

Optimization Algorithms: HW2

Lo Chun, Chou
R13922136

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We're given the following problem:

$$x_{\star} \in \arg \min_{x \in \Delta_d} f(x), \quad f(x) = - \sum_{i=1}^n w_i \log \langle a_i, x \rangle$$

where:

1.

$$x \in \Delta_d = \{x \in \mathbb{R}^d \mid x[i] \geq 0, \sum_{i=1}^d x[i] = 1\} \text{ (probability simplex)}$$

2.

$$w_i \in \mathbb{R}, w_i > 0, \sum_{i=1}^n w_i = 1$$

3.

$$a_i \in \mathbb{R}^d, a_i = \begin{bmatrix} a_i[1] \\ a_i[2] \\ \vdots \\ a_i[d] \end{bmatrix},$$
$$a_i[j] \geq 0 \quad \forall i = 1, \dots, n, j = 1, \dots, d$$
$$a_i \neq 0 \quad \forall i = 1, \dots, n$$

We're asked to show that:

f is 1-smooth relative to the log-barrier, which is defined as:

$$h(x) = - \sum_{i=1}^d \log x[i]$$

Solution. By the following proposition ¹:

PROPOSITION 1.1. *The following conditions are equivalent:*
 (a-i) $f(\cdot)$ is L -smooth relative to $h(\cdot)$;
 (a-ii) $Lh(\cdot) - f(\cdot)$ is a convex function on Q ;
 (a-iii) under twice differentiability $\nabla^2 f(x) \preceq L\nabla^2 h(x)$ for any $x \in \text{int } Q$;
 (a-iv) $\langle \nabla f(x) - \nabla f(y), x - y \rangle \leq L \langle \nabla h(x) - \nabla h(y), x - y \rangle$ for all $x, y \in \text{int } Q$.

we could prove the required condition (which is (a-i), with $L = 1$) by proving its equivalent condition (a-iii, with $L = 1$).

First calculate $\nabla f(x)$:

$$\begin{aligned}\nabla f(x) &= \frac{d}{dx} \left(- \sum_{i=1}^n w_i \log \langle a_i, x \rangle \right) \\ &= - \sum_{i=1}^n w_i \cdot \frac{d}{dx} (\log \langle a_i, x \rangle) \\ &= - \sum_{i=1}^n w_i \cdot \left(\frac{a_i}{\langle a_i, x \rangle} \right) \\ &= - \sum_{i=1}^n w_i \frac{a_i}{\langle a_i, x \rangle}\end{aligned}$$

Then the Hessian of f is:

$$\begin{aligned}\nabla^2 f(x) &= \frac{d}{dx} \left(- \sum_{i=1}^n w_i \frac{a_i}{\langle a_i, x \rangle} \right) \\ &= - \sum_{i=1}^n w_i \cdot \frac{d}{dx} \left(\frac{a_i}{\langle a_i, x \rangle} \right) \\ &= - \sum_{i=1}^n w_i \cdot \left(\frac{a_i}{\langle a_i, x \rangle^2} a_i^T \right)\end{aligned}$$

Expanding the expression and writing in another form, we have:

$$\nabla^2 f(x) = - \sum_{i=1}^n \frac{w_i}{\langle a_i, x \rangle^2} a_i a_i^T \quad (1)$$

¹Relative Smooth Convex Optimization by First-Order Methods, and Applications, MIT Lecture Notes, available at: <https://dspace.mit.edu/bitstream/handle/1721.1/120867/16m1099546.pdf>, accessed: May. 9, 2025, p. 336.

Then we shall do the same to $h(x)$

$$\begin{aligned}\nabla h(x) &= \frac{d}{dx} \left(- \sum_{i=1}^d \log x[i] \right) \\ &= \begin{bmatrix} -\frac{1}{x[1]} \\ -\frac{1}{x[2]} \\ \vdots \\ -\frac{1}{x[d]} \end{bmatrix}\end{aligned}$$

