2. (1) $\int_{-\infty}^{+\infty} f(\tau) e^{-j\omega\tau} d\tau = \int_{-1}^{1} f(\tau) e^{-j\omega\tau} d\tau$

$f(t)$ 为偶函数.

$$f(t) = \frac{2}{\pi}\int_0^{+\infty}\left[\int_0^{+\infty} f(\tau)\cos\omega\tau d\tau\right]_{\cos\omega t} d\omega$$

$$= \frac{2}{\pi}\int_0^{+\infty}\left[\int_0^{1}(1-\tau^2)\cos\omega\tau \cdot d\tau\right]^{\cos\omega t} d\omega$$

$$= \frac{4}{\pi}\int_0^{+\infty}\frac{\sin\omega - \omega\cos\omega}{\omega^3}\cos\omega t d\omega$$

(2) $f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\tau) e^{-j\omega\tau} d\tau\right] e^{j\omega t} d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_0^{\infty} e^{-\tau}\sin 2\tau \cdot e^{-j\omega\tau} d\tau\right] e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_0^{\infty}\sin 2\tau \cdot e^{-(j\omega+1)\tau} d\tau\right] e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{2}{5-\omega^2+2j\omega} e^{j\omega t} d\omega$$

$$= \frac{2}{\pi}\int_0^{+\infty}\frac{2\omega\sin\omega t + (5-\omega^2)\cos\omega t}{25-6\omega^2+\omega^4} d\omega$$

3. (1) $f(t)$ 为偶函数

$$f(t) = e^{-\beta|t|} = \frac{2}{\pi}\int_0^{+\infty}\left[\int_0^{+\infty} e^{-\beta\tau}\cos\omega\tau d\tau\right]^{\cos\omega t} d\omega$$

$$= \frac{2}{\pi}\int_0^{+\infty}\frac{\beta}{\beta^2+\omega^2}\cos\omega t d\omega$$

即 $\int_0^{+\infty}\frac{\beta}{\beta^2+\omega^2}\cos\omega t d\omega = \frac{\pi}{2\beta} e^{-\beta|t|}$

3.(3) $f(t)$为奇函数

$$f(t)=\frac{2}{\pi}\int_0^{+\infty}\left[\int_0^{+\infty}f(\tau)\cos w\tau\, d\tau\right]\sin wt\, dw$$

$$=\frac{2}{\pi}\int_0^{+\infty}\left[\int_0^{\pi}\sin\tau\cos w\tau\, d\tau\right]\sin wt\, dw$$

$$=\frac{2}{\pi}\int_0^{+\infty}\frac{\sin w\pi}{1-w^2}\sin wt\, dw$$

$$\therefore \int_0^{+\infty}\frac{\sin w\pi\ \sin wt}{1-w^2}dw=\frac{\pi}{2}f(t)=\begin{cases}0, & |t|>\pi\\ \frac{\pi}{2}\sin t, & |t|\le\pi\end{cases}$$

习题

二. 1. $F(w)=\int_{-\infty}^{+\infty}f(t)e^{-jwt}dt$

$$=\int_0^{\tau}A\cdot e^{-jwt}dt$$

$$=(1-e^{-jw\tau})\frac{A}{jw}$$

$$=\frac{A}{w}(e^{-jw\tau}-1)j$$

2. $F(w)=\int_{-\infty}^{+\infty}f(t)e^{-jwt}dt$

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(w)e^{jwt}dw$$

①$f(t)$为偶函数时. $f(t)=f(-t)$, $F(-w)=\int_{-\infty}^{+\infty}f(t)e^{-jwt}dt$

$$=\int_{-\infty}^{+\infty}f(-t)e^{+jwt}dt=\int_{-\infty}^{+\infty}f(x)e^{-jwx}dx=F(w)$$

即 $F(-w)=F(w)$ ② $F(w)$为偶函数时. $F(w)=F(-w)$

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(w)e^{-jwt}dw=\frac{1}{2\pi}\int^{+\infty}F(x)e^{jxt}dx=f(t)$$

3.(1) $F(w)=\int_{-\infty}^{+\infty}f(t)e^{-jwt}dt$

$$=\int_0^{+\infty}\alpha e^{-\beta t}e^{-jwt}dt$$

$$=\int_0^{+\infty}\alpha e^{-(\beta+jw)t}dt$$

$$=\frac{\alpha}{\beta+jw}=\frac{\alpha(\beta-jw)}{\beta^2+w^2}=\frac{\alpha\beta}{\beta^2+w^2}-\frac{\alpha w}{\beta^2+w^2}j$$

