V. Homomorphic Signal Processing

同 質

◉ 5-A Homomorphism

Homomorphism is a way of "carrying over" operations from one algebra system into another.

Fourier transform 就是一種 homomorphic signal processing

Ex. convulution $\xrightarrow[Fourier]{}$ multiplication $\xrightarrow[\log]{}$ addition

把複雜的運算，變成效能相同但較簡單的運算

主要的目的：解決 multipath problem ，
而非只是把乘法變加法

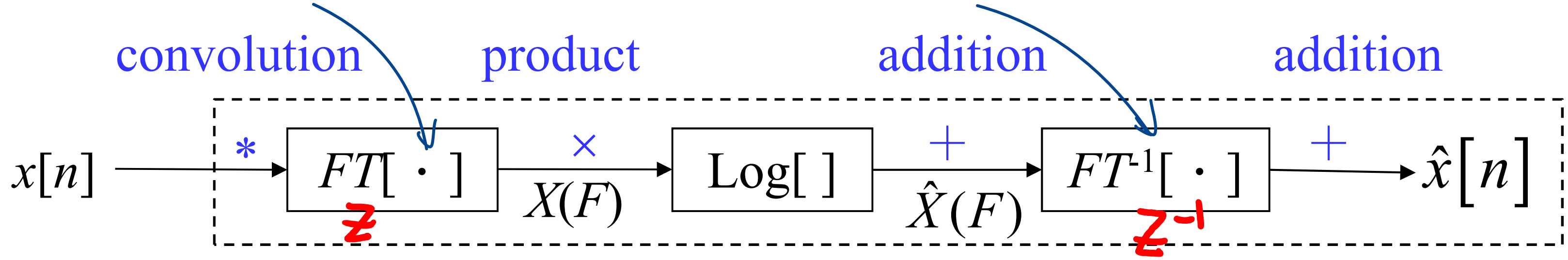
◉ 5-B Cepstrum 倒頻譜

$$\hat{X}(Z)\Big|_{z=e^{i2\pi F}} = \log X(Z)\Big|_{z=e^{i2\pi F}} = \log\left|X(Z)\right|_{z=e^{i2\pi F}} + j\arg[X(e^{i2\pi F})]$$

For the process of cepstrum (denoted by $D_*[\ \cdot\ ]$)

實務上經常用 z-transform 來取代 Fourier transform，inverse z-transform 來取代 inverse Fourier transform

相反



$x[n]*h[n]$ $\quad X(F)\times H(F)$ $\quad \hat{X}(F)+\hat{H}(F)$ $\quad \hat{x}[n]+\hat{h}[n]$

Convolution 經過 Fourier transform 變成相乘

$\hat{X}(F) = \log(X(F))$ $\quad \hat{H}(F) = \log(H(F))$

FT: discrete-time Fourier transform

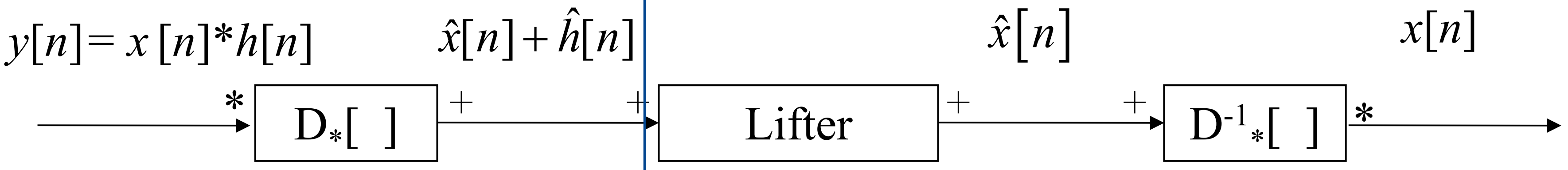
$\hat{x}[n] = IFT(\log(X(F)))$

$\hat{h}[n] = IFT(\log(H(F)))$

可以用 cepstrum 的方式來解決 multipath problem，因為
multipath problem input 和 output 的關係：convolution
(y[n] = x[n] * k[n])

