8. target function: $f(x) = 1-2x^2$

sample x uniformly from $[0,1]$ $\rightarrow$ $h(x) = w_0 + w_1 x$, squared error

training set $D = \{(x_1, f(x_1)), (x_2, f(x_2))\}$ x_1, x_2 uniformly sampled from $[0,1]$

$$\Rightarrow E_{in}(g) = \sum_{i=1}^{2} (h(x_i) - f(x_i))^2$$

$$= \sum_{i=1}^{2} \left[w_0 + w_1 x_i - (1-2x_i^2)\right]^2$$

$$= \sum_{i=1}^{2} \left[w_0 + w_1 x_i - 1 + 2x_i^2\right]^2$$

$\because$ we're fitting 2 points to a line

$\therefore$ can find such line that can fit both 2 points $\forall$ possible D.

$\rightarrow$ Therefore, $E_{in}(g) = 0$

$$\begin{cases} \frac{\partial}{\partial w_0} E_{in}(g) = \sum_{i=1}^{2} 2(w_0 + w_1 x_i - 1 + 2x_i^2)\cdot 1 = 0 & \text{— ①} \\ \frac{\partial}{\partial w_1} E_{in}(g) = \sum_{i=1}^{2} 2(w_0 + w_1 x_i - 1 + 2x_i^2)\, x_i = 0 & \text{— ②} \end{cases}$$

$$\Rightarrow \begin{cases} ① \; (w_0 + w_1 x_1 - 1 + 2x_1^2) + (w_0 + w_1 x_2 - 1 + 2x_2^2) = 0 \\ ② \; (w_0 + w_1 x_1 - 1 + 2x_1^2)\, x_1 + (w_0 + w_1 x_2 - 1 + 2x_2^2)\, x_2 = 0 \end{cases}$$

$① \times x_1$: $(w_0 + w_1 x_1 - 1 + 2x_1^2)\, x_1 + (w_0 + w_1 x_2 - 1 + 2x_2^2)\, x_1 = 0$

$\rightarrow ① \times x_1 - ②$: $(w_0 + w_1 x_2 - 1 + 2x_2^2)(x_1 - x_2) = 0$

$① \times x_2$: $(w_0 + w_1 x_1 - 1 + 2x_1^2)\, x_2 + (w_0 + w_1 x_2 - 1 + 2x_2^2)\, x_2 = 0$

$\rightarrow ② - ① \times x_2$: $(w_0 + w_1 x_1 - 1 + 2x_1^2)(x_1 - x_2) = 0$

If we ignore degenerate cases that $x_1 = x_2$, then

$$\begin{cases} w_0 + w_1 x_2 - 1 + 2x_2^2 = 0 \\ w_0 + w_1 x_1 - 1 + 2x_1^2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} w_0 = -w_1 x_2 + 1 - 2x_2^2 \\ -w_1 x_2 + \cancel{1} - 2x_2^2 + w_1 x_1 - \cancel{1} + 2x_1^2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} w_0 = -w_1 x_2 + 1 - 2x_2^2 \\ w_1 (x_1 - x_2) + 2(x_1^2 - x_2^2) = 0 \end{cases}$$

$\rightarrow$ $w_1(x_1 - x_2) + 2(x_1 - x_2)(x_1 + x_2) = 0$

$[w_1 + 2(x_1 + x_2)](x_1 - x_2) = 0$

$\Downarrow$

$w_1 + 2(x_1 + x_2) = 0 \Rightarrow$ $w_1 = -2(x_1 + x_2)$

$w_0 = 2(x_1 + x_2)x_2 + 1 - 2x_2^2$

$= 2x_1 x_2 + \cancel{2x_2^2} + 1 - \cancel{2x_2^2}$

$$\Rightarrow \begin{cases} w_0 = 2x_1 x_2 + 1 \\ w_1 = -2(x_1 + x_2) \end{cases}$$

