彭·高数

高等数学下期末试题答案

（2010-2021）

彭康书院学业辅导与发展中心

目录

2021 高数下期末答案

一、填空题

1. $-x+y+z-\dfrac{\pi}{6}=0$.

令 $F(x,y,z)=\sin^2 x+\cos(y+z)-\dfrac{3}{4}$，则 $\nabla F=(2\cos x\sin x,-\sin(y+z),-\sin(y+z))^{\top}$，于是 $\nabla F\Big|_{\left(\frac{\pi}{6},\frac{\pi}{3},0\right)}=-\dfrac{\sqrt{3}}{2}(-1,1,1)$。由平面点法式方程得 $-(x-\dfrac{\pi}{6})+(y-\dfrac{\pi}{3})+z=0$，化简可得答案。

2. $R=1$.

因为级数 $\sum\limits_{n=1}^{\infty}a_n$ 条件收敛，由级数收敛的必要条件，有 $\lim\limits_{n\to\infty}a_n=0$，则有 $\lim\limits_{n\to\infty}\sqrt[n]{a_n+\dfrac{1}{n}}=\lim\limits_{n\to\infty}\sqrt[n]{\dfrac{1}{n}}=1$。

3. $u=x^2+e^x\sin y+2$.

$\dfrac{\partial u}{\partial x}=2x+e^x\sin y$，作偏积分 $u=x^2+e^x\sin y+f(y)$，则 $u_y=e^x\cos y+f'(y)$，对比 $\mathbf{grand}u$ 的值可得 $f'(y)=0\Rightarrow f(y)=\mathrm{C}$，代入 $u(0,\pi)=2$ 可得 $\mathrm{C}=2$。

4. $\dfrac{2\sqrt{2}}{3}\pi^3$.

$\mathrm{d}s=\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\cdot\mathrm{d}t=2\sqrt{2}\mathrm{d}t$，代入积分式有 $I=\int_0^{\pi}\dfrac{4t^2\cdot 2\sqrt{2}}{4}\mathrm{d}t=\dfrac{2\sqrt{2}}{3}\pi^3$.

5. $\dfrac{5}{4}$.

由 Dirichlet 定理，且 $\dfrac{1}{2}$ 是 f 的连续点，则 $S\left(-\dfrac{15}{2}\right)=S\left(\dfrac{1}{2}\right)=\dfrac{5}{4}$.

二、选择题

1. **B**

$\lim\limits_{(x,y)\xrightarrow{x=k^2y}(0,0)}f(x,y)=\lim\limits_{x\to0}\dfrac{2ky^4}{(k^2+1)y^4}=\dfrac{2k}{k^2+1}$ 随 k 的取值变化，故 $f(x)$ 不连续，即 $f(x)$ 不可微。当 $x=0$ 时，$f(x,y)\equiv0$，故 $f_y=0$；同理 $f_x=0$，故 $f(x,y)$ 在原点偏导数存在。$\dfrac{\partial f}{\partial \vec{e_l}}=\lim\limits_{x\to0}\dfrac{f(x,kx)-f(0,0)}{\sqrt{1+k^2}}$

$=\dfrac{k^2}{\sqrt{1+k^2}}\lim\limits_{x\to0}\dfrac{x}{1+k^4x}=0$，故 $f(x,y)$ 沿各个方向导数均存在。

2. **A**

令 $x=r\cos\theta,y=r\sin\theta,z=z$，则积分域可表示为 $r\in(0,1),\theta\in(0,2\pi),z\in(0,\sqrt{4-r^2})$，故 $V=\iiint\limits_{\Omega}\mathrm{d}V$

$=\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r\mathrm{d}r\int_0^{\sqrt{4-r^2}}\mathrm{d}z=4\int_0^{\pi/2}\mathrm{d}\theta\int_0^1 r\sqrt{4-r^2}\mathrm{d}r$.

3. **D**

注意当曲面Σ按Oyz或Oxz分成两片时解出的x或y相差一个符号，由于前侧和后侧的不同，在投影域上的二重积分也相差一个符号。对于被积函数时x^2或y^2，代入后相同，故只相差一个符号，则积分为 0，对于x带入后则不为 0。

4. **B**

积分域可以表示为$y\in\left(\frac{1}{2},1\right),x\in\left(\frac{1}{y},2\right)$，故$I=\int_{1/2}^1 y\mathrm{d}y\int_{1/y}^2 \mathrm{e}^{xy}\mathrm{d}x=\int_{1/2}^1\left(\mathrm{e}^{2y}-\mathrm{e}\right)\mathrm{d}y=\frac{\mathrm{e}^2}{2}-\mathrm{e}$。

5. **C**

A. $a_n=\frac{1}{n^2}$时，$\sum_{n=1}^{\infty}a_n$收敛，且$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$存在，但$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=1$；B. 取$a_n=\begin{cases}\frac{1}{n^2}, & n\text{为奇数}\\ \frac{1}{n}, & n\text{为偶数}\end{cases}$，$a_n$发散，但存在$a_n<\frac{1}{n}$的情况；C. $|a_n|=\left|\frac{1}{\sqrt{n}}-\sin\frac{1}{\sqrt{n}}\right|=\left|\frac{1}{\sqrt{n}}-\left[\frac{1}{\sqrt{n}}-\frac{1}{3!}\left(\frac{1}{\sqrt{n}}\right)^3\right]+o\left[\left(\frac{1}{\sqrt{n}}\right)^3\right]\right|$，$n\to+\infty$时，$|a_n|\sim\frac{1}{3!}\left(\frac{1}{\sqrt{n}}\right)^3$，由$p$级数收敛性质可知，$\sum_{n=1}^{\infty}|a_n|$收敛，则原级数绝对收敛；D. 级数绝对收敛时才满足交换次序和不变。

三、计算题

1. $\frac{\partial f}{\partial x}=yf_1+\frac{1}{y}f_2$，$\frac{\partial f}{\partial y}=xf_1+\frac{x}{y^2}f_2$，$\frac{\partial z}{\partial x}=f+x\left(\frac{\partial f}{\partial x}\right)=f+xyf_1+\frac{x}{y}f_2$，

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial y}=xf_1-\frac{x}{y^2}f_2+x\left[f_1+y\left(xf_{11}-\frac{x}{y^2}f_{12}\right)\right]+\frac{x}{y^2}\left(x_y f_{21}-\frac{x}{y}f_{22}-f_2\right)=2xf_1-\frac{2x}{y^2}f_2+x^2yf_{11}-\frac{x^3}{y^3}f_{22}$$

2. 令$x=r\cos\theta,y=r\sin\theta,z=r$，易知$\|r_\theta\times r_r\|=\sqrt{2}r$，故$\mathrm{d}S=\sqrt{2}r\mathrm{d}\theta\mathrm{d}r$，则$I=\iint(r\cos\theta)r^2\sqrt{2}\mathrm{d}r\mathrm{d}\theta$

$=\int_{-\pi/2}^{\pi/2}\mathrm{d}\theta\int_0^{2a\cos\theta}\sqrt{2}r^3\cos\theta\mathrm{d}r=\int_{-\pi/2}^{\pi/2}4\sqrt{2}a^4\cos^5\theta\mathrm{d}\theta=\frac{64}{15}\sqrt{2}a^4$。

3. $z_x=x^2-y,z_y=-x+y-2,z_{xx}=2x,z_{xy}=-1,z_{yy}=1$，令$z_{x_0}=z_{y_0}=0$，得$(x_0,y_0)=(2,4)$或$(-1,1)$。

(1) $(x_0,y_0)=(2,4)$时，$z(x,y)$在$(2,4)$点的 Hesse 矩阵为$\begin{bmatrix}4 & -1\\ -1 & 1\end{bmatrix}$，$AC-B^2>0$且$A>0$，故$(2,4)$为极小值点，极小值为$z(2,4)=-\frac{16}{3}$；

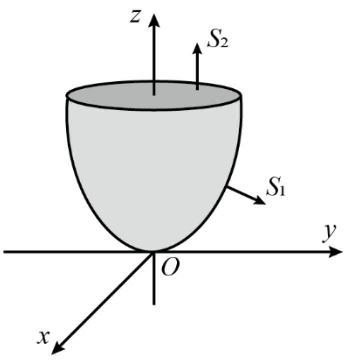
(2) $(x_0,y_0)=(-1,1)$ 时，$z(x,y)$ 在 $(-1,1)$ 点的 Hesse 矩阵为 $\begin{bmatrix}-2 & -1\\ -1 & 1\end{bmatrix}$，$AC-B^2<0$，故 $(-1,1)$ 不是极值点.

4.

$$I=\oint_{C\cup\overline{OA}}\vec{A}\cdot\mathrm{d}\vec{S}+\int_{\overline{AO}}\vec{A}\cdot\mathrm{d}\vec{S}\overset{\text{Green}}{=\!=\!=}\iint\left(-\frac{\partial\left(\sqrt{x^2-y^2}\right)}{\partial y}+\frac{\partial\left[2x+y\cdot\ln\left(x+\sqrt{x^2+y^2}\right)\right]}{\partial x}\right)\mathrm{d}\delta+\int_{\overline{AO}}\vec{A}\cdot\mathrm{d}\vec{S}=2\int_0^{\pi}\mathrm{d}x\int_0^{x\sin x}\mathrm{d}y$$

$$+\int_{\pi}^{0}x\mathrm{d}x=2\pi-\frac{\pi^2}{2}\text{。}$$

5. 补出如图所示的平面，方向向上。



$$I=\oiint_{S_1\cup S_2}\vec{A}\cdot\mathrm{d}\vec{S}-\iint_{S_2}\vec{A}\cdot\mathrm{d}\vec{S}\overset{\text{Gauss}}{=\!=\!=}\iiint_V 3z^2\mathrm{d}V+\iint_{S_2}\mathrm{d}x\mathrm{d}y$$，令 $x=\rho\cos\theta,y=\rho\sin\theta,z=z$，则

$$I=\int_0^{2\pi}\mathrm{d}\theta\int_0^1 3z^2\mathrm{d}z\int_0^{\sqrt{z}}\rho\mathrm{d}\rho+\pi=\frac{7}{4}\pi\text{。}$$

6. (1) $\ln(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}+\cdots+(-1)^{n-1}\dfrac{x^n}{n}+\cdots,x\in(-1,1]$；

$$\frac{\ln(1+x)}{x}=1-\frac{x}{2}+\frac{x^3}{3}+\cdots+(-1)^{n-1}\frac{x^{n-1}}{n}+\cdots,x\in(-1,0)\cup(0,1]\text{。}$$

(2) $\displaystyle\int\frac{\ln(1+x)}{x}=\sum_{n=1}^{\infty}\int_0^x(-1)^{n-1}\frac{x^{n-1}}{n}\mathrm{d}x=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}-2\sum_{n=1}^{\infty}\frac{1}{(2n)^2}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{12}$。

五、将 $f(x)$ 偶延拓，$f(x)=\begin{cases}-2x-\pi, & x\in(-\pi,-\pi/2)\\ 0, & x\in(-\pi/2,\pi/2)\\ 2x+\pi, & x\in(\pi/2,\pi)\end{cases}$，$a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x)\mathrm{d}x=\pi$，

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot\cos(nx)\mathrm{d}x=\frac{2}{\pi}\int_{\pi/2}^{\pi}(2x-\pi)\cdot\cos(nx)\mathrm{d}x=\frac{4}{n^2\pi}\left(\cos\pi n-\cos\frac{\pi}{2}n\right),$$

故 $f(x)=\pi+\displaystyle\sum_{n=1}^{\infty}\frac{4}{n^2\pi}\left(\cos\pi n-\cos\frac{\pi}{2}n\right)\cdot\cos nx$。

六、$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\dfrac{n^2-1}{(n+1)^2-1}=1$，故收敛半径$R=1$，收敛区间为$(-1,1)$，$x=\pm1$时也收敛，则收敛域为$[-1,1]$。

$S(x)=\sum\limits_{n=2}^{\infty}\dfrac{x^n}{n^2-1}=\sum\limits_{n=2}^{\infty}\dfrac{1}{2}\left(\dfrac{1}{n-1}-\dfrac{1}{n+1}\right)x^n$，其中$\sum\limits_{n=2}^{\infty}\dfrac{x^n}{n-1}=x\sum\limits_{n=1}^{\infty}\dfrac{x^n}{n};\sum\limits_{n=2}^{\infty}\dfrac{x^n}{n+1}=\dfrac{1}{x}\sum\limits_{n=2}^{\infty}\dfrac{x^n}{n^3}(x\neq0)$。

