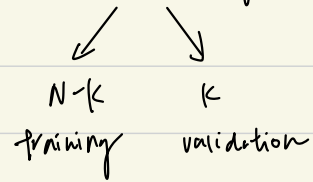


1.  $N$  labels:  $y_1, \dots, y_N \sim \text{i.i.d. } \mathcal{M}(0, \sigma^2)$



Expected validation error:  $E\left(\frac{1}{k} \sum_{n=N-k+1}^N (y_n - 0)^2\right) = \sigma^2 \Rightarrow \frac{1}{k} E\left[\sum_{n=N-k+1}^N y_n^2\right] = \sigma^2 \Rightarrow E\left[\sum_{n=N-k+1}^N y_n^2\right] = k\sigma^2$

$\uparrow$   
 $E[(X - E[X])^2] = \sigma^2$

$N-k$  samples estimate mean:  $\bar{y} = \frac{1}{N-k} \sum_{n=1}^{N-k} y_n$

Q: expected validation error:  $E\left(\frac{1}{k} \sum_{n=N-k+1}^N (y_n - \bar{y})^2\right)$

expansion

$$\begin{aligned}
 E\left(\frac{1}{k} \sum_{n=N-k+1}^N (y_n - \bar{y})^2\right) &= \frac{1}{k} E\left[\sum_{n=N-k+1}^N (y_n^2 - 2y_n\bar{y} + \bar{y}^2)\right] \quad \text{linearity of expectation} \\
 &= \frac{1}{k} \left( E\left[\sum_{n=N-k+1}^N y_n^2\right] - 2E\left[\sum_{n=N-k+1}^N y_n\bar{y}\right] + E\left[\sum_{n=N-k+1}^N \bar{y}^2\right] \right) \quad \text{linearity of expectation} \\
 &= \frac{1}{k} \left( k\sigma^2 - 2 \sum_{n=N-k+1}^N E[y_n\bar{y}] + E\left[\sum_{n=N-k+1}^N \bar{y}^2\right] \right) \\
 &= \frac{1}{k} \left( k\sigma^2 - 2 \sum_{n=N-k+1}^N \cancel{E[y_n]E[\bar{y}]} + E\left[\sum_{n=N-k+1}^N \bar{y}^2\right] \right) \quad \begin{array}{l} \text{if } X, Y: \text{independent} \\ \text{then } E[XY] = E[X]E[Y] \end{array} \\
 &\quad \begin{array}{l} \because E[y_n] = 0 \\ \text{(labels generated from distribution with mean } = 0) \end{array} \\
 &\quad \begin{array}{l} \because \text{values of } y_n \text{ range from } N-k+1 \text{ to } N \\ \text{and } \bar{y} = \frac{1}{N-k} \sum_{n=1}^{N-k} y_n \text{ depends on } y_n \\ \text{that range from } 1 \text{ to } N-k \end{array} \\
 &\quad \therefore y_n, \bar{y}: \text{independent} \\
 &\quad \text{Thus, } E[y_n\bar{y}] = E[y_n]E[\bar{y}] \\
 &= \frac{1}{k} \left( k\sigma^2 + E\left[\sum_{n=N-k+1}^N \bar{y}^2\right] \right)
 \end{aligned}$$

$$\begin{aligned}
 E[\bar{y}] &= E\left[\frac{1}{N-k} \sum_{n=1}^{N-k} y_n\right] = \frac{1}{N-k} \sum_{n=1}^{N-k} E[y_n] = 0 \quad \because E[y_n] = 0 \\
 E[\bar{y}^2] &= E\left[\sum_{n=N-k+1}^N (\bar{y} - E[\bar{y}])^2\right] \quad \because E[\bar{y}] = 0 \\
 &= \sum_{n=N-k+1}^N E[(\bar{y} - E[\bar{y}])^2] \\
 &= k \cdot \text{Var}(\bar{y}) \quad \begin{array}{l} \text{def of variance,} \\ \text{expected value: constant} \Rightarrow \sum_{n=N-k+1}^N c = kc \end{array} \\
 &= k \cdot \text{Var}\left(\frac{1}{N-k} \sum_{n=1}^{N-k} y_n\right) \\
 &= k \cdot \left(\frac{1}{N-k}\right)^2 \text{Var}\left(\sum_{n=1}^{N-k} y_n\right) \quad \text{Binomaymé's identity} \\
 &= k \cdot \left(\frac{1}{N-k}\right)^2 \left[ \sum_{n=1}^{N-k} \text{Var}(y_n) + \sum_{i \neq j} \text{Cov}(y_i, y_j) \right]
 \end{aligned}$$

$$\begin{aligned}
 &= K \cdot \left( \frac{1}{N-K} \right)^2 \sum_{n=1}^{N-K} \text{Var}(y_n) \quad \swarrow \text{ ' Cov}(y_i, y_j) = 0 \quad \forall i, j \in \{1, \dots, N-K\} \\
 &\quad \text{since } y_1, \dots, y_{N-K} \text{ drawn independently} \\
 &= K \cdot \left( \frac{1}{N-K} \right)^2 \sum_{n=1}^{N-K} \sigma^2 \quad \swarrow \text{ given by the problem} \\
 &= K \cdot \left( \frac{1}{N-K} \right)^2 (N-K) \sigma^2 \\
 &= \frac{K \sigma^2}{N-K}
 \end{aligned}$$

↓

$$\begin{aligned}
 &= \frac{1}{K} \left( K \sigma^2 + \frac{K \sigma^2}{N-K} \right) \\
 &= \sigma^2 + \frac{\sigma^2}{N-K} \\
 &= \sigma^2 \left( 1 + \frac{1}{N-K} \right) \quad \square
 \end{aligned}$$