# Optimization Algorithms: HW1

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## 1

## 2

### (1)

Given a twice differentiable function $\varphi : \mathbb{R}^d \to [-\infty, \infty]$, assume that it is logarithmically homogeneous, then by the definition, the following holds:

$$\varphi(\gamma x) = \varphi(x) - \log \gamma, \quad \forall x \in \mathbb{R}^d, \gamma > 0 \tag{1}$$

Claim: $\langle \nabla \varphi(x), x \rangle = -1$

To derive the first equation, we first define the following:

$$F(\gamma) = \varphi(\gamma x)$$

Then the original equation (1) would become:

$$F(\gamma) = \varphi(x) - \log \gamma$$

Taking the derivative w.r.t. $\gamma$ on both sides, we get:

$$\frac{dF}{d\gamma} = \frac{d}{d\gamma}\varphi(\gamma x) = \nabla \varphi(\gamma x) \cdot x = \langle \nabla \varphi(\gamma x), x \rangle \tag{2}$$

$$\frac{dF}{d\gamma} = \frac{d}{d\gamma}(\varphi(x) - \log \gamma) = -\frac{1}{\gamma} \tag{3}$$

Thus by (2) and (3), we have:

$$\langle \nabla \varphi(\gamma x), x \rangle = -\frac{1}{\gamma}$$

Then by plugging in $\gamma = 1$, we have:

$$\langle \nabla \varphi(x), x \rangle = -1 \qquad \square$$

Claim: $\nabla \varphi(x) = -\nabla^2 \varphi(x) x$

From the previous part, we have:

$$\nabla \varphi(x)^T x = -1$$

Compute the gradient of both sides, for the left hand side, we have:

$$\begin{aligned} \nabla(\nabla \varphi(x)^T x) &= \nabla(\nabla \varphi(x))^T x + \nabla \varphi(x)^T \nabla x \\ &= \nabla^2 \varphi(x) x + \nabla \varphi(x)^T \nabla x \end{aligned}$$

For the right hand side, we have:

$$\nabla(-1) = 0$$

Thus we have:

$$\begin{aligned} &\nabla^2 \varphi(x) x + \nabla \varphi(x)^T \nabla x = 0 \\ \Rightarrow &\nabla \varphi(x)^T \nabla x = -\nabla^2 \varphi(x) x \\ \Rightarrow &\nabla \varphi(x)^T I_d = -\nabla^2 \varphi(x) x \\ \Rightarrow &\nabla \varphi(x) = -\nabla^2 \varphi(x) x \qquad \square \end{aligned}$$

Claim: $\langle x, \nabla^2 \varphi(x) x \rangle = 1$

From the previous part, we have:

$$\nabla \varphi(x) = -\nabla^2 \varphi(x) x$$

Multiply both sides by $x^T$, we have:

$$x^T \nabla \varphi(x) = -x^T \nabla^2 \varphi(x) x$$

Which is equivalent to the following by using $\langle \nabla \varphi(x), x \rangle = -1$:

$$\langle x, \nabla^2 \varphi(x) x \rangle = -\langle x, \nabla \varphi(x) \rangle = (-1) \times (-1) = 1 \qquad \square$$

## (2)

Suppose that $\varphi : \mathbb{R}^d \to [-\infty, \infty]$ is a twice differentiable function, and is strictly convex and logarithmically homogeneous, then the following holds by the definition:

$$\begin{aligned}
&\nabla^2 \varphi(x) > 0 \quad \forall x \in \mathbb{R}^d \\
&\varphi(\gamma x) = \varphi(x) - \log \gamma, \quad \forall x \in \mathbb{R}^d, \gamma > 0.
\end{aligned}$$

Also, we have the following properties from the previous subsection:

$$\langle \nabla \varphi(x), x \rangle = -1 \tag{1}$$
$$\nabla \varphi(x) = -\nabla^2 \varphi(x) x \tag{2}$$
$$\langle x, \nabla^2 \varphi(x) x \rangle = 1 \tag{3}$$

<u>Claim: $\nabla^2 \varphi(x) \geq \nabla \varphi(x) \left(\nabla \varphi(x)\right)^T, \quad \forall x \in \text{dom } \varphi$</u>

The claim is equivalent to proving that:

$$\nabla^2 \varphi(x) - \nabla \varphi(x) \left(\nabla \varphi(x)\right)^T \succeq 0$$

where $\succeq 0$ denotes positive semidefinite.
Let $z$ be any vector in $\mathbb{R}^d$, then we have:

$$\begin{aligned}
z^T \left(\nabla^2 \varphi(x) - \nabla \varphi(x) \left(\nabla \varphi(x)\right)^T\right) z &= z^T \nabla^2 \varphi(x) z - z^T \nabla \varphi(x) \left(\nabla \varphi(x)\right)^T z \\
&= z^T \nabla^2 \varphi(x) z - \left(\nabla \varphi(x)^T z\right)^2 \qquad (*)
\end{aligned}$$

