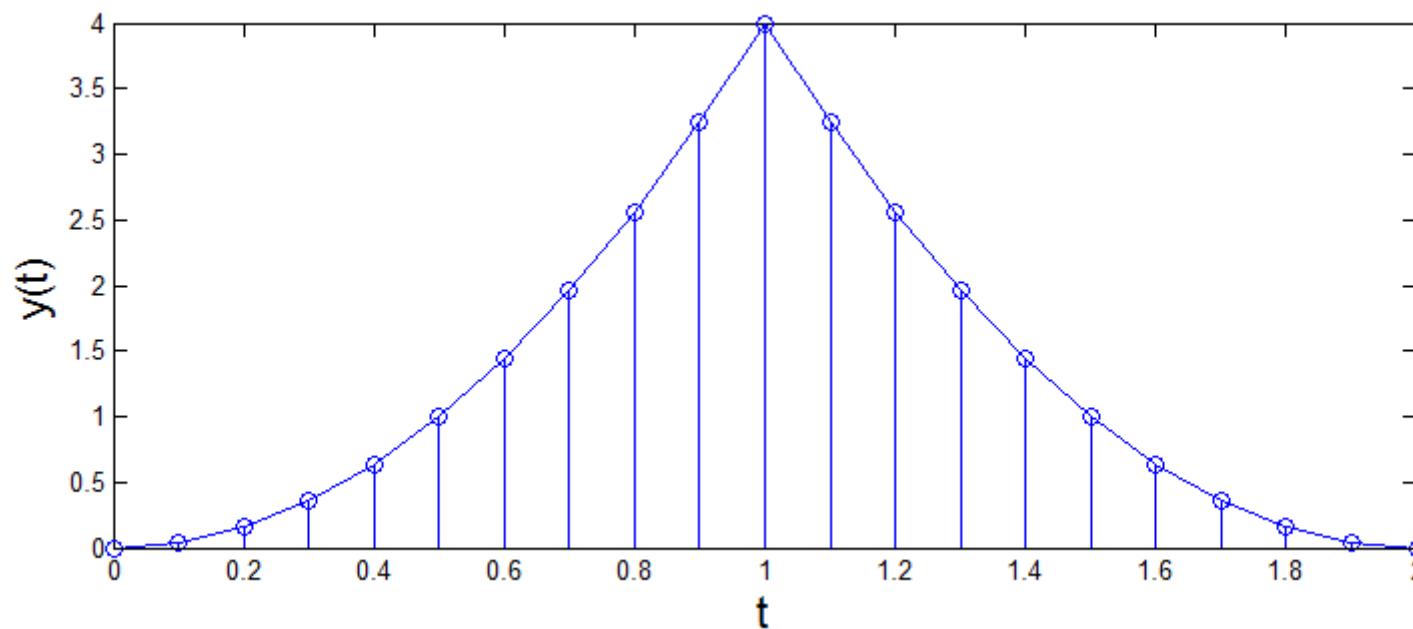
$$y(t) = (2t)^2 \quad \text{for } 0 \leq t \leq 1 \quad y(t) = (4-2t)^2 \quad \text{for } 1 \leq t \leq 2$$

