Digital Systems Design and Laboratory
[3. Boolean Algebra (Continued)]

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Outline

- ❑ **Multiplying Out and Factoring Expressions**
- ❑ Exclusive OR and Equivalence Operations
- ❑ Consensus Theorem
- ❑ Algebraic Simplification of Switching Expressions
- ❑ Proving Validity of an Equation

Multiplying Out and Factoring

- Distributive laws
  - $X(Y + Z) = XY + XZ$
  - $(X + Y)(X + Z) = X + YZ$
- Another useful theorem
  - $(X + Y)(X' + Z) = XZ + X'Y$
    - Proof? Check two cases: $X = 0$ or $X = 1$
- Multiplying out
  - Apply them from left terms to right terms
- Factoring
  - Apply them from right terms to left terms

Multiplying Out: from POS to SOP

- From product-of-sums (POS) to sum-of-products (SOP)
- Example

$$\underbrace{(\underline{A + B} + C')(\underline{A + B} + D)}_{2.}(A + B + E)\underbrace{(\underline{A} + D' + E)(\underline{A'} + C)}_{3.}$$

$$= \underbrace{(\underline{A + B} + C'D)(\underline{A + B} + E)}_{2.}[AC + A'(D' + E)]$$

$$= (A + B + C'DE)(AC + A'D' + A'E)$$

$$= \underline{AC + ABC} + A'BD' + A'BE + A'C'DE$$

$$= \underline{AC} + A'BD' + A'BE + A'C'DE$$

1. $X(Y + Z) = XY + XZ$
2. $(X + Y)(X + Z) = X + YZ$
3. $(X + Y)(X' + Z) = XZ + X'Y$

AC 就包含了 ABC

$\because$ $B=1$: $ABC = AC \rightarrow AC + ABC = AC + AC = AC$

$B=0$: $ABC = 0 \rightarrow AC + ABC = AC + 0 = AC$

Factoring

- From sum-of-products (SOP) to product-of-sums (POS)
- Example

$AC + A'BD' + A'BE + A'C'DE$

$= \underbrace{\underline{A}C + \underline{A'}(BD' + BE + C'DE)}_{3.}$

$= (A + BD' + BE + C'DE)(A' + C)$

$= \underbrace{[\underline{A + C'DE} + B(D' + E)]}_{2.}(A' + C)$

可删

$= (A + C'DE + B)(A + \underline{C'DE + D' + E})(A' + C)$

$= \underbrace{(\underline{A + B} + C'DE)}_{2.}(A + \underline{D' + E})(A' + C)$

$= (A + B + C')(A + B + D)(A + B + E)(A + D' + E)(A' + C)$

1. $X(Y + Z) = XY + XZ$
2. $(X + Y)(X + Z) = X + YZ$
3. $(X + Y)(X' + Z) = XZ + X'Y$

$C'DE + D' + E$

$= D \cdot C'E + D' \cdot 1 + E$

$= (D' + C'E)(D + 1) + E$

$= (D' + C'E) + E$

$= D' + E$

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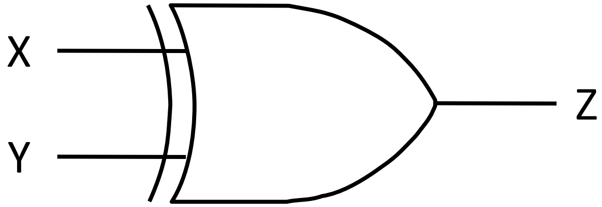
Exclusive-OR (1/2)

❑ Exclusive-OR (XOR) ($\oplus$)

- $0 \oplus 0 = 0$, $0 \oplus 1 = 1$, $1 \oplus 0 = 1$, $1 \oplus 1 = 0$
- Symbol (XOR gate)



- Truth table

X	Y	$Z = X \oplus Y$
0	0	0
0	1	1
1	0	1
1	1	0

Exclusive-OR (2/2)

- $X \oplus Y = X'Y + XY'$ 不一樣時 $\xrightarrow{\quad}$ 1
  一樣時 $\xrightarrow{\text{output}}$ 0
- Theorems
  - ~~$X \oplus 0 = X$~~
  - ~~$X \oplus 1 = X'$~~
  - $X \oplus X = 0$
  - $X \oplus X' = 1$
  - $X \oplus Y = Y \oplus X$ (commutative)
  - $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$ (associative)
  - $X(Y \oplus Z) = XY \oplus XZ$ (distributive)
  - ~~$(X \oplus Y)' = X'Y' \oplus XY$~~
    - X and Y must be the same