<u>**Case 1:** $x = z$</u>

If $x = z$, then $(*)$ becomes the following using (1) and (2):

$$\begin{aligned}
z^T \left(\nabla^2 \varphi(x) - \nabla \varphi(x) \left(\nabla \varphi(x)\right)^T\right) z &= z^T \nabla^2 \varphi(x) x - \left(\langle \nabla \varphi(x), x \rangle\right)^2 \\
&= x^T \left(-\nabla \varphi(x)\right) - (-1)^2 \\
&= x^T \left(-\nabla \varphi(x)\right) - 1 \\
&= -\langle \nabla \varphi(x), x \rangle - 1 \\
&= -(-1) - 1 \\
&= 0 \geq 0
\end{aligned}$$

<u>**Case 2:** $x \neq z$</u>

Using (2) to replace $\nabla\varphi(x)$ with $-\nabla^2\varphi(x)x$ in $(*)$, and using the fact that $(\nabla^2\varphi(x))^T = \nabla^2\varphi(x)$ (the Hessian is symmetric):

$$\begin{aligned}
z^T\left(\nabla^2\varphi(x) - \nabla\varphi(x)\left(\nabla\varphi(x)\right)^T\right)z &= z^T\nabla^2\varphi(x)z - \left(\nabla\varphi(x)^T z\right)^2 \\
&= z^T\nabla^2\varphi(x)z - \left((-\nabla^2\varphi(x)x)^T z\right)^2 \\
&= z^T\nabla^2\varphi(x)z - \left(-\underbrace{x^T}_{A^T}\underbrace{(\nabla^2\varphi(x))^T z}_{B}\right)^2 \\
&= z^T\nabla^2\varphi(x)z - \underbrace{[(\nabla^2\varphi(x))^T z]^T x x^T [(\nabla^2\varphi(x))^T z]}_{B^T A A^T B} \\
&= z^T\nabla^2\varphi(x)z - [x^T(\nabla^2\varphi(x))^T z]^T[x^T(\nabla^2\varphi(x))^T z] \\
&= z^T\nabla^2\varphi(x)z - ||x^T(\nabla^2\varphi(x))^T z||^2 \\
&= z^T\nabla^2\varphi(x)z - ||x^T(\nabla^2\varphi(x))z||^2
\end{aligned}$$

Let $H = \nabla^2\varphi(x)$, then the above expression is equivalent to:

$$z^T Hz - (x^T Hz)^2$$

Let's first check that we can define the function

$$h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, \quad h(u,v) = u^T Hv$$

as the inner product on $\mathbb{R}^d$. [1]

Using the fact that we assumed that $\nabla^2\varphi(x) > 0$, so $H$ is positive definite, thus by theorem [2], there exists a one and only one positive definite matrix $H^{1/2}$ (also symmetric) such that $H = H^{1/2}H^{1/2}$.

- **Symmetry:** For any $u, v \in \mathbb{R}^d$, we have:

$$\begin{aligned}
h(u,v) &= u^T Hv \\
&= u^T H^{1/2}H^{1/2}v \\
&= (H^{1/2}u)^T(H^{1/2}v) \\
&= (H^{1/2}v)^T(H^{1/2}u) \\
&= v^T H^{1/2}H^{1/2}u \\
&= v^T Hu \\
&= h(v,u)
\end{aligned}$$

[1] H. Amann and J. Escher, *Analysis I*, 1st ed., Birkhäuser Basel, 2005, p. 153.

[2] "Square root of a matrix", Wikipedia, `https://en.wikipedia.org/wiki/Square_root_of_a_matrix`

- **Linearity:** For any $\lambda, \mu \in \mathbb{R}$ and $t, u, v \in \mathbb{R}^d$, we have:

$$\begin{aligned} h(t, \lambda u + \mu v) &= t^T H(\lambda u + \mu v) \\ &= t^T H(\lambda u + \mu v) \\ &= t^T H(\lambda u) + t^T H(\mu v) \\ &= \lambda t^T H u + \mu t^T H v \\ &= \lambda h(t, u) + \mu h(t, v) \end{aligned}$$

- **Positive definiteness:** For any $u \in \mathbb{R}^d$, we have:

$$h(u, u) = u^T H u > 0 \quad \text{since } H \text{ is positive definite}$$

[3]

