Digital Systems Design and Laboratory
[6. Quine-McCluskey Method]

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Two-Level Logic Minimization

- Karnaugh map vs. Quine-McCluskey method
  - Same input
    - Minterm expansion
  - Same output
    - A minimum SOP
  - Same high-level procedure
    - Find all Prime Implicants (PIs)
    - Find a minimum SOP

Finding All Prime Implicants (1/5)

- **Start with all minterms** 列 output = 1 的 minterms
- Group pairs of adjacent minterms
  - Mark and remove all covered terms
- Repeat grouping until no more grouping possible

CD \ AB	00	01	11	10
00	1 m_0	m_4	m_{12}	1 m_8
01	1 m_1	1 m_5	m_{13}	1 m_9
11	m_3	1 m_7	m_{15}	m_{11}
10	1 m_2	1 m_6	1 m_{14}	1 m_{10}

用這個 minterm 有幾個一樣到

#1s	Minterm	Binary
0	0	0000
1	1	0001
	2	0010
	8	1000
2	5	0101
	6	0110
	9	1001
	10	1010
3	7	0111
	14	1110

Finding All Prime Implicants (2/5)

- Start with all minterms
- **Group pairs of adjacent minterms**
  - Mark and remove all covered terms
- Repeat grouping until no more grouping possible

CD \ AB	00	01	11	10
00	1 m_0	m_4	m_{12}	1 m_8
01	1 m_1	1 m_5	m_{13}	1 m_9
11	m_3	1 m_7	m_{15}	m_{11}
10	1 m_2	1 m_6	1 m_{14}	1 m_{10}

#1s	Minterm	Size 1
0	0	0000
	1	0001
1	2	0010
	8	1000
2	5	0101
	6	0110
	9	1001
	10	1010
3	7	0111
	14	1110

Minterm	Size 2
0, 1	000-
0, 2	00-0
0, 8	-000
1, 5	0-01
1, 9	-001
2, 6	0-10
2, 10	-010
8, 9	100-
8, 10	10-0
5, 7	01-1
6, 7	011-
6, 14	-110
10,14	1-10

0, 1, 2, 4, 5, 6, 9, 10, 13, 14 都至少在它们的 minterm 出现一次
→ 代表它们都是 PI (prime implicant)

把格图圈接成一格.

Finding All Prime Implicants (3/5)

" PI: 没办法被其它 implicant cover 的 implicants

- Start with all minterms
- Group pairs of adjacent minterms
  - **Mark and remove all covered terms**
- Repeat grouping until no more grouping possible

∴ 要移除这些 terms

再继续做下去.

CD \ AB	00	01	11	10
00	1 m_0	m_4	m_{12}	1 m_8
01	1 m_1	1 m_5	m_{13}	1 m_9
11	m_3	1 m_7	m_{15}	m_{11}
10	1 m_2	1 m_6	1 m_{14}	1 m_{10}

#1s	Minterm	Size 1
0	0	0000
1	1	0001
	2	0010
	8	1000
2	5	0101
	6	0110
	9	1001
	10	1010
3	7	0111
	14	1110

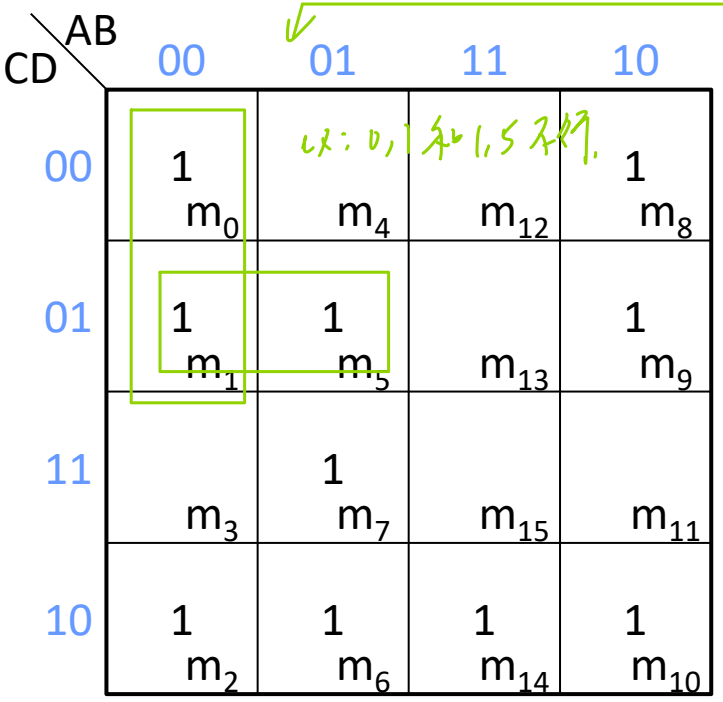
size 1 的 minterm 拿来跟 size > 的 minterm.

