8. N labels, $y_1, \dots, y_N \overset{i.i.d.}{\sim} \mu < \infty$

w_0: estimate mean

$$E_{in}[w_0] = \frac{1}{N}\sum_{n=1}^{N}(w_0 - y_n)^2 \qquad w_0^* = \frac{1}{N}\sum_{n=1}^{N} y_n \qquad E_{in}[w_0^*] = \frac{1}{N}\sum_{n=1}^{N}(w_0^* - y_n)^2$$

$$= \frac{1}{N}\sum_{n=1}^{N}\left[(w_0^*)^2 - 2w_0^* y_n + y_n^2\right] \text{ ——— } *$$

<u>Claim:</u>

for $N \geq 2$

$$E_{loocv}(A_{avg}) = \left(\frac{N}{N-1}\right)^2 E_{in}(w_0^*)$$

<u>Prove:</u> For $N \geq 2$, each time we can partition the N examples into $\begin{cases} N-1 : \text{training} \\ 1 : \text{validation} \end{cases}$

Let D_n be the new training set, which is:

$$D_n = \{(\vec{x}_1, y_1), (\vec{x}_2, y_2), \dots, \cancel{(\vec{x}_n, y_n)}, \dots, (\vec{x}_N, y_N)\}$$

Also, let g_n^- be the hypothesis learned from D_n

→ Let error on validation set $\{(\vec{x}_n, y_n)\}$: e_n

where $e_n = E_{val}[g_n^-] = e(g_n^-(x_n), y_n)$

$$w_0^* = \frac{1}{N}\sum_{i=1}^{N} y_i$$

$$\to N w_0^* = \sum_{i=1}^{N} y_i$$

$$\to N w_0^* - y_n = \sum_{i=1}^{N} y_i - y_n = \sum_{\substack{i=1 \\ i \neq N}}^{N} y_i$$

$$\to \boxed{\frac{N w_0^* - y_n}{N-1} = \frac{1}{N-1}\sum_{\substack{i=1 \\ i \neq n}}^{N} y_i}$$

$$E_{loocv}(A_{avg}) = \frac{1}{N}\sum_{n=1}^{N} e_n$$

$$= \frac{1}{N}\sum_{n=1}^{N}\left[\left(\boxed{\frac{1}{N-1}\sum_{\substack{i=1 \\ i \neq n}}^{N} y_i}\right) - y_n\right]^2$$

$$= \frac{1}{N}\sum_{n=1}^{N}\left(\frac{N w_0^* - y_n}{N-1} - y_n\right)^2$$

$$= \frac{1}{N}\sum_{n=1}^{N}\left[\frac{N w_0^* - y_n - y_n(N-1)}{N-1}\right]^2$$

$$= \frac{1}{N(N-1)^2}\sum_{n=1}^{N}\left[N w_0^* - y_n - y_n(N-1)\right]^2$$

$$= \frac{N^2}{N(N-1)^2}\underline{\sum_{n=1}^{N}\left[(w_0^*)^2 - 2w_0^* y_n + \overset{// \, y_n^2}{1}\right]} \; *$$

$$= \frac{N^2}{N(N-1)^2} \underline{N \cdot E_{in}[w_0^*]}$$

$$= \frac{N^2}{(N-1)^2} E_{in}[w_0^*] \qquad \square$$

$$[N w_0^* - y_n - y_n(N-1)]^2$$

$$= [(N w_0^* - y_n) - y_n(N-1)]^2$$

$$= (N w_0^* - y_n)^2 - \underline{2(N w_0^* - y_n) y_n (N-1)} + y_n^2 (N-1)^2$$

$$(2N w_0^* - 2y_n) y_n (N-1)$$
$$= (2N w_0^* y_n - 2y_n^2)(N-1)$$
$$= 2N(N-1) w_0^* y_n - 2(N-1)$$

$$= N^2 (w_0^*)^2 - 2N w_0^* y_n + \underline{1} - 2N(N-1) w_0^* y_n + \underline{2(N-1)} + \underline{(N-1)^2}$$

$$[(N-1)+1]^2 = N^2$$

$$= N^2 (w_0^*)^2 - 2N w_0^* y_n [1 + (N-1)] + N^2$$

$$= N^2 (w_0^*)^2 - 2N^2 w_0^* y_n + N^2$$

$$= N^2\left[(w_0^*)^2 - 2w_0^* y_n + 1\right]$$