Therefore, we have:

$$z^T H z - (x^T H z)^2 = \langle z, z \rangle_H - \langle x, z \rangle_H^2$$

Using the Cauchy-Schwarz inequality [4]:

**Cauchy-Schwarz inequality**

Let $(E, (\cdot \mid \cdot))$ be an inner product space. Then

$$|(x \mid y)|^2 \leq (x \mid x)(y \mid y), \qquad x, y \in E$$

We can derive the later equation using the fact that:

$$\langle x, x \rangle_H = x^T H x = x^T \nabla^2 \varphi(x) x = 1$$

(this is because property (3))
So we have:

$$\begin{aligned} \langle x, z \rangle_H^2 &\leq \langle x, x \rangle_H \langle z, z \rangle_H \\ &= 1 \times \langle z, z \rangle_H \\ &= \langle z, z \rangle_H \end{aligned}$$

Thus we have:

$$z^T H z - (x^T H z)^2 \geq 0 \qquad \square$$

[3] I later found that we have "A function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is an inner product on $\mathbb{R}^n$ if and only if there exists a symmetric positive-definite matrix $\mathbf{M}$ such that $\langle x, y \rangle = x^\top \mathbf{M} y$ for all $x, y \in \mathbb{R}^n$." on "Inner product space", Wikipedia, `https://en.wikipedia.org/wiki/Inner_product_space`

[4] H. Amann and J. Escher, *Analysis I*, 1st ed., Birkhäuser Basel, 2005, p. 154.

## (3)

We need to prove the following equivalence:

$$
\begin{aligned}
& (1) \quad e^{-\varphi(x)} \text{ is concave} \\
\iff & (2) \quad \varphi(y) \geq \varphi(x) - \log(1 - \langle \nabla\varphi(x), y - x\rangle), \quad \forall x, y \in \operatorname{dom}(\varphi) \\
\iff & (3) \quad \nabla^2\varphi(x) \succeq \nabla\varphi(x)\nabla\varphi(x)^\top, \quad \forall x \in \operatorname{dom}(\varphi)
\end{aligned}
$$

**(1) $\implies$ (2)**

Let $f : \mathbb{R}^d \to \mathbb{R}$ be defined as $f(x) = e^{-\varphi(x)}$.

Suppose that $f(x) = e^{-\varphi(x)}$ is concave, then by the definition of concavity [5]:

> Convex
>
> A continuously differentiable function $f(x)$ is called convex on $\mathbb{R}^n$ if for any $x, y \in \mathbb{R}^n$, we have:
>
> $$f(y) \geq f(x) + \langle \nabla f(x), y - x\rangle$$
>
> If $-f(x)$ is convex, then $f(x)$ is concave.

this means that our assumption is equivalent to saying that $-e^{-\varphi(x)}$ is convex. Let $g(x) = -f(x) = -e^{-\varphi(x)}$ a convex function, using the fact that:

$$\nabla g(x) = \frac{d}{dx}(-e^{-\varphi(x)}) = e^{-\varphi(x)}\nabla\varphi(x)$$

we have the following:

For any $x, y \in \mathbb{R}^d$:

$$
\begin{aligned}
& g(y) \geq g(x) + \langle \nabla g(x), y - x\rangle \\
\Rightarrow & -e^{-\varphi(y)} \geq -e^{-\varphi(x)} + \langle e^{-\varphi(x)}\nabla\varphi(x), y - x\rangle \\
\Rightarrow & e^{-\varphi(y)} \leq e^{-\varphi(x)} - e^{-\varphi(x)}\langle \nabla\varphi(x), y - x\rangle \\
\Rightarrow & e^{-\varphi(y)} \leq e^{-\varphi(x)}(1 - \langle \nabla\varphi(x), y - x\rangle) \\
\Rightarrow & -\varphi(y) \leq -\varphi(x) + \log(1 - \langle \nabla\varphi(x), y - x\rangle) \\
\Rightarrow & \varphi(y) \geq \varphi(x) - \log(1 - \langle \nabla\varphi(x), y - x\rangle)
\end{aligned}
$$

**(2) $\implies$ (3)**

[5] Y. Nesterov, *Introductory Lectures on Convex Optimization: A Basic Course*, 1st ed., Springer, New York, NY, 2004, p. 52.

