◎ 2-J Relations among Filter Length N, Transition Band, and Accuracy

♦ Suppose that we want:

① passband ripple $\leq \delta_1$,

② stopband ripple $\leq \delta_2$,

③ width of transition band $\leq \Delta F$ (expressed by normalized frequency)

$$\Delta F = (f_1 - f_2)/f_s = (f_1 - f_2)T \quad (f_s\text{: sampling frequency, } T\text{: sampling interval})$$

Then, the estimated length N of the digital filter is:

$$N = \frac{2}{3}\frac{1}{\Delta F}\log_{10}\left(\frac{1}{10\delta_1\delta_2}\right)$$

.

- When there are two transition bands, $\Delta F = \min(\Delta F_1, \Delta F_2)$
- 犧牲 transition band 的 frequency response, 換取較高的 passband and stopband accuracies

$$N = \frac{2}{3}\frac{1}{\Delta F}\log_{10}\left(\frac{1}{10\delta_1\delta_2}\right) \qquad \frac{3}{2}N\Delta F = \log_{10}\left(\frac{1}{10\delta_1\delta_2}\right)$$

$$-\frac{3}{2}N\Delta F = \log_{10}(10\delta_1\delta_2)$$

$$10\delta_1\delta_2 = 10^{-\frac{3}{2}N\Delta F}$$

$$\delta_1\delta_2 = 10^{-3N\Delta F/2-1}$$

if $\delta_1 = \delta_2 = \delta$, $\delta^2 = 10^{-3N\Delta F/2-1}$

[Ref] F. Mintzer and L. Bede, "Practical design rules for optimum FIR bandpass digital filter", *IEEE Trans. ASSP*, vol. 27, no. 2, pp. 204-206, Apr. 1979.

問題：假設 $\sqrt{10}\delta_1 = \sqrt{10}\delta_2 = \delta$，$N$ 為固定，

當 ΔF 變為 A 倍時，δ 變為多少？

If δ is the original error

δ_0 is the new error

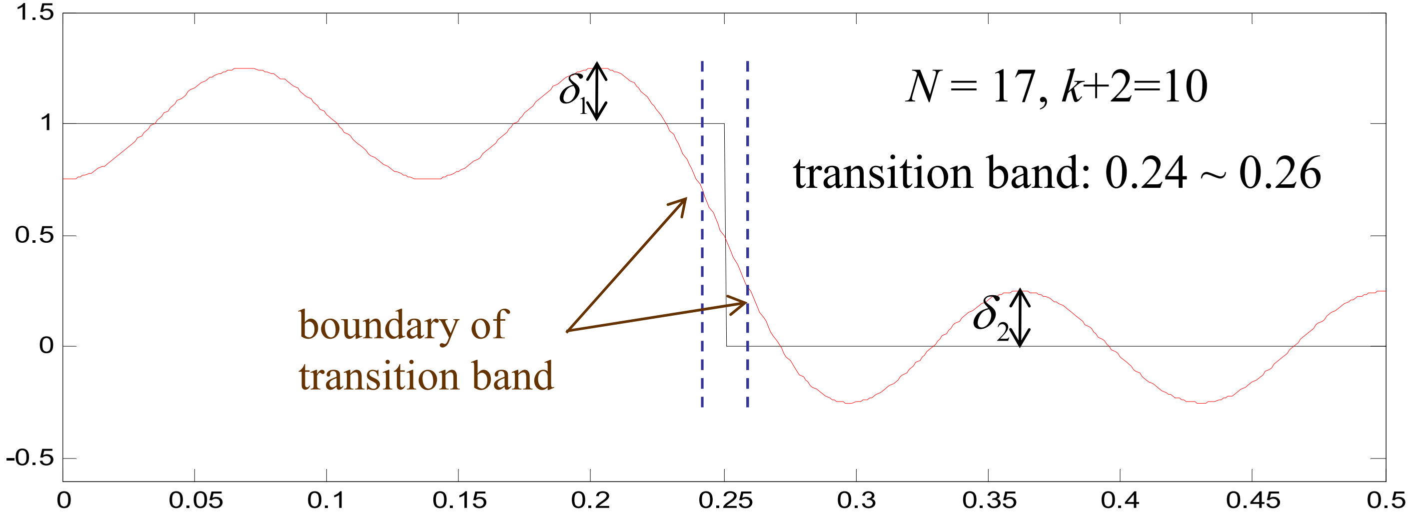
$$10\delta^2 = 10^{-\frac{3}{2}N\Delta F}$$

$$10\delta_0^2 = 10^{(-\frac{3}{2}N\Delta F)A} = (10\delta^2)^A = 10^A\delta^{2A}$$

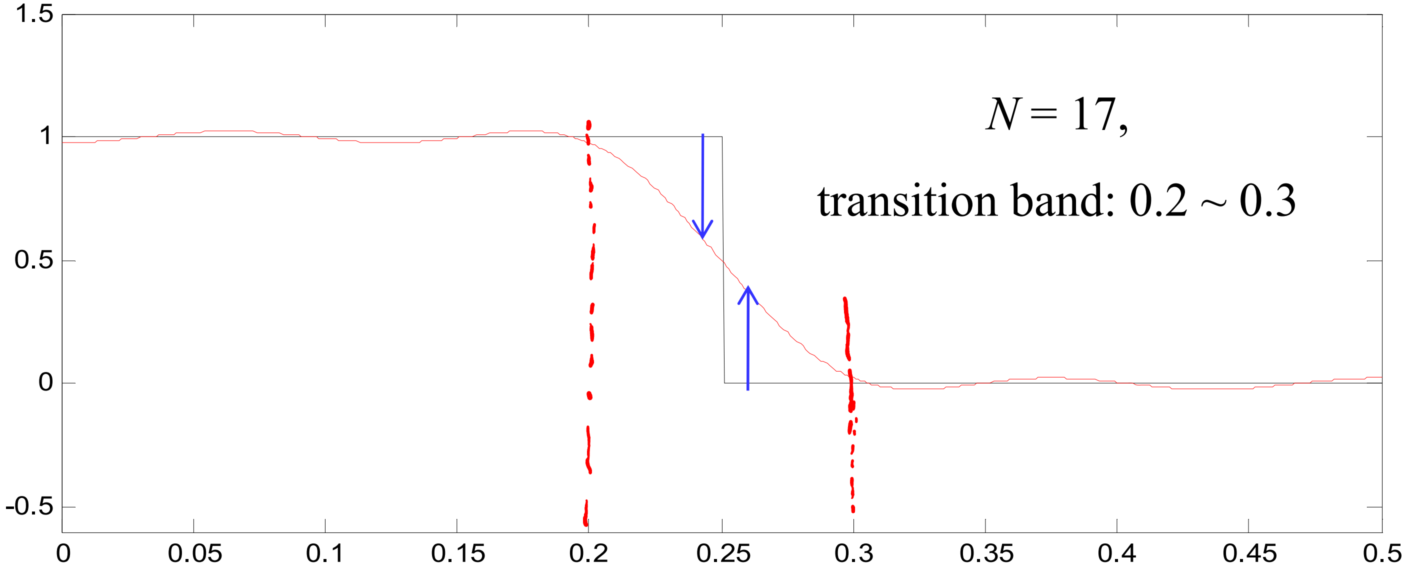
$$\delta_0 = 10^{\frac{A-1}{2}}\delta^A$$

If $\delta = 0.1$, $A = 5$

$$\delta_0 = 100(0.1)^5 = 10^{-3}$$



$N = 17$, $k+2=10$
transition band: 0.24 ~ 0.26
δ_1
δ_2
boundary of
transition band
1.5
1
0.5
0
-0.5
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

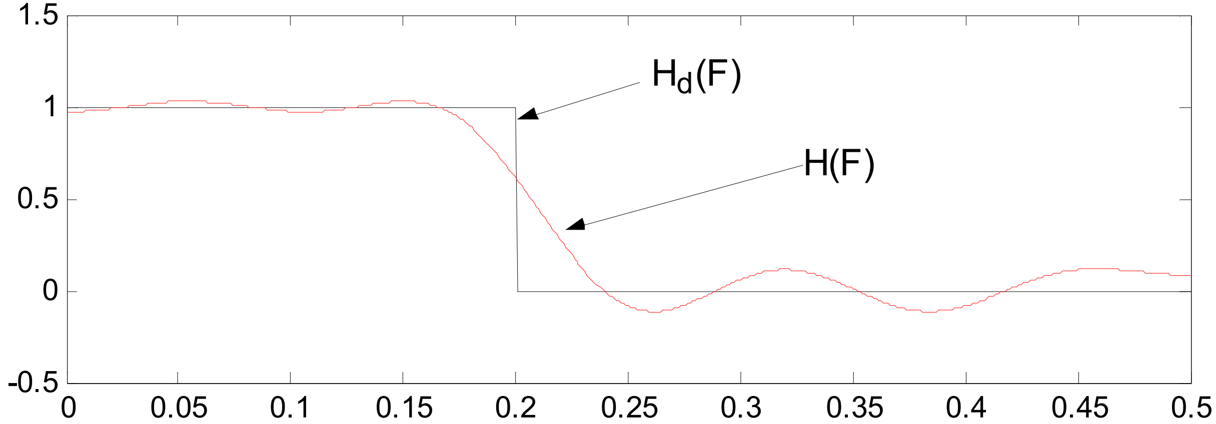


$N = 17$,
transition band: 0.2 ~ 0.3
1.5
1
0.5
0
-0.5
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

◉ 2-K Relations between Weight Functions and Accuracy

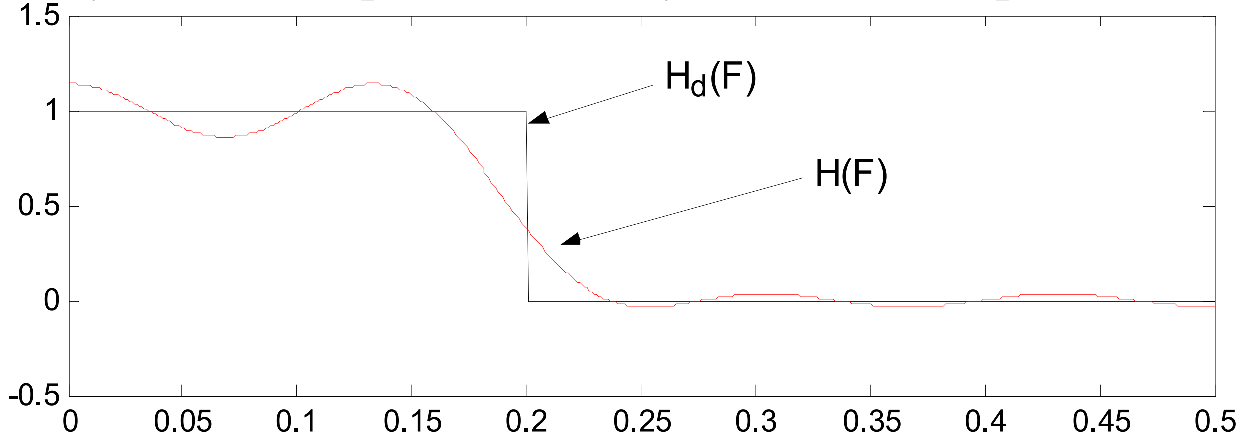
If we treat the passband more important than the stop band

$W(f)$ = 1 in the passband, $0 < W(f) < 1$ in the stopband

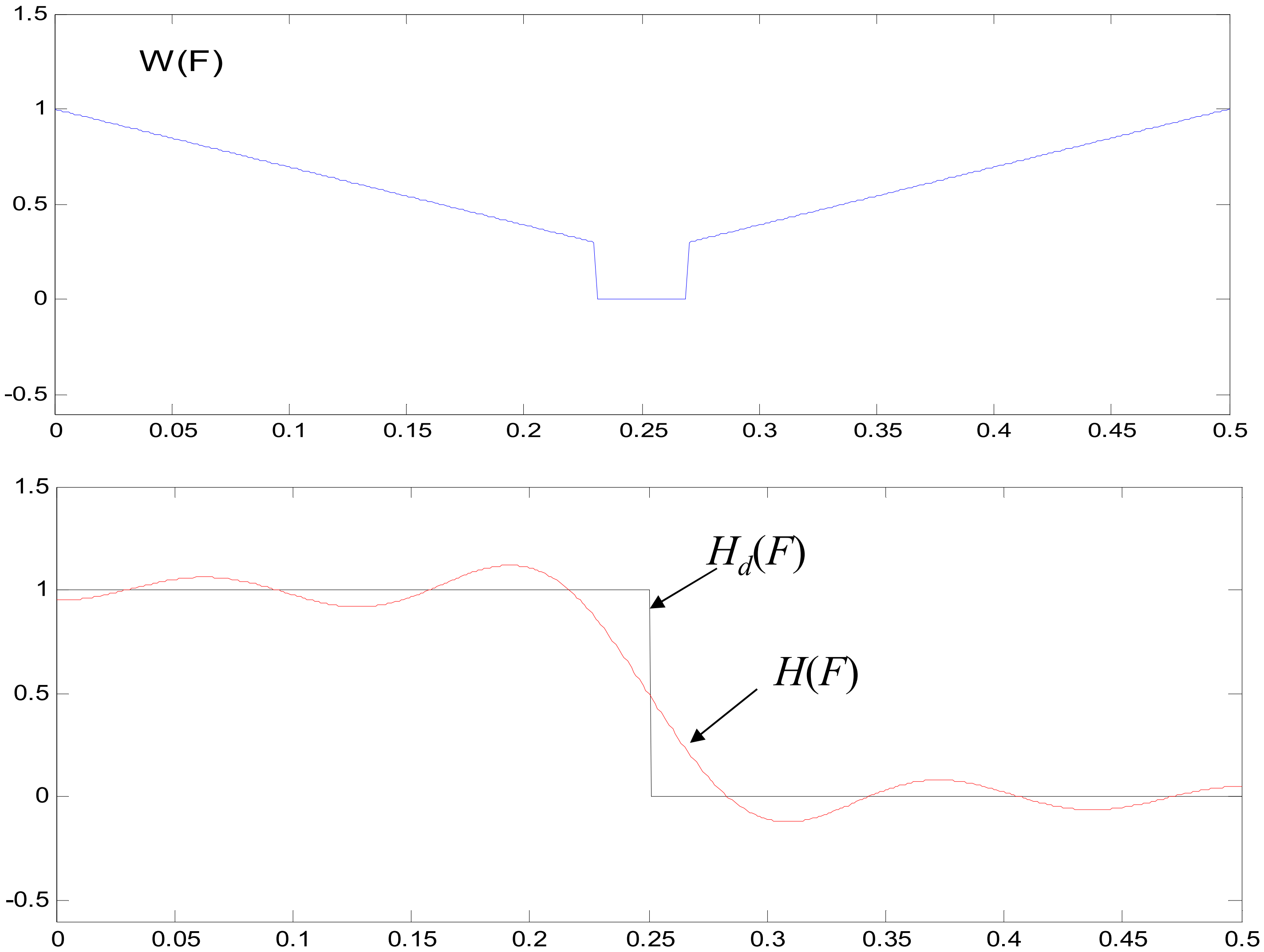


If we treat the stop band more important than the pass band

$0 < W(f) < 1$ in the passband, $W(f)$ = 1 in the stopband



Larger error near the transition band



◎ 2-L FIR Filter in MSE Sense with Weight Functions

$$R(F)=\sum_{n=0}^{k} s[n]\cos(2\pi nF)$$

可對照 pages 49~51

$$MSE=\int_{-1/2}^{1/2} W(F)\left|R(F)-H_d(F)\right|^2 dF$$

$F=f/f_s$

$$=\int_{-1/2}^{1/2} W(F)\left(\sum_{\tau=0}^{k} s[\tau]\cos(2\pi\tau F)-H_d(F)\right)^2 dF$$

$$\frac{\partial MSE}{\partial s[n]}=2\int_{-1/2}^{1/2} W(F)\cos(2\pi nF)\left(\sum_{\tau=0}^{k} s[\tau]\cos(2\pi\tau F)-H_d(F)\right)dF=0$$

