g. M machines, each with probability $\begin{cases} \mu_m & 1 \text{ coin} \\ 1-\mu_m & \text{no coin} \end{cases}$, where μ_m: unknown

at time t, pull machine $m = ((t-1) \bmod M) + 1$

i.e. at time $t=1$, pull machine $m = ((1-1) \bmod M) + 1 = 1$

$t=2$, pull machine $m = ((2-1) \bmod M) + 1 = 2$

after $t > M$, machine m being pulled N_m times, collect c_m coins

$(N_m \geq 1 \because t > M)$

<u>claim</u>: for $M \geq 2$, $\forall m = 1, \ldots, M$, $\forall t = M+1, M+2, \ldots$

$$P\left[\mu_m \leq \frac{c_m}{N_m} + \sqrt{\frac{\ln t + \ln M - \frac{1}{2}\ln \delta}{N_m}}\right] \geq 1 - \delta$$

<u>pf</u>: Using union bound, the total probability that μ_m exceeds the bound for any machine $(m = 1, \ldots, M)$ at any time step $(t = M+1, M+2, \ldots)$ is

$$\sum_{m=1}^{M} \sum_{t=M+1}^{\infty} P\left[\mu_m > \frac{c_m}{N_m} + \sqrt{\frac{\ln t - \frac{1}{2}\ln \delta}{N_m}}\right] \leq \sum_{m=1}^{M} \sum_{t=M+1}^{\infty} \delta t^{-2}$$

$$\leq \sum_{m=1}^{M} \sum_{t=1}^{\infty} \delta t^{-2}$$

$$= \sum_{m=1}^{M} \delta \cdot \frac{\pi^2}{6}$$

$$= M \cdot \delta \cdot \frac{\pi^2}{6}$$

Therefore, the probability for the bound $\frac{c_m}{N_m} + \sqrt{\frac{\ln t - \frac{1}{2}\ln \delta}{N_m}}$ is exceeded by any machine $m \in \{1, \ldots, M\}$ at any time step $\geq M+1$ will $\leq M \cdot \delta \cdot \frac{\pi^2}{6}$

$\therefore\ \forall m \in \{1, \dots, M\},\ \forall t \in \{M+1, M+2, \dots\}$ where $M \geq 2$

$$P\left[\mu_m > \frac{c_m}{N_m} + \sqrt{\frac{\ln t - \frac{1}{2}\ln \delta}{N_m}}\right] \leq M \cdot \delta \cdot \frac{\pi^2}{6}$$

$$\Rightarrow \quad P\left[\mu_m \leq \frac{c_m}{N_m} + \sqrt{\frac{\ln t - \frac{1}{2}\ln \delta}{N_m}}\right] \geq 1 - M \cdot \delta \cdot \frac{\pi^2}{6}$$

By replacing δ by $\frac{6\delta}{\pi^2 M}$, we get:

$$\Rightarrow \quad P\left[\mu_m \leq \frac{c_m}{N_m} + \sqrt{\frac{\ln t - \frac{1}{2}\ln\left(\frac{6\delta}{\pi^2 M}\right)}{N_m}}\right] \geq 1 - \delta$$

$$-\frac{1}{2}(\ln 6\delta - \ln \pi^2 M)$$

$$= -\frac{1}{2}(\ln 6 + \ln \delta - \ln \pi^2 - \ln M)$$

$$= \underline{-\frac{1}{2}\ln 6} - \frac{1}{2}\ln \delta + \underline{\frac{1}{2}\ln \pi^2} + \frac{1}{2}\ln M$$

omit constant term ($\because$ constant does not affect much as $t \uparrow$)

$$\doteq -\frac{1}{2}\ln \delta + \frac{1}{2}\ln M.$$

$$\Rightarrow P\left[\mu_m \leq \frac{c_m}{N_m} + \sqrt{\frac{\ln t - \frac{1}{2}\ln \delta + \frac{1}{2}\ln M}{N_m}}\right] \geq 1 - \delta$$

$$\Rightarrow P\left[\mu_m \leq \frac{c_m}{N_m} + \sqrt{\frac{\ln t - \frac{1}{2}\ln \delta + \ln M}{N_m}}\right] \geq 1 - \delta$$

$\because \sqrt{\frac{\ln t - \frac{1}{2}\ln \delta + \frac{1}{2}\ln M}{N_m}} < \sqrt{\frac{\ln t - \frac{1}{2}\ln \delta + \ln M}{N_m}}$

$\square$