- 由 $y[n]= x[n]*h[n]$ 重建 $x[n]$

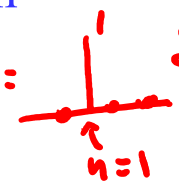
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

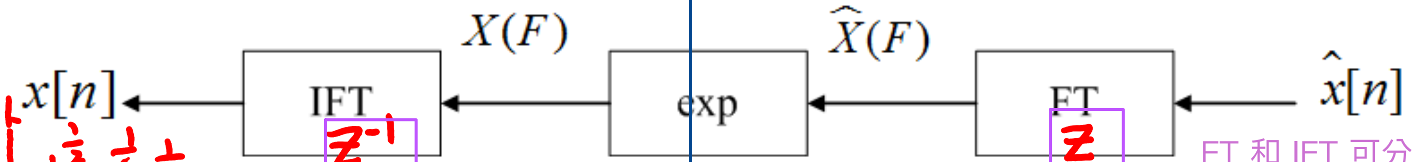


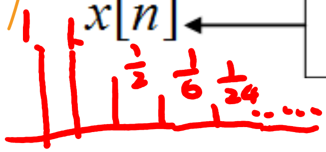
$$l[n]\{\hat{x}[n]+\hat{h}[n]\} = \hat{x}[n]$$

For the process of the inverse cepstrum $D^{-1}_*[\,\cdot\,]$

if $\hat{x}[n] = \delta[n-1]$







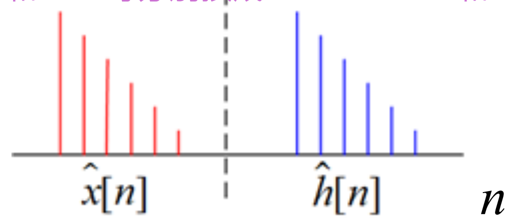
$\sum_{n=0} \frac{z^{-n}}{n!} = \exp(z^{-1})$

用泰勒展開

$\hat{X}(z) = \sum_n \hat{x}[n] z^{-n}$

z^{-1}

FT 和 IFT 可分別換成 z transform 和 z^{−1}



把 x 和 h 的 cepstrum 的位置分開來

- 有趣的名詞

$\hat{x}[n]$	cepstrum	spectrum	倒頻譜
n	quefrency	frequency	倒頻率
$l[n]$	lifter	filter	倒濾波器

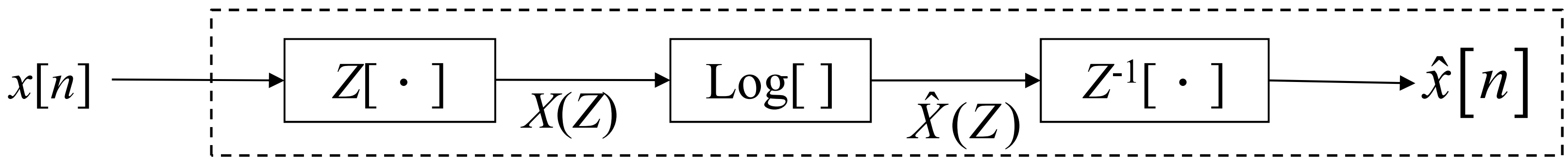
lifter

$$l[n]\{\hat{x}[n]+\hat{h}[n]\} = \hat{x}[n]$$

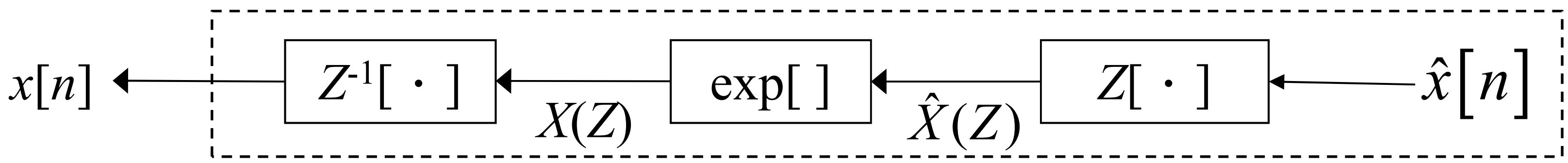
最簡單的 multipath 問題：回音（見 p.194）

Using the Z transforms instead of the Fourier transforms:

