目录

2018 年高数下期中

一、单选题

1． C

解析：曲面方程 $F(x,y,z)=\sin x\sin y\sin(x+y)-z$　　$\therefore$ 任意一点法向量 $\vec{n}=(F_x,F_y,F_z)$

$F_x=\sin y\left[\sin x\cos(x+y)+\cos x\sin(x+y)\right]$　　　$F_y=\sin x\left[\sin y\cos(x+y)+\cos y\sin(x+y)\right]$

$F_{z=}=-1$　　$\therefore \vec{n}=(F_x,F_y,F_z)|_{(\frac{\pi}{6},\frac{\pi}{3},\frac{\sqrt{3}}{4})}=(\frac{3}{4},\frac{1}{4},-1)$　　　$\vec{z}=(0,0,1)$

$\cos\theta=\dfrac{\vec{n}\cdot\vec{z}}{|\vec{n}|\cdot|\vec{z}|}=\dfrac{-2\sqrt{2}}{\sqrt{13}}$　　$\therefore \sin\theta=\sqrt{1-\cos^2\theta}=\dfrac{\sqrt{65}}{13}$

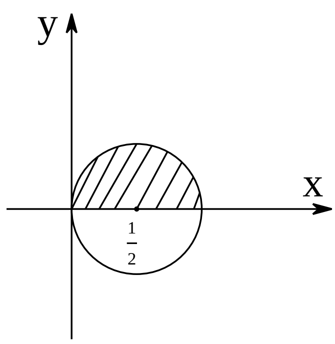
2． C

解析：由中值定理知 D 内存在一点 (x_0,y_0) 使 $\iint\limits_D e^{x^2-y^2}\cos(x+y)dxdy=\pi r^2 e^{{x_0}^2-{y_0}^2}\cos(x_0+y_0)$

$\therefore$ 原式 $=\lim\limits_{r\to 0}e^{x_0^2-y_0^2}\cos(x_0+y_0)=1$

3． D

解析：$\begin{cases}\dfrac{y}{x}=\tan\theta\\ \sqrt{x^2+y^2}=\cos\theta\end{cases}$ $\Rightarrow$ 积分域边界 $x^2+y^2=x$　　结合图：



$(x-\frac{1}{2})^2+y^2=\frac{1}{4}$　　$(x-\frac{1}{2})^2+y^2=\frac{1}{4}$

$\therefore$ 也可以写为 $\int_0^1 dx\int_0^{\sqrt{x-x^2}} f(x,y)dy$

4． B

解析：$f(x,1)=e^{x+1}(x-1)^{\frac{2}{3}}$　　$f_x(x,1)=e^{x+1}\left[(x-1)^{\frac{2}{3}}+\frac{2}{3}(x-1)^{-\frac{1}{3}}\right]$　　$\therefore f_x(0,1)=\frac{1}{3}e$

$f(0,y)=e^y y^{\frac{1}{3}}$　　$f_y(0,y)=e^y\left[\frac{1}{3}y^{-\frac{2}{3}}+y^{\frac{1}{3}}\right]$　　$f_y(0,1)=\frac{4}{3}e$

5． B

解析：曲线在 t 的切线向量 $\vec{a}=(1,-2t,3t^2)$　　平面法向量 $\vec{n}=(1,2,1)$

切平面平行于平面 $\vec{a}\cdot\vec{n}=0$　　$\therefore 1-4t+3t^2=0$　　$\therefore t=\frac{1}{3}$ 或 1，有 2 解

二、填空题

1． $(\frac{2}{9},\frac{4}{9},-\frac{4}{9})$

解析：$gradu=(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z})=(\frac{2x}{x^2+y^2+z^2},\frac{2y}{x^2+y^2+z^2},\frac{2z}{x^2+y^2+z^2})$

代入$M(1,2,-2)$ $\quad\therefore gradu|_M=(\frac{2}{9},\frac{4}{9},-\frac{4}{9})$

2． $\frac{1}{2(\mathrm{x}+1)\sqrt{x}}$

解析： $f(x,1)=\arctan\sqrt{x}$，求导可得 $f_x(x,1)=\frac{\frac{1}{2\sqrt{x}}}{1+x}=\frac{1}{2(1+x)\sqrt{x}}$

3． 0

解析： $F(x,y,z)=\frac{x}{z}-\ln(\frac{z}{x})=0$ 隐函数由 $F(x,y,z)=0$ 确定

$F_x=\frac{1}{z}-\frac{x}{z}\cdot(-\frac{z}{x^2})=\frac{z+x}{zx}$ $\qquad F_z=-\frac{x}{z^2}-\frac{x}{z}\cdot\frac{1}{x}=-\frac{z+x}{z^2}$ $\qquad\therefore\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{z}{x}$

$\therefore\frac{\partial^2 z}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{z}{x}\right)=\frac{\frac{\partial z}{\partial x}x-1\cdot z}{x^2}=\frac{\frac{z}{x}x-z}{x^2}=0$ $\qquad\therefore\frac{\partial^2 z}{\partial x^2}=0$

4． $2\sqrt{5}$

解析： $gradu|_M=(0,2z,2y-2z)|_{(2,-1,1)}=(0,2,-4)$ $\qquad$ 取方向 $\vec{\alpha}=(0,\frac{1}{\sqrt{5}},-\frac{2}{\sqrt{5}})$

$\therefore$ 方向导数 $MAX=gradu|_M\cdot\vec{\alpha}=2\sqrt{5}$

5． $x+2y-z=2$

解析：椭球面 $F(x,y,z)=x^2+2y^2+z^2-1=0$ $\qquad$ 每点法向量 $\vec{n}=(F_x,F_y,F_z)=(2x,4y,2z)$