$n=0,1,\cdots k$

Compared to page 49

$$2\sum_{\tau=0}^{k} s[\tau]\int_{-1/2}^{1/2} W(F)\cos(2\pi nF)\cos(2\pi\tau F)dF-2\int_{-1/2}^{1/2} W(F)H_d(F)\cos(2\pi nF)dF=0$$

$$n=0\sim k$$

問題： $\int_{-1/2}^{1/2} W(F)\cos(2\pi nF)\cos(2\pi\tau F)dF\neq 0$ when $n\neq\tau$

(not orthogonal)

$$\sum_{\tau=0}^{k} s[\tau]\int_{-1/2}^{1/2} W(F)\cos(2\pi n F)\cos(2\pi \tau F)\,dF = \int_{-1/2}^{1/2} W(F)H_d(F)\cos(2\pi n F)\,dF$$

$\tau = 0 \sim k,\quad n = 0 \sim k$

可以表示成 $(k+1) \times (k+1)$ matrix operation

$$\begin{array}{c} \\ n=0 \\ n=1 \\ n=2 \\ \\ n=k \end{array}
\begin{array}{c}
\begin{array}{ccccc} \tau=0 & \tau=1 & \tau=2 & & \tau=k \end{array} \\
\begin{bmatrix}
B[0,0] & B[0,1] & B[0,2] & \cdots & B[0,k] \\
B[1,0] & B[1,1] & B[1,2] & \cdots & B[1,k] \\
B[2,0] & B[2,1] & B[2,2] & \cdots & B[2,k] \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B[k,0] & B[k,1] & B[k,2] & \cdots & B[k,k]
\end{bmatrix}
\end{array}
\begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ \vdots \\ s[k] \end{bmatrix}
=
\begin{bmatrix} C[0] \\ C[1] \\ C[2] \\ \vdots \\ C[k] \end{bmatrix}$$

$$\mathbf{B} \qquad \mathbf{S} = \mathbf{C} \qquad \therefore\ \mathbf{S} = \mathbf{B}^{-1}\mathbf{C}$$

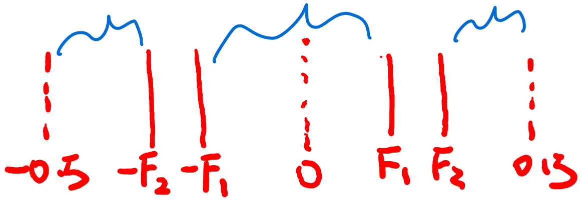
$$B[n,\tau] = \int_{-1/2}^{1/2} W(F)\cos(2\pi n F)\cos(2\pi \tau F)\,dF$$

$$C[n] = \int_{-1/2}^{1/2} W(F)H_d(F)\cos(2\pi n F)\,dF$$

When $W(F)=1$

$$B[n,\tau] = \begin{bmatrix} 1 & & & & 0 \\ & 1/2 & & & \\ & & 1/2 & & \\ & & & \ddots & \\ 0 & & & & 1/2 \end{bmatrix}$$

when we have weight function, we cannot guarantee orthogonality



犧牲 $F_1 \sim F_2$ 和 $-F_1 \sim -F_2$ 的精準性
來換取其他部分和 desired filter 之間的差距

Q： Is it possible to apply the transition band to the FIR filter in the MSE sense?

改變積分範圍，跳過 transition band 即可.

$$MSE = ? \int_{-0.5}^{-F_2} W(F)\left|R(F)-H_d(F)\right|^2 dF + \int_{-F_1}^{F_1} W(F)\left|R(F)-H_d(F)\right|^2 dF + \int_{F_2}^{0.5} W(F)\left|R(F)-H_d(F)\right|^2 dF$$

引入 weight function $W(F)$
能夠決定哪邊誤差可以大一點
哪邊可以小一點.

for $B[n,\tau] = ?$
$C[n]$

$$\int_{\boxed{-\frac{1}{2}}}^{\boxed{\frac{1}{2}}} \cdots dF \Rightarrow \int_{-0.5}^{-F_2} \cdots dF + \int_{-F_1}^{F_1} \cdots dF + \int_{F_2}^{0.5} \cdots dF$$

原本的積分範圍.

◉ 2-M Four Types of FIR Filter

$h[n] = 0$ for $n < 0$ and $n \geq N$ 點數為 N

$$H(F) = \sum_{n=0}^{N-1} h[n] \exp(-j2\pi n F)$$

- Type 1 $R(F) = \sum_{n=0}^{k} s[n] \cos(2\pi n F)$ ← 之前的方法只討論到 Type 1

$h[n_1] = h[n_2 - n]$ and N is odd.

(even symmetric)

$k = (N-1)/2$

根據 $\begin{cases}\text{even} \\ \text{odd}\end{cases}$ symmetric, $N \begin{cases}\text{even} \\ \text{odd}\end{cases}$ 分為四種 type

用MSE方法設計的話 → 代換到 p.49 的 R(F)
minimax → 則須將 type 2~4 轉為 type 1 (p.93)

以前的假設
→ 但我們希望更 general

- Type 1: $R(F) = \sum_{n=0}^{k} s[n]\cos(2\pi n F)$

  $\underline{h[n] = h[N-1-n]}$ (even symmetric) and $\underline{N \text{ is odd.}}$

- Type 2: $R(F) = \sum_{n=1}^{k+1/2} s[n]\cos(2\pi (n-1/2) F)$ ← p.92 推導

  $\underline{h[n] = h[N-1-n]}$ (even symmetric) and $\underline{N \text{ is even.}}$

even symmetric
改 odd symmetric
↓
cos 即改 sin

- Type 3: $R(F) = \sum_{n=1}^{k} s[n]\sin(2\pi n F)$

  ∵ n=0 時 sin(2πnF) = sin(0) = 0 ∴ 可從 1 開始加.

  $\underline{h[n] = -h[N-1-n]}$ (odd symmetric) and $\underline{N \text{ is odd.}}$

- Type 4: $R(F) = \sum_{n=1}^{k+1/2} s[n]\sin(2\pi (n-1/2) F)$

  $\underline{h[n] = -h[N-1-n]}$ (odd symmetric) and $\underline{N \text{ is even.}}$

$k = (N-1)/2$

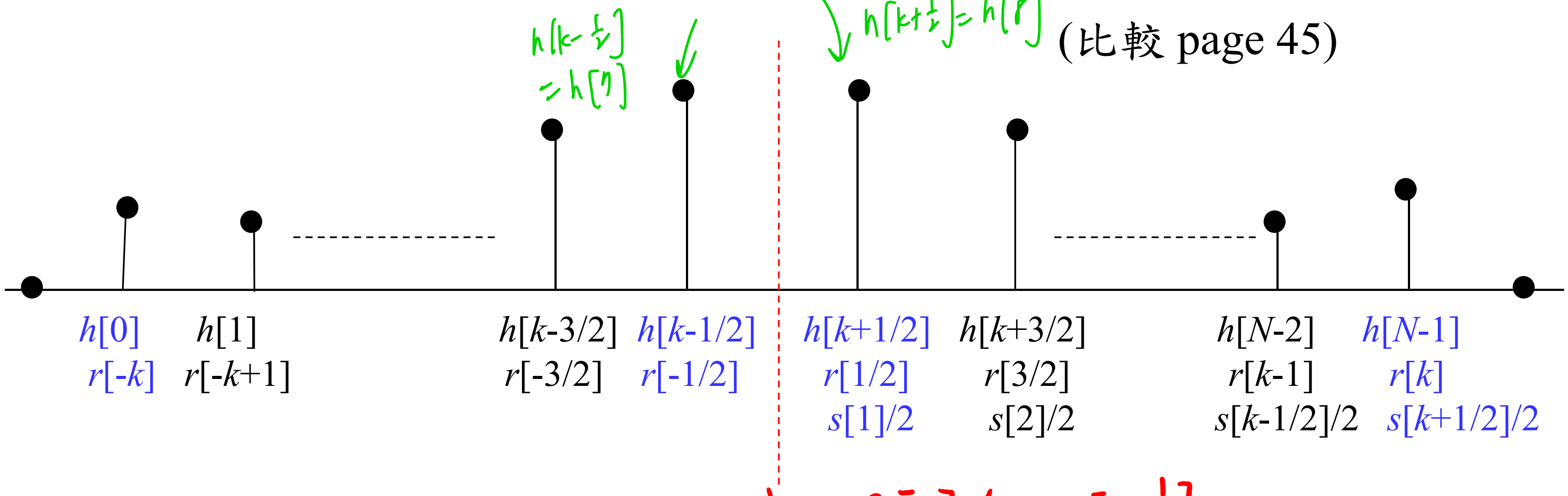
for MSE
substitute R(F) on page 49
by the corresponding functions

- Type 2: When $h[n] = h[N-1-n]$ and N is even:
  (even symmetric)

令 $r[n] = h[n+k]$, where $k = (N-1)/2$ (注意此時 k 不為整數)

ex1 $N=16 \rightarrow k=7.5$

(比較 page 45)



$s[n]/2 = r[n-\frac{1}{2}]$

$$= \sum_{n=-k}^{-1/2} r[n]\, e^{-j2\pi nF} + \sum_{n=1/2}^{k} r[n]\, e^{-j2\pi nF}$$

當 $R(F) = \sum_{n=-k}^{k} r[n]\exp(-j2\pi nF)$

$$R(F) = e^{j2\pi Fk} H(F)$$

$$2\sum_{n=1/2}^{k} r[n]\cos(2\pi nF)$$

$$R(F) = \sum_{n=1/2}^{k} \{r[n]\exp(-j2\pi nF) + r[-n]\exp(j2\pi nF)\}$$

$r[n]=r[-n]$

$$= \sum_{n=1/2}^{k} r[n]\{\exp(-j2\pi nF) + \exp(j2\pi nF)\} = \sum_{n=1/2}^{k} 2r[\boxed{n}]\cos(2\pi \boxed{n} F)$$

$n-\frac{1}{2}$ $\quad$ $n-\frac{1}{2}$ $\quad$ 代入

$\therefore n = \frac{1}{2}+\frac{1}{2} \sim n=k+\frac{1}{2}$

$$\boxed{R(F) = \sum_{n=1}^{k+1/2} s[n]\cos(2\pi(n-1/2)F)}$$

$$n_{(new)} = n_{(old)} + \frac{1}{2} \qquad n_{(old)} = n_{(new)} - \frac{1}{2}$$

$s[n] = 2r[n-1/2]$ $\quad n = 1, 2, \ldots, k+1/2$

設計出 $s[n]$ 之後

$r[n] = s[n+1/2]/2, \qquad h[n] = r[n-k],$

Design Method for Type 2

(for minimax)

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta)) = \cos\alpha\cos\beta$

$$R(F) = \sum_{n=1}^{k+1/2} s[n]\cos\left(2\pi(n-1/2)F\right)$$

由於 n 和 n+1 兩項相加可得

$\alpha = 2\pi nF$, $\beta = \pi F$

$$\cos\left(2\pi(n-1/2)F\right)+\cos\left(2\pi(n+1/2)F\right) = 2\cos(\pi F)\cos(2\pi nF)$$

所以可以「判斷」 $R(F)$ 能被改寫成

係數令為 $s_1[\,]$

$$\boxed{R(F) = \cos(\pi F)\sum_{n=0}^{k_1} s_1[n]\cos(2\pi nF)}$$

← 原 R(F) (p.49) 為:

$R(F) = \sum_{n=0}^{k} s[n]\cos(2\pi nF)$

∴ 這裡的 R(F) 只多了外面的 $\cos(\pi F)$

→ 想辦法去除就能轉換成原本的 type 1

求 $s_1[n]$ 和 $s[n]$ 之間的關係

$\cos(2\pi nF)$ 的 linear combination

$\cos(\pi F)$ 乘進來

$$R(F) = \sum_{n=0}^{k_1} s_1[n]\boxed{\cos(\underbrace{\pi F}_{\beta})\cos(\underbrace{2\pi nF}_{\alpha})}$$

利用 $\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta))$

$$= \sum_{n=0}^{k_1}\frac{1}{2}s_1[n]\cos\left(2\pi(n-1/2)F\right)+\sum_{n=0}^{k_1}\frac{1}{2}s_1[n]\cos\left(2\pi(n+1/2)F\right)$$

$$= \sum_{n=0}^{k_1}\frac{1}{2}s_1[n]\cos\left(2\pi(n-1/2)F\right)+\sum_{n=1}^{k_1+1}\frac{1}{2}s_1[n-1]\cos\left(2\pi(n-1/2)F\right)$$

$n_{new} = n_{old}+1$

$n_{old} = n_{new}-1$

$$R(F)=\sum_{n=0}^{k_1}\frac{1}{2}s_1[n]\cos(2\pi(n-1/2)F)+\sum_{n=1}^{k_1+1}\frac{1}{2}s_1[n-1]\cos(2\pi(n-1/2)F)$$

$n = 0$ $n = 1\sim k_1$ $n = 1\sim k_1$ $n = k_1+1$

$$R(F)=\frac{1}{2}s_1[0]\cos(\pi F)+\sum_{n=1}^{k_1}\frac{1}{2}(s_1[n]+s_1[n-1])\cos(2\pi(n-1/2)F)$$

$$+\frac{1}{2}s_1[k_1]\cos(2\pi(k_1+1/2)F)$$

$n = 1$ $n = 2\sim k_1$

$$R(F)=\left(s_1[0]+\frac{1}{2}s_1[1]\right)\cos(\pi F)+\sum_{n=2}^{k-1/2}\frac{1}{2}(s_1[n]+s_1[n-1])\cos(2\pi(n-1/2)F)$$