取樣間隔 :  $\Delta_t = 0.1$

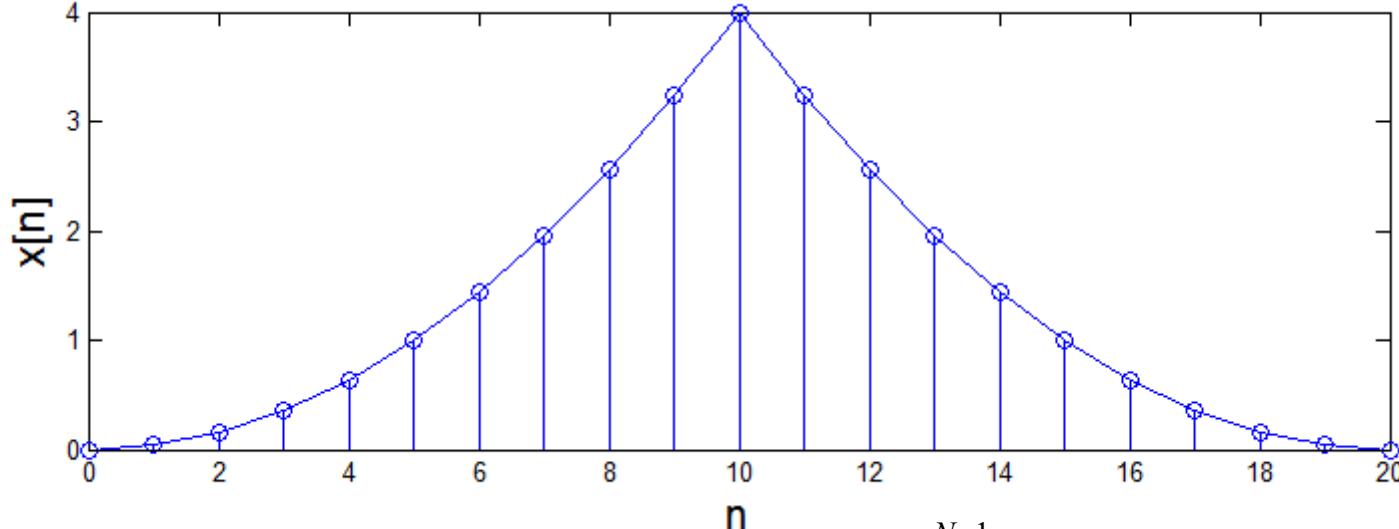
$$f_s = \frac{1}{\Delta_t} = 10$$

$$x[n] = y(n \Delta_t) \text{ for } 0 \leq n \leq 20 \quad N=21 \text{ 個取樣點.}$$

如何用 DFT 來正確的畫出  $y(t)$  的頻譜？



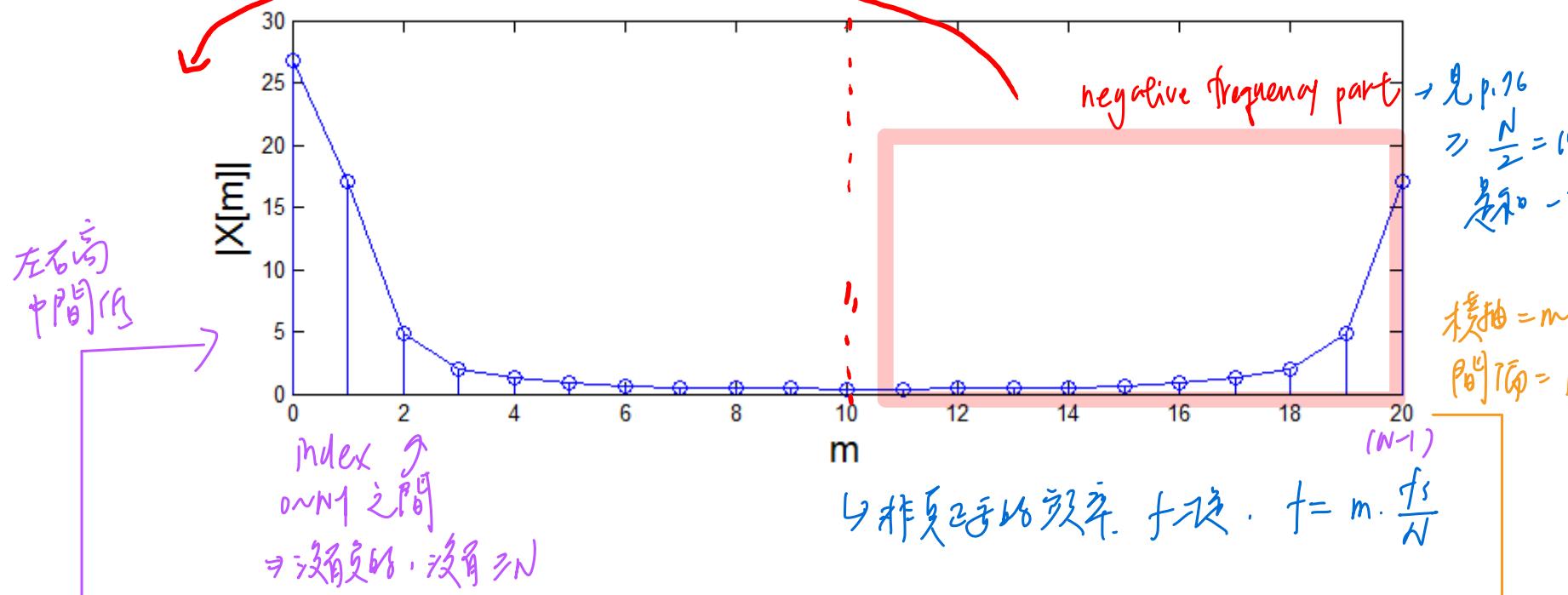
$$x[n] = y(n \Delta_t) \text{ for } 0 \leq n \leq 20$$



