Number Theoretic Transform and Its Applications in Lattice-based Cryptosystems: A Survey

ZHICHUANG LIANG, Fudan University, Shanghai, China.
YUNLEI ZHAO, Fudan University, Shanghai, China; State Key Laboratory of Cryptology, Beijing, China.

Number theoretic transform (NTT) is the most efficient method for multiplying two polynomials of high degree with integer coefficients, due to its series of advantages in terms of algorithm and implementation, and is consequently widely-used and particularly fundamental in the practical implementations of lattice-based cryptographic schemes. Especially, recent works have shown that NTT can be utilized in those schemes without NTT-friendly rings, and can outperform other multiplication algorithms. In this paper, we first review the basic concepts of polynomial multiplication, convolution and NTT. Subsequently, we systematically introduce basic radix-2 fast NTT algorithms in an algebraic way via Chinese Remainder Theorem. And then, we elaborate recent advances about the methods to weaken restrictions on parameter conditions of NTT. Furthermore, we systematically introduce how to choose appropriate strategy of NTT algorithms for the various given rings. Later, we introduce the applications of NTT in the lattice-based cryptographic schemes of NIST post-quantum cryptography standardization competition. Finally, we try to present some possible future research directions.

Additional Key Words and Phrases: Lattice-based cryptography, Polynomial multiplication, Number theoretic transform, FFT trick, Chinese Remainder Theorem, NIST PQC.

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1 INTRODUCTION

Most current public-key cryptographic schemes in use, which are based on the hardness assumptions of factoring large integers and solving (elliptic curve) discrete logarithms, will suffer from quantum attack, if practical quantum computers are built. These cryptosystems play an important role in ensuring the confidentiality and authenticity of communications on the Internet. With the increasingly cryptographic security risks about quantum computing in recent years, post-quantum cryptography (PQC) has become a research focus for the crypto community. There are five main types of post-quantum cryptographic schemes: the hash-based, code-based, lattice-based, multivariable-based, and isogeny-based schemes, among which lattice-based cryptography is the most promising one due to its outstandingly balanced performance in security, communication bandwidth and computing efficiency. Most cryptographic primitives, such as public key encryption (PKE), key encapsulation mechanism (KEM), digital signature, key exchange, homomorphic encryption, etc, can be constructed based on lattices.

In the post-quantum cryptography standardization competition held by the US National Institute of Standards and Technology (NIST), lattice-based schemes account for 26 out of 64 schemes in the first round [NIS16], 12 out of 26 in the second round [NIS19] and 7 out of 15 in the third round [NIS20]. NIST announced 4 candidates to be standardized [NIS22], among which 3 schemes are based on lattices. Most of these lattice-based schemes are based on one of the following types: standard lattice, ideal lattice, NTRU lattice and module lattice. They are also instantiated based on the following hardness assumptions: Learning With Errors (LWE) [Reg09] and its variants such as Ring-Learning With Errors (RLWE) [LPR10] and Module-Learning With Errors (MLWE) [LS15], as well as the derandomized version of {R,M}LWE: Learning With Rounding (LWR) [BPR12], Ring-Learning With Rounding (RLWR) [BPR12] and Module-Learning With Rounding (MLWR) [AA16, DKRV18].

From a computational point of view, the fundamental and also time-consuming operation in many schemes based on standard lattices is matrix (vector) multiplication over a finite field, while that in schemes based on lattices with algebraic structures like the ideal, module, NTRU lattices is the multiplication of elements in the polynomial ring $\mathbb{Z}_q[x]/(\phi(x))$, where $\phi(x)$ is a polynomial with integer coefficients and q is a positive integer. There are some schemes based on lattices with algebraic structures in NIST PQC competition: Kyber KEM [BDK+18, ABD+20], Dilithium signature [DKL+18, BDK+20] and Saber KEM [DKRV18, BMD+20] are based on module lattices, while Falcon signature [FHK+20], NTRU KEM [CDH+20] and NTRU Prime KEM [BCLvV17, BBC+20] are based on NTRU lattices. Among them, Kyber KEM, Dilithium signature and Falcon signature are standardized by NIST [NIS22].

To compute polynomial multiplication, there are schoolbook algorithm [Knu14], Karatsuba [KO62, WP06]/ Toom-Cook [Too63, CA69] algorithm and DFT/FFT [CG99, CT65, GS66]/NTT [Pol71, AB75] algorithm, among which the schoolbook algorithm is the most trivial and simplest, but with the quadratic complexity of $O(n^2)$, where n is the length of polynomials. Karatsuba algorithm follows the "divide and conquer" technique, by dividing the original polynomial into two parts, resulting in the complexity of $O(n^{1.58})$. Toom-Cook algorithm is a generalization of the Karatsuba algorithm. Different from the Karatsuba algorithm, Toom-Cook algorithm divides the original polynomial into k parts, giving the complexity of $O(n^r)$ where $r = \log_k(2k-1)$.

Discrete Fourier transform (DFT) can be utilized to multiply two polynomials, but the complexity of directly-computing DFT is $O(n^2)$, similar to that of schoolbook algorithm. In the past few decades, lots of fast algorithms to compute DFT have been proposed, which are called fast Fourier transform (FFT) at present. Actually, the idea of FFT was first developed by Carl Friedrich Gauss in his unpublished work in 1805 [Gau05]. It was not until 1965 that FFT attracted widespread attention, when Cooley and Tukey independently proposed Cooley-Tukey algorithm [CT65], a fast algorithm of DFT. Number theoretic transform (NTT) is a special case of DFT over a finite field [Pol71]. Similarly, most FFT techniques can be applied to NTT, resulting with the corresponding fast algorithms for number theoretic transform. From an implementation point of view, FFT and NTT are the most efficient methods for computing polynomial multiplication of high degree, due to its quasilinear complexity $O(n \log n)$, which takes a significant advantage over all other multiplication algorithms. However, since FFT is performed over the complex field, the floating point operations during the computing process might cause errors about rounding precision when multiplying two polynomials with integer coefficients. In contrast, the operations during NTT computation are all performed with integers, which could avoid those precision errors. As most polynomials in lattice-based schemes are generated with integer coefficients, it is clear that NTT is especially suited for the implementation of those schemes.

Most schemes based on lattices with algebraic structures prefer to NTT-based multiplication due to its many advantages (more details in section 3.5), apart from the quasilinear complexity and integer arithmetic. However, not all lattice-based schemes can use NTT directly, since NTT puts some restrictions on its parameter conditions. For example, NTT based on negative wrapped convolution (NWC) requires its length n being a power of two and its modulus q being a prime satisfying $q \equiv 1 \pmod{2n}$. Among the standardized signature schemes in NIST PQC, both Dilithium and Falcon can meet the condition. The parameters of Kyber that is currently the only standardized KEM by NIST at most satisfy $q \equiv 1 \pmod{n}$, so it fails to utilize full NWC-based NTT. Neither does Saber, for the reason that Saber uses a power-of-two modulus q. The module reductions and rounding operations in Saber benefit from its power-of-two modulus, but its previous implementations also fail to use NTT, and Saber has to fall back to Karatsuba/Toom-Cook algorithm which is relatively less efficient in general. As for NTRU and NTRU Prime, they have NTT-unfriendly ring $\mathbb{Z}_q[x]/(\phi(x))$, where $\phi(x)$ has a prime degree, for which NWC-based NTT was not utilized for them at first. Therefore, it is very meaningful to generalize NTT methods to the case of NTT-unfriendly rings, so as to implement the underlying polynomial multiplication and obtain superior performance compared to the state-of-the-art implementations based on other multiplication algorithms.

Moreover, as we observe, most NTT algorithms and their recent improvements contained in this paper have appeared in the literatures before, but until now they are not well summarized in other works and none has introduced their applications in lattice-based schemes. Besides, few literatures give systematic study of the mathematical principles, theoretic derivation, recent advances, and practical applications with respect to NTT. Those motivate our comprehensive work and systematization of NTT. In this work, our contributions are listed as follows.

- We introduce the systematical knowledge of NTT, including the basic concepts, basic fast algorithms, computing advantages and computational complexity.
- We review the recent advances about the methods to weaken restrictions on parameter conditions of NTT, present a taxonomy based on their strategies and make a comparison.
- We propose high-level descriptions of some NTT techniques in a mathematically algebraic way for better understanding of the classification.
- We classify the polynomial rings into three categories mainly, and systematically introduce how to choose appropriate strategy of NTT for the given ring.
- As an important application, we introduce the recent advances of utilizing NTT in NIST PQC Round 3, especially the candidates with NTT-unfriendly rings.

We hope that this paper will also provide researchers in relevant fields with basic but comprehensive mathematical knowledge of NTT. However, as for the limitations of this paper, we mainly focus on those schemes based on lattices with algebraic structures from NIST PQC, because those schemes based on standard lattices don't need NTT. In addition, recent optimized techniques for different platforms (e.g., software/hardware) are not covered yet. But we leave it as a future work.

2 PRELIMINARIES

In this section, we will define some notations, and give a brief introduction of polynomial multiplication and convolution.

2.1 Notations and Definitions

Let $\mathbb{Z}$ be the ring of rational integers, n and q be some positive integers, and $\mathbb{Z}_q \cong \{0, 1, \ldots, q-1\}$. We write $x' \equiv x \pmod{q}$ to mean that $x' - x$ is a multiple of q. We define $x' = x \bmod q$ to be the unique element x' in $\mathbb{Z}_q$ satisfying $x' \equiv x \pmod{q}$. Polynomials are written as bold $\boldsymbol{a}$ or unbold $a(x)$. As for $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{r}$, we also write $\boldsymbol{r} = \boldsymbol{a} \bmod \boldsymbol{b}$ to mean that $\boldsymbol{r}$ is the polynomial remainder of $\boldsymbol{a}$ divided by $\boldsymbol{b}$. For any $n_1, n_2 \in \mathbb{Z}$, $n_1 > n_2$, and any a_i, define $\sum_{i=n_1}^{n_2} a_i = 0$. The symbol "$\circ$" denotes the point-wise multiplication, i.e., multiplication of corresponding components.

Definition 2.1 (Primitive and principal root of unity). Let R be a commutative ring with multiplicative identity 1, k be a positive integer, and ψ be an element in R. Define ψ is the **primitive** k-th root of unity in R, if and only if $\psi^k = 1$, and $\psi^i \neq 1, i = 1, 2, \ldots, k-1$. Define ψ is the **principal** k-th root of unity in R, if and only if $\psi^k = 1$, and $\sum_{j=0}^{k-1} \psi^{jl} = 0, l = 1, 2, \ldots, k-1$. Notice that primitive and principle k-th root of unity coincide if $R = \mathbb{Z}_q$ where q is a prime number. See [Für09, ACC+22] for more details.

Definition 2.2 (Bitreversal). Let n be a power of two, and b be a non-negative integer satisfying $b < n$. The bitreversal of b with respect to n is defined as

$$\mathrm{brv}_n(b_{\log n-1}2^{\log n-1} + \ldots + b_1 2 + b_0) = b_0 2^{\log n-1} + \ldots + b_{\log n-2}2 + b_{\log n-1},$$

where b_i is the i-th bit of the binary expansion of b .

2.2 Polynomial Rings

Let $\mathbb{Z}[x]$ and $\mathbb{Z}_q[x]$ be the polynomial rings over $\mathbb{Z}$ and $\mathbb{Z}_q$ respectively, with corresponding quotient rings $\mathbb{Z}[x]/(\phi(x))$ and $\mathbb{Z}_q[x]/(\phi(x))$, where $\phi(x)$ is a polynomial with integer coefficients. Especially, $\phi(x)$ is chosen to be a cyclotomic polynomial [Was97]. In this paper, we only take account of operations over $\mathbb{Z}_q$ unless otherwise noted, because the corresponding results over $\mathbb{Z}$ can be obtained easily. If $\deg\phi(x) = n$ holds, the element in $\mathbb{Z}_q[x]/(\phi(x))$, for example, $\boldsymbol{a}$, can be represented in the form of $\boldsymbol{a} = \sum_{i=0}^{n-1} a_i x^i$, or in the form of $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1})$, where $a_i \in \mathbb{Z}_q$. We will mainly focuses on $\mathbb{Z}_q[x]/(x^n - 1)$ and $\mathbb{Z}_q[x]/(x^n + 1)$, since they are widely used in lattice-based schemes. Let n be a power of two, then $x^n + 1$ is the $2n$-th cyclotomic polynomial.

2.3 Polynomial Multiplication and Convolution

Without loss of generality, we always considers $\boldsymbol{a}$ and $\boldsymbol{b}$ of degree $n-1$ in this paper. Pad them with zero if their lengths are less than n.

- **Linear convolution.** Consider $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]$, then $\boldsymbol{c} = \sum_{k=0}^{2n-2} c_k x^k \in \mathbb{Z}_q[x]$, where $c_k = \sum_{i+j=k} a_i b_j \bmod q, k = 0, 1, \ldots, 2n-2$. Here, $\boldsymbol{c}$ is referred to as the linear convolution of $\boldsymbol{a}$ and $\boldsymbol{b}$. Consider $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]/(\phi(x))$. One can first compute $\boldsymbol{c}' = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]$, then $\boldsymbol{c} = \boldsymbol{c}' \bmod \phi(x)$.
- **Cyclic convolution.** Consider $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]/(x^n - 1)$, then $\boldsymbol{c} = \sum_{k=0}^{n-1} c_k x^k$, where $c_k = \sum_{i=0}^{k} a_i b_{k-i} + \sum_{i=k+1}^{n-1} a_i b_{k+n-i} \bmod q, k = 0, 1, \ldots, n-1$. And $\boldsymbol{c}$ is referred to as the cyclic convolution (CC for short)[1] of $\boldsymbol{a}$ and $\boldsymbol{b}$.
- **Negative wrapped convolution.** Consider $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]/(x^n + 1)$, then $\boldsymbol{c} = \sum_{k=0}^{n-1} c_k x^k$, where $c_k = \sum_{i=0}^{k} a_i b_{k-i} - \sum_{i=k+1}^{n-1} a_i b_{k+n-i} \bmod q, k = 0, 1, \ldots, n-1$. Here, $\boldsymbol{c}$ is referred to as their negative wrapped convolution (NWC for short)[2].

3 NUMBER THEORETIC TRANSFORM (NTT)

In this section, we will introduce some basic concepts of number theoretic transform (NTT). NTT is the special case of discrete Fourier transform (DFT) over a finite field [Pol71, AB75].

3.1 Cyclic Convolution-based NTT

Here we introduce cyclic convolution-based NTT (CC-based NTT for short). n-point CC-based NTT has two parameters: the length or the point n, and the modulus q, where n is a power of two and q is a prime number satisfying $q \equiv 1 \pmod n$. It implies that the primitive n-th root of unity ω_n in $\mathbb{Z}_q$ exists. The forward transform, denoted by NTT, is defined as: $\hat{\boldsymbol{a}} = \mathsf{NTT}(\boldsymbol{a})$, where

$$\hat{a}_j = \sum_{i=0}^{n-1} a_i \omega_n^{ij} \bmod q, j = 0, 1, \ldots, n-1. \tag{1}$$

The inverse transform, denoted by INTT, is defined as $\boldsymbol{a} = \mathsf{INTT}(\hat{\boldsymbol{a}})$, where

$$a_i = n^{-1} \sum_{j=0}^{n-1} \hat{a}_j \omega_n^{-ij} \bmod q, i = 0, 1, \ldots, n-1. \tag{2}$$

Note that the inverse transform can be implemented by replacing the ω_n in NTT procedure with ω_n^{-1}, followed by multiplying by a scale factor n^{-1}. NTT and DFT share the same formula and similar properties, except that DFT has complex twiddle factors $\exp(-2\pi i/n)$, while NTT uses integer primitive root of unity ω_n. Some properties of NTT and INTT are listed as follows.

Proposition 3.1. *It always holds that $\boldsymbol{a} = \mathsf{INTT}(\mathsf{NTT}(\boldsymbol{a}))$.*

[1] It is sometimes referred to as positive wrapped convolution.

[2] It is sometimes referred to as negacyclic convolution, but here we refer to it as negative wrapped convolution to clearly distinguish it from cyclic convolution.

Proposition 3.2 (Cyclic convolution property [SZS80]). *Let $\boldsymbol{c}$ be the cyclic convolution of $\boldsymbol{a}$ and $\boldsymbol{b}$, then it holds that*

$$\mathsf{NTT}(\boldsymbol{c}) = \mathsf{NTT}\left(\boldsymbol{a}\right) \circ \mathsf{NTT}(\boldsymbol{b}).$$

3.2 Negative Wrapped Convolution-based NTT

Here we introduce negative wrapped convolution-based NTT (NWC-based NTT for short). Moreover, the modulus q is set to be a prime number satisfying $q \equiv 1 \pmod{2n}$ such that the primitive $2n$-th root of unity ψ_{2n} in $\mathbb{Z}_q$ exits. Take $\omega_n = \psi_{2n}^2 \bmod q$, and write $\boldsymbol{\psi} = (1, \psi_{2n}, \psi_{2n}^2, \ldots, \psi_{2n}^{n-1})$, $\boldsymbol{\psi}^{-1} = (1, \psi_{2n}^{-1}, \psi_{2n}^{-2}, \ldots, \psi_{2n}^{-(n-1)})$. Define $\bar{\boldsymbol{a}} = \boldsymbol{\psi} \circ \boldsymbol{a}$, where $\bar{a}_i = \psi_{2n}^i a_i$ in detail, which implies $\boldsymbol{a} = \boldsymbol{\psi}^{-1} \circ \bar{\boldsymbol{a}}$, where $a_i = \psi_{2n}^{-i} \bar{a}_i$. n-point NWC-based NTT is to integrate $\boldsymbol{\psi}$ (resp., $\boldsymbol{\psi}^{-1}$) into NTT (resp., INTT), and denote them by NTT^{ψ} (resp., $\mathsf{INTT}^{\psi^{-1}}$), that is

$$\hat{\boldsymbol{a}} = \mathsf{NTT}^{\psi}(\boldsymbol{a}) = \mathsf{NTT}\left(\boldsymbol{\psi} \circ \boldsymbol{a}\right), \tag{3}$$

$$\boldsymbol{a} = \mathsf{INTT}^{\psi^{-1}}(\hat{\boldsymbol{a}}) = \boldsymbol{\psi}^{-1} \circ \mathsf{INTT}(\hat{\boldsymbol{a}}). \tag{4}$$

More specifically, the forward transform $\hat{\boldsymbol{a}} = \mathsf{NTT}^{\psi}(\boldsymbol{a})$ can be written as:

$$\hat{a}_j = \sum_{i=0}^{n-1} a_i \psi_{2n}^i \omega_n^{ij} \bmod q, j = 0, 1, \ldots, n-1. \tag{5}$$

The inverse transform $\boldsymbol{a} = \mathsf{INTT}^{\psi^{-1}}(\hat{\boldsymbol{a}})$ can be written as:

$$a_i = n^{-1} \psi_{2n}^{-i} \sum_{j=0}^{n-1} \hat{a}_j \omega_n^{-ij} \bmod q, i = 0, 1, \ldots, n-1. \tag{6}$$

Some properties of NTT^{ψ} and $\mathsf{INTT}^{\psi^{-1}}$ are listed as follows.

Proposition 3.3. *It always holds that $\boldsymbol{a} = \mathsf{INTT}^{\psi^{-1}}(\mathsf{NTT}^{\psi}(\boldsymbol{a}))$.*

Proposition 3.4 (Negative wrapped convolution property [CG99]). *Let $\boldsymbol{c}$ be the negative wrapped convolution of $\boldsymbol{a}$ and $\boldsymbol{b}$, then it holds that*

$$\mathsf{NTT}^{\psi}(\boldsymbol{c}) = \mathsf{NTT}^{\psi}\left(\boldsymbol{a}\right) \circ \mathsf{NTT}^{\psi}(\boldsymbol{b}).$$

3.3 NTT-based Polynomial Multiplication

NTT can be used to compute {linear, cyclic, negative wrapped} convolutions [Win96], which are equivalent to corresponding polynomial multiplications.

- **Linear convolution-based polynomial multiplication.** To compute the linear convolution $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]$, first, pad them to the length of $2n$ with zeros, resulting with $\boldsymbol{a}' = (a_0, \ldots, a_{n-1}, 0, \ldots, 0)$ and $\boldsymbol{b}' = (b_0, \ldots, b_{n-1}, 0, \ldots, 0)$. Second, use $2n$-point NTT/INTT for $\boldsymbol{c} = \mathsf{INTT}(\mathsf{NTT}(\boldsymbol{a}') \circ \mathsf{NTT}(\boldsymbol{b}'))$. Moreover, to compute $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]/(\phi(x))$, one can compute $\boldsymbol{c}' = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]$ with $2n$-point NTT/INTT, followed by computing $\boldsymbol{c} = \boldsymbol{c}' \bmod \phi(x)$.
- **Cyclic convolution-based polynomial multiplication.** To compute the cyclic convolution $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]/(x^n - 1)$, one can straightly use n-point NTT/INTT, according to cyclic convolution property:

$$\boldsymbol{c} = \mathsf{INTT}\left(\mathsf{NTT}\left(\boldsymbol{a}\right) \circ \mathsf{NTT}(\boldsymbol{b})\right). \tag{7}$$

- **Negative wrapped convolution-based polynomial multiplication.** To compute the negative wrapped convolution $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]/(x^n + 1)$, one can use the negative wrapped convolution property:

$$\boldsymbol{c} = \mathsf{INTT}^{\psi^{-1}}\left(\mathsf{NTT}^{\psi}\left(\boldsymbol{a}\right) \circ \mathsf{NTT}^{\psi}(\boldsymbol{b})\right). \tag{8}$$

3.4 Complexity

The complexity of directly-computing NTT/INTT is $O(n^2)$. There are two NTTs, one point-wise multiplication and one INTT for NTT-based multiplication. Therefore, the complexity of NTT-based multiplication without fast algorithms is $O(n^2)$.