XOR取NOT

→ 一樣時 → 1
不一樣時 → 0

$X=Y=1 \rightarrow X'Y' \oplus XY = 0\cdot0 \oplus 1\cdot1 = 0 \oplus 1 = 1$

$X=Y=0 \rightarrow X'Y' \oplus XY = 1\cdot1 \oplus 0\cdot0 = 1 \oplus 0 = 1$

$X=1, Y=0 \rightarrow X'Y' \oplus XY = 0\cdot1 \oplus 1\cdot0 = 0 \oplus 0 = 0$

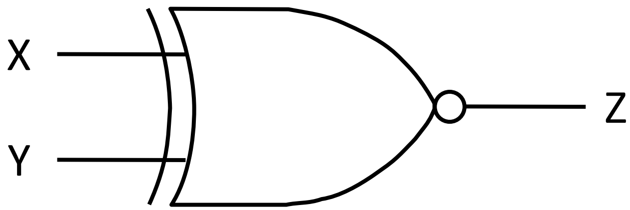
$X=0, Y=1 \rightarrow X'Y' \oplus XY = 1\cdot0 \oplus 0\cdot1 = 0 \oplus 0 = 0$

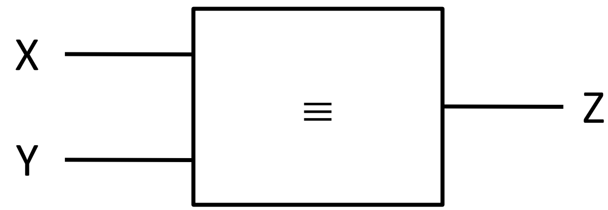
Exclusive-NOR

- **Exclusive-NOR (XNOR) (≡)**

  一樣時 → 1
  不一樣時 → 0

  - $0 \equiv 0 = 1,\ 0 \equiv 1 = 0,\ 1 \equiv 0 = 0,\ 1 \equiv 1 = 1$
  - Symbol (XNOR gate)





  - Truth table

X	Y	Z = X ≡ Y
0	0	1
0	1	0
1	0	0
1	1	1

Simplification of XOR and XNOR

- $X \oplus Y = X'Y + XY'$
- $X \equiv Y = X'Y' + XY$
- Examples

$$
\begin{aligned}
F &= (A'B \equiv C) + (B \oplus AC') \\
&= [A'BC + (A'B)'C'] + [B'AC' + B(AC')'] \\
&= A'BC + (A + B')C' + B'AC' + B(A' + C) \\
&= B(A'C + A' + C) + C'(A + B' + AB') \\
&= B(A' + C) + C'(A + B') \quad = BA' + BC + AC' + B'C' \\
&= \text{... can be further simplified} \quad = (A'B + AC') + (BC + B'C') \\
&\qquad = (A+B)(A'+C') + (B+C')(B'+C) \\
&= B + C'
\end{aligned}
$$

直接拿掉

$$
\begin{aligned}
F &= A' \oplus B \oplus C \\
&= (A'B' + AB) \oplus C \\
&= (A'B' + AB)C' + (A'B' + AB)'C \\
&= A'B'C' + ABC' + A'BC + AB'C \\
&= \text{... can be further simplified?}
\end{aligned}
$$

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- Multiplying Out and Factoring Expressions
- Exclusive OR and Equivalence Operations
- **Consensus Theorem**
- Algebraic Simplification of Switching Expressions
- Proving Validity of an Equation

Consensus Theorem

看到X, X'
各自余的项又乘在一起 consensus term 可直接删掉.

- $\boxed{X}Y + \boxed{X}'Z + YZ = XY + X'Z$
  - YZ is redundant, referred to as the **consensus term**

- Example

consensus term

A'C'D + A'BD + BCD + ABC + ACD'

= A'C'D + **A'**BD + BCD + **A**BC + ACD'

= A'C'D + A'BD + ABC + ACD'

- Example

A'B' + AC + BC' + B'C + AB

= **A'**B' + **A**C + BC' + B'C + AB

= A'B' + AC + BC' + AB

= A'B' + A**C** + B**C'** + AB

= A'B' + AC + BC'

用consensus Thm做簡化時，

Ordering Does Matter

結果不同，甚至項數也不同，

- Example (one ordering)

  A'C'D + A'BD + BCD + ABC + ACD'

  = A'C'D + **A'**BD + BCD + **A**BC + ACD'

  = A'C'D + A'BD + ABC + ACD'

- Example (another ordering)

  A'**C'**D + A'BD + B**C**D + ABC + ACD'

  = A'C'D + BCD + ABC + ACD'

  = A'C'D + BC**D** + ABC + AC**D'**

  = A'C'D + BCD + ACD'

Dual Form of Consensus Theorem

- XY + X'Z + YZ = XY + X'Z
  - YZ is redundant 可直接删.
- (X + Y)(X' + Z)(Y + Z) = (X + Y)(X' + Z)
  - (Y + Z) is redundant
- Example