$\therefore\vec{n}|_{(\frac{1}{2},\frac{1}{2},-\frac{1}{2})}=(1,2,-1)$ $\qquad$ 又 $\vec{n}=(1,2,-1)$ 为切平面的法向量

故 $(x-\frac{1}{2})+2(y-\frac{1}{2})-(z+\frac{1}{2})=0$ $\qquad$ 化简后得： $x+2y-z=2$

三、解答题

1． $\frac{\partial z}{\partial x}=f_1e^{x+y}+f_2\frac{1}{y}$ $\qquad\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}(\frac{\partial z}{\partial x})=\frac{\partial}{\partial y}(f_1e^{x+y}+f_2\frac{1}{y})$

$\frac{\partial}{\partial y}(f_1e^{x+y})=f_1e^{x+y}+e^{x+y}(f_{11}e^{x+y}+f_{12}(-\frac{x}{y^2}))=e^{x+y}\left(f_1+f_{11}e^{x+y}-\frac{x}{y^2}f_{12}\right)\cdots\cdots$（1）

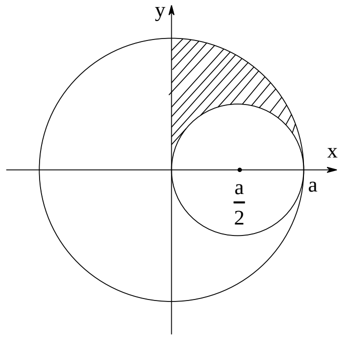
$\frac{\partial}{\partial y}(f_2\frac{1}{y})=f_2(-\frac{1}{y^2})+\frac{1}{y}(f_{21}e^{x+y}+f_{22}(-\frac{x}{y^2}))=-\frac{1}{y^2}f_2+\frac{1}{y}f_{21}e^{x+y}-\frac{x}{y^3}f_{22}\cdots\cdots$（2）

合并（1）（2）可得 $\frac{\partial^2 z}{\partial x\partial y}$

经整合 $\frac{\partial z}{\partial x}=f_1e^{x+y}+f_2\frac{1}{y}$

$$\frac{\partial^2 z}{\partial x\partial y}=f_1e^{x+y}-\frac{1}{y^2}f_2+e^{2(x+y)}f_{11}+e^{x+y}\frac{y-x}{y^2}f_{21}-\frac{x}{y^3}f_{22}$$

2. $\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta\end{cases}$ $\therefore dxdy=\rho d\rho d\theta$ 图：



$$\therefore\iint\limits_D(1-\sqrt{x^2+y^2})dxdy=\iint\limits_D(1-\rho)\rho d\rho d\theta=\int_0^{\frac{\pi}{2}}d\theta\int_{a\cos\theta}^{a}(1-\rho)\rho d\rho=\frac{3\pi}{8}a^2-\frac{\pi}{6}a^3+\frac{2}{9}a^3$$

3. 偏导数存在问题：$f(0,0)=0$ $f(x,0)=x^2\sin\frac{1}{x^2}$

$$f_x(0,0)=\lim_{x\to0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{x\to0}\frac{\Delta x^2\sin\frac{1}{\Delta x^2}}{\Delta x}=0$$ 同理 $f_y(0,0)=0$

偏导数连续问题：$f(0,0)=x^2\sin\frac{1}{x^2}$ $f_x(x,0)=2x\sin\frac{1}{x^2}-\frac{2}{x}\cos\frac{1}{x^2}$ $\lim\limits_{x\to0}f_x(0,0)$ 不存在

同理可得 f_y 情况 $\therefore f_x$，f_x 在（0, 0）处不连续

可微与否问题：$df=\lim\limits_{\substack{\Delta x\to0\\ \Delta y\to0}}\left[f(\Delta x,\Delta y)-f(0,0)\right]=\lim\limits_{\substack{\Delta x\to0\\ \Delta y\to0}}\left[(\Delta x)^2+(\Delta y)^2\right]\sin\frac{1}{(\Delta x)^2+(\Delta y)^2}$

$$=0\Delta x+0\Delta y+\left[(\Delta x)^2+(\Delta y)^2\right]\sin\frac{1}{(\Delta x)^2+(\Delta y)^2}$$

$\therefore$ 只要证明余项是否是 $\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$ 的高阶无穷小

而 $\lim\limits_{\substack{\Delta x\to0\\ \Delta y\to0}}\frac{\left[(\Delta x)^2+(\Delta y)^2\right]\sin\frac{1}{(\Delta x)^2+(\Delta y)^2}}{\rho}=0$ $\therefore$ 是 ρ 的高阶无穷小

$\therefore f(x,y)$ 在（0, 0）处可微，$df=0\Delta x+0\Delta y+o(\rho)$

4. 设切线的方向向量为 $\vec{n}$，则它与 2 个面在该点的法向量皆垂直

平面法向量 $\vec{\alpha}\,|_{(1,1,1)}=(2,-3,5)$ 球面法向量 $\vec{\beta}\,|_{(1,1,1)}=(2x,2y,2z-3)=(2,2,-1)$

$\begin{cases}\vec{n}\cdot\vec{\alpha}=0\\ \vec{n}\cdot\vec{\beta}=0\end{cases}$ $\therefore$ 可解得 $\vec{n}=(7,-12,-10)$ $\therefore$ 切线方程 $\frac{x-1}{7}=-\frac{y-1}{12}=-\frac{z-1}{10}$

