Digital Systems Design and Laboratory
[5. Karnaugh Maps]

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Outline

- **Minimum Forms of Switching Functions**
- Two- and Three-Variable Karnaugh Maps
- Four-Variable Karnaugh Maps
- Determination of Minimum Expressions Using Essential Prime Implicants
- Five-Variable Karnaugh Maps
- Other Forms of Karnaugh Maps

Recap: Logic Design

* $\boxed{\text{sequential circuit}}$ = combinational logic circuit + 記憶過去 value 的功能

- Design a combinational logic circuit starting with a word description of the desired circuit behavior

  設計 combinational circuit
  1. 有一般自然語言描述
  2. 轉成 circuit

- Steps
  - Translate the word description into a switching function (Unit 4)
    - Truth table
    - Boolean expression
      - SOP/POS derived from minterm or maxterm expansion (Unit 4)

    ex.

  - Simplify the function
    - Boolean algebra (Units 2 and 3)
    - Karnaugh map (Unit 5)
    - Quine-McCluskey (Unit 6) (QM method)
    - Other methods
  - Realize it using available logic gates

對應 logic gate
↓
希望 gate 越少越好.
↓
simplification (not systematic ⇒ 試看看)
↑
前幾章 ⟷ 本章: systematic 方法 ⇒ K-map
(做 Boolean function 的 simplification)

Difficulties in Algebraic Simplification

❑ Problems

- ➢ Difficult to apply in a systematic way
- ➢ Difficult to tell when you have arrived at a **minimum** solution

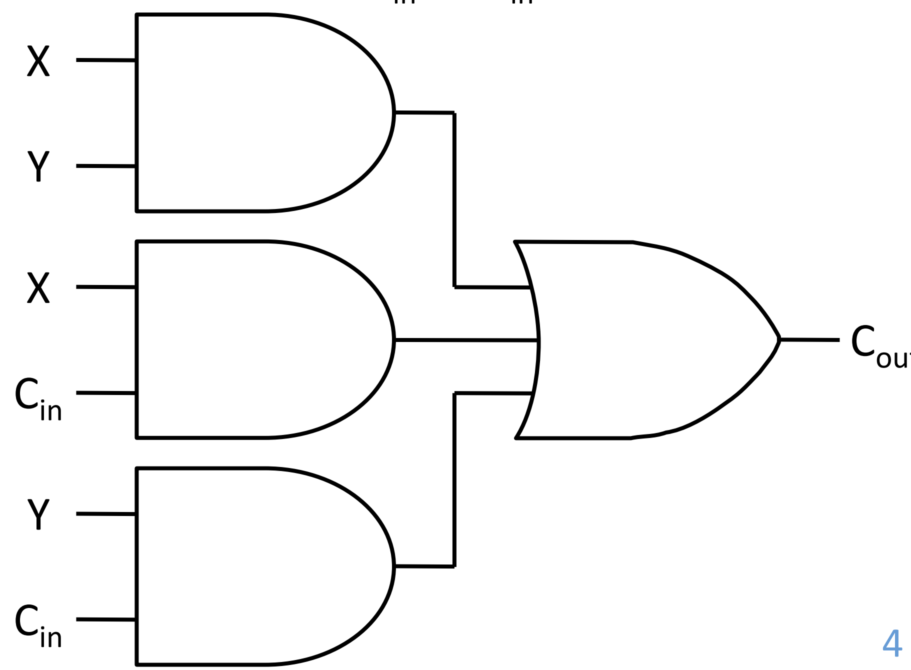
  literals
  → minimum number of gates

  - Minimum SOP/POS
    - Minimum # of terms (i.e., # of gates)
    - Minimum # of literals (i.e., # of gate inputs)

$$C_{out} = X'YC_{in} + XY'C_{in} + XYC_{in}' + XYC_{in}$$
$$= XY + XC_{in} + YC_{in}$$

❑ Solutions: systematic methods

- ➢ Karnaugh map (K-map) (Unit 5)
  - Especially useful for 3 or 4 variables
- ➢ Quine-McCluskey (Unit 6)
- ➢ Other methods



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- ❑ Minimum Forms of Switching Functions
- ❑ **Two- and Three-Variable Karnaugh Maps**
- ❑ Four-Variable Karnaugh Maps
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Two-Variable Karnaugh Maps (1/2)

- Truth table = minterm expansion = Karnaugh map
  - Each square of the K-map corresponds to a combination of values of inputs
  - Each square = a minterm = a row in truth table

A=0, B=0

A=0, B=1

B \ A	0	1
0	a_0	a_2
1	a_1	a_3

A=1, B=0

A=1, B=1

⟺

B \ A	0	1
0	m_0 = A'B'	m_2 = AB'
1	m_1 = A'B	m_3 = AB

⟺

m_i	A B	F
0	0 0	a_0
1	0 1	a_1
2	1 0	a_2
3	1 1	a_3

↳ 每格 = truth table 中的一個 row
= 一種 case

Two-Variable Karnaugh Maps (2/2)

❑ Example

m_i	A B	F
0	0 0	1
1	0 1	1
2	1 0	0
3	1 1	0

B \ A	0	1
0	1	0
1	1	0

B \ A	0	1
0	1	0
1	1	0

F = A'

把1圈起來，希望圈越大越好

→不能再圈以後，再來研究圈代表的意思

Three-Variable Karnaugh Maps (1/2)

↳ $2^3=8$ 種 cases → 8個 square

- ❑ Minterms in adjacent squares of K-map differ in only ONE bit
  - ➢ Combine them: XY'+XY = X(Y'+Y) = X

注意順序

∴3在2前面

BC \ A	0	1
00	A'B'C'	AB'C'
01	A'B'C	AB'C
11	A'BC	ABC
10	A'BC'	ABC'

→

BC \ A	0	1
00	m_0	m_4
01	m_1	m_5
11	m_3	m_7
10	m_2	m_6

A

B

C

C是1的狀況

* 相鄰的兩個square只能有一個bit不同

Three-Variable Karnaugh Maps (2/2)

❑ Example

m_i	A B C	F
0	0 0 0	0
1	0 0 1	1
2	0 1 0	0
3	0 1 1	1
4	1 0 0	0
5	1 0 1	1
6	1 1 0	0
7	1 1 1	0