$n=k+\frac{1}{2}$

$$+\frac{1}{2}s_1[k-1/2]\cos(2\pi(k)F)$$

(令 $k_1 + 1/2 = k$)

比較係數可得

$$s[1]=s_1[0]+\frac{1}{2}s_1[1]$$

$$s[n]=\frac{1}{2}(s_1[n]+s_1[n-1])$$ for $n = 2,\ 3,\ \ldots,\ k-1/2$

$$s[k+1/2]=\frac{1}{2}s_1[k-1/2]$$

$$err(F) = [R(F) - H_d(F)]W(F)$$

$$= \left[\cos(\pi F)\sum_{n=0}^{k-1/2} s_1[n]\cos(2\pi nF) - H_d(F)\right]W(F)$$

$\frac{1}{\cos(\pi F)}$ 提 $\cos(\pi F)$

$$= \left[\sum_{n=0}^{k-1/2} s_1[n]\cos(2\pi nF) - \underline{\sec(\pi F)H_d(F)}\right]\underline{\cos(\pi F)W(F)}$$

只需將 pages 58-61 的方法當中，$H_d(F)$ 換成 $\underline{\sec(\pi F)H_d(F)}$

$$\left[\sum_{n=0}^{k} s[n]\cos(2\pi nF) - H_d(F)\right]W(F)$$

$W(F)$ 換成 $\underline{\cos(\pi F)W(F)}$

k 換成 $k - 1/2 = N/2 - 1$

注意 $s_1[n]$ 和 $s[n]$ 之間的關係即可

Design Method for Type 3

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\sin\beta\cos\alpha$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\sin\beta\cos\alpha$

$\alpha=2\pi nF$

$\beta=2\pi F$

$$R(F)=\sum_{n=1}^{k} s[n]\sin(2\pi nF)$$

由於 $n-1$ 和 $n+1$ 兩項相減可得

$$\sin(2\pi(n+1)F)-\sin(2\pi(n-1)F)=2\sin(2\pi F)\cos(2\pi nF)$$

$$=\sin(\underbrace{2\pi nF}_{\alpha}+\underbrace{2\pi F}_{\beta})-\sin(\underbrace{2\pi nF}_{\alpha}-\underbrace{2\pi F}_{\beta})=[\cancel{\sin\alpha\cos\beta}+\sin\beta\cos\alpha]-[\cancel{\sin\alpha\cos\beta}-\sin\beta\cos\alpha]$$

$$=2\sin\beta\cos\alpha$$

所以「判斷」可將 $R(F)$ 改寫為

$$\boxed{R(F)=\sin(2\pi F)\sum_{n=0}^{k_1} s_1[n]\cos(2\pi nF)}$$

求 $s_1[n]$ 和 $s[n]$ 之間的關係

$$R(F)=\frac{1}{2}\sum_{n=0}^{k_1} s_1[n]\sin(2\pi(n+1)F)-\frac{1}{2}\sum_{n=0}^{k_1} s_1[n]\sin(2\pi(n-1)F)$$

$$=\frac{1}{2}\sum_{n=0}^{k_1} s_1[n]\sin(2\pi(n+1)F)+\frac{s_1[0]}{2}\sin(2\pi F)-\frac{1}{2}\sum_{n=2}^{k_1} s_1[n]\sin(2\pi(n-1)F)$$

$$=\frac{1}{2}\sum_{n=1}^{k_1+1} s_1[n-1]\sin(2\pi nF)+\frac{s_1[0]}{2}\sin(2\pi F)-\frac{1}{2}\sum_{n=1}^{k_1-1} s_1[n+1]\sin(2\pi nF)$$

$$R(F) = \frac{s_1[0]}{2}\sin(2\pi\ F) + \frac{1}{2}(s_1[0] - s_1[2])\sin(2\pi\ F)$$
$$+\frac{1}{2}\sum_{n=2}^{k_1-1}(s_1[n-1] - s_1[n+1])\sin(2\pi n F)$$
$$+\frac{1}{2}s_1[k_1-1]\sin(2\pi k_1 F) + \frac{1}{2}s_1[k_1]\sin(2\pi(k_1+1)F)$$

令 $k_1 = k - 1$, 比較係數可得

$$s[1] = s_1[0] - \frac{1}{2}s_1[2]$$

$$s[n] = \frac{1}{2}s_1[n-1] - \frac{1}{2}s_1[n+1] \qquad \text{for } n = 2,\ 3,\ \ldots.,\ k-2$$

$$s[k-1] = \frac{1}{2}s_1[k-2]$$

$$s[k] = \frac{1}{2}s_1[k-1]$$

$$err(F) = [R(F) - H_d(F)]W(F)$$

$$= \left[\sin\left(2\pi F\right) \sum_{n=0}^{k-1} s_1[n] \cos\left(2\pi nF\right) - H_d(F) \right] W(F)$$

$$= \left[\sum_{n=0}^{k-1} s_1[n] \cos\left(2\pi nF\right) - \csc\left(2\pi F\right) H_d(F) \right] \sin\left(2\pi F\right) W(F)$$

將 pages 58-61 的方法當中，$H_d(F)$ 換成 $\csc\left(2\pi F\right) H_d(F)$

$W(F)$ 換成 $\sin\left(2\pi F\right) W(F)$

k 換成 $k-1$

注意 $s_1[n]$ 和 $s[n]$ 之間的關係即可

(Think)：Design the Method for Type 4

附錄三：寫 Matlab / Python 程式需注意的地方

一、各種程式語言寫程式共通的原則

(1) 能夠不在迴圈內做的運算，則移到迴圈外，以節省運算時間

(2) 寫一部分即測試，不要全部寫完再測試 (縮小範圍比較容易 debug)

(3) 先測試簡單的例子，成功後再測試複雜的例子

二、Matlab 寫程式特有的技巧

(1) 迴圈能避免就儘量避免

(2) 儘可能使用 Matrix 及 Vector operation

Example: 由 1 加 到100，用 Matlab 一行就可以了

```
sum([1:100])
```

完全不需迴圈

三、一些重要的 Matlab 指令

(1) function: 放在第一行，可以將整個程式函式化

(2) tic, toc: 計算時間

tic 為開始計時，toc 為顯示時間

(3) find: 找尋一個 vector 當中不等於 0 的entry 的位置

範例：find([1 0 0 1]) = [1, 4]

find(abs([-5:5])<=2) = [4, 5, 6, 7, 8]

(因為 abs([-5:5])<=2 = [0 0 0 1 1 1 1 1 0 0 0])

(4) ′ : Hermitian (transpose + conjugation)， .′ : transpose

(5) imread: 讀圖， image, imshow, imagesc: 將圖顯示出來，

(註： 較老的 Matlab 版本 imread 要和 double 並用

A=double(imread('Lena.bmp'));

(6) imwrite: 製做圖檔

(7) xlsread: 由 Excel 檔讀取資料

A = xlsread(‘檔名’, ‘工作表名’, 範圍);

例如

A = xlsread(‘test. xlsx’, ‘工作表1’, A1:D50);

(8) xlswrite: 將資料寫成 Excel 檔

(9) aviread: 讀取 video 檔

(10) dlmread: 讀取 *.txt 或其他類型檔案的資料

(11) dlmwrite: 將資料寫成 *.txt 或其他類型檔案

四、寫 Python 版本程式可能會用到的重要指令

建議必安裝模組

```
pip install numpy
pip install scipy
pip install opencv-python
pip install openpyxl   # for Excel files
```

(1) 定義函式：使用def

(2) 計算時間

```
import time
start_time = time.time()   #獲取當前時間
end_time = time.time()
total_time = end_time - start_time   #計算時間差來得到總執行時間
```

感謝2021年擔任助教的蔡昌廷同學

(3) 讀取圖檔、輸出圖檔(建議使用opencv)

```
import cv2
image = cv2.imread(file_name)  #預設color channel為BGR
cv2.imwrite(file_name, image)   #需將color channel轉為BGR
```

(4) 尋找array中滿足特定條件的值的位置

(相當於 Matlab 的 find 指令)

```
import numpy as np
a = np.array([0, 1, 2, 3, 4, 5])
index = np.where(a > 3) # 回傳array([4, 5])
print(index)
    (array([4, 5], dtype=int64),)
index[0][0]
    4
index[0][1]
   5
```

$$\mathbf{A_1} = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 4 & 5 \end{bmatrix}$$

```
A1= np.array([[1,3,6],[2,4,5]])
index = np.where(A1 > 3)
print(index)
    (array([0, 1, 1], dtype=int64), array([2, 1, 2], dtype=int64))
```

(代表滿足 A1 > 3 的點的位置座標為 [0, 2], [1, 1], [1, 2]

```
[index[0][0], index[1][0]]
   [0, 2]
[index[0][1], index[1][1]]
   [1, 1]
[index[0][2], index[1][2]]
   [1, 2]
```

(5) Hermitian、transpose

```
import numpy as np
result = np.conj(matrix.T)     # Hermitian
result = matrix.T    # transpose
```

(6) 在 Python 當中讀取 Matlab 當中的 mat 檔

```
data = scipy.io.loadmat('***.mat')
y = np.array(data['y'])   # 假設 y 是 ***.mat 當中儲存的資料
```

(7) 在 Python 當中讀取 Excel 檔

```
import openpyxl
data = openpyxl.load_workbook(‘filename’)
data1 = data[‘工作表名’]
A  = [row for row in data1.values]
A1 = np.array(A)
A1 = np.double(A1)  # 資料數值化
```

◉ 2-M Frequency Sampling Method

假設 designed filter $h[n]$ 的區間為 $n \in [0\ , N{-}1]$

filter 的點數為 N, $k = (N-1)/2$

remember:

$H_d(f) = H_d(f{+}f_s)$

- **Frequency Sampling 基本精神：**

若 $H_d(f)$ 是 desired filter 的 discrete-time Fourier transform

我們設計的 filter

$R(f)$ 是 $r[n] = h[n{+}P]$ 的 discrete-time Fourier transform

要求 $\boxed{R\left(\frac{m}{N}f_s\right) = H_d\left(\frac{m}{N}f_s\right)}$ for $m = 0, 1, 2, 3, \ldots\ldots, N-1$

f_s : sampling frequency

我們設計的 filter 和 desired filter 在取樣點的值相同.

若以 normalized frequency $F = f/f_s$ 表示

$\boxed{R\left(\frac{m}{N}\right) = H_d\left(\frac{m}{N}\right)}$ for $m = 0, 1, 2, 3, \ldots\ldots, N-1$

(see page 110)

References：

- L. R. Rabiner and B. Gold, *Theory and Application of Digital Signal Processing*, Prentice-Hall, N. J., 1975.
- B. Gold and K. Jordan, “A note on digital filter synthesis,” *Proc. IEEE*, vol. 56, no. 10, pp. 1717-1718, 1969.
- L. R. Rabiner and R. W. Schafer, “Recursive and nonrecursive realizations of digital filters designed by frequency sampling techniques,” *IEEE Trans. Audio and Electroacoust.*, vol. 19, no. 3, pp. 200-207. Sept. 1971.

設計方法：

Step 1 Sampling $H_d\left(\frac{m}{N}\right)$ for $m = 0, 1, 2, 3, \ldots\ldots, N-1$

Step 2 $$\boxed{r_1[n] = \frac{1}{N}\sum_{m=0}^{N-1} H_d\left(\frac{m}{N}\right)\exp\left(j\frac{2\pi m}{N}n\right)}$$ $n = 0, 1, \ldots., N-1$

換句話說，$r_1[n]$ 是 $H_d(m/N)$ 的 inverse discrete Fourier transform (IDFT)

Step 3 When N is odd

$r[n] = r_1[n]$ for $n = 0, 1, \ldots., k$ $k = (N-1)/2$

$r[n] = r_1[n+N]$ for $n = -k, -k+1, \ldots., -1$

注意：$r[n]$ 的區間為 $n \in [-(N-1)/2, (N-1)/2]$

Step 4 $h[n] = r[n-k]$ $k = (N-1)/2$

Proof:

注意，若 $R(F)$是 $r[n]$ 的 discrete-time Fourier transform

$$R(F)=\sum_{n=-\infty}^{\infty} r[n]e^{-j2\pi Fn}=\sum_{n=-k}^{k} r[n]e^{-j2\pi Fn}=\sum_{n=0}^{k} r[n]e^{-j2\pi Fn}+\sum_{n=-k}^{-1} r[n]e^{-j2\pi Fn}$$

$$=\sum_{n=0}^{k} r[n]e^{-j2\pi Fn}+\sum_{n=-k}^{-1} r_1[n+N]e^{-j2\pi F(n+N)}=\sum_{n=0}^{N-1} r_1[n]e^{-j2\pi Fn}$$

when $F=m/N$

(We apply the fact where $e^{-j2\pi Fn}=e^{-j2\pi F(n+N)}$ when $F=m/N$)

$$R(m/N)=\sum_{n=0}^{N-1} r_1[n]\exp\left(-j\frac{2\pi m}{N}n\right)$$

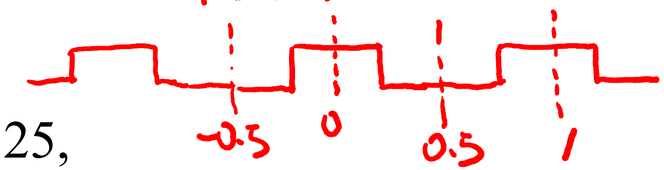
又由於 $r_1[n]$ 是 $H_d(m/N)$ 的 inverse discrete Fourier transform (IDFT)

$$H_d\left(\frac{m}{N}\right)=DFT\{r_1[n]\}=\sum_{m=0}^{N-1} r_1[n]\exp\left(-j\frac{2\pi m}{N}n\right)$$