设$g(x)=\sum\limits_{n=1}^{\infty}\dfrac{x^n}{n}$，则$g'(x)=\sum\limits_{n=1}^{\infty}x^{n-1}=\dfrac{1}{x-1}\left(|x|<1\right)$，$g(x)=\int_0^x\dfrac{1}{1-t}\mathrm{d}t=-\ln(1-x)$，而$\sum\limits_{n=3}^{\infty}\dfrac{x^n}{n}=g(x)-x-\dfrac{x^2}{2}=-\ln(1-x)-x-\dfrac{x^2}{2}$，$S(x)=\dfrac{x}{2}\left[-\ln(1-x)\right]-\dfrac{1}{2x}\left[-\ln(1-x)-x-\dfrac{x^2}{2}\right]=\dfrac{2+x}{4}+\dfrac{\ln(1-x)}{2x}(1-x^2)$，此时$|x|<1且x\neq0$，且$x=0,S(0)=0$，则$S(x)=\begin{cases}\dfrac{2+x}{4}+\dfrac{\ln(1-x)}{2x}(1-x^2), & |x|<1且x\neq0\\ 0, & x=0\end{cases}$。令$x=\dfrac{1}{2}$，

$S(x)=\sum\limits_{n=2}^{\infty}\dfrac{1}{n^2-1}\cdot\dfrac{1}{2^n}=\dfrac{5}{8}-\dfrac{3}{4}\ln2$。

七、$\dfrac{\partial f(x,y)}{\partial x}=(2x+1)\mathrm{e}^{2x-y}\Rightarrow f(x,y)=x\mathrm{e}^{2x-y}+f(y)$，$f(0,y)=f(y)=y+1\Rightarrow f(x,y)=x\mathrm{e}^{2x-y}+y+1$，由全微分，$I=\int_{L_t}\mathrm{d}\big(f(x,y)\big)=f(1,t)-f(0,0)=\mathrm{e}^{2-t}+t$，$I'(t)=1-\mathrm{e}^{2-t}$，令$I'(t)=0\Rightarrow t=2$，易证$t=2$是$I(t)$的极小值点，极小值为$I(2)=3$。

八、由单调有界判别准则可知，函数$f(x)$收敛于某常数a，故由Cauchy收敛原理，$\forall\varepsilon=\dfrac{\varepsilon_0}{p},\exists N,\forall x_1,x_2>N,\left|f(x_1)-f(x_2)\right|<\dfrac{\varepsilon_0}{p}$。对于级数$\sum\limits_{n=1}^{\infty}\left[f(n)-\int_{n-1}^{n}f(x)\mathrm{d}x\right],M=\left|\sum\limits_{k=n+1}^{n+p}a_k\right|=\left|\sum\limits_{k=n+1}^{n+p}f(n)-\int_{n-1}^{n}f(x)\mathrm{d}x\right|$，由微分中值定理，$\exists n'\in(n-1,n)$使$\int_{n-1}^{n}f(x)=f(n')$，故$M=\left|\sum\limits_{k=n+1}^{n+p}f(n)-f(n')\right|$，令$n>N$，则$M<\left|\sum\limits_{k=n+1}^{n+p}\dfrac{\varepsilon_0}{p}\right|=\varepsilon_0$，由Cauchy收敛原理，原级数收敛。

2020 高数下期末答案

一、选择题

1. **A**

两个偏导数连续$\Rightarrow$可微$\Rightarrow$连续。

2. **C**

积分域为第一象限内与 x 轴正方向夹角为$\dfrac{\pi}{4}$的圆弧，易化为$\int_0^{\sqrt{2}/2}\mathrm{d}y\int_y^{\sqrt{1-y^2}} f(x,y)\mathrm{d}x$。

3. **D**

设 $x=a\cos t, y=a\sin t(0\le t\le 2\pi)$，则$\oint_L \dfrac{(x+y)\mathrm{d}x-(x-y)\mathrm{d}y}{x^2+y^2}=\int_0^{2\pi}\dfrac{1}{a^2}[a^2(\sin t+\cos t)(-\sin t)-a^2(\cos t$

$-\sin t)\cos t]\mathrm{d}t=-\int_0^{2\pi}\mathrm{d}t=-2\pi$。

4. **D**

$\sum\limits_{n=1}^{+\infty}u_n$ 收敛，$\sum\limits_{n=1}^{+\infty}u_{n+1}$ 收敛，可知$\sum\limits_{n=1}^{+\infty}(u_n+u_{n+1})$ 收敛。

5. **B**

$$a_0=\frac{1}{\pi}\int_{-\pi}^{0}-x\mathrm{d}x=\frac{\pi}{2}，\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\mathrm{d}x=\frac{1}{\pi}\int_{-\pi}^{0}-x\cos nx\mathrm{d}x=\begin{cases}-\dfrac{2}{n^2-\pi}, & n\text{为奇数}\\ 0, & n\text{为偶数}\end{cases},$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{0}-x\sin nx\mathrm{d}x=\begin{cases}-\dfrac{1}{n}, & n\text{为奇数}\\ \dfrac{1}{n}, & n\text{为偶数}\end{cases},$$

$f(x)=\dfrac{\pi}{4}-\left(\dfrac{2}{\pi}\cos x-\sin x\right)+\dfrac{\sin 2x}{2}-\left(\dfrac{2}{3^2\pi}\cos 3x-\dfrac{1}{3}\sin 3x\right)-\cdots$，$x=-\pi$时，傅里叶级数收敛于

$\dfrac{1}{2}\left[f(-\pi-0)+f(\pi-0)\right]=\dfrac{\pi}{2}$。

二、填空题

1. $\dfrac{\pi}{2}$. 法向量$\vec{n}$为$\left(1, f'(y-z),-1-f'(y-z)\right)$，$\vec{n}\cdot(1,1,1)=0$，夹角为$\dfrac{\pi}{2}$。

2. $\ln(\cos 1)$. 交换积分次序，$I=\int_0^1\mathrm{d}x\int_0^x\dfrac{\tan x}{x}\mathrm{d}y=\int_0^1\tan x\mathrm{d}x=-\ln(\cos x)\Big|_0^1=-\ln(\cos 1)$。

3. 4π. 补上 xOy 平面上的圆面 $S: x^2+y^2\le 4$，法线方向向下。使用 Gauss 公式，得

$$I=\iint\limits_{\Sigma}+\iint\limits_{S下}+\iint\limits_{S上}=\iiint\limits_{V}y\mathrm{d}V+\iint\limits_{S上}x^2\mathrm{d}x\mathrm{d}y=0+\int_0^{2\pi}\mathrm{d}\theta\int_0^2\rho^3\cos^2\theta\mathrm{d}\rho=4\pi。$$

4. 8. $\sum_{n=1}^{\infty}(-1)^{n-1}a_n-\sum_{n=1}^{\infty}a_{2n-1}=-\sum_{n=1}^{\infty}a_{2n}=2-5=-3$，故有$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}a_{2n-1}+\sum_{n=1}^{\infty}a_{2n}=5+3=8$。

5. $x^4e^{x^3}$. $\mathrm{e}^{x^3}=\sum_{n=0}^{\infty}\frac{1}{n!}x^{3n},x^4\mathrm{e}^{x^3}=\sum_{n=0}^{\infty}\frac{1}{n!}x^{3n+4}$ 。

三、计算题

1. $\varphi_1\cdot 2x\mathrm{d}x+\varphi_2\mathrm{e}^y\cos x\mathrm{d}x+\varphi_3\mathrm{d}z=0$，$\frac{\mathrm{d}z}{\mathrm{d}x}=-\frac{2x\varphi_1+\varphi_2\mathrm{e}^{\sin x}\cos x}{\varphi_3}$，$\frac{\mathrm{d}u}{\mathrm{d}x}=f_1+f_2\cos x-\frac{2x\varphi_1+\varphi_2\mathrm{e}^{\sin x}\cos x}{\varphi_3}f_3$。

2. 令$\begin{cases}f_x=2x-2xy^2=0\\ f_y=4y-2x^2y=0\end{cases}\Rightarrow\begin{cases}x=0\\ y=0\end{cases}$或$\begin{cases}x=\pm\sqrt{2}\\ y=\pm 1\end{cases}$，在 D 内的驻点有$M_1(0,0),M_2(\sqrt{2},1),M_3(-\sqrt{2},1)$，

$f(M_1)=0,f(M_2)=f(M_3)=2$。

(1) 在边界$y=0,-2\le x\le 2$上，$f(x,y)=x^2$，最小值 0，最大值 4；

(2) 在边界$x^2+y^2=4,y\ge 0$上，$f(x,y)=y^4-3y^2+4$，最小值$\frac{7}{4}$，最大 8。

3. 两端求微分，得$\begin{cases}2x\mathrm{d}x+2y\mathrm{d}y+2z\mathrm{d}z=0\\ \mathrm{d}x+\mathrm{d}y+\mathrm{d}z=0\end{cases}$，代入$(1,-2,1)$得$\begin{cases}\mathrm{d}x-2\mathrm{d}y+\mathrm{d}z=0\\ \mathrm{d}x+\mathrm{d}y+\mathrm{d}z=0\end{cases}$，易求得一组非零解为

$(\mathrm{d}x,\mathrm{d}y,\mathrm{d}z)\Big|_{p_0}=\left(-\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right)$是所求单位切向量。

$\nabla f\big|_{(0,1,2)}=\frac{2}{x^2+y^2+z^2}(x,y,z)\big|_{(0,1,2)}=\left(0,\frac{2}{5},\frac{4}{5}\right)$，$\frac{\partial f(0,1,2)}{\partial\vec{n}}=\left\langle\nabla f\big|_{(0,1,2)},\vec{n}\right\rangle=\frac{2\sqrt{2}}{5}$。

4. 在 xOy 平面投影区域$(\sigma)=\{(x,y)\,|\,x^2+y^2\le 1\}$，$V=\int_0^{2\pi}\mathrm{d}\theta\int_0^1\mathrm{d}\rho\int_{\rho}^{2-\rho^2}\mathrm{d}z=2\pi\int_0^1(2\rho-\rho^3-\rho^2)\mathrm{d}\rho=\frac{5}{6}\pi$。

(1) 对$z=\sqrt{x^2+y^2},z_x=\frac{x}{\sqrt{x^2+y^2}},z_y=\frac{y}{\sqrt{x^2+y^2}}$，$\sqrt{1+z_x^2+z_y^2}=\sqrt{2}$，$S_1=\iint\limits_{(\sigma)}\sqrt{1+z_x^2+z_y^2}\mathrm{d}\sigma=\sqrt{2}\pi$；

(2) 对$z=2-x^2-y^2,z_x=-2x,z_y=-2y$，$\sqrt{1+z_x^2+z_y^2}=\sqrt{1+4x^2+4y^2}$，$S_2=\int_0^{2\pi}\mathrm{d}\theta\int_0^1\sqrt{1+4\rho^2}\rho\mathrm{d}\rho$

$=\frac{5\sqrt{5}-1}{6}\pi$。

表面积$S=S_1+S_2=\left(\sqrt{2}+\frac{5\sqrt{5}-1}{6}\right)\pi$。

5. $\frac{\partial P}{\partial y}=6xy^2-2y\cos x=\frac{\partial Q}{\partial x}$，积分与路径无关。用折线$(0,0)\to\left(\frac{\pi}{2}\right)\to\left(\frac{\pi}{2},1\right)$代替弧线，得

$I=\int_0^{\pi/2}0\cdot\mathrm{d}x+\int_0^1\left(1-2y+\frac{3}{4}\pi^2y^2\right)\mathrm{d}y=\frac{\pi^2}{4}$。

6. 由 Gauss 公式，$0=\iiint\limits_{(V)}\left[f(x)+xf'(x)-xf(x)-e^{2x}\right]dV$。由$V$任意，知 $f(x)+xf'(x)-xf(x)-e^{2x}=0$，

解得 $f(x)=\dfrac{e^x}{x}(e^x+C)$，由于 $\lim\limits_{x\to0^+}f(x)=1$，得$C=-1$，故 $f(x)=\dfrac{e^{2x}-e^x}{x}$。

7. 由奇偶性，知 $\iiint\limits_{(V)}\left(\dfrac{2xy}{ab}+\dfrac{2yz}{bc}+\dfrac{2xz}{ac}\right)dV=0$，故 $I=\iiint\limits_{(V)}\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\right)dV$，由对称性，得

$$I=\frac{1}{3}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\iiint\limits_{(V)}(x^2+y^2+z^2)dV=\frac{1}{3}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi d\varphi\int_0^R r^4dr=\frac{4\pi}{15}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)R^5。$$