法平面方程 $7(x-1)-12(y-1)-10(z-1)=0$ 得 $7x-12y-10z+15=0$

5. $y_{AB}=-\frac{1}{3}x+\frac{10}{3}$ c 点范围在 $x\geq0,y\geq0$，ΔABC 面积最大化，则一定取两端

若是过端点（0, 2），则 $y=-\frac{1}{3}x+2$，$S_{ABC}=\frac{2}{3}|AB|$

$\therefore$ 取端点（3, 0），$\therefore y=-\frac{1}{3}x+1$　　$S_{ABC}=\frac{7}{6}|AB|$　　$\therefore$ 可求得点 C 是（3, 0）时，ΔABC 面积最大化

6.　$f(0)=e^0+0=1$　　$f(t)=e^{4\pi t^2}+2\pi\int_0^{2t}f(\frac{1}{2}\rho)\rho d\rho$，两边求导得 $f'(t)=8\pi te^{4\pi t^2}+8\pi tf(t)$

先解齐次：　$f'(t)=8\pi tf(t)$，即 $\frac{df}{dt}=8\pi tf(t)$　　即 $\frac{1}{f}df=8\pi tdt$　　积分得 $\ln f=4\pi t^2+C$

$\therefore f(t)=Ce^{4\pi t^2}$　　又 $\because$ 是非齐次方程　　令 $f(t)=h(t)e^{4\pi t^2}$　　$f'(t)=8\pi te^{4\pi t^2}h(t)+h'(t)e^{4\pi t^2}$ 代入

$\therefore h'(t)e^{4\pi t^2}=8\pi te^{4\pi t^2}$　　$\therefore h'(t)=8\pi t$　　$\therefore h(t)=4\pi t^2+C$　　$\therefore f(t)=(4\pi t^2+C)e^{4\pi t^2}$

又 $f(0)=1$ 解得 C=1　　$\therefore f(t)=(4\pi t^2+1)e^{4\pi t^2}$

2017 年高数下期中

一、单选题

1.　D

解析：对于一元函数，可导 $\Rightarrow$ 连续；对于二元函数，在一点处可偏导并不能保证 f 在此点连续

2.　C

解析：$gradf|_M=(1,-1)$　　沿着梯度方向，方向导数最大，该处增长最快

3.　D

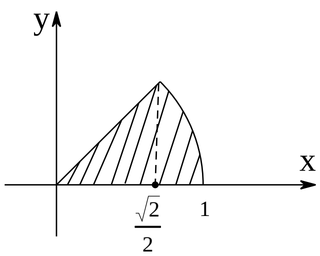
解析：$\therefore \lim\limits_{\substack{\Delta x\to0\\ \Delta y\to0}}f(\Delta x,\Delta y)=\lim\limits_{\substack{\Delta x\to0\\ \Delta y\to0}}\sqrt{|\Delta x\Delta y|}=0=f(x,y)\Big|_{x=0,y=0}$　　$\therefore$ 连续

$\because f_x(0,0)=\lim\limits_{\Delta x\to0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0$　　$f_y(0,0)=\lim\limits_{\Delta y\to0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=0$　　$\therefore$ 偏导数皆存在

4.　C

解析：由三角形式可得到积分区域如图：

$$I=\int_0^{\frac{\sqrt2}{2}}dy\int_y^{\sqrt{1-y^2}}f(x,y)dx$$



5.　D

解析：$A=\iint\limits_D\frac{a\sqrt{f(x)}+b\sqrt{f(y)}}{\sqrt{f(x)}+\sqrt{f(y)}}d\delta$　　由对称性 $A=\iint\limits_D\frac{a\sqrt{f(y)}+b\sqrt{f(x)}}{\sqrt{f(x)}+\sqrt{f(y)}}d\delta$

$\therefore 2A=\iint\limits_D(a+b)d\delta=(a+b)\times\frac{1}{4}\times\pi\times4=(a+b)\pi$　　$\therefore A=\frac{1}{2}(a+b)\pi$

二、填空题

1.　$dz|_{(1,0,-1)}=dx-\sqrt{2}dy$

解析：$F(x,y,z)=xyz+\sqrt{x^2+y^2+z^2}-\sqrt{2}=0$

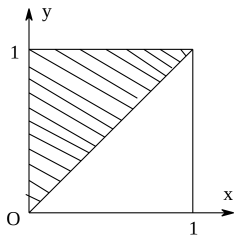
$F_x=yz+\dfrac{x}{\sqrt{x^2+y^2+z^2}}$ $\quad$ $F_y=xz+\dfrac{y}{\sqrt{x^2+y^2+z^2}}$ $\quad$ $F_z=xy+\dfrac{z}{\sqrt{x^2+y^2+z^2}}$

$\dfrac{\partial z}{\partial x}|_M=-\dfrac{F_x}{F_z}|_M=1$ $\quad$ $\dfrac{\partial z}{\partial y}|_M=-\dfrac{F_y}{F_z}|_M=-\sqrt{2}$ $\quad$ $\therefore dz|_{(1,0-1)}=dx-\sqrt{2}dy$

2. （$\dfrac{2}{9},\dfrac{4}{9},-\dfrac{4}{9}$）

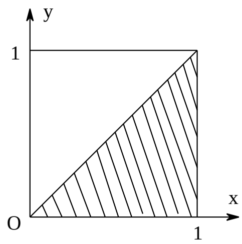
解析：同 2018 年填空题第一题

3. $\dfrac{1}{2}A^2$



解析：利用对称性 $I=\int_0^1 dx\int_x^1 f(x)dy$ 积分区间如右图所示

利用对称性，积分与标识无关，故可以互换 x,y



$I=\int_0^1 dy\int_y^1 f(y)dx$ 积分区间如右图所示

$\therefore 2I=\int_0^1\int_0^1 f(x)f(y)dxdy=\int_0^1 dyf(y)\int_0^1 f(x)dx=A^2$ $\quad$ $\therefore I=\dfrac{1}{2}A^2$