3.5 Advantages of NTT

Here, let NTT/INTT be any kind of forward/inverse transforms.

- Firstly, both NTT and INTT are linear transformations, based on which it can save INTTs in lattice-based schemes (e.g., Kyber [BDK⁺18, ABD⁺20] and Dilithium [DKL⁺18, BDK⁺20]), i.e.,

$$\sum_{i=0}^{l} \boldsymbol{a}_i \boldsymbol{b}_i = \sum_{i=0}^{l} \mathrm{INTT}\left(\mathrm{NTT}\left(\boldsymbol{a}_i\right) \circ \mathrm{NTT}(\boldsymbol{b}_i)\right) = \mathrm{INTT}\Big(\sum_{i=0}^{l} \mathrm{NTT}(\boldsymbol{a}_i) \circ \mathrm{NTT}(\boldsymbol{b}_i)\Big).$$

- Additionally, consider $\boldsymbol{c} = \mathrm{INTT}(\mathrm{NTT}(\boldsymbol{a}) \circ \mathrm{NTT}(\boldsymbol{b}))$, where $\boldsymbol{a}$ is random. Since the NTT transforms keep the randomness of a random polynomial, i.e., $\hat{\boldsymbol{a}} = \mathrm{NTT}(\boldsymbol{a})$ is also random, one can directly generate a random polynomial, and view it as random $\hat{\boldsymbol{a}}$ already in the NTT domain, and compute $\boldsymbol{c} = \mathrm{INTT}(\hat{\boldsymbol{a}} \circ \mathrm{NTT}(\boldsymbol{b}))$, thus eliminating the need for the forward transform.
- Besides, NTT and INTT preserve the dimension and bit length of all individual coefficients of a polynomial, i.e., $\hat{\boldsymbol{a}}$ and $\boldsymbol{a}$ share the same dimension and bit length of any coefficient. Thus, $\hat{\boldsymbol{a}}$ can be stored where $\boldsymbol{a}$ is originally placed.
- Finally, in some case where $\boldsymbol{a}$ involves in multiple multiplications, $\hat{\boldsymbol{a}}$ is computed once and stored for its use in subsequent multiplications, which can save forward transforms without any extra requirement of storage.

4 SOME TRICKS FOR POLYNOMIAL MULTIPLICATIONS

In this subsection, we will show some useful tricks about polynomial multiplications.

Definition 4.1 (One-iteration Karatsuba algorithm [WP06]). Let a, b, c, d be any numbers or polynomials. Briefly speaking, one-iteration Karatsuba algorithm implies that, to compute $t_1 = a \cdot c$, $t_2 = a \cdot d + b \cdot c$ and $t_3 = b \cdot d$, first compute t_1 and t_3, and then compute t_2 by $t_2 = (a + b) \cdot (c + d) - t_1 - t_3$. One-iteration Karatsuba algorithm saves one multiplication at the cost of three extra additions (subtractions).

Definition 4.2 (Good's trick [Ber01, Goo51]). As for the ring $\mathbb{Z}_q[x]/(x^{h\cdot 2^k} - 1)$ where h is an odd number, Good's trick maps $\mathbb{Z}_q[x]/(x^{h\cdot 2^k} - 1)$ to $(\mathbb{Z}_q[z]/(z^{2^k} - 1))[y]/(y^h - 1)$, where $\boldsymbol{a} = \sum_{l=0}^{h\cdot 2^k - 1} a_l x^l$ is mapped to $\sum_{l=0}^{h\cdot 2^k-1} a_l y^{(l \bmod h)} z^{(l \bmod 2^k)} = \sum_{i=0}^{h-1} \sum_{j=0}^{2^k-1} \tilde{a}_{i,j} y^i z^j$. Write the coefficients $\tilde{a}_{ij}$ into a matrix $\tilde{A} = (\tilde{a}_{i,j})_{h\times 2^k}$, and do h parallel 2^k-point NTT over $\mathbb{Z}_q[z]/(z^{2^k} - 1)$ with each row. The corresponding point-wise multiplications are 2^k parallel degree-$(h - 1)$ polynomial multiplications in the ring $\mathbb{Z}_q[y]/(y^h - 1)$ with each column from $\tilde{A}$. Then we do h parallel 2^k-point INTT with each row. Denote the resulting matrix by $\tilde{C} = (\tilde{c}_{i,j})_{h\times 2^k}$. Map $\sum_{i=0}^{h-1} \sum_{j=0}^{2^k-1} \tilde{c}_{i,j} y^i z^j$ back to $\boldsymbol{c} = \sum_{l=0}^{h\cdot 2^k-1} c_l x^l \in \mathbb{Z}_q[x]/(x^{h\cdot 2^k} - 1)$ according to the CRT formula $l = ((2^k)^{-1} \bmod h) \cdot 2^k \cdot i + (h^{-1} \bmod 2^k) \cdot h \cdot j \bmod h \cdot 2^k$ to obtain $c_l x^l$ from $\tilde{c}_{i,j} y^i z^j$.

Definition 4.3 (Schönhage's trick [Ber01, Sch77]). Map the multiplicand $\boldsymbol{a} = \sum_{i=0}^{2mn-1} a_i x^i \in \mathbb{Z}_q[x]/(x^{2mn} - 1)$ to $\sum_{j=0}^{2n-1} (\sum_{i=0}^{m-1} a_{m\cdot j+i} x^i) y^j \in (\mathbb{Z}_q[x][y]/(y^{2n} - 1))/(x^m - y)$ with $y = x^m$. To compute multiplication in $(\mathbb{Z}_q[x][y]/(y^{2n} - 1))/(x^m - y)$, one can first compute that in $\mathbb{Z}_q[x][y]/(y^{2n} - 1)$, and then obtain the result modulo $(x^m - y)$. And to compute multiplication in $\mathbb{Z}_q[x]$ with multiplicands of degree less than m, we can do it in $\mathbb{Z}_q[x]/(x^{2m} + 1)$ without modulo $(x^{2m} + 1)$. Therefore, multiplication in $\mathbb{Z}_q[x][y]/(y^{2n} - 1)$ can be computed in $(\mathbb{Z}_q[x]/(x^{2m} + 1))[y]/(y^{2n} - 1)$. Note that it is an NTT-friendly ring and x is the primitive $4m$-th root of unity in $\mathbb{Z}_q[x]/(x^{2m} + 1)$.

Definition 4.4 (Nussbaumer's trick [Ber01, Nus80]). Nussbaumer's trick is similar to Schönhage's trick. It maps $\boldsymbol{a} = \sum_{i=0}^{2mn-1} a_i x^i \in \mathbb{Z}_q[x]/(x^{2mn} + 1)$ to $\sum_{i=0}^{m-1} (\sum_{j=0}^{2n-1} a_{m\cdot j+i} y^j) x^i \in (\mathbb{Z}_q[y]/(y^{2n} + 1))[x]/(x^m - y)$ with $y = x^m$. To compute multiplication in $(\mathbb{Z}_q[y]/(y^{2n} + 1))[x]/(x^m - y)$, first we compute multiplication in $(\mathbb{Z}_q[y]/(y^{2n} + 1))[x]$, and then obtain the result modulo $(x^m - y)$. And to compute multiplication in $(\mathbb{Z}_q[y]/(y^{2n} + 1))[x]$ with multiplicands of degree less than n, one can do it in $(\mathbb{Z}_q[y]/(y^{2n} + 1))[x]/(x^{2n} - 1)$ for $n \geq m$ without modulo $(x^{2n} - 1)$. Note that it is an NTT-friendly ring and y is the primitive $4n$-th root of unity in $\mathbb{Z}_q[y]/(y^{2n} + 1)$.

5 BASIC RADIX-2 FAST NUMBER THEORETIC TRANSFORM

In this section, we will introduce the basic radix-2 fast number theoretic transform algorithms (radix-2 NTT for short), which are generalization of fast Fourier transform algorithms over a finite field. Here, "radix-2" means the length n of NTT has a

描述 NTT 的兩種角度：
1. FFT
2. CRT (Chinese Remainder Theorem)
>> CRT （本文主要採取的角度）其實是 FFT 的 algebraic view

factor as a power of two, resulting that original algorithm can be divided into two parts of less length. During the practical computing process, there exists multiple different radix-2 NTT algorithms, among which we mainly focus on those widely used in lattice-based cryptographic schemes. All the radix-2 NTT algorithms in this section are summarized in Table 1 and their signal flows ($n = 8$) are shown in Appendix A. To describe NTT algorithms, there are two optional ways. One is from FFT perspectives, such as [CT65, GS66, RVM+14, ZYC+20]. The other is from algebraical perspectives by Chinese Remainder Theorem, such as [Ber01, CHK+21, ACC+22]. In fact, they are equivalent and the latter is the algebraic view of the former. We mainly follow algebraical perspectives in this section, since based on it we can describe recent improvements of NTT more succinctly. Interested readers can learn more details about FFT perspectives in Appendix D, or the works [CG99, Ber01, CT65, GS66, RVM+14, ZYC+20].

Table 1. Radix-2 fast NTT algorithms

Transforms	Cooley-Tukey algorithm	Gentleman-Sande algorithm
NTT	$\mathrm{NTT}^{CT}_{bo\to no}$, $\mathrm{NTT}^{CT}_{no\to bo}$	$\mathrm{NTT}^{GS}_{bo\to no}$, $\mathrm{NTT}^{GS}_{no\to bo}$
INTT	$\mathrm{INTT}^{CT}_{bo\to no}$, $\mathrm{INTT}^{CT}_{no\to bo}$	$\mathrm{INTT}^{GS}_{bo\to no}$, $\mathrm{INTT}^{GS}_{no\to bo}$
NTT^{ψ}	$\mathrm{NTT}^{CT,\psi}_{bo\to no}$, $\mathrm{NTT}^{CT,\psi}_{no\to bo}$	
$\mathrm{INTT}^{\psi^{-1}}$		$\mathrm{INTT}^{GS,\psi^{-1}}_{bo\to no}$, $\mathrm{INTT}^{GS,\psi^{-1}}_{no\to bo}$

5.1 FFT Trick

Bernstein [Ber01] summarized and generalized FFT techniques from algebraical perspectives and used Chinese Remainder Theorem in ring form (or CRT for short, see Theorem 5.1) to describe FFT techniques that is referred to as FFT trick.

Theorem 5.1 (Chinese Remainder Theorem in ring form [Ber01]). *Let R be a commutative ring with multiplicative identity, $I_1, I_2, \ldots, I_k$ be ideals in R that are pairwise co-prime, and I be their intersection. Then there is a ring isomorphism:*

$$\Phi : R/I \cong R/I_1 \times R/I_2 \times \cdots \times R/I_k. \tag{9}$$

In the work [Ber01], FFT trick means that according to Theorem 5.1, for polynomial rings $\mathbb{Z}_q[x]/(x^{2m} - \omega^2)$ where $m > 0$ and invertible $\omega \in \mathbb{Z}_q$, we have the following isomorphism:

$$\begin{aligned} \Phi : \mathbb{Z}_q[x]/(x^{2m} - \omega^2) \ &\cong \ \mathbb{Z}_q[x]/(x^m - \omega) \times \mathbb{Z}_q[x]/(x^m + \omega) \\ \boldsymbol{a} &\mapsto \left(\boldsymbol{a}' = \boldsymbol{a} \bmod x^m - \omega, \boldsymbol{a}'' = \boldsymbol{a} \bmod x^m + \omega\right) \end{aligned}$$

and the detailed mapping process:

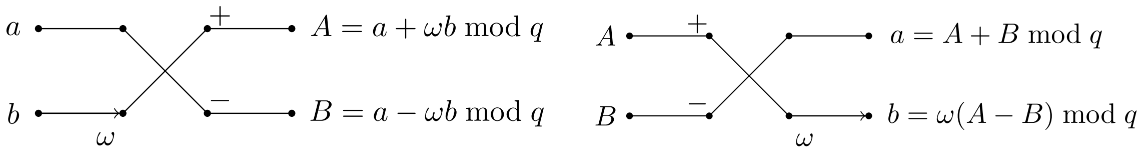
$$\Phi\left(\sum_{i=0}^{2m-1} a_i x^i\right) = \left(\sum_{i=0}^{m-1} (a_i + \omega \cdot a_{i+m}) x^i, \sum_{i=0}^{m-1} (a_i - \omega \cdot a_{i+m}) x^i\right) \tag{10}$$

$$\Phi^{-1}\left(\sum_{i=0}^{m-1} a_i' x^i, \sum_{i=0}^{m-1} a_i'' x^i\right) = \sum_{i=0}^{m-1} \frac{1}{2}(a_i' + a_i'') x^i + \sum_{i=0}^{m-1} \frac{\omega^{-1}}{2}(a_i' - a_i'') x^{i+m}. \tag{11}$$

As for the forward FFT trick (see formula (10)), it is very effective to compute $\boldsymbol{a}'$ and $\boldsymbol{a}'$. Their i-th coefficient can be computed via $a_i' = a_i + \omega \cdot a_{i+m}, a_i'' = a_i - \omega \cdot a_{i+m}$, where $\omega \cdot a_{i+m}$ is computed once but used twice, $i = 0, 1, \ldots, m-1$. This type of operation is known as Cooley-Tukey butterfly or CT butterfly for short, which is illustrated in Figure 1(a). It is the algorithm that first proposed by Cooley and Tukey [CT65]. The forward FFT trick totally takes m multiplications, m additions and m subtractions.

As for the inverse FFT trick (see formula (11)), the i-th and $(i + \frac{n}{2})$-th coefficient of $\boldsymbol{a}$ can be derived from the i-th coefficient of $\boldsymbol{a}'$ and $\boldsymbol{a}''$. The process is detailed below: $a_i = (a_i' + a_i'')/2, a_{i+m} = \omega^{-1}(a_i' - a_i'')/2, i = 0, 1, \ldots, m-1$. In the practical applications, the scale factor 2 can be omitted, with multiplying a total factor in the end (more details will be given below). This type of operation is known as Gentlemen-Sande butterfly or GS butterfly for short (see Figure 1(b)). It was first proposed by Gentlemen and Sande [GS66].

Based on FFT trick, we will introduce the fast algorithms to compute NTT and INTT over $\mathbb{Z}_q[x]/(x^n - 1)$, as well as NTT^{ψ} and $\mathrm{INTT}^{\psi^{-1}}$ over $\mathbb{Z}_q[x]/(x^n + 1)$ in the subsequent two subsections.


a
b
ω
$+$
$-$
$A = a + \omega b \bmod q$
$B = a - \omega b \bmod q$
A
B
$+$
$-$
ω
$a = A + B \bmod q$
$b = \omega(A - B) \bmod q$

(a) Cooley-Tukey butterfly

(b) Gentleman-Sande butterfly

Fig. 1. Cooley-Tukey/Gentleman-Sande butterfly

5.2 FFT Trick for CC-based NTT over $\mathbb{Z}_q[x]/(x^n - 1)$

The work [Ber01] introduced how to use FFT trick to compute NTT and INTT over $\mathbb{Z}_q[x]/(x^n - 1)$, where n is a power of two and q is a prime number satisfying $q \equiv 1 \pmod n$. Denote by ω_n the primitive n-th root of unity in $\mathbb{Z}_q$. Here is the fast algorithm to compute NTT by using the forward FFT trick. It implies that for $\mathbb{Z}_q[x]/(x^n - 1)$, we first have the following isomorphism:

$$\mathbb{Z}_q[x]/(x^n - 1) \cong \mathbb{Z}_q[x]/(x^{\frac{n}{2}} - 1) \times \mathbb{Z}_q[x]/(x^{\frac{n}{2}} + 1)$$

Notice that the forward FFT trick can be applied repeatedly to map $\mathbb{Z}_q[x]/(x^{\frac{n}{2}} \pm 1)$, according to the fact $x^{\frac{n}{2}} + 1 = x^{\frac{n}{2}} - \omega_n^{\frac{n}{2}}$. In fact, $x^n - 1$ has n distinct roots in $\mathbb{Z}_q$, i.e., $\omega_n^i, i = 0, 1, \ldots, n-1$. Therefore, forward FFT trick can be applied recursively from $\mathbb{Z}_q[x]/(x^n - 1)$ all the way down to linear terms with the detailed mapping process of formula (10), which can be described via CRT mapping as:

$$\mathbb{Z}_q[x]/(x^n - 1) \cong \prod_{i=0}^{n-1} \mathbb{Z}_q[x]/(x - \omega_n^{\mathrm{brv}_n(i)}). \tag{12}$$

Finally, $\boldsymbol{a}$ generates its images in $\mathbb{Z}_q[x]/(x - \omega_n^{\mathrm{brv}_n(i)})$, i.e., $\hat{a}_{\mathrm{brv}_n(i)}$, which turns out to be the coefficient of $\hat{\boldsymbol{a}}$ indexed by $\mathrm{brv}_n(i)$. Notice that using Cooley-Tukey algorithm, the coefficients of the input polynomials are indexed under natural order, while the coefficients of the output polynomials are indexed under bit-reversed order. In this paper, we follow the notations as used in [POG15] which denotes this Cooley-Tukey NTT algorithm by $\mathrm{NTT}^{CT}_{no\to bo}$ where the subscripts $no \to bo$ indicates the input coefficients are under natural order and output coefficients are under bit-reversed order. The signal flow of $\mathrm{NTT}^{CT}_{no\to bo}$ for $n = 8$ can be seen in Figure 6(c) in Appendix A. Adjust the input to bit-reversed order, then the Cooley-Tukey butterflies in the signal flow is changed elsewhere, as in Figure 6(a). The output will be under natural order. This type of NTT is denoted by $\mathrm{NTT}^{CT}_{bo\to no}$. Figure 2 shows the detailed process of using FFT trick to map $\mathbb{Z}_q[x]/(x^n - 1)$. The mapping process takes on the shape of a binary tree, with the root node being the 0-th level and the leaf nodes being the $(\log n)$-th level. After the k-th level, $0 \le k < \log n$, there are 2^{k+1} nodes.

The fast algorithm to compute INTT can be obtained by iteratively inverting the CRT mappings mentioned above with formula (11). In this case, the coefficients of the input polynomials are indexed under bit-reversed order, i.e., $\hat{a}_{\mathrm{brv}_n(i)}, i = 0, 1, \ldots, n-1$. Apply the inverse FFT trick (see formula (11)) to the computation from the $(k+1)$-th level to the k-th level, where $1 \le k < \log n$. Note that the scale factor 2 in each level of Gentleman-Sande butterfly can be omitted, with multiplying the final result by n^{-1} in the end. The coefficients of the output polynomials are indexed under natural order. This type of Gentlemen-Sande INTT algorithm is denoted by $\mathrm{INTT}^{GS}_{bo\to no}$ (see Figure 7(d) for $n = 8$). Adjust its input/output order and we can obtain $\mathrm{INTT}^{GS}_{no\to bo}$ (see Figure 7(b)).

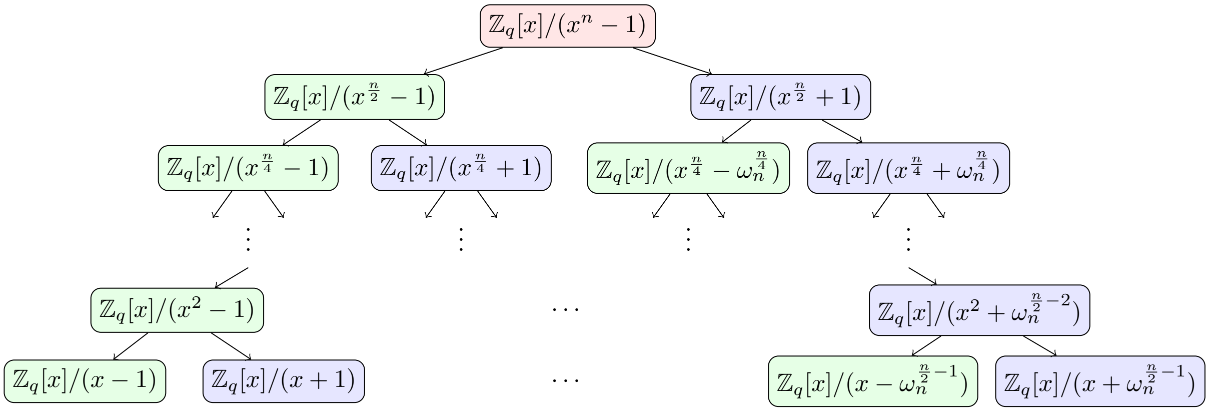

$\mathbb{Z}_q[x]/(x^n - 1)$
$\mathbb{Z}_q[x]/(x^{\frac{n}{2}} - 1)$
$\mathbb{Z}_q[x]/(x^{\frac{n}{2}} + 1)$
$\mathbb{Z}_q[x]/(x^{\frac{n}{4}} - 1)$
$\mathbb{Z}_q[x]/(x^{\frac{n}{4}} + 1)$
$\mathbb{Z}_q[x]/(x^{\frac{n}{4}} - \omega_n^{\frac{n}{4}})$
$\mathbb{Z}_q[x]/(x^{\frac{n}{4}} + \omega_n^{\frac{n}{4}})$
$\mathbb{Z}_q[x]/(x^2 - 1)$
$\mathbb{Z}_q[x]/(x^2 + \omega_n^{\frac{n}{2}-2})$
$\mathbb{Z}_q[x]/(x - 1)$
$\mathbb{Z}_q[x]/(x + 1)$
$\mathbb{Z}_q[x]/(x - \omega_n^{\frac{n}{2}-1})$
$\mathbb{Z}_q[x]/(x + \omega_n^{\frac{n}{2}-1})$

Fig. 2. CRT map of FFT trick over $\mathbb{Z}_q[x]/(x^n - 1)$

5.3 FFT Trick for NWC-based NTT over $\mathbb{Z}_q[x]/(x^n + 1)$

The work [Sei18] proposed the way to use FFT trick to compute NTT^{ψ} and $\mathsf{INTT}^{\psi^{-1}}$ over $\mathbb{Z}_q[x]/(x^n + 1)$, where n is a power of two and q is a prime number satisfying $q \equiv 1 \pmod{2n}$. Since $\psi_{2n}^{n} = -1 \bmod q$, it holds that $x^n + 1 = x^n - \psi_{2n}^{n} = (x^{\frac{n}{2}} - \psi_{2n}^{\frac{n}{2}})(x^{\frac{n}{2}} + \psi_{2n}^{\frac{n}{2}}) \bmod q$. As for NTT^{ψ}, the forward FFT trick implies that we have the following isomorphism:

$$\mathbb{Z}_q[x]/(x^n + 1) \cong \mathbb{Z}_q[x]/(x^{\frac{n}{2}} - \psi_{2n}^{\frac{n}{2}}) \times \mathbb{Z}_q[x]/(x^{\frac{n}{2}} + \psi_{2n}^{\frac{n}{2}})$$

FFT trick can be applied repeatedly. Notice that $x^n + 1$ has n distinct roots in $\mathbb{Z}_q$, i.e., $\psi_{2n}^{2i+1}, i = 0, 1, \ldots, n-1$. Therefore, there is a CRT isomorphism similarly.

$$\mathbb{Z}_q[x]/(x^n + 1) \cong \prod_{i=0}^{n-1} \mathbb{Z}_q[x]/(x - \psi_{2n}^{2\mathrm{brv}_n(i)+1}). \tag{13}$$

Figure 3 shows the detailed process of using FFT trick to map $\mathbb{Z}_q[x]/(x^n + 1)$. After the k-th level, where $0 \le k < \log n$, it produces $\mathbb{Z}_q[x]/(x^{n/2^{k+1}} \pm \psi_{2n}^{\mathrm{brv}_n(2^k+i)}), i = 0, 1, \ldots, 2^k - 1$ with pairs of rings. Such fast algorithm of NTT^{ψ} is denoted by $\mathsf{NTT}_{no\to bo}^{CT,\psi}$ (see Figure 8(c)). Similarly, by adjusting its input/output order, we can get $\mathsf{NTT}_{bo\to no}^{CT,\psi}$ (see Figure 8(a)).