→ 继续看能不能拿来跟 size 大的 minterm

Minterm	Size 2
0, 1	000-
0, 2	00-0
0, 8	-000
1, 5	0-01
1, 9	-001
2, 6	0-10
2, 10	-010
8, 9	100-
8, 10	10-0
5, 7	01-1
6, 7	011-
6, 14	-110
10,14	1-10

Finding All Prime Implicants (4/5)

- Start with all minterms
- Group pairs of adjacent minterms
  - Mark and remove all covered terms
- **Repeat grouping until no more grouping possible**



#1s	Minterm	Size 1
0	0	0000
1	1	0001
	2	0010
	8	1000
2	5	0101
	6	0110
	9	1001
	10	1010
3	7	0111
	14	1110

Minterm	Size 2
0, 1	000-
0, 2	00-0
0, 8	-000
1, 5	0-01
1, 9	-001
2, 6	0-10
2, 10	-010
8, 9	100-
8, 10	10-0
5, 7	01-1
6, 7	011-
6, 14	-110
10, 14	1-10

Minterm	Size 4
0,1,8,9	-00-
0,2,8,10	-0-0
0,8,1,9	-00-
0,8,2,10	-0-0
2,6,10,14	--10
2,10,6,14	--10

Finding All Prime Implicants (5/5)

- Start with all minterms
- Group pairs of adjacent minterms
  - Mark and remove all covered terms
- **Repeat grouping until no more grouping possible**

CD \ AB	00	01	11	10
00	1 m_0	m_4	m_{12}	1 m_8
01	1 m_1	1 m_5	m_{13}	1 m_9
11	m_3	1 m_7	m_{15}	m_{11}
10	1 m_2	1 m_6	1 m_{14}	1 m_{10}

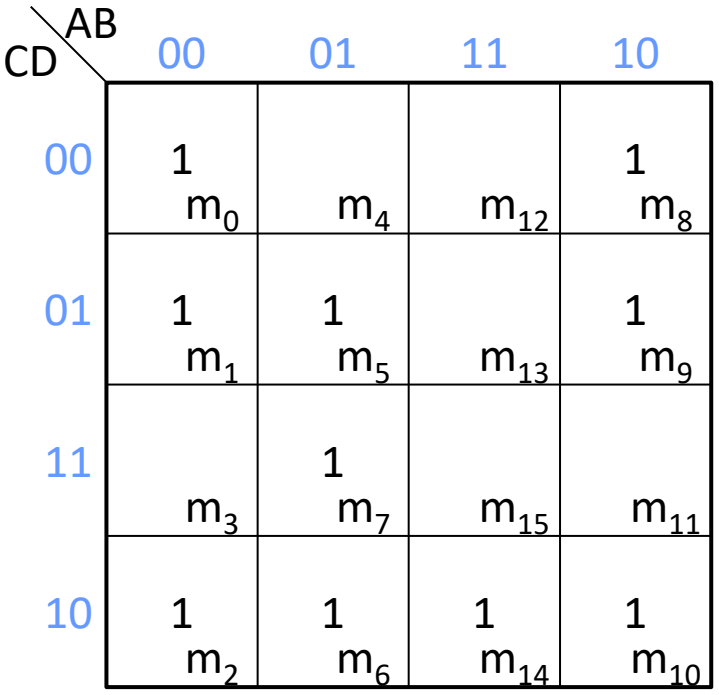
#1s	Minterm	Size 1
0	0	0000
1	1	0001
	2	0010
	8	1000
2	5	0101
	6	0110
	9	1001
	10	1010
3	7	0111
	14	1110