(Step 1) Perform the DFT for  $x[n]$

$$X[m] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nm/N}$$

$$N = 21$$



但頻率是變動的.

和已的頻率是相同的.  
但 phase 不同

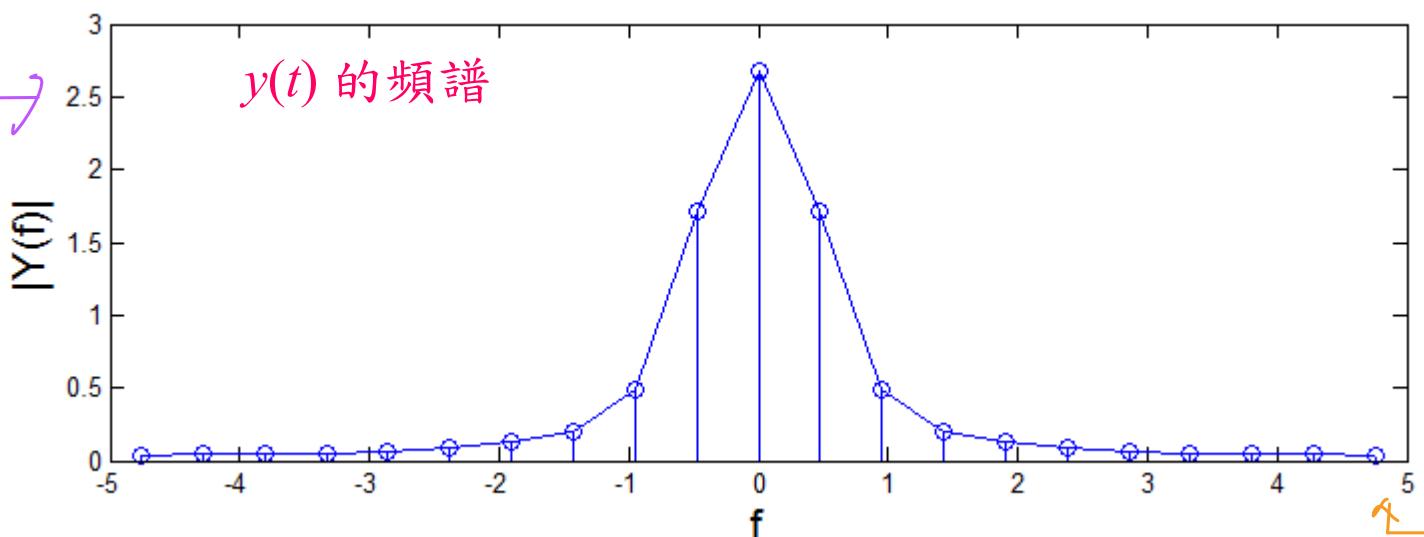
圖的  
形狀不同.

$$(Step\ 2-1) \quad Y\left(m \frac{f_s}{N}\right) \cong X[m] \Delta_t \quad \text{for } m \leq N/2$$

$$(Step\ 2-2) \quad Y\left((m-N) \frac{f_s}{N}\right) \cong X[m] \Delta_t \quad \text{for } m > N/2$$

$$\text{In this example, } \frac{f_s}{N} = \frac{1}{N\Delta_t} = \frac{1}{21 \cdot 0.1} = 0.4762$$

左高低  
中間高.