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(w)e^{jwt}dw$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\alpha\beta-\alpha wj}{\beta^2+w^2}\cdot e^{jwt}dw$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\alpha\beta-\alpha wj}{\beta^2+w^2}(\cos wt+j\sin wt)dw$$

$$=\frac{\alpha}{\pi}\int_0^{+\infty}\frac{\beta\cos wt+w\sin wt}{\beta^2+w^2}dw$$

$$\text{即}\int_0^{+\infty}\frac{\beta\cos wt+w\sin wt}{\beta^2+w^2}dw=\frac{\pi}{\alpha}f(t)=\begin{cases}\pi e^{-\beta t}, & t>0\\ \frac{\pi}{2}, & t=0\\ 0, & t<0\end{cases}$$

当$f(t)$为奇函数时, 同理可得$F(w)$为奇函数.

当$F(w)$为奇函数时, 同理可得$f(t)$为奇函数.

(2) $F(w)=\int_{-\infty}^{+\infty} f(t) e^{-jwt} dw$

$=2\int_0^{+\infty} f(t)\cos wt\, dt$

$=2\int_0^{\pi} \cos t \cos wt\, dt$

$=\int_0^{\pi}[\cos(1-w)+\cos(1+w)t]dt$

$=\dfrac{2w\sin w\pi}{1-w^2}$

$f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(w) e^{jwt} dt$

$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\dfrac{2w\sin w\pi}{1-w^2}(\cos wt+j\sin wt)dt$

$=\frac{2}{\pi}\int_0^{+\infty}\dfrac{w\sin w\pi}{1-w^2}\cos wt\, dw$

$|t|=\pi$ 时. $f(t)=\frac{1}{2}\dfrac{f(t-0)+f(t+0)}{2}$

$\therefore \int_0^{+\infty}\dfrac{w\sin w\pi\cos wt}{1-w^2}dw=\begin{cases}\frac{\pi}{2}\cos t & |t|\le\pi\\ -\frac{\pi}{4}, & |t|=\pi\\ 0 & |t|>\pi\end{cases}$

6. $f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\dfrac{\sin w}{w} e^{jwt}dw$

$=\frac{1}{\pi}\int_0^{+\infty}\dfrac{\sin w}{w}(\cos wt+j\sin wt)dw$

$=\frac{1}{\pi}\int_0^{+\infty}\dfrac{\sin w}{w}\cos wt\, dw$

$=\frac{1}{2\pi}\int_0^{+\infty}\dfrac{\sin(1+t)\cdot w}{w}dw+\frac{1}{2\pi}\int_0^{+\infty}\dfrac{\sin(1-t)w}{w}dw$

由 $\int_0^{+\infty}\dfrac{\sin wt}{w}dw=\pi\left(u(t)-\frac{1}{2}\right)\ (t\neq 0)$

$u(t)$ 为单位阶跃函数.

则 $f(t)=\frac{1}{2}[u(t+1)+u(1-t)-1]\ ,|t|\neq 1$

$t=\pm 1$ 时. $f(t)=\frac{1}{2}[f(t+0)+f(t-0)]$
$=\frac{1}{4}$

$\therefore f(t)=\begin{cases}\frac{1}{2}[u(1+t)+u(1-t)-1] & |t|\neq 1\\ \frac{1}{4} & |t|=1\end{cases}$

7. $f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\pi[\delta(w+w_0)+\delta(w-w_0)]e^{jwt}dw$

$=\frac{1}{2}\left[\int_{-\infty}^{+\infty}\delta(w+w_0)e^{jwt}dw+\int_{-\infty}^{+\infty}\delta(w-w_0)e^{jwt}dw\right]$

$=\frac{1}{2}\left[\int_{-\infty}^{+\infty}\delta(w+w_0)e^{j(w+w_0)t}e^{-jw_0t}+\int_{-\infty}^{+\infty}\delta(w-w_0)e^{j(w-w_0)t}e^{jw_0t}dw\right]$

$=\frac{1}{2}\left[e^{-jw_0t}+e^{jw_0t}\right]$

$=\cos w_0 t$

8. $f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}-\frac{\pi}{a}e^{a|\omega|}\cdot e^{j\omega t}d\omega$

