6. L2 regularization, twice differentiable, convex $E_{in}(\vec{w})$

$$\min_{\vec{w}\in\mathbb{R}^{d+1}} E_{aug}(\vec{w})$$

$$E_{aug}(\vec{w}) = E_{in}(\vec{w}) + \frac{\lambda}{N}\|\vec{w}\|_2^2 \qquad \nabla E_{in}(\vec{w}^*) = 0$$

$$\tilde{E}_{in}(\vec{w}) = E_{in}(\vec{w}^*) + \underbrace{(\vec{w}-\vec{w}^*)^T \nabla E_{in}(\vec{w}^*)}_{0} + \frac{1}{2}(\vec{w}-\vec{w}^*)^T H (\vec{w}-\vec{w}^*)$$

$H \in \mathbb{R}^{(d+1)\times(d+1)}$: Hessian, positive semi-definite ($\because$ convex)

approx $E_{aug}(\vec{w})$: $\tilde{E}_{aug}(\vec{w}) = \tilde{E}_{in}(\vec{w}) + \frac{\lambda}{N}\|\vec{w}\|^2$

$$E_{aug}(\vec{w}) = E_{in}(\vec{w}^*) + \cancelto{0}{(\vec{w}-\vec{w}^*)^T \nabla E_{in}(\vec{w}^*)} + \frac{1}{2}(\vec{w}-\vec{w}^*)^T H (\vec{w}-\vec{w}^*) + \frac{\lambda}{N}\vec{w}^T\vec{w}$$

$$\frac{d}{d\vec{w}}\vec{w}^T A \vec{w} = (A^T + A)\vec{w}$$

$$\frac{\partial}{\partial\vec{w}} E_{aug}(\vec{w}) = \frac{\partial}{\partial\vec{w}} E_{in}(\vec{w}^*) + \frac{1}{2}\frac{\partial}{\partial\vec{w}}(\vec{w}-\vec{w}^*)^T H (\vec{w}-\vec{w}^*) + \frac{\partial}{\partial\vec{w}}\left(\frac{\lambda}{N}\vec{w}^T\vec{w}\right)$$

$$= 0 + \frac{1}{2}(H^T+H)(\vec{w}-\vec{w}^*) + \frac{\lambda}{N}\cdot 2\vec{w}$$

$$= H\cdot(\vec{w}-\vec{w}^*) + \frac{2\lambda}{N}\vec{w}$$

$\vec{w}^T\vec{w} = \vec{w}^T I \vec{w}$ $\quad\therefore \frac{\partial}{\partial\vec{w}} = (I^T + I)\vec{w} = 2\vec{w}$

$\because$ $E_{in}(\vec{w})$ twice differentiable

$\therefore$ the second partial derivatives $\Rightarrow$ continuous

$\Rightarrow$ Hessian: symmetric

$\Rightarrow \frac{1}{2}(H^T+H)(\vec{w}-\vec{w}^*) = \frac{1}{2}\cdot 2H\cdot(\vec{w}-\vec{w}^*)$
$= H(\vec{w}-\vec{w}^*)$

Set $\frac{\partial}{\partial\vec{w}} E_{aug}(\vec{w}) = 0$

$$H\cdot(\vec{w}-\vec{w}^*) + \frac{2\lambda}{N}\vec{w} = 0$$

$$\Rightarrow H\vec{w} - H\vec{w}^* + \frac{2\lambda}{N}\vec{w} = 0$$

$$\Rightarrow \left(H + \frac{2\lambda}{N}I\right)\vec{w} = H\vec{w}^*$$

$$\Rightarrow \vec{w} = \left(H + \frac{2\lambda}{N}I\right)^{-1} H\vec{w}^* \quad \square$$

$\{h_1, h_2, \ldots, h_{d+1}\}$ $\|$ let

since H: positive semi-definite, $\forall h_i \in \lambda(H)$, $h_i \geq 0$

also, $\frac{2\lambda}{N} > 0$

Therefore, $\lambda\left(H + \frac{2\lambda}{N}I\right) = \{\underset{>0}{h_1 + \frac{2\lambda}{N}}, \ldots, \underset{>0}{h_{d+1} + \frac{2\lambda}{N}}\}$

$\Rightarrow \left(H + \frac{2\lambda}{N}I\right)$: invertible