13. By the definition of $B(N,k)$, it is the max number of dichotomies on $N$ points,

& no subset of size $k$ of the $N$ points can be shattered

by these dichotomies

If we consider the case that we have a set $S$, containing dichotomies of $N$ points

that have number of $\{-1\}$s $\leq k-1$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $\cdots$ | $x_{N-1}$ | $x_N$ | containing $\#\{-1\}$ |
|---|---|---|---|---|---|---|---|---|
| | +1 | +1 | +1 | +1 | | +1 | +1 | 0 |
| | -1 | +1 | +1 | +1 | | +1 | +1 | $1 \to \binom{N}{1}$ dichotomies |
| | +1 | -1 | +1 | +1 | | +1 | +1 | |
| | $\vdots$ | | -1 | | $\ddots$ | $\vdots$ | $\vdots$ | |
| | | | | | | -1 | | |
| | +1 | +1 | | | $\cdots$ | | -1 | |
| | -1 | -1 | | | | | | $2 \to \binom{N}{2}$ dichotomies |
| | -1 | | -1 | | | | | |
| | $\vdots$ | | | | | | | $\vdots$ |

Similarly, it is easy to see that we will have $\sum_{i=0}^{k-1} \binom{N}{i}$ dichotomies in $S$

Suppose $\exists$ subset of size $k$ of the $N$ points:

$$X' = \underbrace{\{\vec{x}_j, \vec{x}_{j+1}, \dots, \vec{x}_{j+k-1}\}}_{\text{size } k} \subseteq \{\vec{x}_1, \dots, \vec{x}_N\}$$

$\nexists$ $X'$ can be shattered by these $\sum_{i=0}^{N} \binom{N}{i}$ dichotomies,

$\rightarrow$ then $[\vec{x}_j, \vec{x}_{j+1}, \dots, \vec{x}_{j+k-1}] = [\underbrace{-1, -1, \cdots -1}_{k\{-1\}s}] \in S$ $\cancel{\times}$

$\because$ By our assumption, $S$ has dichotomies with number of $\{-1\}s$ $\leq k-1$

Thus, we obtain the result:

$$B(n,k) \geq \sum_{i=0}^{k-1} \binom{N}{i} \qquad \square$$