Optimization Algorithms: HW1

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Given a twice differentiable function $\varphi : \mathbb{R}^d \to [-\infty, \infty]$, assume that it is logarithmically homogeneous, then by the definition, the following holds:

$$\varphi(\gamma x) = \varphi(x) - \log \gamma, \quad \forall x \in \mathbb{R}^d, \gamma > 0 \tag{1}$$

Claim: $\langle \nabla \varphi(x), x \rangle = -1$

To derive the first equation, we first define the following:

$$F(\gamma) = \varphi(\gamma x)$$

Then the original equation (1) would become:

$$F(\gamma) = \varphi(x) - \log \gamma$$

Taking the derivative w.r.t. γ on both sides, we get:

$$\frac{dF}{d\gamma} = \frac{d}{d\gamma}\varphi(\gamma x) = \nabla \varphi(\gamma x) \cdot x = \langle \nabla \varphi(\gamma x), x \rangle \tag{2}$$

$$\frac{dF}{d\gamma} = \frac{d}{d\gamma}(\varphi(x) - \log \gamma) = -\frac{1}{\gamma} \tag{3}$$

Thus by (2) and (3), we have:

$$\langle \nabla \varphi(\gamma x), x \rangle = -\frac{1}{\gamma}$$

Then by plugging in $\gamma = 1$, we have:

$$\langle \nabla\varphi(x), x\rangle = -1 \qquad \square$$

Claim: $\nabla\varphi(x) = -\nabla^2\varphi(x)x$

From the previous part, we have:

$$\nabla\varphi(x)^T x = -1$$

Compute the gradient of both sides, for the left hand side, we have:

$$\begin{aligned} \nabla(\nabla\varphi(x)^T x) &= \nabla(\nabla\varphi(x))^T x + \nabla\varphi(x)^T \nabla x \\ &= \nabla^2\varphi(x)x + \nabla\varphi(x)^T \nabla x \end{aligned}$$

For the right hand side, we have:

$$\nabla(-1) = 0$$

Thus we have:

$$\begin{aligned} &\nabla^2\varphi(x)x + \nabla\varphi(x)^T\nabla x = 0 \\ \Rightarrow &\nabla\varphi(x)^T\nabla x = -\nabla^2\varphi(x)x \\ \Rightarrow &\nabla\varphi(x)^T I_d = -\nabla^2\varphi(x)x \\ \Rightarrow &\nabla\varphi(x) = -\nabla^2\varphi(x)x \qquad \square \end{aligned}$$

Claim: $\langle x, \nabla^2\varphi(x)x\rangle = 1$

From the previous part, we have:

$$\nabla\varphi(x) = -\nabla^2\varphi(x)x$$

Multiply both sides by x^T, we have:

$$x^T\nabla\varphi(x) = -x^T\nabla^2\varphi(x)x$$

Which is equivalent to the following by using $\langle \nabla\varphi(x), x\rangle = -1$:

$$\langle x, \nabla^2\varphi(x)x\rangle = -\langle x, \nabla\varphi(x)\rangle = (-1)\times(-1) = 1 \qquad \square$$