For the process of cepstrum



For the process of the inverse cepstrum



◉ 5-C Methods for Computing the Cepstrum

- **Method 1**: Compute the inverse discrete time Fourier transform:

$$\hat{x}[n] = \int_{-1/2}^{1/2} \hat{X}(F) e^{i2\pi nF} dF \qquad \text{: inverse F.T}$$

$\hat{X}(F) = \log X(F)$

where $\hat{X}(F) = \log|X(F)| + j\arg[X(F)]$

取log

$X(F) = |X(F)| e^{j\arg X(F)}$

ex: $X(F) = j$

通解?

$|X(F)| = 1$, $\arg(X(F)) = \frac{\pi}{2}, \boxed{\frac{-3\pi}{2}}, \ldots$ ambiguity for phase

通常會有無限多解

用 log 把 convolution 變成加法不再是最主要的應用，因為有兩個問題

如果 X(F) 在某些地方 =0 或趨近於零

Problems: (1) $\log|X(F)| \to -\infty$ if $|X(F)| \to 0$ 對負無限大的東西做 inverse FT 比較不方便

(2) $\arg(X(F))$ has infinite number of solutions.

Actually, the COMPLEX Cepstrum is REAL for real input

$\frac{-3\pi}{2} = -270°$ $\cos(-270°) = 0$, $\sin(-270°) = \sin(90°) = 1$

$\cos(-270°) + j\sin(-270°) = 0 + j = j$

原本: $1-\frac{1}{b_1}z^{-1} \ni b_1^{-1}z^{-1} \ni$ 抽出 $1-b_1 z$

$\frac{1}{b_1} = b_1^{-1}$, $b_1 > 1$

• Method 2 (From Poles and Zeros of the Z Transform)

實際上計算 cepstrum的方法

time delay

算好這個直接代 p.188 公式

$$X(Z) = \frac{A\cancel{Z^r}\prod_{k=1}^{m_i}(1-a_k Z^{-1}) \prod_{k=1}^{m_0}(1-b_k Z)}{\prod_{k=1}^{P_i}(1-c_k Z^{-1}) \prod_{k=1}^{P_0}(1-d_k Z)}$$

where

$$|a_k|, |b_k|, |c_k|, |d_k| \leq 1$$

a_k: zeros inside unit circle

c_k: poles inside unit circle

b_k^{-1}: zeros outside unit circle

d_k^{-1}: poles outside unit circle

$\prod$: 連乘

Σ: 連加

zero, pole 分成兩組改寫原式

$$\therefore \hat{X}(Z) = \log X(Z) = \log \boxed{A} + \cancel{r \cdot \log Z} + \sum_{k=1}^{m_i}\log(1-a_k Z^{-1}) + \sum_{k=1}^{m_0}\log(1-b_k Z)$$

振幅

p.184 cepstrum 第二步：取 log

$$-\sum_{k=1}^{P_i}\log(1-c_k Z^{-1}) - \sum_{k=1}^{P_0}\log(1-d_k Z)$$

在 time domain 乘上 Z^{r} = 在 time domain 做位移
>> 一個濾波器經過 delay 性質仍然相同，只是 output 的時間會延遲
>> 因此我們把 r 這項去除掉，方便做分析

ex: $z^2-0.7z+0.1$
$=(z-0.5)(z-0.2)$
$=z^2(1-0.5z^{-1})(1-0.2z^{-1})$

$$\therefore \hat{X}(Z) = \log X(Z) = \log A + \cancel{r \cdot \log Z} + \sum_{k=1}^{m_i} \log\left(1 - a_k Z^{-1}\right) + \sum_{k=1}^{m_0} \log\left(1 - b_k Z\right)$$

$$- \sum_{k=1}^{P_i} \log\left(1 - c_k Z^{-1}\right) - \sum_{k=1}^{P_0} \log\left(1 - d_k Z\right)$$

$$\Big\downarrow \; Z^{-1} \text{ (inverse Z transform)}$$

?