Then $\nabla^2 h(x)$ is:

$$\begin{aligned}\nabla^2 h(x) &= \nabla \begin{bmatrix} -\frac{1}{x[1]} \\ -\frac{1}{x[2]} \\ \vdots \\ -\frac{1}{x[d]} \end{bmatrix} \\ &= \begin{bmatrix} \frac{d}{dx[1]} \left(-\frac{1}{x[1]} \right) & \frac{d}{dx[2]} \left(-\frac{1}{x[1]} \right) & \cdots & \frac{d}{dx[d]} \left(-\frac{1}{x[1]} \right) \\ \frac{d}{dx[1]} \left(-\frac{1}{x[2]} \right) & \frac{d}{dx[2]} \left(-\frac{1}{x[2]} \right) & \cdots & \frac{d}{dx[d]} \left(-\frac{1}{x[2]} \right) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d}{dx[1]} \left(-\frac{1}{x[d]} \right) & \frac{d}{dx[2]} \left(-\frac{1}{x[d]} \right) & \cdots & \frac{d}{dx[d]} \left(-\frac{1}{x[d]} \right) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{x[1]^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{x[2]^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{x[d]^2} \end{bmatrix} \tag{2}\end{aligned}$$

Observe $\nabla^2 f(x)$ in (1), since we're given $w_i > 0$, $x \in \Delta_d$, $a_i \neq 0$, and with proposition (a-iii) only requires dealing with $\text{int } \Delta_d$, we can guarantee $x[i] > 0$, so the scalar $\frac{w_i}{\langle a_i, x \rangle^2} > 0$.

Also, we knew that for any $a_i \neq 0$, $a_i a_i^T$ is positive semidefinite, thus, each term in the summation is positive semidefinite, by summing up the n terms and adding a negative sign, we have $\nabla^2 f(x) \preceq 0$ as follows:

$$\nabla^2 f(x) = - \sum_{i=1}^n \frac{w_i}{\langle a_i, x \rangle^2} a_i a_i^T \preceq 0$$

Then, since $\nabla^2 h(x)$ is a diagonal matrix, and we're given that $x[i] \geq 0$, same as above, with proposition (a-iii) only requires dealing with $\text{int } \Delta_d$, we can guarantee $x[i] > 0$ (so for each $\frac{1}{x[i]}$), and $\nabla^2 h(x)$ is positive definite.

Therefore, we have:

$$\nabla^2 f(x) \preceq 1 \cdot \nabla^2 h(x) \quad \text{for any } x \in \text{int } \Delta_d$$

which means that (a-iii) is proved, and its equivalent condition (a-i) is also proved, and we have:

f is 1-smooth relative to the log-barrier h

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Denote the Bregman divergence associated with h as D_h , i.e.,

$$D_h(y, x) = h(y) - [h(x) + \langle \nabla h(x), (y - x) \rangle]$$

Consider solving the optimization problem (1) by the following algorithm:

- Let $x_1 = \begin{bmatrix} \frac{1}{d} \\ \vdots \\ \frac{1}{d} \end{bmatrix} \in \Delta_d$
- For every $t \in \mathbb{N}$, compute:

$$x_{t+1} \in \arg \min_{x \in \Delta_d} [\langle \nabla f(x_t), x - x_t \rangle + D_h(x, x_t)]$$

Note: I use $\begin{bmatrix} \frac{1}{d} \\ \vdots \\ \frac{1}{d} \end{bmatrix}$ to represent the vector $(1/d, \dots, 1/d)$ (which is the notation used in the HW spec) in the following solution.

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Show that for any $x \in \Delta_d$ and $0 \leq \alpha < 1$,

$$f(x_\alpha) \leq f(x) + \frac{\alpha}{1 - \alpha}, \quad \text{where } x_\alpha = (1 - \alpha)x + \alpha \begin{bmatrix} \frac{1}{d} \\ \vdots \\ \frac{1}{d} \end{bmatrix}$$

Solution. From the previous subproblem, we knew that f is 1-smooth relative to the log-barrier, so we have:

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + D_h(y, x) \quad \forall x, y \in \text{int } \Delta_d$$

To bound $f(x_\alpha)$, we first show that $x_\alpha \in \text{int } \Delta_d$, and then let $y = x_\alpha$, $x = x$ so that we would have:

$$f(x_\alpha) \leq f(x) + \langle \nabla f(x), x_\alpha - x \rangle + D_h(x_\alpha, x)$$

By the definition of x_α , we knew that it is the convex combination of x and $\begin{bmatrix} \frac{1}{d} \\ \vdots \\ \frac{1}{d} \end{bmatrix}$, where $x \in \Delta_d$ and $\begin{bmatrix} \frac{1}{d} \\ \vdots \\ \frac{1}{d} \end{bmatrix} = x_1 \in \Delta_d$ as stated in the algorithm. Also, for

each element in x_α , we have:

$$x_\alpha[i] = (1 - \alpha)x[i] + \alpha \left(\frac{1}{d} \right) \quad \forall i = 1, \dots, d$$

Since $x[i] \geq 0$ and α is strictly smaller than 1, consider the case that $0 < \alpha < 1$, then we have $x_\alpha[i] > 0$ for all $i = 1, \dots, d$. For $\alpha = 0$, we have $x_\alpha[i] = x[i] \geq 0$ for all $i = 1, \dots, d$, and since in order to use the previous inequality, we need $x \in \text{int } \Delta_d$, thus each $x[i]$ is strictly positive, so we have $x_\alpha \in \text{int } \Delta_d$ for $\alpha = 0$ (and also for $0 < \alpha < 1$).