四、(1) $\dfrac{1}{1+x^2}=\sum\limits_{n=0}^{\infty}(-1)^n x^{2n}$，积分，得 $\arctan x=\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{2n+1}x^{2n+1}$。

$x\neq0$时，$f(x)=\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{2n+1}x^{2n}(x^2+1)=\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{2n+1}x^{2n+1}+\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{2n+1}x^{2n+3}=x+\sum\limits_{n=1}^{\infty}\dfrac{2\cdot(-1)^n}{1-4n^2}x^{2n+1}$；

$x=0$时，$f(x)=1$。

(2) 令$x=1$，得$1+\sum\limits_{n=1}^{\infty}\dfrac{2\cdot(-1)^n}{1-4n^2}=f(1)=\dfrac{\pi}{2}$，$\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{1-4n^2}=\dfrac{1}{2}\left(\dfrac{\pi}{2}-1\right)=\dfrac{\pi}{4}-\dfrac{1}{2}$。

五、两次分部积分，$a_n=\dfrac{2}{\pi}\int_0^{\pi}f(x)\cos nx dx=\dfrac{2}{\pi}\int_0^{\pi}\dfrac{1}{n}f(x)d(\sin nx)=-\dfrac{2}{\pi}\int_0^{\pi}\dfrac{f'(x)}{n}\sin nx dx$

$$=\frac{2}{\pi}\left[-\frac{f'(0)}{n^2}+\frac{f'(\pi)}{n^2}\cos n\pi-\int_0^{\pi}\frac{f''(x)}{n^2}\cos nx dx\right]。$$

$f''(x)$连续，故在$[-\pi,\pi]$上有界，$|a_n|\le\dfrac{1}{n^2}\cdot\dfrac{2}{\pi}\left[|f'(0)|+|f'(\pi)\cos n\pi|+\pi|f'(x)|_{\max}\right]=C\cdot\dfrac{1}{n^2}$，

C 为一正的常数。

$\sum\limits_{n=1}^{\infty}C\cdot\dfrac{1}{n^2}$收敛$\Rightarrow\sum\limits_{n=1}^{\infty}|a_n|$收敛$\Rightarrow\sum\limits_{n=0}^{\infty}a_n$绝对收敛

2019 年高等数学下册期末试题答案

一、填空题

1. $\frac{10}{3}$

   详解: 函数在某方向的方向导数为函数在该点的梯度向量在该方向的投影, 取该方向的单位向量 $\vec{e} = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$, 梯度向量 $= (2y, 2x, -2z)|_{(2,-1,1)} = (-2, 4, -2)$，故方向导数为 $= \frac{1}{3} \times (-2) + \frac{2}{3} \times 4 + \left(-\frac{2}{3}\right) \times (-2) = \frac{10}{3}$。

2. $(-3, 1)$

   详解: $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{2\ln n}{\ln(n+1)} = 2$, 故 $x + 1 \in (-2, 2)$ 解得收敛域为 $(-3, 1)$。

3. $4x + 2y - z - 6 = 0$

   详解: 根据曲面的方程可知在 M_0 处法向量为 $(4, 2, -1)$，故切平面的法向量为 $(4, 2, -1)$，进而求得切平面方程为 $4(x - 2) + 2(y - 1) - (z - 4) = 0$

4. $\frac{3}{8}\left(e^4 - 1\right)$

   详解: 记积分的直线段为 $(t, 2t, 2t)(0 \le t \le 1)$ 则对应的曲线积分转化为 $\int_0^1 te^{2t\cdot 2t}\sqrt{1^2 + 2^2 + 2^2}dt = \frac{3}{8}\left(e^4 - 1\right)$

5. $\frac{3}{4}$

   详解: 由 $f(x)$ 展开的 Fourier 级数的形式可以看出采用的偶延拓的方式，故延拓函数 $F(x)$ 的周期为 2，且易得该函数在点 $x = -\frac{1}{2}$ 处间断，由 Dirichlet 定理:

$$S\left(-\frac{5}{2}\right) = S\left(-\frac{1}{2}\right) = \frac{1}{2}\left(F\left(-\frac{1}{2} - 0\right) + F\left(-\frac{1}{2} + 0\right)\right) = \frac{3}{4}$$

二、计算题

1. 解:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= f_x + f_z\frac{\partial z}{\partial x} = f_x + f_z \cdot e^x \sin y \\
\frac{\partial^2 u}{\partial x\partial y} &= f_{xy} + f_{xz} \cdot \frac{\partial z}{\partial y} + \left(f_{zy} + f_{zz}\frac{\partial z}{\partial y}\right) e^x \sin y + f_z e^x \cos y \\
&= f_{xy} + f_{xz}e^x \cos y + f_z e^x \cos y + e^x \sin y \left(f_{zy} f_{zy} + f_{zz} \cdot e^x \cos y\right)
\end{aligned}$$

2. 解: 根据交线的方程可以设交线的参数方程为 $\begin{cases} x = \cos t \\ y = 1 + \sin t \\ z = 3 - \sin t \end{cases}, t \in (0, 2\pi)$

$$\begin{aligned}
\int_C -y^2\,\mathrm{d}x + x dy + z^2\,\mathrm{d}z &= \int_0^{2\pi} \left[-(1+\sin t)^2(-\sin t) + \cos t \cdot \cos t + (3-\sin t)^2(-\cos t)\right] \mathrm{d}t \\
&= \int_0^{2\pi} \left(\sin^3 t + 2\sin^2 t + \sin t + \cos^2 t - 9\cos t + 6\sin t\cos t - \sin^2 t\cos t\right) \mathrm{d}t \\
&= \int_0^{2\pi} \left(\sin^3 t + \cos^3 t + \sin^2 t + \sin t - 10\cos t + 6\sin t\cos t + 1\right) \mathrm{d}t \\
&= 3\pi
\end{aligned}$$

除了通过引入 t 直接对线积分进行转换以外，也可以通过 Stokes 公式将线积分转换为面积分来求解，具体求解过程如下所示：

$$\begin{aligned}
I &= \iint_{\Sigma_{上}} (1+2y)\mathrm{d}x \wedge \mathrm{d}y \\
&= \iint_{x^2+y^2 \le 2y} (1+2y)\mathrm{d}x\mathrm{d}y \\
&= \int_0^{\pi} d\varphi \int_0^{2\sin\varphi} (1 + 2\rho\sin\varphi)\rho d\rho \\
&= 3\pi
\end{aligned}$$

3. 解:Σ 在 xOy 平面上的投影为 $D_{xy} = \left\{(x,y) | x^2 + y^2 \le 4\right\}$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}, dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y = \sqrt{2}\,\mathrm{d}x\,\mathrm{d}y$$

题目中所求积分为:

$$\begin{aligned}
\iiint_{\Sigma} \left(x^2+y^2\right)\,\mathrm{d}S &= \iiint_{D_{xy}} \sqrt{2}\left(x^2+y^2\right)\,\mathrm{d}x\,\mathrm{d}y \\
&= \sqrt{2}\int_0^{\pi}\,\mathrm{d}\theta \int_0^2 \rho^3\,\mathrm{d}\rho \\
&= 8\sqrt{2}\pi
\end{aligned}$$

三、计算题

1. 解:Ω 是旋转抛物面圆锥面的所围成的闭区域, 在 xOy 平面的投影域为 $\begin{cases} x^2 + y^2 \le 1 \\ z = 1 \end{cases}$。观察方程的形式，这

里采取“先单后重”的积分方式:

$$V=\iiint_{\Omega} dV=\int_0^{2\pi} \mathrm{d}\theta \int_0^1 \rho\, \mathrm{d}\rho \int_{\rho^2}^{1-\rho} \mathrm{d}z=$$
$$=2\pi\int_0^1 \left(2-\rho-\rho^2\right)\mathrm{d}\rho$$
$$=\frac{5}{6}\pi$$

2. 解:

$$\sum_{n=0}^{\infty}\frac{(2n+1)x^{2n}}{n!}=\sum_{n=0}^{\infty}\frac{\left(x^{2n+1}\right)'}{n!}=\left(x\sum_{n=0}^{\infty}\frac{\left(x^2\right)^n}{n!}\right)'=\left(xe^{x^2}\right)'=\left(1+2x^2\right)e^{x^2}$$

又 $\because R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|\to\infty$
故上述级数的收敛域为 $(-\infty,\infty)$, 该幂级数的和函数为 $S(x)=\left(1+2x^2\right)e^{x^2}\ (x\in(-\infty,\infty))$

3. 解:$\iiint_{\Omega} 2\sin y\,\mathrm{d}V$ 的积分函数是关于 y 的奇函数，积分域 Ω 关于 xOz 平面对称，由对称性可得 $\iiint_{\Omega} 2\sin y\,\mathrm{d}V=0$。

$$\iiint_{\Omega} zdV=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{4}}\mathrm{d}\varphi\int_0^{2\cos\varphi} r^3\cos\varphi\sin\varphi dr$$
$$=8\pi\int_0^{\frac{\pi}{4}}\cos^5\varphi\sin\varphi\mathrm{d}\varphi$$
$$=\frac{8}{6}\pi\left(-\cos^6\varphi\right)\Big|_0^{\frac{\pi}{4}}$$
$$=\frac{7\pi}{6}$$

四、解答题

1. 解：记 $P(x,y)=\frac{-y}{x^2+y^2}\quad Q(x,y)=\frac{x}{x^2+y^2}$, 由 $\frac{\partial P}{\partial y}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}=\frac{\partial Q}{\partial x}$ 故曲线积分的值与路径无关，考虑到积分函数的分母，这里取路径：$x=\pi\cos t, y=\pi\sin t, t$ 从 π 到 0

$$\int_L\frac{-ydx+xdy}{x^2+y^2}=\frac{1}{\pi^2}\int_{\pi}^{0}[-\pi\sin t(-\pi\sin t)+\pi\cos t(\pi\sin t)]dt=-\pi$$

2. 解：椭圆上一点 $\mathrm{P}(x_0,y_0,0)$ 到点 M 距离的平方 $d^2=x^2+y^2+4$，点 P 的坐标满足 $5x_0-6x_0y_0+5y_0^2=4$。
问题转化为函数的有约束极值问题: 取 $F(x,y,\lambda)=x^2+y^2+4+\lambda\left(5x^2-6xy+5y^2-4\right)$

令 $\begin{cases} F_x=2x+10\pi x-6\lambda y=0 \\ F_y=2y+10\pi y-6\pi x=0 \\ F_\lambda=5x^2-6xy+5y^2-4=0 \end{cases}$

解得：$M_1(1,1,0), M_2(-1,-1,0), M_3(\frac{1}{2},-\frac{1}{2},0), M_4(-\frac{1}{2},\frac{1}{2},0)$。

$$d|_{M_1}=d|_{M_2}=\sqrt{6},\ d|_{M_3}=d|_{M_4}=\frac{3\sqrt{2}}{2}$$

故椭圆上的点到 M 的最长距离为 $\sqrt{6}$, 最短距离为 $\dfrac{3\sqrt{2}}{2}$

3. 解：向量场通过曲面的通量可通过第二型面积分计算。针对本题可将面积分写成坐标形式，然后借助高斯公式求解。

通量 $\Phi=\iint\limits_{\Sigma}(2x+z)\mathrm{d}y\wedge\mathrm{d}z+y^2\,\mathrm{d}z\wedge\mathrm{d}x+z\,\mathrm{d}x\wedge\mathrm{d}y$，作有向曲面 $\Sigma_1: z=1(x^2+y^2\le 1)$，并取上侧，设曲面 Σ 和 Σ_1 所围成的闭区域为 Ω，

记 $D_{xy}=\left\{(x,y)\mid x^2+y^2\le 1\right\}$, 由高斯公式, 得：

$$
\begin{aligned}
\Phi&=\oiint\limits_{\Sigma+\Sigma_1}-\iint\limits_{\Sigma_1}(2x+z)\mathrm{d}y\wedge\mathrm{d}z+y^2\,\mathrm{d}z\wedge\mathrm{d}x+z\,\mathrm{d}x\wedge\mathrm{d}y\\
&=\iiint\limits_{\Omega}(3+2y)\mathrm{d}V-\iint\limits_{\Sigma_1}z\,\mathrm{d}x\wedge\mathrm{d}y=\iiint\limits_{\Omega}3\,\mathrm{d}V-\iint\limits_{\Sigma_1}z\,\mathrm{d}x\wedge\mathrm{d}y\\
&=3\int_0^{2\pi}\mathrm{d}\theta\int_0^1\rho\mathrm{d}\rho\int_{\rho^2}^1\mathrm{d}z-\iint\limits_{D_{xy}}\mathrm{d}x\,\mathrm{d}y=6\pi\int_0^1\rho\left(1-\rho^2\right)\mathrm{d}\rho-\pi=\frac{3\pi}{2}-\pi=\frac{\pi}{2}
\end{aligned}
$$