$F = A'B'C + A'BC + AB'C$

001 011 101

BC \ A	0	1
00	0	0
01	001 1	101 1
11	011 1	0
10	0	0

$F = A'C + B'C$

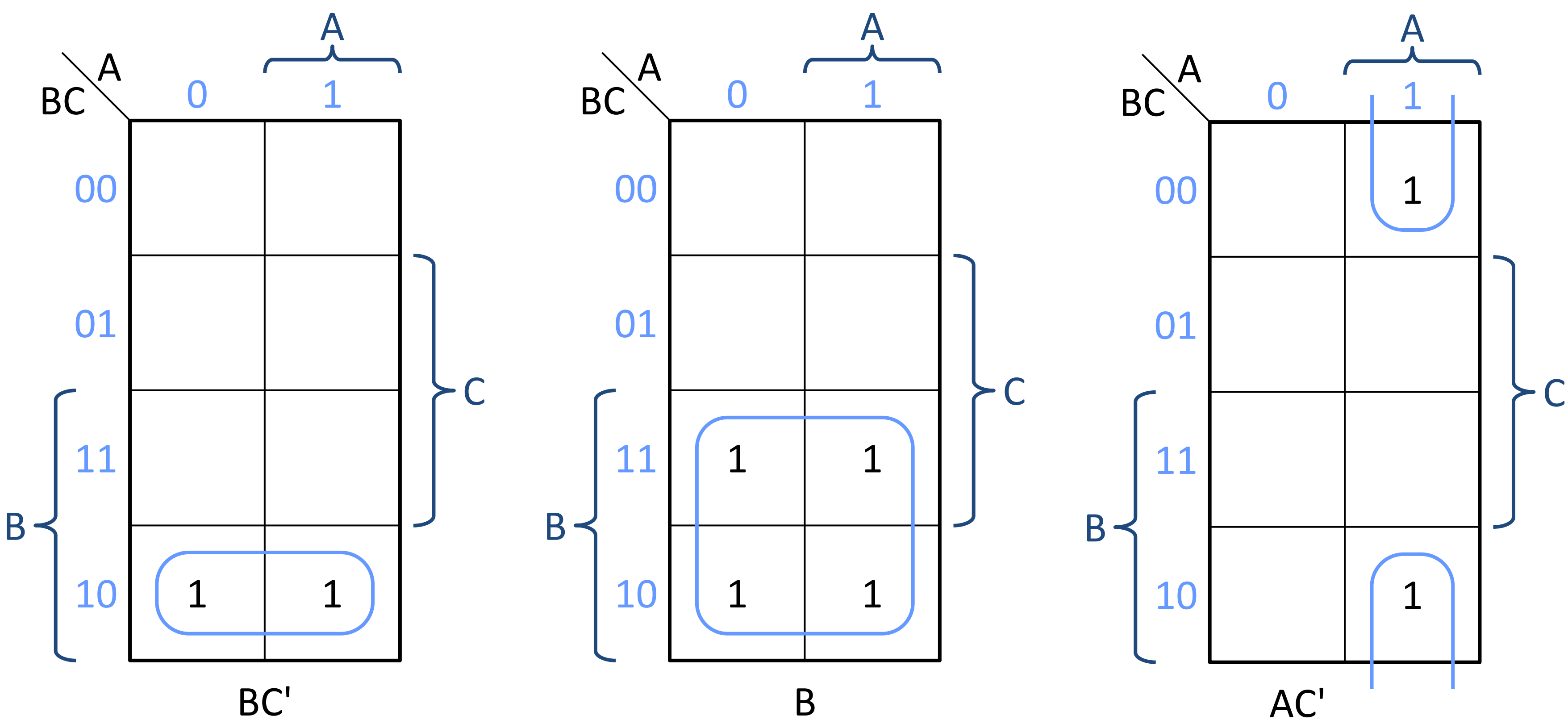
A須為0，B可為0/1，C須為1

∴此框的項式：$A=0, C=1 \rightarrow A'C$

$A'B'C + A'BC = A'(B'C+BC) = A'(B'+B)C = A'C$

Product Terms in Karnaugh Maps

❑ Examples



1. 框住的正方形數必為2的次方 (ex: 框一個, 2個, 4個, 8個)
6個 ✗

2. 必為長方形 (正方形也算長方形的一種)

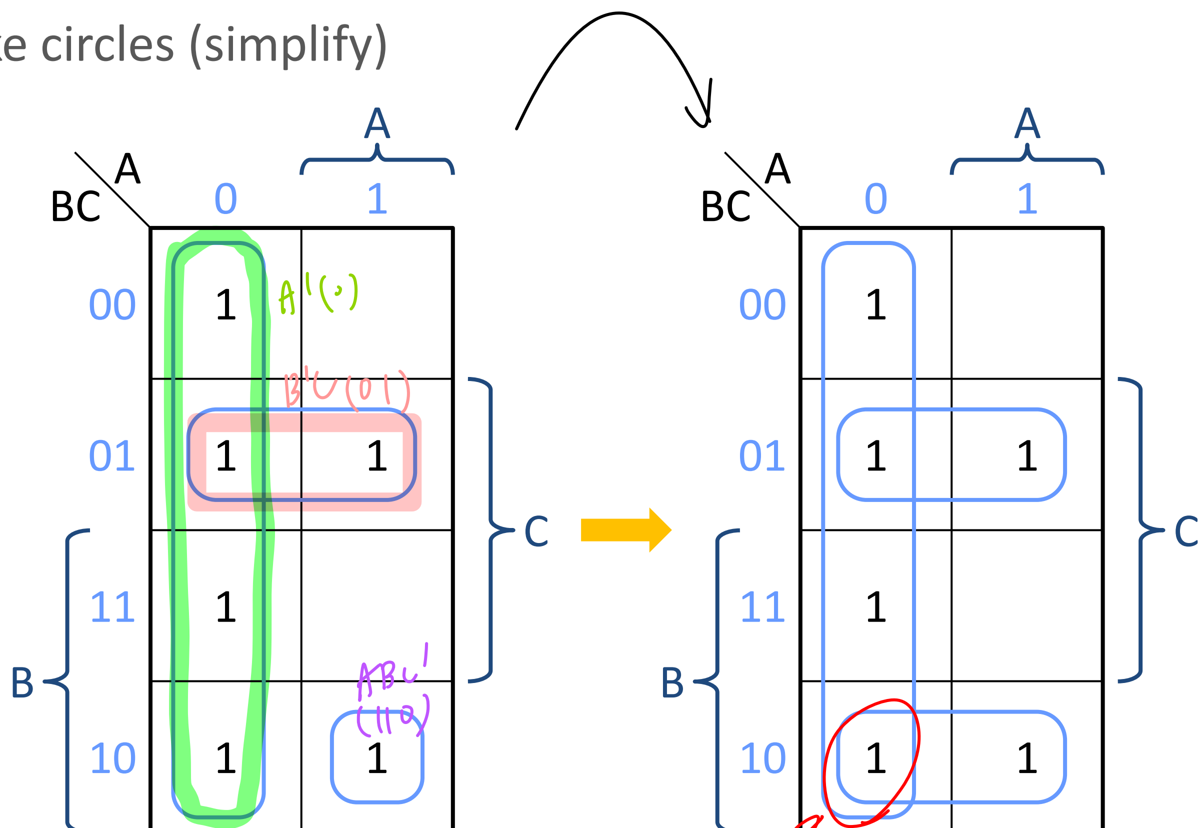
3. 框可跨过边界.

