第五章

1. (2) $\dfrac{\sin z}{z^3}$ ，$z=0$为奇点

$\lim\limits_{\substack{z\to z_0\\(z_0=0)}} \dfrac{\sin z}{z^3}\times z^2=1$ $z=0$为极点,为二级极点

(4) $\dfrac{\ln(z+1)}{z}$ $z=0$为奇点

$\lim\limits_{z\to0}\dfrac{\ln(1+z)}{z}=1$ $z=0$为可去奇点

(5) $\dfrac{z}{(1+z^2)(1+e^{\pi z})}=\dfrac{z}{(z-i)(z+i)(1+e^{\pi z})}$

$z=i$ 为一级极点 $1+e^{\pi z}=0$

$z=-i$ 为一级极点 $\Rightarrow z=i(2k-1),k=0,\pm1,\pm2\ldots$

$z=(2k+1)i$ $(k=1,\pm2,\pm3\cdots)$

为一级极点

(9) $\dfrac{1}{\sin z^2}$ $\sin z^2=0$ $z^2=k\pi$ $(k=0,\pm1,\pm2\cdots)$

$z=\pm\sqrt{k\pi}$ $(k=0,1,2\cdots)$

$z=\pm i\sqrt{k\pi}$ $(k=0,1,2\cdots)$

$(\sin z^2)'=2z\cos z^2\neq0$

$z=\pm i\sqrt{k\pi},z=\pm\sqrt{k\pi}$ $(k=1,2,3\cdots)$为一级极点

$z=0$ 为二级极点

9 (2) $\dfrac{1-e^{2z}}{z^4}$

$f(z)=\dfrac{1-e^{2z}}{z^4}=-\dfrac{e^{2z}-1}{z^4}=-\dfrac{-1+1+2z+\frac{4z^2}{2!}+\frac{8z^3}{3!}+\frac{16}{4!}z^4+\frac{32z^5}{5!}+\cdots}{z^4}$

$\text{Res}\left[\dfrac{1-e^{2z}}{z^4},0\right]=-\dfrac{8}{3!}=-\dfrac{8}{6}=-\dfrac{4}{3}$

(5) $\cos\dfrac{1}{1-z}$

$\text{Res}\left[\cos\dfrac{1}{1-z},1\right]=0$

(6) $z^2\sin\dfrac{1}{z}$

$f(z)=z^2\sin\dfrac{1}{z}=z^2\left(\dfrac{1}{z}-\dfrac{1}{3!z^3}+\dfrac{1}{5!z^5}\cdots\right)$

$\text{Res}\left(z^2\sin\dfrac{1}{z},0\right)=-\dfrac{1}{6}$

11. (1) 由题意 $f(z)=(z-z_0)^n\psi(z)$ $\psi(z_0)\neq0$.

$f'(z)=(z-z_0)^n\psi'(z)+n(z-z_0)^{n-1}\psi(z)$

$\varphi(z)\dfrac{f'(z)}{f(z)}=\varphi(z)\dfrac{(z-z_0)^n\psi'(z)+n(z-z_0)^{n-1}\psi(z)}{(z-z_0)^n\psi(z)}$

$=\varphi(z)\dfrac{(z-z_0)\psi'(z)+n\psi(z)}{(z-z_0)\psi(z)}$

$\text{Res}\left[\varphi(z)\dfrac{f'(z)}{f(z)},z_0\right]=\lim\limits_{z\to z_0}\varphi(z)\dfrac{(z-z_0)\psi'(z)+n\psi(z)}{\psi(z)}$

$=n\varphi(z_0)$

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(2) 令 $f(z)=\frac{\psi(z)}{(z-z_0)^n}$ $\psi(z_0)\neq 0$

$$f'(z)=\frac{\psi'(z)\cdot(z-z_0)^n-\psi(z)\cdot n(z-z_0)^{n-1}}{(z-z_0)^{2n}}$$

$$\varphi(z)\frac{f'(z)}{f(z)}=\varphi(z)\frac{\psi'(z)\cdot(z-z_0)^n-\psi(z)\cdot n(z-z_0)^{n-1}}{(z-z_0)^n\psi(z)}$$

$$=\varphi(z)\left[\frac{\psi'(z)}{\psi(z)}-\frac{n}{(z-z_0)}\right]$$

$$Res\left[\varphi(z)\frac{f'(z)}{f(z)},z_0\right]=\lim_{z\to z_0}\varphi(z)\left[\frac{\psi'(z)(z-z_0)}{\psi(z)}-n\right]=-n\varphi(z_0)$$

12(2) $\oint_{|z|=2}\frac{e^{2z}}{(z-1)^2}dz=2\pi i\,Res\left[\frac{e^{2z}}{(z-1)^2},1\right]$

$$Res\left[\frac{e^{2z}}{(z-1)^2},1\right]=(e^{2z})'=2e^{2z}\Big|_{z=1}=2e^2$$

$$\oint_{|z|=2}\frac{e^{2z}}{(z-1)^2}dz=2\pi i\times 2e^2=4\pi e^2 i$$

(3) $\oint_{|z|=\frac{3}{2}}\frac{1-\cos z}{z^m}dz$

① $m\leq 2$时，原式$=0$.