Thus, $g(x) = 2x_1 x_2 + 1 - 2(x_1 + x_2)\, x$

$$E_D[E_{out}(g)]$$

$$= E_D\left[E_x\left[(f(x) - g(x))^2\right]\right]$$

$$= E_D\left[E_x\left[\left(\cancel{1} - 2x^2 - 2x_1x_2 - \cancel{1} + 2(x_1+x_2)x\right)^2\right]\right]$$

$$= E_D\left[\int_0^1 (2x_1x_2 - 2(x_1+x_2)x + 2x^2)^2 \, dx\right]$$

$$= E_D\Big[\frac{4}{5} + \frac{4}{3}x_1x_2 - \frac{4}{3}x_1 - \frac{4}{3}x_2$$
$$+ \frac{4}{3}x_1x_2 + 4x_1^2x_2^2 - 2x_1^2x_2 - 2x_1x_2^2$$
$$- x_1 - 2x_1^2x_2 + \frac{4}{3}x_1^2 + \frac{4}{3}x_1x_2$$
$$- x_2 - 2x_1x_2^2 + \frac{4}{3}x_1x_2 + \frac{4}{3}x_2^2\Big]$$

$$(-2x^2 - 2x_1x_2 + 2x_1x + 2x_2x)^2$$
$$= 4x^4 + 4x_1x_2x^2 - 4x_1x^2 - 4x_2x^2$$
$$+ 4x_1x_2x^2 + 4x_1^2x_2^2 - 4x_1^2x_2x - 4x_1x_2^2x$$
$$- 4x_1x^3 - 4x_1^2x_2x + 4x_1^2x^2 + 4x_1x_2x^2$$
$$- 4x_2x^3 - 4x_1x_2^2x + 4x_1x_2x^2 + 4x_2^2x^2$$

$$= \int_0^1\int_0^1 \Big[\frac{4}{5} + \frac{4}{3}x_1x_2 - \frac{4}{3}x_1 - \frac{4}{3}x_2$$
$$+ \frac{4}{3}x_1x_2 + 4x_1^2x_2^2 - 2x_1^2x_2 - 2x_1x_2^2$$
$$- x_1 - 2x_1^2x_2 + \frac{4}{3}x_1^2 + \frac{4}{3}x_1x_2$$
$$- x_2 - 2x_1x_2^2 + \frac{4}{3}x_1x_2 + \frac{4}{3}x_2^2\Big] dx_1\, dx_2$$

$$= \int_0^1 \Big[\frac{4}{5} + \frac{4}{3}x_2 - \frac{2}{3} - \frac{4}{3}x_2 + \frac{2}{3}x_2 + \frac{4}{3}x_2^2 - \frac{2}{3}x_2 - x_2^2 - \frac{1}{2} - \frac{2}{3}x_2 + \frac{4}{9} + \frac{2}{3}x_2$$
$$- x_2 - x_2^2 + \frac{2}{3}x_2 + \frac{4}{3}x_2^2\Big] dx_2$$

$$= \frac{4}{5} + \cancel{\frac{2}{3}} - \cancel{\frac{2}{3}} - \frac{2}{3} + \cancel{\frac{1}{3}} + \frac{4}{9} - \cancel{\frac{1}{3}} - \cancel{\frac{1}{3}} - \frac{1}{2} - \cancel{\frac{1}{3}} + \frac{4}{9} + \cancel{\frac{1}{3}} - \frac{1}{2} - \frac{1}{3} + \cancel{\frac{1}{3}} + \frac{4}{9}$$

$$= \frac{4}{5} - \frac{2}{3} + \frac{12}{9} - 1 - \frac{1}{3} \qquad -2 \qquad \frac{4}{3} + \frac{4}{5} = \frac{20+12}{15} = \frac{32}{15}$$

$$= \frac{32}{15} - 2$$

$$= \frac{2}{15} \qquad \square$$