8.

We need to show that $K(x, x') = \exp(2\cos(x - x') - 2)$ is a valid kernel.

By the definition of a valid kernel, it should be a symmetric and positive semidefinite function, so it is equivalent to show that $\mathbf{K}$ is a valid (i.e. symmetric and positive semidefinite) Gram matrix, where $\mathbf{K}$ is defined as follows:

$$\mathbf{K} = \begin{bmatrix} K(x_1, x_1) & K(x_1, x_2) & \cdots & K(x_1, x_N) \\ K(x_2, x_1) & K(x_2, x_2) & \cdots & K(x_2, x_N) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_N, x_1) & K(x_N, x_2) & \cdots & K(x_N, x_N) \end{bmatrix}$$

To prove the symmetric property, we need to show that for arbitrary $x_i, x_j, \quad K(x_i, x_j) = K(x_j, x_i)$.

Plug in to the definition, and using the property $\cos(x) = \cos(-x)$, we have:

$$K(x_i, x_j) = \exp(2\cos(x_i - x_j) - 2) = \exp(2\cos(x_j - x_i) - 2) = K(x_j, x_i)$$

Thus, $K(x_i, x_j) = K(x_j, x_i)$ holds.

To prove the positive semidefinite property, we need to show that for arbitrary $\mathbf{z} \in \mathbb{R}^N$, $\mathbf{z}^\top \mathbf{K} \mathbf{z} \geq 0$.

Let $\mathbf{z} = \begin{bmatrix} z_1 & z_2 & \cdots & z_N \end{bmatrix}^\top$, then we have:

$$\mathbf{z}^\top \mathbf{K} \mathbf{z} = \sum_{i=1}^{N} \sum_{j=1}^{N} z_i z_j K(x_i, x_j)$$

Substitute the definition of $K(x_i, x_j)$, we have:

$$\begin{aligned} \mathbf{z}^\top \mathbf{K} \mathbf{z} &= \sum_{i=1}^{N} \sum_{j=1}^{N} z_i z_j \exp(2\cos(x_i - x_j) - 2) \\ &= e^{-2} \sum_{i=1}^{N} \sum_{j=1}^{N} z_i z_j \exp(2\cos(x_i - x_j)) \end{aligned}$$

Since $e^{-2} > 0$, we only need to show that:

$$\sum_{i=1}^{N} \sum_{j=1}^{N} z_i z_j \exp(2\cos(x_i - x_j)) \geq 0$$

This is equivalent to:

$$\sum_{i=1}^{N}\sum_{j=1}^{N} z_i z_j \exp(2(\cos x_i \cos x_j + \sin x_i \sin x_j))$$

If we define the vectors:

$$\mathbf{v}_i = \begin{bmatrix} \cos x_i \\ \sin x_i \end{bmatrix}, \quad \mathbf{v}_j = \begin{bmatrix} \cos x_j \\ \sin x_j \end{bmatrix}$$

Then the above equation can be written as:

$$\sum_{i=1}^{N}\sum_{j=1}^{N} z_i z_j \exp(2\mathbf{v}_i^\top \mathbf{v}_j)$$

We further define the matrix $\mathbf{V}$ with:

$$\mathbf{V}_{ij} = \exp(2\mathbf{v}_i^\top \mathbf{v}_j)$$

We can easily see that $\mathbf{V}$ is a symmetric matrix, and since $\exp(2\mathbf{v}_i^\top \mathbf{v}_j)$ is a positive definite function of the inner product $\mathbf{v}_i^\top \mathbf{v}_j$, $\mathbf{V}$ is a positive (semi)definite matrix.

And we can rewrite the above equation:

$$\mathbf{z}^\top \mathbf{K}\mathbf{z} = e^{-2}\mathbf{z}^\top \mathbf{V}\mathbf{z}$$

Since $\mathbf{V}$ is a positive semidefinite matrix, $e^{-2}\mathbf{z}^\top \mathbf{V}\mathbf{z} \geq 0$ holds for arbitrary $\mathbf{z} \in \mathbb{R}^N$.

Thus, $\mathbf{z}^\top \mathbf{K}\mathbf{z} \geq 0$ holds for arbitrary $\mathbf{z} \in \mathbb{R}^N$, and $\mathbf{K}$ is also a positive semidefinite matrix.

Therefore, we have shown that $K(x, x')$ is a valid kernel.