  (A + B + C')(A + B + D')(B + C + D')

  = (A + B + $\underline{\textbf{C'}}$)(A + B + D')(B + $\underline{\textbf{C}}$ + D')

  = (A + B + C')(B + C + D')

  = … can be further simplified

  = B + AC + C'D'

Redundancy Insertion

❑ Example

- ABCD + B'CDE + A'B' + BCE'
- Consensus terms

$\binom{4}{2}$組
  - A**B**CD + **B'**CDE → ACDE
  - **A**BCD + **A'**B' → 0; A**B**CD + A'**B'** → 0
  - **B'**CDE + **B**CE' → 0; B'CD**E** + BC**E'** → 0
  - A'**B'** + **B**CE → A'CE
- No redundancy
- Redundancy insertion

ABCD, B'CDE 的 consensus term = ACDE
刪掉沒影响 → 多加也沒影响.

ABCD + B'CDE + A'B' + BCE' + ACDE

= ABCD + B'CDE + A'B' + BC**E'** + ACD**E**

= B'CDE + A'B' + BCE' + ACDE

= B'CDE + **A'**B' + BCE' + **A**CDE

= A'B' + BCE' + ACDE

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- **Algebraic Simplification of Switching Expressions**
- Proving Validity of an Equation

Three Basic Ways

- **Goal** 為什麼要簡化 Boolean expression ? ∵ Boolean expression 對應到 circuit ∴ 簡化 Boolean expression ⇒ 簡化 circuit
  - Simplifying an expression reduces the cost of realizing the expression using gates
- **Combining terms:** $XY + XY' = X(Y + Y') = X$
  - $ABC'D' + ABCD' = ABD'$ $= (C+C')ABD'$ $= ABD'$
  - $AB'C + ABC + A'BC = AB'C + ABC + ABC + A'BC = AC + BC$
  - $(A + BC)(D + E') + \underline{A'(B' + C')}(D + E') = D + E'$; $A'(B'+C') = (A+BC)'$
    - DeMorgan's Law
- **Eliminating terms:** $X + XY = X$ and $XY + X'Z + YZ = XY + X'Z$
  - $A'B + A'BC = A'B$
  - $A'BC' + BCD + A'BD = A'BC' + BCD$
- **Eliminating literals:** $X + X'Y = X + Y$
  - $A'B + A'B'C'D' + ABCD' = A'B + A'C'D' + ABCD' = A'B + A'C'D' + BCD'$

$X \cdot 1 + X'Y = (X+Y)(X'+1) = (X+Y) \cdot 1 = X+Y$

Adding Redundant Terms

- $Y = Y + XX'$
- $Y = Y(X + X')$
- $XY + X'Z = XY + X'Z + YZ$
- $X = X + XY$
- Example

$$W\underline{\mathbf{X}} + XY + \underline{\mathbf{X'}}Z' + WY'Z'$$

$$= WX + XY + X'Z' + \underline{WY'Z' + WZ'}$$

$$= W\underline{\mathbf{X}} + XY + \underline{\mathbf{X'}}Z' + \underline{WZ'}$$

$= WX + XY + X'Z'$ ...can you derive this directly?

Quick Note

- No easy way to determine if a Boolean expression has a minimum number of terms or literals
  - Systematic (graphical) method will be discussed

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- ❑ **Proving Validity of an Equation**

Equation Validity (1/2)

- Several methods
  - Construct a truth table
    - Proof by cases (equivalent for all combinations of values of variables)
  - Manipulate one side until it is identical to the other side
  - Reduce both sides independently to the same expression
  - Perform the same operation on both sides if the operation is reversible
    - Complement is reversible
    - AND is not reversible: XY = XZ does not imply Y = Z X=0
    - OR is not reversible: X + Y = X + Z does not imply Y = Z X=1
    - $X \oplus Y = X \oplus Z$ implies Y = Z? Yes

(相同) X
X和Y不同

X和Z不同 (相同)

- Prove invalidity
  - Try to find **one** combination of values of variables such that two sides have different values

Equation Validity (2/2)

- Example: prove A'BD' + BCD + ABC' + AB'D = AD + A'BC + BC'D'

  A'BD' + BCD + ABC' + AB'D

  = A'BD' + BCD + ABC' + AB'D + A'BC + BC'D' + ABD

  = AD + A'BD' + BCD + ABC' + A'BC + BC'D'

  = AD + A'BC + BC'D'

- Example: prove the equivalence between
  - A'BC'D + (A' + BC)(A + C'D') + BC'D + A'BC'
  - ABCD + A'C'D' + ABD + ABCD' + BC'D

Q&A