4. 解: 考虑到 $f(x)$ 在对应区间上是奇函数，所以将函数展开成傅里叶级数后的 $a_n=0(n=0,1,2,\cdots)$，

$$
\begin{aligned}
b_n&=\frac{2}{\pi}\int_0^\pi\sin\frac{x}{2}\sin nx\,\mathrm{d}x=\frac{1}{\pi}\int_0^\pi\left[\cos\left(n-\frac{1}{2}\right)x-\cos\left(n+\frac{1}{2}\right)x\right]\mathrm{d}x\\
&=\frac{1}{\pi}\left[\frac{2}{2n-1}\sin\left(n-\frac{1}{2}\right)x-\frac{2}{2n+1}\sin\left(n+\frac{1}{2}\right)x\right]_0^\pi=\frac{(-1)^{n-1}8n}{(4n^2-1)\,\pi},
\end{aligned}
$$

当 $x=\pm\pi, S(\pm\pi)=0$, 故 $f(x)=\dfrac{8}{\pi}\sum\limits_{n=1}^{\infty}\dfrac{n(-1)^{n-1}}{4n^2-1}\sin nx(-\pi<x<\pi)$

五、

解: 观察所求级数的和，我们希望得到类似 $\sum\dfrac{x^n}{n(n-1)}$ 的级数求和形式，因为观察到 n 在分母上，因此应该考虑先求导再做积分:

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n, x\in(-1,1]$$

$$f'(x)=1+\ln(1+x)=1+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n, x\in(-1,1]$$

又 $\because f(0)=0$

$$\therefore f(x)=f(0)+\int_0^x f'(t)\mathrm{d}t=x+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n(n+1)}x^{n+1}=x+\sum_{n=2}^{\infty}\frac{(-1)^n}{n(n-1)}x^n, x\in(-1,1].$$

且 $\sum\limits_{n=2}^{\infty}\dfrac{(-1)^n}{n(n-1)}=f(1)-1=2\ln 2-1.$

六、

证明：由格林公式, 得

$$I=\oint_L xe^{\sin y}\,\mathrm{d}y-ye^{-\sin x}\,\mathrm{d}x=\iint_D\left(e^{\sin y}+e^{-\sin x}\right)\mathrm{d}x\,\mathrm{d}y.$$

D 关于直线 $y = x$ 对称, 由轮换对称性,

$$\iint_D e^{\sin y}\mathrm{d}x\mathrm{d}y = \iint_D e^{\sin x}\mathrm{d}x\mathrm{d}y$$

于是 $I = \iint_D \left(\mathrm{e}^{\sin y} + \mathrm{e}^{-\sin x}\right)\mathrm{d}x\,\mathrm{d}y = \int_0^\pi \mathrm{d}y \int_0^\pi \left(\mathrm{e}^{\sin x} + \mathrm{e}^{-\sin x}\right)\mathrm{d}x = \pi\int_0^\pi \left(\mathrm{e}^{\sin x} + \mathrm{e}^{-\sin x}\right)\mathrm{d}x.$

由于 $f(u) = \mathrm{e}^u + \mathrm{e}^{-u} = \sum\limits_{n=0}^{\infty} \dfrac{2}{(2n)!}x^{2n} \geq 2\left(1 + \dfrac{1}{2}u^2\right) = 2 + u^2$

故 $I = \pi\int_0^\pi \left(\mathrm{e}^{\sin x} + \mathrm{e}^{-\sin x}\right)\mathrm{d}x \geq \pi\int_0^\pi \left(2 + \sin^2 x\right)\mathrm{d}x = \dfrac{5}{2}\pi^2$

2018 年高数下期末答案

一、单选题

1. D

解析：偏导数连续 $\Rightarrow$ 可微 $\Rightarrow \begin{cases} \text{可偏导} \\ \text{连续} \end{cases}$

以上均为充分条件，反之均无法推出，且可偏导与连续无法互相推出

2. B

解析：$\iint\limits_D \dfrac{\partial^2 f(x,y)}{\partial x \partial y} dxdy = \int_c^d \int_a^b \dfrac{\partial^2 f(x,y)}{\partial x \partial y} dxdy = \int_c^d \left[\dfrac{\partial f(b,y)}{\partial y} - \dfrac{\partial f(a,y)}{\partial y} \right] dy$

$= f(b,d) - f(a,d) - f(b,c) + f(a,c)$

3. A

解析：$\because$ 球面与平面关于原点中心对称 $\quad \therefore$ L 关于原点中心对称

$\therefore I = \oint_L (x+1)^2 ds = \dfrac{1}{3}\oint_L \left[(x+1)^2 + (y+1)^2 + (z+1)^2\right] ds = \dfrac{1}{3}\oint_L (x^2+y^2+z^2+3+2x+2y+2z) ds$

$= \dfrac{1}{3}\oint_L (x^2+y^2+z^2+3)^2 ds = \dfrac{1}{3}\oint_L (4+3) ds = \dfrac{1}{3} \times 7 \times 4\pi \quad \dfrac{28}{3}\pi$

4. B

解析：$F(t) = \int_1^t dy \int_y^t f(x) dx = \int_1^t dx \int_1^x f(x) dy = \int_1^t (x-1) f(x) dx \qquad F'(t) = (t-1) f(t) \Rightarrow F'(2) = f(2)$

二、 计算题

1. 设 $F(x,y,z) = e^z - z + xy - 3 \qquad F_x = y \qquad F_y = x \qquad F_z = e^z - 1$

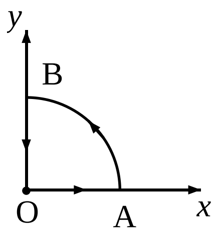
$\therefore (F_x, F_y, F_z)|_{(z,1,0)} = (1,2,0)$ 切平面：$x - 2 + 2(y-1) = 0 \Rightarrow x + 2y - 4 = 0$ 法线：$x - 2 = \dfrac{y-1}{2} = \dfrac{z}{0}$

2. $I = \iiint\limits_V (x^2+y^2)^2 dV = \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{z}} \rho^2 \cdot \rho d\rho d\theta dz = \dfrac{\pi}{6}$

3. $z = \sqrt{4 - r^2} \qquad z_x = \dfrac{-x}{\sqrt{4-r^2}} \qquad z_y = \dfrac{-y}{\sqrt{4-r^2}}$

原式 $= \iint\limits_{(S)} (x+y+z) ds = \iint\limits_{(S)} z ds = \iint\limits_{(S)} \sqrt{4-r^2} \bullet \sqrt{1 + {z_x}^2 + {z_y}^2} d\sigma = \iint\limits_D 2 d\sigma = 2 \times 4\pi = 8\pi$

4. 设 $O(0,0)$，则 $I + I_{BO} + I_{OA} = \oint \left[y^2 + \sin^2(x+y)\right] dx + \left[x^2 - \cos^2(x+y)\right] dy$

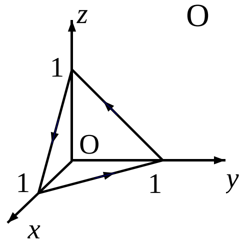


由格林公式：

上式 $= \iint\limits_D [2x + 2\cos(x+y)\sin(x+y) - 2y - 2\sin(x+y)\cos(x+y)] d\sigma$

$= 2\iint\limits_D (x-y) d\sigma = 0$ （如右图，积分域 D 关于 $y = x$ 对称）

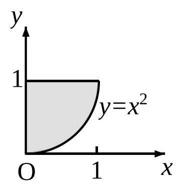
$I = -I_{BO} - I_{OA} = \int_1^0 \cos^2 y dy - \int_0^1 \sin^2 x dx = -\int_0^1 (\sin^2 x + \cos^2) dx = -1$



5. 由 Stokes 公式：$I = \iint\limits_{(S)} dy \wedge dz + dz \wedge dx + dx \wedge dy = 3 \times \dfrac{1}{2} = \dfrac{3}{2}$

6. $\vec{A} = gradf(x,y,z) = (\dfrac{2x}{r^2}, \dfrac{2y}{r^2}, \dfrac{2z}{r^2}) \qquad \text{div}\,\vec{A} = \dfrac{2}{r^2} = \dfrac{2}{x^2+y^2+z^2} \qquad \text{rot}\,\vec{A} = 0$

7. $I = \int_0^1 \int_0^{\sqrt{y^2}} \dfrac{xy}{\sqrt{1+y^3}} dxdy = \int_0^1 \dfrac{y^2}{2\sqrt{1+y^3}} = \dfrac{1}{3}\sqrt{1+y^3}\,|_0^1 = \dfrac{\sqrt{2}-1}{3}$



三、解答题

1. $\lim\limits_{\substack{x\to0\\y\to0}} f(x,y)=\lim\limits_{\substack{x\to0\\y\to0}} xy\bullet\frac{1}{\sqrt{x^2+y^2}}\le\lim\limits_{\substack{x\to0\\y\to0}}|x|\bullet|\frac{y}{x^2+y^2}|\le\lim\limits_{x\to0}|x|=0=f(0,0)$ $\quad\therefore$连续

$f_x(0,0)=\lim\limits_{\Delta x\to0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0\qquad f_y(0,0)=\lim\limits_{\Delta y\to0}\frac{f(0,\Delta y,)-f(0,0)}{\Delta y}=0$ $\quad\therefore$偏导数存在

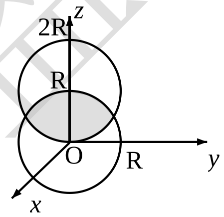
$$\lim_{\substack{\Delta x\to0\\ \Delta y\to0}}\frac{f(\Delta x,\Delta y)-f_x(0,0)\Delta x-f_y(0,0)\Delta y-f(0,0)}{\sqrt{\Delta x^2+\Delta y^2}}=\lim_{\substack{\Delta x\to0\\ \Delta y\to0}}\frac{\Delta x\Delta y\arctan\frac{1}{\sqrt{\Delta x^2+\Delta y^2}}}{\sqrt{\Delta x^2+\Delta y^2}}=\lim_{\substack{\Delta x\to0\\ \Delta y\to0}}\frac{\Delta x\Delta y}{\Delta x^2+\Delta y^2}$$

令 $\Delta y=k\Delta x$ 则上式 $=\frac{k}{1+k^2}$ 故极限不存在 $\quad\therefore f(x,y)$ 在 $(0,0)$ 不可微

2. $\frac{\partial u}{\partial\vec n}=(2x,2y,2z)\bullet(\frac{1}{\sqrt2},-\frac{1}{\sqrt2},0)=\sqrt2(x-y)$

$2x^2+2y^2+z^2=1\Rightarrow x^2+y^2\le\frac12\Rightarrow|x-y|\le1\qquad\therefore(\frac{\partial u}{\partial\vec n})|_{\max}=\sqrt2$

3. 由高斯公式，积分域如右图：



$$I=\iiint\limits_V(1+1+2z)dV=\int_0^{\frac R2}\pi\left[R^2-(z-R)^2\right](2+2z)dz+\int_{\frac R2}^{R}\pi(R^2-z^2)(2+2z)dz$$

$=(\frac{R^4}{12}+R^3)\pi$

4. 令 $P=\frac{y}{(2-x)^2+y^2}+\frac{y}{(2+x)^2+y^2}\qquad Q=\frac{2-x}{(2-x)^2+y^2}-\frac{2+x}{(2+x)^2+y^2}$

当 L 不包含（2，0）和（−2，0）时 $I=\iint\limits_{(\sigma)}(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})d\sigma=0$

当 L 包含（2，0）时，取 $(x-2)^2+y=\varepsilon_1{}^2$ 为 $L_1(-)$，其中 $\varepsilon_1>0$ 且足够小

$x=\varepsilon_1\cos\theta+2\qquad y=\varepsilon_1\sin\theta\qquad$ 则 $I+I_1=0\Rightarrow I=\int\limits_{(+L_1)}\frac{ydx+(2-x)dy}{(2-x)^2+y^2}=\int_0^{2\pi}(-1)d\theta=-2\pi$

当 L 包含（−2，0）时取 $(x+2)^2+y=\varepsilon_2{}^2$ 为 $L_2(-)$，其中 $\varepsilon_2>0$ 且足够小