$=-\frac{1}{2a}\int_{-\infty}^{+\infty}e^{a|\omega|}e^{j\omega t}d\omega$

$=-\frac{1}{2a}\left[\int_{0}^{+\infty}e^{(a+jt)\omega}d\omega+\int_{-\infty}^{0}e^{(jt-a)\omega}d\omega\right]$

$=\frac{1}{a^2+t^2}.$

9. $sgn(t)=2u(t)-1$

$F(\omega)=\mathcal{F}[sgnt]=2\mathcal{F}[u(t)]-\mathcal{F}[1]$

$=2\left(\frac{1}{j\omega}+\pi\delta(\omega)\right)-2\pi\delta(\omega)$

$=\frac{2}{j\omega}.$

10. $F(\omega)=\int_{-\infty}^{+\infty}f(t)\,e^{-j\omega t}dt$

$=\frac{1}{2}\int_{-\infty}^{+\infty}\left[\delta(t+a)+\delta(t-a)+\delta(t+\frac{a}{2})+\delta(t-\frac{a}{2})\right]e^{-j\omega t}dt$

$=\frac{1}{2}\left(e^{j\omega a}+e^{-j\omega a}+e^{j\omega\frac{a}{2}}+e^{-j\omega\frac{a}{2}}\right)$

$=\cos\omega a+\cos\frac{\omega a}{2}$

11(1) $F(\omega)=\mathcal{F}[f(t)]=\mathcal{F}[\cos t\sin t]$

$=\frac{1}{2}\mathcal{F}[\sin 2t]=\frac{1}{2}j\pi[\delta(\omega+2)-\delta(\omega-2)]$

(4) $F(\omega)=\int_{-\infty}^{+\infty}e^{-a|t|}e^{-j\omega t}dt$

$=\int_{0}^{+\infty}e^{-(a+j\omega)t}dt+\int_{-\infty}^{0}e^{-j\omega t}\cdot e^{at}dt$

$=\frac{1}{a-j\omega}+\frac{1}{a+j\omega}=\frac{2a}{a^2+\omega^2}$

(6) $F(\omega)=\int_{-\infty}^{+\infty}f(t)e^{-j\omega t}dt$

$=\int_{-\pi}^{\pi}\sin t\,e^{-j\omega t}dt$

$=\int_{-\pi}^{\pi}\sin t(\cos\omega t-j\sin\omega t)dt$

$=j\frac{-2}{1-\omega^2}\sin\omega\pi$

习题三. 2.

证明: 由
$$\begin{cases} f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(w)e^{jwt}dw \\ f(-t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(w)e^{-jwt}dw \end{cases}$$

则
$$\begin{cases} f(w)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(t)e^{jwt}dt \\ f(-w)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(t)e^{-jwt}dt \end{cases}$$

$$\Rightarrow \begin{cases} 2\pi f(w)=\int_{-\infty}^{+\infty}F(t)e^{jwt}dt \\ 2\pi f(-w)=\int_{-\infty}^{+\infty}F(t)e^{-jwt}dt \end{cases}$$

令 $t=-k$

$$\Rightarrow \begin{cases} 2\pi f(w)=-\int_{+\infty}^{-\infty}F(-k)e^{-jwk}dk \\ 2\pi f(-w)=-\int_{+\infty}^{-\infty}F(-k)e^{jwk}dk \end{cases}$$

$$\Rightarrow 2\pi f(w)=\int_{-\infty}^{+\infty}F(-k)e^{-jwk}dk$$

$$2\pi f(-w)=\int_{-\infty}^{+\infty}F(-k)e^{jwk}dk$$

即 $2\pi f(w)=\int_{-\infty}^{+\infty}F(-t)e^{-jwt}dt$

$\Rightarrow 2\pi f(w)=\mathcal{F}[F(-t)]$

$2\pi f(-w)=\int_{-\infty}^{+\infty}F(t)e^{-jwt}dt$

$\Rightarrow 2\pi f(-w)=\mathcal{F}[F(t)]$

即证. $\mathcal{F}[F(\mp t)]=2\pi f(\pm w)$

3. $\mathcal{F}[f(at)]=\int_{-\infty}^{+\infty}f(at)e^{-jwt}dt$

① $a>0$时. $=\frac{1}{a}\int_{-\infty}^{+\infty}f(at)e^{-j\frac{w}{a}(at)}d(at)$

$=\frac{1}{a}F(\frac{w}{a})$

② $a<0$时 $=\frac{1}{-a}\int_{-\infty}^{+\infty}f(at)e^{-j\frac{w}{a}(at)}d(at)$

$=\frac{1}{-a}F(\frac{w}{a})$

即 $\mathcal{F}[f(at)]=\frac{1}{|a|}F(\frac{w}{a})$

4. 证明: $\mathcal{F}[e^{\pm jw_0t}f(t)]=\int_{-\infty}^{+\infty}e^{-jwt}e^{\pm jw_0t}f(t)\cdot dt$

