8.

Let $\vec{x}_n^{orig} = [x_{n1}\ x_{n2}\ \cdots\ x_{nd}] \in \mathbb{R}^d$

$$\Rightarrow \begin{cases} \vec{x}_n = [1\ x_{n1}\ x_{n2}\ \cdots\ x_{nd}] \in \mathbb{R}^{d+1} \\ \vec{x}_n' = [3\ \ 3x_{n1}\ 3x_{n2} \cdots 3x_{nd}] \in \mathbb{R}^{d+1} \end{cases}$$

For arbitrary $(\vec{x}_n, y_n) \in \{(\vec{x}_n, y_n)\}_{n=1}^{N}$, and assume $\text{sign}(0) = 1$

if $\quad \text{sign}(\vec{w}_0^T \vec{x}_p) \neq y_p$, which means $\vec{x}_p$ is the first mispredicted input

then $y_p = 1$

and

$$\vec{w}_1 = \vec{w}_0 + y_n \vec{x}_p = \vec{x}_p$$

If we further assume the second misprediction occurs at $(\vec{x}_q, y_q)$, then

$$\text{sign}(\vec{w}_1^T \vec{x}_q) \neq y_q$$

$$\Rightarrow \text{sign}(\vec{x}_p^T \vec{x}_q) \neq y_q$$

$$\Rightarrow \text{sign}\left([1\ \ x_{p1}\ \ x_{p2} \cdots\ x_{pd}] \begin{bmatrix} 1 \\ x_{q1} \\ x_{q2} \\ \vdots \\ x_{qd} \end{bmatrix} \right) \neq y_q$$

$$\Rightarrow \text{sign}\left(1 + \sum_{i=1}^{d} x_{pi} x_{qi}\right) \neq y_q$$

$\because$ $\vec{x}_0 \sim \vec{x}_{p-1}$ is predicted correctly by $\vec{w}_0$

$\therefore$ $\text{sign}(\vec{w}_0^T \vec{x}_i) = y_i$ for $i = 1 \sim p-1$

$\Rightarrow$ $\text{sign}(\vec{w}_0^T \vec{x}_i) = y_i$

$\Rightarrow$ $\text{sign}\left([0\ 0\ \cdots\ 0] \begin{bmatrix} 1 \\ x_{i1} \\ x_{i2} \\ \vdots \\ x_{id} \end{bmatrix} \right) = y_i$

$\Rightarrow$ $\text{sign}(0) = -1 = y_i$ for $i = 1 \sim p-1$

$\because$ $\text{sign}(\vec{w}_0^T \vec{x}_p) \neq y_p$ as we assumed $\therefore$ $y_p = 1$

Thus, $\vec{w}_1 = \vec{w}_0 + y_p \vec{x}_p = \vec{0} + 1 \cdot \begin{bmatrix} 1 \\ x_{p1} \\ x_{p2} \\ \vdots \\ x_{pd} \end{bmatrix} = \begin{bmatrix} 1 \\ x_{p1} \\ x_{p2} \\ \vdots \\ x_{pd} \end{bmatrix}$

Consider the same situation but replace all $\vec{x}_n$ to $\vec{x}_n'$:

For $\vec{x}_i'$ $i = 1 \sim p-1$, since $\text{sign}(\vec{w}_0'^T \vec{x}_i') = -1$ and y_i remain unchanged

$\therefore$ $\vec{x}_1' \sim \vec{x}_{p-1}'$ is predicted correctly

For $\vec{x}_p'$, we have also have $\text{sign}(\vec{w}_0'^T \vec{x}_p') = -1 \neq y_p = 1$

Therefore,

$$\vec{w}_1' = \vec{w}_0' + y_p \vec{x}_p' = \vec{0} + 1 \cdot \begin{bmatrix} 3 \\ 3x_{p1} \\ 3x_{p2} \\ \vdots \\ 3x_{pd} \end{bmatrix} = \begin{bmatrix} 3 \\ 3x_{p1} \\ 3x_{p2} \\ \vdots \\ 3x_{pd} \end{bmatrix}$$

$\because$ $\vec{x}_{p+1} \sim \vec{x}_{q-1}$ is predicted correctly

$\therefore$ $\text{sign}(\vec{x}_p^T \vec{x}_i) = y_i$ for $i = p+1 \sim q-1$

We want to know if $\vec{x}\,'_{p+1} \sim \vec{x}\,'_{q-1}$ will be predicted correctly or not

as $\vec{w}_1\,' = [3 \quad 3x_{p1} \cdots 3x_{pd}]^T$

$$\Rightarrow \text{sign}(\vec{w}_1\,'^T \vec{x}\,'_{p+1}) = \text{sign}\left([3 \quad 3x_{p1} \cdots 3x_{pd}] \begin{bmatrix} 3 \\ 3x_{p+1,1} \\ 3x_{p+1,2} \\ \vdots \\ 3x_{p+1,d} \end{bmatrix} \right)$$

$$= \text{sign}(9(1 + x_{p1}x_{p+1,1} + \cdots + x_{pd}x_{p+1,d}))$$

$$= \text{sign}(1 + x_{p1}x_{p+1,1} + \cdots + x_{pd}x_{p+1,d})$$

$$= \text{sign}(\vec{x}_p^T \vec{x}_{p+1})$$

$$= \text{sign}(\vec{w}_1^T \vec{x}_{p+1})$$

Similarly, we know that:

$$\text{sign}(\vec{w}_1\,'^T \vec{x}_i\,') = \text{sign}(\vec{w}_1^T \vec{x}_i) \quad \text{for } i = p+1 \sim q-1$$

Thus, $\vec{x}_i\,'$ is predicted correctly by $\vec{w}_1\,'$ for $i = p+1 \sim q-1$

Next we want to check if $\text{sign}(\vec{w}_1\,'^T \vec{x}_q\,') = y_q$ or not:

$$\text{sign}(\vec{w}_1\,'^T \vec{x}_q\,') = \text{sign}\left([3 \quad 3x_{p1} \cdots 3x_{pd}] \begin{bmatrix} 3 \\ 3x_{q1} \\ 3x_{q2} \\ \vdots \\ 3x_{qd} \end{bmatrix} \right)$$

$$= \text{sign}(9(1 + x_{p1}x_{q1} + \cdots + x_{pd}x_{qd}))$$

$$= \text{sign}\left(9\left(1 + \sum_{i=1}^{d} x_{pi}x_{qi}\right)\right)$$

$$= \text{sign}\left(1 + \sum_{i=1}^{d} x_{pi}x_{qi}\right)$$

$$= \text{sign}(\vec{x_p}^T \vec{x_q})$$

$$= \text{sign}(\vec{w_1}^T \vec{x_q})$$

Thus, if $\text{sign}(\vec{w_1}^T \vec{x_q}) \neq y_q$, then $\text{sign}(\vec{w_1}'^T \vec{x_q}') \neq y_q$

Therefore, we can know that $\vec{w}_{PLA} = \vec{w}_{PLA}'$.