② $m>2$时，$\oint_{|z|=\frac{3}{2}}\frac{1-\cos z}{z^m}dz=2\pi i\,Res\left[\frac{1-\cos z}{z^m},0\right]$

$$Res\left[\frac{1-\cos z}{z^m},0\right]=\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(1-\cos z)$$

$$=\frac{-1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\cos z$$

① $m-1=4k-3,k\in N^+$. 原式$=\frac{-2\pi i}{(m-1)!}\sin z\Big|_{z=0}=0$.

② $m-1=4k-2$, $k\in N^+$, 原式$=\frac{-2\pi i}{(m-1)!}(-\cos z|_{z=0})=\frac{+2\pi i}{(m-1)!}$

③ $m-1=4k-1$, $k\in N^+$. 原式$=\frac{-2\pi i}{(m-1)!}(-\sin z)\Big|_{z=0}=0$

④ $m-1=4k$, $k\in N^+$. 原式$=\frac{-2\pi i}{(m-1)!}\cos z\Big|_{z=0}=\frac{-2\pi i}{(m-1)!}$

14. (1) 令 $z=\frac{1}{t}$

$$f(z)=e^{\frac{1}{z^2}}=e^{t^2}=\varphi(t)$$

$\varphi(t)=e^{t^2}$ $t=0$为可去奇点.

$z=\infty$为可去奇点，留数为0.

(2) $z=\frac{1}{t}$

$$\cos z-\sin z=\cos\frac{1}{t}-\sin\frac{1}{t}$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}t^{-2n}-\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}t^{-(2n+1)}$$

$$Res\left[(\cos\tfrac{1}{t}\cdot\sin\tfrac{1}{t})\tfrac{1}{t^2},0\right]$$

$=0$ 本性奇点，留数为0.

(3) $\frac{2z}{3+z^2}$ 令 $z=\frac{1}{t}$

$$\frac{2z}{3+z^2}=\frac{2\frac{1}{t}}{3+\frac{1}{t^2}}=\frac{2t}{3t^2+1}$$

$z=\infty$ 是 $f(z)$ 的可去奇点。

$$\mathrm{Res}\left[\frac{2z}{3+z^2},\infty\right]=-\mathrm{Res}\left(\frac{2t}{3t^2+1}\cdot\frac{1}{t^2},0\right)$$
$$=-2$$

15. (1) $f(z)=\frac{e^z}{z^2-1}=\frac{e^{\frac{1}{t}}}{\frac{1}{t^2}-1}=\frac{t^2e^{\frac{1}{t}}}{1-t^2}=\varphi(t)$

$$\mathrm{Res}[f(z),\infty]=-\mathrm{Res}\left[\frac{t^2e^{\frac{1}{t}}}{1-t^2}\cdot\frac{1}{t^2},0\right]$$
$$=-\mathrm{Res}\left[\frac{e^{\frac{1}{t}}}{1-t^2},0\right]$$

$$\mathrm{Res}[f(z),\infty]=-(\mathrm{Res}[f(z),1]+\mathrm{Res}[f(z),-1])$$
$$=-\left(\frac{e}{2}+\frac{e^{-1}}{-2}\right)$$
$$=-\left(\frac{e}{2}-\frac{e^{-1}}{2}\right)=\frac{e^{-1}-e}{2}$$

(2) $f(z)=\frac{1}{z(z+1)^4(z-4)}=t\frac{1}{(\frac{1}{t}+1)^4(\frac{1}{t}-4)}=\frac{t^6}{(1+t)^4(1-4t)}$

$$\mathrm{Res}[f(z),\infty]=-\mathrm{Res}\left[\frac{t^6}{(1+t^4)(1-4t)t^2},0\right]$$

$$=0$$

16. (1) 原式 $=-2\pi i\,\mathrm{Res}[f(z),\infty]$

$$f(z)=\frac{z^{15}}{(z^2+1)^2(z^4+2)^3}=\frac{1}{t^{15}(\frac{1}{t^2}+1)^2(\frac{1}{t^4}+2)^3}$$
$$=\frac{t}{(1+t^2)^2(1+2t^4)^3}=\varphi(t)$$

$$\mathrm{Res}[f(z),\infty]=-\mathrm{Res}\left[\varphi(t)\cdot\frac{1}{t^2},0\right]$$
$$=-\mathrm{Res}\left[\frac{1}{(1+t^2)^2(1+2t^4)^3t},0\right]$$
$$=-1$$

