Boolean Algebra Properties

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2.4 Basic Theorems

Idempotent Laws

$$X + X = X$$
$$X \cdot X = X$$

Involution Law

$$(X')' = X$$

Laws of Complementarity

$$X + X' = 1$$
$$X \cdot X' = 0$$

2.5 Commutative, Associative, Distributive and DeMorgan's Laws

Distributive Laws

$$X(Y + Z) = XY + XZ$$
$$X + YZ = (X + Y)(X + Z)$$

Example.

$$(A + B + C')(A + B + D) = A + B + C'D$$

Proof.

$$\begin{aligned}
(X + Y)(X + Z) &= X(X + Z) + Y(X + Z) \\
&= XX + XZ + XY + YZ \\
&= X + XZ + XY + YZ \\
&= X \cdot 1 + XZ + XY + YZ \\
&= X(1 + Z + Y) + YZ \\
&= X \cdot 1 + YZ \\
&= X + YZ
\end{aligned}$$

□

DeMorgan's Laws

$$\begin{aligned}
(X + Y)' &= X' \cdot Y' \\
(X \cdot Y)' &= X' + Y'
\end{aligned}$$

2.6 Simplification Theorems

Uniting

$$\begin{aligned}
XY + XY' &= X \\
(X + Y)(X + Y') &= X
\end{aligned}$$

Proof.

$$\begin{aligned}
XY + XY' &= X(Y + Y') \\
&= X \cdot 1 \\
&= X
\end{aligned}$$

□

Proof.

$$
\begin{aligned}
(X+Y)(X+Y') &= XX + XY' + XY + YY' \\
&= X + XY' + XY \\
&= X(1 + Y' + Y) \\
&= X \cdot 1 \\
&= X
\end{aligned}
$$

□

Absorption

$$
\begin{aligned}
X + XY &= X \\
X(X+Y) &= X
\end{aligned}
$$

Proof.

$$
\begin{aligned}
X(X+Y) &= (X + 0)(X+Y) \qquad \text{(distributive)} \\
&= X + 0 \cdot Y \\
&= X
\end{aligned}
$$

□

Elimination

$$
\begin{aligned}
X + X'Y &= X + Y \\
X(X'+Y) &= XY
\end{aligned}
$$

Proof. By distributive laws, we have $X + YZ = (X+Y)(X+Z)$

Thus,

$$
\begin{aligned}
X + X'Y &= (X + X')(X+Y) \\
&= 1 \cdot (X+Y) \\
&= X + Y
\end{aligned}
$$

□

Consensus

$$
\begin{aligned}
XY + X'Z + YZ &= XY + X'Z \\
(X+Y)(X'+Z)(Y+Z) &= (X+Y)(X'+Z)
\end{aligned}
$$

Proof.

$$
\begin{aligned}
XY + X'Z + YZ &= XY + X'Z + 1 \cdot YZ \\
&= XY + X'Z + (X + X')YZ \\
&= XY + X'Z + XYZ + X'YZ \qquad \text{(absorption)} \\
&= XY + X'Z + X'YZ \qquad \text{(absorption)} \\
&= XY + X'Z
\end{aligned}
$$

□

2.8 Complementing Boolean Expressions

$$
\begin{aligned}
(X_1 + X_2 + \cdots + X_n)' &= X_1' \cdot X_2' \cdots X_n' \\
(X_1 \cdot X_2 \cdots X_n)' &= X_1' + X_2' + \cdots + X_n'
\end{aligned}
$$

3.2 Exclusive-OR and Equivalence Operations

Exclusive-OR

$$X \oplus Y = XY' + X'Y$$

Because X exclusive-OR Y is 1 only when $X = 1$ and $Y = 0$ (XY') or $X = 0$ and $Y = 1$ $(X'Y)$.

Another way to think:
X exclusive-OR Y is 1 only when $X = 1$ or $Y = 1$, and X, Y not both 1, therefore:

$$
\begin{aligned}
X \oplus Y &= (X + Y)(XY)' \\
&= (X + Y)(X' + Y') \\
&= XY' + X'Y
\end{aligned}
$$

Properties of Exclusive-OR

$$
\begin{aligned}
&X \oplus 0 = X \\
&X \oplus 1 = X' \\
&X \oplus X = 0 \\
&X \oplus X' = 1 \\
&X \oplus Y = Y \oplus X \text{ (commutative)} \\
&X \oplus (Y \oplus Z) = (X \oplus Y) \oplus Z = X \oplus Y \oplus Z \text{ (associative)} \\
&X(Y \oplus Z) = XY \oplus XZ \text{ (distributive)} \\
&(X \oplus Y)' = X \oplus Y' = X' \oplus Y = XY + X'Y'
\end{aligned}
$$

Proof.

$$
\begin{aligned}
(X \oplus Y)' &= (XY' + X'Y)' \\
&= (XY')' \cdot (X'Y)' \\
&= (X' + Y)(X + Y') \\
&= X'Y' + XY
\end{aligned}
$$

□

Equivalence Relation

$$
\begin{aligned}
&(X \equiv Y) = 1 \iff X = Y \\
\Rightarrow\ &(X \equiv Y) = XY + X'Y'
\end{aligned}
$$

Because $X \equiv Y$ is 1 only when $X = Y = 1$ or $X = Y = 0$.

Therefore, combining the above equations, we have:

Equivalence is the complement of exclusive-OR

$$(X \equiv Y) = XY + X'Y' = (X \oplus Y)'$$

Hence, the equivalence gate is also called the exclusive-NOR gate.
Also, note that:

$$
\begin{aligned}
&X \oplus Y = XY' + X'Y \\
\Rightarrow\ &(X \oplus Y)' = (XY' + X'Y)' = XY + X'Y'
\end{aligned}
$$

m This is useful, for example:

$$\begin{aligned} A' \oplus B \oplus C &= (A'B' + AB) \oplus C \\ &= (A'B' + AB)'C + (A'B' + AB)C' \\ &= (A'B + AB')C + (A'B' + AB)C' \\ &= A'BC + AB'C + A'B'C' + ABC' \end{aligned}$$