9. boolean function $h: \{-1,1\}^k \to \{-1,1\}$

↳ symmetric $\Rightarrow$ value only depends on $\#\{1\}$
if

Let $H = \{h \mid h:$ symmetric boolean functions$\}$

Suppose for any $h \in H$, $S_h = \{\vec{x} \in \{0,1\}^k \mid h(\vec{x}) = 1\}$

(i.e. S contains all input vectors that will be mapped to 1 by hypothesis h)

Let $\vec{v_i} = [\underbrace{1\,1 \cdots 1}_{i \text{ bits}}\underbrace{0 \cdots 0}_{k-i \text{ bits}}]^T$

$S = \{\vec{v_i} \mid 0 \le i \le k\}$ (i.e. the set that contains vectors of all possible number of 1s)

$S' = \{\vec{v}_{p1}, \vec{v}_{p2}, \ldots, \vec{v}_{pd}\} \subseteq S$ (i.e. the set that contains vector $h(\vec{v}_{pi}) = 1$ $i = 1, \ldots, d$, which is S_h eliminating equivalent ones under permutation)

$\Rightarrow S \cap S_h = S'$

$\because |S| = k+1 \quad \therefore |S'| \le k+1$

This means that $\nexists$ a set more than $k+1$ points that can be shattered

To show that the set of $k+1$ point can be shattered, we can construct symmetric Boolean function:

$$h(\vec{x}) = \begin{cases} y_0 & \text{if } \vec{x} \text{ equiv. to } \vec{v_0} \\ y_1 & \text{if } \vec{x} \text{ equiv. to } \vec{v_1} \\ \vdots & \vdots \\ y_k & \text{if } \vec{x} \text{ equiv. to } \vec{v_k} \end{cases} \qquad \text{where } y_i \in \{0,1\} \text{ for } i = 0, \ldots, k$$

Therefore, $d_{vc}(H) = k+1$

$\square$.