同理得 $I=\int\limits_{(+L_2)}\frac{ydx+(2+x)dy}{(2+x)^2+y^2}=-2\pi$

当 L 包含（2，0）和（2，0）时取 $L_1(-)$，$L_2(-)$，则 $I+I_1+I_2=0\Rightarrow I=-I_1-I_2=-4\pi$

2017 年高数下期末答案

一、计算题

1. $\mathrm{grad}\,u=(8x,2y,2z)=(8,0,4)\qquad(\frac{\partial f}{\partial\vec l})=4\sqrt5$

2. $\frac{a_{n+1}}{a_n}=\frac{a_{n+1}\sin\frac{\pi}{2^{n+1}}}{a_n\sin\frac{\pi}{2^n}}=\frac a2\qquad$ 当 $a=2$ 时 $a_n=2^n\sin\frac{\pi}{2^n}=\pi$

$\therefore$发散 $\qquad$ 故 $0<a<2$ 时收敛，$a\ge2$ 时发散

3. 偶延拓：$F(x)=\begin{cases}x,0\le x\le\pi\\-x,-\pi\le x\le0\end{cases}$ 则 $f(x)=\frac{a_0}{2}+\sum\limits_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$

$a_n=\frac{2}{\pi}\int_0^{\pi}x\cos dx=\frac{2[(-1)^n-1]}{n^2\pi}$ $\quad a_0=\frac{2}{\pi}\int_0^{\pi}xdx=\pi$ $\quad \therefore f(x)=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2[(-1)^n-1]}{n^2\pi}\cos n\pi$

4. $\frac{\partial u}{\partial x}=f'(t)(y\varphi_1+\varphi_2)$

$$\frac{\partial^2 u}{\partial x\partial y}=(y\varphi_1+\varphi_2)f''(t)\times x\varphi_1+f'(t)(\varphi_1+xy\varphi_{11}+x\varphi_{21})=x\varphi_1(u\varphi_1+\varphi_2)f''(t)+(\varphi_1+xy\varphi_{11}+x\varphi_{21})f'(t)$$

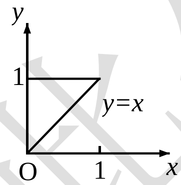
5. $\vec{n}=(1,-2t,3t^2)$ $\quad (1,-2t,3t^2)\cdot(1,2,1)=0\Rightarrow 1-4t+3t^2=0\Rightarrow t_1=1,t_2=\frac{1}{3}$

切点$(1,-1,1),B(\frac{1}{3},-\frac{1}{9},\frac{1}{27})$均不在平面上 故切线$\frac{x-1}{1}=\frac{y+\frac{1}{2}}{-2}=\frac{z-1}{3}$或$\frac{x-\frac{1}{3}}{1}=\frac{y+\frac{1}{9}}{-\frac{2}{3}}=\frac{z-\frac{1}{27}}{\frac{1}{3}}$

6. $f_x=3x^2+6x-9=0$ $\quad f_y=-3y^2+6y=0\Rightarrow$ 驻点$(1,0),(1,2),(-3,0),(-3,2)$

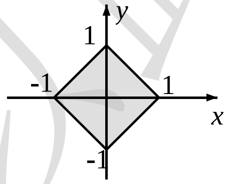
$A=f_{xx}=x+6,B=f_{xy}=0,C=f_{yy}=-6y+6$ 代入得$(1,0)$处取得极小值−5，$(-3,2)$处取得极大值 31

7. $\int_0^1 dy\int_0^y x^2e^{-y^2}dx=\int_0^1\frac{y^3}{3}e^{-y^2}dy$ $\quad$ 令$u=y^2$



则$I=\int_0^1\frac{u}{6}e^{-u}du=-\frac{1}{6}\int_0^1 ude^{-u}=-\frac{1}{6}[ue^{-u}|_0^1-\int_0^1 e^{-u}du]=\frac{1-2e^{-1}}{6}$

8. 由对称性知$\iint\limits_D xydxdy=0$ 设D_1为第一象限的区域



$\therefore I=\iint\limits_D |y|dxdy=4\iint\limits_{D_1}ydxdy=4\int_0^1dx\int_0^{1-x}ydy=\frac{2}{3}$

9. 由高斯公式：

$$I=\iiint\limits_{(V)}\frac{3r^3-3r(x^4+y^4+z^4)}{r^6}dV=3\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\frac{(x^2+y^2+z^2)^2-(x^4+y^4+z^4)}{r^5}r^2\sin\theta drd\theta d\varphi$$

$$=3\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\frac{2(x^2y^2+x^2z^2+y^2z^2)}{r^3}\sin\theta drd\theta d\varphi=9\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\frac{z^2(x^2+y^2)}{r^3}\sin\theta drd\theta d\varphi$$

$$=9\int_0^{2\pi}\int_0^{\pi}\int_0^{a}r\sin^3\theta\cos^2\theta drd\theta d\varphi=9\int_0^{2\pi}d\varphi\int_0^{\pi}\sin^3\theta\cos^2\theta d\theta\int_0^a rdr=\frac{12}{5}\pi a^2$$

10. $I=\int\limits_L\sqrt{2y^2+z^2}ds=\int\limits_L\sqrt{x^2+y^2+z^2}ds=\int\limits_L ads=a\times 2a\pi=4\pi a^2$

11. $S=\iint\limits_{(S)}ds=\iint\limits_{(S)}\sqrt{1+y^2+x^2}dxdy=\int_0^{2\pi}\int_0^R\rho\sqrt{1+\rho^2}d\rho d\theta=\frac{2}{3}\pi[(1+R^2)^{\frac{3}{2}}-1]$

二．解答题

1. $f_x(0,0)=\lim\limits_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\Delta x\sin\frac{1}{\Delta x^2}=0$

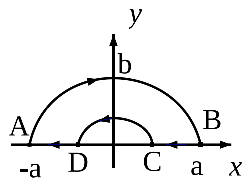
$f_y(0,0)=\lim\limits_{\Delta y\to 0}\frac{f(\Delta y,0)-f(0,0)}{\Delta y}=\lim\limits_{\Delta y\to 0}\Delta y\sin\frac{1}{\Delta y^2}=0$ $\quad\therefore$偏导数存在

$f_x(0,0)=\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(\Delta x,\Delta y)-f_x(0,0)\Delta x-f_y(0,0)\Delta y-f\left(0,0\right)}{\sqrt{\Delta x^2+\Delta y^2}}=\sqrt{\Delta x^2+\Delta y^2}\sin\frac{1}{\Delta x^2+\Delta y^2}=0$ $\quad\therefore$可微

$\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}f_x=\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}[2x\sin\frac{1}{x^2+y^2}+\frac{-2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}]=\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{-2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}$不存在

$\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}f_y=\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}[2y\sin\frac{1}{x^2+y^2}+\frac{-2y}{x^2+y^2}\cos\frac{1}{x^2+y^2}]$不存在 $\quad$ 偏导数不连续

2. $P=\dfrac{x-y}{x^2+y^2}$， $Q=\dfrac{x+y}{x^2+y^2}$ ， $\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$ 取 $x^2+y^2=\varepsilon^2$， $\varepsilon>0$ 足够小



$x=\varepsilon\cos\theta, y=\varepsilon\sin\theta$ 则 $I+I_{\overline{BC}}+I_{CD}+I_{\overline{DA}}=0$

$$I=I_{\overline{CB}}+I_{DC}+I_{\overline{AD}}=\int_\varepsilon^a\frac{1}{x}dx-\int_0^\pi d\theta+\int_{-a}^{-\varepsilon}\frac{1}{x}dx=-\pi$$

3.（1）$\lambda(x)=\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\dfrac{(n+1)^2+1}{3^{n+1}(n+1)!}x^{n+1}/(\dfrac{n^2+1}{3^n n!}x^n)=\lim\limits_{n\to\infty}\dfrac{(n+1)^2+1}{3(n^2+1)(n+1)}x=0$ 收敛域为 $(-\infty,\infty)$

$$\sum_{n=0}^{\infty}\frac{n^2+1}{3^n n!}x^n=\sum_{n=0}^{\infty}\frac{n^2}{3^n n!}x^n+\frac{x^n}{3^n n!}=\sum_{n=1}^{\infty}\frac{nx^n}{3^n(n-1)!}+\sum_{n=1}^{\infty}\frac{x^n}{3^n n!}=\sum_{n=1}^{\infty}\frac{(n-1)x^n+x^n}{3^n(n-1)!}+\sum_{n=0}^{\infty}\frac{x^n}{3^n n!}$$

$$=\sum_{n=2}^{\infty}\frac{x^n}{3^n(n-1)!}+\sum_{n=0}^{\infty}\frac{x^n}{3^n n!}=[(\frac{x}{3})^2+\frac{x}{3}+1]e^{\frac{x}{3}}=\frac{x^2+3x+9}{9}e^{\frac{x}{3}}$$

（2）$a_n=|\dfrac{\cos nx}{1+n^{\frac{3}{2}}}|\le\dfrac{1}{1+n^{\frac{3}{2}}}<\dfrac{1}{n^{\frac{3}{2}}}$ $\because\sum\limits_1^{\infty}\dfrac{1}{n^{\frac{3}{2}}}$ 收敛 $\therefore$ 原级数一致收敛

4. 由 Stoke 公式：$I=\iint\limits_{(S)}(2y-2z)dy\wedge dz+(2z-2x)dz\wedge dx+$（$2x-2y)dx\wedge dy$

由对称性知：$I=\iint\limits_{(S)}2xdx\wedge dy=4\iint\limits_{(x^2+y^2\le 2x)}xdxdy=8\int_0^{\frac{\pi}{2}}d\varphi\int_0^{2\cos\theta}(\rho\cos\theta+1)\rho d\rho=8\pi$

2016 年高数下期末答案

一、填空题

1. 1

解析：$\dfrac{\partial f}{\partial y}=x^2+y^2+1\Rightarrow f(x,y)=\dfrac{x^3}{3}+xy+x+g(y)$ $\dfrac{\partial f}{\partial y}=x+g'(y)\Rightarrow a=1$

2. $dx+dy+dz$

解析：$f_x=f_y=f_z=\cos^2(x+y+z)^2=1$

3. $e^{\frac{1}{2}}-1$

解析：$\int_0^1\int_x^1 e^{\frac{y^2}{2}}dydx=\int_0^1\int_0^y e^{\frac{y^2}{2}}dxdy=\int_0^1 ye^{\frac{y^2}{2}}dy=e^{\frac{1}{2}}-1$

4．5

解析：$L(x,y,\lambda)=3x+4y+\lambda(x^2+y^2-1)$

$L_x=2\lambda x+3=0, L_y=2\lambda y+4=0, L_\lambda=x^2+y^2=0$ $\Rightarrow x=\pm\dfrac{3}{5}, y=\pm\dfrac{4}{5}, \Rightarrow z_{\max}=5$

5. $\dfrac{\pi^2-\pi+1}{2}$

解析：$S(-\pi)=\dfrac{f(-\pi)+f(\pi)}{2}=\dfrac{1-\pi+\pi^2}{2}$

二. 单选题

1. C

解析：两个偏导数均连续是可微的充分条件，C 为其逆否命题，显然正确

2.D

解析：假设 $\exists f(x,y)>0$，则一定存在一点 $A(x_1,y_1)$ 为极值点且极值大于 0

$f_x(x_1,y_1)=f_y(x_1,y_1)=0\Rightarrow f(x_1,y_1)=-f_x(x_1,y_1)-2f_y(x_1,y_1)=0$ 与假设矛盾

故不存在 $f(x,y)>0$ 的点，同理也不存在 $f(x,y)<0$ 的点 故 $f(x,y)=0$

3. A

解析：$\because e^{xyz}>0$ $\therefore$ 积分域越大则 I 越大，而积分域 $V_3<V_1<V_2$ 故 $I_3<I_1<I_2$

4. B

解析：$W=\int_L \vec{F}\bullet d\vec{S}=\int_L Pdx+Qdy=\int_L P(x,y)dx$

5. C

解析 $\int_L (x+y)^2 ds=\int_L (x^2+y^2+2xy)ds=\int_L (x^2+y^2)ds=\int_L a^2 ds=2\pi a^3$

6. C

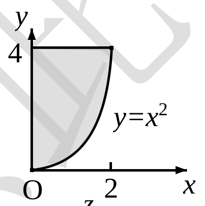
解析：$\sum_{n=1}^{\infty}(|a_n|+|b_n|)\ge\sum_{n=1}^{\infty}|b_n|\ge\sum_{n=1}^{\infty}b_n$ $\therefore C$ 发散

取 $a_n=0$，则 A，B 均为 0，收敛　取 $a_n=0$，$b_n=\dfrac{1}{n}$ 则 $\sum_{n=1}^{\infty}\dfrac{1}{n^2}$ 收敛