Minterm	Size 2
0, 1	000-
0, 2	00-0
0, 8	-000
1, 5	**0-01**
1, 9	-001
2, 6	0-10
2, 10	-010
8, 9	100-
8, 10	10-0
5, 7	**01-1**
6, 7	**011-**
6, 14	-110
10, 14	1-10

Minterm	Size 4
0,1,8,9	**-00-**
0,2,8,10	**-0-0**
~~0,8,1,9~~	~~-00-~~
~~0,8,2,10~~	~~-0-0~~
2,6,10,14	**--10**
~~2,10,6,14~~	~~--10~~

Comparing with Karnaugh Map

- Try to find all prime implicants directly in the Karnaugh map



#1s	Minterm	Size 1
0	0	0000
1	1	0001
	2	0010
	8	1000
2	5	0101
	6	0110
	9	1001
	10	1010
3	7	0111
	14	1110

Minterm	Size 2
0, 1	000-
0, 2	00-0
0, 8	-000
1, 5	**0-01**
1, 9	-001
2, 6	0-10
2, 10	-010
8, 9	100-
8, 10	10-0
5, 7	**01-1**
6, 7	**011-**
6, 14	-110
10, 14	1-10

Minterm	Size 4
0,1,8,9	**-00-**
0,2,8,10	**-0-0**
~~0,8,1,9~~	~~-00-~~
~~0,8,2,10~~	~~-0-0~~
2,6,10,14	**--10**
~~2,10,6,14~~	~~--10~~

→不能再合了
紅色的都是PI.

Finding a Minimum SOP (1/4)

K-map：化簡迷圖圖

- **Build the prime implicant chart (table)**
- Simplify the table using essential
  - Select an essential, delete covered minterms, and repeat
- Optimally select PIs by column covering

CD \ AB	00	01	11	10
00	1 m_0	m_4	m_{12}	1 m_8
01	1 m_1	1 m_5	m_{13}	1 m_9
11	m_3	1 m_7	m_{15}	m_{11}
10	1 m_2	1 m_6	1 m_{14}	1 m_{10}

		Minterms									
PIs		0	1	2	5	6	7	8	9	10	14
0,1,8,9	-00-	x	x					x	x		
0,2,8,10	-0-0	x		x				x		x	
2,6,10,14	--10			x		x				x	x
1,5	0-01		x		x						
5,7	01-1				x		x				
6,7	011-					x	x				

Finding a Minimum SOP (2/4)

- Build the prime implicant chart (table)
- **Simplify the table using essential prime implicants**
  - **Select essential prime implicants** and delete covered minterms
- Optimally select PIs by column covering

CD \ AB	00	01	11	10
00	1 m_0	m_4	m_{12}	1 m_8
01	1 m_1	1 m_5	m_{13}	1 m_9
11	m_3	1 m_7	m_{15}	m_{11}
10	1 m_2	1 m_6	1 m_{14}	1 m_{10}

PIs		Minterms 0	1	2	5	6	7	8	9	10	14
0,1,8,9	-00-	x	x					x	x		
0,2,8,10	-0-0	x		x				x		x	
2,6,10,14	--10			x		x				x	x
1,5	0-01		x		x						
5,7	01-1				x		x				
6,7	011-					x	x				

Finding a Minimum SOP (3/4)

- Build the prime implicant chart (table)
- **Simplify the table using essential prime implicants**
  - Select essential prime implicants and **delete covered minterms**
- Optimally select PIs by column covering

CD \ AB	00	01	11	10
00	1 m_0	m_4	m_{12}	1 m_8
01	1 m_1	1 m_5	m_{13}	1 m_9
11	m_3	1 m_7	m_{15}	m_{11}
10	1 m_2	1 m_6	1 m_{14}	1 m_{10}

		Minterms									
PIs		0	1	2	5	6	7	8	9	10	14
0,1,8,9	-00-	x	x					x	x		
0,2,8,10	-0-0	x		x				x		x	
2,6,10,14	--10			x		x				x	x
1,5	0-01		x		x						
5,7	01-1				x		x				
6,7	011-					x	x				

