8. Linear regression on dataset: $\{(\vec{x}_n, y_n)\}_{n=1}^{N}$ $\vec{x}_n \in \mathbb{R}^{d+1}$ with $x_{n0} = 1$ $\forall n \in \{1, \dots, N\}$

Assume: X^TX: invertible

unique sol $\vec{w}_{lin}$ acquired on $\{(\vec{x}_n, y_n)\}_{n=1}^{N}$

+ change $x_{n0} = 1126$ $\forall n \in \{1, \dots, N\}$

Prove: $\vec{w}_{LIN} = D\vec{w}_{lucky}$ where D: diagonal matrix

---

$\because X^TX$: invertible $\therefore$ $\vec{w}_{LIN} = \overbrace{(X^TX)^{-1}X^T}^{X^\dagger} y$

Write:

$$X' = \underbrace{\begin{bmatrix} 1126 & x_{11} & x_{12} & \cdots & x_{1d} \\ 1126 & x_{21} & x_{22} & \cdots & x_{2d} \\ \vdots & \vdots & & \ddots & \\ 1126 & x_{N1} & x_{N2} & \cdots & x_{Nd} \end{bmatrix}}_{N \times (d+1)} = \underbrace{\begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1d} \\ 1 & x_{21} & x_{22} & \cdots & x_{2d} \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & x_{N1} & x_{N2} & \cdots & x_{Nd} \end{bmatrix}}_{N \times (d+1)} \underbrace{\begin{bmatrix} 1126 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{bmatrix}}_{\substack{= D' \ \text{det} \\ (d+1)\times(d+1)}} = XD'$$

$\rightarrow X'^TX' = (XD')^T(XD') = D'^TX^TXD' = D'X^TXD'$

$\because D'$: diagonal $\therefore D'^T = D'$

$\because \det(D') = 1126$

and $\exists (X^TX)^{-1} \therefore \det(X^TX) \neq 0$

$\downarrow$

$\det(X'^TX') = \det(D'X^TXD') = \det(D')\det(X^TX)\det(D') = \det(D')^2\det(X^TX) \neq 0$

$\therefore X'^TX'$: invertible

Thus, $\vec{w}_{lucky} = (X'^T X')^{-1} X'^T \vec{y} = (\underbrace{D'}_{A}\underbrace{X^T X}_{B}\underbrace{D'}_{C})^{-1} (X D')^T \vec{y} = D'^{-1}(X^T X)^{-1} \underline{D'^{-1} D'^T} X^T \vec{y}$

$\because D^T = D$

$\therefore D^{-1} D^T = D^{-1} D = I$

$\because \det(D') \neq 0 \quad \therefore D'$: invertible

$(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$

$$= D'^{-1} \underbrace{(X^T X)^{-1} X^T \vec{y}}_{= \vec{w}_{LIN}}$$

Therefore, $\vec{w}_{LIN} = D' \vec{w}_{lucky}$

$$\rightarrow D = D' = \begin{bmatrix} 1126 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \quad \square$$