\rightarrow need to be true for all $x \in \Delta_d$

Then, we have:

$$f(x_\alpha) \leq f(x) + \langle \nabla f(x), x_\alpha - x \rangle + D_h(x_\alpha, x)$$

To further simplify, we have:

$$x_\alpha - x = \left[(1 - \alpha)x + \alpha \begin{bmatrix} \frac{1}{d} \\ \vdots \\ \frac{1}{d} \end{bmatrix} \right] - x = \alpha \left[\begin{bmatrix} \frac{1}{d} \\ \vdots \\ \frac{1}{d} \end{bmatrix} - x \right]$$

So we could expand the following expressions:

$$\begin{aligned}
\langle \nabla f(x), x_\alpha - x \rangle &= \left\langle -\sum_{i=1}^n w_i \frac{a_i}{\langle a_i, x \rangle}, \alpha \left(\begin{bmatrix} \frac{1}{d} \\ \vdots \\ \frac{1}{d} \end{bmatrix} - x \right) \right\rangle \\
&= -\frac{\alpha}{d} \sum_{i=1}^n \frac{w_i}{\langle a_i, x \rangle} [a_i[1] \cdots a_i[d]] \begin{bmatrix} 1 - x[1] \\ \vdots \\ 1 - x[d] \end{bmatrix} \\
&= -\frac{\alpha}{d} \sum_{i=1}^n \frac{w_i}{\langle a_i, x \rangle} \left(\sum_{j=1}^d a_i[j] - \sum_{j=1}^d a_i[j]x[j] \right) \\
&= -\frac{\alpha}{d} \sum_{i=1}^n \frac{w_i}{\sum_{k=1}^d a_i[k]x[k]} \left(\sum_{j=1}^d a_i[j] - \sum_{j=1}^d a_i[j]x[j] \right) \\
&= -\frac{\alpha}{d} \sum_{i=1}^n \frac{w_i \sum_{j=1}^d a_i[j]}{\sum_{k=1}^d a_i[k]x[k]} + \frac{\alpha}{d} \sum_{i=1}^n w_i \\
&= \frac{\alpha}{d} \left(1 - \sum_{i=1}^n w_i \sum_{j=1}^d \frac{a_i[j]}{a_i[j]x[j]} \right) \\
&= \frac{\alpha}{d} \left(1 - \sum_{i=1}^n w_i \sum_{j=1}^d \frac{1}{x[j]} \right) \tag{1}
\end{aligned}$$

By the definition of D_h , we have:

$$\begin{aligned}
D_h(x_\alpha, x) &= h(x_\alpha) - (h(x) + \langle \nabla h(x), (x_\alpha - x) \rangle) \\
&= h(x_\alpha) - \left(h(x) + \left\langle \begin{bmatrix} -\frac{1}{x[1]} \\ -\frac{1}{x[2]} \\ \vdots \\ -\frac{1}{x[d]} \end{bmatrix}, \alpha \left(\begin{bmatrix} \frac{1}{d} \\ \vdots \\ \frac{1}{d} \end{bmatrix} - x \right) \right\rangle \right) \\
&= -\sum_{i=1}^d \log x_\alpha[i] - \left(-\sum_{i=1}^d \log x[i] + \left\langle \begin{bmatrix} -\frac{1}{x[1]} \\ -\frac{1}{x[2]} \\ \vdots \\ -\frac{1}{x[d]} \end{bmatrix}, \alpha \left(\begin{bmatrix} \frac{1}{d} \\ \vdots \\ \frac{1}{d} \end{bmatrix} - x \right) \right\rangle \right) \\
&= -\sum_{i=1}^d \log x_\alpha[i] + \sum_{i=1}^d \log x[i] + \alpha \left[-\frac{1}{x[1]} \cdots -\frac{1}{x[d]} \right] \begin{bmatrix} \frac{1-dx[1]}{d} \\ \vdots \\ \frac{1-dx[d]}{d} \end{bmatrix} \\
&= \sum_{i=1}^d (\log x[i] - \log x_\alpha[i]) - \sum_{i=1}^d \frac{\alpha}{dx[i]} + \alpha d
\end{aligned} \tag{2}$$