Another Example

- ❑ F = ABC' + B'C + A'
  - ➢ Mark 1's
  - ➢ Make circles (simplify)

110 01 0

ABC' → BC' (少一個AND Gate → 少繞一條線)



F = ABC' + B'C + A'

F = BC' + B'C + A'

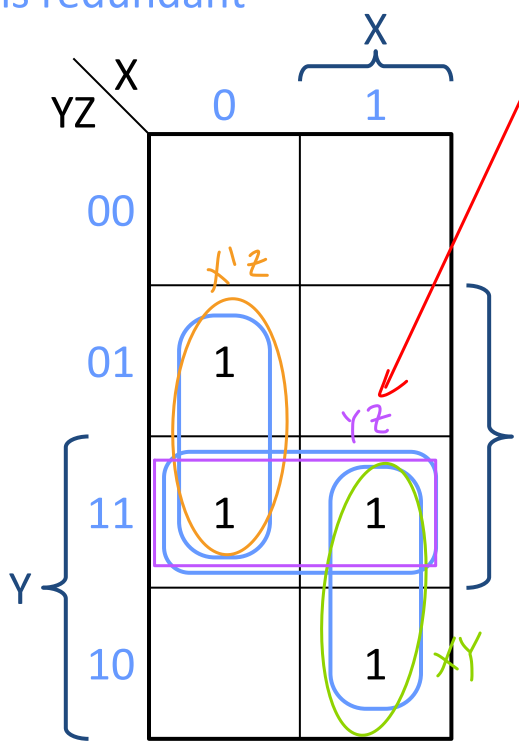
往左擴展一格: A'BC'

(1) $F = \boxed{ABC' + A'BC'} + B'C + A' = (A+A')BC' + B'C + A'$
$= BC' + B'C + A'$

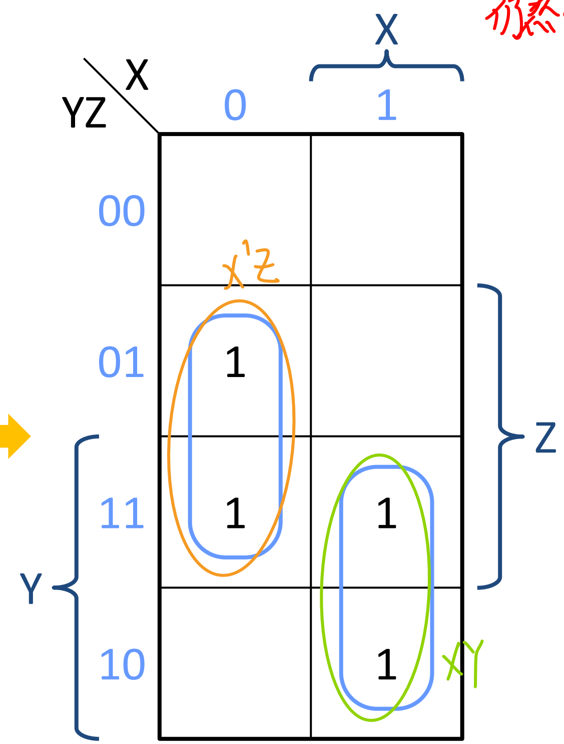
Consensus Theorem in Karnaugh Maps

- ❑ Overlapped circles imply redundant terms
- ❑ Consensus theorem
  - ➢ $X\boxed{Y} + X'\boxed{Z} + \boxed{YZ} = XY + X'Z$ ← consensus term
    - YZ is redundant

YZ redundant ∵ 拿掉 YZ 的框
→ 仍有 2 個面積=2 的框，所有 1 的地方
仍然有包住



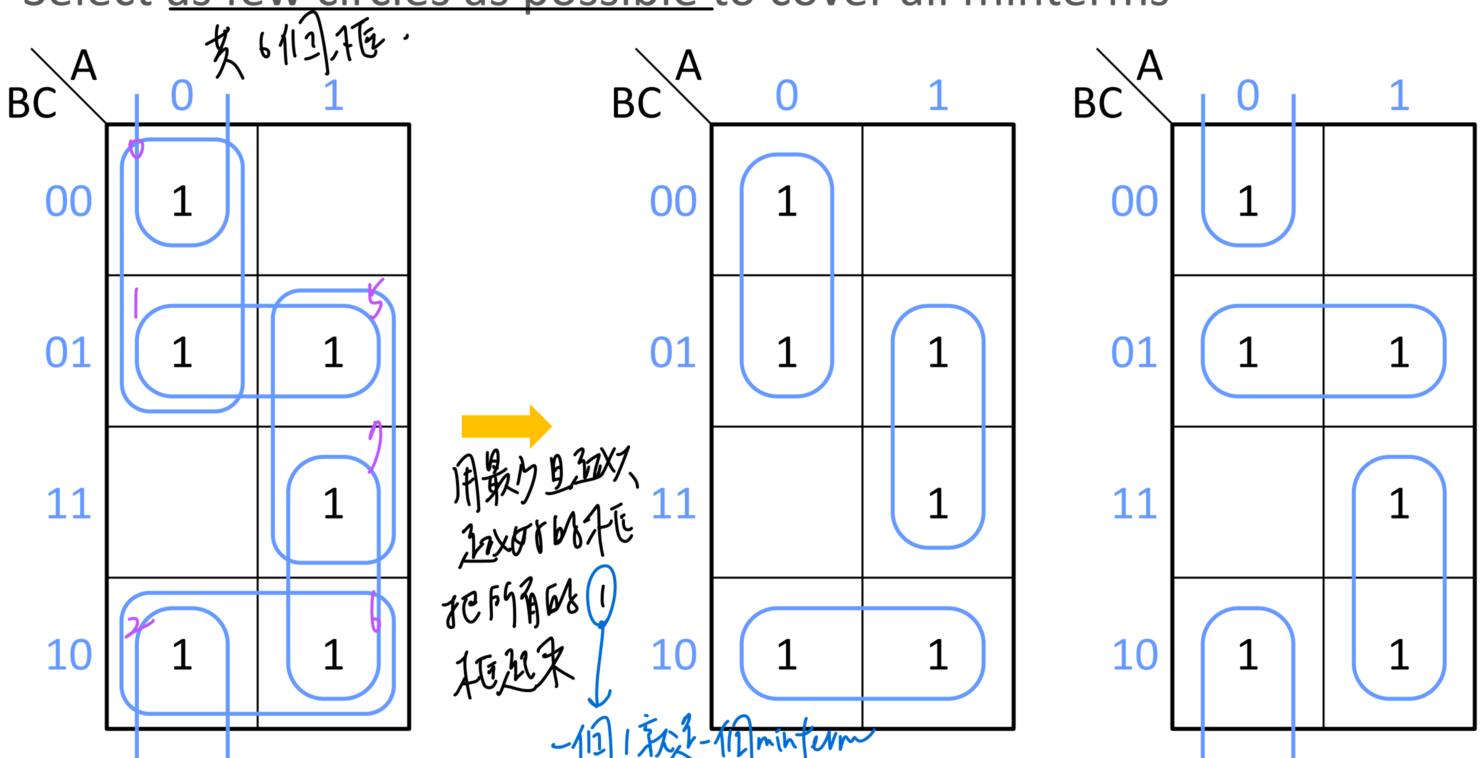
F = XY + X'Z + YZ



F = XY + X'Z

All Solutions in Karnaugh Maps

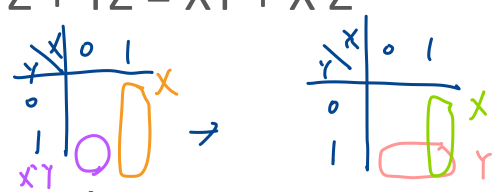
- All possible minimum SOPs can be determined from K-map
  - # of terms and # of literals
- Example: F = ∑ m(0, 1, 2, 5, 6, 7)
  - Make each circle as large as possible
  - Select as few circles as possible to cover all minterms