4. $-\pi$

解析：$\lim\limits_{c\to 0}\iint\limits_{c\le x^2+y^2\le 1}\ln(x^2+y^2)dxdy=\lim\limits_{c\to 0}\int_0^{2\pi}\int_{\sqrt{c}}^1\ln\rho^2\rho d\rho d\theta=\lim\limits_{c\to 0}2\pi\int_{\sqrt{c}}^1\left(\ln\rho^2\right)\rho d\rho=\lim\limits_{c\to 0}\pi(\rho^2\ln\rho^2-\rho^2)|_{\sqrt{c}}^1$

$=\lim\limits_{c\to 0}\pi\left[-1-c\ln c+c\right]=-\pi$

5. $\dfrac{xF_u}{zF_v}$

解析：$F_x=F_u\cdot 2x=2xF_u$ $\quad$ $F_z=F_v(-2z)=-2zF_z$ $\quad$ $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}=\dfrac{xF_u}{zF_v}$

三、计算题

1. 曲线在 t 处的切线向量 $\vec{\alpha}=(-\sin t,\cos t,\dfrac{\frac{1}{2}}{\cos^2\frac{t}{2}})$ $\quad$ $\vec{\alpha}|_{(0,1,1)}=\vec{\alpha}|_{t=\frac{\pi}{2}}=(-1,0,1)$

此也是法平面的法向量 $\quad$ $\therefore -1(x-0)+0+(z-1)=0$ $\quad$ $\therefore$法平面方程是 $x-z+1=0$

2. $F(x,y,z)=z-e^z+2xy-3=0$ $\quad$ $F_x=2y$ $\quad$ $F_y=2x$ $\quad$ $F_x=1-e^z$

曲面在（1, 2, 0）处的法向量 $\vec{n}=(F_x,F_y,F_z)|_{(1,2,0)}=(4,2,0)$ $\quad$ $\vec{n}$ 为切平面的法向量，亦为法线方向向量

$\therefore 4(x-1)+2(y-2)=0$ 即 $2x+y-4=0$ $\quad$ 切线方程 $\dfrac{x-1}{4}=\dfrac{y-2}{2}=\dfrac{z}{0}$

3. $\frac{\partial z}{\partial x}=2xf+x^2\frac{\partial f}{\partial x}$ $\frac{\partial f}{\partial x}=f_1-\frac{y^2}{x^2}f_2$……（1） 代入（1）得 $\frac{\partial z}{\partial x}=2xf+x^2f_1-y^2f_2$

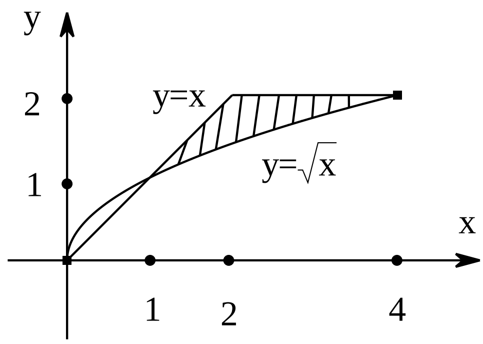
$$\frac{\partial^2 z}{\partial x}=\frac{\partial}{\partial x}(\frac{\partial z}{\partial x})=2f+2x\frac{\partial f}{\partial x}+2xf_1+x^2\frac{\partial f_1}{\partial x}-y^2\frac{\partial f_2}{\partial x}$$

$\frac{\partial f_1}{\partial x}=f_{11}-\frac{y^2}{x^2}f_{12}$……（2） $\frac{\partial f_2}{\partial x}=f_{21}-\frac{y^2}{x^2}f_{22}$……（3）

代入（1）（2）（3）：$\frac{\partial^2 z}{\partial x}=2f+2xf_1-\frac{2y^2}{x}f_2+2xf_1+x^2f_{11}-y^2f_{12}-y^2f_{21}+\frac{y^4}{x^2}f_{22}$

又∵ f 具有连续二阶偏导，化简上式得：$\frac{\partial^2 z}{\partial x}=2f+4xf_1-\frac{2y^2}{x}f_2+x^2f_{11}-2y^2f_{12}+\frac{y^4}{x^2}f_{22}$

4. 画出积分区域，交换积分次序：



$$I=\int_1^2 dy\int_y^{y^2}\sin\frac{\pi x}{2y}dx=\int_1^2-\frac{2y}{\pi}\cos(\frac{\pi y}{2})dy=\frac{4(\pi+2)}{\pi^3}$$

5. $I=\iint_D xdxdy=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a\cos\theta}\rho\cos\theta\cdot\rho d\rho d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\int_0^{a\cos\theta}\rho^2 d\rho d\theta$

$$=\frac{1}{3}a^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta d\theta=\frac{a^3\pi}{8}$$

6. $\vec{\alpha}=(2,0,0)$ 设该选定点为 M $gradf|_M=(2x,2y,2z)$

方向导数最大 $\Rightarrow gradf|_M \mathbin{//} \vec{\alpha}$ $\therefore y=z=0$ 代入球面 $\therefore x=\frac{\sqrt{2}}{2}\Rightarrow$ 该点为 $(\frac{\sqrt{2}}{2},0,0)$