移動 =  $f$  ( $m$  級  $\frac{f_s}{N} = 0.4762$ )

$$\begin{aligned} \text{階距} &= 1 \times 0.4762 \\ &= 0.4762 \end{aligned}$$

## ◎ 2-H Relations among Filter Length N, Transition Band, and Accuracy

◆ Suppose that we want:

$$\textcircled{1} \text{ passband ripple } \leq \delta_1, \quad \text{passband 誤差}$$

$$\textcircled{2} \text{ stopband ripple } \leq \delta_2, \quad \text{stopband 誤差}$$

$$\textcircled{3} \text{ width of transition band } \leq \underline{\Delta F} \quad (\text{expressed by } \underline{\text{normalized frequency}})$$

$$\underline{\Delta F} = (f_1 - f_2)/f_s = (f_1 - f_2)T \quad (f_s: \text{sampling frequency}, T: \text{sampling interval})$$

Then, the estimated length N of the digital filter is:

$$N = \frac{2}{3} \frac{1}{\Delta F} \log_{10} \left( \frac{1}{10\delta_1\delta_2} \right)$$

- When there are two transition bands,  $\Delta F = \min(\Delta F_1, \Delta F_2)$
- 牺牲 transition band 的 frequency response, 換取較高的 passband and stopband accuracies

$$N = \frac{2}{3} \frac{1}{\Delta F} \log_{10} \left( \frac{1}{10\delta_1\delta_2} \right)$$

$$\underline{\underline{\delta_1\delta_2}} = \underline{\underline{10^{-3N\Delta F/2-1}}}$$

$$\frac{3}{2} N \Delta F = \log_{10} \left( \frac{1}{10\delta_1\delta_2} \right)$$

$$\begin{aligned} \Rightarrow -\frac{3}{2} N \Delta F &= \log_{10}(10\delta_1\delta_2) \\ \Rightarrow -\frac{3}{2} N \Delta F &= 1 + \log_{10}\delta_1\delta_2 \end{aligned}$$

$$\begin{aligned} \Rightarrow \log \delta_1\delta_2 &= -\frac{3}{2} N \Delta F - 1 \\ \Rightarrow \delta_1\delta_2 &= 10^{-\frac{3}{2} N \Delta F - 1} \end{aligned}$$

[Ref] F. Mintzer and L. Bede, "Practical design rules for optimum FIR bandpass digital filter", *IEEE Trans. ASSP*, vol. 27, no. 2, pp. 204-206, Apr. 1979.

問題：假設  $\sqrt{10}\delta_1 = \sqrt{10}\delta_2 = \delta$ ， $N$  為固定，

當  $\Delta F$  變為  $A$  倍時， $\delta$  變為多少？

可用這方法來減少誤差

但 computation 太大

且可調整 transition band 的寬度。

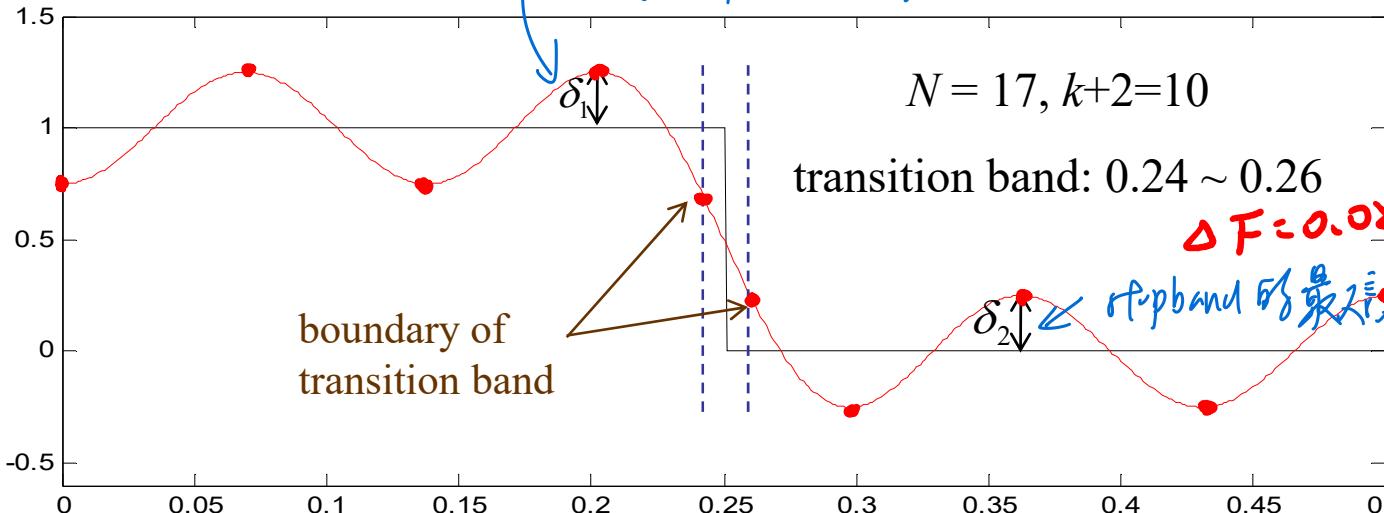
$$\delta^2 = 10^{-3N\Delta F/2} \quad \text{如果 \Delta F 變成 } A \text{ 倍。}$$