Combining (1) and (2), we have:

$$\begin{aligned}
&\langle \nabla f(x), x_\alpha - x \rangle + D_h(x_\alpha, x) \\
&= \frac{\alpha}{d} \left(1 - \sum_{i=1}^n w_i \sum_{j=1}^d \frac{1}{x[j]} \right) + \sum_{i=1}^d (\log x[i] - \log x_\alpha[i]) - \sum_{i=1}^d \frac{\alpha}{dx[i]} + \alpha d \\
&=
\end{aligned}$$

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Solution. We need to show that:

$$\begin{aligned}
x_{t+1} &= \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \oslash \left\{ \nabla f(x_t) + \begin{bmatrix} \frac{1}{x_t[1]} \\ \vdots \\ \frac{1}{x_t[d]} \end{bmatrix} + \begin{bmatrix} \lambda \\ \vdots \\ \lambda \end{bmatrix} \right\} \\
&= \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \oslash \begin{bmatrix} \frac{\nabla f(x_t)x_t[1]+1+\lambda x_t[1]}{x_t[1]} \\ \vdots \\ \frac{\nabla f(x_t)x_t[d]+1+\lambda x_t[d]}{x_t[d]} \end{bmatrix} \\
&= \begin{bmatrix} \frac{x_t[1]}{\nabla f(x_t)x_t[1]+1+\lambda x_t[1]} \\ \vdots \\ \frac{x_t[d]}{\nabla f(x_t)x_t[d]+1+\lambda x_t[d]} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\nabla f(x_t) + \frac{1}{x_t[1]} + \lambda} \\ \vdots \\ \frac{1}{\nabla f(x_t) + \frac{1}{x_t[d]} + \lambda} \end{bmatrix} \tag{*}
\end{aligned}$$

By the updating rule, we have:

$$\begin{aligned}
x_{t+1} &\in \arg \min_{x \in \Delta_d} \{ \langle \nabla f(x_t), x - x_t \rangle + D_h(x, x_t) \} \\
\rightarrow x_{t+1} &\in \arg \min_{x \in \Delta_d} \{ \langle \nabla f(x_t), x - x_t \rangle + h(x) - [h(x_t) + \langle \nabla h(x_t), x - x_t \rangle] \} \\
\rightarrow x_{t+1} &\in \arg \min_{x \in \Delta_d} \{ \langle \nabla f(x_t), x - x_t \rangle + h(x) - h(x_t) - \langle \nabla h(x_t), x - x_t \rangle \} \\
\rightarrow x_{t+1} &\in \arg \min_{x \in \Delta_d} \{ \langle \nabla f(x_t), x - x_t \rangle + h(x) - h(x_t) - \langle \nabla h(x_t), x \rangle + \langle \nabla h(x_t), x_t \rangle \} \tag{1}
\end{aligned}$$

Recall we previously derived that:

$$\nabla h(x_t) = \begin{bmatrix} -\frac{1}{x_t[1]} \\ -\frac{1}{x_t[2]} \\ \vdots \\ -\frac{1}{x_t[d]} \end{bmatrix}$$

So:

$$\langle \nabla h(x_t), x_t \rangle = \left[-\frac{1}{x_t[1]} \cdots -\frac{1}{x_t[d]} \right] \begin{bmatrix} x_t[1] \\ \vdots \\ x_t[d] \end{bmatrix} = -\sum_{i=1}^d \frac{1}{x_t[i]} x_t[i] = -d$$

Plugging this into (1), we have:

$$x_{t+1} \in \arg \min_{x \in \Delta_d} \{ \langle \nabla f(x_t), x - x_t \rangle + h(x) - h(x_t) - \langle \nabla h(x_t), x - x_t \rangle \}$$

(Similarly, $\langle \nabla f(x_t), x_t \rangle$ would also be a constant.)

