7. N labels: $y_1, \dots, y_N \overset{i.i.d.}{\sim} \mu = 0, \sigma^2$

$N-K$ training　　　K validation

<u>expected validation error</u>: $E\left(\frac{1}{K}\underbrace{\sum_{n=N-K+1}^{N}(y_n - 0)^2}\right) = \sigma^2 \Rightarrow \frac{1}{K}E\left[\sum_{n=N-K+1}^{N} y_n^2\right] = \sigma^2 \Rightarrow E\left[\sum_{n=N-K+1}^{N} y_n^2\right] = K\sigma^2$

$E[(X - E[X])^2] = \sigma^2$

<u>N-K examples estimate mean</u>: $\bar{y} = \frac{1}{N-K}\sum_{n=1}^{N-K} y_n$

Q: <u>expected validation error</u>: $E\left(\frac{1}{K}\sum_{n=N-K+1}^{N}(y_n - \bar{y})^2\right)$

expansion

$$E\left(\frac{1}{K}\sum_{n=N-K+1}^{N}(y_n - \bar{y})^2\right) = \frac{1}{K}E\left[\sum_{n=N-K+1}^{N}(y_n^2 - 2y_n\bar{y} + \bar{y}^2)\right]$$

linearity of expectation

$$= \frac{1}{K}\left(E\left[\sum_{n=N-K+1}^{N} y_n^2\right] - 2E\left[\sum_{n=N-K+1}^{N} y_n\bar{y}\right] + E\left[\sum_{n=N-K+1}^{N}\bar{y}^2\right]\right)$$

linearity of expectation

$$= \frac{1}{K}\left(K\sigma^2 - 2\sum_{n=N-K+1}^{N} E[y_n\bar{y}] + E\left[\sum_{n=N-K+1}^{N}\bar{y}^2\right]\right)$$

$\boxed{\text{if } X, Y: \text{independent then } E[XY] = E[X]E[Y]}$

$$= \frac{1}{K}\left(K\sigma^2 - \cancel{2\sum_{n=N-K+1}^{N} E[y_n]E[\bar{y}]}^{\,0} + E\left[\sum_{n=N-K+1}^{N}\bar{y}^2\right]\right)$$

$\because E[y_n] = 0$

(labels generated from distribution with mean = 0)

$\because$ values of y_n range from $N-K+1$ to N

and $\bar{y} = \frac{1}{N-K}\sum_{n=1}^{N-K} y_n$ depends on y_n

that range from 1 to $N-K$

$\therefore y_n, \bar{y}$: independent

Thus, $E[y_n\bar{y}] = E[y_n]E[\bar{y}]$

$$= \frac{1}{K}\left(K\sigma^2 + E\left[\sum_{n=N-K+1}^{N}\bar{y}^2\right]\right)$$

$E[\bar{y}] = E\left[\frac{1}{N-K}\sum_{n=1}^{N-K} y_n\right] = \frac{1}{N-K}\sum_{n=1}^{N-K} E[y_n] = 0$　$\because E[y_n] = 0$

$E[\bar{y}] = 0$

$$E\left[\sum_{n=N-K+1}^{N}\bar{y}^2\right] = E\left[\sum_{n=N-K+1}^{N}(\bar{y} - E[\bar{y}])^2\right]$$

$$= \sum_{n=N-K+1}^{N} E[(\bar{y} - E[\bar{y}])^2]$$

$$= K \cdot \mathrm{Var}(\bar{y})$$

def of variance, expected value: constant $\Rightarrow \sum_{n=N-K+1}^{N} c = Kc$

$$= K \cdot \mathrm{Var}\left(\frac{1}{N-K}\sum_{n=1}^{N-K} y_n\right)$$

$$= K \cdot \left(\frac{1}{N-K}\right)^2 \mathrm{Var}\left(\sum_{n=1}^{N-K} y_n\right)$$

Bienaymé's identity

$$= K \cdot \left(\frac{1}{N-K}\right)^2\left[\sum_{n=1}^{N-K}\mathrm{Var}(y_n) + \sum_{i}\sum_{j \neq i}\mathrm{Cov}(y_i, y_j)\right]$$

$i, j \in \{1, \dots, N-K\}$

$$= K \cdot \left(\frac{1}{N-K}\right)^2 \sum_{n=1}^{N-K} Var(y_n)$$ $\because Cov(y_i, y_j) = 0 \quad \forall i,j \in \{1, \dots, N-K\}$ since $y_1, \dots, y_N$ drawn independently

$$= K \cdot \left(\frac{1}{N-K}\right)^2 \sum_{n=1}^{N-K} \sigma^2$$ given by the problem

$$= K \cdot \left(\frac{1}{N-K}\right)^2 (N-K)\sigma^2$$

$$= \frac{K\sigma^2}{N-K}$$

$$= \frac{1}{K}\left(K\sigma^2 + \frac{K\sigma^2}{N-K}\right)$$

$$= \sigma^2 + \frac{\sigma^2}{N-K}$$

$$= \sigma^2\left(1 + \frac{1}{N-K}\right) \quad \square$$