Summary

每個square = 一個minterm

- ❑ Truth table = minterm expansion = Karnaugh map
- ❑ Simplification in Karnaugh maps
  - ➢ Minimum SOP = (min # of terms, min # of literals)
  - ➢ Steps (make adjacent squares different in only one bit)
    - Mark 1's
    - Make circles
      - Make each circle as large as possible (# of literals)
      - Select as few circles as possible to cover all 1's (# of terms)
- ❑ Algebraic simplification also holds in Karnaugh maps
  - ➢ Combining terms: XY + XY' = X
  - ➢ Eliminating terms: X + XY = X; XY + X'Z + YZ = XY + X'Z
  - ➢ Eliminating literals: X + X'Y = X + Y



X+Y

  - ➢ Adding redundant terms:
    Y = Y+ XX'; Y = Y(X + X'); XY + X'Z = XY + X'Z + YZ; X = X + XY

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Four-Variable Karnaugh Maps

- Minterms in adjacent squares of K-map differ in only ONE bit

非10免

CD \ AB	00	01	11	10
00	m_0	m_4	m_{12}	m_8
01	m_1	m_5	m_{13}	m_9
11	m_3	m_7	m_{15}	m_{11}
10	m_2	m_6	m_{14}	m_{10}

CD \ AB	00	01	11	10
00	1	1	1	1
01		1		
11		1	1	1
10	1	1	1	1

8個
→ D必為0
∴ D'

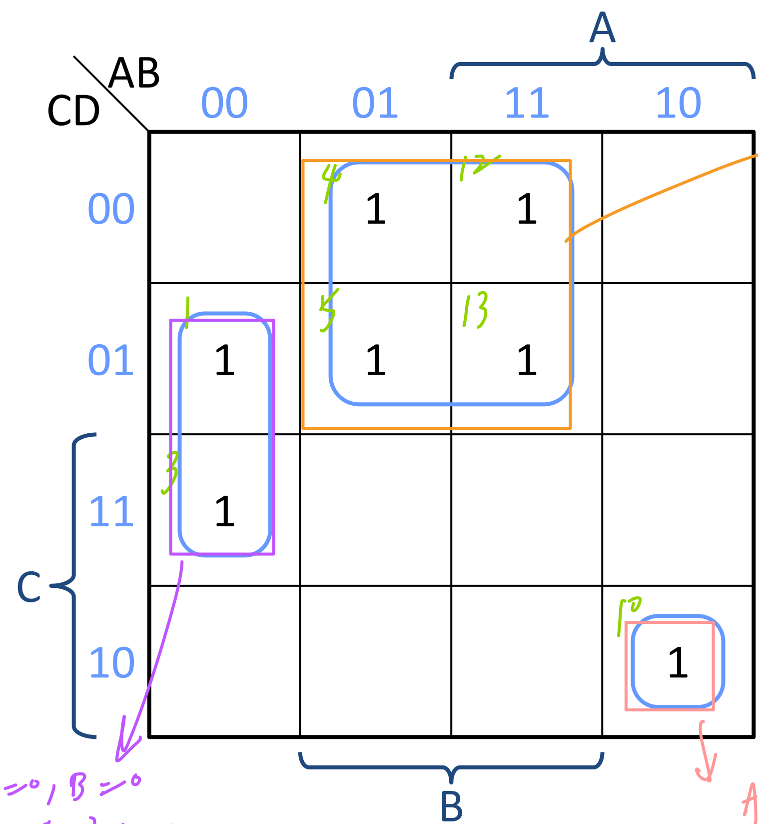
F = AC + A'B + D'

Two More Examples

- $F_1 = \sum m(1, 3, 4, 5, 10, 12, 13)$
- $F_2 = \sum m(0, 2, 3, 5, 6, 7, 8, 10, 11, 14, 15)$

$A \in \{0,1\}, B = 0$
$C \in \{0,1\}, D = 0$
$\therefore B'D'$

K-map是循環的概念，
∴ square boundary 會繞回來(接起來)

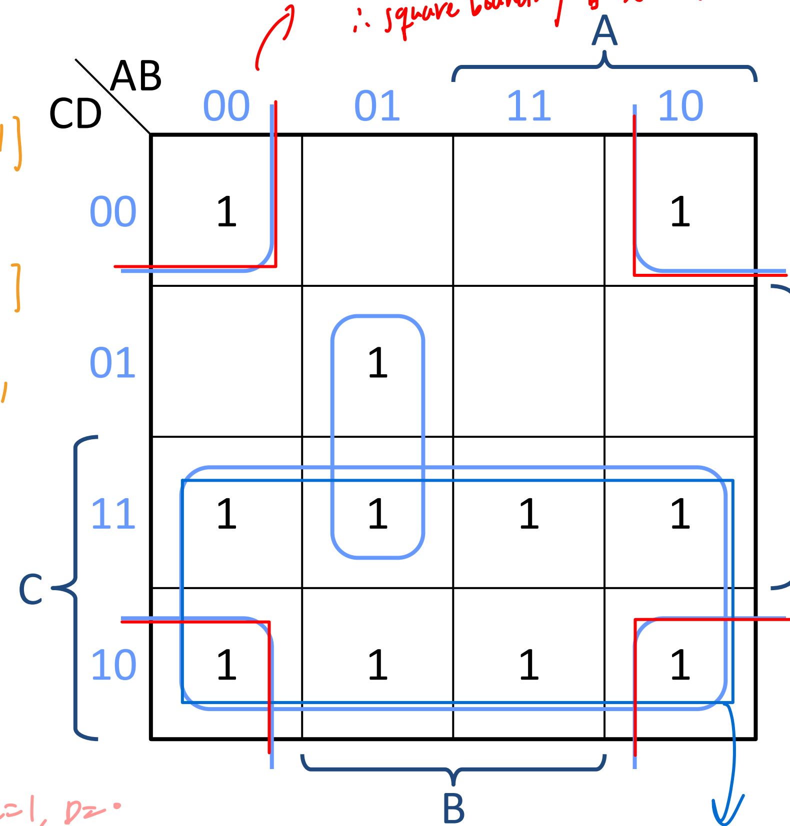


$A \in \{0,1\}$
$B = 1$
$C = 0$
$D \in \{0,1\}$
$\Downarrow$
BC'