Since we need to find the x that gives the minimum, we can drop the terms that are constant or independent of x (the Green terms), so we have:

$$x_{t+1} \in \arg \min_{x \in \Delta_d} \{ \langle \nabla f(x_t), x \rangle + h(x) - \langle \nabla h(x_t), x \rangle \}$$

Combining the inner product terms, we have:

$$x_{t+1} \in \arg \min_{x \in \Delta_d} \{ \langle \nabla f(x_t) - \nabla h(x_t), x \rangle + h(x) \}$$

Expand this equation by what we previously derived:

$$\begin{aligned} \nabla h(x_t) &= \begin{bmatrix} -\frac{1}{x_t[1]} \\ -\frac{1}{x_t[2]} \\ \vdots \\ -\frac{1}{x_t[d]} \end{bmatrix} \\ h(x) &= -\sum_{i=1}^d \log x[i] \end{aligned}$$

we have:

$$x_{t+1} \in \arg \min_{x \in \Delta_d} \sum_{i=1}^d \left(\nabla f(x_t)[i] + \frac{1}{x_t[i]} \right) x[i] - \sum_{i=1}^d \log x[i]$$

In order to deal with the constraint $x \in \Delta_d$, we can use Lagrange multiplier λ and write the Lagrangian as:

$$L(x, \lambda) = \sum_{i=1}^d \left(\nabla f(x_t)[i] + \frac{1}{x_t[i]} \right) x[i] - \sum_{i=1}^d \log x[i] + \lambda \left(\sum_{i=1}^d x[i] - 1 \right)$$

Then taking the derivative w.r.t. $x[i]$ and set the result to 0 to find the optimal x , we have:

$$\frac{\partial L}{\partial x[i]} = \nabla f(x_t)[i] + \frac{1}{x_t[i]} - \frac{1}{x[i]} + \lambda = 0$$

Rearrange to solve for $x[i]$, we have:

$$\begin{aligned}\frac{1}{x[i]} &= \nabla f(x_t)[i] + \frac{1}{x_t[i]} + \lambda \\ \rightarrow x[i] &= \frac{1}{\nabla f(x_t)[i] + \frac{1}{x_t[i]} + \lambda}\end{aligned}$$

And this matches with (*), which is the given updating rule. ■

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We need to show that the following function is self-concordant:

$$\varphi(u) = u - \sum_{i=1}^d \log(u + \nabla f(x_t)[i] + \frac{1}{x_t[i]})$$

Solution. In order to show that $\varphi(u)$ is self-concordant, since $\varphi(u)$ is univariate, we can directly use the following definition ²:

Self-concordant for univariate functions

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is self-concordant on \mathbb{R} if :

$$|f'''(x)| \leq 2f''(x)^{3/2}$$

Claim:

$$|\varphi'''(u)| \leq 2\varphi''(u)^{3/2}$$

Proof: Let us define:

$$y_i := u + \nabla f(x_t)[i] + \frac{1}{x_t[i]}, \quad \forall i = 1, \dots, d$$

Then, the original function $\varphi(u)$ can be rewritten as:

$$\varphi(u) = u - \sum_{i=1}^d \log y_i = u + \sum_{i=1}^d (-\log y_i)$$

Now we can compute the derivatives of $\varphi(u)$:

²*Self-concordant function*, available at: https://en.wikipedia.org/wiki/Self-concordant_function#Univariate_self-concordant_function, accessed: May. 29, 2025.

$$\varphi'(u) = 1 - \sum_{i=1}^d \frac{1}{y_i}$$

and the second derivative:

$$\varphi''(u) = \sum_{i=1}^d \frac{1}{y_i^2}$$

and the third derivative:

$$\varphi'''(u) = -2 \sum_{i=1}^d \frac{1}{y_i^3}$$

Now we have:

$$|\varphi'''(u)| = 2 \sum_{i=1}^d \frac{1}{y_i^3}$$

$$\varphi''(u) = \sum_{i=1}^d \frac{1}{y_i^2}$$

In order to let the original definition of $\varphi(u)$ be valid, $y_i \in (0, \infty)$ must hold, thus, if we further define $g(y_i) = -\log y_i$, then

$$g : \{y_i \in \mathbb{R} \mid y_i > 0\} \rightarrow \mathbb{R}$$

, and we have:

$$g'(y_i) = \frac{d}{dy_i}(-\log y_i) = -\frac{1}{y_i}$$

$$g''(y_i) = \frac{d}{dy_i} \left(-\frac{1}{y_i} \right) = \frac{1}{y_i^2}$$

$$g'''(y_i) = \frac{d}{dy_i} \left(\frac{1}{y_i^2} \right) = -\frac{2}{y_i^3}$$

And we have:

$$|g'''(y_i)| = \left| -\frac{2}{y_i^3} \right| = \frac{2}{y_i^3} \leq 2 \left(\frac{1}{y_i^2} \right)^{3/2} = 2 \left(\frac{1}{y_i^3} \right)$$

Which shows that $g(y_i)$ is self-concordant.

Then, using the following property ³:

■ **Sum of self-concordant functions.** The set of self-concordant functions is closed under addition.