Suppose (2) holds, so we have:

$$\varphi(y) \geq \varphi(x) - \log(1 - \langle \nabla \varphi(x), y - x \rangle), \quad \forall x, y \in \mathrm{dom}(\varphi)$$

By plugging in $y = x + h$ ($h = y - x$), with $||h|| \to 0$, we have:

$$\varphi(x + h) \geq \varphi(x) - \log(1 - \langle \nabla \varphi(x), h \rangle) \tag{1}$$

Then by using the second-order approximation [6]:

**Second-order approximation**

Let $f$ be twice differentiable at $\bar{x}$. Then

$$f(y) = f(\bar{x}) + \langle \nabla f(\bar{x}), y - \bar{x} \rangle + \frac{1}{2} \langle \nabla^2 f(\bar{x})(y - \bar{x}), y - \bar{x} \rangle + o(||y - \bar{x}||^2)$$

Since $\varphi$ is twice differentiable on its domain, we have:

$$\varphi(x + h) = \varphi(x) + \langle \nabla \varphi(x), h \rangle + \frac{1}{2} \langle \nabla^2 \varphi(x) h, h \rangle + o(||h||^2) \tag{2}$$

Combining (1) and (2), we have:

$$\begin{aligned}
& \varphi(x) + \langle \nabla \varphi(x), h \rangle + \frac{1}{2} \langle \nabla^2 \varphi(x) h, h \rangle + o(||h||^2) \geq \varphi(x) - \log(1 - \langle \nabla \varphi(x), h \rangle) \\
\Rightarrow\ & \langle \nabla \varphi(x), h \rangle + \frac{1}{2} \langle \nabla^2 \varphi(x) h, h \rangle + o(||h||^2) \geq -\log(1 - \langle \nabla \varphi(x), h \rangle) \\
\Rightarrow\ & \langle \nabla \varphi(x), h \rangle + \frac{1}{2} \langle \nabla^2 \varphi(x) h, h \rangle + o(||h||^2) \geq -(-\sum_{n=1}^{\infty} \frac{\langle \nabla \varphi(x), h \rangle^n}{n}) \\
\Rightarrow\ & \langle \nabla \varphi(x), h \rangle + \frac{1}{2} \langle \nabla^2 \varphi(x) h, h \rangle + o(||h||^2) \geq \langle \nabla \varphi(x), h \rangle + \sum_{n=2}^{\infty} \frac{\langle \nabla \varphi(x), h \rangle^n}{n} \\
\Rightarrow\ & \frac{1}{2} \langle \nabla^2 \varphi(x) h, h \rangle + o(||h||^2) \geq \sum_{n=2}^{\infty} \frac{\langle \nabla \varphi(x), h \rangle^n}{n} \\
\Rightarrow\ & \frac{1}{2} \langle \nabla^2 \varphi(x) h, h \rangle + o(||h||^2) \geq \frac{\langle \nabla \varphi(x), h \rangle^2}{2} + \frac{\langle \nabla \varphi(x), h \rangle^3}{3} + \cdots \qquad (*)
\end{aligned}$$

Examine the terms on the right hand side by Cauchy-Schwarz inequality:

[6] Y. Nesterov, *Introductory Lectures on Convex Optimization: A Basic Course*, 1st ed., Springer, New York, NY, 2004, p. 19.

$$\frac{(\langle \nabla\varphi(x), h\rangle)^3}{3} \leq \frac{(||\nabla\varphi(x)|| \cdot ||h||)^3}{3}$$

Since $||h|| \to 0$ by our assumption, we can write:

$$\frac{\langle \nabla\varphi(x), h\rangle^2}{2} + \frac{\langle \nabla\varphi(x), h\rangle^3}{3} + \cdots = o(||h||^2)$$

Substituting this bound back into $(*)$, we have:

$$\begin{aligned}
&\frac{1}{2}\langle \nabla^2\varphi(x)h, h\rangle + o(||h||^2) \geq \frac{\langle \nabla\varphi(x), h\rangle^2}{2} + o(||h||^2) \\
\Rightarrow\ &\frac{1}{2}\langle \nabla^2\varphi(x)h, h\rangle \geq \frac{\langle \nabla\varphi(x), h\rangle^2}{2} \\
\Rightarrow\ &\langle \nabla^2\varphi(x)h, h\rangle \geq \langle \nabla\varphi(x), h\rangle^2 \\
\Rightarrow\ &(\nabla^2\varphi(x)h)^T h \geq (\nabla\varphi(x)^T h)^T(\nabla\varphi(x)^T h) \\
\Rightarrow\ &h^T(\nabla^2\varphi(x))^T h \geq h^T\nabla\varphi(x)(\nabla\varphi(x))^T h \\
\Rightarrow\ &h^T((\nabla^2\varphi(x))^T - \nabla\varphi(x)(\nabla\varphi(x))^T)h \geq 0 \\
\Rightarrow\ &\nabla^2\varphi(x) - \nabla\varphi(x)(\nabla\varphi(x))^T \succeq 0 \qquad \text{(since the Hessian is symmetric)}
\end{aligned}$$