所以 $R\left(\frac{m}{N}\right)=H_d\left(\frac{m}{N}\right)$

HW2：因為 discrete Fourier transform 的 index 永遠都是正的，所以 –0.5~0，根據週期的關係就變成 0.5~1 之間

$H_d(F) = H_d(F+1)$



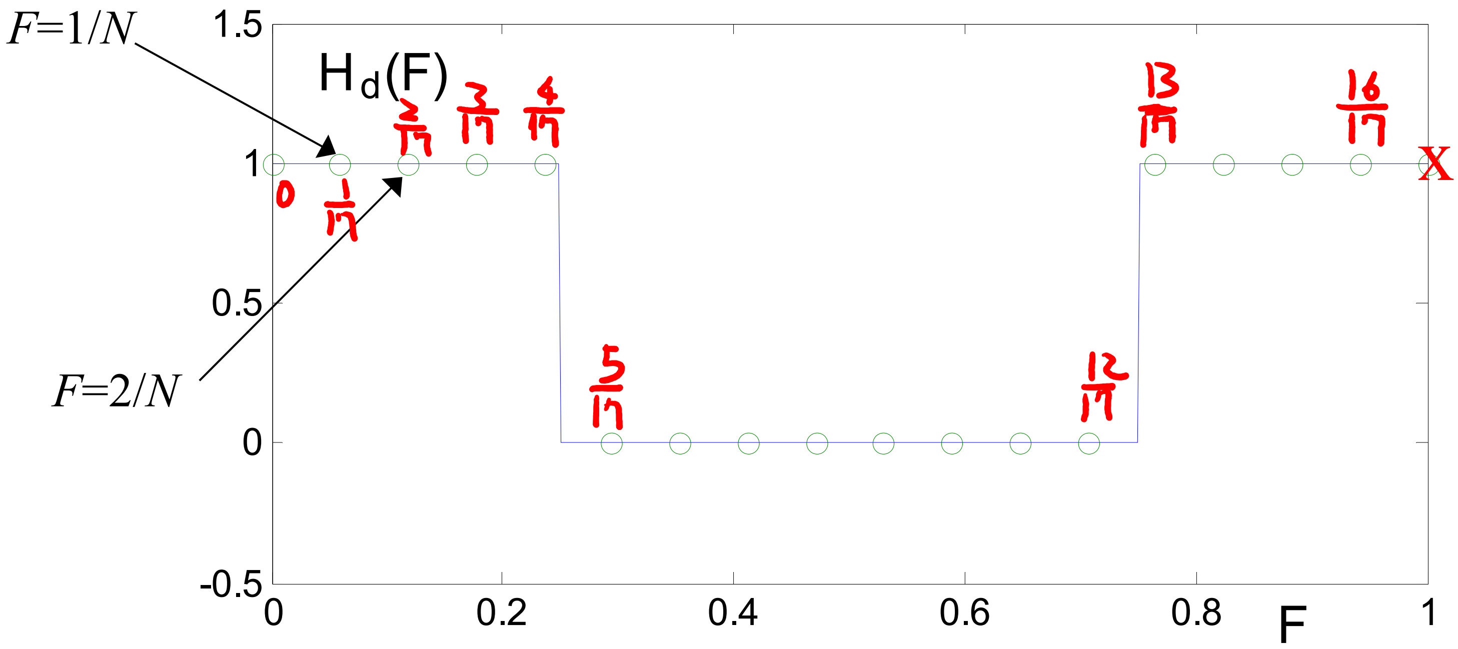
Example: $N = 17$

$H_d(F) = 1$ for $-0.25 < F < 0.25$,

$H_d(F) = 0$ for $-0.5 < F < -0.25$, $0.25 < F < 0.5$

(Step 1) 取樣

[1 , 1 , 1 , 1 , 1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 1 , 1 , 1 , 1]



inverse fourier transform

(Step 2)

$r_1[n] = \text{ifft}([1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1])$

$= [0.529\ \ 0.319\ \ -0.030\ \ -0.107\ \ 0.032\ \ 0.066\ \ -0.035\ \ -0.049\ \ 0.040$
$\underline{0.040\ \ -0.049\ \ -0.035\ \ 0.066\ \ 0.032\ \ -0.107\ \ -0.030\ \ 0.319}]$ $\quad n = 0 \sim 16$

第9~16個點搬到左边(下頁圖)

(Step 3)

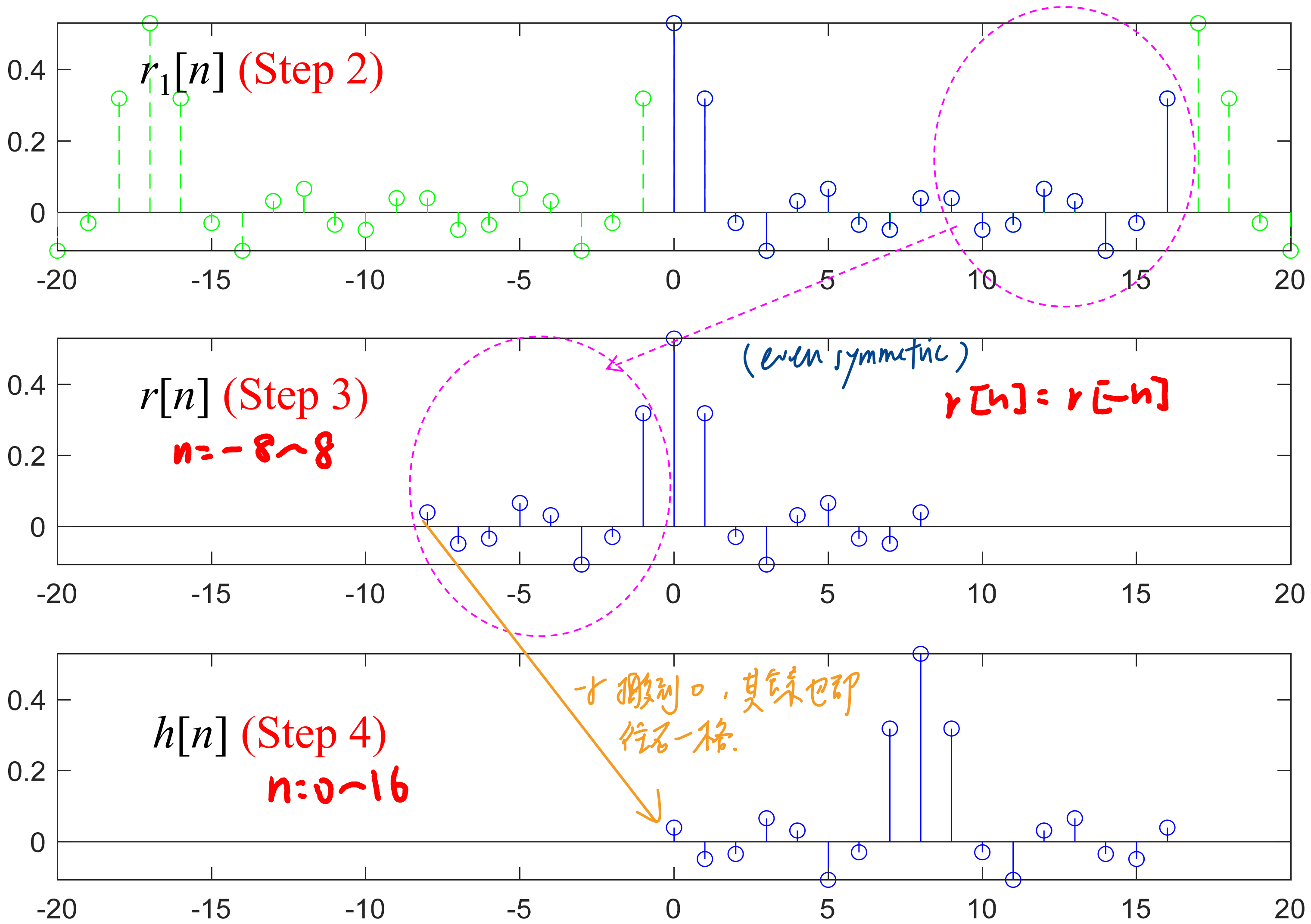
$r[n] = [\underline{0.040\ \ -0.049\ \ -0.035\ \ 0.066\ \ 0.032\ \ -0.107\ \ -0.030\ \ 0.319}\ \ 0.529$
$0.319\ \ -0.030\ \ -0.107\ \ 0.032\ \ 0.066\ \ -0.035\ \ -0.049\ \ 0.040]$ $\quad n = -8 \sim 8$

(Step 4)

若我們希望所設計出來的 filter $h[n]$ 有值的區域為 $n \in [0, 16]$

$h[n] = r[n-8]$

$= [0.040\ \ -0.049\ \ -0.035\ \ 0.066\ \ 0.032\ \ -0.107\ \ -0.030\ \ 0.319\ \ 0.529$
$0.319\ \ -0.030\ \ -0.107\ \ 0.032\ \ 0.066\ \ -0.035\ \ -0.049\ \ 0.040]$ $\quad n = 0 \sim 16$



$r_1[n]$ (Step 2)
0.4
0.2
0
-20
-15
-10
-5
0
5
10
15
20
$r[n]$ (Step 3)
n=-8~8
(even symmetric)
r[n]=r[-n]
$h[n]$ (Step 4)
n:0~16
-8 搬到 0，其餘也都往右一格.

Frequency Response in terms of $R(F)$

$$R\left(\frac{m}{N}\right) = H_d\left(\frac{m}{N}\right)$$

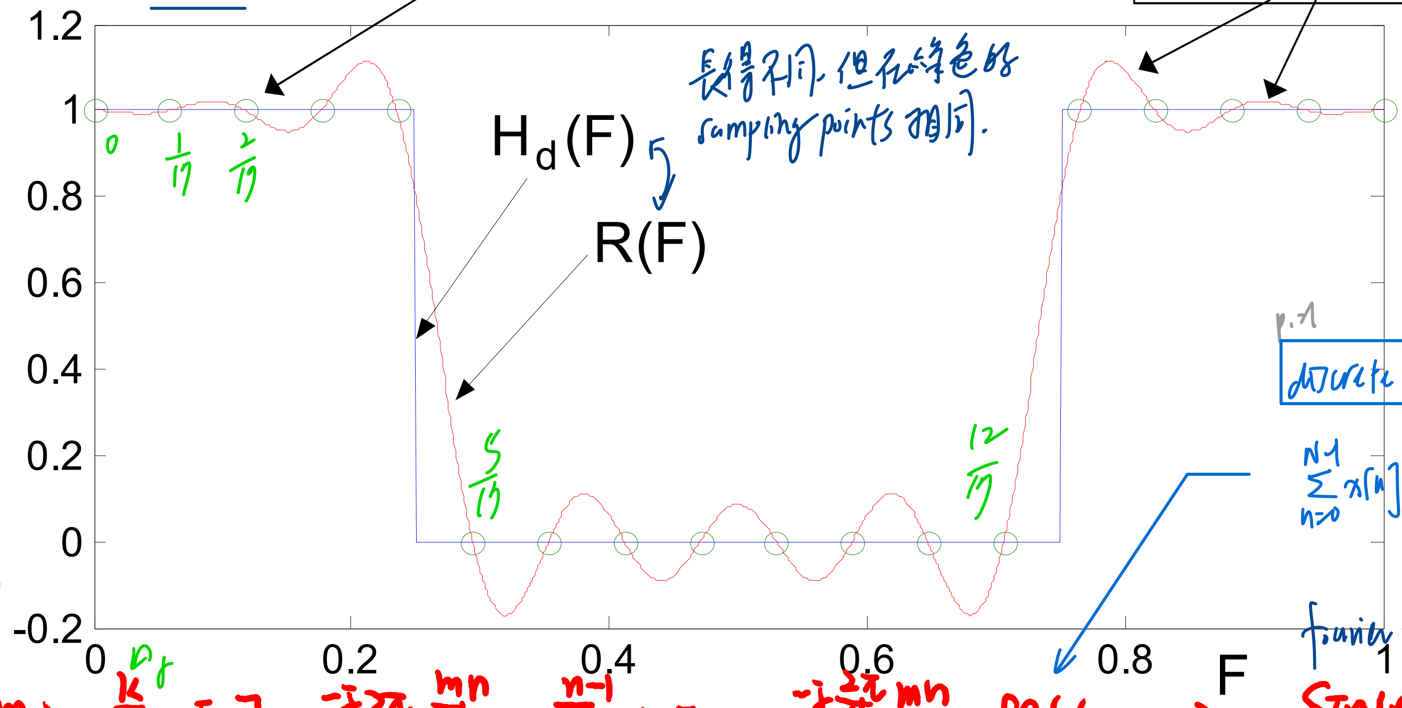
$$R(F) = \sum_{n=-\infty}^{\infty} r[n] e^{-j2\pi Fn}$$

($r[n]$ is from Step 3)

∵ $r[n] = r[-n]$ 在數學上此條件符合 even symmetric ∴ 是 $r[n]$ 而不是上一頁最後的 $h[n]$

$R(F)$在sample frequency 等於$H_d(F)$

Error 非 equal-ripple



p.71

$\boxed{\text{discrete fourier transform (DFT)}}$

$$\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi mn}{N}}$$

fourier transform有唯一性,

When $F = \frac{m}{N}$

$$R\left(\frac{m}{N}\right) = \sum_{n=-k}^{k} r[n] e^{-j2\pi\frac{mn}{N}} = \sum_{n=0}^{N-1} r_1[n] e^{-j\frac{2\pi}{N}mn} = fft(r_1[n])$$

$k = \frac{N-1}{2}$

Since $r_1[n] = ifft\left(H_d\left(\frac{m}{N}\right)\right)$

- The approximation error tends to be highest around the transition band and smaller in the passband and stopband regions.