Theorem 2.2. Let $f_1 : \Omega_1 \rightarrow \mathbb{R}$ and $f_2 : \Omega_2 \rightarrow \mathbb{R}$ be self-concordant functions whose domains satisfy $\Omega_1 \cap \Omega_2 \neq \emptyset$. Then, the function $f + g : \Omega_1 \cap \Omega_2 \rightarrow \mathbb{R}$ is self-concordant.

Since $g(y_i)$ is self-concordant for all $i = 1, \dots, d$, and they have the same domain, so $\bigcap_{i=1}^d \text{dom } g(y_i) \neq \emptyset$, thus, their sum:

$$\sum_{i=1}^d g(y_i) = \sum_{i=1}^d (-\log y_i)$$

is also self-concordant.

Then, using another property:

■ **Addition of an affine function.** Addition of an affine function to a self-concordant functions does not affect the self-concordance property, since self-concordance depends only on the Hessian of the function, and the addition of affine functions does not affect the Hessian.

Theorem 2.3. Let $f : \Omega \rightarrow \mathbb{R}$ be self-concordant function. Then, the function $g(x) := f(x) + \langle a, x \rangle + b$ is self-concordant on Ω .

If we let $h(u) = u$, then h is an affine function, then our self concordant function $\sum_{i=1}^d (-\log y_i)$ plussing the affine function h :

$$\varphi(u) = u + \sum_{i=1}^d (-\log y_i)$$

is also self-concordant. ■

³G. Farina, *Lecture 14A–B: Self-concordant functions*, MIT 6.7220/15.084 — Nonlinear Optimization, Apr. 16–18th 2024. Available at: https://www.mit.edu/~gfarina/2024/67220s24_L14B_self_concordance/L14.pdf, p. 4.

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We're given that:

$$g_\mu(w) = \max_{v \in \mathcal{B}_\infty} \langle w, v \rangle - \frac{\mu}{2} \|v\|_2^2$$

where \mathcal{B}_∞ is the unit l_∞ norm ball.

We need to show that g_μ is differentiable and:

$$\nabla g_\mu(w) = \begin{cases} 1 & \text{if } w[i] \geq \mu \\ \frac{w[i]}{\mu} & \text{if } -\mu \leq w[i] \leq \mu \\ -1 & \text{if } w[i] < -\mu \end{cases}$$

Solution. By the definition of l_∞ norm, we have:

$$\|v\|_\infty \leq 1 \iff \max_{i=1, \dots, d} |v[i]| \leq 1$$

Then the original $g_\mu(w)$ can be rewritten as:

$$g_\mu(w) = \max_{v \in \mathcal{B}_\infty} \sum_{i=1}^d \left(w[i]v[i] - \frac{\mu}{2} v[i]^2 \right), \quad \text{where } \|v\|_\infty \leq 1$$

Since to find the v that maximizes the above expression, we can independently find each $v[i]$ that maximizes the component in the summation, so we can further define:

$$h_i(w[i]) = \max_{|v[i]| \leq 1} \left(w[i]v[i] - \frac{\mu}{2} v[i]^2 \right)$$

Then the original $g_\mu(w)$ can be rewritten as:

$$g_\mu(w) = \sum_{i=1}^d h_i(w[i])$$

Now we can prove the differentiability of $g_\mu(w)$ by proving the differentiability of each $h_i(w[i])$. Let:

$$f_{w[i]}(v[i]) = w[i]v[i] - \frac{\mu}{2} v[i]^2$$

Since $w[i]v[i]$ is linear in $v[i]$, and the quadratic term $-\frac{\mu}{2}v[i]^2 < 0$ (for $\mu > 0$), $f_{w[i]}(v[i])$ is concave in $v[i]$, which means that exists a unique $v^*[i]$ that maximizes $f_{w[i]}(v[i])$, and we have:

$$\frac{d}{dv[i]} f_{w[i]}(v[i]) = w[i] - \mu v[i] = 0 \iff v[i]^* = \frac{w[i]}{\mu}$$

Thus, if we do not restrict the solution to be in the unit ball, the v that maximizes $\langle w, v \rangle - \frac{\mu}{2} \|v\|_2^2$ is:

$$v^* = \begin{bmatrix} \frac{w[1]}{\mu} \\ \vdots \\ \frac{w[d]}{\mu} \end{bmatrix} = \begin{bmatrix} v[1] \\ \vdots \\ v[d] \end{bmatrix}$$

To further impose the restriction that $\max_{i=1,\dots,d} |v[i]| \leq 1$, the optimal v need to satisfy:

$$v[i] \in [-1, 1]$$

Thus, we need to project $v[i]$ to the interval $[-1, 1]$, by the following definition of Euclidean projection ⁴:

- The Euclidean projection of x_0 on a rectangle $C = \{x \mid l \preceq x \preceq u\}$ (where $l \prec u$) is given by

$$P_C(x_0)_k = \begin{cases} l_k & x_{0k} \leq l_k \\ x_{0k} & l_k \leq x_{0k} \leq u_k \\ u_k & x_{0k} \geq u_k \end{cases}$$

We have:

$$\text{proj}_{[-1,1]}(v[i]) = \begin{cases} -1 & \text{if } v[i] < -1 \\ v[i] & \text{if } -1 \leq v[i] \leq 1 \\ 1 & \text{if } v[i] > 1 \end{cases}$$

or equivalently:

$$\text{proj}_{[-1,1]} \left(\frac{w[i]}{\mu} \right) = v^*(w[i]) = \begin{cases} -1 & \text{if } w[i] < -\mu \\ \frac{w[i]}{\mu} & \text{if } |w[i]| \leq \mu \\ 1 & \text{if } w[i] > \mu \end{cases} \quad (1)$$

⁴S. Boyd, *Convex Optimization*, 1st ed., Cambridge University Press, Cambridge, UK, 2004, p. 399.

And this matches the given $\nabla g_\mu(w)[i]$.

Then getting back to the part of proving differentiability, we have $h_i(w[i])$:

$$\begin{aligned}
h_i(w[i]) &= \max_{|v[i]| \leq 1} \left(w[i]v[i] - \frac{\mu}{2}v[i]^2 \right) \\
&= \max_{|v[i]| \leq 1} (f_{w[i]}(v[i])) \\
&= f_{w[i]}(v^\star(w[i])) \\
&= w[i]v^\star(w[i]) - \frac{\mu}{2}(v^\star(w[i]))^2
\end{aligned} \tag{2}$$

Consider the three cases of $\text{proj}_{[-1,1]}(v[i])$ in (1):

- Case 1: $w[i] < -\mu$

Then $v^\star(w[i]) = -1$, and by plugging it into (2):

$$\begin{aligned}
h_i(w[i]) &= w[i](-1) - \frac{\mu}{2}(-1)^2 = -w[i] - \frac{\mu}{2} \\
\rightarrow h'_i(w[i]) &= \frac{d}{dw[i]} \left(-w[i] - \frac{\mu}{2} \right) = -1
\end{aligned}$$

- Case 2: $-\mu \leq w[i] \leq \mu$

Then $v^\star(w[i]) = \frac{w[i]}{\mu}$, and by plugging it into (2):

$$\begin{aligned}
h_i(w[i]) &= w[i] \frac{w[i]}{\mu} - \frac{\mu}{2} \left(\frac{w[i]}{\mu} \right)^2 = \frac{w[i]^2}{\mu} - \frac{\mu}{2} \frac{w[i]^2}{\mu^2} = \frac{w[i]^2}{2\mu} \\
\rightarrow h'_i(w[i]) &= \frac{d}{dw[i]} \left(\frac{w[i]^2}{2\mu} \right) = \frac{w[i]}{\mu}
\end{aligned}$$

- Case 3: $w[i] > \mu$

Then $v^\star(w[i]) = 1$, and by plugging it into (2):

$$\begin{aligned}
h_i(w[i]) &= w[i](1) - \frac{\mu}{2}(1)^2 = w[i] - \frac{\mu}{2} \\
\rightarrow h'_i(w[i]) &= \frac{d}{dw[i]} \left(w[i] - \frac{\mu}{2} \right) = 1
\end{aligned}$$

Thus, at the boundaries:

- $w[i] = \mu$

Left derivative:

$$\lim_{w[i] \rightarrow \mu^-} h'_i(w[i]) = \lim_{w[i] \rightarrow \mu^-} \frac{w[i]}{\mu} = \frac{\mu}{\mu} = 1$$

Right derivative:

$$\lim_{w[i] \rightarrow \mu^+} h'_i(w[i]) = 1$$

- $w[i] = -\mu$

Left derivative:

$$\lim_{w[i] \rightarrow -\mu^-} h'_i(w[i]) = -1$$

Right derivative:

$$\lim_{w[i] \rightarrow -\mu^+} h'_i(w[i]) = \lim_{w[i] \rightarrow -\mu^+} \frac{w[i]}{\mu} = \frac{-\mu}{\mu} = -1$$

And in the interior:

$$\begin{aligned} h'_i(w[i]) &= w[i] \frac{w[i]}{\mu} - \frac{\mu}{2} \left(\frac{w[i]}{\mu} \right)^2 \\ &= \frac{w[i]^2}{\mu} - \frac{w[i]^2}{2\mu} \\ &= \frac{w[i]^2}{2\mu} \end{aligned}$$

Which always exists and is unique.