$A = 0, B = 0$
$C \in \{0,1\}, D = 1$
$\therefore A'B'D$

$A = 1, B = 0, C = 1, D = 0$
$\therefore AB'CD'$

$F_1 = BC' + A'B'D + AB'CD'$

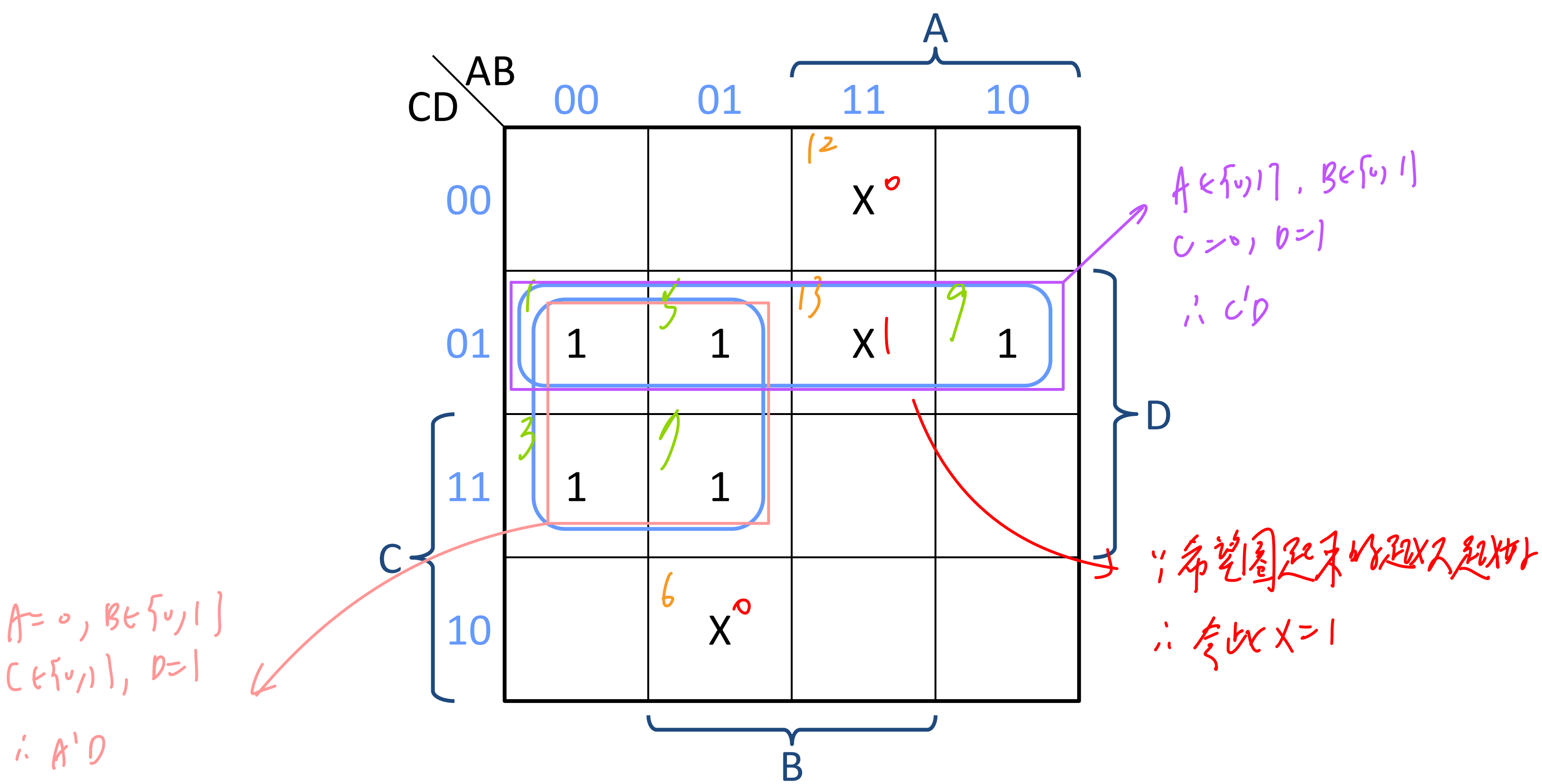


$A \in \{0,1\}, B \in \{0,1\}$
$C = 1, D = \{0,1\}$
$\therefore C$

$F_2 = C + B'D' + A'BD$

Karnaugh Maps with Don't Cares

- ❑ Don't cares can be assigned with 0's or 1's
  - ➢ After assignment, the function becomes completely specified
- ❑ $F = \sum m(1, 3, 5, 7, 9) + \sum d(6, 12, 13)$



$F = A'D + C'D$

Minimum POS

↳ 即 1. 用0做 SOP
2. 取 complement

- Minimum SOP = circle 1's of F
- Minimum POS = circle 0's of F
  - Find minimum SOP of F' and then complement it
  - Example: F = X'Z' + WYZ + W'Y'Z' + X'Y

1. 圈0，得到的 minimum sop 是F'
2. 对F'取 complement

YZ \ WX	00	01	11	10
00	1	1	0	1
01	0	0	0	0
11	1	0	1	1
10	1	0	0	1

W, X, Y, Z

W'XY   WXZ'

W∈{0,1}, X∈{0,1}
Y=0, Z=1
∴ Y'Z

F' = Y'Z + W'XY + WXZ'

By DeMorgan's law:

$\boxed{F = (Y + Z')(W + X' + Y')(W' + X' + Z)}$ minimum pos

$F = (F')' = (Y'Z + W'XY + WXZ')'$

$= (Y'Z)'(W'XY)'(WXZ')'$

$= (Y+Z')(W+X'+Y')(W'+X'+Z)$

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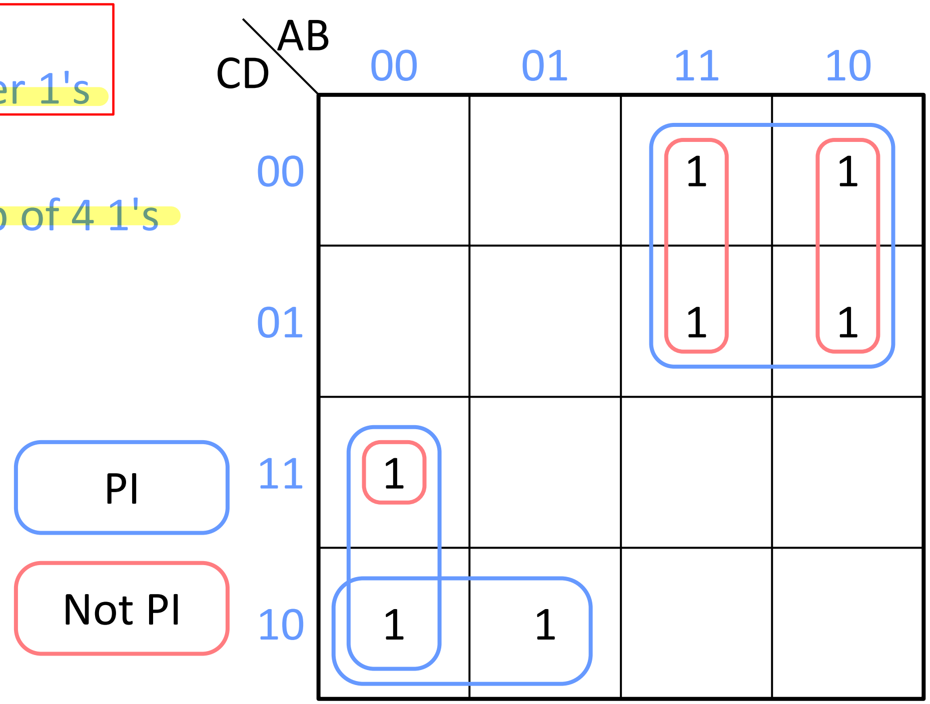
Prime Implicants (1/2)

→ 一個框框 → 1, 2, 4, 8, ... 個圈起來的框

- **Implicant**: a product term
  - Any single 1 or any group of 1's in the K-map
- **Prime implicant** (PI): an implicant that cannot be covered by other implicants ↳ 這個 implicant 不能被其他 implicant 包住 = 不能再長大的 implicant (框框)
  - A circle that cannot be enlarged any more
    - A single 1 is a PI if not adjacent to any other 1's
    - Two adjacent 1's is a PI if not contained in a group of 4 1's



Prime Implicants (2/2)

只能用prime implicants

- **Cover**: a set of prime implicants which covers all 1's
- A minimum SOP contains only prime implicants (why?)
  - Minimum cover = (min # of PIs, min # of literals)
- Don't cares are treated just like 1's here

①如果非PI
→ 可再長大框框，代表
換成Boolean expression 會變少.
∵希望 Boolean expression minimum
∴用PI → 不能再長大 → Boolean expression 不能
再變短.