7. 见 2018 年大题第 3 题

2016 年高数下期中

一、选择题

1. A

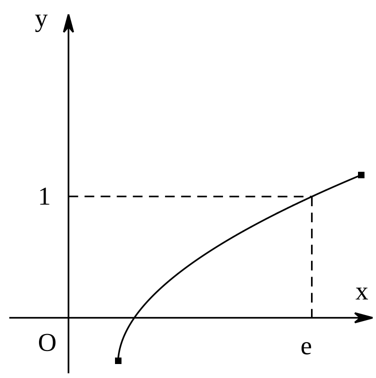
解析：$f(x,x^2)=x^3$ 对两边的 x 求导

$\therefore f_1+2xf_2=3x^2$ $f_1=x^2-2x^4$

$\therefore 2xf_2=3x^2-x^2+2x^4=2x^2+2x^4$ $\therefore f_2=x+x^3$

2. D

解析：如图



3. C

解析：A 中 $df(x,y)=3dx+dy$；B 中法向量为 $(3,1,-1)$

4. B

解析：$(0,0)$ 处有定义，为 2　又 $\lim\limits_{\substack{x\to0\\y\to0}}\frac{\sin 2(x^2+y^2)}{x^2+y^2}=2$　$\therefore$ 连续

二、填空题

1. $z=0$

解析：曲面 $F(x,y,z)=\sin xy+\sin yz+\sin xz-1$　$F_x=y\cos xy+z\cos xz$　$F_y=x\cos xy+z\cos yz$

$F_z=y\cos yz+x\cos xz$　切平面法向量 $\vec{n}=(F_x,F_y,F_z)|_{(1,\frac{\pi}{2},0)}=(0,0,\frac{\pi}{2}+1)$　$\therefore(\frac{\pi}{2}+1)(z-0)=0$　$\therefore z=0$

2. $(\frac{2}{9},\frac{4}{9},-\frac{4}{9})$

解析：同 2018 年填空题第 1 题

3. $-\frac{11}{5}$

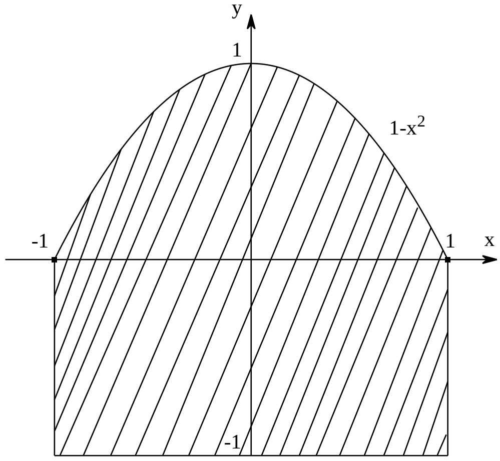
解析：$gradf|_{(1,-1)}=(3,-5)$　$\therefore k=(3,-5)\cdot(\frac{3}{5},\frac{4}{5})=\frac{9}{5}-\frac{20}{5}=-\frac{11}{5}$

4. $\frac{1}{2}A^2$

解析：同 2017 年填空题第三题

5. $I=\int_{-1}^{0}dy\int_{-1}^{1}f(x,y)dx+\int_{0}^{1}dy\int_{-\sqrt{1-y}}^{\sqrt{1-y}}f(x,y)dx$

解析：如图



三、计算题

1. $\frac{\partial z}{\partial y}=f_1e^{x+y}-\frac{x}{y^2}f_2$

$$\frac{\partial^2 z}{\partial x\partial y}=f_1e^{x+y}+e^{x+y}(f_{11}e^{x+y}+f_{12}\frac{1}{y})-\frac{1}{y^2}f_2-\frac{x}{y^2}(f_{21}e^{x+y}+f_{22}\frac{1}{y})$$

2. 同 2018 年解答题第 2 题

3. 对于 (x_0,y_0,z_0) 来说切平面方程为：$\frac{1}{2\sqrt{x_0}}(x-x_0)+\frac{1}{2\sqrt{y_0}}(y-y_0)+\frac{1}{2\sqrt{z_0}}(z-z_0)=0$

$\frac{x}{\sqrt{x_0}}+\frac{y}{\sqrt{y_0}}+\frac{z}{\sqrt{z_0}}=1$　$\therefore$ 截距为 $\sqrt{x_0},\sqrt{y_0},\sqrt{z_0}$，截距之积为 $\sqrt{x_0y_0z_0}$ 最大 $\Rightarrow x_0y_0z_0$ 最大

又 $\because$ 满足条件 $\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}=1$　$\therefore L(x_0,y_0,z_0,\pi)=x_0y_0z_0+\pi(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}-1)$

又$\because$取最大值 $\therefore\dfrac{\partial L}{\partial x_0}=0$ $\dfrac{\partial L}{\partial y_0}=0$ $\dfrac{\partial L}{\partial z_0}=0$ $\dfrac{\partial L}{\partial \pi}=0$ $\therefore x_0=y_0=z_0=\dfrac{1}{9}$

$\therefore$取点$(\dfrac{1}{9},\dfrac{1}{9},\dfrac{1}{9})$ 切平面方程为$x+y+z=\dfrac{1}{3}$

4. $\dfrac{\partial z}{\partial x}=\dfrac{x}{\sqrt{1-x^2+y}\sqrt{x^2-y}}$ $\dfrac{\partial z}{\partial y}=\dfrac{-\frac{1}{2}}{\sqrt{1-x^2+y}\sqrt{x^2-y}}$ $\therefore$全微分$dz=\dfrac{1}{\sqrt{1-x^2+y}}(\dfrac{xdx-\frac{1}{2}dy}{\sqrt{x^2-y}})$

5. 两个方程取全微分，并带入$(1,1,1)$ $\begin{cases}-dx+2dy+2dz=0\\2dx-3dy+5dz=0\end{cases}$ $\therefore$可以得一解$(dx,dy,dz)=(16,9,-1)$