Similarly, the fast algorithm to compute $\mathsf{INTT}^{\psi^{-1}}$ can be obtained by iteratively inverting the CRT mapping process with formula (11). This type of fast algorithm for $\mathsf{INTT}^{\psi^{-1}}$ is denoted by $\mathsf{INTT}_{bo\to no}^{GS,\psi^{-1}}$. Adjust the input/output order and get $\mathsf{INTT}_{no\to bo}^{GS,\psi^{-1}}$. See Figure 8(b) and Figure 8(d). Omitting the scale factor 2 in each level, we can multiply the final result by n^{-1} in the end.

There is an alternative way to deal with the total factor n^{-1}. Zhang et al. [ZYC+20] noticed that n^{-1} can both be integrated into the computing process of each level, based on the fact that the scale factor 2 will be dealt with directly, by using addition and displacement (i.e., ">>") to compute $x/2 \bmod q$. When x is even, $x/2 \equiv (x >> 1) \bmod q$. When x is odd, $x/2 \equiv (x >> 1) + (q+1)/2 \bmod q$.

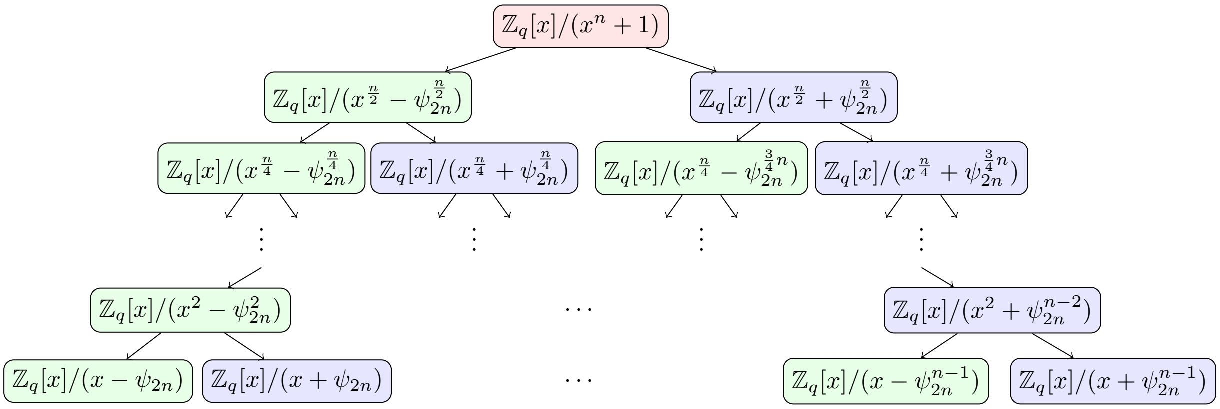

$\mathbb{Z}_q[x]/(x^n+1)$
$\mathbb{Z}_q[x]/(x^{\frac{n}{2}}-\psi_{2n}^{\frac{n}{2}})$
$\mathbb{Z}_q[x]/(x^{\frac{n}{2}}+\psi_{2n}^{\frac{n}{2}})$
$\mathbb{Z}_q[x]/(x^{\frac{n}{4}}-\psi_{2n}^{\frac{n}{4}})$
$\mathbb{Z}_q[x]/(x^{\frac{n}{4}}+\psi_{2n}^{\frac{n}{4}})$
$\mathbb{Z}_q[x]/(x^{\frac{n}{4}}-\psi_{2n}^{\frac{3}{4}n})$
$\mathbb{Z}_q[x]/(x^{\frac{n}{4}}+\psi_{2n}^{\frac{3}{4}n})$
$\mathbb{Z}_q[x]/(x^2-\psi_{2n}^{2})$
$\mathbb{Z}_q[x]/(x^2+\psi_{2n}^{n-2})$
$\mathbb{Z}_q[x]/(x-\psi_{2n})$
$\mathbb{Z}_q[x]/(x+\psi_{2n})$
$\mathbb{Z}_q[x]/(x-\psi_{2n}^{n-1})$
$\mathbb{Z}_q[x]/(x+\psi_{2n}^{n-1})$

Fig. 3. CRT map of FFT trick over $\mathbb{Z}_q[x]/(x^n+1)$

5.4 Twisted FFT Trick

The work [Ber01] also summarized a variant of FFT trick, referred to as twisted FFT trick. It shows that Gentleman-Sande algorithm can be applied to compute NTT, and Cooley-Tukey algorithm can be applied to compute INTT. It means, mapping $\mathbb{Z}_q[x]/(x^n-1) \cong \mathbb{Z}_q[x]/(x^{\frac{n}{2}}-1) \times \mathbb{Z}_q[x]/(x^{\frac{n}{2}}+1)$ by CRT, followed by mapping $\mathbb{Z}_q[x]/(x^{\frac{n}{2}}+1)$ via following isomorphism:

$$\begin{aligned} \Psi : \mathbb{Z}_q[x]/(x^{\frac{n}{2}}+1) \ &\cong\ \mathbb{Z}_q[x]/(x^{\frac{n}{2}}-1) \\ x \ &\mapsto\ \omega_n x \end{aligned} \tag{14}$$

Thus, as for $\mathbb{Z}_q[x]/(x^n-1)$, there is

$$\begin{aligned} \Psi\Phi : \mathbb{Z}_q[x]/(x^n-1) \ &\cong\ \mathbb{Z}_q[x]/(x^{\frac{n}{2}}-1) \times \mathbb{Z}_q[x]/(x^{\frac{n}{2}}-1) \\ \boldsymbol{a} \ &\mapsto\ (\boldsymbol{a}',\ \boldsymbol{a}'') \end{aligned} \tag{15}$$

and the detailed functioning process:

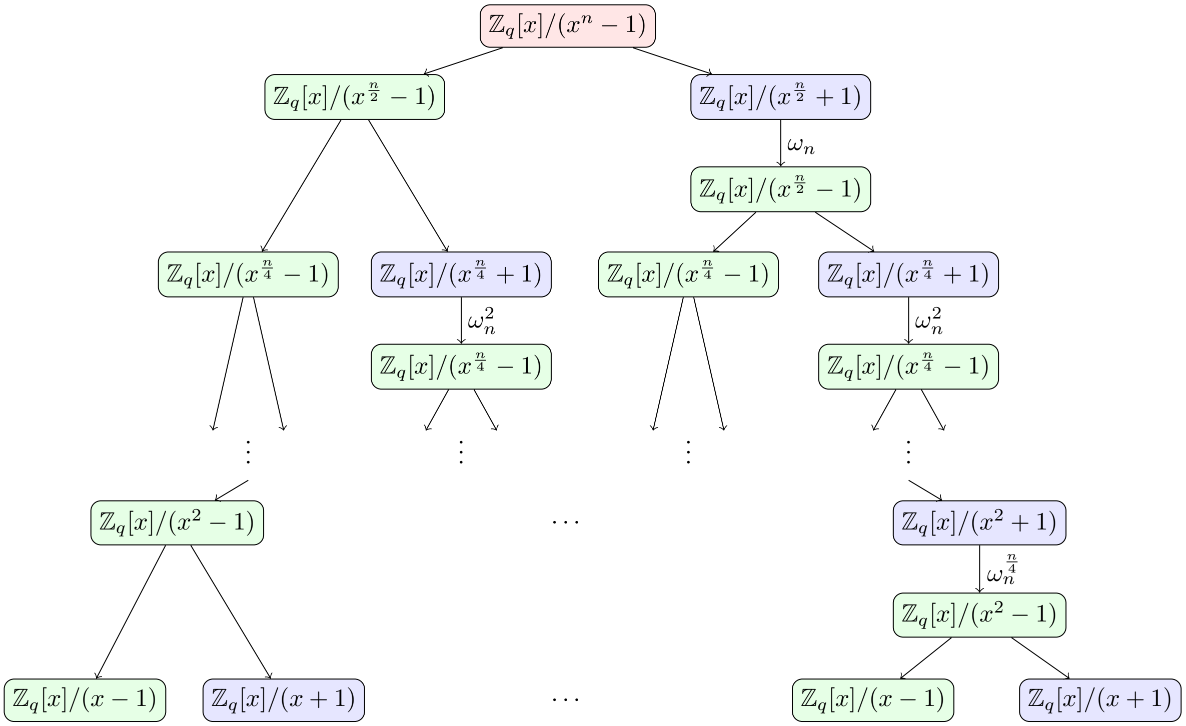
$$(\Psi\Phi)\left(\sum_{i=0}^{n-1} a_i x^i\right) = \left(\sum_{i=0}^{\frac{n}{2}-1} (a_i + a_{i+\frac{n}{2}})x^i,\ \sum_{i=0}^{\frac{n}{2}-1} \omega_n^i \cdot (a_i - a_{i+\frac{n}{2}})x^i\right) \tag{16}$$

$$(\Psi\Phi)^{-1}\left(\sum_{i=0}^{\frac{n}{2}-1} a_i' x^i,\ \sum_{i=0}^{\frac{n}{2}-1} a_i'' x^i\right) = \sum_{i=0}^{\frac{n}{2}-1} \frac{a_i' + \omega_n^{-i} \cdot a_i''}{2} x^i + \sum_{i=0}^{\frac{n}{2}-1} \frac{a_i' - \omega_n^{-i} \cdot a_i''}{2} x^{i+\frac{n}{2}}. \tag{17}$$

As for the forward twisted FFT trick, for example, $\boldsymbol{a} \in \mathbb{Z}_q[x]/(x^n-1)$ in the 0-th level generates $\boldsymbol{a}'$ and $\boldsymbol{a}''$, where $a_i' = a_i + a_{i+\frac{n}{2}}, a_i'' = \omega_n^i \cdot (a_i - a_{i+\frac{n}{2}})$, $i = 0, 1, \ldots, n/2-1$, where Gentleman-Sande algorithm are used. It can be applied repeatedly to map $\mathbb{Z}_q[x]/(x^{\frac{n}{2}}-1)$, and down to linear terms such as $\mathbb{Z}_q[x]/(x \mp 1)$. Specifically, in the k-th level, the similar isomorphism $\Psi : x \mapsto \omega_n^{2^{k-1}} x$ is applied from $\mathbb{Z}_q[x]/(x^{n/2^k}+1)$ to $\mathbb{Z}_q[x]/(x^{n/2^k}-1)$, $1 \le k < \log n$. The complete process of twisted FFT trick on mapping $\mathbb{Z}_q[x]/(x^n-1)$ is shown in Figure 4. Such GS NTT algorithm is denoted by $\text{NTT}^{GS}_{no\to bo}$. Adjust the input/output order, and we obtain $\text{NTT}^{GS}_{bo\to no}$. See Figure 6(d) and Figure 6(b).

The inverse twisted FFT trick is computed in much the same way by iteratively inverting the above process, which is specified in formula (17). For example, the process of computing $\boldsymbol{a}$ in the 0-th level from $\boldsymbol{a}'$ and $\boldsymbol{a}''$ in the first level with Cooley-Tukey butterflies is as follows: $a_i = (a_i' + \omega_n^{-i} \cdot a_i'')/2, a_{i+\frac{n}{2}} = (a_i' - \omega_n^{-i} \cdot a_i'')/2$, $i = 0, 1, \ldots, n/2-1$. Such computing from the $(k+1)$-th level to the k-th level can be achieved in the same way, where $1 \le k < \log n$. The scale factor 2 in each level can be omitted, with multiplying the final result by n^{-1} in the end. We denote this type of CT INTT by $\text{INTT}^{CT}_{bo\to no}$. Adjust its input/output order and the new transform is denoted by $\text{INTT}^{CT}_{no\to bo}$. See Figure 7(a) and Figure 7(c).

Notice that there only exists fast algorithm based on Cooley-Tukey butterfly for NTT^{ψ} , and that based on Gentleman-Sande butterfly for $\mathsf{INTT}^{\psi^{-1}}$. This is because, once we use Gentleman-Sande butterfly to compute NTT^{ψ} or use Cooley-Tukey butterfly to compute $\mathsf{INTT}^{\psi^{-1}}$, the ψ_{2n} term can not be further processed.



$\mathbb{Z}_q[x]/(x^n-1)$
$\mathbb{Z}_q[x]/(x^{\frac{n}{2}}-1)$
$\mathbb{Z}_q[x]/(x^{\frac{n}{2}}+1)$
ω_n
$\mathbb{Z}_q[x]/(x^{\frac{n}{2}}-1)$
$\mathbb{Z}_q[x]/(x^{\frac{n}{4}}-1)$
$\mathbb{Z}_q[x]/(x^{\frac{n}{4}}+1)$
$\mathbb{Z}_q[x]/(x^{\frac{n}{4}}-1)$
$\mathbb{Z}_q[x]/(x^{\frac{n}{4}}+1)$
ω_n^2
ω_n^2
$\mathbb{Z}_q[x]/(x^{\frac{n}{4}}-1)$
$\mathbb{Z}_q[x]/(x^{\frac{n}{4}}-1)$
$\mathbb{Z}_q[x]/(x^2-1)$
$\mathbb{Z}_q[x]/(x^2+1)$
$\omega_n^{\frac{n}{4}}$
$\mathbb{Z}_q[x]/(x^2-1)$
$\mathbb{Z}_q[x]/(x-1)$
$\mathbb{Z}_q[x]/(x+1)$
$\mathbb{Z}_q[x]/(x-1)$
$\mathbb{Z}_q[x]/(x+1)$

Fig. 4. CRT map of twisted FFT trick over $\mathbb{Z}_q[x]/(x^n-1)$

5.5 In-place Operation, Reordering and Complexity

5.5.1 In-place operation. One can see from Figure 1(a) - 1(b), Cooley-Tukey butterfly and Gentleman-Sande butterfly store the input and output data in the same address before and after the computing process, i.e., read data from some storage address for the computation, where the computing results are stored. This kind of operation is referred to as in-place operation. Obviously, there is no need of extra storage for in-place operation. In-place operation can be applied to all the radix-2 NTTs/INTTs as in Figure 6(a) - 8(d).

5.5.2 Reordering. The input and output order of the polynomial coefficients have to be taken into consideration. Although the coefficients of the practical input polynomial are under natural order, the output after $\mathsf{NTT}^{CT}_{no\to bo}$/$\mathsf{NTT}^{GS}_{no\to bo}$ will end under bit-reversed order, which is the required order for input of $\mathsf{INTT}^{CT}_{bo\to no}$/$\mathsf{INTT}^{GS}_{bo\to no}$, but not the required one for input of $\mathsf{INTT}^{CT}_{no\to bo}$/$\mathsf{INTT}^{GS}_{no\to bo}$. In this case, extra reordering is needed from bit-reversed order to natural order. Besides, if NTT is conducted via $\mathsf{NTT}^{CT}_{bo\to no}$ and $\mathsf{NTT}^{GS}_{bo\to no}$, the input polynomial is supposed to be reordered from natural order to bit-reversed order. Similarly, there is a requirement on reordering the output polynomial of $\mathsf{INTT}^{CT}_{no\to bo}$ and $\mathsf{INTT}^{GS}_{no\to bo}$. In a word, there is a way for cyclic convolution-based polynomial multiplication without reordering, i.e., for $\ddagger, \natural \in \{\text{CT,GS}\}$:

$$\boldsymbol{c} = \mathsf{INTT}^{\ddagger}_{bo\to no}\left(\mathsf{NTT}^{\natural}_{no\to bo}(\boldsymbol{a}) \circ \mathsf{NTT}^{\natural}_{no\to bo}(\boldsymbol{b})\right). \tag{18}$$

Similarly, there is a way for NWC-based polynomial multiplication without extra reordering:

$$\boldsymbol{c} = \mathsf{INTT}^{GS,\psi^{-1}}_{bo\to no}\left(\mathsf{NTT}^{CT,\psi}_{no\to bo}(\boldsymbol{a}) \circ \mathsf{NTT}^{CT,\psi}_{no\to bo}(\boldsymbol{b})\right). \tag{19}$$

5.5.3 Complexity. The complexities of NTT/INTT are given in Table 2. One can learn from Figure 1(a) that, each Cooley-Tukey butterfly consumes one multiplication and two additions (subtractions), where ωb is computed once and can be used twice. Similar analysis can also be applied to Gentleman-Sande butterfly. All the fast algorithms consist of $\log n$ levels, where there are $\frac{n}{2}$ butterfly operations on each level. As for the inverse transforms, their complexities require extra n multiplications because of dealing with the scale factor n^{-1}. All the complexity of the polynomial multiplication based on these NTT fast algorithms is $O(n \log n)$, which has a significant advantage over that of polynomial multiplication based on directly-computing NTT/INTT, or any other polynomial multiplication algorithms such as the schoolbook algorithm and Karatsuba/Toom-Cook algorithm.

Table 2. Multiplication complexities of NTT algorithms. $\ddagger, \natural \in \{\text{CT}, \text{GS}\}$. $\boldsymbol{\psi}_{bo}$ and $\boldsymbol{\psi}_{bo}^{-1}$ mean that the coefficients are under bit-reversed order. n is the length of NTT.

NTT algorithms	Multiplication complexities
$\mathsf{NTT}, \mathsf{INTT}, \mathsf{NTT}^{\psi}, \mathsf{INTT}^{\psi^{-1}}$	$O(n^2)$
$\mathsf{NTT}^{\natural}_{no\to bo}, \mathsf{NTT}^{\natural}_{bo\to no}, \mathsf{NTT}^{CT,\psi}_{no\to bo}, \mathsf{NTT}^{CT,\psi}_{bo\to no}$	$\frac{1}{2} n \log n$
$\mathsf{INTT}^{\ddagger}_{no\to bo}, \mathsf{INTT}^{\ddagger}_{bo\to no}, \mathsf{INTT}^{GS,\psi^{-1}}_{no\to bo}, \mathsf{INTT}^{GS,\psi^{-1}}_{bo\to no}$	$\frac{1}{2} n \log n + n$
$\mathsf{NTT}^{\natural}_{no\to bo} \circ \boldsymbol{\psi}, \mathsf{NTT}^{\natural}_{bo\to no} \circ \boldsymbol{\psi}_{bo}$	$\frac{1}{2} n \log n + n$
$\boldsymbol{\psi}_{bo}^{-1} \circ \mathsf{INTT}^{\ddagger}_{no\to bo}, \boldsymbol{\psi}^{-1} \circ \mathsf{INTT}^{\ddagger}_{bo\to no}$	$\frac{1}{2} n \log n + 2n$

6 METHODS TO WEAKEN RESTRICTIONS ON PARAMETER CONDITIONS OF NTT

The full CC-based NTT requires that the parameter n is a power of two and q is prime satisfying $q \equiv 1 \pmod{n}$, while the full NWC-based NTT requires that the parameter n is a power of two and q is prime satisfying $q \equiv 1 \pmod{2n}$. Traditionally, NTT puts some restrictions on its parameters. In recent years, many research efforts are made for NTT's restrictions on parameters and a series of methods have been proposed to weaken them.

In this section, we mainly introduce the recent advances of weakening parameter restrictions with respect to $\mathbb{Z}_q[x]/(x^n \pm 1)$ where n is a power of two. Those methods can be mainly classified into the following three categories:

- Method based on incomplete FFT trick;
- Method based on splitting polynomial ring;
- Method based on large modulus.

The first two methods are applied for the case that the modulus q is an NTT-friendly prime of the form $q = q' \cdot 2^e + 1$ but q can not lead to a full NTT. The last method is applied for the case that the modulus q is an NTT-unfriendly prime. The further classification can be found in Figure 5.

6.1 Method Based on Incomplete FFT trick

We mainly introduce the method based on incomplete FFT trick over $\mathbb{Z}_q[x]/(x^n + 1)$ where n is a power of two and q is NTT-friendly prime but does not satisfy $q \equiv 1 \pmod{2n}$. Then we extend the ring to $\mathbb{Z}_q[x]/(x^n - 1)$ where n is a power of two and q is NTT-friendly prime but does not satisfy $q \equiv 1 \pmod{n}$.

