6.

If we only consider the original constraint (i.e. the constraints for example 1 to N), the lagrange function with lagrange multipliers α_n and β_n is:

$$\mathcal{L}(b, \mathbf{w}, \xi, \alpha, \beta) = \frac{1}{2}\mathbf{w}^T\mathbf{w} + C\sum_{n=1}^{N}\xi_n + \sum_{n=1}^{N}\alpha_n\left(1 - \xi_n - y_n(b + \mathbf{w}^T\mathbf{\Phi}(\mathbf{x_n}))\right) + \sum_{n=1}^{N}\beta_n(-\xi_n)$$

For the anchor pseudo-example, we have the constraint:

$$y_0(\mathbf{w}^T\mathbf{\Phi}(\mathbf{x_0}) + b) \geq 1$$

Since we have $\mathbf{x}_0 = \mathbf{0}$ and $y_0 = -1$, the constraint becomes:

$$-b \geq 1 \Rightarrow b \leq -1$$

Convert into canonical form:

$$b + 1 \leq 0$$

So we can add the term $\gamma_0(b+1)$, where γ_0 is the corresponding lagrange multiplier.

Thus we have the new lagrange function:

$$\mathcal{L}(b, \mathbf{w}, \xi, \alpha, \beta, \gamma_0) = \frac{1}{2}\mathbf{w}^T\mathbf{w} + C\sum_{n=1}^{N}\xi_n + \sum_{n=1}^{N}\alpha_n\left(1 - \xi_n - y_n(b + \mathbf{w}^T\mathbf{\Phi}(\mathbf{x_n}))\right) + \sum_{n=1}^{N}\beta_n(-\xi_n) + \gamma_0(b+1)$$

For the lagrange dual, we need to solve:

$$\max_{\alpha,\beta,\gamma_0 \geq 0} \min_{b,\mathbf{w},\xi} \mathcal{L}(b, \mathbf{w}, \xi, \alpha, \beta, \gamma_0)$$

Taking the partial derivative of each variable, we get:

$$\frac{\partial \mathcal{L}}{\partial b} = 0 \Rightarrow -\sum_{n=1}^{N}\alpha_n y_n + \gamma_0 = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{w}} = 0 \Rightarrow \mathbf{w} = \sum_{n=1}^{N}\alpha_n y_n \mathbf{\Phi}(\mathbf{x_n}) \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \xi_n} = 0 \Rightarrow C - \alpha_n - \beta_n = 0 \tag{3}$$

Since $\alpha_n \geq 0$ and $\beta_n \geq 0$, we have $0 \leq \alpha_n \leq C$.

From equation (2), we can substitute $\mathbf{w}$ into the Lagrangian and obtain the dual:

$$\frac{1}{2}\left\|\sum_{n=1}^{N}\alpha_n y_n \Phi(\mathbf{x}_n)\right\|^2 + C\sum_{n=1}^{N}\xi_n + \sum_{n=1}^{N}\alpha_n\left(1 - \xi_n - y_n(b + \mathbf{w}^T\mathbf{\Phi}(\mathbf{x_n}))\right) + \sum_{n=1}^{N}\beta_n(-\xi_n) + \gamma_0(b+1)$$

Converting $||\mathbf{w}||^2$ to the proper form, can write:

$$||\mathbf{w}||^2 = \sum_{n=1}^{N}\sum_{m=1}^{N}\alpha_n\alpha_m y_n y_m \mathbf{\Phi}(\mathbf{x_n})^T\mathbf{\Phi}(\mathbf{x_m})$$

Plug in back to the dual, we get:

$$\frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N}\alpha_n\alpha_m y_n y_m K(x_n, x_m) + C\sum_{n=1}^{N}\xi_n + \sum_{n=1}^{N}\alpha_n\left(1 - \xi_n - y_n(b + \mathbf{w}^T\mathbf{\Phi}(\mathbf{x_n}))\right) + \sum_{n=1}^{N}\beta_n(-\xi_n) + \gamma_0(b+1)$$

Expand the terms and we get:

$$\begin{aligned}
&\frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N}\alpha_n\alpha_m y_n y_m K(x_n, x_m) + C\sum_{n=1}^{N}\xi_n \\
&+ \sum_{n=1}^{N}\alpha_n - \sum_{n=1}^{N}\alpha_n\xi_n - \sum_{n=1}^{N}\alpha_n y_n b - \sum_{n=1}^{N}\alpha_n y_n \mathbf{w}^T\mathbf{\Phi}(\mathbf{x_n}) \\
&- \sum_{n=1}^{N}\beta_n\xi_n + \gamma_0 b + \gamma_0
\end{aligned}$$