Thus, we have proved that:

$$\nabla^2\varphi(x) \geq \nabla\varphi(x)(\nabla\varphi(x))^T, \quad \forall x, y \in \operatorname{dom}\varphi$$

**(3) $\implies$ (1)**

Suppose (3) holds, so we have:

$$\nabla^2\varphi(x) \geq \nabla\varphi(x)(\nabla\varphi(x))^T, \quad \forall x, y \in \operatorname{dom}\varphi$$

Since we need to show that $e^{-\varphi(x)}$ is concave, similar to the previous proof, we can define $g(x) = -f(x) = -e^{-\varphi(x)}$ ( where $f(x) = e^{-\varphi(x)}$), and show that $g(x)$ is convex.

By theorem [7], we have:

[7] Y. Nesterov, *Introductory Lectures on Convex Optimization: A Basic Course*, 1st ed., Springer, New York, NY, 2004, p. 55.

Theorem 2.1.4

Two times continuously differentiable function $f \in \mathcal{F}^2(\mathbb{R}^n)$ iff for any $x \in \mathbb{R}^n$, we have:

$$f''(x) \succeq 0$$

Therefore, we need to show that $\nabla^2 g(x) \succeq 0$. We derive the following using the Scalar-by-vector identity [8]:

If $u = u(x)$ and $v = v(x)$ are vector functions of $x$, then:

$$\nabla(u \cdot v) = (\nabla u)v^T + u^T(\nabla v)$$

Hence, we have:

$$\begin{aligned}
\nabla^2 g(x) &= \nabla(e^{-\varphi(x)}\nabla\varphi(x)) \\
&= \left[\frac{d}{dx}(e^{-\varphi(x)})\right](\nabla\varphi(x))^T + e^{-\varphi(x)}\nabla^2\varphi(x) \\
&= -e^{-\varphi(x)}(\nabla\varphi(x))(\nabla\varphi(x))^T + e^{-\varphi(x)}\nabla^2\varphi(x) \\
&= e^{-\varphi(x)}\left[\nabla^2\varphi(x) - (\nabla\varphi(x))(\nabla\varphi(x))^T\right]
\end{aligned}$$

By our assumption, we knew that $\nabla^2\varphi(x) - \nabla\varphi(x)(\nabla\varphi(x))^T \succeq 0$, and multiplying by $e^{-\varphi(x)} > 0$ would not change the sign, therefore we have:

$$\nabla^2 g(x) \succeq 0$$

And the equivalence of the three statements is proved. $\square$

## (4)

We're given:

$$\begin{aligned}
f: \mathbb{R}^d \to \mathbb{R} &\qquad \text{differentiable, may be non-convex} \\
\nabla f: \ L\text{-Lipschitz},\ L > 0 &\quad i.e. \\
&\|\nabla f(y) - \nabla f(x)\|_* \leq L\|y - x\|, \quad \forall x, y \in \mathbb{R}^d \\
&\text{where } \|u\|_* := \max_{x \in \mathbb{R}^d, \|x\| \leq 1} \langle u, x \rangle
\end{aligned}$$

And the definition of a point $x$ being $\epsilon$-stationary for some $\epsilon > 0$ if:

$$\|\nabla f(x)\|_* \leq \varepsilon$$

[8]"Matrix calculus", Wikipedia, `https://en.wikipedia.org/wiki/Matrix_calculus`

**(1)**

Need to show:

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2, \quad \forall x, y \in \mathbb{R}^d$$

The thought is to use the proof process of Lemma 1.2.3 [9]:

> Lemma 1.2.3
>
> Let $f \in C_L^{1,1}(\mathbb{R}^n)$. Then for any $x, y \in \mathbb{R}^n$, we have:
>
> $$|f(y) - f(x) - \langle \nabla f(x), y - x \rangle| \leq \frac{L}{2} \|y - x\|^2$$

Let $g(\tau) = x + \tau(y - x), \quad$ where $\tau \in [0, 1]$, which means that $g(0) = x$ and $g(1) = y$. Then we have:

$$\begin{aligned} \frac{d}{d\tau} g(\tau) &= y - x \\ \nabla f(g(\tau)) &= \nabla f(x + \tau(y - x)) \end{aligned}$$

Then, for all $x, y \in \mathbb{R}^d$, we have:

$$\begin{aligned} f(y) - f(x) &= \int_x^y \nabla f(g(\tau)) \cdot dg(\tau) \\ &= \int_0^1 \nabla f(x + \tau(y - x)) \cdot (y - x) d\tau \\ &= \int_0^1 \langle \nabla f(x + \tau(y - x)), y - x \rangle d\tau \end{aligned}$$