CD \ AB	00	01	11	10
00		X	1	
01	1	1	1	
11	1		1	1
10			1	X

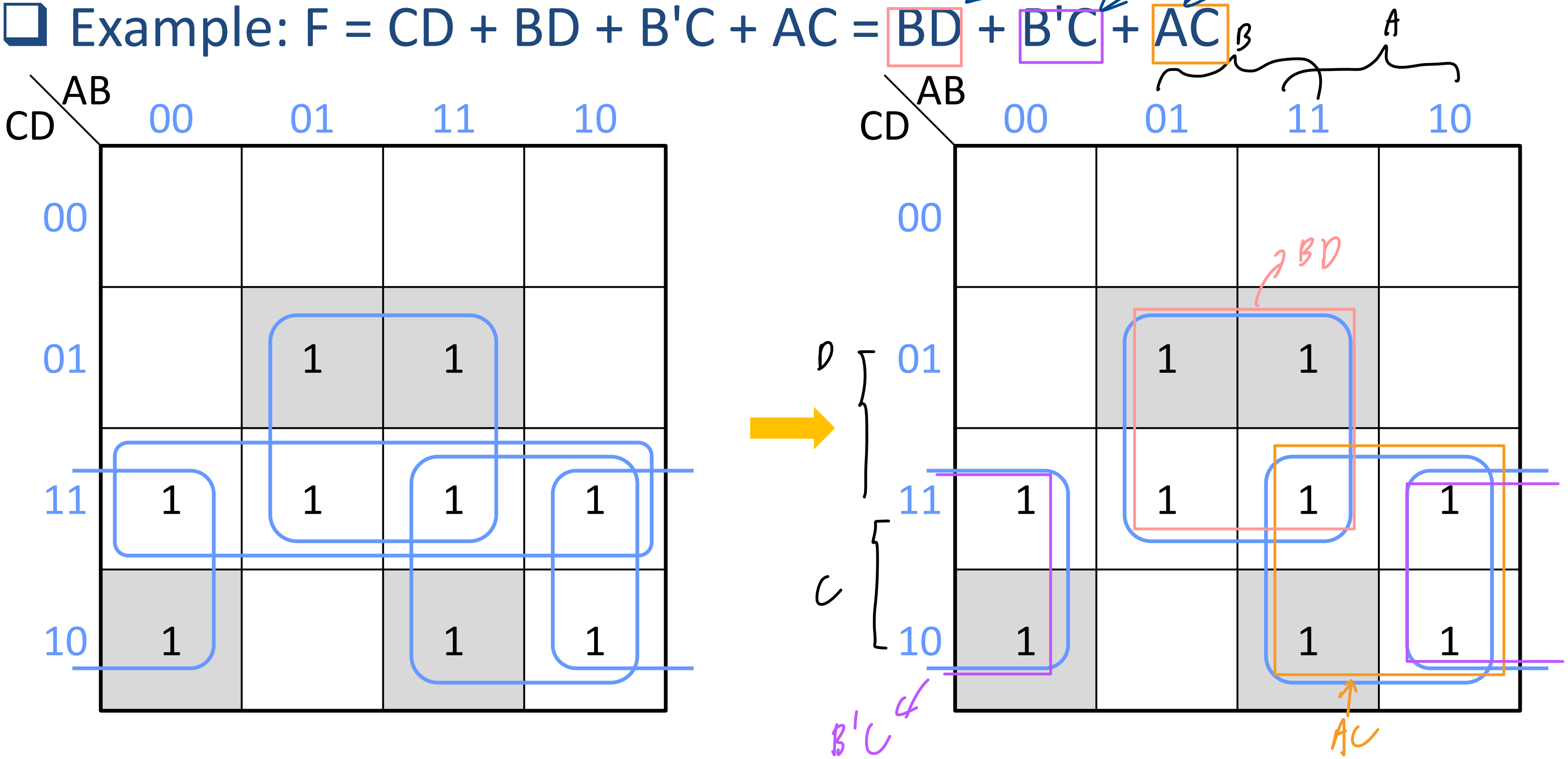
F = A'B'D + BC' + AC

F = A'C'D + AB + B'CD

∵有1只被這個 essential PI 包到 ∴在做SOP時必包含它

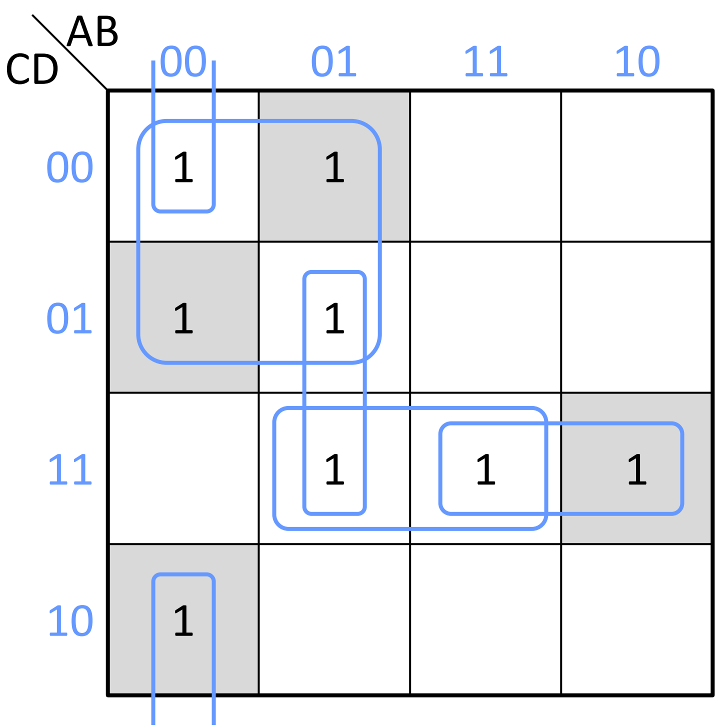
Essential Prime Implicants

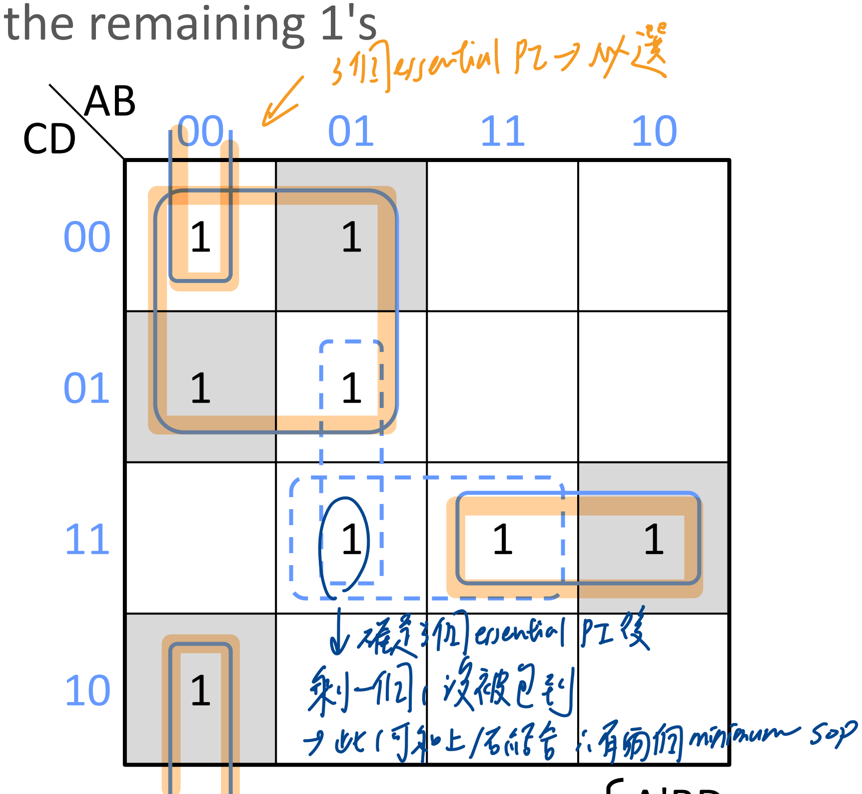
- **Essential prime implicant**: if a minterm is covered by only one PI, the PI is essential ↳如果有一個minterm，只被PI包到，則此PI為 Essential PI (一個框框中的1)
  - Essential PI must be included in minimum SOP
  - Find essential PI's = find the 1's circled only once
- Example: F = CD + BD + B'C + AC = $\boxed{BD}$ + $\boxed{B'C}$ + $\boxed{AC}$ 皆為 essential PI



Another Example

- ❑ Find minimum cover
  - ➢ Find all PI's
  - ➢ Find essential PI's
