1. 性能指标 $\min J=\sum_{k=0}^{2}\left[3x^2(k)+4u^2(k)\right]$

$N=3$，$L[x(k),u(k)]=3x^2(k)+4u^2(k)$

$\Phi[x(N),N]=0$.

① 从最后一级开始：$J^*[x(N)]=\Phi[x(3),3]=0$

② 向前倒推一级：$k=3-1=2$

$$J^*[x(2)]=\min_{u(2)}\left\{L[x(2),u(2),2]+J^*[x(3)]\right\}$$

$$=\min_{u(2)}\left[3x^2(2)+4u^2(2)\right]$$

$$\frac{\partial J^*[x(2)]}{\partial u(2)}=8u(2)=0\Rightarrow u^*(2)=0$$

$J^*[x(2)]=3x^2(2)$，$x^*(3)=x(2)+2u^*(2)$
$=x(2)$

③ 再向前倒推一级：$k=1$

$$J^*[x(1)]=\min_{u(1)}\left[L(x(1),u(1),1)+J^*(x(2))\right]$$

$$=\min_{u(1)}\left[3x^2(1)+4u^2(1)+3x^2(2)\right]$$ $\left(x(2)=x(1)+2u(1)\right.$

$$=\min_{u(1)}\left[3x^2(1)+4u^2(1)+3\left(x(1)+2u(1)\right)^2\right]$$

$$\frac{\partial J^*(x(1))}{\partial u(1)} = 8u(1) + 3(x(1)+2u(1)) \times 2 \times 2$$
$$= 8u(1) + 12(x(1)+2u(1)) = 0$$

$$\Rightarrow u^*(1) = -\frac{3}{8}x(1)$$

$$x^*(2) = x(1) + 2u^*(1) = x(1) - \frac{3}{4}x(1) = \frac{1}{4}x(1)$$

$$J^*(x(1)) = 3x^2(1) + 4 \times \frac{9}{64}x^2(1) + 3 \times \frac{1}{16}x^2(1)$$
$$= \left(3 + \frac{9}{16} + \frac{3}{16}\right)x^2(1)$$
$$= \frac{15}{4} \cdot x^2(1)$$

④ 再向前倒推一级 $k=0$

$$J^*(x(0)) = \min_{u(0)}\left[L(x(0),u(0),0) + J^*(x(1))\right]$$
$$= \min_{u(0)}\left[3x^2(0) + 4u^2(0) + \frac{15}{4}x^2(1)\right]$$
$$= \min_{u(0)}\left[3x^2(0) + 4u^2(0) + \frac{15}{4}(x(0)+2u(0))^2\right]$$

$$\frac{\partial J^*(x(0))}{\partial u(0)} = 8u(0) + \frac{15}{4}(x(0)+2u(0)) \times 4 = 0$$

$$\Rightarrow u^*(0) = -\frac{15}{38}x(0)$$

$$J^*(x(0)) = \frac{2043}{361}x(0)^2$$

$$u^*(0) = -\frac{15}{38}x(0)$$
$$u^*(1) = -\frac{3}{38}x(0)$$
$$u^*(2) = 0$$

$$\Rightarrow x^*(1) = \frac{4}{19}x(0)$$

$$x^*(3) = x(2) = \frac{1}{19}x(0)$$

$$x^*(2) = \frac{1}{4}x(1) = \frac{1}{19}x(0)$$

2. $\begin{bmatrix} 216 \\ 220 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \theta + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$

$H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $Z = \begin{bmatrix} 216 \\ 220 \end{bmatrix}$, $R = \begin{bmatrix} 4^2 & 0 \\ 0 & 2^2 \end{bmatrix}$

① 最小二乘估计：$\hat{\theta}_{LS} = (H^T H)^{-1} H^T Z = 218$

$P_{LS} = (H^T H)^{-1} H^T R H (H^T H)^{-1}$
$= 5$

② 加权最小二乘估计. $\hat{\theta}_{WLS} = (H^T W H)^{-1} H^T W Z$

$P_{WLS} = (H^T W H)^{-1} H^T W R W H (H^T W H)^{-1}$

选择马尔科夫估计：令 $W = R^{-1}$

则 $\hat{\theta}_M = (H^T R^{-1} H)^{-1} H^T R^{-1} Z$

$= 219.2$

$P_M = (H^T R^{-1} H)^{-1} = \frac{16}{5}$

3. $\dot{x}=\begin{bmatrix}0&1\\0&-1\end{bmatrix}x+\begin{bmatrix}0\\1\end{bmatrix}u$

$$J=\int_0^{+\infty}\left\{x^T\begin{bmatrix}1&0\\0&\mu\end{bmatrix}x+u^2\right\}dt$$

$$=\frac{1}{2}\int_0^{+\infty}\left\{x^T\begin{bmatrix}2&0\\0&2\mu\end{bmatrix}x+2u^2\right\}dt$$

解. $A=\begin{bmatrix}0&1\\0&-1\end{bmatrix}$ $B=\begin{bmatrix}0\\1\end{bmatrix}$ $Q=\begin{bmatrix}2&0\\0&2\mu\end{bmatrix}$ $R=2$

$u^*=-R^{-1}B^TPx=-\frac{1}{2}\begin{bmatrix}0&1\end{bmatrix}Px$

P 满足 $A^TP+PA-PBR^{-1}B^TP+Q=0$

令 $P=\begin{bmatrix}a&b\\b&c\end{bmatrix}$, 且 $a>0$, $ac-b^2>0$

$\Rightarrow$ $b=2$, $c=-2+\sqrt{12+4\mu}$, $a=\sqrt{12+4\mu}=2\sqrt{3+\mu}$

$u^*=-\frac{1}{2}\begin{bmatrix}b&c\end{bmatrix}x=-x_1+(1-\sqrt{3+\mu})x_2$

$J=\frac{1}{2}x^T(0)Px(0)=\frac{1}{2}a=\sqrt{3+\mu}$