Which is the same as the following, using the fact that the integral is linear, and $f(x), y - x$ are not functions of $\tau$:

[9] Y. Nesterov, *Introductory Lectures on Convex Optimization: A Basic Course*, 1st ed., Springer, New York, NY, 2004, p. 22-23.

$$
\begin{aligned}
&f(y) = f(x) + \int_0^1 \langle \nabla f(x + \tau(y-x)), y - x\rangle d\tau \\
\Rightarrow\ &f(y) = f(x) + \int_0^1 \langle \nabla f(x + \tau(y-x)) + \nabla f(x) - \nabla f(x), y - x\rangle d\tau \\
\Rightarrow\ &f(y) = f(x) + \int_0^1 \langle \nabla f(x), y - x\rangle d\tau + \int_0^1 \langle \nabla f(x + \tau(y-x)) - \nabla f(x), y - x\rangle d\tau \\
\Rightarrow\ &f(y) = f(x) + \langle \nabla f(x), y - x\rangle + \int_0^1 \langle \nabla f(x + \tau(y-x)) - \nabla f(x), y - x\rangle d\tau
\end{aligned}
\tag{*}
$$

And we're given for any $x, y \in \mathbb{R}^d$:

$$
\begin{aligned}
&\|\underbrace{\nabla f(y) - \nabla f(x)}_{\text{u}}\|_* \le L\|y - x\| \\
\Rightarrow\ &\max_{z \in \mathbb{R}^d, \|z\| \le 1} \langle \nabla f(y) - \nabla f(x), z\rangle \le L\|y - x\|
\end{aligned}
$$

Let $y = x + \tau(y - x)$, then we have:

$$
\begin{aligned}
&\|\nabla f(y) - \nabla f(x)\|_* = \|\nabla f(x + \tau(y-x)) - \nabla f(x)\|_* \le L\|x + \tau(y-x) - x\| = L\tau\|y - x\| \\
\Rightarrow\ &\|\nabla f(x + \tau(y-x)) - \nabla f(x)\|_* \le L\tau\|y - x\| \qquad (1)
\end{aligned}
$$

Going back to the definition $\|u\|_* := \max_{x \in \mathbb{R}^d, \|x\| \le 1} \langle u, x\rangle$, this means that for any $z \in \mathbb{R}^d, \|z\| \le 1$:

$$
\langle u, z\rangle \le \|u\|_*
$$

If we want to expand the definition to arbitrary $v \in \mathbb{R}^d$ (not necessarily $\|v\| \le 1$), we can let $z = \frac{v}{\|v\|}$, then we have:

$$
\begin{aligned}
&\langle u, z\rangle = \langle u, \frac{v}{\|v\|}\rangle = \frac{\langle u, v\rangle}{\|v\|} \le \|u\|_* \\
\Rightarrow\ &\langle u, v\rangle \le \|u\|_*\|v\| \qquad \forall u, v \in \mathbb{R}^d
\end{aligned}
$$

Let $u = \nabla f(x + \tau(y - x)) - \nabla f(x)$, $v = y - x$, then we have:

$$
\langle \nabla f(x + \tau(y-x)) - \nabla f(x), y - x\rangle \le \|\nabla f(x + \tau(y-x)) - \nabla f(x)\|_*\|y - x\| \tag{2}
$$

Multiply the result of (1) by $\|y - x\|$, we have:

$$\begin{aligned} &\|\nabla f(x + \tau(y - x)) - \nabla f(x)\|_* \leq L\tau\|y - x\| \\ \Rightarrow\ &\|\nabla f(x + \tau(y - x)) - \nabla f(x)\|_*\|y - x\| \leq L\tau\|y - x\|^2 \end{aligned} \tag{3}$$

Combining (2) and (3), we have:

$$\langle \nabla f(x + \tau(y - x)) - \nabla f(x), y - x\rangle \leq L\tau\|y - x\|^2$$

Substituting this back into $(*)$, we have:

$$\begin{aligned} &f(y) \leq f(x) + \langle \nabla f(x), y - x\rangle + \int_0^1 L\tau\|y - x\|^2 d\tau \\ \Rightarrow\ &f(y) \leq f(x) + \langle \nabla f(x), y - x\rangle + L\|y - x\|^2 \int_0^1 \tau d\tau \\ \Rightarrow\ &f(y) \leq f(x) + \langle \nabla f(x), y - x\rangle + \frac{L}{2}\|y - x\|^2 \qquad \square \end{aligned}$$