$\therefore$ 原式 $=2\pi i$

(2) 原式 $=-2\pi i\,\mathrm{Res}[f(z),\infty]$

$$=2\pi i\,\mathrm{Res}\left[\frac{e^t}{t^3(1+\frac{1}{t})}\cdot\frac{1}{t^2},0\right]$$
$$=2\pi i\,\mathrm{Res}\left[\frac{e^t}{t^4(t+1)},0\right]$$
$$=2\pi i\times\left(-\frac{1}{3}\right)$$
$$=-\frac{2}{3}\pi i$$

(3) $\oint \frac{z^{2n}}{1+z^n} dz = -2\pi i \, Res\left[\frac{z^{2n}}{1+z^n}, \infty\right]$

$= -2\pi i \, Res\left[\frac{1}{t^{2n+2}+t^{n+2}}, 0\right]$

$= \begin{cases} 2\pi i & (n=1) \\ 0 & (n \in N, n>1) \end{cases}$

17. 1) $\int_0^{2\pi} \frac{1}{5+3\sin\theta} d\theta$

令 $z=e^{i\theta}$. $dz = i e^{i\theta} d\theta = i z d\theta$

$d\theta = \frac{dz}{iz}$

$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2i} = \frac{z^2-1}{2iz}$

原式 $= \oint_{|z|=1} \frac{1}{5+\frac{3z^2-3}{2iz}} \cdot \frac{dz}{iz}$

$= \oint_{|z|=1} \frac{1}{5iz+\frac{3z^2-3}{2}} dz = \oint_{|z|=1} \frac{2}{3z^2-3+10iz} dz$

$= \oint_{|z|=1} \frac{1}{(3z+i)(z+3i)} dz$

$= \frac{2\pi i \times 2 \times}{3} \frac{-i}{\frac{8}{3}} = \frac{3}{4}\pi \times \frac{2}{3}$

$= \frac{\pi}{2}$

(2) $\int_0^{\pi} \frac{\sin^2\theta}{a+b\cos\theta} d\theta$

$= \oint_{|z|=1} \frac{i(z^2-1)^2}{2z^2(bz^2+2za+b)} dz$

$bz^2+2za+b=0 \Rightarrow z = \frac{-2a\pm\sqrt{4a^2-4b^2}}{2b} = \frac{-a\pm\sqrt{a^2-b^2}}{b}$

$z_1 = \frac{-a+\sqrt{a^2-b^2}}{b}$, $z_2 = \frac{-a-\sqrt{a^2-b^2}}{b}$

$|z|=1$ 内 原式 $= 2\pi i[f(z),0] + 2\pi i[f(z), \frac{-a+\sqrt{a^2-b^2}}{b}]$

$= \frac{2\pi}{b^2}(a-\sqrt{a^2-b^2})$

(4) $\int_0^{+\infty} \frac{x^2}{1+x^4} dx$

$= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2}{1+x^4} dx$

$= \frac{2\pi i}{2} \times \left(Res\left[\frac{x^2}{1+x^4}, e^{\frac{\pi}{4}}\right] + Res\left[\frac{x^2}{1+x^4}, e^{\frac{3\pi}{4}}\right]\right)$

$= \pi i \times \left(\frac{x^2}{4x^3}\Big|_{x=e^{\frac{\pi}{4}i}} + \frac{x^2}{4x^3}\Big|_{x=e^{\frac{3}{4}\pi i}}\right)$

$= \frac{\sqrt{2}}{4}\pi$

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(6) $\displaystyle\int_{-\infty}^{+\infty} \frac{x\sin x}{1+x^2}\,dx$

$$\int_{-\infty}^{+\infty} \frac{x e^{ix}}{1+x^2}\,dx = 2\pi i \cdot \frac{x}{1+x} e^{ix}\Big|_{x=i}$$

$$= \frac{\pi}{e}(-1+i)$$

$$\therefore \int_{-\infty}^{+\infty} \frac{x\sin x}{1+x^2}\,dx = \frac{\pi}{e}$$