5. Prove / disprove:

For any $H_1, H_2 \neq \phi$ (for binary classification, operate on same input space)

$d_{vc}(H_1 \cup H_2) \leq d_{vc}(H_1) + d_{vc}(H_2)$

Pf:

Suppose $\exists H_1, H_2 \neq \phi$ $\ni$ $d_{vc}(H_1 \cup H_2) > d_{vc}(H_1) + d_{vc}(H_2)$

Let $d_{vc}(H_1) = k_1$, $d_{vc}(H_2) = k_2$, then

$$\begin{cases} m_{H_1}(k_1) = 2^{k_1}, & m_{H_1}(k_1+1) < 2^{k_1+1} \\ m_{H_2}(k_2) = 2^{k_2}, & m_{H_2}(k_2+1) < 2^{k_2+1} \end{cases}$$

$\Rightarrow d_{vc}(H_1 \cup H_2) > k_1 + k_2$, which means

$\exists$ some $k_1 + k_2 + 1$ points that can be shattered by $H_1 \cup H_2$

Let $d_{vc}(H_1 \cup H_2) = k$, $k > k_1 + k_2$

$m_{H_1 \cup H_2}(k_1 + k_2 + 1) \leq (k_1 + k_2 + 1)^k + 1 = 2^k$

$\because k_1, k_2 \in \mathbb{Z}^+ \cup \{0\}$

$\therefore (k_1, k_2) \in \{(0,1), (1,0)\} \Rightarrow (k_1 + k_2 + 1)^k + 1 = 2^k + 1 > 2^k$ ($\times$)

Thus, $k_1 = k_2 = 0$, then $(k_1 + k_2 + 1)^k + 1 = |H| = 2$

By Thm, $d_{vc}(H) = 0 \Leftrightarrow H = \{y = a, a \in \mathbb{R}\}$ (H has exactly one hypothesis, which is a constant function)

Therefore, let $H_1 = \{y = a_1\}$, where $a_1 \in \mathbb{R}$, $H_2 = \{y = a_2\}$ where $a_2 \in \mathbb{R}$

$\Rightarrow H_1 \cup H_2 = \{y = a_1, y = a_2\}$ where $a_1, a_2 \in \mathbb{R}$.

However, $d_{vc}(H_1 \cup H_2) = 0 \not> 0$ $\times$

Thus, $d_{vc}(H_1 \cup H_2) \leq d_{vc}(H_1) + d_{vc}(H_2)$ $\square$