$\therefore$切线方程为$\dfrac{x-1}{16}=\dfrac{y-1}{9}=\dfrac{z-1}{-1}$ 法平面方程为$16(x-1)+9(y-1)-(z-1)=0$

6. $F(x,y,z)=\dfrac{x}{z}-\ln\dfrac{z}{y}$ $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}=\dfrac{z}{z+x}$ $\therefore\dfrac{\partial^2 z}{\partial x^2}=\dfrac{(z+x)\frac{\partial z}{\partial x}-z(\frac{\partial z}{\partial x}+1)}{(z+x)^2}=-\dfrac{z^2}{(z+x)^3}$

四、综合题

1. $\begin{cases}dx=vdu+udv\\dy=udu-vdv\end{cases}\Rightarrow\begin{cases}du=\dfrac{vdx+udy}{u^2+v^2}\\dv=\dfrac{udx-vdy}{u^2+v^2}\end{cases}$ $4=(\dfrac{\partial f}{\partial x})^2+(\dfrac{\partial f}{\partial y})^2=\dfrac{1}{u^2+v^2}[(\dfrac{\partial g}{\partial u})^2+(\dfrac{\partial g}{\partial v})^2]$

$\left.\begin{array}{l}\dfrac{1}{4}[(\dfrac{\partial g}{\partial u})^2+(\dfrac{\partial g}{\partial v})^2]=u^2+v^2\\a(\dfrac{\partial g}{\partial u})^2-b(\dfrac{\partial g}{\partial v})^2=u^2+v^2\end{array}\right\}\Rightarrow\begin{cases}a=\dfrac{1}{4}\\b=-\dfrac{1}{4}\end{cases}$ $\therefore a=\dfrac{1}{4},b=-\dfrac{1}{4}$

2. 设直线 L 的方向向量为(l,m,n) $Fi(x,y,z)=0$在(x_0,y_0,z_0)的法向量为$(\dfrac{\partial Fi}{\partial x},\dfrac{\partial Fi}{\partial y},\dfrac{\partial Fi}{\partial z})|_{(x_0,y_0,z_0)}$

同时相切于(x_0,y_0,z_0)的 L $\begin{bmatrix}\dfrac{\partial F_1}{\partial x}&\dfrac{\partial F_1}{\partial y}&\dfrac{\partial F_1}{\partial z}\\\dfrac{\partial F_2}{\partial x}&\dfrac{\partial F_2}{\partial y}&\dfrac{\partial F_2}{\partial z}\\\dfrac{\partial F_3}{\partial x}&\dfrac{\partial F_3}{\partial y}&\dfrac{\partial F_3}{\partial z}\end{bmatrix}\begin{bmatrix}l\\m\\n\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ 又$\therefore(l,m,n)\neq(0,0,0)$

$\therefore$左边矩阵行列式为 0 $\therefore\dfrac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)}=0$

3. 同 2018 年解答题第 6 题

2015 年高数下期中

一、单选题

1. D

解析：$\frac{\partial f}{\partial x}=\frac{y(x^2+y^2)-2x\cdot xy}{(x^2+y^2)^2}=\frac{y(y^2-x^2)}{(x^2+y^2)^2}\Big|_{y=0}=0$ $\qquad$ $\frac{\partial f}{\partial y}\Big|_{x=0}=0$

按定义 $\lim\limits_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0$ $\qquad$ 故 x,y 偏导数存在

2. C

解析：$\left(\frac{\partial u}{\partial l}\right)_{\max}=\left|\nabla u\right|=\left\|\left(2x-2z,2y+2,-2x\right)\right\|=\left\|(-2,0,-2)\right\|=2\sqrt{2}$

3. C

解析：切平面法向量 $(-y,-x,2z)$ 与 $x-y+2z-1=0$ 的法向量平行 $\Rightarrow \frac{-y}{1}=\frac{-x}{-1}=\frac{2z}{2}$

代入 $z^2-xy=8\Rightarrow(2,-2,2)$

4. C

解析：$F(t)=\int_0^{\pi}d\theta\int_0^t f(\rho)\rho d\rho=\pi\int_0^t f(\rho)\rho d\rho$ $\qquad$ $F'(t)=\pi f(t)\cdot t$

5. C 和 D

解析：$f_x(x,y)=\frac{1+a}{2}\cdot\left(x^2+y^2\right)^{\frac{a-1}{2}}\cdot 2x$ $\qquad$ $f_y(x,y)=\frac{1+a}{2}\cdot\left(x^2+y^2\right)^{\frac{a-1}{2}}\cdot 2y$

$f_x(0,0)=\lim\limits_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{\Delta x^{1+a}-0}{\Delta x}=0$ $\qquad$ $f_y(0,0)=\lim\limits_{\Delta x\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim\limits_{\Delta x\to 0}\frac{\Delta y^{1+a}-0}{\Delta y}=0$

$\therefore df\big|_{(0,0)}=f_x(0,0)dx+f_y(0,0)dy=0$ 故 C 正确

$\lim\limits_{\substack{x\to 0\\ y\to 0}}f_x(x,y)=\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{1+a}{2}\cdot\left(x^2+y^2\right)^{\frac{a-1}{2}}\cdot 2x=0$ 且 $\lim\limits_{\substack{x\to 0\\ y\to 0}}f_y(x,y)=0$ $\qquad$ 故 $f_x(x,y)$， $f_y(x,y)$ 连续，D 正确

二、填空题

1. z

解析：$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\frac{x}{\sqrt{x^2+y^2}}+y\frac{y}{\sqrt{x^2+y^2}}=\sqrt{x^2+y^2}=z$