6.1.1 Method based on incomplete FFT trick over $\mathbb{Z}_q[x]/(x^n + 1)$. Fully-mapping FFT trick means to map $\mathbb{Z}_q[x]/(x^n + 1)$ down to linear terms, e.g., $\mathbb{Z}_q[x]/(x - \psi_{2n}^{2i+1})$. See Figure 3. The condition $q \equiv 1 \pmod{2n}$ is required such that the primitive $2n$-th root of unity ψ_{2n} exits. Moenck [Moe76] noticed that FFT trick does not have to map down to linear terms, and one can stop its mapping before the last β level, $\beta = 0, 1, \ldots, \log n - 1$. Applying his method to FFT, Moenck named it mixed-basis FFT multiplications algorithm. Some recent researches introduced it to NTT [ABD+20, LSS+20, ABC19, LS19, CHK+21]. In detail, its CRT map of $\mathbb{Z}_q[x]/(x^n + 1)$ is as follows:

$$\mathbb{Z}_q[x]/(x^n + 1) \cong \prod_{i=0}^{n/2^{\beta}-1} \mathbb{Z}_q[x]/(x^{2^{\beta}} - \psi_{2n/2^{\beta}}^{2\mathrm{brv}_{n/2^{\beta}}(i)+1}).$$

It is referred to as "Incomplete NTT" in [CHK+21] or "Truncated-NTT" in [LSS+20]. The reason is that its CRT tree map is obtained by cropping the last β levels from fully-mapping ($\log n$)-level FFT trick tree map in Figure 3. Note that after forward

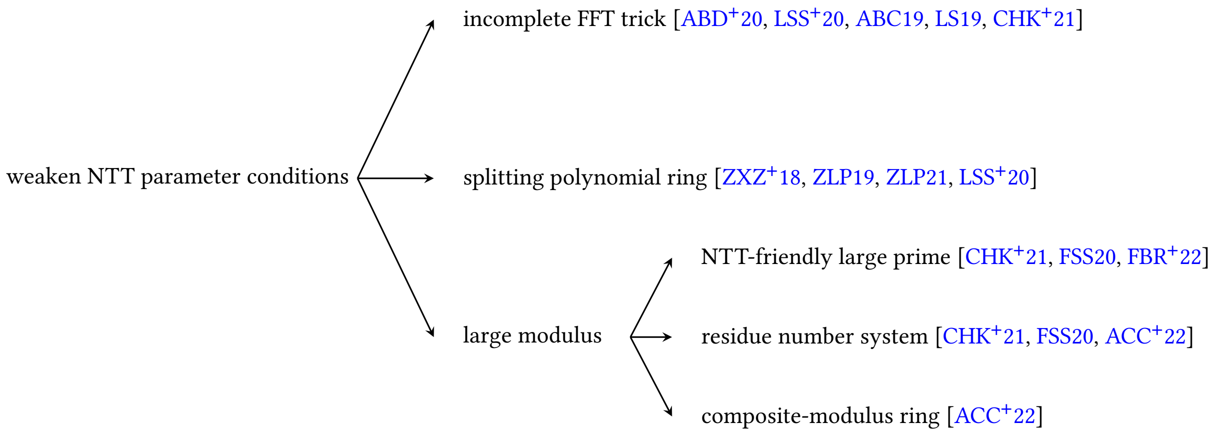

weaken NTT parameter conditions
incomplete FFT trick [ABD+20, LSS+20, ABC19, LS19, CHK+21]
splitting polynomial ring [ZXZ+18, ZLP19, ZLP21, LSS+20]
large modulus
NTT-friendly large prime [CHK+21, FSS20, FBR+22]
residue number system [CHK+21, FSS20, ACC+22]
composite-modulus ring [ACC+22]

Fig. 5. Overview of methods to weaken NTT parameter conditions

transforms, $\boldsymbol{a}$'s images in $\mathbb{Z}_q[x]/(x^{2^\beta} - \psi_{2n/2^\beta}^{2\mathrm{brv}_{n/2^\beta}(i)+1})$ are degree-$(2^\beta - 1)$ polynomials. The point-wise multiplication is performed about the corresponding degree-$(2^\beta - 1)$ polynomials in $\mathbb{Z}_q[x]/(x^{2^\beta} - \psi_{2n/2^\beta}^{2\mathrm{brv}_{n/2^\beta}(i)+1})$. As for the inverse transforms, the scale factor 2 is omitted in every level, followed by multiplying by a total scalar $(n/2^\beta)^{-1}$ in the end.

The forward/inverse transforms with β levels cropped are denoted by $\mathsf{NTT}^{CT,\psi}_{no\to bo,\beta}/\mathsf{INTT}^{GS,\psi^{-1}}_{bo\to no,\beta}$ respectively, where $\beta = 0, 1, \ldots, \log n - 1$. Obviously, they are exactly $\mathsf{NTT}^{CT,\psi}_{no\to bo}$ and $\mathsf{INTT}^{GS,\psi^{-1}}_{bo\to no}$ if $\beta = 0$. The restriction on n and q can be weakened to $q \equiv 1 \pmod{\frac{2n}{2^\beta}}$. The way to compute polynomial multiplication is the general form of formula (19), that is

$$\boldsymbol{c} = \mathsf{INTT}^{GS,\psi^{-1}}_{bo\to no,\beta}\left(\mathsf{NTT}^{CT,\psi}_{no\to bo,\beta}(\boldsymbol{a}) \circ \mathsf{NTT}^{CT,\psi}_{no\to bo,\beta}(\boldsymbol{b})\right). \tag{20}$$

A high-level description of incomplete FFT trick over $\mathbb{Z}_q[x]/(x^n+1)$. Here, we propose a high-level description of incomplete FFT trick. We map $\mathbb{Z}_q[x]/(x^n+1)$ to $\left(\mathbb{Z}_q[x]/(x^{2^\beta} - y)\right)[y]/(y^{\frac{n}{2^\beta}} + 1)$, along with rewriting $\boldsymbol{a} \in \mathbb{Z}_q[x]/(x^n+1)$ as $\boldsymbol{a} = \sum_{i=0}^{\frac{n}{2^\beta}-1} \tilde{\boldsymbol{a}}_i y^i$, where $y = x^{2^\beta}$ and $\tilde{\boldsymbol{a}}_i = \sum_{j=0}^{2^\beta-1} a_{2^\beta \cdot i+j} x^j \in \mathbb{Z}_q[x]/(x^{2^\beta} - y)$. Thus, $\boldsymbol{a}$ can be seen as a polynomial of degree $(\frac{n}{2^\beta} - 1)$ with respect to y. FFT trick will map

$$\left(\mathbb{Z}_q[x]/(x^{2^\beta} - y)\right)[y]/(y^{\frac{n}{2^\beta}} + 1) \cong \prod_{i=0}^{n/2^\beta - 1} \left(\mathbb{Z}_q[x]/(x^{2^\beta} - y)\right)[y]/(y - \psi_{2n/2^\beta}^{2\mathrm{brv}_{n/2^\beta}(i)+1}).$$

Its forward transform (resp., inverse transform) is treated as radix-2 $\frac{n}{2^\beta}$-point full NWC-based $\mathsf{NTT}^{CT,\psi}_{no\to bo}$ (resp., $\mathsf{INTT}^{GS,\psi^{-1}}_{bo\to no}$) with respect to y. And the point-wise multiplication is performed in $\left(\mathbb{Z}_q[x]/(x^{2^\beta} - y)\right)[y]/(y - \psi_{2n/2^\beta}^{2\mathrm{brv}_{n/2^\beta}(i)+1})$.

6.1.2 *Method based on incomplete FFT trick over $\mathbb{Z}_q[x]/(x^n-1)$.*

We follow the concepts from section 6.1.1. The method based on incomplete FFT trick over $\mathbb{Z}_q[x]/(x^n-1)$ achieves its CRT map as follows:

$$\mathbb{Z}_q[x]/(x^n - 1) \cong \prod_{i=0}^{n/2^\beta - 1} \mathbb{Z}_q[x]/(x^{2^\beta} - \omega_{n/2^\beta}^{\mathrm{brv}_{n/2^\beta}(i)}).$$

where $\beta = 0, 1, \ldots, \log n - 1$. Similarly, denote by $\mathsf{NTT}^{CT}_{no\to bo,\beta}$ and $\mathsf{INTT}^{GS}_{bo\to no,\beta}$ the forward and the inverse transform. The restriction on q can be weakened to $q \equiv 1 \pmod{\frac{n}{2^\beta}}$. Its way to compute $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]/(x^n-1)$ is

$$\boldsymbol{c} = \mathsf{INTT}^{GS}_{bo\to no,\beta}\left(\mathsf{NTT}^{CT}_{no\to bo,\beta}(\boldsymbol{a}) \circ \mathsf{NTT}^{CT}_{no\to bo,\beta}(\boldsymbol{b})\right).$$

Its high-level description can be written as: mapping $\mathbb{Z}_q[x]/(x^n-1)$ to $\left(\mathbb{Z}_q[x]/(x^{2^\beta}-y)\right)[y]/(y^{\frac{n}{2^\beta}}-1)$, followed by CRT isomorphism:

$$\left(\mathbb{Z}_q[x]/(x^{2^\beta}-y)\right)[y]/(y^{\frac{n}{2^\beta}}-1) \cong \prod_{i=0}^{n/2^\beta-1}\left(\mathbb{Z}_q[x]/(x^{2^\beta}-y)\right)[y]/(y-\omega_{n/2^\beta}^{\mathrm{brv}_{n/2^\beta}(i)}).$$

6.2 Method Based on Splitting Polynomial Ring

We found that the computing strategies of Pt-NTT [ZXZ+18], K-NTT [ZLP19, ZLP21] and H-NTT [LSS+20] are similarly dependent on splitting initial polynomial ring, based on which we classify them into the category named the **method based on splitting polynomial ring**. And their basic idea can be traced back to Nussbaumer's trick [Ber01, Nus80] (see Definition 4.4). Following the description of Nussbaumer's trick, they can essentially be described by the following isomorphism. Let α be a non-negative integer.

$$\begin{aligned}\Psi_\alpha: \mathbb{Z}_q[x]/(x^n\pm1) &\cong \left(\mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}}\pm1)\right)[x]/(x^{2^\alpha}-y)\\ \boldsymbol{a}=\sum_{i=0}^{n-1}a_ix^i &\mapsto \Psi_\alpha(\boldsymbol{a})=\sum_{i=0}^{2^\alpha-1}\left(\sum_{j=0}^{\frac{n}{2^\alpha}-1}a_{2^\alpha\cdot j+i}y^j\right)x^i\end{aligned} \tag{21}$$

Obviously, the isomorphism Ψ_α and its inverse Ψ_α^{-1} only perform simple reordering of the polynomial coefficients. Ψ_α is the identity mapping if $\alpha=0$. Briefly speaking, the general form of α-round method based on splitting polynomial ring to compute $\boldsymbol{c}=\boldsymbol{a}\cdot\boldsymbol{b}\in\mathbb{Z}_q[x]/(x^n\pm1)$ mainly contains the following three steps, where n is a power of two and q is a prime number (more details about q can be seen below).

- **Step 1, Splitting.** The polynomials $\boldsymbol{a}$ and $\boldsymbol{b}$ are split by Ψ_α into:

$$\Psi_\alpha(\boldsymbol{a})=\sum_{i=0}^{2^\alpha-1}\tilde{\boldsymbol{a}}_i\cdot x^i, \Psi_\alpha(\boldsymbol{b})=\sum_{i=0}^{2^\alpha-1}\tilde{\boldsymbol{b}}_i\cdot x^i\in\left(\mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}}\pm1)\right)[x]/(x^{2^\alpha}-y),$$

  where $y=x^{2^\alpha}$, and

$$\tilde{\boldsymbol{a}}_i=\sum_{j=0}^{\frac{n}{2^\alpha}-1}a_{2^\alpha\cdot j+i}y^j, \tilde{\boldsymbol{b}}_i=\sum_{j=0}^{\frac{n}{2^\alpha}-1}b_{2^\alpha\cdot j+i}y^j\in\mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}}\pm1).$$

- **Step 2, Multiplication.** The product of $\Psi_\alpha(\boldsymbol{a})$ and $\Psi_\alpha(\boldsymbol{b})$ is obtained by $(\sum_{i=0}^{2^\alpha-1}\tilde{\boldsymbol{a}}_i\cdot x^i)(\sum_{i=0}^{2^\alpha-1}\tilde{\boldsymbol{b}}_i\cdot x^i) \bmod x^{2^\alpha}-y$, which means that one need to compute $\tilde{\boldsymbol{c}}_i\in\mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}}\pm1)$ for $i=0,1,\ldots,2^\alpha-1$ as follows:

$$\tilde{\boldsymbol{c}}_i=\sum_{l=0}^{i}\tilde{\boldsymbol{a}}_l\cdot\tilde{\boldsymbol{b}}_{i-l}+\sum_{l=i+1}^{2^\alpha-1}y\cdot\tilde{\boldsymbol{a}}_l\cdot\tilde{\boldsymbol{b}}_{2^\alpha+i-l}\in\mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}}\pm1). \tag{22}$$

- **Step 3, Gatheration.** Gather all the $\tilde{\boldsymbol{c}}_i$ by Ψ_α^{-1}, and obtain $\boldsymbol{c}=\Psi_\alpha^{-1}\left(\sum_{i=0}^{2^\alpha-1}\tilde{\boldsymbol{c}}_i\cdot x^i\right)$.

Step 1 and Step 3 are simple and easy. Essentially, Ψ_α transforms NTT/INTT over $\mathbb{Z}_q[x]/(x^n\pm1)$ into those over $\mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}}\pm1)$ which requires only $\frac{n}{2^\alpha}$-point NTT/INTT with arbitrary appropriate modulus q, but the point-wise multiplication needs to be adapted to NTT/INTT. There are three variants based on the method based on splitting polynomial ring, including Pt-NTT, K-NTT, H-NTT. The main difference between them is that they use different skills and NTTs to compute $\tilde{\boldsymbol{c}}_i$ in formula (22) of Step 2.

6.2.1 Pt-NTT. Preprocess-then-NTT (Pt-NTT) proposed by Zhou et al. [ZXZ+18] improves formula (22) as follows:

$$\begin{aligned}\tilde{\boldsymbol{c}}_i&=\sum_{l=0}^{i}\tilde{\boldsymbol{a}}_l\cdot\tilde{\boldsymbol{b}}_{i-l}+\sum_{l=i+1}^{2^\alpha-1}\vec{\boldsymbol{a}}_l\cdot\tilde{\boldsymbol{b}}_{2^\alpha+i-l}\\&=\mathrm{INTT}\left(\sum_{l=0}^{i}\mathrm{NTT}(\tilde{\boldsymbol{a}}_l)\circ\mathrm{NTT}(\tilde{\boldsymbol{b}}_{i-l})+\sum_{l=i+1}^{2^\alpha-1}\mathrm{NTT}(\vec{\boldsymbol{a}}_l)\circ\mathrm{NTT}(\tilde{\boldsymbol{b}}_{2^\alpha+i-l})\right),\end{aligned}$$

where "$\circ$" is the corresponding point-wise multiplication, and for $\mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}}+1)$, there is

$$\vec{\boldsymbol{a}}_l = y \cdot \tilde{\boldsymbol{a}}_l = -a_{n-2^\alpha+l} + \sum_{j=0}^{\frac{n}{2^\alpha}-2} a_{2^\alpha \cdot j+l} y^{j+1} \in \mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}}+1),$$

or, for $\mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}}-1)$, there is

$$\vec{\boldsymbol{a}}_l = y \cdot \tilde{\boldsymbol{a}}_l = a_{n-2^\alpha+l} + \sum_{j=0}^{\frac{n}{2^\alpha}-2} a_{2^\alpha \cdot j+l} y^{j+1} \in \mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}}-1).$$

Here, Pt-NTT uses $\frac{n}{2^\alpha}$-point full NWC-based NTT/INTT over $\mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}}+1)$, or $\frac{n}{2^\alpha}$-point full CC-based NTT/INTT over $\mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}}-1)$.

6.2.2 K-NTT. Later, Zhu et al. [ZLP19, ZLP21] proposed Karatsuba-NTT (K-NTT) based on Pt-NTT, equipping with one-iteration Karatsuba algorithm (see Definition 4.1). Its Step 2 is given as:

$$\begin{aligned}\tilde{\boldsymbol{c}}_i &= \sum_{l=0}^{i} \tilde{\boldsymbol{a}}_l \cdot \tilde{\boldsymbol{b}}_{i-l} + \sum_{l=i+1}^{2^\alpha-1} y \cdot \tilde{\boldsymbol{a}}_l \cdot \tilde{\boldsymbol{b}}_{2^\alpha+i-l} \\ &= \mathrm{INTT}\left(\sum_{l=0}^{i} \mathrm{NTT}(\tilde{\boldsymbol{a}}_l) \circ \mathrm{NTT}(\tilde{\boldsymbol{b}}_{i-l}) + \sum_{l=i+1}^{2^\alpha-1} \mathrm{NTT}(y) \circ \mathrm{NTT}(\tilde{\boldsymbol{a}}_l) \circ \mathrm{NTT}(\tilde{\boldsymbol{b}}_{2^\alpha+i-l})\right).\end{aligned}$$

Here, K-NTT uses $\frac{n}{2^\alpha}$-point full NWC-based NTT/INTT over $\mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}}+1)$, or $\frac{n}{2^\alpha}$-point full CC-based NTT/INTT over $\mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}}-1)$. Since y has been known, $\mathrm{NTT}(y)$ can be computed and stored offline in advance. In Step 2, one-iteration Karatsuba algorithm is used in such a manner: First compute and store $\mathrm{NTT}(\tilde{\boldsymbol{a}}_i) \circ \mathrm{NTT}(\tilde{\boldsymbol{b}}_i)$ for any $i=j$, and then we have $\mathrm{NTT}(\tilde{\boldsymbol{a}}_i) \circ \mathrm{NTT}(\tilde{\boldsymbol{b}}_j) + \mathrm{NTT}(\tilde{\boldsymbol{a}}_j) \circ \mathrm{NTT}(\tilde{\boldsymbol{b}}_i) = (\mathrm{NTT}(\tilde{\boldsymbol{a}}_i) + \mathrm{NTT}(\tilde{\boldsymbol{a}}_j)) \circ (\mathrm{NTT}(\tilde{\boldsymbol{b}}_i) + \mathrm{NTT}(\tilde{\boldsymbol{b}}_j)) - \mathrm{NTT}(\tilde{\boldsymbol{a}}_i) \circ \mathrm{NTT}(\tilde{\boldsymbol{b}}_i) - \mathrm{NTT}(\tilde{\boldsymbol{a}}_j) \circ \mathrm{NTT}(\tilde{\boldsymbol{b}}_j)$ for any $i \neq j$.

6.2.3 H-NTT. Furthermore, Liang et al. [LSS+20] improved K-NTT and proposed a new variant of NTT referred to as Hybrid-NTT (H-NTT), by applying truncated-NTT in Step 2 and one-iteration Karatsuba algorithm in its point-wise multiplication. H-NTT uses truncated-NTT with β levels cropped, instead of those full NTT in K-NTT. Its Step 2 is given as:

$$\begin{aligned}\tilde{\boldsymbol{c}}_i &= \sum_{l=0}^{i} \tilde{\boldsymbol{a}}_l \cdot \tilde{\boldsymbol{b}}_{i-l} + \sum_{l=i+1}^{2^\alpha-1} y \cdot \tilde{\boldsymbol{a}}_l \cdot \tilde{\boldsymbol{b}}_{2^\alpha+i-l} \\ &= \mathsf{INTT}_\beta\left(\sum_{l=0}^{i} \mathsf{NTT}_\beta(\tilde{\boldsymbol{a}}_l) \circ \mathsf{NTT}_\beta(\tilde{\boldsymbol{b}}_{i-l}) + \sum_{l=i+1}^{2^\alpha-1} \mathsf{NTT}_\beta(y) \circ \mathsf{NTT}_\beta(\tilde{\boldsymbol{a}}_l) \circ \mathsf{NTT}_\beta(\tilde{\boldsymbol{b}}_{2^\alpha+i-l})\right),\end{aligned}$$

where $\mathsf{NTT}_\beta/\mathsf{INTT}_\beta$ means NWC-based $\mathsf{NTT}^{CT,\psi}_{no\to bo,\beta}/\mathsf{INTT}^{GS,\psi^{-1}}_{bo\to no,\beta}$ over $\mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}}+1)$, or CC-based $\mathsf{NTT}^{CT}_{no\to bo,\beta}/\mathsf{INTT}^{GS}_{bo\to no,\beta}$ over $\mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}}-1)$. One-iteration Karatsuba algorithm is also used in its point-wise multiplication. For example, to compute $(\sum_{i=0}^{2^\beta-1} \hat{a}_i x^i)(\sum_{i=0}^{2^\beta-1} \hat{b}_i x^i) \bmod x^{2^\beta} - \psi$, one can compute $\hat{a}_i \hat{b}_j$ for any $i=j$ first and then compute $\hat{a}_i \hat{b}_j + \hat{a}_j \hat{b}_i = (\hat{a}_i + \hat{a}_j)(\hat{b}_i + \hat{b}_j) - \hat{a}_i \hat{b}_i - \hat{a}_j \hat{b}_j$ for any $i \neq j$.

6.2.4 Comparisons and discussions. Based on the high-level description of incomplete FFT trick, we can see that it shares some similarities with the method based on splitting polynomial ring. In fact, there is an isomorphism between $(\mathbb{Z}_q[x]/(x^{2^\beta}-y))[y]/(y^{\frac{n}{2^\beta}} \pm 1)$ and $(\mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}} \pm 1))[x]/(x^{2^\alpha}-y)$ for any $\alpha = \beta$. And they have been proved computationally equivalent [LSS+20], which implies that their efficiencies are the same theoretically.