$$10^{-3N(A\Delta F)/2} = (\delta^2)^A = \delta^{2A} = (\delta^A)^2$$

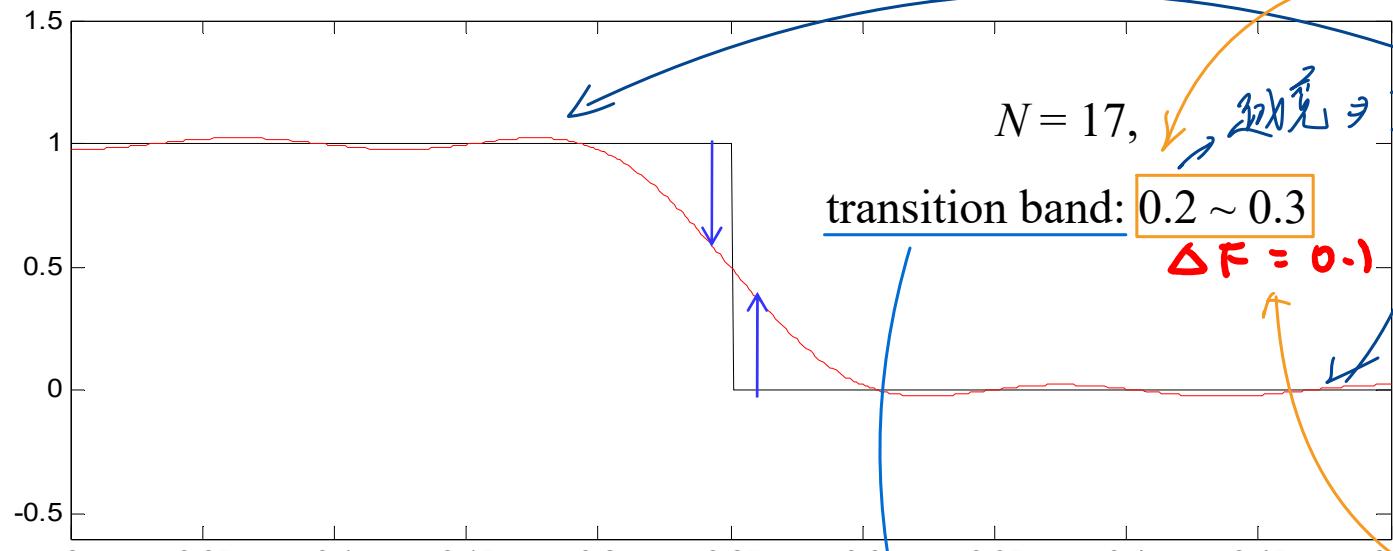
$$\text{original error: } \frac{\delta}{\sqrt{10}}$$

$$\text{new error } \frac{\delta^A}{\sqrt{10}}$$

紅色和黑色間的最大誤差,  
即  $f_1 \Rightarrow$  passband 的最大誤差



問這個的  $N$   
 transition band  
 有幾多倍.  
 $\Delta F = 0.02$   
 $\downarrow$   
 $0.1$