2. $yz\cdot x^{yz-1}dx+\ln x\cdot x^{yz}\cdot z+\ln x\cdot x^{yz}\cdot y$

解析：$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=yz\cdot x^{yz-1}dx+\ln x\cdot x^{yz}\cdot z+\ln x\cdot x^{yz}\cdot y$

3. $(3,-2,-1)$

解析：$(2x,2y,2z)\times(1,1,1)\big|_{(1,2,-1)}=(3,-2,-1)$

4. $\frac{3}{2}+\sqrt{2}$

解析：$\nabla u=\left(y^2-yz,2xy-xz,3z^2-xy\right)\Big|_{(1,-1,1)}=(2,1,2)$ $\nabla u\cdot(\frac{1}{2},\frac{1}{2},\frac{\sqrt{2}}{2})=1+\sqrt{2}$

5．$\int_{-1}^{0}dy\int_{-1}^{1}f(x,y)dx+\int_{0}^{1}dy\int_{-\sqrt{1-y}}^{\sqrt{1-y}}f(x,y)dx$

解析：同 2016 年填空题第 5 题

三、计算题

1．$\frac{\partial z}{\partial x}=2xf_1+ye^{xy}f_2-\frac{2xy}{g^2(x^2+y^2)}g'$

$$\frac{\partial^2 z}{\partial y\partial x}=2x\frac{\partial f_1}{\partial y}+\left(e^{xy}+xye^{xy}\right)f_2+ye^{xy}\frac{\partial f_2}{\partial y}-\frac{(2xg'+2xy\cdot 2y\cdot g'')g^2(x^2+y^2)-2g(x^2+y^2)\cdot 2y\cdot 2xyg'}{g^4(x^2+y^2)}$$

$\frac{\partial f_1}{\partial y}=-2yf_{11}+xe^{xy}f_{12}$ $\frac{\partial f_2}{\partial y}=-2yf_{21}+xe^{xy}f_{22}$

$$\frac{\partial^2 z}{\partial y\partial x}=-4xyf_{11}+2x^2e^{xy}f_{12}+(e^{xy}+xye^{xy})f_2-2y^2e^{xy}f_{21}+xye^{2xy}f_{22}$$

$$+\frac{2xg'g^2(x^2+y^2)+4xy^2g''g^2(x^2+y^2)-8xy^2g'g(x^2+y^2)}{g^4(x^2+y^2)}$$

综上可得 $\frac{\partial z}{\partial x}=2xf_1+ye^{xy}f_2-\frac{2xy}{g^2(x^2+y^2)}g'$

$$\frac{\partial^2 z}{\partial y\partial x}=e^{xy}(1+xy)f_2-4xyf_{11}+2e^{xy}(x^2-y^2)f_{12}+xye^{2xy}f_{22}+\frac{2xg'g^2+4xy^2g''g^2-8xy^2g'g}{g^4(x^2+y^2)}$$

2．$F_x=\frac{F_1}{z}$，$F_y=\frac{F_2}{z}$，$F_z=-\frac{xF_1}{z^2}-\frac{yF_2}{z^2}$

$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{zF_1}{xF_1+yF_2}$，$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{zF_2}{xF_1+yF_2}$ $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{z(xF_1+yF_2)}{xF_1+yF_2}=z$

3．$\iint\limits_D\ln(1+x^2+y^2)d\sigma=\int_0^{\frac{\pi}{2}}\int_0^2\ln(1+\rho^2)\rho d\rho d\theta=\frac{\pi}{2}\int_0^2\ln(1+\rho^2)\rho d\rho$ 令 $\rho^2=t$

则 $\frac{\pi}{4}\int_0^4\ln(1+t)dt=\frac{\pi}{4}\left[(1+t)\ln(1+t)-(t+1)\right]\Big|_0^4=\frac{\pi}{4}\left[5\ln 5-4\right]$

4．（1）$\frac{\partial u}{\partial x}=\frac{df}{dr}\cdot\frac{\partial r}{\partial x}=f'(r)\frac{x}{\sqrt{x^2+y^2+z^2}}=f'(r)\frac{x}{r}$

$$\frac{\partial^2 u}{\partial x^2}=\frac{f'(r)}{r}+x\frac{\partial\frac{f'(r)}{r}}{\partial x}=\frac{f'(r)}{r}+x\frac{d\frac{f'(r)}{r}}{dr}\cdot\frac{\partial r}{\partial x}=\frac{f'(r)}{r}+\frac{rf''(r)-f'(r)}{r^2}\frac{x}{r}\cdot x$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{3f'(r)}{r}+\frac{rf''(r)-f'(r)}{r^3}r^2=\frac{3f'(r)}{r}+f''(r)-\frac{f'(r)}{r}=f''(r)+\frac{2f'(r)}{r}$$

$\therefore \Delta u=f''(r)+\dfrac{2f'(r)}{r}$

（2） $\Delta u=0$，即 $f''(r)+\dfrac{2f'(r)}{r}=0$　　设 $f'(r)=p(r)$　　$\therefore \dfrac{2p(r)}{r}+\dfrac{dp}{dr}=0$

解得 $p=\dfrac{k}{r^2}$　　$\therefore f(r)=\dfrac{C_1}{r}+C_2$

5. $\dfrac{\partial f_1}{\partial x}=\sin x+x\cos x$　　$\dfrac{\partial f_1}{\partial y}=\dfrac{\partial f_1}{\partial z}=0$　　$\dfrac{\partial f_2}{\partial x}=0$　　$\dfrac{\partial f_2}{\partial y}=e^z$　　$\dfrac{\partial f_2}{\partial z}=ye^z$