Therefore, $h_i(w[i])$ is differentiable, and $g_\mu(w) = \sum_{i=1}^d h_i(w[i])$ is a sum of differentiable functions, so $g_\mu(w)$ is also differentiable. ■

7

We need to further prove that g_μ is $\frac{1}{\mu}$ -smooth.

Solution. By the definition in the following image ⁵:

Definition 2. Differentiable² $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is L -smooth if and only for all $x, y \in \mathbb{R}^n$ we have that

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq L \cdot \|x - y\|_2$$

Since we have already proved that $g_\mu(w)$ is differentiable, proving the following claim is equivalent to proving that $g_\mu(w)$ is $\frac{1}{\mu}$ -smooth.

Claim:

$$\|\nabla g_\mu(w_1) - \nabla g_\mu(w_2)\|_2 \leq \frac{1}{\mu} \|w_1 - w_2\|_2, \quad \forall w_1, w_2 \in \mathbb{R}^d$$

Proof: Following the notation in the previous subproblem, we have:

$$\nabla g_\mu(w) = \begin{bmatrix} \nabla g_\mu(w)[1] \\ \vdots \\ \nabla g_\mu(w)[d] \end{bmatrix}$$

Let:

$$w_1 = \begin{bmatrix} w_1[1] \\ \vdots \\ w_1[d] \end{bmatrix}, \quad w_2 = \begin{bmatrix} w_2[1] \\ \vdots \\ w_2[d] \end{bmatrix}$$

Then we have:

$$\nabla g_\mu(w_1) - \nabla g_\mu(w_2) = \begin{bmatrix} \nabla g_\mu(w_1)[1] - \nabla g_\mu(w_2)[1] \\ \vdots \\ \nabla g_\mu(w_1)[d] - \nabla g_\mu(w_2)[d] \end{bmatrix}$$

The maximum of $\|\nabla g_\mu(w_1) - \nabla g_\mu(w_2)\|_2$ happens when:

$$\nabla g_\mu(w_1) = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \quad \text{and} \quad \nabla g_\mu(w_2) = \begin{bmatrix} -1 \\ \vdots \\ -1 \end{bmatrix}$$

⁵A. Sidford, *MS&E 213 / CS 269O: Chapter 2 — Smooth Functions*, Stanford University, Oct. 17, 2020. Available at: https://web.stanford.edu/~sidford/courses/20fa_opt_theory/sidford_mse213_2020fa_chap_2_smoothness.pdf, p. 2.

which implies that:

$$w_1[i] \geq \mu, w_2[i] \leq -\mu \quad \forall i = 1, \dots, d \quad (1)$$

and we'll have:

$$\|\nabla g_\mu(w_1) - \nabla g_\mu(w_2)\|_2 = \left\| \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ \vdots \\ -1 \end{bmatrix} \right\|_2 = 2\sqrt{d}$$

Under the condition in (1), if we want to find the minimum of $L\|w_1 - w_2\|_2$, we can set $w_1[i] = \mu$ and $w_2[i] = -\mu$ for all $i = 1, \dots, d$, and we'll have:

$$L\|w_1 - w_2\|_2 = L \left\| \begin{bmatrix} \mu \\ \vdots \\ \mu \end{bmatrix} - \begin{bmatrix} -\mu \\ \vdots \\ -\mu \end{bmatrix} \right\|_2 = L\sqrt{4\mu^2 d} = L2\mu\sqrt{d}$$

Thus, we can set $L = \frac{1}{\mu}$, and we'll have:

$$\|\nabla g_\mu(w_1) - \nabla g_\mu(w_2)\|_2 = 2\sqrt{d} \leq 2\sqrt{d} = \frac{1}{\mu} 2\mu\sqrt{d} = \frac{1}{\mu} \|w_1 - w_2\|_2$$

■

8

We're asked to show that:

$$g_\mu(w) \leq g(w) \leq g_\mu(w) + \frac{\mu d}{2}$$

Solution. Since $g(w)$ is defined as $\|w\|_1$, it is equivalent to show that:

$$g_\mu(w) \leq \|w\|_1 = \sum_{i=1}^d |w[i]| \leq g_\mu(w) + \frac{\mu d}{2}$$

■