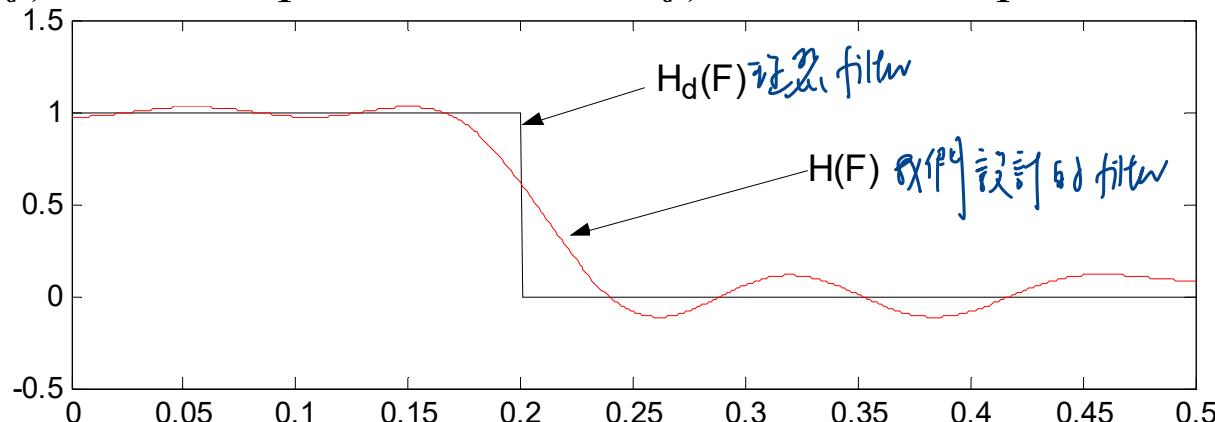


但在 transition band 中  $\Rightarrow$  這兩地方等同被浪費掉  
 無法適用

## ◎ 2-I Relations between Weight Functions and Accuracy

If we treat the passband more important than the stop band

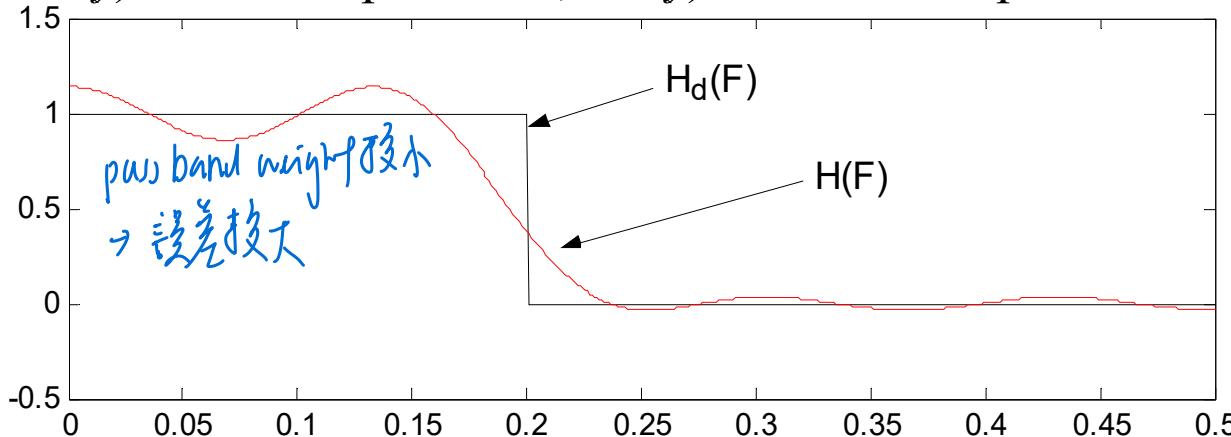
$W(f) = 1$  in the passband,  $0 < W(f) < 1$  in the stopband



$W(F)$  and  
 $\propto W(F)$  have  
the same designed  
filter

If we treat the stop band more important than the pass band

$0 < W(f) < 1$  in the passband,  $W(f) = 1$  in the stopband



## Larger error near the transition band