## (2)

We're given the algorithm (generalization of gradient descent):

$$\begin{aligned} &x_1 \in \mathbb{R}^d \\ &\text{for every } t \in \mathbb{N} \\ &\qquad x_{t+1} \in \operatorname*{argmin}_{x \in \mathbb{R}^d} \langle \nabla f(x_t), x - x_t\rangle + \frac{L}{2}\|x - x_t\|^2 \end{aligned}$$

Need to show:

$$f(x_{t+1}) - f(x_t) \leq -\frac{1}{2L}\|\nabla f(x_t)\|_*^2, \quad \forall t \in \mathbb{N}$$

Let the function to minimize in the update rule be $g$:

$$\begin{aligned} &g : \mathbb{R}^d \to \mathbb{R} \\ &g(x) = \langle \nabla f(x_t), x - x_t\rangle + \frac{L}{2}\|x - x_t\|^2 \end{aligned}$$

Let $z = x - x_t$, then we have:

$$g(x) = \langle \nabla f(x_t), z\rangle + \frac{L}{2}\|z\|^2$$

So:

$$\arg\min_x g(x) = x_t + \arg\min_z \{\langle \nabla f(x_t), z \rangle + \frac{L}{2}\|z\|^2\}$$

Let $h(z) = \langle \nabla f(x_t), z \rangle + \frac{L}{2}\|z\|^2$, we can rearrange the equation as:

$$h(z) - \frac{L}{2}\|z\|^2 = \langle \nabla f(x_t), z \rangle$$

Using the following proposition [10]:

**Proposition: Equivalent conditions of strong convexity**

A differentiable function $f$ is strongly convex with constant $\mu > 0$

$$\Leftrightarrow g(x) = f(x) - \frac{\mu}{2}\|x\|^2 \text{ is convex}, \forall x$$

Since $\langle \nabla f(x_t), z \rangle$ is affine, $h(z) - \frac{L}{2}\|z\|^2$ is convex, so $h(z)$ is strongly convex with convexity parameter $L$.

Then, taking the gradient of $h(z)$ with respect to $z$, we have:

$$\begin{aligned} \partial h(z) &= \frac{d}{dz}\left(\langle \nabla f(x_t), z \rangle + \frac{L}{2}\|z\|^2\right) \\ &= \nabla f(x_t) + \partial\left(\frac{L}{2}\|z\|^2\right) \end{aligned}$$

Here we can be sure that the derivative of $\langle \nabla f(x_t), z \rangle$ is $\nabla f(x_t)$, since this term is linear in $z$, and a lienar map is differentiable, however, we need to take the subdifferential for $\frac{L}{2}\|z\|^2$, since the norm is uncertain. [11]

Using the following theorem [12]:

**Theorem 3.1.15**

We have $f(x^*) = \min_{x \in \mathrm{dom}\, f} f(x)$ iff.

$$0 \in \partial f(x^*)$$

[10] *Strong Convexity*, available at: `https://xingyuzhou.org/blog/notes/strong-convexity`, accessed: Apr. 21, 2025.

[11] The definition of subdifferential is from Nesterov, Y. N., *Introductory Lectures on Convex Optimization: A Basic Course*, 1st ed., Springer, New York, NY, 2004, p. 126.

[12] Nesterov, Y. N., *Introductory Lectures on Convex Optimization: A Basic Course*, 1st ed., Springer, New York, NY, 2004, p. 129.

Since we knew that $h(z)$ is strongly convex, this means that the above equation is equivalent to saying there exists an unique minimizer $z^*$ for $h(z)$ such that:

$$0 \in \partial h(z^*)$$

so there exists $u \in \partial(\frac{1}{2}\|z\|^2)$ such that:

$$\begin{aligned} \nabla f(x_t) + Lu &= 0 \\ \Rightarrow u &= -\frac{1}{L}\nabla f(x_t) \end{aligned}$$

---

Define:

$$\begin{aligned} &\phi : \mathbb{R}^d \to \mathbb{R} \\ &\phi(z) = \frac{L}{2}\|z\|^2 \end{aligned}$$

Then the conjugate [13] of $\phi$ is defined as:

$$\begin{aligned} \phi^*(v) &= \sup_{z \in \mathbb{R}^d} \left( \langle v, z \rangle - \phi(z) \right) \\ &= \sup_{z \in \mathbb{R}^d} \left( \langle v, z \rangle - \frac{L}{2}\|z\|^2 \right) \end{aligned}$$

Let $z = \alpha z'$, where $|\|z'\| = 1$, then:

$$\begin{aligned} \phi^*(v) &= \sup_{z \in \mathbb{R}^d} \left( \alpha\langle v, z' \rangle - \frac{L}{2}\langle \alpha z', \alpha z' \rangle \right) \\ &= \sup_{\alpha \in \mathbb{R}} \left( \alpha\langle v, z' \rangle - \frac{L\alpha^2}{2} \right) \end{aligned}$$

And by the definition of dual norm, we can derive the inequality (generalization of Cauchy-Schwarz inequality) [14] :

$$\alpha\langle v, z' \rangle \le \alpha\|v\|_*\|z'\| = \alpha\|v\|_*$$

And the original conjugate can be rewritten as:

[13] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004, p. 91. Available online at `https://web.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf`.