These two methods can expand the value range of modulus q, because q can only satisfy $q \equiv 1 \pmod{\frac{2n}{2^{\alpha+\beta}}}$ for some α, β, instead of $q \equiv 1 \pmod{2n}$ when n is fixed, for NWC-based NTT; besides, q can only satisfy $q \equiv 1 \pmod{\frac{n}{2^{\alpha+\beta}}}$ for some α, β, instead of $q \equiv 1 \pmod{n}$ when n is fixed, for CC-based NTT.

However, the limitations on q can not be ignored, since q must be an NTT-friendly prime such that $\mathbb{Z}_q$ is a finite field and $x^n \pm 1$ can be split into polynomials of small degree over $\mathbb{Z}_q$. In addition, as for the method based on splitting polynomial ring, the polynomial operations over $\mathbb{Z}_q[x]/(x^n \pm 1)$ can be transformed into those over a smaller ring $\mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}} \pm 1)$. It leads to more modular implementation, since NTT over $\mathbb{Z}_q[y]/(y^{\frac{n}{2^\alpha}} \pm 1)$ can be implemented as a "black box" (notice that we don't limit the types of NTT here), and be re-called when required in Step 2.

Both Pt-NTT and K-NTT use $\frac{n}{2^\alpha}$-point full NTT in their Step 2. It requires that the primitive $\frac{n}{2^{\alpha-1}}$-th root of unity exits for NWC-based NTT, or the primitive $\frac{n}{2^\alpha}$-th root of unity exits for CC-based NTT. Therefore, the parameter condition of Pt-NTT and K-NTT can be weakened to $q \equiv 1 \pmod{\frac{2n}{2^\alpha}}$ for the initial ring $\mathbb{Z}_q[x]/(x^n+1)$, or $q \equiv 1 \pmod{\frac{n}{2^\alpha}}$ for the initial ring $\mathbb{Z}_q[x]/(x^n-1)$. H-NTT can further weaken the parameter condition to $q \equiv 1 \pmod{\frac{2n}{2^{\alpha+\beta}}}$ for the ring $\mathbb{Z}_q[x]/(x^n+1)$ by using truncated-NTT, or $q \equiv 1 \pmod{\frac{n}{2^{\alpha+\beta}}}$ for the ring $\mathbb{Z}_q[x]/(x^n-1)$. By comparison, it can be seen that α-round K-NTT and truncated-NTT are the special cases of H-NTT when $\beta = 0$ and $\alpha = 0$ respectively. More specially, $\mathsf{NTT}^{CT,\psi}_{no\to bo}$ and $\mathsf{INTT}^{GS,\psi^{-1}}_{bo\to no}$ are the cases of H-NTT over $\mathbb{Z}_q[x]/(x^n+1)$ when $\alpha = 0, \beta = 0$, and $\mathsf{NTT}^{CT}_{no\to bo}$ and $\mathsf{INTT}^{GS}_{bo\to no}$ are the cases of H-NTT over $\mathbb{Z}_q[x]/(x^n-1)$ when $\alpha = 0, \beta = 0$.

6.3 Method Based on Large Modulus

Here we consider $\mathbb{Z}_q[x]/(x^n \pm 1)$, where n is a power of two and the modulus q is an NTT-unfriendly prime, but q can be actually any positive integer. Recent researches [CHK+21, FSS20, FBR+22, ACC+22] indicate that one can use NTTs in this case. In detail, the process of $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]/(x^n \pm 1)$ is divided into two steps.

- **Step 1.** Compute $\boldsymbol{c}' = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_N[x]/(x^n \pm 1)$, where N is a positive integer and larger than the maximum absolute value of the coefficients during the computation over $\mathbb{Z}$.
- **Step 2.** One can recover the result in $\mathbb{Z}_q[x]/(x^n \pm 1)$ through reduction module q, i.e., $\boldsymbol{c} = \boldsymbol{c}' \bmod q$.

It is called the **method based on large modulus** in this paper, because its key step is to select a large enough modulus N in Step 1. Obviously, N could be chosen such that $N > nq^2$. When N is large enough, the product of $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{Z}[x]/(x^n \pm 1)$ is identical to that in $\mathbb{Z}_N[x]/(x^n \pm 1)$. In order to apply NTTs, we consider two sub-cases of the values of N. One is that N is an NTT-friendly prime. The other is that N is the product of some NTT-friendly primes. When N is the product of some NTT-friendly primes, i.e., $N = \prod_{i=1}^{l} q_i$ where each q_i is an NTT-friendly prime, in this case, it can further be classified into two sub-methods. One is the method based on residue number system (RNS). The other is the method based on composite-modulus ring.

6.3.1 Method based on NTT-friendly large prime. The works [CHK+21, FSS20, FBR+22] show that, when N is an NTT-friendly prime, NTT can be performed in $\mathbb{Z}_N[x]/(x^n \pm 1)$ directly in Step 1. Notice that if N is set sufficiently large and NTT-friendly, this method always works regardless of the value of the original modulus q.

For example, if N is a prime satisfying $N > nq^2$ and $N \equiv 1 \pmod{2n}$, n-point full NWC-based NTT of modulus N can always be used over $\mathbb{Z}_N[x]/(x^n+1)$. Besides, if N is a prime satisfying $N > nq^2$ and $N \equiv 1 \pmod{n}$, n-point full CC-based NTT of modulus N can always be used over $\mathbb{Z}_N[x]/(x^n-1)$.

6.3.2 Method based on residue number system. Residue number system (RNS) is widely used in the context of homomorphic encryption, e.g., [BGV14, Bra12, FV12], for computing NTTs over many primes. The works [CHK+21, FSS20, ACC+22] show that, based on RNS, negative wrapped convolution with a modulus that is the product of some primes, can be transformed into ones with smaller moduli by Chinese Remainder Theorem, that is

$$\mathbb{Z}_N[x]/(x^n \pm 1) \cong \prod_{i=1}^{l} \mathbb{Z}_{q_i}[x]/(x^n \pm 1).$$

Denote by $\boldsymbol{c}_i$ the product of $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{Z}_{q_i}[x]/(x^n \pm 1), i = 1, \ldots, l$. After using NTTs to compute $\boldsymbol{c}_i, i = 1, \ldots, l$, the original product $\boldsymbol{c}$ in $\mathbb{Z}_N[x]/(x^n \pm 1)$ can be recovered from $\boldsymbol{c}_i, i = 1, \ldots, l$, by Chinese Remainder Theorem in number theoretic form [CHK+21].

6.3.3 Method based on composite-modulus ring. The work [ACC+22] shows that NTT can be performed over a polynomial ring with a composite modulus directly. Specifically, the work [ACC+22] generalizes the concepts of NTT from a finite field to an integer ring, the basic idea of which was first developed in terms of FFT by Fürer [Für09].

Consider $\mathbb{Z}_N[x]/(x^n+1)$, where $N=\prod_{i=1}^{l} q_i$ and each q_i is NTT-friendly prime. If $2n|\gcd(q_1-1,\ldots,q_l-1)$, there exits a principal $2n$-th root of unity ψ_{2n} in $\mathbb{Z}_N$. ψ_{2n} is a principle $2n$-th root of unity in $\mathbb{Z}_N$, iff (ψ_{2n} mod q_i) is a principle $2n$-th root of unity in $\mathbb{Z}_{q_i}$ for any i. FFT trick over a composite-modulus polynomial ring is similar to that mentioned in section 5.1, i.e.,

$$\mathbb{Z}_N[x]/(x^n+1) \cong \prod_{i=0}^{n-1} \mathbb{Z}_N[x]/(x-\psi_{2n}^{2\mathrm{brv}_n(i)+1}).$$

Its forward transform and inverse transform are illustrated as $\mathsf{NTT}_{no\to bo}^{CT,\psi}$ and $\mathsf{INTT}_{bo\to no}^{GS,\psi^{-1}}$ over $\mathbb{Z}_N[x]/(x^n+1)$. The CRT map of truncated-NTT with β levels cropped is as follows:

$$\mathbb{Z}_N[x]/(x^n+1) \cong \prod_{i=0}^{n/2^{\beta}-1} \mathbb{Z}_N[x]/(x^{2^{\beta}}-\psi_{2n/2^{\beta}}^{2\mathrm{brv}_{n/2^{\beta}}(i)+1}),$$

where $\psi_{2n/2^{\beta}}$ is a principal $2n/2^{\beta}$-th root of unity in $\mathbb{Z}_N$. Its forward transform and inverse transform are illustrated as $\mathsf{NTT}_{no\to bo,\beta}^{CT,\psi}$ and $\mathsf{INTT}_{bo\to no,\beta}^{GS,\psi^{-1}}$ over $\mathbb{Z}_N[x]/(x^n+1)$, respectively.

As for $\mathbb{Z}_N[x]/(x^n-1)$, where $N=\prod_{i=1}^{l} q_i$ and each q_i is NTT-friendly prime. If $n|\gcd(q_1-1,\ldots,q_l-1)$, there exits a principal n-th root of unity ω_n in $\mathbb{Z}_N$. ω_n is a principle n-th root of unity in $\mathbb{Z}_N$, iff (ω_n mod q_i) is a principle n-th root of unity in $\mathbb{Z}_{q_i}$ for any i. The CRT map of full-mapping FFT trick and truncated-NTT with β levels cropped over $\mathbb{Z}_N[x]/(x^n-1)$ can be derived similarly to those over $\mathbb{Z}_N[x]/(x^n+1)$.

6.3.4 Comparisons and discussions. We compare the method based on large modulus and those based on incomplete FFT trick/splitting polynomial ring. The three sub-methods of the method based on large modulus are valid for any original modulus q (including NTT-friendly ones), and can completely remove the restriction on q. But, their shortcomings are also obvious.

Specifically, the method based on incomplete FFT trick and based on splitting polynomial ring still choose original q as the modulus. But, the modulus N used in the method based on NTT-friendly large prime and method based on composite-modulus ring, is much larger than original modulus q. For examples, N could be chosen as nq^2. Although N be smaller if one of the multiplicands is small, it will still be several orders of magnitude larger than q. It causes that the storage of coefficients and the computing-resource consume will be more than the cases of q during the computation. Besides, as for the method based on residue number system, there are more than one NTT computation needed to be computed. Traditionally, it is more time-consuming and resource-consuming, because full NWC-based NTT, the method based on incomplete FFT trick and the method based on splitting polynomial ring only need one NTT computation. Therefore, if q is an NTT-friendly prime number, the methods based on incomplete FFT trick and based on splitting polynomial ring are recommended strongly for an efficient implementation, instead of the method based on large modulus. But, when q is an NTT-unfriendly prime number, one can turn to the method based on large modulus.

Note that, there are some connections between the method based on residue number system and composite-modulus ring. Here we review $\mathbb{Z}_N[x]/(x^n \pm 1)$. Both of them could choose $N=\prod_{i=1}^{l} q_i$ where each q_i is NTT-friendly prime. If we split N via CRT (Theorem 5.1) and keep $x^n \pm 1$ unchanged, it implies the method based on residue number system. If we split $x^n \pm 1$ via CRT and keep N unchanged, it implies the method based on composite-modulus ring. Therefore, these two methods are derived from different splitting forms of moduli via CRT.

7 CHOOSING NUMBER THEORETIC TRANSFORM FOR GIVEN RINGS

In this section, we will introduce how to choose appropriate NTT for the given polynomial ring. We mainly classify the rings into three categories for the convenience of understanding:

- $\mathbb{Z}_q[x]/(x^n \pm 1)$ with respect to power-of-two n;
- $\mathbb{Z}_q[x]/(x^n \pm 1)$ with respect to non-power-of-two n;
- $\mathbb{Z}_q[x]/(\phi(x))$ with respect to general $\phi(x)$ of degree n.

We mainly focus on the special rings of the form $\mathbb{Z}_q[x]/((x^n \pm 1)$ in the first two categories, and then we extend the results to the general rings of the form $\mathbb{Z}_q[x]/(\phi(x))$ with respect to general $\phi(x)$ of degree n.

7.1 $\mathbb{Z}_q[x]/(x^n \pm 1)$ with Respect to Power-of-two n

7.1.1 NTT-friendly q. In this case, q is an NTT-friendly prime of the form $q = q' \cdot 2^e + 1$. Furthermore, we classify it into two sub-cases. The first sub-case is that q can lead to a full NTT. The another sub-case is that q can not lead to a full NTT.

We first consider the first sub-case of n being a power of two and q leading to a full NTT. As for $\mathbb{Z}_q[x]/(x^n + 1)$, there exits $q \equiv 1 \pmod{2n}$, then full NWC-based NTT (e.g., $\mathsf{NTT}^{CT,\psi}_{no\to bo}$ and $\mathsf{INTT}^{GS,\psi^{-1}}_{bo\to no}$) can be used to multiply two polynomials in $\mathbb{Z}_q[x]/(x^n + 1)$. Restate that one can compute $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]/(x^n + 1)$ by

$$\boldsymbol{c} = \mathsf{INTT}^{GS,\psi^{-1}}_{bo\to no}\left(\mathsf{NTT}^{CT,\psi}_{no\to bo}(\boldsymbol{a}) \circ \mathsf{NTT}^{CT,\psi}_{no\to bo}(\boldsymbol{b})\right).$$

As for $\mathbb{Z}_q[x]/(x^n - 1)$, there exits $q \equiv 1 \pmod{n}$, then full CC-based NTT (e.g., $\mathsf{NTT}^{CT}_{no\to bo}$ and $\mathsf{INTT}^{GS}_{bo\to no}$) can be used to multiply two polynomials in $\mathbb{Z}_q[x]/(x^n - 1)$. Restate that one can compute $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]/(x^n - 1)$ by

$$\boldsymbol{c} = \mathsf{INTT}^{GS}_{bo\to no}\left(\mathsf{NTT}^{CT}_{no\to bo}(\boldsymbol{a}) \circ \mathsf{NTT}^{CT}_{no\to bo}(\boldsymbol{b})\right).$$

We then consider the another sub-case of n being a power of two and q not leading to a full NTT. It means that, for $\mathbb{Z}_q[x]/(x^n+1)$, q does not satisfy $q \equiv 1 \pmod{2n}$, or, for $\mathbb{Z}_q[x]/(x^n - 1)$, q does not satisfy $q \equiv 1 \pmod{n}$. In this sub-case, the method based on incomplete FFT trick and the method splitting polynomial ring are highly recommended.

For example, as for $\mathbb{Z}_q[x]/(x^n + 1)$, if q only satisfies $q \equiv 1 \pmod{\frac{2n}{2^\alpha}}$ (resp., $q \equiv 1 \pmod{\frac{2n}{2^\beta}}$), then one can use α-round method based on splitting polynomial ring (resp., method based on incomplete FFT trick with β levels cropped) can be used to compute $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]/(x^n + 1)$. Besides, as for $\mathbb{Z}_q[x]/(x^n - 1)$, if q only satisfies $q \equiv 1 \pmod{\frac{n}{2^\alpha}}$ (resp., $q \equiv 1 \pmod{\frac{n}{2^\beta}}$), then one can use α-round method based on splitting polynomial ring (resp., method based on incomplete FFT trick with β levels cropped) can be used to compute $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]/(x^n - 1)$.

Furthermore, if q only satisfies $q \equiv 1 \pmod{\frac{2n}{2^{\alpha+\beta}}}$ for $\mathbb{Z}_q[x]/(x^n + 1)$, one can use H-NTT with α-round splitting and β levels cropped; similarly, if q only satisfies $q \equiv 1 \pmod{\frac{n}{2^{\alpha+\beta}}}$ for $\mathbb{Z}_q[x]/(x^n - 1)$, one can use H-NTT with α-round splitting and β levels cropped.

7.1.2 NTT-unfriendly q. In this case, q is an NTT-unfriendly prime. In order to compute $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]/(x^n \pm 1)$, one can use the method based on large modulus. Firstly, compute $\boldsymbol{c}' = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_N[x]/(x^n \pm 1)$, where N is a positive integer and larger than the maximum absolute value of the coefficients during the computation over $\mathbb{Z}$, and then compute $\boldsymbol{c} = \boldsymbol{c}' \bmod q$.

To compute $\boldsymbol{c}' = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_N[x]/(x^n \pm 1)$, one can use the method based on NTT-friendly large prime, the method based on residue number system or the method based on composite-modulus ring, as described in section 6.3.

7.2 $\mathbb{Z}_q[x]/(x^n \pm 1)$ with Respect to Non-power-of-two n

Here we turn to consider the case of non-power-of-two n, but actually n can be a general integer. The works [CHK+21, FBR+22] show that as for radix-2 fast NTT algorithms, there are some useful technical methods. The essential idea about the computation of $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]/(x^n \pm 1)$ is described through two steps as follows.

- **Step 1.** Compute $\boldsymbol{c}' = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]/(x^{n'} \pm 1)$, where n' is a larger integer and $n' \geq 2n$.
- **Step 2.** One can recover the result in $\mathbb{Z}_q[x]/(x^n \pm 1)$ through reduction modulo $x^{n'} \pm 1$, i.e., $\boldsymbol{c} = \boldsymbol{c}' \bmod x^{n'} \pm 1$.

As for the ring $\mathbb{Z}_q[x]/(x^{n'} \pm 1)$, the works [CHK+21, FBR+22] furthermore classify it into two cases. The first case is that n' is a power of two, which is denoted by 2^k. The other case is that n' is of the form $h \cdot 2^k$, where h is an odd number.

7.2.1 Power-of-two n'. As for the first case, the computation is transformed into that with respect to $\boldsymbol{c}' = \boldsymbol{a}\cdot\boldsymbol{b} \in \mathbb{Z}_q[x]/(x^{n'}\pm 1)$, where n' is a power of two. The methods to compute polynomial multiplication over $\mathbb{Z}_q[x]/(x^{n'} \pm 1)$ can be referred to section 7.1.

7.2.2 Use Good's trick. As for the other case, the computation is transformed into that with respect to $\boldsymbol{c}' = \boldsymbol{a}\cdot\boldsymbol{b} \in \mathbb{Z}_q[x]/(x^{n'}\pm 1)$, where $n' = h \cdot 2^k$ and h is an odd number. One can use Good's trick (see Definition 4.2) to compute polynomial multiplication over $\mathbb{Z}_q[x]/(x^{h\cdot 2^k} - 1)$.

In this paper, Good's trick is recommended, since there are more freedom to select an odd number h and power-of-two 2^k. If we expand the length to power-of-two n', it is inconvenient to find a suitable polynomial of some particular degree up to the next power of two.

7.3 $\mathbb{Z}_q[x]/(\phi(x))$ with Respect to General $\phi(x)$ of Degree n

Here we consider the more general ring $\mathbb{Z}_q[x]/(\phi(x))$ where $\phi(x)$ is an arbitrary polynomial of degree n. In order to apply fast NTT algorithm, there are complex but efficient methods. The computation of $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]/(\phi(x))$ can be described through two steps as follows.

- **Step 1.** Compute $\boldsymbol{c}' = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_N[x]/(x^{n'} \pm 1)$, where N is a positive integer and larger than the maximum absolute value of the coefficients during the computation over $\mathbb{Z}$, n' is a larger integer and $n' \geq 2n$.
- **Step 2.** And then compute $\boldsymbol{c} = (\boldsymbol{c}' \bmod \phi(x)) \bmod q$.

Therefore, the computation of $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_q[x]/(\phi(x))$ is transformed into that of $\boldsymbol{c}' = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_N[x]/(x^{n'} \pm 1)$. In this case, one can use the methods in section 7.2 to compute polynomial multiplications over $\mathbb{Z}_N[x]/(x^{n'} \pm 1)$.

Besides, the works [ACC+21, BBCT22] also use Schönhage's trick and Nussbaumer's trick (see Definition 4.3 and 4.4) to compute polynomial multiplications over $\mathbb{Z}_N[x]/(x^{n'} \pm 1)$.

8 NUMBER THEORETIC TRANSFORM IN NIST PQC

In this section, we will introduce NTT and its variants for NIST PQC Round 3 candidates. Their parameter sets can be seen in Table 3. Among the schemes, key encapsulation mechanisms (KEM) include Kyber [BDK+18, ABD+20], Saber [DKRV18, BMD+20], NTRU [CDH+20], NTRU Prime [BCLvV17, BBC+20]. Digital signature schemes include Dilithium [DKL+18, BDK+20] and Falcon [FHK+20]. Among them, Kyber, Dilithium and Falcon are standardized by NIST [NIS22]. All these KEMs are constructed from passive secure (IND-CPA secure or OW-CPA secure) PKEs via tweak variants of Fujisaki-Okamoto transform [FO99, FO13, HHK17].

We mainly focus on their NTT-based implementations over their underlying rings $\mathbb{Z}_q[x]/(\phi(x))$, where $\mathbb{Z}_q[x]/(\phi(x))$ is instantiated as in Table 3. Notice that not all the schemes can directly use full NTT to multiply polynomials. For example, full NWC-based NTT over $\mathbb{Z}_q[x]/(x^n + 1)$ further requires that the prime q satisfies $q \equiv 1 \pmod{2n}$. Only Dilithium and Falcon can meet the situation. Besides, full NTT can not be directly used in those lattice-based schemes with power-of-two moduli, such as Saber and NTRU.

Table 3. Parameter sets of algebraically-structural lattice-based schemes in NIST PQC. Recommended parameter sets of NTRU Prime are given here. Kyber KEM, Dilithium signature and Falcon signature are standardized by NIST [NIS22]. Saber KEM and NTRU KEM were NIST PQC Round 3 finalists. NTRU Prime KEM was a alternate candidate in NIST PQC Round 3.