$\therefore H_d\left(\frac{m}{N}\right) = R\left(\frac{m}{N}\right)$

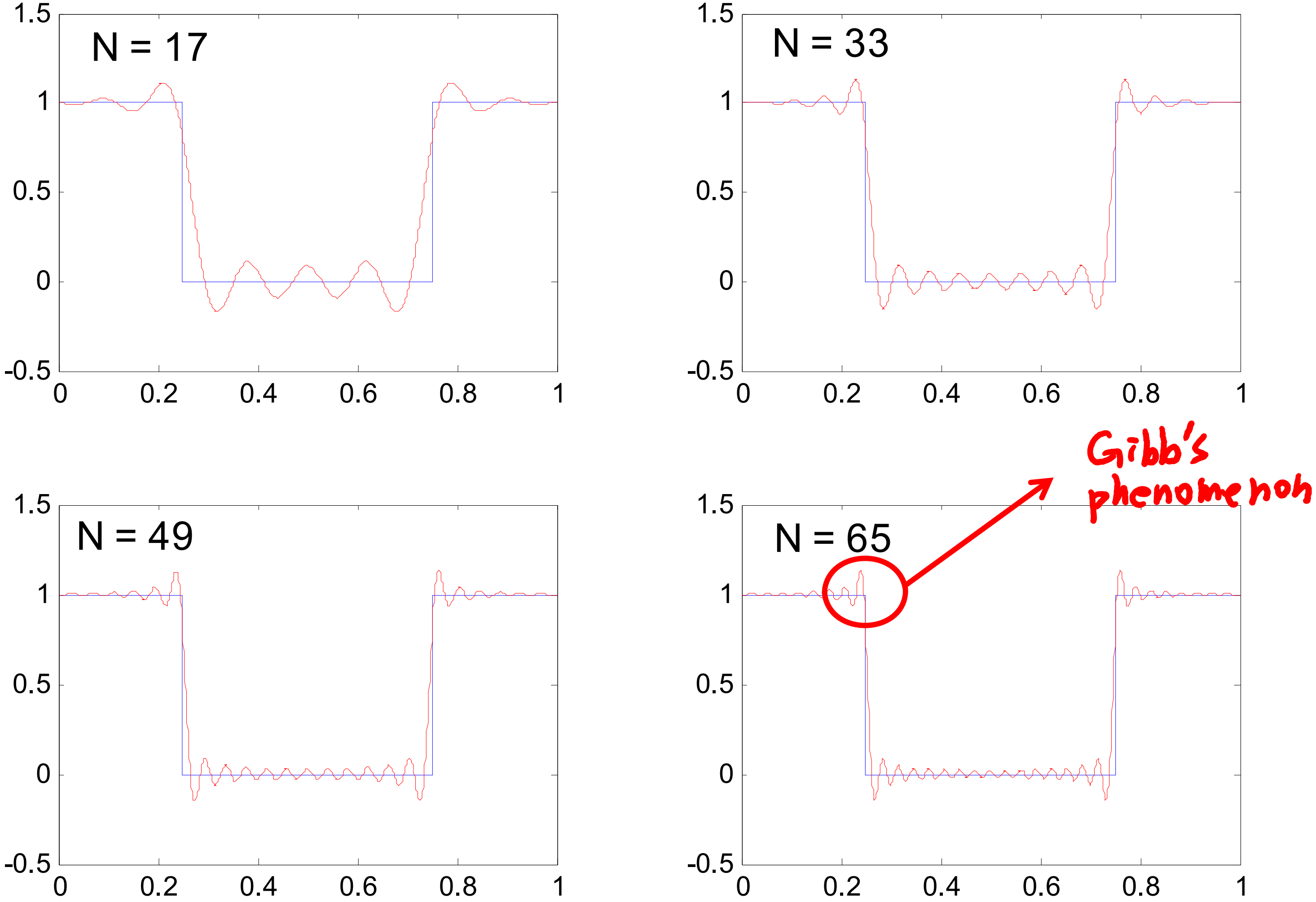
如果把N取大點

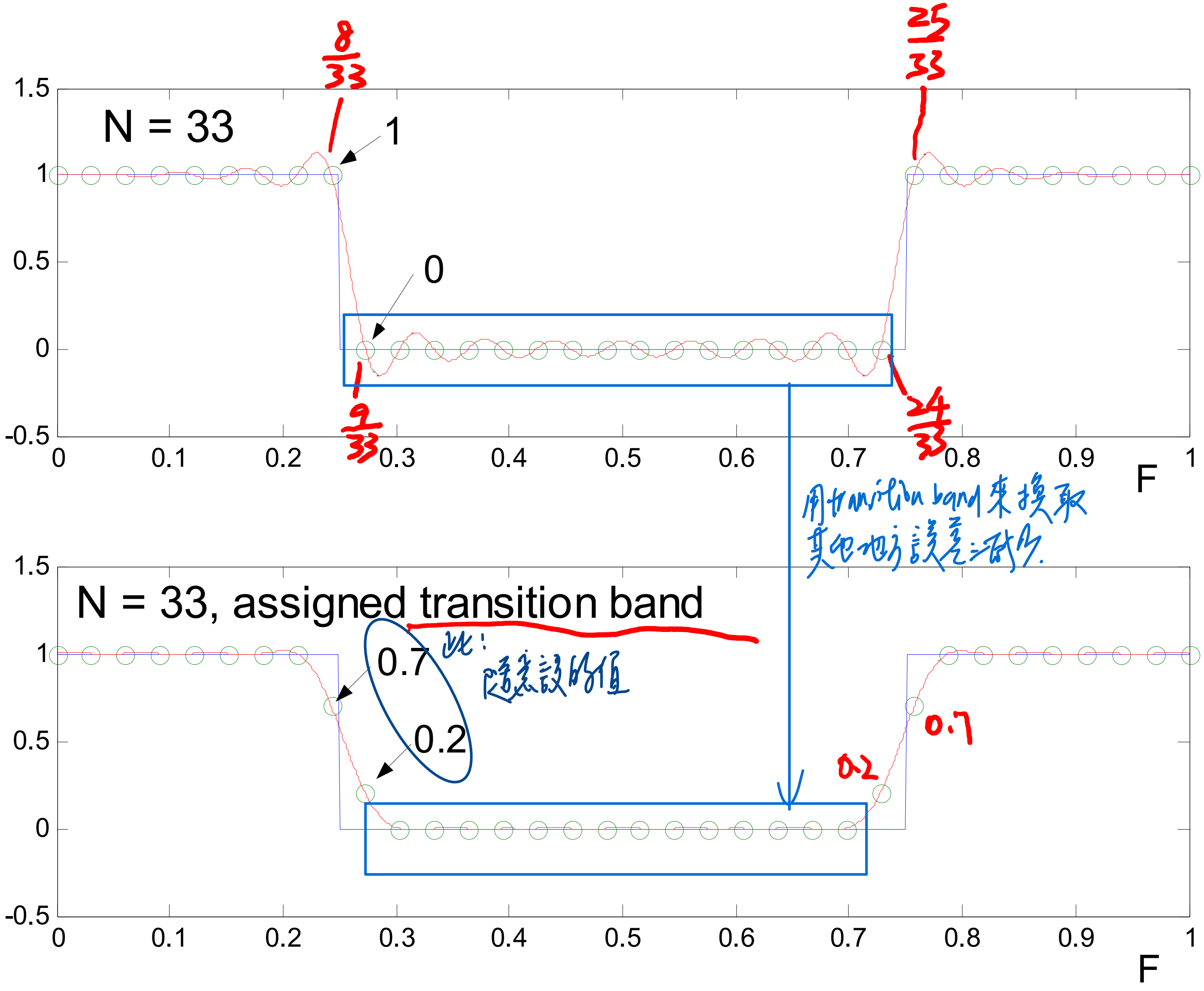
→ 4 sampling point 處的 R() 和 Hd() 值相同.

∴ 紅色線就會更貼近藍色

(R(F)) (Hd(F))

Error is larger at the edge




N = 33
1
0
8/33
25/33
9/33
24/33
F
用transition band來換取
其他地方誤差減少.
N = 33, assigned transition band
此!
隨意設的值
0.7
0.2
F

討論：

(1) Frequency sampling 的方法頗為簡單且直觀，

但得出來的 filter 不為 optimal

(2) Ripple 大小變化的情形，介於 MSE 和 Minimax 之間

(3) 可以用設定 transition band 的方式，來減少 passband 和 stopband 的 ripple。(In transition band, $R(m/N) \neq H_d(m/N)$).

然而，如何設定 transition band $R(m/N)$ 的值，讓 passband 和 stopband 的 ripple 變為最小 ……… 需要作 linear programming。

(運算時間不少)

為何要設 0.7, 0.2

為何要設兩個點. . . .

→ 不一定，須做最佳化

◉ 2-N 三種 FIR Digital Filter 設計方法的比較

- 以設計方法而論

  MSE : integrals, matrix

  現在沒有HW限制，就用得比較多.

  Minimax : most complicated (recursive)

  frequency sampling : simplest (ifft) ⇒ 運算較快速 (早期比較需要) ← HW限制.

  只須做.

- 以方法的限制而論

  沒有transition band的話，minimax的誤差永遠都不可能<0.5 (見p.549圖)

  MSE : no constraint

  Minimax : often used in pass-stop band filters; transition band is necessary

  frequency sampling : weight function cannot be applied

- 以效果而論

  可達到 optimal { MSE : minimize the mean square error
  Minimax : minimize the maximal error }

  frequency sampling : not optimal

把高頻的濾掉留低頻的.

The 4th Method for the FIR Filter Design?

$$x[n] \xrightarrow{\text{DFT}} X[m] \longrightarrow Y[m] = X[m]\underline{H[m]} \xrightarrow{\text{IDFT}} y[n]$$

$$\boxed{\begin{array}{l} H[m] = 1 \text{ for passband} \\ H[m] = 0 \text{ for stopband} \end{array}}$$

乘上 H[m] 來留下 passband
去除 stopband.

為什麼實務上不用這個看起來很簡單的方法？

complexity of DFT : $O(N\log_2 N)$

complexity of other FIR methods $O(N)$

N: length(x) if N >> filter length

ex: 音訊每秒至少要取 2000 → 1 sec 時才能做 FT.

fs > 2B (p.96)

Q: Why do we not apply the method?

⦿ 2-O Implementation of the FIR Filter

$$y[n] = x[n] * h[n]$$

convolution

(1) 使用 FFT

$$y[n] = IFFT[\ FFT\{x[n]\} \times FFT\{h[n]\}\ ]$$

(2) 直接作 summation 即可

(3) Sectioned FFT

$$y[n] = x[n] * h[n]$$

(2) 直接作 summation

假設 $h[n] = 0$ for $n < 0$ and $n \geq N$

$$y[n] = h[0]x[n] + h[1]x[n-1] + \ldots.. + h[N-2]x[n-N+2] + h[N-1]x[n-N+1]$$

- 若 $h[n] = h[N-1-n]$ (even symmetric), N 為odd

$$y[n] = h[0](x[n] + x[n-N+1]) + h[1](x[n-1] + x[n-N+2])$$
$$+ \ \ldots.. + h[k-1](x[n-k+1] + x[n-N+k]) + h[k]\, x[n-k]$$

$$k = (N-1)/2$$

3. Theories about IIR Filters

⦿ 3-A Minimum-Phase Filter

• FIR filter: The length of the impulse response is **finite**

usually **linear phase** (i.e., even or odd impulse response)

always stable

但是今天如果可以用一些方法來讓 IIR filter 變得穩定？

• IIR filter: (i) May be unstable

(ii) The length of the impulse response is **infinite**.

(Question): Is the implementation also a problem?

Advantages of the IIR filter:

References

- A. Antoniou, *Digital Filters: Analysis and Design*, McGraw-Hill, New York, 1979.
- T. W. Parks and C. S. Burrus, *Digital Filter Design*, John Wiley, New York, 1989.
- O. Herrmann and W. Schussler, 'Design of nonrecursive digital filters with minimum phase,' *Elec. Lett.*, vol. 6, no. 11, pp. 329-330, 1970.
- C. M. Rader and B. Gold, 'Digital filter design techniques in the frequency domain,' *Proc. IEEE*, vol. 55, pp. 149-171, Feb. 1967.
- R. W. Hamming, *Digital Filters*, Prentice-Hall, Englewood Cliffs, NJ, 1988.
- F. W. Isen, *DSP for MATLAB and LabVIEW*, Morgan & Claypool Publishers, 2009.

- IIR filter: The length of the impulse response is **infinite**.

  → try to make the energy concentrating on the region near to $n = 0$

.

  → try to make both the forward and the inverse transforms stable

using the **minimum phase filter**.

Advantages of the minimum phase filter

① (All the poles and all the zeros are within the unit circle.)

It makes both the forward and the inverse transforms stable

② It makes the impluse response concentrated around 0.

IIR 還是一個無限長的濾波器，但是如果 impulse response 都集中在零附近，假如因為記憶體空間的限制不得不把尾巴砍掉，所造成的影響也不會很大

Z transform $\quad H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$

分子提 R 個 z，分母提 S 個 z

$H(z)$ can be expressed as $\quad C\dfrac{(z-z_1)(z-z_2)(z-z_3)\cdots\cdots(z-z_R)}{(z-p_1)(z-p_2)(z-p_3)\cdots\cdots(z-p_S)}$

$$= C z^{R-S} \frac{(1-z_1 z^{-1})(1-z_2 z^{-1})(1-z_3 z^{-1})\cdots\cdots(1-z_R z^{-1})}{(1-p_1 z^{-1})(1-p_2 z^{-1})(1-p_3 z^{-1})\cdots\cdots(1-p_S z^{-1})}$$

$p_1, p_2, p_3, \ldots\ldots, p_S$: **poles** $\qquad z_1, z_2, z_3, \ldots\ldots, z_R$: **zeros**

如果所有的 pole 都在單位圓中，那這個 filter 就是穩定了

- **Stable filter**: All the poles are within the unit circle.

- **Minimum phase filter**: All the poles and all the zeros are within the unit circle.

  i.e., $|p_s| \le 1$ and $|z_r| \le 1$

Minimum phase filter 不只要求 poles 要在單位圓內，zeros 也要
>> 這樣才能確保不管是 inverse 還是 forward transform 都會是穩定的

If any pole falls outside the unit circle ($|p_s| > 1$), then the impulse response of the filter is not convergent.

$$H(z) = C\frac{(z-z_1)(z-z_2)(z-z_3)\cdots\cdots(z-z_R)}{(z-p_1)(z-p_2)(z-p_3)\cdots\cdots(z-p_S)}$$

$$= C\,z^{R-S}\frac{(1-z_1z^{-1})(1-z_2z^{-1})(1-z_3z^{-1})\cdots\cdots(1-z_Rz^{-1})}{(1-p_1z^{-1})(1-p_2z^{-1})(1-p_3z^{-1})\cdots\cdots(1-p_Sz^{-1})}$$

跟 z^{−1} 有關的多項式

$$= C\,z^{R-S}\left(\boxed{Q(z^{-1})} + \frac{A_1}{1-p_1z^{-1}} + \frac{A_2}{1-p_2z^{-1}} + \cdots\cdots + \frac{A_S}{1-p_Sz^{-1}}\right)$$

If $R \geq S$, $Q(z^{-1})$ is a polynomial of z^{-1} with degree $R–S$.

$$Q(z^{-1}) = q_0 + q_1z^{-1} + \cdots + q_{R-S}z^{-(R-S)}$$

If $R < S$, $Q(z^{-1}) = 0$.

$$Z\left(A_s p_s^n u[n]\right) = \sum_{n=0}^{\infty} A_s p_s^n z^{-n} = A_s \sum_{n=0}^{\infty} (p_s z^{-1})^n = A_s \frac{1}{1-p_sz^{-1}}$$

$$h_s[n] = Z^{-1}\left(\frac{A_s}{1-p_sz^{-1}}\right) = A_s p_s^n u[n]$$

inverse z transform

Z^{-1} : inverse Z transform

~~for n = 1, 2, 3, 4,~~ .....