  - ➢ Find a minimum set of PI's to cover the remaining 1's





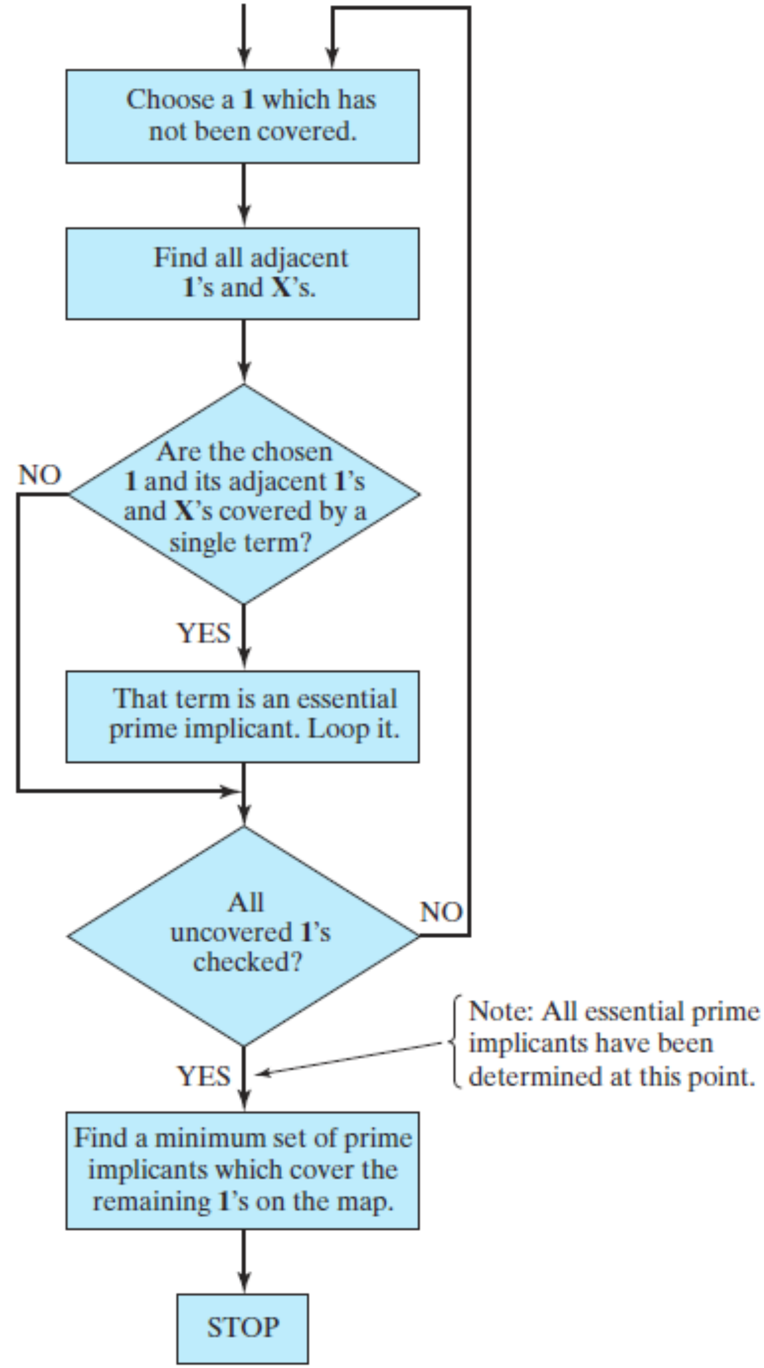
$$F = A'C' + A'B'D' + ACD + \begin{cases} A'BD \\ BCD \end{cases}$$

Summary

- Minimum SOP = minimum cover = a minimum set of PI's which cover all 1's
  - Minimum cover = (min # of PIs, min # of literals)
- Steps
  - Find all PI's
  - Find essential PI's 看哪些1只能被一個PI包到
  - Find a minimum set of PI's to cover the remaining 1's 剩下的1再從剩下的PI找
- Recap: steps of simplification in Karnaugh maps
  - Mark 1's
  - Make circles
    - Make each circle as large as possible **= find PI**
    - Select as few circles as possible to cover all 1's **= find minimum cover**

Flowchart

FIGURE 5-19
Flowchart for Determining a Minimum Sum of Products Using a Karnaugh Map