Schemes		n	q	Rings	Types	Methods & Algorithms
Kyber	Round 1 [ABD+17]	256	7681	$\mathbb{Z}_q[x]/(x^n+1)$	NTT-friendly $q \equiv 1 \pmod{2n}$	n-point full NWC-based NTT [ABD+17, BDK+18]
	Round 2 [ABD+19] Round 3 [ABD+20]	256	3329		NTT-friendly $q \equiv 1 \pmod{n}$	Incomplete FFT trick [ABD+19, ABD+20] Splitting polynomial ring [LSS+20, ZXZ+18]
Dilithium	Round 3 [BDK+20]	256	8380417	$\mathbb{Z}_q[x]/(x^n+1)$	NTT-friendly $q \equiv 1 \pmod{2n}$	n-point full NWC-based NTT [BDK+20]
Falcon	Round 3 [FHK+20]	512 1024	12289	$\mathbb{Z}_q[x]/(x^n+1)$	NTT-friendly $q \equiv 1 \pmod{2n}$	n-point full NWC-based NTT [FHK+20]
Saber	Round 3 [BMD+20]	256	8192	$\mathbb{Z}_q[x]/(x^n+1)$	NTT-unfriendly power-of-two q	Method based on large modulus [CHK+21, FSS20, FBR+22, ACC+22]
NTRU	Round 3 [CDH+20]	509	2048	$\mathbb{Z}_q[x]/(x^n-1)$	NTT-unfriendly prime n	Power-of-two n' + Method based on large modulus [FBR+22] Good's trick (+ Method based on large modulus) [CHK+21]
		677				
		701	8192			
		821	4096			
NTRU Prime	Round 3 [BBC+20]	653	4621	$\mathbb{Z}_q[x]/(x^n-x-1)$	NTT-unfriendly prime n and q	Power-of-two n' + Method based on large modulus [ACC+21] Good's trick (+ Method based on large modulus) [ACC+21, PMT+21] Schönhage's trick + Nussbaumer's trick [BBCT22]
		761	4591			
		857	5167			

8.1 Kyber

Kyber [ABD+17, ABD+19, ABD+20, BDK+18] is an IND-CCA secure MLWE-based KEM from the lattice-based cryptography suite called "Cryptographic Suite for Algebraic Lattices" (CRYSTALS for short). It is currently the only standardized KEM by NIST PQC [NIS22].

The Kyber submission in NIST PQC Round 1, named Kyber Round 1 [ABD+17], uses $n = 256$ and $q = 7681$ which satisfies $q \equiv 1 \pmod{2n}$. Its way to compute $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathcal{R}_q$ in C reference implementation is elaborated as

$$\boldsymbol{c} = \boldsymbol{\psi}^{-1} \circ \mathsf{INTT}^{GS}_{bo \to no}\left(\mathsf{NTT}^{CT,\psi}_{no \to bo}(\boldsymbol{a}) \circ \mathsf{NTT}^{CT,\psi}_{no \to bo}(\boldsymbol{b})\right),$$

where the inverse transform is $\boldsymbol{\psi}^{-1} \circ \mathsf{INTT}^{GS}_{bo \to no}$, the output of which is the same as that of $\mathsf{INTT}^{GS,\psi^{-1}}_{bo \to no}$ according to formula (4). As for its AVX2 optimized implementation, Kyber Round 1 uses $\mathsf{NTT}^{CT,\psi}_{no \to bo}$ and $\mathsf{INTT}^{GS,\psi^{-1}}_{bo \to no}$, and the computation is restated as

$$\boldsymbol{c} = \mathsf{INTT}^{GS,\psi^{-1}}_{bo \to no}\left(\mathsf{NTT}^{CT,\psi}_{no \to bo}(\boldsymbol{a}) \circ \mathsf{NTT}^{CT,\psi}_{no \to bo}(\boldsymbol{b})\right).$$

The Kyber submissions in the second and third round of NIST PQC competition use a smaller prime number $q = 3329$ which no longer satisfies $q \equiv 1 \pmod{2n}$, but $q \equiv 1 \pmod{n}$. Kyber Round 2 and Round 3 use truncated-NTT with one level cropped: $\mathsf{NTT}^{CT,\psi}_{no \to bo,\beta=1}$ and $\mathsf{INTT}^{GS,\psi^{-1}}_{bo \to no,\beta=1}$. The point-wise multiplication is treated as the corresponding polynomial multiplications in $\mathbb{Z}_q[x]/(x^2 - \omega_n^{2\mathrm{brv}_{n/2}(i)+1})$. Its way to compute $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathcal{R}_q$ is restated as

$$\boldsymbol{c} = \mathsf{INTT}^{GS,\psi^{-1}}_{bo \to no,\beta=1}\left(\mathsf{NTT}^{CT,\psi}_{no \to bo,\beta=1}(\boldsymbol{a}) \circ \mathsf{NTT}^{CT,\psi}_{no \to bo,\beta=1}(\boldsymbol{b})\right).$$

Further, Liang et al. [LSS+20] improve its truncated-NTT by using H-NTT with $\alpha = 0$, which can be seen as $\mathsf{NTT}^{CT,\psi}_{no \to bo,\beta=1}$ and $\mathsf{INTT}^{GS,\psi^{-1}}_{bo \to no,\beta=1}$ with one-iteration Karatsuba algorithm in its point-wise multiplication. Actually, the idea of decreasing the modulus q of Kyber Round 1 from $q = 7681$ to $q = 3329$ was first proposed by Zhou et al. [ZXZ+18], and they denoted it by "small-Kyber". Different from truncated-NTT used in Kyber Round 2 and Round 3, they used 1-round Pt-NTT with $\frac{n}{2}$-point full NWC-based NTT for "small-Kyber". But, their implementation had a slightly worser performance than the initial one.

8.2 Dilithium

Dilithium [BDK+20] is a signature scheme based on module lattice and is one of the algorithms from CRYSTALS. It is one of the standardized signature scheme in NIST PQC [NIS22]. Its parameter sets satisfy the condition $q \equiv 1 \pmod{2n}$ such that n-point full NWC-based $\mathsf{NTT}^{CT,\psi}_{no \to bo}$ and $\mathsf{INTT}^{GS,\psi^{-1}}_{bo \to no}$ can be utilized. Its way to compute $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathcal{R}_q$ is restated as

$$\boldsymbol{c} = \mathsf{INTT}^{GS,\psi^{-1}}_{bo \to no}\left(\mathsf{NTT}^{CT,\psi}_{no \to bo}(\boldsymbol{a}) \circ \mathsf{NTT}^{CT,\psi}_{no \to bo}(\boldsymbol{b})\right).$$

8.3 Falcon

Falcon [FHK+20] is a signature scheme based on NTRU lattice. It is the another standardized signature scheme in NIST PQC [NIS22]. Falcon also uses n-point full NWC-based $\mathsf{NTT}^{CT,\psi}_{no \to bo}$ and $\mathsf{INTT}^{GS,\psi^{-1}}_{bo \to no}$ to compute polynomial multiplication in its verification algorithm. Its way to compute $\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathcal{R}_q$ is also written as

$$\boldsymbol{c} = \mathsf{INTT}^{GS,\psi^{-1}}_{bo \to no}\left(\mathsf{NTT}^{CT,\psi}_{no \to bo}(\boldsymbol{a}) \circ \mathsf{NTT}^{CT,\psi}_{no \to bo}(\boldsymbol{b})\right).$$

8.4 Saber

Saber [DKRV18, BMD+20] is an IND-CCA secure KEM based on MLWR, and was one of the Finalists in NIST PQC Round 3. Since the modulus $q = 2^{13}$ chosen by Saber is not a prime number, but a power of two, it does not satisfy the parameter conditions of NTT. The previous works [DKRV18] used Toom-Cook and Karatsuba algorithm, instead of NTT. Recent researches [CHK+21, FSS20, FBR+22, ACC+22] successfully applies NTT in Saber via the method based on large modulus.

The polynomial multiplications in Saber mainly contain matrix-vector polynomial multiplication $\mathbf{As}$ and vector-vector polynomial multiplication $\mathbf{b}^T\mathbf{s}$ where $\mathbf{A} \in \mathcal{R}_q^{k \times k}$, $\mathbf{b}, \mathbf{s} \in \mathcal{R}_q^{k \times 1}$, $|s_i| \leq \mu$ and μ is the parameter of the central binomial distribution.

In [CHK+21, FSS20, FBR+22, ACC+22], the coefficients of $\mathbf{A}$ and $\mathbf{b}$ are represented in $[-q/2, q/2)$, and the coefficients of $\mathbf{s}$ are represented in $[-\mu/2, \mu/2]$. The coefficients obtained by matrix-vector and vector-vector multiplication range in $[-knq\mu/4, knq\mu/4]$. Therefore, the modulus N used in the method based on large modulus, must be chosen carefully such that $N > knq\mu/2$ holds. There are three parameter sets, referred to as LightSaber-KEM, Saber-KEM and FireSaber-KEM, corresponding to $k = 2, 3, 4$ and $\mu = 10, 8, 6$, respectively.

Chung et al. [CHK+21] use different N and NTT methods for their ARM Cortex-M4 and AVX2 implementations in Saber. Specifically, they use the method based on NTT-friendly large prime in ARM Cortex-M4 implementations, and choose $\mathsf{NTT}^{CT,\psi}_{no\to bo,\beta=2}$ and $\mathsf{INTT}^{GS,\psi^{-1}}_{bo\to no,\beta=2}$. LightSaber-KEM chooses $N = 20972417$, while Saber-KEM and Firesaber-KEM choose $N = 25166081$. All of them are NTT-friendly primes and satisfy the condition $N \equiv 1 \pmod{\frac{n}{2}}$. As for their AVX2 implementations, they use the method based on residue number system. $\mathsf{NTT}^{CT,\psi}_{no\to bo}$ and $\mathsf{INTT}^{GS,\psi^{-1}}_{bo\to no}$ are used in all the three parameter sets. The modulus $N = q_1q_2$ is chosen where $q_1 = 7681$ and $q_2 = 10753$ are NTT-friendly primes such that $q_i \equiv 1 \pmod{2n}, i = 1, 2$ holds.

Abdulrahman et al. [ACC+22] use the method based on composite-modulus ring in their ARM Cortex-M3 and Cortex-M4 implementations of Saber. They choose a large composite number $N = q_1q_2$, where $q_1 = 7681$ and $q_2 = 3329$. In order to achieve speed and memory optimized implementation, the NTT method they use $\mathsf{NTT}^{CT,\psi}_{no\to bo,\beta=2}$ and $\mathsf{INTT}^{GS,\psi^{-1}}_{bo\to no,\beta=2}$ over $\mathbb{Z}_N[x]/(x^n+1)$.

8.5 NTRU

NTRU [CDH+20] ia an IND-CCA secure KEMs based on NTRU cryptosystem [HPS98], and was one of the Finalists in NIST PQC Round 3. NTRU actually includes two suit of schemes named NTRU-HRSS and NTRUEncrypt, which two teams independently submitted to the first round of NIST PQC competition. But they merged their schemes, and gave a new name "NTRU" after the first round. NTRU uses the polynomial ring $\mathbb{Z}_q[x]/(x^n-1)$ where n is a prime number and q is a power of two. There are two ways to utilize NTT in NTRU. The first way was applied by Fritzmann et al. [FBR+22]. To allow a modular NTT arithmetic architecture with RISC-V instruction set extensions, they first map $\mathbb{Z}_q[x]/(x^n-1)$ to $\mathbb{Z}_q[x]/(x^{n'}-1)$ where n' is a power of two and $n' \geq 2n$. n' could be set to be 2048, for $n \in \{509, 677, 821, 701\}$. Then they utilize the method based on NTT-friendly large prime over $\mathbb{Z}_q[x]/(x^{n'}-1)$. Specifically, they choose an NTT-friendly sufficiently large prime $N = 549755809793$, such that $N \equiv 1 \pmod{n'}$ holds. Firstly, compute $\boldsymbol{c'} = \boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{Z}_N[x]/(x^{n'}-1)$ by using n'-point full cyclic convolution-based NTT, and then compute $\boldsymbol{c} = (\boldsymbol{c'} \bmod x^n - 1) \bmod q$.

The other way is to use Good's trick in Chung et al.'s work [CHK+21]. As for $n \in \{677, 701\}$, they choose $h = 3$ and $k = 9$ for Good's trick in their ARM Cortex-M4 implementation. For h parallel 2^k-point NTT over $\mathbb{Z}_q[z]/(z^{2^k}-1)$, they apply the method based NTT-friendly large prime, where it maps $\mathbb{Z}_q[z]/(z^{2^k}-1)$ to $\mathbb{Z}_N[z]/(z^{2^k}-1)$, and use $N = 5747201$ and $N = 1389569$ for the cases of $n = 701$ and $n = 677$ respectively.

8.6 NTRU Prime

NTRU Prime [BCLvV17, BBC+20] is an IND-CCA secure KEMs based on NTRU crytosystem [HPS98], and was one of the Alternates in NIST PQC Round 3. It was proposed for the aim "an efficient implementation of high security prime-degree large-Galois-group inert-modulus ideal-lattice-based cryptography" [BCLvV17]. The NTRU Prime submission to the NIST PQC competition [BBC+20] offers two KEMs: Streamlined NTRU Prime and NTRU LPRime. NTRU Prime tweaks the classic NTRU scheme to use rings with less special structures, i.e., $\mathbb{Z}_q[x]/(x^n-x-1)$, where both n and q are primes.

For radix-2 NTT-based implementation of NTRU Prime, one can map $\mathbb{Z}_q[x]/(x^n-x-1)$ to $\mathbb{Z}_N[x]/(x^{n'}-1)$ where $n' \geq 2n$, and choose an NTT-friendly sufficiently large prime N such that $N > 2nq^2$. After computing NTT over $\mathbb{Z}_N[x]/(x^{n'}-1)$, the final result can be obtained module q and x^n-x-1.

Besides, the works [ACC+21, PMT+21] show that Good's trick can also be utilized over $\mathbb{Z}_q[x]/(x^n-x-1)$ in NTRU Prime. It maps $\mathbb{Z}_q[x]/(x^n-x-1)$ to $\mathbb{Z}_N[x]/(x^{n'}-1)$ with $n' = 1536$ for the case of $n = 761$. The work [ACC+21] chooses $N = 6984193$, while [PMT+21] chooses $N = q_1q_2q_3$ where $q_1 = 7681, q_2 = 12289, q_3 = 15361$. As for the parameters in Good's trick, they use $h = 3$ and $k = 9$.

The work [ACC+21] also presented an optional idea that one can manufacture roots of unity for radix-2 NTT in $\mathbb{Z}_q[x]/(x^n-x-1)$ by using Schönhage's trick and Nussbaumer's trick, which was later implemented with an improvement by Bernstein et al. [BBCT22]. More specifically, since the original polynomials of NTRU Prime have degree $n < 1024$, Bernstein et al. [BBCT22] map $\mathbb{Z}_q[x]/(x^n - x - 1)$ to $\mathbb{Z}_q[x]/(x^{2048} - 1)$. They use Schönhage's trick in order to transform the operations to multiplications in $\mathbb{Z}_q[x]/(x^{64} + 1)$. Then, they use Nussbaumer's trick to compute the multiplications in $\mathbb{Z}_q[x]/(x^{64} + 1)$.

9 CONCLUSION AND FUTURE WORK

This paper makes a mathematically systematic study of NTT, including its history, basic concepts, basic radix-2 fast computing algorithms, methods to weaken restrictions on parameter conditions, selections for the given rings and applications in the lattice-based schemes of NIST PQC.

Here are some future works listed as follows.

- Notice that the concepts of fast NTT algorithms and incomplete FFT trick, are first invented in terms of FFT. In most cases, one can learn from skills and techniques of FFT, and apply them into NTT methods. There are still numerous FFT techniques, such as radix-2^l, mixed-radix, split-radix algorithms, Stockham butterfly FFT [CCF67], WFTA [Win76] etc. Therefore, one could choose a best one for the NTT method from the implementation point of view.
- For efficient and secure implementation of NTT, resistance against implementation attacks such as side-channel attacks have been increasingly considered as an important criteria for NIST PQC. Side-channel attack can make use of the information leaked from the target devices to recover some secrets of cryptographic schemes. For example, in order to avoiding timing attack [Koc96], all the operations should be under the strategy of constant implementation. There are some types of attacks for the implementation of NTT, including single trace attack, simple power analysis, fault attack and so on [RPBC20, HPA21, NDR$^+$19]. The correlative strategies against side-channel attack on NTT have been under development. We leave it as a future work of comprehensive survey on implementing NTT against side-channel attacks.

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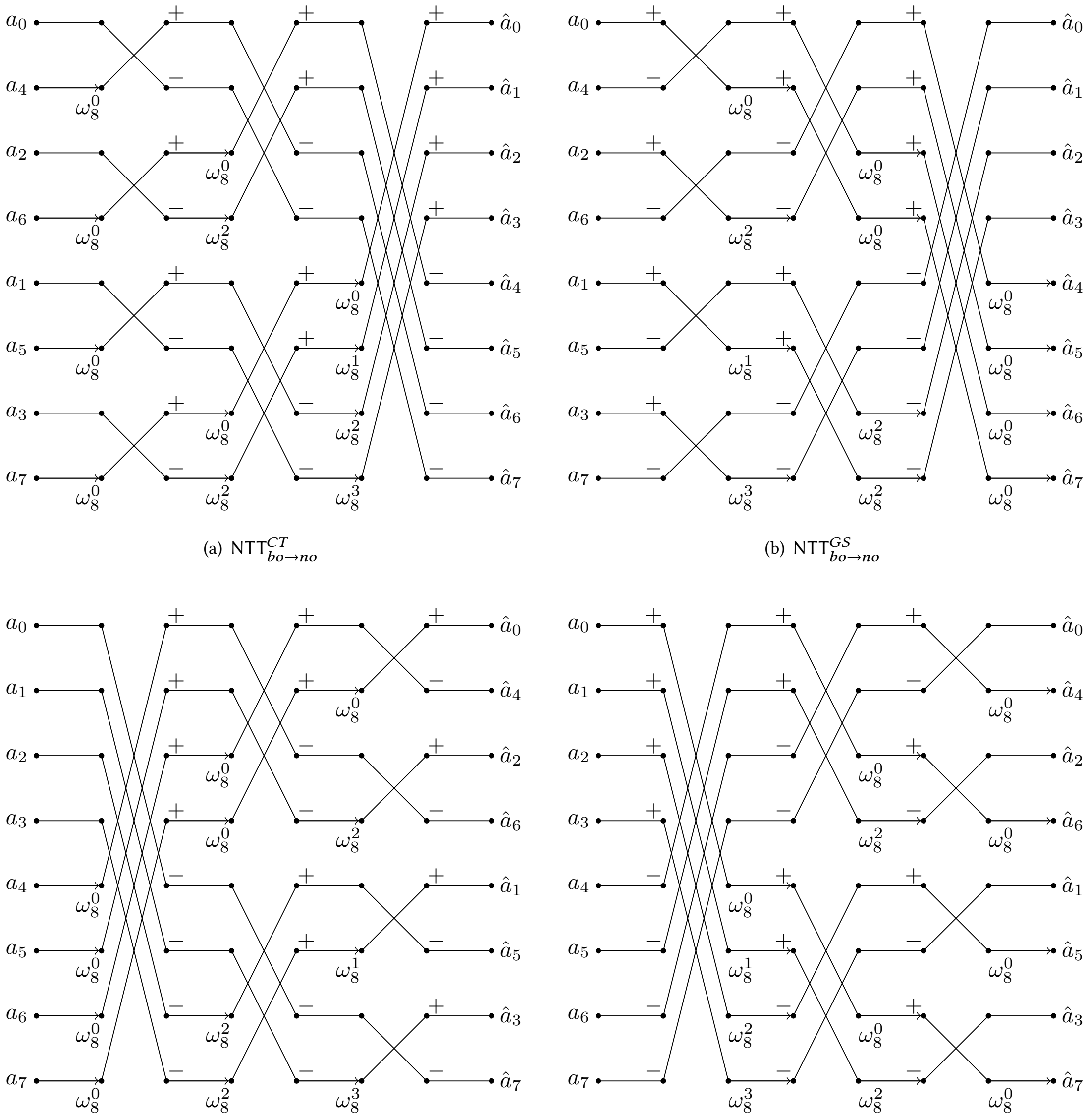
A SIGNAL FLOW OF RADIX-2 NTT


a_0 a_4 a_2 a_6 a_1 a_5 a_3 a_7
ω_8^0 ω_8^0 ω_8^0 ω_8^0
ω_8^0 ω_8^2 ω_8^0 ω_8^2
ω_8^0 ω_8^1 ω_8^2 ω_8^3
+ − + − + − + −
$\hat{a}_0$ $\hat{a}_1$ $\hat{a}_2$ $\hat{a}_3$ $\hat{a}_4$ $\hat{a}_5$ $\hat{a}_6$ $\hat{a}_7$
a_0 a_1 a_2 a_3 a_4 a_5 a_6 a_7
ω_8^0 ω_8^1 ω_8^2 ω_8^3
ω_8^0 ω_8^2 ω_8^0 ω_8^2
$\hat{a}_0$ $\hat{a}_4$ $\hat{a}_2$ $\hat{a}_6$ $\hat{a}_1$ $\hat{a}_5$ $\hat{a}_3$ $\hat{a}_7$