$u[n] = 1$ *for* $n \geq 0$,

$u[n] = 0$ *otherwise.*

If $|p_s| < 1$, $\lim_{n\to\infty} h_s[n] = 0$

If $|p_s| > 1$, $\lim_{n\to\infty} h_s[n] \to \pm\infty$

Inverse z transform 以後到底是 stable / unstable ，取決於 p_s 的絕對值到底是 <1 還是 >1

Therefore,

$$h[n] = C\left(\underbrace{q[n+R-S]}_{\text{FIR filter}} + \underbrace{\sum_{s=1}^{S} h_s[n+R-S]}_{\text{geometric series}} \right)$$

where

$$q[n] = q_n \quad for\ n = 1, 2, \cdots R-S$$

$$h_s[n] = A_s p_s^n u[n] \qquad \text{for } s = 1,\ 2,\ \ldots..,\ S$$

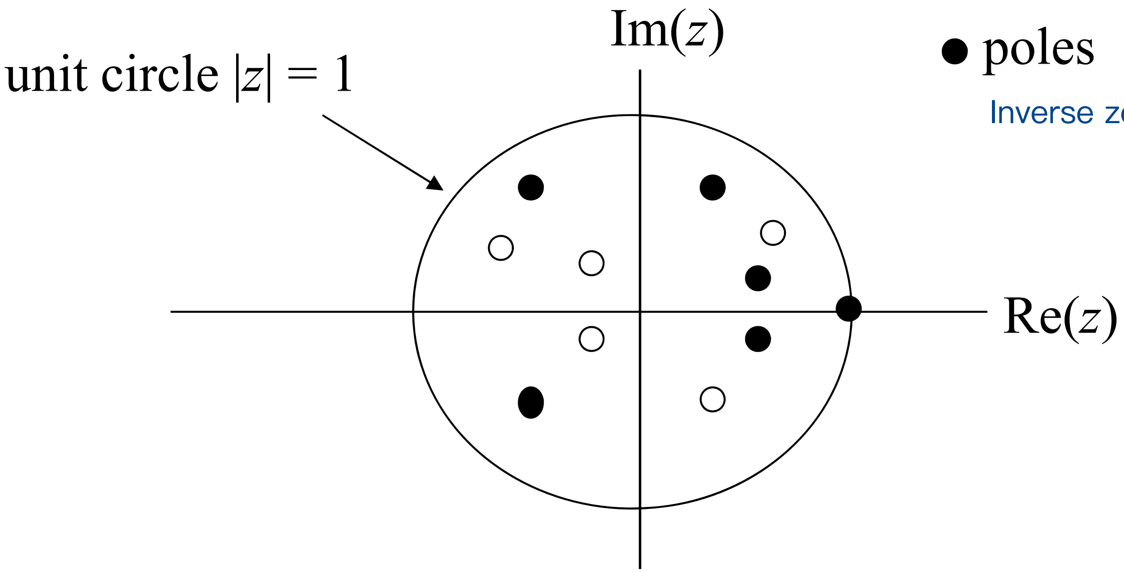
Thus, **the minimum phase filter is stable and causal**.

The **inverse** of the **minimum phase filter is stable and causal**.

$$H(z)=C\frac{(z-z_1)(z-z_2)(z-z_3)\cdots\cdots(z-z_R)}{(z-p_1)(z-p_2)(z-p_3)\cdots\cdots(z-p_S)}$$

要反推原本的信號
>> 需要做 H^{-1}()

$$H^{-1}(z)=C^{-1}z^{S-R}\frac{(1-p_1z^{-1})(1-p_2z^{-1})(1-p_3z^{-1})\cdots\cdots(1-p_Sz^{-1})}{(1-z_1z^{-1})(1-z_2z^{-1})(1-z_3z^{-1})\cdots\cdots(1-z_Rz^{-1})}$$



Inverse zero = pole

◉ 3-B Converting an IIR Filter into a Minimum Phase Filter

zeros

$$H(z)=C\frac{(z-\boxed{z_1})(z-\boxed{z_2})(z-\boxed{z_3})\cdots\cdots(z-\boxed{z_R})}{(z-\boxed{p_1})(z-\boxed{p_2})(z-\boxed{p_3})\cdots\cdots(z-\boxed{p_S})}$$

poles

不影響 amplitude

代表非 Minimum phase

Suppose that z_2 is not within the unit circle, $|z_2|>1$

$$H_1(z)=C\frac{(z-z_1)(z-z_2)(z-z_3)\cdots\cdots(z-z_R)}{(z-p_1)(z-p_2)(z-p_3)\cdots\cdots(z-p_S)}\times \boxed{z_2\frac{z-(\overline{z_2^{-1}})}{z-z_2}}$$

把原本是 IIR 且 stable ，但不是 minimum phase 的 filter 轉換成 minimum phase 只需要乘上這個東西

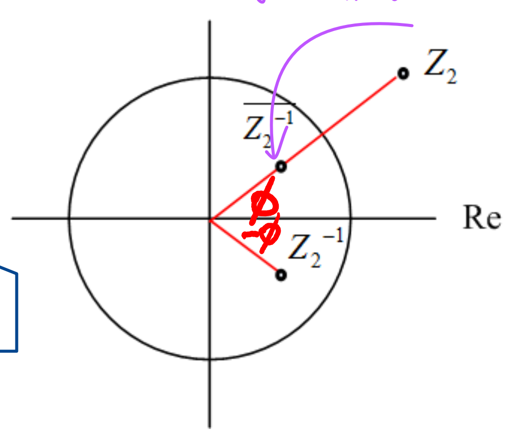
$$=\boxed{z_2}C\frac{(z-z_1)\boxed{\left(z-(\overline{z_2^{-1}})\right)}(z-z_3)\cdots\cdots(z-z_R)}{(z-p_1)(z-p_2)(z-p_3)\cdots\cdots(z-p_S)}$$

replace z_2 by $\overline{z_2^{-1}}$

把 z_2 移到單位圓內.

The upper bar means conjugation.

$$\begin{cases} Z_2 = Ae^{j\phi} \\ \overline{Z_2^{-1}} = \dfrac{1}{A}e^{j\phi} \\ Z_2^{-1} = \dfrac{1}{A}e^{-j\phi} \end{cases}$$



$\overline{z_2^{-1}}$ 和 z_2 在同一條線上,
只是 $A \to \frac{1}{A}$

新的 zero 和舊的角度（phase）一樣，但振幅相反（取倒數）

$\overline{z_2^{-1}}$ z_2

$\boxed{\text{Z-transform}}$ $G(z)=\sum_{n=-\infty}^{\infty} g[n]z^{-n}$

$\boxed{\text{DTFT}}$ $G(f)=\sum_{n=-\infty}^{\infty} g[n]e^{-j2\pi f n\Delta t}$ $(z=e^{j2\pi f\Delta t}) \to e^{j2\pi F}$, $|z|=1$

$= \sum_{n=-\infty}^{\infty} g[n]e^{-j2\pi F n}$

In fact, if $z=e^{j2\pi F}$ (see page 29), then $H(z)$ and $H_1(z)$ only differ in phase,

$$|H_1(F)|=|H(F)|$$

(proof):

$z-(\overline{z_2^{-1}}) = z(1-(\overline{z_2^{-1}})z^{-1}) = z(\overline{z_2^{-1}})(\overline{z_2}-z^{-1})$

提出.

amplitudes are 1

$$\left|z_2\frac{z-(\overline{z_2^{-1}})}{z-z_2}\right| = \left|z_2\boxed{(\overline{z_2^{-1}})z}\frac{\overline{z_2}-z^{-1}}{z-z_2}\right| = \left|z_2(\overline{z_2^{-1}})z\frac{\overline{z_2}-\overline{z}}{z-z_2}\right| = 1$$

when $z=e^{j2\pi F}$, $z^{-1}=\overline{z}$ when $z=e^{j2\pi F}$

(單位圓上)

- We call the filter whose amplitude response is always 1 as the **all-pass filter**.

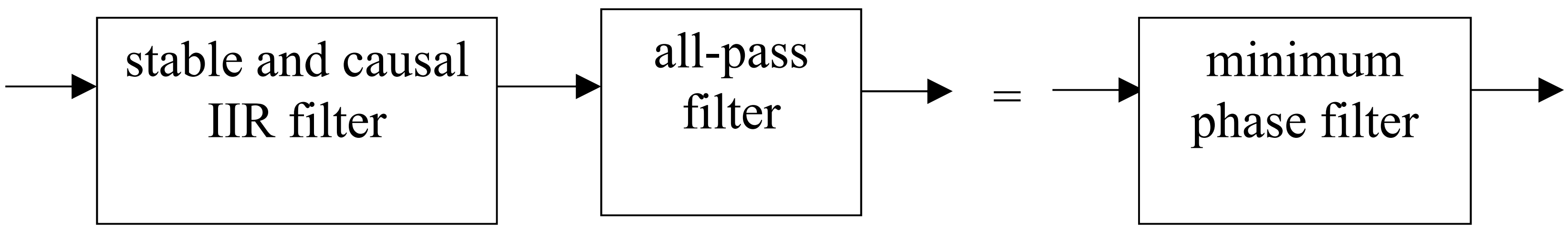
$$z_2\frac{z-(\overline{z_2^{-1}})}{z-z_2}$$ **is an all-pass filter**

- One can also use the similar way to move poles from the outside of the unit circle into the inside of the unit circle.

Any stable IIR filter can be expressed as a cascade of the **minimum phase filter** and an **all-pass filter**.

$H(z)$:IIR filter, $H_{mp}(z)$: minimum phase filter, $H_{ap}(z)$: allpass filter

$$H(z)H_{ap}(z) = H_{mp}(z)$$



Example:

但 zero：−0.6, 1.6 + 1.2j
>> 有 zero 在單位圓之外

$$H(z)=\frac{(z+0.6)[z-(1.6+1.2j)]}{z-0.9}$$

pole = 0.9 在單位圓內

stable ($|\text{pole}|=0.9<1$)
but not minimum phase
$|1.6+1.2j|=2>1$

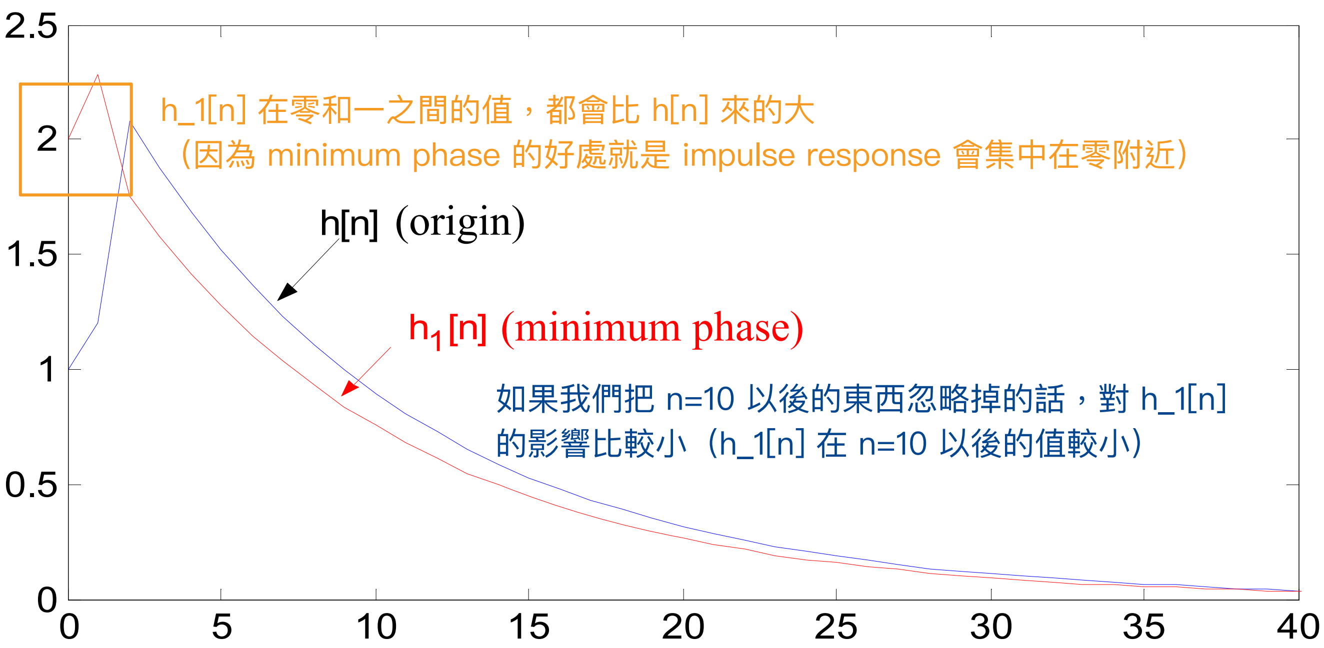
$$\times z_2\cdot\frac{z-\overline{(z_2^{-1})}}{z-z_2}$$

$$\frac{1}{1.6+1.2j}=0.4-0.3j \text{ conjugates with } \underbrace{0.4+0.3j}_{\overline{z_2^{-1}}}$$

$$H_1(z)=\underbrace{\boxed{(1.6+1.2j)}}_{z_2}\frac{(z+0.6)[z-(0.4+0.3j)]}{z-0.9}$$

$z_2=1.6+1.2j$
$=2(0.8+0.6j)$
$z_2^{-1}=\frac{1}{2}(0.8-0.6j)$
$\overline{z_2^{-1}}=\frac{1}{2}(0.8+0.6j)$
$=0.4+j0.3$

$h[n]$, $h_1[n]$ are the impulse response of the two filters $H(z)$ and $H_1(z)$



如果我們設計出來的 filter 是 stable 但非 minimum phase （pole in unit circle, but exists zero not in unit circle），如何轉換成 minimum phase？

◉ 3-C The Meaning of Minimum Phase

Another important advantage of the minimum phase filter：

The energy concentrating on the region near to *n* = 0.

$$H(z) = C\frac{(z-z_1)(z-z_2)(z-z_3)\cdots\cdots(z-z_R)}{(z-p_1)(z-p_2)(z-p_3)\cdots\cdots(z-p_S)}$$

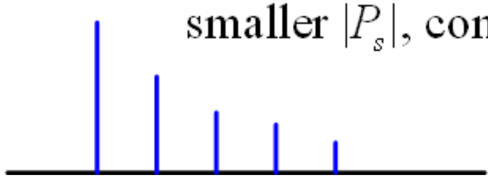
在 frequency domain 做相乘 = 在 time domain 做 convolution

$$= Cz^{R-S}\frac{(1-z_1z^{-1})(1-z_2z^{-1})(1-z_3z^{-1})\cdots\cdots(1-z_Rz^{-1})}{(1-p_1z^{-1})(1-p_2z^{-1})(1-p_3z^{-1})\cdots\cdots(1-p_Sz^{-1})}$$

$Z^{-1}\left[\dfrac{1}{1-p_sz^{-1}}\right] = a_s[n]$ $\quad a_s[n] = 0$ when $n < 0$ $\quad a_s[n] = \boxed{p_s^n}$ when $n \geq 0$

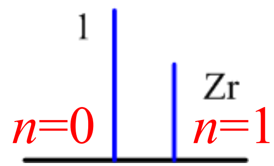
smaller $|P_s|$, converge faster

越往零集中



$Z^{-1}\left[1-z_rz^{-1}\right] = b_r[n]$ $\quad b_r[0] = 1,\ b_r[1] = -z_r,\ b_r[n] = 0$ otherwise

$r = 1, \ldots, R$



如果希望 impulse response 在零附近
>> 希望 z_r 越小越好（p_s 也越小越好）
>> 代表他們都在單位圓中且越接近零

Phase is related to delay

$$x[n-\tau] \xrightarrow[\text{}]{\text{discrete time Fourier transform}} e^{-j2\pi f\tau\Delta_t}X(f)$$

Minimum phase $\rightarrow$ Minimum delay

$$H(z)=Cz^{R-S}\frac{(1-z_1z^{-1})(1-z_2z^{-1})(1-z_3z^{-1})\cdots\cdots(1-z_Rz^{-1})}{(1-p_1z^{-1})(1-p_2z^{-1})(1-p_3z^{-1})\cdots\cdots(1-p_Sz^{-1})}$$

The multiplications in the Z domain (frequency domain) are equivalent to the convolutions in the time domain, so we could analyze each term individually in the previous page!!

