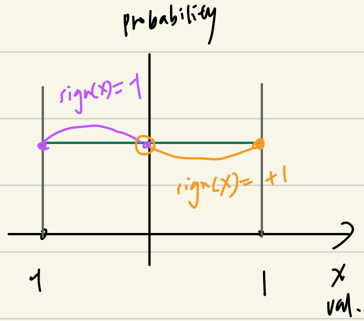
10.



decision stump model: $h_{s,\theta}(x) = s \cdot \text{sign}(x-\theta)$

Let $\text{sign}(0) = -1$ for simplicity

Growth function $m_H(N) = 2N$, VC dimension $= 2$

For $x^t \in [-1, 0]$ probability

$$y^t = \begin{cases} -1 & 1-p \\ +1 & p \text{ (flip)} \end{cases}$$

where $0 \le p < \frac{1}{2}$

For $x^t \in (0, 1]$ probability

$$y^t = \begin{cases} +1 & 1-p \\ -1 & p \text{ (flip)} \end{cases}$$

where $0 \le p < \frac{1}{2}$

By definition,

$$E_{out}(h_{s,\theta}) = P(h_{s,\theta}(x) \ne y)$$

$$= P(h_{s,\theta}(x) = y \cap \text{flipped}) + P(h_{s,\theta}(x) \ne y \cap \text{not flipped})$$

$$= p \cdot P(h_{s,\theta}(x) = y) + (1-p) P(h_{s,\theta}(x) \ne y)$$

$$= p \cdot [1 - P(h_{s,\theta}(x) \ne y)] + (1-p) P(h_{s,\theta}(x) \ne y)$$

$$= p - p \cdot \boxed{P(h_{s,\theta}(x) \ne y)} + \boxed{P(h_{s,\theta}(x) \ne y)} - p \cdot \boxed{P(h_{s,\theta}(x) \ne y)}$$

$$= p + (1-2p) \cdot P(h_{s,\theta}(x) \ne y)$$

$P(h_{s,\theta}(x) \neq y)$

$= \boxed{P(h_{s,\theta}(x) \neq y \cap s=-1)} + \boxed{p(h_{s,\theta}(x) \neq y \cap s=+1)}$

$= ①\ P(h_{s,\theta}(x) \neq y \mid s=-1, x > \theta) + ②\ P(h_{s,\theta}(x) \neq y \mid s=1, x \leq \theta)$

$③\ P(h_{s,\theta}(x) \neq y \mid s=1, x > \theta) + ④\ P(h_{s,\theta}(x) \neq y \mid s=1, x \leq \theta)$

① Prediction made by $h_{s,\theta}(x) = -1$

$s=-1$, $x > \theta$

For $x > 0$, $y = +1$ $\therefore h_{s,\theta}(x) \neq y$

For $x \leq 0$, $y = -1$ $\therefore h_{s,\theta}(x) = y$

probability

$\because x > 0,\ x > \theta$

$\therefore x \in (\theta, 1]$

$1 - \frac{|\theta|}{2}$

$(s=-1)$

② Prediction made by $h_{s,\theta}(x) = +1$

$s=1$, $x \leq \theta$

For $x > 0$, $y = 1$ $\therefore h_{s,\theta}(x) = y$

For $x \leq 0$, $y = 1$ $\therefore h_{s,\theta}(x) \neq y$

$\because x \leq 0,\ x \leq \theta$

$\therefore x \in [-1, \theta]$

③ Prediction made by $h_{s,\theta}(x) = +1$

$s=1$, $x > \theta$

For $x > 0$, $y = 1$ $\therefore h_{s,\theta}(x) = y$

For $x \leq 0$, $y = -1$ $\therefore h_{s,\theta}(x) \neq y$

$\because x \leq 0,\ x > \theta$

$\therefore x \in (\theta, 0]$

$\frac{|\theta|}{2}$

$(s=1)$

④ Prediction made by $h_{s,\theta}(x) = -1$

$s=1$, $x \leq \theta$

For $x > 0$, $y = 1$ $\therefore h_{s,\theta}(x) \neq y$

For $x \leq 0$, $y = -1$ $\therefore h_{s,\theta}(x) = y$

$\because x > 0,\ x \leq \theta$

$\therefore x \in (0, \theta]$

$\therefore$ Continue the process on the previous 2 page, we get:

$$E_{out}(h_{s,\theta}) = p + (1-2p)\cdot P(h_{s,\theta}(x) \neq y)$$

$$\Rightarrow \begin{cases} E_{out}(h_{-1,\theta}) = p + (1-2p)\cdot \underline{P(h_{-1,\theta}(x) \neq y)} = p + (1-2p)\cdot 1 - \frac{|\theta|}{2} \\ E_{out}(h_{1,\theta}) = p + (1-2p)\cdot \underline{P(h_{1,\theta}(x) \neq y)} = p + (1-2p)\cdot \frac{|\theta|}{2} \end{cases}$$

If $s=+1$, then $h_{s,\theta} = \text{sign}(x-\theta)$

$s=-1$, then $h_{s,\theta} = -\text{sign}(x-\theta)$

Thus we can modify $h_{s,\theta}$ to consider the 2 cons:

$$\boxed{h_{s,\theta} = \frac{1+s}{2}\,\text{sign}(x-\theta) - \frac{1-s}{2}\,\text{sign}(x-\theta)}$$

$\Rightarrow$ $h_{s,\theta} = \frac{1+1}{2}\,\text{sign}(x-\theta) - \frac{1-1}{2}\,\text{sign}(x-\theta) = \text{sign}(x-\theta)$ if $s=+1$

$h_{s,\theta} = \frac{1+(-1)}{2}\,\text{sign}(x-\theta) - \frac{1-(-1)}{2}\,\text{sign}(x-\theta) = -\text{sign}(x-\theta)$ if $s=-1$

Or we can write equivalently:

$$\boxed{h_{s,\theta} = \underbrace{\frac{1+s}{2}\,\text{sign}(x-\theta)}_{s=1} + \frac{1-s}{2}\underbrace{\left[-\text{sign}(x-\theta)\right]}_{s=-1}}$$

$$
\begin{aligned}
\therefore\ E_{out}(h_{s,\theta}) &= \frac{1-s}{2}\left[p+(1-2p)\left(1-\frac{|\theta|}{2}\right)\right] + \frac{1+s}{2}\left[p+(1-2p)\cdot\frac{|\theta|}{2}\right] \\
&= \frac{p}{2}\left[(1-s)+(1+s)\right] + \frac{1-2p}{2}\left[(1-s)\left(1-\frac{|\theta|}{2}\right)+(1+s)\frac{|\theta|}{2}\right] \\
&= \frac{p}{2}\cdot 2 + \left(\frac{1}{2}-p\right)\left[1-s-\cancel{\frac{|\theta|}{2}}+\frac{s|\theta|}{2}+\cancel{\frac{|\theta|}{2}}+\frac{s|\theta|}{2}\right] \\
&= p + \left(\frac{1}{2}-p\right)\left[1-s+s|\theta|\right] \\
&= \cancel{p} + \frac{1}{2} - \frac{1}{2}s + \frac{1}{2}s|\theta| \cancel{-p} + sp - sp|\theta| \\
&= \underbrace{\frac{1}{2} - s\left(\frac{1}{2}-p\right)}_{=\,u\,=\,\frac{1}{2}-v} + \underbrace{s\left(\frac{1}{2}-p\right)}_{=\,v}|\theta| \qquad \square
\end{aligned}
$$