用 → Taylor series

$$f(t) = f(t_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(t_0)}{n!}(t - t_0)^n$$

$$\frac{d^n}{dt^n}\log(1+t) = \frac{(-1)^{n-1}(n-1)!}{(1+t)^n}$$

$$\log(1+t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!} t^n \qquad (t_0 = 0)$$

以零為中心展開

$$\log(1+t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} t^n$$

$$\log(1 - a_k Z^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n} a_k^n Z^{-n}$$

$$= \sum_{n=1}^{\infty} \boxed{\frac{-1}{n} a_k^n} Z^{-n}$$

inverse z transform >> 看係數

t = −b_k z 代

$$\log(1 - b_k Z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(-1)^n b_k^n Z^n$$

$$= \sum_{n=1}^{\infty} -\frac{1}{n} b_k^n Z^n \qquad n \to -n$$

$$= \sum_{n=-\infty}^{-1} \frac{1}{n} b_k^{-n} Z^{-n}$$

$n: -\infty \sim -1$

$\therefore Z^{-1}$ 做完 $\exists\, n<0$ (下頁)

$\boxed{\text{Taylor series expansion}}$ $\big\downarrow Z^{-1}$ (Suppose that $r = 0$)

$$\hat{x}[n] = \begin{cases} \log(A) & , n = 0 \\ -\sum_{k=1}^{m_i} \frac{a_k^{\ n}}{n} + \sum_{k=1}^{P_i} \frac{c_k^{\ n}}{n} & , n > 0 \\ \sum_{k=1}^{m_0} \frac{b_k^{\ -n}}{n} - \sum_{k=1}^{P_0} \frac{d_k^{\ -n}}{n} & , n < 0 \end{cases}$$

Poles & zeros inside unit circle, right-sided sequence

Poles & zeros outside unit circle, left-sided sequence

可以代入看看
就知道

Note:

所有的 zeros, poles 都在單位圓內 = minimum phase >> 沒有 b_k, d_k >> 所以如果是 minimum phase，則在 n < 0 的地方都沒有值

- maximum phase 則相反

(1) $\hat{x}[n]$ always decays with $|n|$. |n| 越大，倒頻譜就越小

(2) 在 complex cepstrum domain
Minimum phase 及 maximum phase 之貢獻以 $n = 0$ 為分界切開

(3) For FIR case, there is no c_k and d_k

(4) The complex cepstrum is unique and of infinite duration for both positive & negative n, even though $x[n]$ is causal & of finite durations

x[n] 通常是無限長的

$\boxed{\hat{x}[n] \text{ is always IIR}}$

- **Method 3**

$$Z \cdot \hat{X}'(Z) = Z \cdot \frac{X'(Z)}{X(Z)}$$

$$\therefore ZX'(Z) = Z\hat{X}'(Z) \cdot X(Z)$$

$$\Big\downarrow Z^{-1}$$

$$n\,x[n] = \sum_{k=-\infty}^{\infty} k\,\hat{x}[k]\,x[n-k]$$

$$\therefore x[n] = \sum_{k=-\infty}^{\infty} \frac{k}{n}\hat{x}[k]\,x[n-k] \qquad for\ n \neq 0$$