(Question): How about the case of $|p_n| = 1$ or $|z_n| = 1$?

Note:

$$Z^{-1}\left[\frac{1}{1-p_s z^{-1}}\right] = a_s[n] \qquad a_s[n] = 0 \text{ when } n < 0 \qquad a_s[n] = p_s^n \text{ when } n \geq 0$$

When $|p_n| = 1$, the response is finite but the energy is infinite.

附錄四：查資料的方法

(1) Google 學術搜尋 (不可以不知道)

網址： http://scholar.google.com.tw/

(太重要了，不可以不知道) 只要任何的書籍或論文，在網路上有電子版，都可以用這個功能查得到

輸入關鍵字，或標題、作者

再按「搜尋」，就可找到想要的資料

註：由於版權，大部分的論文必需要在學校上網才可以下載

按搜尋之後將出現相關文章

點選後，可找到該學術文章的原始出處和相關的電子檔

可限定要找的文章的刊登時間

若要引用這篇論文，可點選此按鈕，會出現三種不同格式的引用方式

(2) 尋找 IEEE 的論文

http://ieeexplore.ieee.org/Xplore/guesthome.jsp

註：除非你是 IEEE Member，否則必需要在學校上網，才可以下載到 IEEE 論文的電子檔

(3) Google

(4) Wikipedia

(5) ChatGPT

(6) 數學的百科網站

http://eqworld.ipmnet.ru/index.htm

有多個 tables，以及對數學定理的介紹

(7) 傳統方法：去圖書館找資料

台大圖書館首頁 http://www.lib.ntu.edu.tw/

或者去 http://www.lib.ntu.edu.tw/tulips

(8) 查詢其他圖書館有沒有我要找的期刊

台大圖書館首頁 ⟶ 其他聯合目錄 ⟶ 全國期刊聯合目錄資料庫

如果發現其他圖書館有想要找的期刊，可以申請「館際合作」，請台大圖書館幫忙獲取所需要的論文的影印版

台大圖書館首頁 ⟶ 館際合作

(9) 查詢其他圖書館有沒有我要找的書

「台大圖書館首頁」⟶「其他圖書館」

(10) 找尋電子書

「台大圖書館首頁」⟶「電子書」或「免費電子書」

(11) 中文電子學位論文服務

http://www.cetd.com.tw/ec/index.aspx

可以查到多個碩博士論文 (尤其是 2006年以後的碩博士論文) 的電子版

(12) 想要對一個東西作入門但較深入的了解:

看書會比看 journal papers 或 Wikipedia 適宜

如果實在沒有適合的書籍，可以看 “review”， “survey” ，或 “tutorial” 性質的論文

(13) 有了相當基礎之後，再閱讀 journal papers

(以 Paper Title， Abstract， 以及其他 Papers 對這篇文章的描述，

來判斷這篇 journal papers 應該詳讀或大略了解即可)

(14) 積分查詢網站：http://integrals.wolfram.com/index.jsp

(15) 可以查詢數學公式的工具書 (Handbooks)

M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, McGraw-Hill, 3rd Ed., New York, 2009.（已經有電子版）

M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions, with Formula, Graphs and Mathematical Tables*, Dover Publication, New York, 1965.

A. Jeffrey, *Handbook of Mathematical Formulas and Integrals*, Academic Press, San Diego, 2000.

4. Some Popular Filters

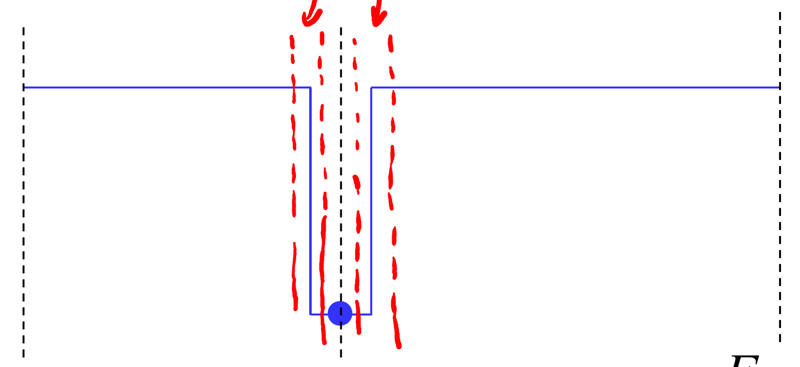
◉ 4-A Popular Filters (1): Pass-Stop Band Filters

highpass　　bandpass　　lowpass　　allpass　　bandstop

very narrow transition band

notch filter: 想濾掉 $F = F_0$ 的 noise，但 stop band 越小越好

Stop band 特別窄的 band stop filter



和 $e^{j2\pi F_0 n}$ 內積，就能知道這個的分量

$x[n]$ — $C_{F_0} e^{j2\pi F_0 n}$

例如：希望濾除汽車喇叭聲 400Hz，又希望 stop band 不要太寬，因為如果把 300~500Hz 都作為 stop band ，就會把很多人的聲音也濾除（人：100~600 Hz）

Question: Why the notch filter is hard to design?

以前：在不增加點數的情況下可以透過 transition band 設計來讓設計出來的 filter 誤差大幅減少

>> 但 notch filter 的 transition band 不能設太寬（因為 transition band 太寬就沒有 stop band 的存在了）

>> notch filter 不得不增加點數來提升效果

References

[1] K. Hirano, S. Nishimura, and S. K. Mitra, “Design of digital notch filters,” *IEEE Trans. Commun.*, vol. 22, no. 7, pp. 964-970, Jul. 1974.

[2] T. H. Yu, S. K. Mitra and H. Babic, “Design of linear phase FIR notch filters,” in *Sadhana*, Springer, vol. 15, issue 3, pp. 133-155, Nov. 1990.

[3] S. C. D. Roy, S. B. Jain, and B. Kumar, "Design of digital FIR notch filters," *Vision, Image and Signal Processing, IEE Proceedings*, vol.141, no. 5, pp.334-338, Oct. 1994.

[4] S. C. Pei and C. C. Tseng, “IIR multiple notch filter design based on allpass filter,” *IEEE Trans. Circuits Syst. II*, vol. 44, no.2, pp. 133-136, Feb. 1997.

[5] C. C. Tseng and S. C. Pei, “Stable IIR notch filter design with optimal pole placement,” *IEEE Trans. Signal Processing*, vol. 49, issue 11, pp. 2673-2681, Nov. 2001.

Smoother 顧名思義就是把信號比較小範圍的擾動和變化去除掉，來看大範圍的趨勢（見 p.145）

◉ 4-B Popular Filters (2): Smoother (Weighted Average)

近似 low-pass filter

最簡單的 smoother:

find the average $y[n]=\dfrac{1}{2L+1}\sum_{\tau=n-L}^{n+L}x[\tau]$

從 n–L 到 n+L 做平均

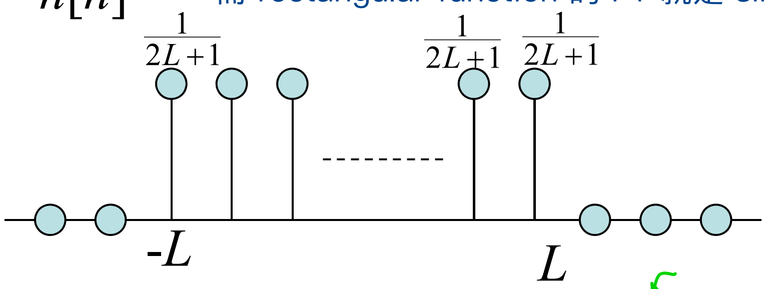
DFT

可改寫成

h[n] 類似 rectangular function 取 sample
而 rectangular function 的 FT 就是 sinc （見下頁證明）

$$y[n]=x[n]*h[n]$$

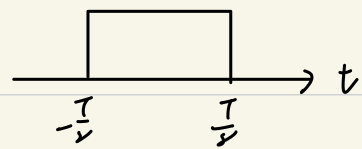
$h[n]$ 如右圖



$$y[n]=\sum_{\tau}x[n-\tau]h[\tau]=\sum_{\tau=-L}^{L}x[n-\tau]\frac{1}{2L+1}=\frac{1}{2L+1}\sum_{\tau=-L}^{L}x[n+\tau]$$

假設我們要算 3/26 的平均溫度，這樣的話應該要是越接近 3/26 加權要越大，但是我們這樣的設定變成 3/21~3/31 的加權都相同（假設我們取前後五天求平均）
>> 我們的 smooth filter 可以改成像 p.144 一樣的型態

$$\mathrm{rect}_T(t) = \begin{cases} 1 & \text{if } |t| \le \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$



continuous FT of $x(t)$ is defined as :

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt \qquad \mathrm{rect}_T(t) \text{ 代入 } x(t)$$

$$\rightarrow X(f) = \int_{-\infty}^{\infty} \mathrm{rect}_T(t)\, e^{-j2\pi f t}\, dt$$

∵ $\mathrm{rect}_T(t)$ 在 t介於 $\pm\frac{T}{2}$ 間 $=1$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi f t}\, dt$$

$$= \left[\frac{e^{-j2\pi f t}}{-j2\pi f} \right]_{-\frac{T}{2}}^{\frac{T}{2}}$$

$$= \frac{1}{-j2\pi f} \left(e^{-j\overset{\theta}{\underline{\pi f T}}} - e^{j\overset{\theta}{\underline{\pi f T}}} \right)$$

$$e^{j\theta} - e^{-j\theta} = 2j\sin(\theta)$$

$$= \frac{1}{-j2\pi f} \left(-2j\sin(\pi f T) \right)$$

normalized sinc

$$= \frac{\sin(\pi f T)}{\pi f} = T \cdot \frac{\sin(\pi \overset{x}{\underline{f T}})}{\underbrace{T \cdot \pi f}_{x}} = T \cdot \mathrm{sinc}(fT) \quad \text{where } \boxed{\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}}$$

一般型態的 smoother

$$y[n] = x[n] * h[n] = \sum_{\tau} x[n-\tau] h[\tau]$$

$= x[n]h[0] + (x[n+1] + x[n-1])h[1]$
$+ (x[n+2] + x[n-2])h[2]$
$+ (x[n+3] + x[n-3]h[3]$
$+ \cdots\cdots$

Choose (1) $h[n] = h[-n]$ smooth filter: even symmetric

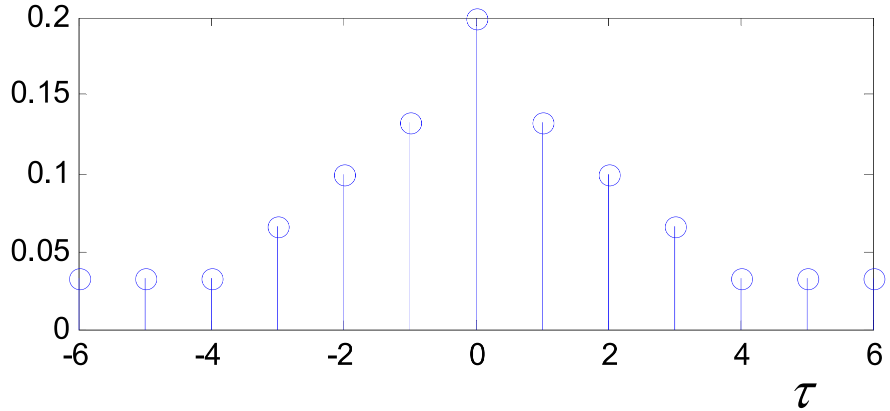
(2) $|h[n_1]| \le |h[n_2]|$ if $|n_1| > |n_2|$ 越遠離中心點，重要性越小