$=\int_{-\infty}^{+\infty}f(t)\cdot e^{-j(w\mp w_0)t}dt$

$=F(w\mp w_0)$

即证 $F(w\mp w_0)=\mathcal{F}[e^{\pm jw_0t}f(t)]$

5. $F(\omega)=\int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$

$\frac{d}{d\omega}F(\omega)=\frac{d}{d\omega}\int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt=\int_{-\infty}^{+\infty}\frac{d}{d\omega}[f(t)e^{-j\omega t}]dt$

$=\int_{-\infty}^{+\infty}(-jt)f(t)e^{-j\omega t}dt$

$=\mathcal{F}[-jtf(t)]$

6. $F(-\omega)=\int_{-\infty}^{+\infty} f(t)e^{-j(-\omega)t}dt$

令$u=t$ $=\int_{\infty}^{+\infty} f(-u)e^{-j\omega u}du$

$=\int_{-\infty}^{+\infty} f(-t)e^{-j\omega t}dt$

$=\mathcal{F}[f(-t)]$

9. 证明: $\mathcal{F}[f(t)]=\int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$

$=\int_{-1}^{1}e^{-j\omega t}dt=\frac{2\sin\omega}{\omega}$

则 $\mathcal{F}[F(t)]=2\pi f(-\omega)$

$\Rightarrow \mathcal{F}[\frac{2\sin\omega}{\omega}]=2\pi f(-\omega)$

即 $\mathcal{F}[\frac{\sin t}{t}]=\pi f(-\omega)$

$=\begin{cases}\pi & (|\omega|<1)\\ 0 & (|\omega|>1)\end{cases}$

10. 已知 $\mathcal{F}[Ae^{-\beta t^2}]=\sqrt{\frac{\pi}{\beta}}Ae^{-\frac{\omega^2}{4\beta}}$

令$A=1,\ \beta=1$, $\mathcal{F}[e^{-t^2}]=\sqrt{\pi}e^{-\frac{\omega^2}{4}}$

$\mathcal{F}[te^{-t^2}]=-\frac{1}{j}\frac{d}{d\omega}[\sqrt{\pi}e^{-\frac{\omega^2}{4}}]$

$=-\frac{1}{j}\sqrt{\pi}\times-\frac{2}{4}\omega\cdot e^{-\frac{\omega^2}{4}}$

$=-\frac{\sqrt{\pi}\omega j}{2}e^{-\frac{\omega^2}{4}}$

11. (1) $\mathcal{F}[f(2t)]=\frac{1}{2}F(\frac{w}{2})$

$\mathcal{F}[tf(2t)]=-\frac{1}{j}\frac{d}{dw}\mathcal{F}[f(2t)]$

$=-\frac{1}{j}\frac{d}{dw}(\frac{1}{2}F(\frac{w}{2}))$

$=\frac{j}{2}\frac{d}{dw}F(\frac{w}{2})$

(3) $\mathcal{F}(g(t))=\mathcal{F}((t-2)f(-2t))$

$=\mathcal{F}(tf(-2t))-2\mathcal{F}(f(-2t))$

$=-\frac{1}{j}\frac{d}{dw}\mathcal{F}[f(-2t)]-2\frac{1}{|-2|}F(\frac{w}{-2})$

$=j\frac{1}{|-2|}\frac{d}{dw}F(\frac{w}{-2})-F(\frac{w}{-2})$

$=\frac{\frac{j}{2}dF(-\frac{w}{2})}{dw}-F(-\frac{w}{2})$

$=\frac{j}{2}\frac{d}{dw}F(-\frac{w}{2})-F(-\frac{w}{2})$

(5) $\mathcal{F}[g(t)]=\mathcal{F}[tf'(t)]=\frac{1}{-j}\frac{d}{dw}\mathcal{F}[f'(t)]$

$=-\frac{1}{j}\frac{d}{dw}(jwF(w))=-\frac{d}{dw}(wF(w))$

$=-(F(w)+w\frac{dF(w)}{dw})=-F(w)-w\frac{d}{dw}F(w)$

习题四.