Suppose that $x[n]$ is causal and has minimum phase, i.e. $x[n] = \hat{x}[n] = 0,\ n < 0$

$$x[n] = \sum_{k=-\infty}^{\infty} \frac{k}{n} \hat{x}[k] x[n-k] \qquad \textit{for } n \neq 0$$

$$\Rightarrow x[n] = \sum_{k=0}^{n} \frac{k}{n} \hat{x}[k] x[n-k] \qquad \textit{for } n > 0 \qquad (\textit{causal sequence})$$

$$x[n] = \hat{x}[n] x[0] + \sum_{k=0}^{n-1} \frac{k}{n} \hat{x}[k] x[n-k]$$

For a minimum phase sequence $x[n]$

$$\hat{x}[n] = \begin{cases} 0 & ,n < 0 \\ \dfrac{x[n]}{x[0]} - \displaystyle\sum_{k=0}^{n-1} \left(\frac{k}{n}\right) \hat{x}[k] \frac{x[n-k]}{x[0]} & ,n > 0 \\ \log A & ,n = 0 \end{cases} \qquad \text{recursive method}$$

Determining $\hat{x}[n]$ from $\hat{x}[0], \hat{x}[1], \cdots, \hat{x}[n-1]$

For anti-causal and maximum phase sequence, $x[n] = \hat{x}[n] = 0$, $n > 0$

$$x[n] = \sum_{k=n}^{0} \frac{k}{n} \hat{x}[k] x[n-k] \qquad , n < 0$$

$$= \hat{x}[n] x[0] + \sum_{k=n+1}^{0} \frac{k}{n} \hat{x}[k] x[n-k]$$

For maximum phase sequence,

$$\hat{x}[n] = \begin{cases} 0 & , n > 0 \\ \log A & , n = 0 \\ \dfrac{x[n]}{x[0]} - \displaystyle\sum_{k=n+1}^{0} \left(\frac{k}{n}\right) \hat{x}[k] \frac{x[n-k]}{x[0]} & , n < 0 \end{cases}$$

◉ 5-D Properties

P.1) The complex cepstrum decays at least as fast as $\frac{1}{n}$

$$|\hat{x}[n]| < c\left|\frac{\alpha^n}{n}\right| \qquad -\infty < n < \infty$$

$$\alpha = \max\left(|a_k|, |b_k|, |c_k|, |d_k|\right)$$

P.2) If $X(Z)$ has no poles and zeros outside the unit circle, i.e. $x[n]$ is minimum phase, then

$$\hat{x}[n] = 0 \qquad \textit{for all } n < 0$$

because of no b_k, d_k

P.3) If $X(Z)$ has no poles and zeros inside the unit circle, i.e. $x[n]$ is maximum phase, then