h[n] 類似 weight ，所以要求 >= 0

(3) $h[n] \ge 0$ for all n

(4) $\sum_{\tau} h[\tau] = 1$

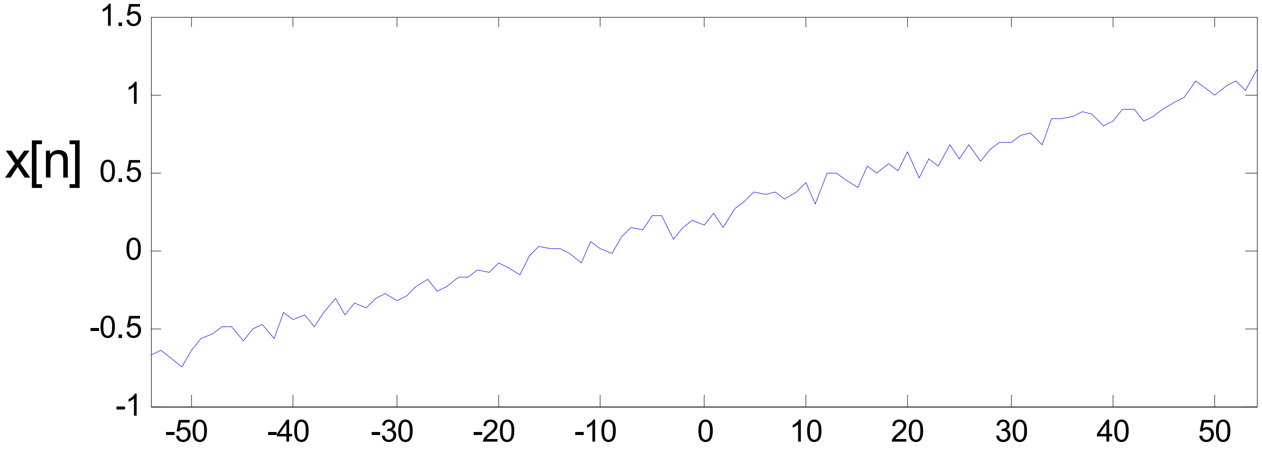
>> 只要滿足這些條件都能當作是 smooth filter



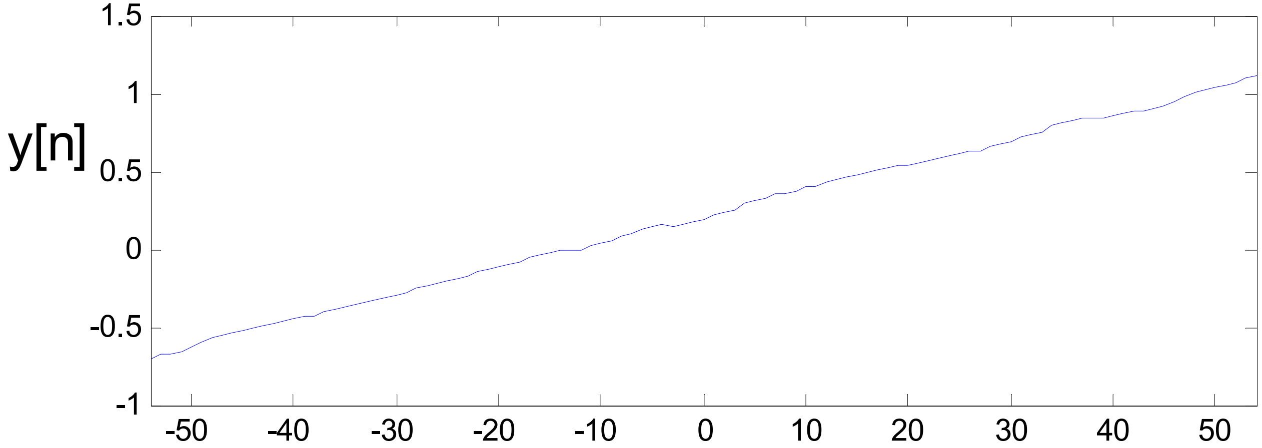
越靠近中間點 impulse response 越大

任何能量隨著 $|n|$ 遞減的 even function，都可以當成 smoother filter

Example:



After applying the smoother filter



Smoother 是一種 lowpass filter (但不為 pass-stop band filter)

思考: smoother 在信號處理上有哪些功用？

(i) extract trend

(ii) extract large-scaled features

(iii) noise removal

smooth filter >> 比較偏向 low pass
Edge filter >> 比較偏向 high pass

◉ 4-C Popular Filters (3): Family of Odd Symmetric Filters

(a) Differentiation $H(f) = j2\pi f$ when $-f_s/2 < f < f_s/2$,

做微分或差分來找出變化

$$H(f) = H(f + f_s)$$

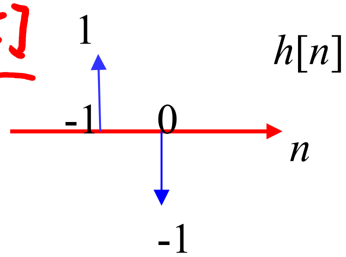
(在離散的世界難做微分)

(b) Difference (一個簡單取代 differentiation 的方法)

$\rightarrow \tau = -1$

$$x_1[n] = x[n] * h[n] = x[\underline{n+1}] - x[n]$$

$\underline{\sum_{\tau} x[n-\tau]\, h[\tau]}$



$$h[n] = 1 \text{ when } n = -1, \quad h[n] = -1 \text{ when } n = 0,$$

$$h[n] = 0 \text{ otherwise}$$

$$H(F) = j2e^{j\pi F} \sin(\pi F)$$

不只 edge 是高頻的，雜訊也是高頻的，
所以也會把雜訊所造成的影響放大

These two filters are equivalent only at low frequencies

和前面用微分和差分做的差異就是他考慮比較多點，不會只考慮兩點間的差異

$x[n] \xrightarrow{DTFT} X(F) \rightarrow X(F)H(F) \xrightarrow{IDTFT} x_H[n]$

(C) Discrete Hilbert Transform

(IIR filter)

Discrete Hilbert Transform 是無限長的濾波器，所以要算 DTFT, IDTFT 運算量都很大

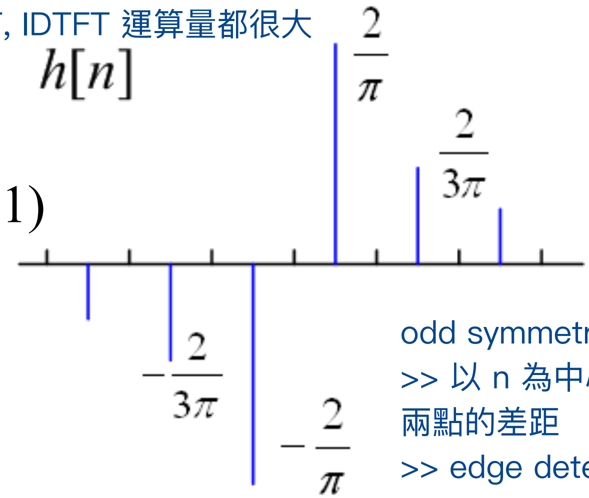
在正的頻率乘上 −j
負的頻率乘上 +j

$H(F) = -j \quad$ for $0 < F < 0.5$

$H(F) = j \quad$ for $-0.5 < F < 0$ $\qquad H(F) = H(F+1)$

$H(0) = H(0.5) = 0$

$h[n] = \dfrac{2}{\pi n}$ when n is odd, $\quad h[n] = 0$ otherwise



odd symmetric
>> 以 n 為中心左右兩點的差距
>> edge detection

single-sided band

Applications: (1) analytic function, (2) instantaneous frequency, (3) edge detection

for real $x[n]$, $X(F) = X^*(-F)$

Analytic function: $\quad x_a[n] = x[n] + jx_H[n]$

where $\quad x_H[n] = x[n] * h[n]$

$X_a(F) = X(F) + jX_H(F)$
$= X(F) + jH(F)X(F)$
$= (1 + jH(F))X(F)$

$1 + jH(F) = \begin{cases} 2 & \text{if } F > 0 \\ 1 & \text{if } F = 0 \\ 0 & \text{if } F < 0 \end{cases}$ ∵ $F>0$: $H(F) = -j$; $F<0$: $H(F) = j$

$x[n] = \mathrm{Re}(x_a[n])$

$y[n] = \sum_{\tau} x[n-\tau]h[\tau]$

if $h[n] = -h[-n]$ Odd symmetric

$y[n] = h[1](x[n-1] - x[n+1])$
$+ h[2](x[n-2] - x[n+2])$
$+ h[3](x[n-3] - x[n+3])$
$+ \cdots\cdots$

Discrete Hilbert Transform 的應用：

(1) 藉由 analytic function 產生single-sided band（實數訊號（when x[n]: real）正的部分跟負的相等，所以只要知道正的，負的就能還原回來，所以我們可以直接把負的頻率忽略掉，只記錄正的，在傳送的時候也只用傳送正的）

(D) Edge Detection ← 近似 high-pass filter

(1) $h[n] = -h[-n]$

(2) $|h[n_1]| \leq |h[n_2]|$ if $|n_1| > |n_2|$

or the shifted version of $h[n]$ satisfies the above two constraints.

Difference 和 discrete Hilbert transform 都可用作 edge detection

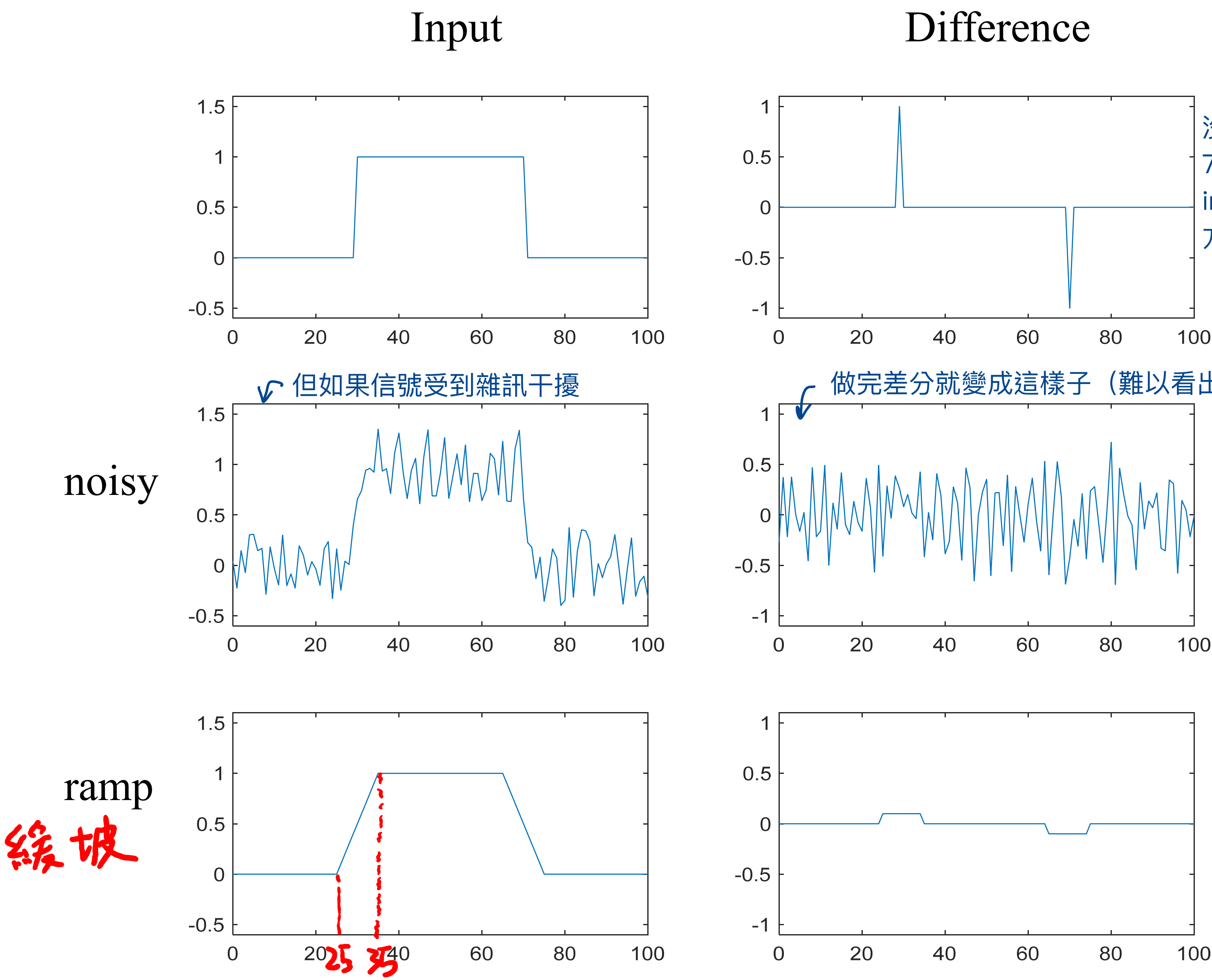
(1) 任何能量隨著 $|n|$ 遞減的 odd function，都可以當成 edge detection filter

(2) The edge detection filter is in fact a matched filter.

Reference

S. C. Pei and J. J. Ding, "Short response Hilbert transform for edge detection," *IEEE Asia Pacific Conference on Circuits and Systems*, Macao, China, pp. 340-343, Dec. 2008.

Difference 的問題：比較難看出 edge 和 noise



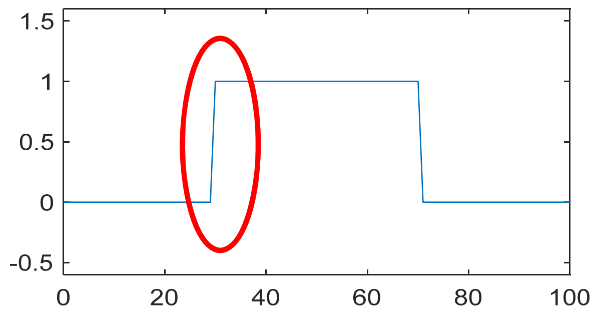
沒有雜訊的情況下確實在 30, 70 的地方得到比較大的 impulse response ，這兩個地方也確實是 edge

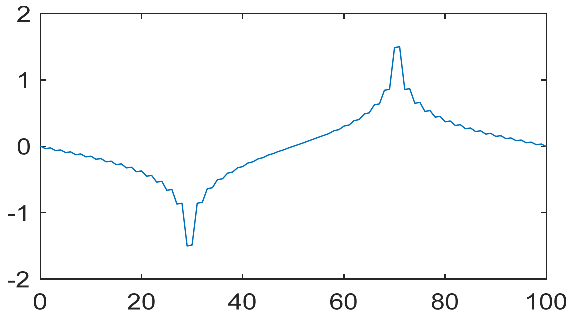
Impulse response 太短的話會有兩個問題：

1. 很容易受到雜訊影響（因為雜訊也會有前後兩點差距比較大的情形）
2. 沒有辦法觀察範圍比較大的特徵

用 Hilbert transform 可以大幅避免受到雜訊影響
(因為 impulse response 比較長)
(但代價是運算量大)

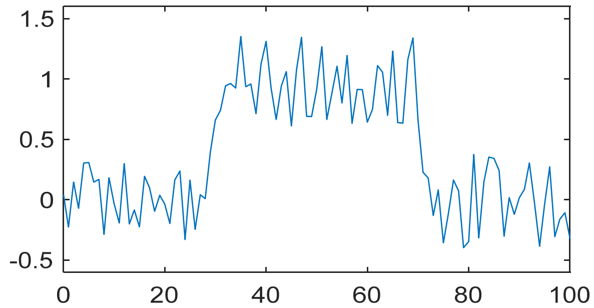
Input　　Discrete Hilbert Transform

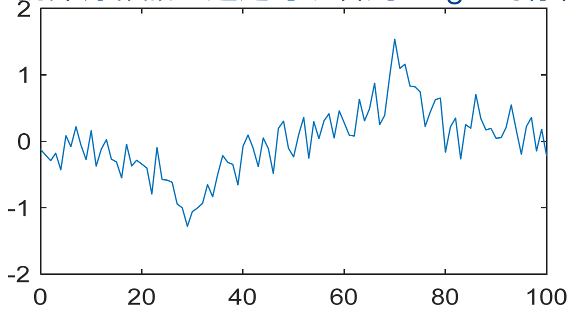




就算有雜訊，還是可以看到 edge 的存在

noisy





ramp