三．简答题

1. $\dfrac{\partial z}{\partial x}=yf_1$ $\dfrac{\partial^2 z}{\partial x\partial y}=f_1+y(xf_{11}+f_{12}\cos y)$

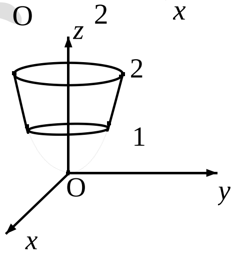
2. 令 $x=t$，则 $y^2=6-3t^2, z=t$， $\dot{x}=1,\dot{y}=-\dfrac{3t}{y}=-\sqrt{3},\dot{z}=1$



$\therefore$ 切线：$\dfrac{x-1}{1}=\dfrac{y-\sqrt{3}}{-\sqrt{3}}=\dfrac{z-1}{1}$

法平面：$x-1-\sqrt{3}(y-\sqrt{3})+z-1=0\Rightarrow x-\sqrt{3}+z+1=0$

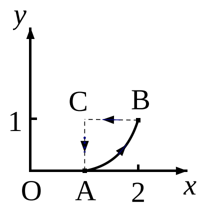
3. 积分域如右图：$I=\int_0^4\int_0^{\sqrt{y}}\dfrac{x\cos y}{y}dxdy=\int_0^4\dfrac{\cos y}{2}dy=\dfrac{\sin 4}{2}$



4. 积分域如右图：$I=\int_1^2\int_0^{2\pi}\int_0^z \rho\cdot\rho d\rho d\theta dz=\dfrac{5}{2}\pi$

5. $f_x=4x-3y-1=0, f_y=-3x+4y+2=0\Rightarrow x=-\dfrac{2}{7}, y=-\dfrac{5}{7}$

$f_{xx}=4, f_{xy}=-3, f_{yy}=4$ $\because f_{xx}\cdot f_{yy}>f_{xy}{}^2$ 且 $f_{xx}>0$ $\therefore$ 极小值为 $-\dfrac{4}{7}$



6. 由格林公式 $I_L+I_{BC}+I_{CA}=\iint\limits_{(\sigma)}(\dfrac{e^y}{x}-1-\dfrac{e^y}{x})d\sigma=-\dfrac{\pi}{4}$

$I_{BC}=\int_2^1(1+\dfrac{e}{x})dx=-e\ln 2-1$　$I_{CA}=\int_1^0 0dx=0$　$\therefore I_L=e\ln 2+1-\dfrac{\pi}{4}$

7. （1）$\lim\limits_{x\to 0}\dfrac{f(x)}{x}=0\Rightarrow f(0)=0$　　$\lim\limits_{x\to 0}\dfrac{f(x)}{x}=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=f'(0)=0$

$\therefore$ 泰勒展开 $f(x)=f(0)+f'(0)x+\dfrac{f''(\xi)}{2}x^2=\dfrac{f''(\xi)}{2}x^2$　$(0<\xi<x)$

$\left|f\left(\dfrac{1}{n}\right)\right|=\dfrac{|f''(\xi)|}{2}\cdot\dfrac{1}{n^2}$　$\because|f''(\xi)|$ 有界，$\sum_{n=1}^{\infty}\dfrac{1}{n^2}$ 收敛　$\therefore\sum_{n=1}^{\infty}f\left(\dfrac{1}{n}\right)$ 绝对收敛

（2）$\dfrac{x^2}{e^{nx}}=\dfrac{x^2}{1+nx+\dfrac{(nx)^2}{2!}+\cdots+\dfrac{(nx)^k}{k!}}<\dfrac{x^2}{\dfrac{(nx)^4}{4!}}=\dfrac{4!}{n^4}\cdot\dfrac{1}{x^2}$

$\sum_{n=1}^{\infty}x^2e^{-nx}<\sum_{n=1}^{\infty}\dfrac{4!}{n^4}\cdot\dfrac{1}{x^2}$　$\because\sum_{n=1}^{\infty}\dfrac{4!}{n^4}\cdot\dfrac{1}{x^2}$ 在 $(0,+\infty)$ 上收敛　$\therefore\sum_{n=1}^{\infty}x^2e^{-nx}$ 在 $(0,+\infty)$ 上一致收敛

当 $x=0$ 时原级数 $=0$ 收敛　故在 $[0,+\infty)$ 上一致收敛

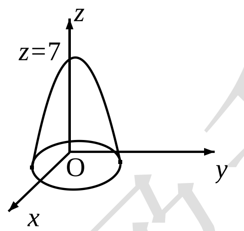
8. $f(x)=\frac{1}{3+x-2}=\frac{1}{3}\cdot\frac{1}{1+\frac{x-2}{3}}=\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n(\frac{x-2}{3})^n$ $\qquad -1<\frac{x-2}{3}<1\Rightarrow x\in(-1,5)$

9. $S(x)=\sum_{0}^{\infty}\frac{2n+1}{n!}x^{2n}\Rightarrow\int S(x)=\sum_{0}^{\infty}\frac{x^{2n+1}}{n!}\Rightarrow\frac{1}{x}\int S(x)=\sum_{0}^{\infty}\frac{(x^2)^n}{n!}=e^{x^2}$ $\quad\therefore S(x)=(2x^2+1)e^{x^2}$ $\quad S(\sqrt{2})=5e^2$

10.（1）存在 u 使 $du=Pdx+Qdy+Rdz$

（2）$I=\int_0^{\pi}[-(a\sin t+t)a\sin t+(t+a\cos t)a\cos t+a(\sin t+\cos t)]dt$

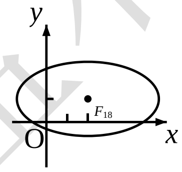
$=\int_0^{\pi}[a^2(\cos^2 t-\sin^2 t)+at(\cos t-\sin t)+a(\sin t+\cos t)]dt=-\pi a$



11. $P=\frac{x}{r^3},Q=\frac{y}{r^3},R=\frac{z}{r^3}$

取平面 $S_1:z=0$，球面 $S_2:x^2+y^2+z^2=\varepsilon^2(z\geq 0)$，$\varepsilon>0$ 足够小

如右图：$I+I_1+I_2=\iiint_V P_x+Q_y+R_z dV=0$ 在 XOY 平面的投影：



又 $I_1=0,I_2=\iint_S\frac{dx\wedge dy}{q^3}=-\int_0^{2\pi}\int_0^{\varepsilon}\frac{\rho\sqrt{\varepsilon^2-\rho^2}}{\varepsilon^3}d\rho d\theta=-\frac{2\pi}{3}$ $\qquad\therefore I=\frac{2\pi}{3}$

2015 年高数期末答案

一、选择题

1. D

解析：令 $y=kx$ 则 $\lim\limits_{\substack{x\to 0\\y\to 0}}f(x)=\lim\limits_{\substack{x\to 0\\y\to 0}}\frac{2x^2}{x^2+k^2x^2}=\frac{2}{k^2+1}$，不存在

2. C

解析：不由于对称性分析知 A,B,D 左侧积分均为 0，右侧积分不为 0

3. D

解析：$I=\iint_{(S)}(2-x^2-1-\frac{y^2}{2})d\sigma=\iint_{(S)}\left[1-(x^2+\frac{y^2}{2})\right]d\sigma=\pi-\frac{3}{4}\iint_{(S)}(x^2+y^2)d\sigma=\pi-\frac{3}{4}\int_0^{2\pi}\int_0^1\rho^3 d\rho d\theta=\frac{5\pi}{8}$

4. A

解析：$f_x=\lim\limits_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0$ $\qquad f_y=\lim\limits_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=0$ $\qquad \left.\frac{\partial f}{\partial\vec{l}}\right|_{(0,0)}=(f_x,f_y)\cdot(\frac{1}{2},\frac{\sqrt{3}}{2})=0$

二、填空题

1. $9-2\cos 8$

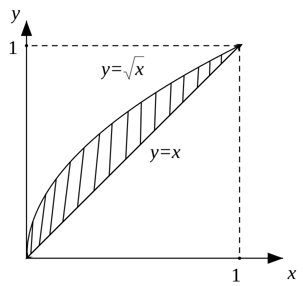
解析：$f_x=3x^2y-2x\cos(x^2-y^2)=9-2\cos 8$

2. $\frac{\pi}{3}$

解析：$x=t$，$y=t$，$z=t^2+1$；$(\dot{x},\dot{y},\dot{z})=(1,0,\sqrt{3})$，$\therefore\alpha=\frac{\pi}{3}$

3. $1-\cos 1$

解析：如右图 $I=\int_0^1 dy\int_{y^2}^{y}\frac{\cos y}{y}dx=\int_0^1(\cos y-y\cos y)dy=1-\cos 1$



4. $\frac{\pi}{2}R^2$

解析：C 为平面 $x+y+z=\dfrac{3R}{2}$ 从球面 $x^2+y^2+z^2=R^2$ 上截下的圆

$$\oint_c yds=\frac{1}{3}\oint_c (x+y+z)ds=\frac{1}{3}\oint_c \frac{3R}{2}ds=\frac{R}{2}\oint_c ds=\frac{R}{2}\pi R=\frac{\pi}{2}R^2$$

三、解答题

1. $\dfrac{\partial z}{\partial x}=yf(e^{xy},xy)=yf$，$\dfrac{\partial^2 z}{\partial x\partial y}=f+y(xe^{xy}f_1+xf_2)$

2. $d(e^z-2x+yz)=0\Rightarrow e^z dz-2dx+ydz+zdy=0$，$\therefore dz\big|_{(0,0)}=\dfrac{2}{e}dx-\dfrac{1}{e}dy$

3. $\lambda(x)=\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\dfrac{(n+1)(\frac{x}{n+1})^{n+1}}{(\frac{x}{n})^n}=\lim\limits_{n\to\infty}(\dfrac{n}{n+1})^n x=\lim\limits_{n\to\infty}\dfrac{x^n}{(1+\frac{1}{n})^n}=\dfrac{x}{e}<1\Rightarrow 0<x<e$

当 $x=e$ 时，$\dfrac{a_{n+1}}{a_n}=\dfrac{e}{(1+\frac{1}{n})^n}>1\Rightarrow a_{n+1}>a_n$，且 $a_1=e$

$\therefore \lim\limits_{n\to\infty}a_n\neq 0$，发散　　　故 $x\in(0,e)$ 时收敛，$x\in[e,+\infty)$ 时发散

4. $f(x)=\dfrac{a_0}{2}+\sum\limits_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$　　$a_n=\dfrac{2}{2\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\dfrac{2}{\pi}\int_0^{\pi}x\cos nxdx=\dfrac{4[(-1)^n-1]}{\pi^2n^2}$

$a_0=2$　　$b_n=\dfrac{2}{2\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=0$　　$\therefore f(x)=1+\sum\limits_{n=1}^{\infty}\dfrac{4\left[(-1)^n-1\right]}{\pi^2n^2}\cos nx$

5. $\alpha=\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left|\dfrac{(n+1)^2+1}{n+1}\cdot\dfrac{n}{n^2+1}x\right|=|x|<1\Rightarrow -1<x<1$

当 $x=1$ 时 $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}n+\dfrac{1}{n}=\infty$ 故级数发散；当 $x=-1$ 时，$\lim\limits_{n\to\infty}|a_n|=\lim\limits_{n\to\infty}n+\dfrac{1}{n}=\infty$，故级数发散

$\therefore$ 收敛域：$(-1,1)$　　$S(x)=\sum\limits_{n=1}^{\infty}(nx^n+\dfrac{x^n}{n})$

设 $H(x)=\sum\limits_{n=1}^{\infty}nx^n\Rightarrow\dfrac{H(x)}{x}=\sum\limits_{n=1}^{\infty}nx^{n-1}\Rightarrow\int_0^x\dfrac{H(t)}{t}dt=\sum\limits_{n=1}^{\infty}x^n=\dfrac{1}{1-x}-1\Rightarrow H(x)=\dfrac{x}{(x-1)^2}$

设 $T(x)=\sum\limits_{n=1}^{\infty}\dfrac{x^n}{n}\Rightarrow T'(x)=\sum\limits_{n=1}^{\infty}x^{n-1}\dfrac{1}{1-x}\Rightarrow T(x)=-\ln(1-x)$　　$\therefore S(x)=\dfrac{x}{(x-1)^2}-\ln(1-x)$

6.（1）　$\forall x\in[\delta,+\infty)$, $\forall n\in N_+$，恒有 $ne^{-nx}\le ne^{-n\delta}$　　$\because \lambda=\lim\limits_{n\to\infty}\dfrac{(n+1)e^{-(n+1)\delta}}{ne^{-n\delta}}=e^{-\delta}<1$