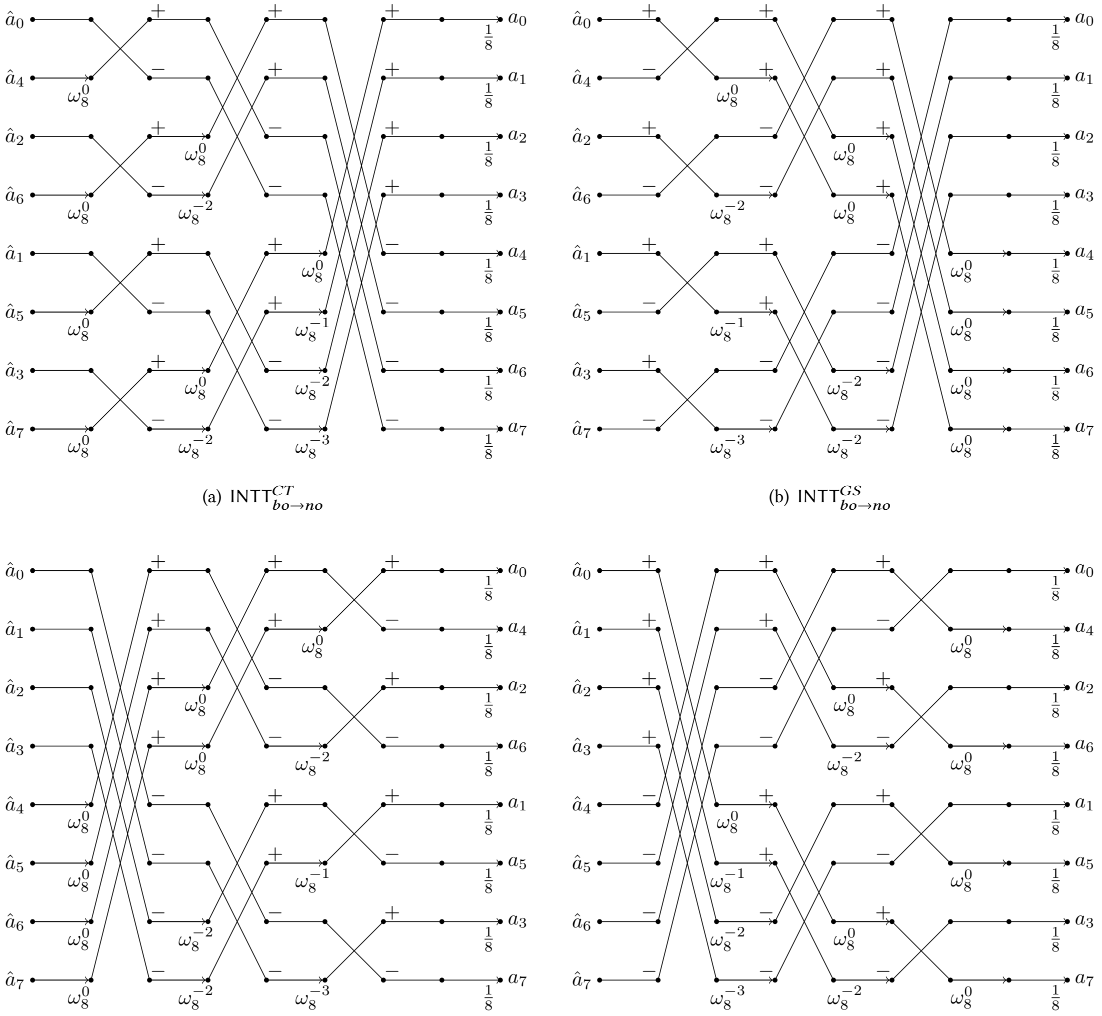
(a) $\mathsf{NTT}^{CT}_{bo\to no}$

(b) $\mathsf{NTT}^{GS}_{bo\to no}$

(c) $\mathsf{NTT}^{CT}_{no\to bo}$

(d) $\mathsf{NTT}^{GS}_{no\to bo}$

Fig. 6. Signal flow of radix-2 NTT for $n = 8$



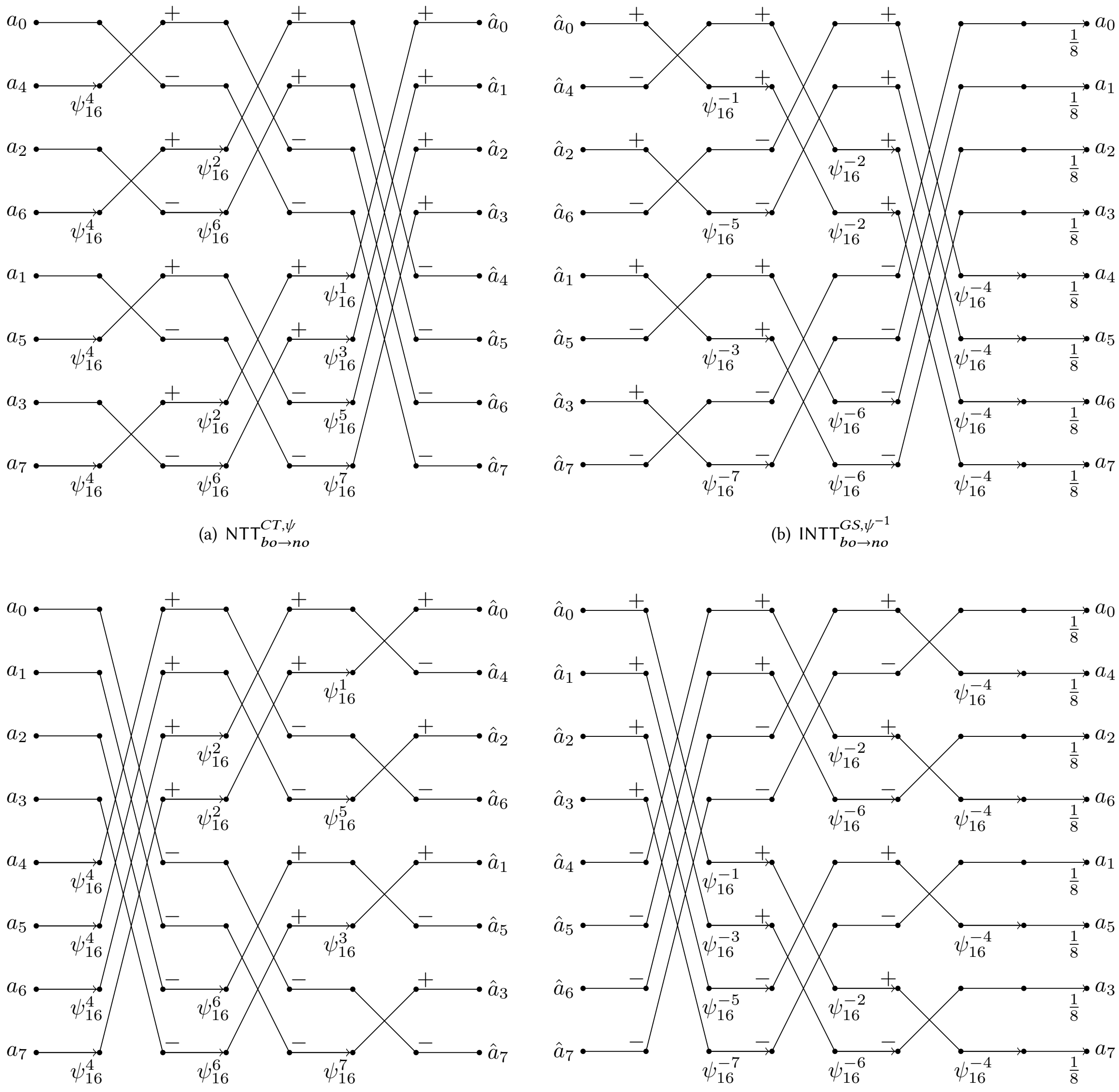
(a) $\mathsf{INTT}_{bo\to no}^{CT}$

(b) $\mathsf{INTT}_{bo\to no}^{GS}$

(c) $\mathsf{INTT}_{no\to bo}^{CT}$

(d) $\mathsf{INTT}_{no\to bo}^{GS}$

Fig. 7. Signal flow of radix-2 INTT for $n = 8$


a_0 a_4 a_2 a_6 a_1 a_5 a_3 a_7
ψ_{16}^4 ψ_{16}^2 ψ_{16}^6 ψ_{16}^1 ψ_{16}^3 ψ_{16}^5 ψ_{16}^7
+ −
$\hat{a}_0$ $\hat{a}_1$ $\hat{a}_2$ $\hat{a}_3$ $\hat{a}_4$ $\hat{a}_5$ $\hat{a}_6$ $\hat{a}_7$
$\hat{a}_0$ $\hat{a}_4$ $\hat{a}_2$ $\hat{a}_6$ $\hat{a}_1$ $\hat{a}_5$ $\hat{a}_3$ $\hat{a}_7$
ψ_{16}^{-1} ψ_{16}^{-5} ψ_{16}^{-3} ψ_{16}^{-7} ψ_{16}^{-2} ψ_{16}^{-6} ψ_{16}^{-4}
$\frac{1}{8}$
a_0 a_1 a_2 a_3 a_4 a_5 a_6 a_7
$\hat{a}_0$ $\hat{a}_4$ $\hat{a}_2$ $\hat{a}_6$ $\hat{a}_1$ $\hat{a}_5$ $\hat{a}_3$ $\hat{a}_7$
a_0 a_4 a_2 a_6 a_1 a_5 a_3 a_7

(a) $\mathsf{NTT}_{bo\to no}^{CT,\psi}$

(b) $\mathsf{INTT}_{bo\to no}^{GS,\psi^{-1}}$

(c) $\mathsf{NTT}_{no\to bo}^{CT,\psi}$

(d) $\mathsf{INTT}_{no\to bo}^{GS,\psi^{-1}}$

Fig. 8. Signal flow of radix-2 $\mathsf{NTT}^{\psi}/\mathsf{INTT}^{\psi^{-1}}$ for $n = 8$

B NUMBER THEORETIC TRANSFORM OVER $\mathbb{Z}_Q[X]/(X^N - X^{N/2} + 1)$

In this section, we introduce some progresses about relaxing the requirement of n being power-of-two such that NTT can be utilized over non-power-of-two rings.

B.1 Incomplete FFT trick over $\mathbb{Z}_q[x]/(x^n - x^{n/2} + 1)$

Some progresses about relaxing the requirement of n being power-of-two are made by Lyubashevsky and Seiler [LS19]. They first introduced a special incomplete NTT to lattice-based cryptographic schemes, by offering a new non-power-of-two ring structure $\mathbb{Z}_q[x]/(x^n - x^{n/2} + 1)$ where $n = 3 \cdot 2^e$ instead of a power of two and q is a prime number satisfying $q \equiv 1 \pmod{n}$ such that ψ_n exits. Actually, $x^n - x^{n/2} + 1$ is the $3n$-th cyclotomic polynomial of degree n where $n = 3^l \cdot 2^e$, $l \geq 0, e \geq 1$, not a power-of-two cyclotomic polynomial any more. The main observation they use is the CRT map as follows:

$$\mathbb{Z}_q[x]/(x^n - x^{n/2} + 1) \cong \mathbb{Z}_q[x]/(x^{n/2} - \zeta_1) \times \mathbb{Z}_q[x]/(x^{n/2} - \zeta_2),$$

where $\zeta_1 + \zeta_2 = 1$ and $\zeta_1 \cdot \zeta_2 = 1$. In their instantiation, they choose $\zeta_1 = \psi_n^{n/6}$ and $\zeta_2 = \zeta_1^5$.

As for its forward transform, $\boldsymbol{a} \in \mathbb{Z}_q[x]/(x^n - x^{n/2} + 1)$ from the 0-th level generates its images $\boldsymbol{a}_l = \boldsymbol{a} \bmod x^{n/2} - \zeta_1$ and $\boldsymbol{a}_r = \boldsymbol{a} \bmod x^{n/2} - (1 - \zeta_1)$ in $\mathbb{Z}_q[x]/(x^{n/2} - \zeta_1)$ and $\mathbb{Z}_q[x]/(x^{n/2} - \zeta_2)$ respectively in the first level, by using the fact that $\zeta_2 = 1 - \zeta_1$. In order to get the coefficients, one can compute $a_{l,i} = a_i + \zeta_1 a_{i+n/2}, a_{r,i} = a_i + a_{i+n/2} - \zeta_1 a_{i+n/2}, i = 0, 1, \ldots, n/2 - 1$. Different from radix-2 Cooley-Tukey algorithm, there are extra $n/2$ additions in this case. These additional additions don't cost much. For a fast NTT algorithm, one can continue with the similar radix-2 ($\log \frac{n}{3}$)-level FFT trick in $\mathbb{Z}_q[x]/(x^{n/2} - \zeta_1)$ and $\mathbb{Z}_q[x]/(x^{n/2} - \zeta_2)$, as in the power-of-two cyclotomic rings above, until the leaf nodes are of the form $\mathbb{Z}_q[x]/(x^3 - \psi_n^j)$ instead of linear terms. The inverse transform can be obtained by inverting the trick mentioned above, where Gentleman-Sande butterflies are used in the radix-2 steps. The point-wise multiplication is performed about the corresponding polynomials of degree 2 in each $\mathbb{Z}_q[x]/(x^3 - \psi_n^j)$. Detailedly, the CRT map can be described as follows:

$$\mathbb{Z}_q[x]/(x^n - x^{n/2} + 1) \;\cong\; \prod_{j \in \mathbb{Z}_n^\times} \mathbb{Z}_q[x]/(x^3 - \psi_n^j),$$

where $\mathbb{Z}_n^\times$ is the group of invertible elements of $\mathbb{Z}_n$.

B.2 Splitting Polynomial Ring over $\mathbb{Z}_q[x]/(x^n - x^{n/2} + 1)$

The method based on splitting polynomial ring can be generalized to the ring $\mathbb{Z}_q[x]/(x^n - x^{n/2} + 1)$ where $n = 3 \cdot 2^e$ and q is a prime number, based on which Liang et al. [LSS+20] proposed a generalized, modular and parallelizable NTT method referred to as Generalized 3-NTT (G3-NTT for simplicity). Similarly, let α, β be non-negative integer. The general α-round G3-NTT with β levels cropped is essentially based on the following isomorphism.

$$\Psi_{\alpha,3} : \mathbb{Z}_q[x]/(x^n - x^{n/2} + 1) \;\cong\; \left(\mathbb{Z}_q[y]/(y^{\frac{n}{3\cdot 2^\alpha}} - y^{\frac{n}{3\cdot 2^{\alpha+1}}} + 1)\right)[x]/(x^{3\cdot 2^\alpha} - y)$$

$$\boldsymbol{a} = \sum_{i=0}^{n-1} a_i x^i \;\mapsto\; \Psi_{\alpha,3}(\boldsymbol{a}) = \sum_{i=0}^{3\cdot 2^\alpha - 1} \left(\sum_{j=0}^{\frac{n}{3\cdot 2^\alpha} - 1} a_{3\cdot 2^\alpha \cdot j + i} y^j \right) x^i$$

where $y^{\frac{n}{3\cdot 2^\alpha}} - y^{\frac{n}{3\cdot 2^{\alpha+1}}} + 1$ is the $\frac{n}{2^\alpha}$-th cyclotomic polynomial of degree $\frac{n}{3\cdot 2^\alpha}$. Similar to NTTRU [LS19], there is a CRT map as follows: $\mathbb{Z}_q[y]/(y^{\frac{n}{3\cdot 2^\alpha}} - y^{\frac{n}{3\cdot 2^{\alpha+1}}} + 1) \cong \mathbb{Z}_q[y]/(y^{\frac{n}{3\cdot 2^{\alpha+1}}} - \zeta_1) \times \mathbb{Z}_q[y]/(y^{\frac{n}{3\cdot 2^{\alpha+1}}} - \zeta_2)$ where $\zeta_1 + \zeta_2 = 1, \zeta_1 \cdot \zeta_2 = 1$. It turns out that radix-2 truncated-NTTs can be performed in $\mathbb{Z}_q[y]/(y^{\frac{n}{3\cdot 2^{\alpha+1}}} - \zeta_1)$ and $\mathbb{Z}_q[y]/(y^{\frac{n}{3\cdot 2^{\alpha+1}}} - \zeta_2)$. If there are β levels to be cropped, $\beta = 0, 1, \ldots, \log \frac{n}{3\cdot 2^\alpha} - 1$, the modulus q can be chosen to satisfy only $q \equiv 1 \pmod{\frac{n}{2^{\alpha+\beta}}}$ such that the primitive $\frac{n}{2^{\alpha+\beta}}$-th root of unity $\psi_{n/2^{\alpha+\beta}}$ exits. The leaf nodes of CRT tree map are degree-2^β polynomials, e.g., $\mathbb{Z}_q[y]/(y^{2^\beta} - \psi_{n/2^{\alpha+\beta}})$. They choose $\zeta_1 = \psi_{n/2^{\alpha+\beta}}^{n/(6\cdot 2^{\alpha+\beta})}$ and $\zeta_2 = \zeta_1^5$. One-iteration Karatsuba algorithm can be used in a same way as H-NTT. Given appropriate and fixed (n, q), the computational complexity of G3-NTT can reach its optimization if $\alpha = 0, \beta = 0$ [LSS+20].

C SOME SKILLS

Those lattice-based schemes fully utilizes the advantages of NTT mentioned in section 3.5 to improve their efficiency.

As for MLWE-based schemes such as Kyber, Dilithium, etc, at a high level, their basic operations can be described as matrix-vector polynomial multiplication $\mathbf{A}^T\mathbf{r}$ and vector-vector polynomial multiplication $\mathbf{s}^T\mathbf{u}$, where $\mathbf{A} \in \mathcal{R}_q^{k\times k}, \mathbf{r}, \mathbf{s}, \mathbf{u} \in \mathcal{R}_q^{k\times 1}$. Those schemes directly generate the public key term $\hat{\mathbf{A}}$ already in the NTT domain by rejection sampling, instead of generating $\mathbf{A}$ followed by applying forward transform on each element. It can save k^2 forward transforms. The linearity of NTT can lead to $\mathrm{INTT}\left(\hat{\mathbf{A}}^T \circ \mathrm{NTT}(\mathbf{r})\right)$ where there are only k forward transforms and k inverse transforms. Once the forward transform result $\hat{\mathbf{s}}$ is computed, $\hat{\mathbf{s}}$ can be stored or transmitted without any extra requirement of storage, for following use in multiple polynomial multiplications. Using $\hat{\mathbf{s}}$ that is stored or transmitted in advance, one can compute $\mathbf{s}^T\mathbf{u}$ by $\mathrm{INTT}\left(\hat{\mathbf{s}}^T \circ \mathrm{NTT}(\mathbf{u})\right)$, within only k forward transforms and k inverse transforms.

As for MLWR-based schemes, the linearity of NTT is helpful in the NTT-based implementation of MLWR-based schemes such as Saber. The number of forward transforms and inverse transforms in Saber's matrix-vector multiplication $\mathbf{As}$ can be reduced from $2k^2$ and k^2 to k^2+k and k, respectively, while the number of inverse transforms $\mathbf{b}^T\mathbf{s}$ in vector-vector multiplication can be reduced from k to 1, where $\mathbf{A} \in \mathcal{R}_q^{k\times k}, \mathbf{b}, \mathbf{s} \in \mathcal{R}_q^{k\times 1}$.

D RADIX-2 FAST NUMBER THEORETIC TRANSFORM FROM FFT PERSPECTIVES

In this section, we will describe NTT algorithms from FFT perspectives. We follow the same notations and make the same requirements on parameters as in section 3. The basic principle of fast NTT algorithms is using "divide and conquer" skill to divide the n-point NTT into two $n/2$-point NTTs, based on the periodicity and symmetry of the primitive root of unity. This section will first introduce the properties of the primitive roots of unity, and then the radix-2 fast NTT algorithms. The properties of primitive roots of unity in NTT are similar to those of twiddle factors in FFT. The primitive n-th root of unity ω_n in $\mathbb{Z}_q$ has the following properties:

$$\begin{aligned} \text{periodicity: } \omega_n^{k+n} &= \omega_n^k \\ \text{symmetry: } \omega_n^{k+n/2} &= -\omega_n^k \end{aligned} \tag{23}$$

where k is a non-negative integer. It is trivial that the primitive $2n$-th root of unity ψ_{2n} shares the similar properties if ψ_{2n} exists.

D.1 Cooley-Tukey Algorithm for CC-based NTT

D.1.1 Cooley-Tukey algorithm for NTT. The fast algorithms of NTT are introduced first. Based on the parity of the indexes of the coefficients a_i in $\boldsymbol{a}$, the terms in the summation in formula (1) can be separated into two parts, for $j = 0, 1, \ldots, n-1$:

$$\begin{aligned} \hat{a}_j &= \sum_{i=0}^{n/2-1} a_{2i}\omega_n^{2ij} + \sum_{i=0}^{n/2-1} a_{2i+1}\omega_n^{(2i+1)j} \bmod q \\ &= \sum_{i=0}^{n/2-1} a_{2i}(\omega_n^2)^{ij} + \omega_n^j \sum_{i=0}^{n/2-1} a_{2i+1}(\omega_n^2)^{ij} \bmod q. \end{aligned}$$

Based on the periodicity and symmetry of the primitive root of unity in fomula (23), for $j = 0, 1, \ldots, n/2-1$, we get:

$$\begin{aligned} \hat{a}_j &= \sum_{i=0}^{n/2-1} a_{2i}(\omega_n^2)^{ij} + \omega_n^j \sum_{i=0}^{n/2-1} a_{2i+1}(\omega_n^2)^{ij} \bmod q \\ \hat{a}_{j+n/2} &= \sum_{i=0}^{n/2-1} a_{2i}(\omega_n^2)^{ij} - \omega_n^j \sum_{i=0}^{n/2-1} a_{2i+1}(\omega_n^2)^{ij} \bmod q. \end{aligned} \tag{24}$$

Let $\hat{a}'_j = \sum_{i=0}^{n/2-1} a_{2i}(\omega_n^2)^{ij} \bmod q$, $\hat{a}''_j = \sum_{i=0}^{n/2-1} a_{2i+1}(\omega_n^2)^{ij} \bmod q$, $j = 0, 1, \ldots, n/2-1$. Formula (24) can be rewritten as:

$$\begin{aligned} \hat{a}_j &= \hat{a}'_j + \omega_n^j \hat{a}''_j \bmod q, \\ \hat{a}_{j+n/2} &= \hat{a}'_j - \omega_n^j \hat{a}''_j \bmod q, j = 0, 1, \ldots, n/2-1. \end{aligned} \tag{25}$$

One can learn from the definition of NTT that, $\hat{a}'_j$ and $\hat{a}''_j$ can be computed by $n/2$-point NTT, from the even-indexed and the odd-indexed coefficients of $\boldsymbol{a}$, respectively. Formula (25) shows that, the original n-point NTT can be divided into two $n/2$-point NTTs by "divide-and-conquer" method. After getting the $n/2$-point NTT results $\hat{a}'_j$ and $\hat{a}''_j$, the original n-point $\hat{\boldsymbol{a}}$ can be easily achieved by multiplying ω_n^j and simple additions/subtractions. This kind of "divide and conquer" skill can also be applied to compute $\hat{a}'_j$ and $\hat{a}''_j$. Since n is a power of two, it can be separated down to 2-point NTTs. Such fast NTT algorithm was first proposed by Cooley and Tukey [CT65], and named as radix-2 Cooley-Tukey NTT algorithm, or radix-2 CT NTT algorithm for short. By the term of FFT, it is also called radix-2 decimation-in-time NTT, or CT decimation-in-time NTT. The process of deriving $\hat{a}_j$ and $\hat{a}_{j+n/2}$ from $\hat{a}'_j$ and $\hat{a}''_j$ is named Cooley-Tukey butterfly, or CT butterfly for short, which is illustrated in Figure 1(a).