From equation (1), we have:

$$-\sum_{n=1}^{N}\alpha_n y_n + \gamma_0 = 0 \Rightarrow (-\sum_{n=1}^{N}\alpha_n y_n + \gamma_0)b = 0$$

And from equation (3), we have:

$$(C - \alpha_n - \beta_n) \sum_{n=1}^{N} \xi_n = 0$$

The Largrangian is simplified to:

$$\frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} \alpha_n \alpha_m y_n y_m K(x_n, x_m) + \sum_{n=1}^{N} \alpha_n - \sum_{n=1}^{N} \alpha_n y_n \mathbf{w}^T \mathbf{\Phi}(\mathbf{x_n}) + \gamma_0 \tag{4}$$

Observe that the term that contain $\mathbf{w}$ can also be expanded by (2) as:

$$\begin{aligned}
-\sum_{n=1}^{N} \alpha_n y_n \mathbf{w}^T \mathbf{\Phi}(\mathbf{x_n}) &= -\sum_{n=1}^{N} \alpha_n y_n \left(\sum_{m=1}^{N} \alpha_m y_m \mathbf{\Phi}(\mathbf{x_m}) \right)^T \mathbf{\Phi}(\mathbf{x_n}) \\
&= -\sum_{n=1}^{N} \sum_{m=1}^{N} \alpha_n \alpha_m y_n y_m \mathbf{\Phi}(\mathbf{x_m})^T \mathbf{\Phi}(\mathbf{x_n}) \\
&= -\sum_{n=1}^{N} \sum_{m=1}^{N} \alpha_n \alpha_m y_n y_m K(x_n, x_m)
\end{aligned}$$

Again, plug into the previous simplified Lagrangian (4), we get:

$$-\frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} \alpha_n \alpha_m y_n y_m K(x_n, x_m) + \sum_{n=1}^{N} \alpha_n + \gamma_0$$

From the problem description, we knew that $y_n = +1 \quad \forall n,\ n \neq 0$, so we have:

$$-\frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} \alpha_n \alpha_m K(x_n, x_m) + \sum_{n=1}^{N} \alpha_n + \gamma_0$$

In order to use the QP solver, we first need to convert the above maximization problem into a minimization problem:

$$\min_{\alpha, \gamma_0 \geq 0} \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} \alpha_n \alpha_m K(x_n, x_m) - \sum_{n=1}^{N} \alpha_n - \gamma_0 \tag{*}$$

With constraints from (1), (3):

$$\sum_{n=1}^{N} \alpha_n - \gamma_0 = 0 \tag{5}$$

$$C - \alpha_n - \beta_n = 0 \tag{6}$$

To convert into the QP form, we define:

$$\begin{aligned}
\alpha &= [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_N]^T \\
\mathbf{u} &= [\gamma_0 \ \alpha_1 \ \alpha_2 \ \cdots \ \alpha_N]^T \\
\mathbf{Q} &= [K(x_n, x_m)]_{N \times N} \quad \text{(Gram matrix)}
\end{aligned}$$

Then the original problem $(*)$ can be written as:

$$\min_{\mathbf{x} \geq 0} \frac{1}{2} \mathbf{u}^T \mathbf{Q} \mathbf{u} - \mathbf{1}^T \mathbf{u}$$

So:

$$Q = [K(x_n, x_m)]_{N \times N} \quad \text{and} \quad \mathbf{p} = -\mathbf{1}$$

subject to:

$$\begin{aligned}
\sum_{n=1}^{N} \alpha_n = \gamma_0 & \qquad \text{by (5)} \\
\alpha_n = C - \beta_n & \qquad \text{by (6)}
\end{aligned}$$

Thus for each row in A,

$$\mathbf{a}_n^T = y_n [1 \ \mathbf{\Phi}(\mathbf{x_n})^\mathbf{T}] = \begin{cases} [1 \ \mathbf{\Phi}(\mathbf{x_n})^\mathbf{T}] & \forall n \neq 0 \\ [1 \ \mathbf{0^T}] & n = 0 \end{cases}$$

Finally, we have each element in $\mathbf{c}$ as $c_n = 0$