[14] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004, p. 637.

$$\phi^*(v) = \sup_{\alpha \in \mathbb{R}} \left( \alpha \|v\|_* - \frac{L}{2}\alpha^2 \right)$$

Taking the derivative:

$$\frac{d}{d\alpha} \left( \alpha \|v\|_* - \frac{L}{2}\alpha^2 \right) = \|v\|_* - L\alpha \implies \alpha = \frac{\|v\|_*}{L}$$

Plugging back in:

$$\begin{aligned}
\phi^*(v) &= \frac{\|v\|_*}{L} \cdot \|v\|_* - \frac{L}{2} \cdot \frac{\|v\|_*^2}{L^2} \\
&= \frac{\|v\|_*^2}{L} - \frac{\|v\|_*^2}{2L} \\
&= \frac{\|v\|_*^2}{2L}
\end{aligned}$$

By Fenchel's inequality [15] , which is stated as follows:

**Fenchel's inequality**

For all $x, y$:

$$f(x) + f^*(y) \geq \langle x, y \rangle$$

Therefore, we have for all $z, v \in \mathbb{R}^d$:

$$\begin{aligned}
&\phi(z) + \phi^*(v) \geq \langle z, v \rangle \\
\Rightarrow\ &\frac{L}{2}\|z\|^2 + \frac{\|v\|_*^2}{2L} \geq \langle z, v \rangle
\end{aligned}$$

Let $v = \nabla f(x_t)$, and since $z = x - x_t$, which means that choosing the optiaml $z$ is equivalent to choosing the optimal $x$, which is $x_{t+1}$, so $z = x_{t+1} - x_t$, and we have:

$$\frac{L}{2}\|x_{t+1} - x_t\|^2 + \frac{1}{2L}\|\nabla f(x_t)\|_*^2 \geq \langle x_{t+1} - x_t, \nabla f(x_t) \rangle$$

[15] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004, p. 94.

By the result of subproblem (1), and plugging in $y = x_{t+1}$ and $x = x_t$, we have:

$$
\begin{aligned}
& f(x_{t+1}) \leq f(x_t) + \langle \nabla f(x_t), x_{t+1} - x_t \rangle + \frac{L}{2}\|x_{t+1} - x_t\|^2 \\
\Rightarrow\ & f(x_{t+1}) - f(x_t) \leq \langle \nabla f(x_t), x_{t+1} - x_t \rangle + \frac{L}{2}\|x_{t+1} - x_t\|^2 \\
\Rightarrow\ & f(x_{t+1}) - f(x_t) \leq \frac{1}{2L}\|\nabla f(x_t)\|_*^2 + \frac{L}{2}\|x_{t+1} - x_t\|^2 + \frac{L}{2}\|x_{t+1} - x_t\|^2
\end{aligned}
$$

For any $u, v \in \mathbb{R}^d,\ \lambda \in \mathbb{R}$, we have:

$$
\begin{aligned}
g(\lambda u + (1-\lambda)v) &= \langle \nabla f(\lambda u + (1-\lambda)v), \lambda u + (1-\lambda)v - x_t \rangle + \frac{L}{2}\|\lambda u + (1-\lambda)v - x_t\|^2 \\
&= \langle \nabla f(\lambda u + (1-\lambda)v), \lambda u \rangle + \langle \nabla f(\lambda u + (1-\lambda)v), (1-\lambda)v \rangle - \langle \nabla f(\lambda u + (1-\lambda)v), x_t \rangle \\
&\quad + \frac{L}{2}\|\lambda u + (1-\lambda)v - x_t\|^2 \\
&= \langle \nabla f(\lambda u + (1-\lambda)v), \lambda u \rangle + \langle \nabla f(\lambda u + (1-\lambda)v), (1-\lambda)v \rangle - \langle \nabla f(\lambda u + (1-\lambda)v), x_t \rangle \\
&= \lambda \langle \nabla f(u), u - x_t \rangle + (1-\lambda)\langle \nabla f(v), v - x_t \rangle \\
&= \lambda g(u) + (1-\lambda) g(v)
\end{aligned}
$$

**(3)**