$\dfrac{\partial f_3}{\partial x}=-z\sin(xz)$　　$\dfrac{\partial f_3}{\partial y}=0$　　$\dfrac{\partial f_3}{\partial z}=-x\sin(xz)$　　$\therefore$ Jacobi 矩阵：$J=\begin{bmatrix}\sin x+x\cos x & 0 & 0\\ 0 & e^z & ye^z\\ -z\sin(xz) & 0 & -x\sin(xz)\end{bmatrix}$

6. $\begin{cases}\dfrac{\partial f}{\partial x}=4x+6y=0\\ \dfrac{\partial f}{\partial y}=6x+2y=0\end{cases}\Rightarrow x=y=0$　　$A=\dfrac{\partial^2 f}{\partial x^2}=4$　　$B=\dfrac{\partial^2 f}{\partial x\partial y}=6$　　$C=\dfrac{\partial^2 f}{\partial y^2}=2$

$\therefore AC-B^2<0$　　$\therefore f(0,0)$ 不是极值　　故最值一定在边界上取到，即满足 $x^2+2y^2-3=0$

设 $L(x,y,\lambda)=2x^2+6xy+y^2+\lambda(x^2+2y^2-3)$

$$\begin{cases}\dfrac{\partial L}{\partial x}=4x+6y+2\lambda x=0\\ \dfrac{\partial L}{\partial y}=6x+2y+4\lambda y=0\\ \dfrac{\partial L}{\partial \lambda}=x^2+2y^2-3=0\end{cases}\Rightarrow \lambda=1\text{或}\lambda=-\frac{7}{2}$$

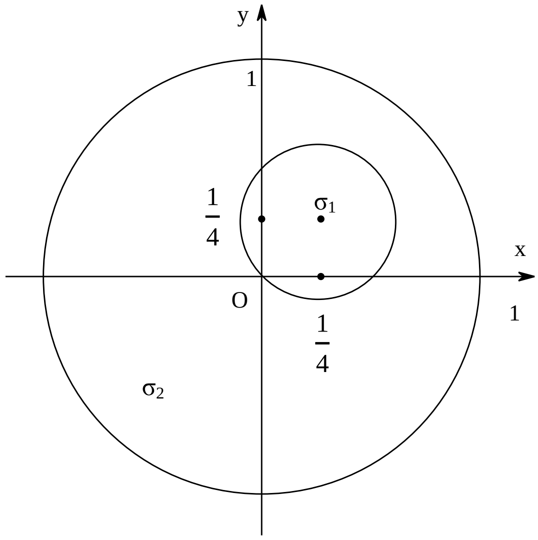
$\therefore x=1$，$y=-1$ 或 $x=-1$，$y=1$ 或 $x=\sqrt{2},y=\dfrac{\sqrt{2}}{2}$ 或 $x=-\sqrt{2},y=-\dfrac{\sqrt{2}}{2}$

$f(1,-1)=-3$　　$f(-1,1)=-3$　　$f(\sqrt{2},\dfrac{\sqrt{2}}{2})=\dfrac{21}{2}$　　$f(-\sqrt{2},-\dfrac{\sqrt{2}}{2})=\dfrac{21}{2}$

$\therefore$ 最小值为-3，$(1,-1)$ 或 $(-1,1)$ 时取到；最大值为 $\dfrac{21}{2}$，$(\sqrt{2},\dfrac{\sqrt{2}}{2})$ 或 $(-\sqrt{2},-\dfrac{\sqrt{2}}{2})$ 时取到

7. 在圆内时，$I_1=\iint\limits_{\sigma_1}(\dfrac{x+y}{2}-x^2-y^2)d\sigma$；在圆外时，$I_2=\iint\limits_{\sigma_2-\sigma_1}(x^2+y^2-\dfrac{x+y}{2})d\sigma$

$\sigma_1:(x-\frac{1}{4})^2+(y-\frac{1}{4})^2\le\frac{1}{8}$ $\sigma_2:x^2+y^2\le 1$



$$I_1=\iint\limits_{\sigma_1}\left[-(x-\frac{1}{4})^2-(y-\frac{1}{4})^2+\frac{1}{8}\right]d\sigma=\iint\limits_{\sigma_1}\frac{1}{8}d\sigma-\iint\limits_{\sigma_1}\rho^2\cdot\rho d\rho d\theta$$

$$=\frac{1}{8}\cdot\pi\cdot(\frac{1}{2\sqrt{2}})^2-\int_0^{2\pi}d\theta\int_0^{\frac{1}{2\sqrt{2}}}\rho^3d\rho=\frac{\pi}{64}-\frac{\pi}{128}=\frac{\pi}{128}$$

$$I_2=\iint\limits_{\sigma_2}\left(x^2+y^2-\frac{x+y}{2}\right)d\sigma-\iint\limits_{\sigma_1}\left[(x-\frac{1}{4})^2+(y-\frac{1}{4})^2-\frac{1}{8}\right]d\sigma$$

$$=\iint\limits_{\sigma_2}(x^2+y^2)d\sigma-\iint\limits_{\sigma_1}\left[(x-\frac{1}{4})^2+(y-\frac{1}{4})^2\right]d\sigma+\iint\limits_{\sigma_1}\frac{1}{8}d\sigma$$

$$\int_0^{2\pi}d\theta\int_0^1\rho^3d\rho-\int_0^{2\pi}d\theta\int_0^{\frac{1}{2\sqrt{2}}}\rho^3d\rho+\iint\limits_{\sigma_1}\frac{1}{8}d\sigma=\frac{\pi}{2}-\frac{\pi}{128}+\frac{\pi}{64}=\frac{65\pi}{128}$$

$$\therefore I=I_1+I_2=\frac{33}{64}\pi$$