# CSIE5410 Optimization Algorithms: HW 2

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Recall that the Kelly criterion, aka the growth-optimal portfolio, yields the optimization problem

$$x_\star \in \operatorname*{arg\,min}_{x \in \Delta_d} f(x), \quad f(x) := -\sum_{i=1}^{n} w_i \log \langle a_i, x \rangle ,$$

where $\Delta_d$ denotes the probability simplex in $\mathbb{R}^d$, $w_i$ are strictly positive numbers satisfying $\sum_i w_i = 1$, and $a_i$ are entrywise non-negative vectors. Suppose that $a_i \neq 0$ for all $1 \leq i \leq n$.

1. (10 points) Show that the function $f$ is 1-smooth relative to the log-barrier and explicitly specify the value of $L$. Recall that the log-barrier is given by

   $$h(x) := -\sum_{i=1}^{d} \log x[i],$$

   where $x[i]$ denotes the $i$-th entry of the vector $x$.

Denote the Bregman divergence associated with $h$ as $D_h$, i.e.,

$$D_h(y, x) := h(y) - [h(x) + \langle \nabla h(x), y - x \rangle] .$$

Consider solving the optimization problem (1) by the following algorithm:

- Let $x_1 = (1/d, \ldots, 1/d) \in \Delta_d$.
- For every $t \in \mathbb{N}$, compute

  $$x_{t+1} \in \operatorname*{arg\,min}_{x \in \Delta_d} \langle \nabla f(x_t), x - x_t \rangle + D_h(x, x_t).$$

2. (10 points) Show that for any $x \in \Delta_d$ and $0 \leq \alpha < 1$,

   $$f(x_\alpha) \leq f(x) + \frac{\alpha}{1 - \alpha},$$

   where $x_\alpha := (1 - \alpha)x + \alpha(1/d, \ldots, 1/d)$.

3. (10 points) Suppose that $t > d$. Show that the algorithm described above satisfies

   $$f(x_{t+1}) - f(x_\star) = O\left( \frac{d \log(t/d)}{t} \right).$$

4. (10 points) Let us now focus on the implementation of the algorithm. Show that

$$x_{t+1} = e \oslash \left[ \nabla f(x_t) + (e \oslash x_t) + \lambda e \right],$$

where $e$ denotes the all-ones vector, $\oslash$ denotes the entrywise division, and $\lambda$ is given by

$$\lambda \in \arg\min_{u \in \mathbb{R}} \varphi(u), \quad \varphi(u) := u - \sum_{i=1}^{d} \log(u + \nabla f(x_t)[i] + 1/x_t[i]).$$

5. (10 points) Show that the function $\varphi$ is self-concordant. Therefore, the iterates $(x_t)_{t \in \mathbb{N}}$ can be computed by the algorithms in Chapter 5.2 of *Lectures on Convex Optimization* by Nesterov.

Consider the problem of learning a linear classifier given data $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^d \times \{\pm 1\}$. The $\ell_1$-regularized lasso yields the optimization problem

$$w_\star \in \arg\min_{w \in \mathbb{R}^d} f(w) + \lambda g(w),$$

for some regularization parameter $\lambda > 0$, where

$$f(x) := \frac{1}{n} \sum_{i=1}^{n} \log\left(1 + \mathrm{e}^{-y_i \langle x_i, w \rangle}\right),$$
$$g(w) := \|w\|_1.$$

Notice that in HW0, we have proved that the function $f$ is convex and $L$-smooth for some $L > 0$. Nevertheless, the $\ell_1$-norm is not differentiable, so gradient descent does not directly apply.

6. (10 points) Show that the function $g_\mu$ is differentiable and

$$\nabla g_\mu(w)[i] = \begin{cases} 1 & \text{if } w[i] \geq \mu; \\ \frac{w[i]}{\mu} & \text{if } -\mu \leq w[i] \leq \mu; \\ -1 & \text{if } w[i] < -\mu. \end{cases}$$

7. (10 points) Show that the function $g_\mu$ is ( $1/\mu$ )-smooth.

8. (10 points) Show that

$$g_\mu(w) \leq g(w) \leq g_\mu(w) + \frac{\mu d}{2}.$$

9. (5 points) Let $F = f + \lambda g$. Show that by gradient descent and appropriately choosing $\mu$, for any pre-specified time horizon $T \in \mathbb{N}$, we can construct a sequence of iterates $w_1, \ldots, w_{T+1}$ such that

$$F(w_{T+1}) - F(w_\star) \leq \frac{\lambda \sqrt{d}}{2\sqrt{T}} \left( \|w_1 - w_\star\|_2^2 + 1 \right) + \frac{L \|w_1 - w_\star\|_2^2}{2T}.$$

10. Show that the optimization error bound can be improved if we replace gradient descent with an accelerated gradient descent. Write down the value of $\mu$ you choose and the corresponding optimization error bound.

11. (5 points) Compare the approach above with FISTA and discuss which is more effective.