$\therefore\sum\limits_{n=1}^{\infty}ne^{-nx}$ 收敛　　由 M 判别法知：$\sum\limits_{n=1}^{\infty}ne^{-nx}$ 在 $[\delta,+\infty)$ 上一致收敛

$u(\dfrac{1}{n})=e^{-1}\nrightarrow 0$　　$\therefore$ 在 $(0,+\infty)$ 内不一致收敛

（2）　$f(x)=\dfrac{x+4}{(2x+1)(x-3)}=\dfrac{1}{x-3}-\dfrac{1}{2x+1}=\dfrac{1}{-2+(x+1)}-\dfrac{1}{3+2(x-1)}=-\dfrac{1}{2}\cdot\dfrac{1}{1-\frac{x-1}{2}}-\dfrac{1}{3}\cdot\dfrac{1}{1+\frac{2(x-1)}{3}}$

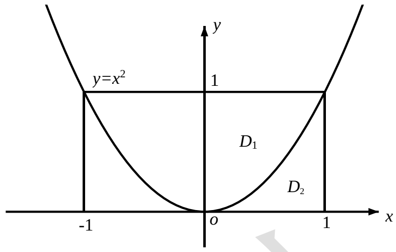
$=-\dfrac{1}{2}\sum\limits_{n=0}^{\infty}(\dfrac{x-1}{2})^n-\dfrac{1}{3}\sum\limits_{n=0}^{\infty}\left[-\dfrac{2}{3}(x-1)^n\right]=\sum\limits_{n=0}^{\infty}\left[-\dfrac{1}{2^{n+1}}-\dfrac{(-2)^n}{3^{n+1}}\right](x-1)^n$　　$\left|\dfrac{x-1}{2}\right|<1\Rightarrow -1<x<3$

$\left|\dfrac{2(x-1)}{3}\right|<1\Rightarrow -\dfrac{1}{2}<x<\dfrac{5}{2}$　　$\therefore x\in(-\dfrac{1}{2},\dfrac{5}{2})$

7. $I=2(\iint\limits_{D_1}\sqrt{y-x^2}dxdy+\iint\limits_{D_2}\sqrt{x^2-y}dxdy)=2(\int_0^1\int_{x^2}^1\sqrt{y-x^2}dydx+\int_0^1\int_0^{x^2}\sqrt{x^2-y}dydx)$

$$=2\left[\int_0^1\frac{2}{3}(1-x^2)^{\frac{3}{2}}dx+\int_0^1\frac{2}{3}x^3dx\right]=\frac{4}{3}\left[\int_0^1(1-x^2)^{\frac{3}{2}}dx+\frac{1}{4}\right]$$

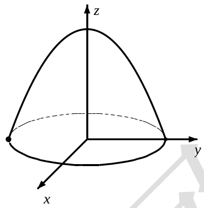
令 $x=\sin\theta$，则 $I=\frac{4}{3}\int_0^{\frac{\pi}{2}}\cos^4\theta d\theta+\frac{1}{3}=\frac{\pi}{4}+\frac{1}{3}$



8．取 S_1 为 $\begin{cases}x^2+y^2=1\\ z=0\end{cases}$ 下侧，由高斯公式：

$$I+I_1=\iiint\limits_V(6x^2+6y^2+6z)dV=6\int_0^1\int_0^{2\pi}\int_0^{\sqrt{1-z}}(r^2+z)rdrd\theta dz=2\pi$$

$$I_1=\iint\limits_{\Sigma}(-3)dx\wedge dy=-\iint\limits_{S_1}(-3)dxdy=3\pi \qquad \therefore I=-\pi$$

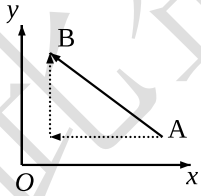


9. $Q=\frac{x}{y^2}-xf(xy)\ \ P=-[\frac{1}{y}+yf(xy)]$ $\qquad \because \frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$

$\therefore$积分与路径无关，如右图：

$I=\int_3^1-\left[\frac{3}{2}+\frac{2}{3}f(\frac{2}{3}x)\right]dx+\int_{\frac{2}{3}}^2\left[\frac{1}{y^2}-f(y)\right]dy$ $\qquad$ 设 $y=\frac{2}{3}x$，则

$$I=\int_2^{\frac{2}{3}}-\left[\frac{9}{4}+f(y)\right]dy+\int_{\frac{2}{3}}^2\left[\frac{1}{y^2}-f(y)\right]dy=\int_{\frac{2}{3}}^2(\frac{9}{4}+\frac{1}{y^2})dy=4$$



10. $(F_x,F_y,F_z)=(2x,2y,1)$

$\therefore$切平面：$2x_0(x-x_0)+2y_0(y-y_0)+z-z_0=0$，则

$$V=\frac{1}{6}(2x_0^2+2y_0^2+z_0)\cdot\frac{2x_0^2+2y_0^2+z_0}{2y_0}\cdot\frac{2x_0^2+2y_0^2+z_0}{2x_0}=\frac{(x_0^2+y_0^2+4)^3}{24x_0y_0},\quad \begin{cases}\frac{\partial v}{\partial x}=0\\ \frac{\partial v}{\partial y}=0\end{cases}\Rightarrow\begin{cases}5x^2y-y^3-4y=0\\ 5xy^3-x^3-4x=0\end{cases}$$

$$\Rightarrow\begin{cases}y^2=5x^2-4\\ x^2=5y^2-4\end{cases}\Rightarrow x=y=1$$

11. $\begin{cases}f_x(0,0)=\lim\limits_{\Delta x\to0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0\\ f_y(0,0)=\lim\limits_{\Delta y\to0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=0\end{cases}\Rightarrow\lim\limits_{\substack{\Delta x\to0\\ \Delta y\to0}}\frac{f(\Delta x,\Delta y)-f_x(0,0)\Delta x-f_y(0,0)\Delta y-f(0,0)}{\sqrt{\Delta x^2+\Delta y^2}}$

$=\lim\limits_{\substack{\Delta x\to0\\ \Delta y\to0}}\frac{|\Delta x\Delta y|}{(\Delta x^2+\Delta y^2)^{\frac{3}{2}}}\sin(\Delta x^2+\Delta y^2)=\lim\limits_{\substack{\Delta x\to0\\ \Delta y\to0}}\frac{|\Delta x\Delta y|}{\sqrt{\Delta x^2+\Delta y^2}}\le\lim\limits_{\Delta x\to0}|\Delta x|=0 \qquad \therefore f(x,y)$ 在 $(0,0)$ 处不可微

12. $I=\int_0^1\oint\limits_{x^2+y^2=r^2}(xf_x+yf_y)dsdr=\int_0^1\oint\limits_{x^2+y^2=r^2}(\frac{x}{r}f_x+\frac{y}{r}f_y)rdsdr=\int_0^1\oint\limits_{x^2+y^2=r^2}(f_xdy-f_ydx)rdr$

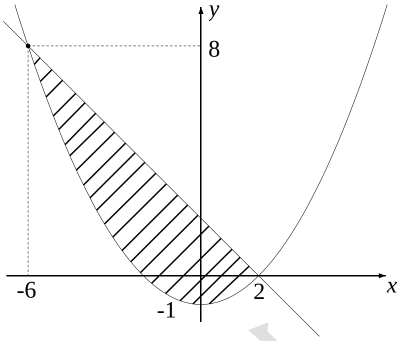
$$=\int_0^1 r\oint\limits_{x^2+y^2\le r^2}(f_{xx}+f_{yy})d\sigma dr=\int_0^1 r\oint\limits_{x^2+y^2\le r^2}e^{-(x^2+y^2)}d\sigma dr=\int_0^1\pi r(1-e^{-r^2})dr=\frac{\pi}{2e}$$

2014 年高数期末答案

一、计算题

1. 切平面：$x_0(x-x_0)+2y_0(y-y_0)-(z-z_0)=0$ $\qquad \frac{x_0}{2}=\frac{2y_0}{2}=\frac{-1}{-1}\Rightarrow x_0=2$，$y_0=1$，$z_0=\frac{3}{2}$

$\therefore 2(x-2)+2(y-1)-(z-\dfrac{3}{2})=0$



2. 如右图：$I=\int_0^8\int_{-2\sqrt{y+1}}^{2-y}f(x,y)dydx+\int_{-1}^0\int_{-2\sqrt{y+1}}^{2\sqrt{y+1}}f(x,y)dydx$

3. $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}=\left(\dfrac{(\lambda-e)^2\lambda^{n+1}(n+1)!}{(n+1)^{n+1}}\right)\Big/\left(\dfrac{(\lambda-e)^2\lambda^n n!}{n^n}\right)=\lim\limits_{n\to\infty}\lambda(\dfrac{n}{n+1})^n=\dfrac{\lambda}{e}$

4. $m=\int_L Pds=\int_0^1 x\sqrt{1+4x^2}dx=\dfrac{5^{\frac{3}{2}}-1}{12}$

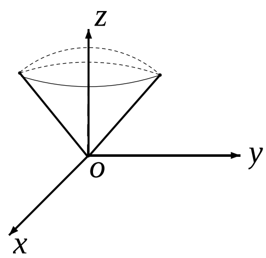
5. 设 $f(x)=\dfrac{a_0}{2}+\sum\limits_{n=1}^{\infty}(a_n\cos\dfrac{n\pi x}{2}+b_n\sin\dfrac{n\pi x}{2})$，在$[-2,2]$上显然满足 Dirichlet 条件

$$\begin{cases}a_n=\dfrac{1}{2}\int_{-2}^2 f(x)\cos\dfrac{n\pi x}{2}dx=\dfrac{1}{2}\int_0^2 x\cos\dfrac{n\pi x}{2}dx=\dfrac{2}{(n\pi)^2}[(-1)^n-1],a_0=\dfrac{1}{2}\int_0^2 xdx=1\\ b_n=\dfrac{1}{2}\int_{-2}^2 f(x)\sin\dfrac{n\pi x}{2}dx=\dfrac{1}{2}\int_0^2 x\sin\dfrac{n\pi x}{2}dx=\dfrac{2}{n\pi}(-1)^{n+1}\end{cases}$$

$$f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\left[\frac{2}{(n\pi)^2}\left[(-1)^n-1\right]\cos\frac{n\pi x}{2}+(-1)^{n+1}\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]\qquad f(4k+2)=1\quad(k=0,\pm1,\pm2,\pm3,\cdots)$$

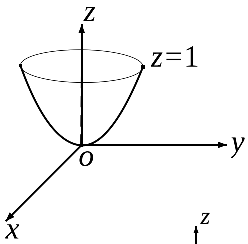
6. $f(x)=\ln\left[4(x-2)+3\right]=\ln3+\ln\left[\dfrac{4}{3}(x-2)+1\right]=\ln3+\sum\limits_{n=1}^{\infty}(-1)^{n-1}(\dfrac{4}{3})^n\dfrac{(x-2)^n}{n}$

$-1<\dfrac{4}{3}(x-2)\le1\Rightarrow\dfrac{5}{4}<x<\dfrac{11}{4}$



7. $\iiint\limits_V zdv=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}}r\cos\theta r^2\sin\theta drd\theta d\varphi=\int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{4}}\sin\theta\cos\theta d\theta\int_0^{\sqrt{2}}r^3dr=\dfrac{\pi}{2}$

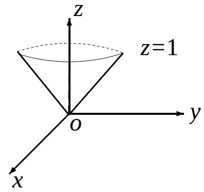
8. $I_1=\iint\limits_{s_1}\vec{A}\cdot d\vec{s}=\iint\limits_{s_1}(z+x^2)dy\wedge dz+xdz\wedge dx+(z^2+3y)dx\wedge dy$



取 $S_2\begin{cases}x^2+y^2=1\\z=1\end{cases}$的上侧，则由高斯公式：

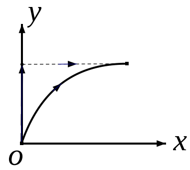
$I_1+I_2=\iiint\limits_V(2x+2z)dV=2\iiint\limits_V zdV=2\int_0^1\int_0^{2\pi}\int_0^{\sqrt{z}}rdrdx\theta dz=\dfrac{2}{3}\pi$

$I_2=\iint\limits_{S_2}(1+3y)dxdy=\iint\limits_{S_2}dxdy=\pi\qquad\therefore I_1=-\dfrac{\pi}{3}$