被essential PI cover到的column可刪除
∴剩5,7考慮

Finding a Minimum SOP (4/4)

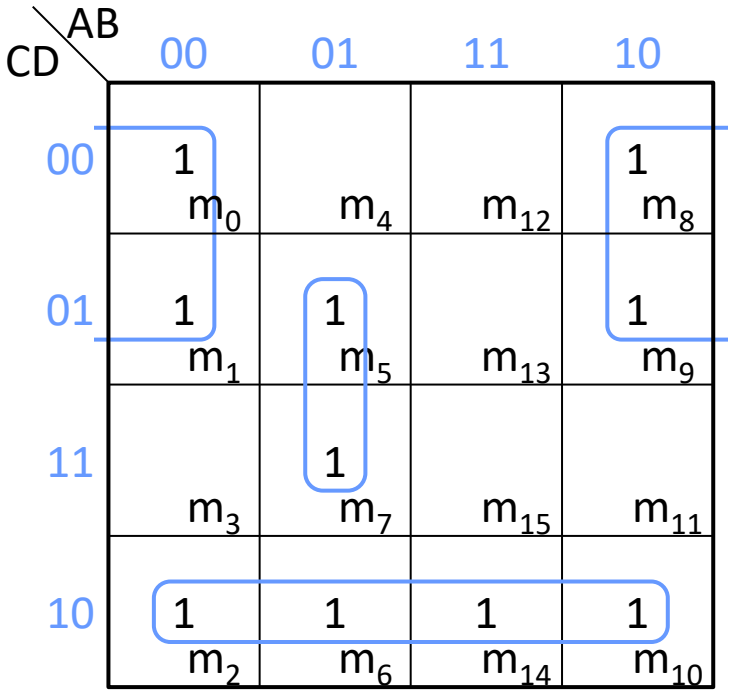
- Build the prime implicant chart (table)
- Simplify the table using essential prime implicants
  - Select essential prime implicants and delete covered minterms
- **Optimally select PIs by column covering**

CD \ AB	00	01	11	10
00	1 m_0	m_4	m_{12}	1 m_8
01	1 m_1	1 m_5	m_{13}	1 m_9
11	m_3	1 m_7	m_{15}	m_{11}
10	1 m_2	1 m_6	1 m_{14}	1 m_{10}

PIs		Minterms 0	1	2	5	6	7	8	9	10	14
0,1,8,9	-00-	x	x					x	x		
0,2,8,10	-0-0	x		x				x		x	
2,6,10,14	--10			x		x				x	x
1,5	0-01		x		x						
5,7	01-1				x		x				
6,7	011-					x	x				

Comparing with Karnaugh Map

- Try to find a minimum SOP directly in the Karnaugh map



PIs		Minterms 0	1	2	5	6	7	8	9	10	14
0,1,8,9	-00-	x	x					x	x		
0,2,8,10	-0-0	x		x				x		x	
2,6,10,14	--10			x		x				x	x
1,5	0-01		x		x						
5,7	01-1				x		x				
6,7	011-					x	x				

Difficulty and Summary

- Column covering is hard
  - NP-complete
  - Consider a Boolean expression with n variables, in general
    - ~2^n minterms
    - ~$3^n/n$ prime implicants
      - The proof is optional: https://core.ac.uk/download/pdf/82016049.pdf
- Summary
  - Karnaugh map ↙多個 Var 不適合
    - Exact and effective as # of variables ≤ 5
  - Quine-McCluskey method ↙ time complexity 高.
    - Exact and realizable for more variables
  - Espresso
    - Heuristic and faster than the Quine-McCluskey method

↑偏 greedy 的方法. 不能保證 optimal 但可能比較快.

More Than Logic Design

- Are Integrated Circuit (IC) designers still doing this?
  - Maybe not…
  - Existing semiconductor intellectual property (IP) core
  - Electronic Design Automation (EDA) tools
- How to decompose a complicated system to components (libraries) which you can implement easily?
  - This decomposition must be correct, i.e., logically equivalent in this case

ex: boolean expression
→ minimum SOP

Q&A