$$\hat{x}[n] = 0 \qquad \textit{for all } n > 0$$

because of no a_k, c_k

P.4) If $x[n]$ is of finite duration, then

$\hat{x}[n]$ has infinite duration

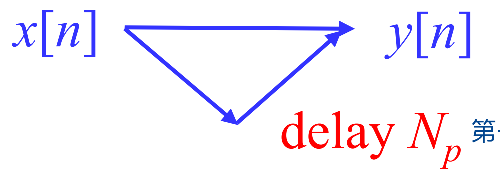
Cepstrum 可以拿來解決 multipath, echo 的問題

◉ 5-E Application of Homomorphic Deconvolution

(1) Equalization for Echo

對比 Equalizer，cepstrum 不需要估計衰減係數 \alpha 的值
（但還是要估計不同路徑的時間差 N_p）

$$y[n] = x[n] + \alpha\, x[n - N_p]$$



第一個 path 和第二個的時間差

衰減係數

Let $p[n]$ be $p[n] = \delta[n] + \boxed{\alpha}\delta[n-N_p]$

$$y[n] = x[n] + \alpha\, x[n-N_p] = x[n] * p[n]$$

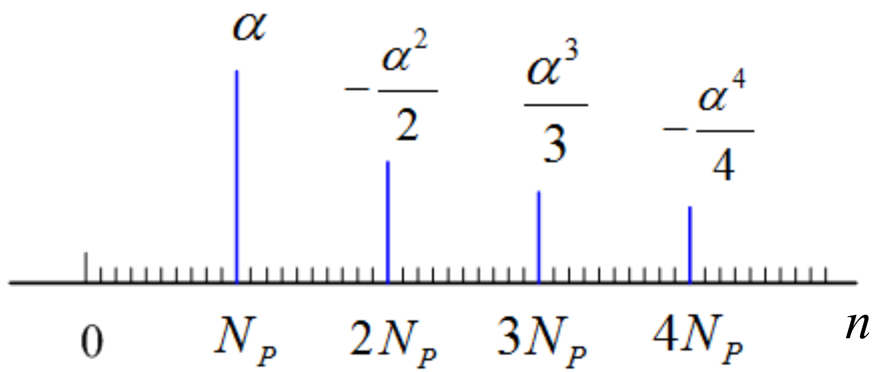
$$\log(1+t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^k$$

$$P(Z) = 1 + \alpha Z^{-N_p}$$

$$\hat{P}(Z) = \log\,(1 + \underbrace{\alpha Z^{-N_p}}_{\to t}) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\alpha^k}{k} Z^{-kN_p}$$

$$\downarrow Z^{-1}$$

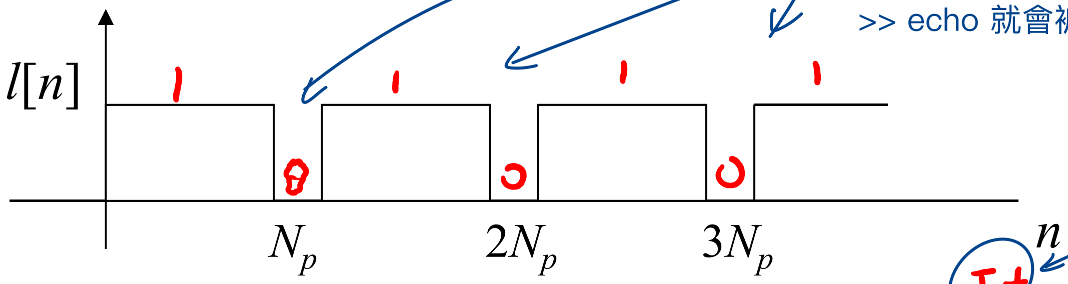
$$\hat{p}[n] = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\alpha^k}{k} \delta(n - k \cdot N_p)$$



回音所造成的 cepstrum 只有
在 N_P 的整數倍時才有值

Filtering out the echo by the following “lifter”:

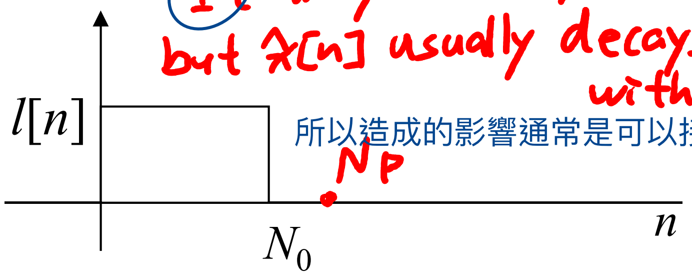
我們就把這些地方的響應設為零就好了
>> echo 就會被去除



把 N_0 以上的東西都砍掉

It may destroy $\hat{x}[n]$
but $\hat{x}[n]$ usually decays fast
with n.

or



所以造成的影響通常是可以接受的

Q: For the case where N_p is unknown

最小的路徑時間差 N_p $\quad Np > N_0$

(2) Representation of acoustic engineering

因為 在 p.188 我們可以看到當 |n| 越大時，因為 |a_k|,…,|d_k| < 1，
所以取 n 次方、除 n 就會越趨近於零

$$y[n] \;=\; x[n] \quad * \quad h[n]$$

Synthesized music — music — building effect：e.g. 羅馬大教堂的 impulse response

如果發射端和接收端都已經知道了（像建築物固定的情況下），就可以預先把所有的 path 都算出來

但像行動通訊就不行（ex: 手機）>> 位置會變來變去