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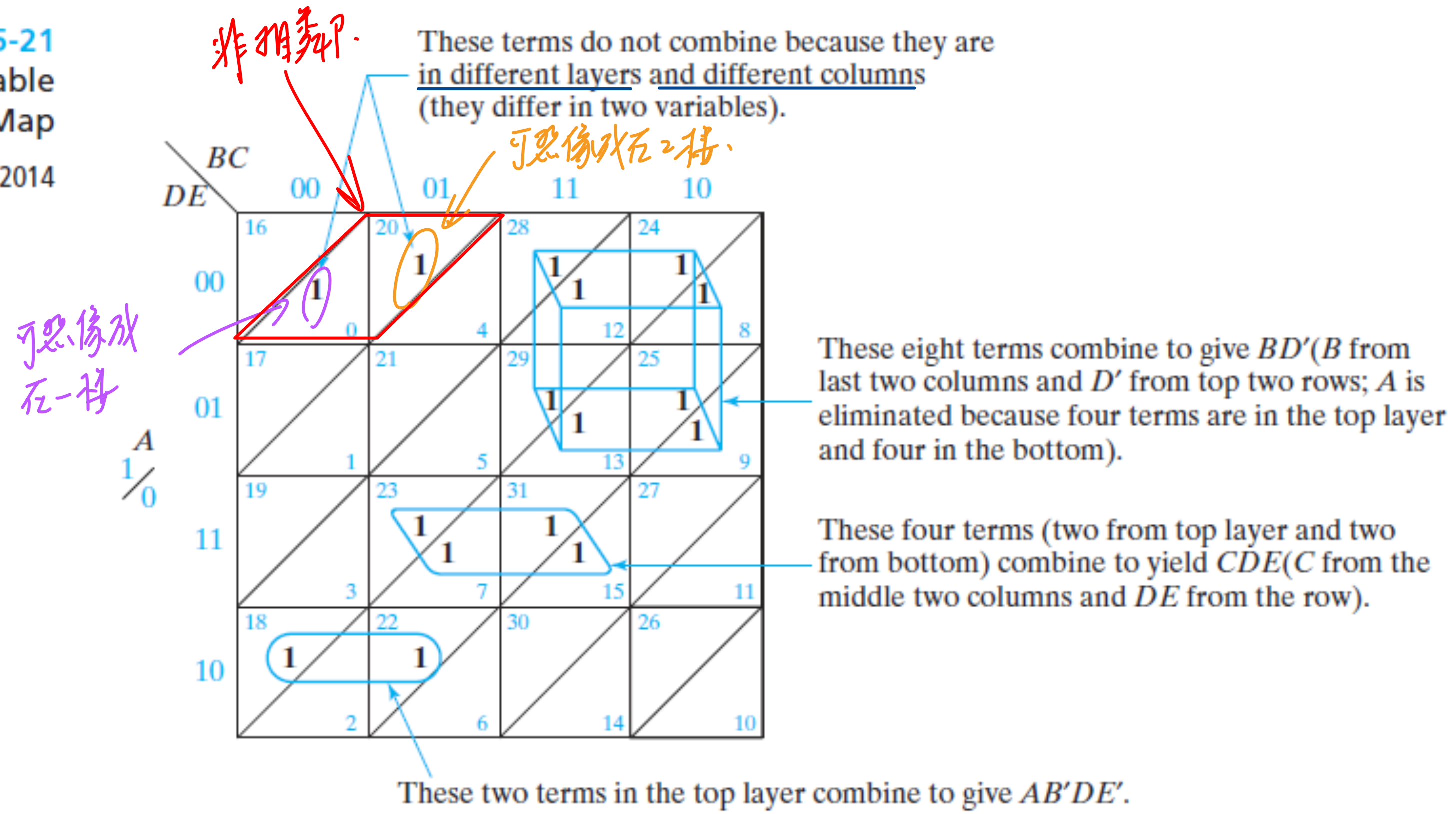
Five-Variable Karnaugh Maps (1/2)

❑ Example

➢ F = BD' + CDE + AB'DE' + AB'CD'E' + A'B'C'D'E'

FIGURE 5-21
A Five-Variable Karnaugh Map



Five-Variable Karnaugh Maps (2/2)

- Example
  - F = A'B' CD' + A'B'CE + A'B'D'E + A'CD'E + B'CD'E

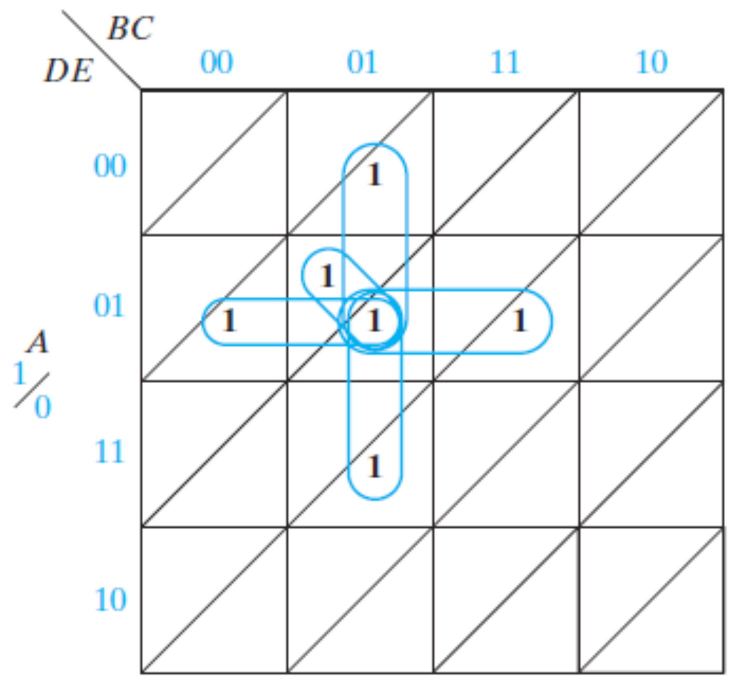


FIGURE 5-22

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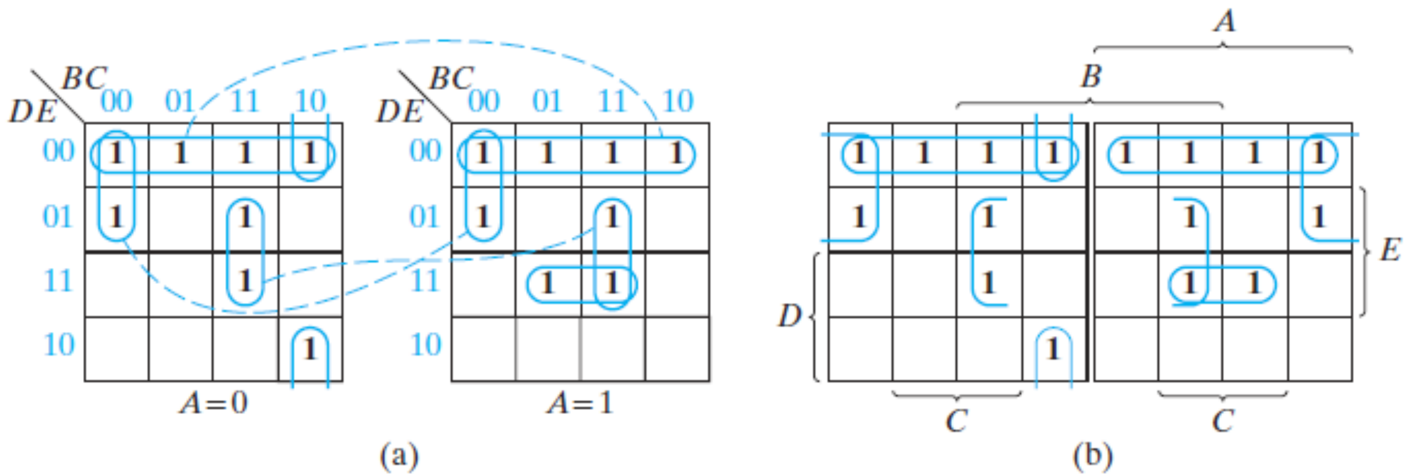
- ❑ **Other Forms of Karnaugh Maps**

Other Forms of Karnaugh Maps

- Side-by-side maps
- Mirror image maps

FIGURE 5-28
Other Forms of Five-Variable Karnaugh Maps



Q&A