

# Boolean Algebra Properties

Lo Chun, Chou  
R13922136

March 30, 2025

## 2.4 Basic Theorems

### Idempotent Laws

$$X + X = X$$

$$X \cdot X = X$$

### Involution Law

$$(X')' = X$$

### Laws of Complementarity

$$X + X' = 1$$

$$X \cdot X' = 0$$

## 2.5 Commutative, Associative, Distributive and DeMorgan's Laws

### Distributive Laws

$$X(Y + Z) = XY + XZ$$

$$X + YZ = (X + Y)(X + Z)$$

**Example.**

$$(A + B + C')(A + B + D) = A + B + C'D$$

*Proof.*

$$\begin{aligned}(X + Y)(X + Z) &= X(X + Z) + Y(X + Z) \\&= XX + XZ + XY + YZ \\&= X + XZ + XY + YZ \\&= X \cdot 1 + XZ + XY + YZ \\&= X(1 + Z + Y) + YZ \\&= X \cdot 1 + YZ \\&= X + YZ\end{aligned}$$

□

#### DeMorgan's Laws

$$\begin{aligned}(X + Y)' &= X' \cdot Y' \\(X \cdot Y)' &= X' + Y'\end{aligned}$$

## 2.6 Simplification Theorems

#### Uniting

$$\begin{aligned}XY + XY' &= X \\(X + Y)(X + Y') &= X\end{aligned}$$

*Proof.*

$$\begin{aligned}XY + XY' &= X(Y + Y') \\&= X \cdot 1 \\&= X\end{aligned}$$

□

*Proof.*

$$\begin{aligned}(X + Y)(X + Y') &= XX + XY' + XY + YY' \\ &= X + XY' + XY \\ &= X(1 + Y' + Y) \\ &= X \cdot 1 \\ &= X\end{aligned}$$

□

#### Absorption

$$\begin{aligned}X + XY &= X \\ X(X + Y) &= X\end{aligned}$$

*Proof.*

$$\begin{aligned}X(X + Y) &= (X + 0)(X + Y) \quad (\text{distributive}) \\ &= X + 0 \cdot Y \\ &= X\end{aligned}$$

□

#### Elimination

$$\begin{aligned}X + X'Y &= X + Y \\ X(X' + Y) &= XY\end{aligned}$$

*Proof.* By distributive laws, we have  $X + YZ = (X + Y)(X + Z)$

Thus,

$$\begin{aligned}X + X'Y &= (X + X')(X + Y) \\ &= 1 \cdot (X + Y) \\ &= X + Y\end{aligned}$$

□

#### Consensus

$$\begin{aligned}XY + X'Z + YZ &= XY + X'Z \\ (X + Y)(X' + Z)(Y + Z) &= (X + Y)(X' + Z)\end{aligned}$$

*Proof.*

$$\begin{aligned}
 XY + X'Z + YZ &= XY + X'Z + 1 \cdot YZ \\
 &= XY + X'Z + (X + X')YZ \\
 &= XY + X'Z + XYZ + X'YZ \quad (\text{absorption}) \\
 &= XY + X'Z + X'YZ \quad (\text{absorption}) \\
 &= XY + X'Z
 \end{aligned}$$

□

## 2.8 Complementing Boolean Expressions

$$\begin{aligned}
 (X_1 + X_2 + \cdots + X_n)' &= X_1' \cdot X_2' \cdots X_n' \\
 (X_1 \cdot X_2 \cdots X_n)' &= X_1' + X_2' + \cdots + X_n'
 \end{aligned}$$

## 3.2 Exclusive-OR and Equivalence Operations

Exclusive-OR

$$X \oplus Y = XY' + X'Y$$

Because  $X$  exclusive-OR  $Y$  is 1 only when  $X = 1$  and  $Y = 0$  ( $XY'$ ) or  $X = 0$  and  $Y = 1$  ( $X'Y$ ).

Another way to think:

$X$  exclusive-OR  $Y$  is 1 only when  $X = 1$  or  $Y = 1$ , and  $X, Y$  not both 1, therefore:

$$\begin{aligned}
 X \oplus Y &= (X + Y)(XY)' \\
 &= (X + Y)(X' + Y') \\
 &= XY' + X'Y
 \end{aligned}$$

### Properties of Exclusive-OR

$$X \oplus 0 = X$$

$$X \oplus 1 = X'$$

$$X \oplus X = 0$$

$$X \oplus X' = 1$$

$$X \oplus Y = Y \oplus X \text{ (commutative)}$$

$$X \oplus (Y \oplus Z) = (X \oplus Y) \oplus Z = X \oplus Y \oplus Z \text{ (associative)}$$

$$X(Y \oplus Z) = XY \oplus XZ \text{ (distributive)}$$

$$(X \oplus Y)' = X \oplus Y' = X' \oplus Y = XY + X'Y'$$

*Proof.*

$$\begin{aligned} (X \oplus Y)' &= (XY' + X'Y)' \\ &= (XY')' \cdot (X'Y)' \\ &= (X' + Y)(X + Y') \\ &= X'Y' + XY \end{aligned}$$

□

### Equivalence Relation

$$\begin{aligned} (X \equiv Y) &= 1 \iff X = Y \\ \Rightarrow (X \equiv Y) &= XY + X'Y' \end{aligned}$$

Because  $X \equiv Y$  is 1 only when  $X = Y = 1$  or  $X = Y = 0$ .

Therefore, combining the above equations, we have:

Equivalence is the complement of exclusive-OR

$$(X \equiv Y) = XY + X'Y' = (X \oplus Y)'$$

Hence, the equivalence gate is also called the exclusive-NOR gate.

Also, note that:

$$\begin{aligned} X \oplus Y &= XY' + X'Y \\ \Rightarrow (X \oplus Y)' &= (XY' + X'Y)' = XY + X'Y' \end{aligned}$$

This is useful, for example:

$$\begin{aligned}A' \oplus B \oplus C &= (A'B' + AB) \oplus C \\&= (A'B' + AB)'C + (A'B' + AB)C' \\&= (A'B + AB')C + (A'B' + AB)C' \\&= A'BC + AB'C + A'B'C' + ABC'\end{aligned}$$