2. $f_1(t)=\begin{cases}e^{-\alpha t} & (t>0)\\ 0 & (t<0)\end{cases}$

$f_2(t)=\begin{cases}\sin t & (t>0)\\ 0 & (t<0)\end{cases}$

$f_1(t)*f_2(t)=\int_{-\infty}^{+\infty}f_2(\tau)\cdot f_1(t-\tau)\,d\tau$

$\begin{cases}\tau>0\\ t-\tau>0\end{cases}\Rightarrow\begin{cases}\tau>0\\ \tau<t\end{cases}\Rightarrow 0<\tau<t$

$\therefore f_1(t)*f_2(t)=\int_0^t f_2(\tau)f_1(t-\tau)d\tau$

$=\int_0^t \sin\tau\, e^{-\alpha(t-\tau)}d\tau$

$=\int_0^t \sin\tau\, e^{(\tau-t)\alpha}d\tau$

$=e^{-t\alpha}\int_0^t \sin\tau\, e^{\tau\alpha}d\tau$

$=\frac{\alpha\sin t-\cos t+e^{-\alpha t}}{\alpha^2+1}$

扫描全能王 创建

3. $f_1 * f_2 = f_2 * f_1 = \int_{-\infty}^{+\infty} f_2(\tau) \times f_1(t-\tau)\,d\tau$

$\begin{cases} 0\le\tau\le\frac{\pi}{2} \\ t-\tau\ge 0\end{cases} \Rightarrow \begin{cases} 0\le\tau\le\frac{\pi}{2}\\ \tau\le t\end{cases}$

① $0<t<\frac{\pi}{2}$时

∴原式 $=\int_0^t f_2(\tau)\times f_1(t-\tau)\,d\tau$

$=\int_0^t \sin\tau\, e^{-(t-\tau)}\,d\tau$

$=\int_0^t \sin\tau\, e^{\tau-t}\,d\tau$

$=e^{-t}\int_0^t \sin\tau\, e^{\tau}\,d\tau$

$=\frac{\sin t-\cos t+e^{-t}}{2}$.

② $t\ge\frac{\pi}{2}$时

原式 $=\int_0^{\frac{\pi}{2}} f_2(\tau)\times f_1(t-\tau)\,d\tau$

$=\int_0^{\frac{\pi}{2}}\sin\tau\, e^{-(t-\tau)}\,d\tau$

$= e^{-t}\int_0^{\frac{\pi}{2}}\sin\tau\cdot e^{\tau}\,d\tau$

$=\frac{1}{2} e^{-t}(1+e^{\frac{\pi}{2}})$

③ $t<0$时，原式=0.

∴综上. $f_1(t)*f_2(t)=\begin{cases}0, & t\le 0\\ \frac{1}{2}(-\cos t+\sin t+e^{-t}), & 0<t\le\frac{\pi}{2}\\ \frac{1}{2} e^{-t}(1+e^{\frac{\pi}{2}}), & t>\frac{\pi}{2}\end{cases}$

5. (3) $\mathcal{F}[f(t)] = \int_{-\infty}^{+\infty} e^{-j\omega t} e^{-\beta t}\cos\omega_0 t\cdot u(t)\,dt$

$=\int_0^{+\infty} e^{-(\beta t+j\omega t)}\cos\omega_0 t\,dt$

$=\int_0^{+\infty} e^{-(\beta+j\omega)t}\cos\omega_0 t\,dt$

$=\frac{\beta+j\omega}{(\beta+j\omega)^2+\omega_0^2}$

(5) $\mathcal{F}[f(t)] = \int_{-\infty}^{+\infty} e^{j\omega_0 t}u(t-t_0)\cdot e^{-j\omega t}\,dt$

$=\int_{t_0}^{+\infty} e^{j(\omega_0-\omega)t}\,dt$

$= e^{-j(\omega-\omega_0)t_0}\left[\frac{1}{j(\omega-\omega_0)}+\pi\delta(\omega-\omega_0)\right]$

(6) $\mathcal{F}[t u(t)] = -\frac{1}{j}\frac{d}{d\omega}\mathcal{F}[u(t)]$

$=-\frac{1}{j}\frac{d}{d\omega}\left(\frac{1}{j\omega}+\pi\delta(\omega)\right)$

$=-\frac{1}{\omega^2}+\pi j\,\delta'(\omega)$

$$\therefore \mathscr{F}[e^{j\omega_0 t} t u(t)] = -\frac{1}{(\omega-\omega_0)^2} + \pi j \, \delta'(\omega-\omega_0)$$