Note that, in the CT NTT derived from formula (25), the coefficients of the input polynomials are indexed under bit-reversed order, while the coefficients of the output polynomials are indexed under natural order. In this paper, we follow the notations as used in [POG15] which denotes this kind of CT NTT by $\mathsf{NTT}^{CT}_{bo\to no}$ where the subscripts $bo \longrightarrow no$ indicates the input coefficients are under bit-reversed order and output coefficients are under natural order. The signal flow of $\mathsf{NTT}^{CT}_{bo\to no}$ for $n = 8$ can be seen in Figure 6(a) in Appendix A. Adjust the input to natural order, then the Cooley-Tukey butterflies in the signal flow is changed elsewhere, as in Figure 6(c). The output will be under bit-reversed order. This new transform is denoted by $\mathsf{NTT}^{CT}_{no\to bo}$. Obviously, the two transforms shares the same number of Cooley-Tukey butterflies, and therefore the same complexity.

D.1.2 Cooley-Tukey algorithm for INTT. Cooley-Tukey butterfly can also be applied to compute INTT. In contrast to NTT, there is extra multiplications by a scale factor n^{-1} in INTT.

With neglecting n^{-1}, the terms in the summation of formula (2) are the same as those in formula (1), except replacing ω_n with ω_n^{-1}. Therefore, the procedure of applying Cooley-Tukey butterfly to compute INTT is basically the same as that in the NTT case, except replacing ω_n with ω_n^{-1} in each step. For the convenience of understanding, its brief computing process is given here. The terms in the summation of formula (2) are divided into two parts based on the parity of the index of $\hat{a}_j$. That is, for $i = 0, 1, \dots, n-1$,

$$a_i = \sum_{j=0}^{n/2-1} \hat{a}_{2j}(\omega_n^2)^{-ij} + \omega_n^{-i} \sum_{j=0}^{n/2-1} \hat{a}_{2j+1}(\omega_n^2)^{-ij} \bmod q. \tag{26}$$

Let $a'_i = \sum_{j=0}^{n/2-1} \hat{a}_{2j}(\omega_n^2)^{-ij} \bmod q$, $a''_i = \sum_{j=0}^{n/2-1} \hat{a}_{2j+1}(\omega_n^2)^{-ij} \bmod q, i = 0, 1, \dots, n/2-1$. Formula (26) can be rewritten as:

$$\begin{aligned} a_i &= a'_i + \omega_n^{-i} a''_i \bmod q \\ a_{i+n/2} &= a'_i - \omega_n^{-i} a''_i \bmod q, i = 0, 1, \dots, n/2-1. \end{aligned} \tag{27}$$

Therefore, the computing of n-point INTT can be divided into two $n/2$-point INTTs via formula (27), which can be done down to 2-point INTTs finally. This kind of INTT is called radix-2 Cooley-Tukey INTT algorithm, or radix-2 decimation-in-time INTT. In this paper it is denoted by $\mathsf{INTT}^{CT}_{bo\to no}$. Its signal flow for $n = 8$ is illustrated in Figure 7(a). Adjust its input to natural order, and then the output is changed under bit-reversed order. This new transform is denoted by $\mathsf{INTT}^{CT}_{no\to bo}$, the signal flow of which for $n = 8$ can be seen in Figure 7(c).

D.2 Gentlemen-Sande Algorithm for CC-based NTT

D.2.1 Gentlemen-Sande algorithm for NTT. Gentlemen-Sande algorithm separates the coefficients of $\boldsymbol{a}$ into the upper half and the lower half. Specifically, for $j = 0, 1, \dots, n-1$, the formula (1) is separated into:

$$\begin{aligned} \hat{a}_j &= \sum_{i=0}^{\frac{n}{2}-1} a_i \omega_n^{ij} + \sum_{i=\frac{n}{2}}^{n-1} a_i \omega_n^{ij} \bmod q \\ &= \sum_{i=0}^{\frac{n}{2}-1} a_i \omega_n^{ij} + \sum_{i=0}^{\frac{n}{2}-1} a_{i+\frac{n}{2}} \omega_n^{(i+\frac{n}{2})j} \bmod q. \end{aligned} \tag{28}$$

Based on the periodicity and symmetry of the primitive root of unity (see formula (23)), the terms $\hat{a}_j$ continue to be dealt with according to the parity of index j. For $j = 0, 1, \ldots, \frac{n}{2} - 1$:

$$\begin{aligned}\hat{a}_{2j} &= \sum_{i=0}^{\frac{n}{2}-1} a_i \omega_n^{2ij} + (-1)^{2j} \cdot \sum_{i=0}^{\frac{n}{2}-1} a_{i+\frac{n}{2}} \omega_n^{2ij} \bmod q \\ &= \sum_{i=0}^{\frac{n}{2}-1} (a_i + a_{i+\frac{n}{2}})(\omega_n^2)^{ij} \bmod q, \\ \hat{a}_{2j+1} &= \sum_{i=0}^{\frac{n}{2}-1} a_i \omega_n^{(2j+1)i} + (-1)^{2j+1} \cdot \sum_{i=0}^{\frac{n}{2}-1} a_{i+\frac{n}{2}} \omega_n^{(2j+1)i} \bmod q \\ &= \sum_{i=0}^{\frac{n}{2}-1} \left[(a_i - a_{i+\frac{n}{2}}) \cdot \omega_n^i \right] (\omega_n^2)^{ij} \bmod q.\end{aligned}$$

Let $b'_i = a_i + a_{i+\frac{n}{2}} \bmod q$, $b''_i = (a_i - a_{i+\frac{n}{2}}) \cdot \omega_n^i \bmod q$, $i = 0, 1, \ldots, \frac{n}{2} - 1$. The above formula can be rewritten as:

$$\begin{aligned}\hat{a}_{2j} &= \sum_{i=0}^{\frac{n}{2}-1} b'_i (\omega_n^2)^{ij} \bmod q, \\ \hat{a}_{2j+1} &= \sum_{i=0}^{\frac{n}{2}-1} b''_i (\omega_n^2)^{ij} \bmod q, j = 0, 1, \ldots, \frac{n}{2} - 1.\end{aligned} \tag{29}$$

One can learn from the definition of NTT that, formula (29) is exact the $n/2$-point NTTs with respect to $b'_i, b''_i, i = 0, 1, \ldots, \frac{n}{2} - 1$. Thus, after deriving $b'_i, b''_i, i = 0, 1, \ldots, \frac{n}{2} - 1$ from $a_i, i = 0, 1, \ldots, n-1$, the original n-point NTT with respect to a_i is transformed into $n/2$-point NTTs with respect to b'_i, b''_i. Similarly, the $n/2$-point NTT in formula (29) can be tranformed into $n/4$-point NTTs, and down to 2-point NTTs. This kind of fast algorithm was first proposed by Gentlemen and Sande [GS66], and named as radix-2 Gentlemen-Sande NTT algorithm, or radix-2 GS NTT algorithm for short, or radix-2 decimation-in-frequency NTT, or else GS decimation-in-frequency NTT. The process of deriving b'_i and b''_i from a_i and $a_{i+\frac{n}{2}}$ is referred to as Gentlemen-Sande butterfly, or GS butterfly for short (see in Figure 1(b)). Such GS NTT algorithm is denoted by $\mathrm{NTT}^{GS}_{no \to bo}$. Its signal flow for $n = 8$ is shown in Figure 6(d). Adjust the input to bit-reversed order and it outputs under natural order, as shown in Figure 6(b). The new transform is denoted by $\mathrm{NTT}^{GS}_{bo \to no}$.

D.2.2 Gentlemen-Sande algorithm for INTT. Gentleman-Sande butterfly can be similarly applied to compute INTT, by neglecting n^{-1} and replacing ω_n with ω_n^{-1} in NTT. That is, for $i = 0, 1, \ldots, \frac{n}{2} - 1$,

$$\begin{aligned}a_{2i} &= \sum_{j=0}^{\frac{n}{2}-1} (\hat{a}_j + \hat{a}_{j+\frac{n}{2}})(\omega_n^2)^{-ij} \bmod q \\ a_{2i+1} &= \sum_{i=0}^{\frac{n}{2}-1} \left[(\hat{a}_j - \hat{a}_{j+\frac{n}{2}}) \cdot \omega_n^{-j} \right] (\omega_n^2)^{-ij} \bmod q.\end{aligned} \tag{30}$$

Let $\hat{b}'_j = \hat{a}_j + \hat{a}_{j+\frac{n}{2}} \bmod q$, $\hat{b}''_j = (\hat{a}_j - \hat{a}_{j+\frac{n}{2}}) \cdot \omega_n^{-j} \bmod q$, $j = 0, 1, \ldots, \frac{n}{2} - 1$. Formula (30) can be rewritten as:

$$\begin{aligned}a_{2i} &= \sum_{j=0}^{\frac{n}{2}-1} \hat{b}'_j (\omega_n^2)^{-ij} \bmod q \\ a_{2i+1} &= \sum_{j=0}^{\frac{n}{2}-1} \hat{b}''_j (\omega_n^2)^{-ij} \bmod q, i = 0, 1, \ldots, \frac{n}{2} - 1.\end{aligned} \tag{31}$$

Similarly, n-point INTT can be divided into two $n/2$-point INTTs according to formula (31), and down to 2-point INTTs. This kind of INTT is called radix-2 Gentlemen-Sande INTT algorithm, or radix-2 decimation-in-frequency INTT, which is denoted by

$\mathsf{INTT}^{GS}_{bo\to no}$. Its signal flow for $n = 8$ is shown in Figure 7(d). Adjust its input to natural order, and then the output will be under bit-reversed order. This new transform is denoted by $\mathsf{INTT}^{GS}_{no\to bo}$, the signal flow of which for $n = 8$ is shown in Figure 7(b).

D.3 Radix-2 Fast NWC-based NTT

In this paper, the process of multiplying the coefficients by ψ^{i}_{2n} before forward transform in formula (3) is referred to as pre-processing, while the process of multiplying the coefficients by ψ^{-i}_{2n} after inverse transform in formula (4) is referred to as post-processing.

Fast algorithms for negative wrapped convolution-based (NWC-based) NTT such as NTT^{ψ} and $\mathsf{INTT}^{\psi^{-1}}$, can be constructed by using radix-2 CT/GS NTT/INTT algorithm with pre-processing and post-processing, according to formula (3)(4). However, such construction requires extra point-wise multiplication with $\boldsymbol{\psi}/\boldsymbol{\psi}^{-1}$ besides radix-2 algorithms, resulting with n extra multiplications. In fact, these additional multiplications are not necessary. Roy et al. [RVM^{+}14] integrate the pre-processing about $\boldsymbol{\psi}$ into $\mathsf{NTT}^{CT}_{bo\to no}$. Pöppelmann et al. [POG15] integrate the post-processing about $\boldsymbol{\psi}^{-1}$ into $\mathsf{INTT}^{GS}_{bo\to no}$. Furthermore, Zhang et al. [ZYC^{+}20] integrate $\boldsymbol{\psi}^{-1}$ and n^{-1} into $\mathsf{INTT}^{GS}_{bo\to no}$. They will be introduced ad follows.

D.3.1 *Cooley-Tukey algorithm for* NTT^{ψ}.

Roy et al. [RVM^{+}14] take advantage of Cooley-Tukey algorithm. According to the definition of NTT^{ψ}, the coefficients of $\hat{\boldsymbol{a}}$ can be written as:

$$\hat{a}_j = \sum_{i=0}^{n-1} a_i \psi_{2n}^{i} \omega_n^{ij} \bmod q, j = 0, 1, \ldots, n-1. \tag{32}$$

Divide the summation into two parts based on the parity of the index of a_i, and for $j = 0, 1, \ldots, n-1$, we get:

$$\begin{aligned}\hat{a}_j &= \sum_{i=0}^{n/2-1} a_{2i}\omega_n^{2ij}\psi_{2n}^{2i} + \sum_{i=0}^{n/2-1} a_{2i+1}\omega_n^{(2i+1)j}\psi_{2n}^{2i+1} \bmod q \\ &= \sum_{i=0}^{n/2-1} a_{2i}(\omega_n^2)^{ij}(\psi_{2n}^2)^i + \omega_n^j\psi_{2n}\sum_{i=0}^{n/2-1} a_{2i+1}(\omega_n^2)^{ij}(\psi_{2n}^2)^i \bmod q.\end{aligned} \tag{33}$$

Based on the periodicity and symmetry of the primitive root of unity, for $j = 0, 1, \ldots, n/2-1$, we get

$$\begin{aligned}\hat{a}_j &= \sum_{i=0}^{n/2-1} a_{2i}(\omega_n^2)^{ij}(\psi_{2n}^2)^i + \omega_n^j\psi_{2n}\sum_{i=0}^{n/2-1} a_{2i+1}(\omega_n^2)^{ij}(\psi_{2n}^2)^i \bmod q \\ \hat{a}_{j+n/2} &= \sum_{i=0}^{n/2-1} a_{2i}(\omega_n^2)^{ij}(\psi_{2n}^2)^i - \omega_n^j\psi_{2n}\sum_{i=0}^{n/2-1} a_{2i+1}(\omega_n^2)^{ij}(\psi_{2n}^2)^i \bmod q.\end{aligned} \tag{34}$$

Let $\hat{a}'_j = \sum_{i=0}^{n/2-1} a_{2i}(\omega_n^2)^{ij}(\psi_{2n}^2)^i \bmod q$, $\hat{a}''_j = \sum_{i=0}^{n/2-1} a_{2i+1}(\omega_n^2)^{ij}(\psi_{2n}^2)^i \bmod q$, $j = 0, 1, \ldots, n/2-1$. With $\omega_n^j\psi_{2n} = \psi_{2n}^{2j+1}$, the above formula can be rewritten as, for $j = 0, 1, \ldots, n/2-1$:

$$\hat{a}_j = \hat{a}'_j + \psi_{2n}^{2j+1}\hat{a}''_j \bmod q, \quad \hat{a}_{j+n/2} = \hat{a}'_j - \psi_{2n}^{2j+1}\hat{a}''_j \bmod q. \tag{35}$$

One can see that, $\hat{a}'_j$ and $\hat{a}''_j$ can be obtained via exact $n/2$-point NTT^{ψ}s. The following analysis is the same as that of radix-2 CT NTT in section D.1, so for briefness the redundant analysis is omitted here. Such fast algorithm of NTT^{ψ} is called radix-2 CT NTT^{ψ}, and is denoted by $\mathsf{NTT}^{CT,\psi}_{bo\to no}$. Its signal flow for $n = 8$ is shown in Figure 8(a). Adjust the input and output order, and we get $\mathsf{NTT}^{CT,\psi}_{no\to bo}$. Its signal flow for $n = 8$ is shown in Figure 8(c).

D.3.2 *Gentleman-Sande algorithm for* $\mathsf{INTT}^{\psi^{-1}}$.

Pöppelmann et al. [POG15] progress the integration with $\boldsymbol{\psi}^{-1}$ by using Gentleman-Sande algorithm. According to the definition of $\mathsf{INTT}^{\psi^{-1}}$, the coefficients of $\boldsymbol{a}$ can be written as:

$$a_i = n^{-1}\psi_{2n}^{-i}\sum_{j=0}^{n-1}\hat{a}_j\omega_n^{-ij} \bmod q, i = 0, 1, \ldots, n-1. \tag{36}$$

With neglecting n^{-1}, the summation can be divided into the upper half and the lower half with respect to the index of $\hat{a}_j$. For $i = 0, 1, \ldots, n-1$,

$$\begin{aligned} a_i &= \psi_{2n}^{-i}\left(\sum_{j=0}^{\frac{n}{2}-1} \hat{a}_j \omega_n^{-ij} + \sum_{j=\frac{n}{2}}^{n-1} \hat{a}_j \omega_n^{-ij}\right) \\ &= \psi_{2n}^{-i}\left[\sum_{j=0}^{\frac{n}{2}-1} \hat{a}_j \omega_n^{ij} + \sum_{j=0}^{\frac{n}{2}-1} \hat{a}_{j+\frac{n}{2}} \omega_n^{-i(j+\frac{n}{2})}\right] \bmod q. \end{aligned}$$

Based on the periodicity and symmetry of the primitive root of unity, for $i = 0, 1, \ldots, \frac{n}{2}-1$ we have

$$\begin{aligned} a_{2i} =& \psi_{2n}^{-2i}\left[\sum_{j=0}^{\frac{n}{2}-1} \hat{a}_j \omega_n^{-2ij} + (-1)^{2i} \cdot \sum_{j=0}^{\frac{n}{2}-1} \hat{a}_{j+\frac{n}{2}} \omega_n^{-2ij}\right] \bmod q \\ =& (\psi_{2n}^2)^{-i} \sum_{j=0}^{\frac{n}{2}-1} (\hat{a}_j + \hat{a}_{j+\frac{n}{2}})(\omega_n^2)^{-ij} \bmod q, \\ a_{2i+1} =& \psi_{2n}^{-(2i+1)}\left[\sum_{j=0}^{\frac{n}{2}-1} \hat{a}_j \omega_n^{-(2i+1)j} + (-1)^{2i+1} \cdot \sum_{j=0}^{\frac{n}{2}-1} \hat{a}_{j+\frac{n}{2}} \omega_n^{-(2i+1)j}\right] \bmod q \\ =& (\psi_{2n}^2)^{-i} \sum_{j=0}^{\frac{n}{2}-1} \left[(\hat{a}_j - \hat{a}_{j+\frac{n}{2}}) \cdot \omega_n^{-j} \psi_{2n}^{-1}\right] (\omega_n^2)^{-ij} \bmod q. \end{aligned}$$

Since $\omega_n^{-j}\psi_{2n}^{-1} = \psi_{2n}^{-(2j+1)}$, letting $\hat{b}'_j = \hat{a}_j + \hat{a}_{j+\frac{n}{2}} \bmod q$, $\hat{b}''_j = (\hat{a}_j - \hat{a}_{j+\frac{n}{2}}) \cdot \psi_{2n}^{-(2j+1)} \bmod q$, $j = 0, 1, \ldots, \frac{n}{2}-1$, the above formula can be rewritten as, for $i = 0, 1, \ldots, \frac{n}{2}-1$:

$$\begin{aligned} a_{2i} &= (\psi_{2n}^2)^{-i} \sum_{j=0}^{\frac{n}{2}-1} \hat{b}'_j (\omega_n^2)^{-ij} \bmod q, \\ a_{2i+1} &= (\psi_{2n}^2)^{-i} \sum_{j=0}^{\frac{n}{2}-1} \hat{b}''_j (\omega_n^2)^{-ij} \bmod q. \end{aligned} \tag{37}$$

Similar to the analysis in section D.2, computing n-point $\mathsf{INTT}^{\psi^{-1}}$ can be transformed into two $n/2$-point $\mathsf{INTT}^{\psi^{-1}}$s with respect to $\hat{b}'_j, \hat{b}''_j$. Such fast algorithm for $\mathsf{INTT}^{\psi^{-1}}$ is named as radix-2 GS $\mathsf{INTT}^{\psi^{-1}}$, and denoted by $\mathsf{INTT}_{no\to bo}^{GS,\psi^{-1}}$. Its signal flow for $n = 8$ is shown in Figure 8(d). Adjust the input/output order and get $\mathsf{INTT}_{bo\to no}^{GS,\psi^{-1}}$ whose signal flow for $n = 8$ is shown in Figure 8(b).

Zhang et al. [ZYC+20] noticed that $\boldsymbol{\psi}^{-1}$ and n^{-1} can both be integrated into $\mathsf{INTT}_{bo\to no}^{GS}$. Thus, n^{-1} is no longer neglected, and one can learn from formula (36), for $i = 0, 1, \ldots, \frac{n}{2}-1$:

$$\begin{aligned} a_{2i} &= (\frac{n}{2})^{-1}(\psi_{2n}^2)^{-i} \sum_{j=0}^{\frac{n}{2}-1} \hat{b}'_j (\omega_n^2)^{-ij} \bmod q, \\ a_{2i+1} &= (\frac{n}{2})^{-1}(\psi_{2n}^2)^{-i} \sum_{j=0}^{\frac{n}{2}-1} \hat{b}''_j (\omega_n^2)^{-ij} \bmod q. \end{aligned} \tag{38}$$

where $\hat{b}'_j = (\hat{a}_j + \hat{a}_{j+\frac{n}{2}})/2 \bmod q$, $\hat{b}''_j = (\hat{a}_j - \hat{a}_{j+\frac{n}{2}})/2 \cdot \psi_{2n}^{-(2j+1)} \bmod q$, $j = 0, 1, \ldots, \frac{n}{2}-1$. Different from formula (37), when computing $\hat{b}'_j$ and $\hat{b}''_j$, the scale factor 2 will be dealt with directly, by using addition and displacement (i.e., ">>") to compute $x/2 \bmod q$. When x is even, $x/2 \equiv (x >> 1) \bmod q$. When x is odd, $x/2 \equiv (x >> 1) + (q+1)/2 \bmod q$.