9. 如右图：$I=\iint\limits_{\sigma}(x^2+y^2)\sqrt{1+\dfrac{x^2+y^2}{x^2+y^2}}dxdy=\sqrt{2}\int_0^{2\pi}\int_0^1\rho^2\rho d\rho d\theta=\dfrac{\sqrt{2}}{2}\pi$

10. $P=ye^{y^2}\qquad Q=xe^{y^2}+2xy^2e^{y^2}$



$\because Q_x=P_y\quad\therefore$和积分路径无关　如右图：$I=\int_0^1 edx=e$

11. $\text{grad}(\sqrt{x^2+y^2+z^2})=(\dfrac{x}{\sqrt{x^2+y^2+z^2}},\dfrac{y}{\sqrt{x^2+y^2+z^2}},\dfrac{z}{\sqrt{x^2+y^2+z^2}})$

$\text{div}\left[\text{grad}(\sqrt{x^2+y^2+z^2})=(\dfrac{x}{\sqrt{x^2+y^2+z^2}},\dfrac{y}{\sqrt{x^2+y^2+z^2}},\dfrac{z}{\sqrt{x^2+y^2+z^2}})\right]=\dfrac{2}{\sqrt{x^2+y^2+z^2}}$

12. （1）$\lambda=\lim\limits_{n\to0}\left|\dfrac{(-1)^n x^{2n+2}}{(2n+3)3^{n+1}}\Big/\dfrac{(-1)^{n-1}x^{2n}}{(2n+1)3^n}\right|=\dfrac{x^2}{3}<1\Rightarrow-\sqrt{3}<x<\sqrt{3}$

当 $x^2=3$ 时，原级数 $=\dfrac{(-1)^{n-1}}{(2n+1)}$，为 Leibniz 型级数，收敛，故收敛域 $[-\sqrt{3},\sqrt{3}]$

$$[xS(x)]'=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n}}{3^n}=\sum_{n=1}^{\infty}(-\frac{x^2}{3})^n=-\left[\frac{1}{1+x^2/3}-1\right]=\frac{x^2}{x^2+3}$$

$$\therefore S(x)=\int_0^x\frac{t^2}{t^2+3}dt/x=\int_0^x\left[1-\frac{1}{(t/\sqrt{3})^2+1}\right]dt/x=1-\frac{\sqrt{3}\arctan\dfrac{x}{\sqrt{3}}}{x}\qquad x\in[-\sqrt{3},\sqrt{3}]$$

（2）$\because\left|\dfrac{\sin(n+\frac{1}{2})x}{\sqrt[3]{n^4+x^4}}\right|\le\dfrac{1}{\sqrt[3]{n^4+x^4}}\le\dfrac{1}{n^{\frac{4}{3}}}$，而 $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^{\frac{4}{3}}}$ 收敛 $\quad\therefore$ 原级数在 $x\in R$ 上一致收敛

$$a_n=\frac{(n+\frac{1}{2})\cos(n+\frac{1}{2})x}{\sqrt[3]{n^4+x^4}}-\frac{4x^3\sin(n+\frac{1}{2})x}{3(n^4+x^4)^{\frac{4}{3}}}$$

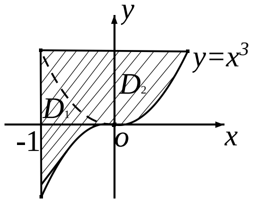
当 $x=2k\pi(\mathrm{k}\in N)$ 时，$a_n'=\dfrac{(n+\frac{1}{2})\cos k\pi}{\sqrt[3]{n^4+4k^2\pi^2}}$

$$|a_n'|=\frac{n+\frac{1}{2}}{\sqrt[3]{n^4+4k^2\pi^2}}\le\frac{n+\frac{1}{2}}{n^{\frac{4}{3}}}=\frac{1}{n^{\frac{1}{3}}}+\frac{1}{2n^{\frac{4}{3}}}$$

$\because\sum\limits_{n=1}^{\infty}\dfrac{1}{n^{\frac{1}{3}}}$ 发散 $\quad\therefore\sum\limits_{n=1}^{\infty}a_n'$ 发散 $\quad$ 故不可逐项求导

13. $\begin{cases}\dfrac{dy}{dx}=f_1+f_2\dfrac{dt}{dx}\\ F_1+F_2\dfrac{dy}{dx}+F_3\dfrac{dt}{dx}=0\end{cases}\Rightarrow\dfrac{dy}{dx}=\dfrac{f_1F_3-F_1f_2}{F_3+F_2f_2}$

14. $I=\iint\limits_D\left[x+xy\sin^2(x^2+y^2)\right]d\sigma$ 将 D 分为 D_1 和 D_2



$$\because\iint\limits_{D_1}xy\sin^2(x^2+y^2)d\sigma=0\quad\iint\limits_{D_2}xy\sin^2(x^2+y^2)d\sigma=0\qquad\iint\limits_{D_2}xd\sigma=0$$

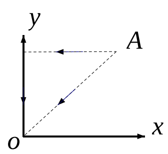
$$\therefore I=\iint\limits_{D_1}xd\sigma=2\int_{-1}^{0}\int_0^{D_2}-x^3xdydx=-\frac{2}{5}$$

15. （1） $P=2[x\varphi(y)+\psi(y)]\qquad Q=x^2\psi(y)+2xy^2+2x\varphi(y)$

$P_y=Q_x\Rightarrow 2x[x\varphi'(y)+\psi'(y)]=2x\psi(y)+2y^2+2\varphi(y)$

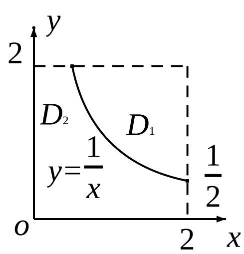
$\therefore\begin{cases}\varphi'(y)=\psi(y)\\ \varphi'(y)=\varphi(y)+y^2\end{cases}\Rightarrow\varphi''(y)=\varphi(y)+y^2$

又 $\because\varphi(0)=-2,\varphi'(0)=0\qquad\therefore\varphi(y)=-y^2-2\qquad\psi(y)=-2y$



（2）如右图：$I=\int_1^0 2[x\varphi(1)+\psi(1)]dx=\int_1^0 2(-3x-2)dx=7$

16. （1）如右图：$A=\iint\limits_{D_1}(xy-1)dxdy+\iint\limits_{D_2}(1-xy)dxdy$



$$=\int_{\frac{1}{2}}^{2}\int_{\frac{1}{x}}^{2}(xy-1)dydx+\int_{\frac{1}{2}}^{2}\int_0^{\frac{1}{x}}(1-xy)dydx+\int_0^{\frac{1}{2}}\int_0^2(1-xy)dydx$$

$$=(\ln 2+\frac{3}{4})+\ln 2+\frac{3}{4}=2\ln 2+\frac{3}{2}$$

（2） $\iint\limits_D xyf(x,y)dxdy-\iint\limits_D f(x,y)dxdy=\iint\limits_D(xy-1)f(x,y)dxdy=1$

$$\because\left|\iint\limits_D(xy-1)f(x,y)dxdy\right|\le\iint\limits_D|xy-1||f(x,y)|dxdy\le|f_{\max}(x,y)|\iint\limits_D|xy-1|dxdy\le A|f_{\max}(x,y)|$$

$\therefore \left|f_{\max}(x,y)\right| \ge \dfrac{1}{A}$, 即 $\exists(\xi,\eta)\in D$，使得 $\left|f(\xi,\eta)\right| \ge A$

2013 年高数期末答案

1. $u_x = 2x-3z \quad u_y = 2y \quad u_z = 4z^3-3x \qquad \text{grad}\, u = (-1,2,1) \qquad \dfrac{\partial f}{\partial \vec{l}} = (-1,2,1)\cdot(\dfrac{1}{3},\dfrac{2}{3},\dfrac{2}{3}) = \dfrac{5}{3}$

2. $F(x,y,z) = 3x^2+y^2+z^2-16 \qquad F_x = 6x \quad F_y = 2y \quad F_z = 2z \qquad 12(x-2)+4(y-2)=0 \Rightarrow 3x+y=8$

3. $2zz_x y-(z^3+3z^2xz_y)=0 \Rightarrow z_x = \dfrac{z^3}{2zy-3zx} \qquad \left.\dfrac{\partial z}{\partial x}\right|_{(1,2,1)} = 1$

4. $\lambda = \lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}(n+1)\sin\dfrac{\pi}{3^{n+1}}/n\sin\dfrac{\pi}{3^n} = \lim\limits_{n\to\infty}\dfrac{n+1}{n}\cdot\dfrac{\pi}{3^{n+1}}/\dfrac{\pi}{3^n} = \dfrac{1}{3} < 1 \qquad \therefore \sum\limits_{n=1}^{\infty} n\sin\dfrac{\pi}{3^n}$ 收敛

5. 设 $f(x) = a_0 + \sum\limits_{n=1}^{\infty}(a_n\cos\dfrac{n\pi x}{\pi} + b_n\sin\dfrac{n\pi x}{2})$

$$\begin{cases} a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos n\pi x dx = \dfrac{2}{\pi}\int_0^{\pi} x\cos n\pi x dx = \dfrac{2}{\pi n^2}[(-1)^n-1], a_0 = \dfrac{2}{\pi}\int_0^{\pi} x dx = \pi \\ b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin n\pi x dx = 0 \end{cases}$$

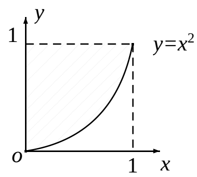
故 $f(x) = \pi + \sum\limits_{n=1}^{\infty}\dfrac{2}{\pi n^2}\left[(-1)^n-1\right]\cos nx$

6. $f(x) = -\dfrac{2}{3}\dfrac{1}{2x-1}+\dfrac{1}{3}\dfrac{1}{x+1} = \dfrac{2}{3}\dfrac{1}{1-2x}+\dfrac{1}{3}\dfrac{1}{1+x} = \dfrac{2}{3}\sum\limits_{n=0}^{\infty}(2x)^n + \dfrac{1}{3}\sum\limits_{n=0}^{\infty}(-x)^n = \sum\limits_{n=0}^{\infty}\dfrac{2^{n+1}+(-1)^n}{3}x^n$

7. $L_{AB}: x+y=1 \qquad \int_L (x+y)ds = \int_L ds = 2\sqrt{2}$

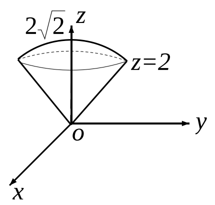
8. $\dfrac{\partial z}{\partial x} = f + x(f_1 y + f_2\dfrac{1}{y}) \qquad \dfrac{\partial^2 z}{\partial x\partial y} = xf_1 - \dfrac{x}{y^2}f_2 + xf_1 + xy(f_{11}x - f_{12}\dfrac{x}{y}) - \dfrac{x}{y^2}f_2 + \dfrac{x}{y}(f_{21}x - f_{22}\dfrac{x}{y_2})$

$= 2xf_1 - \dfrac{2x}{y^2}f_2 + x^2yf_{11} - \dfrac{x^2}{y}f_{12} + \dfrac{x^2}{y}f_{21} - \dfrac{x^2}{y^3}f_{22}$



9. 交换积分次序，积分域如图：

$I = \int_0^1 dy\int_0^{\sqrt{y}}\dfrac{xy}{\sqrt{1+y^3}}dx = \int_0^1\dfrac{y^2}{2\sqrt{1+y^3}}dy = \dfrac{1}{3}\left.\sqrt{1+y^3}\right|_0^1 = \dfrac{\sqrt{2}-1}{3}$

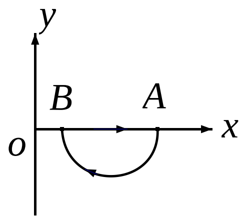


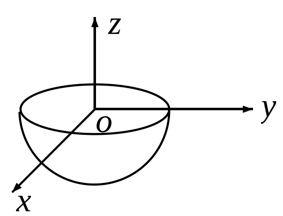
10. 如右图：$m = \iiint\limits_V \rho dV = \int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{4}}d\theta\int_0^{2\sqrt{2}} r\cos\theta r^2\sin\theta dr = 8\pi$

11. $I_1 + I_2 = -\iint\limits_D [e^x\cos y - 1 - (e^x\cos y + 1)]d\sigma$

$= 2\iint\limits_D d\sigma = 2\cdot\dfrac{1}{2}\cdot\pi\cdot 3^2 = 9\pi$

$I_2 = \int_1^7 dx = 6 \qquad \therefore I_1 = 9\pi - 6$





12. 取 $S_1: \begin{cases} x^2+y^2 \le 1 \\ z = 0 \end{cases}$ 的上侧

$-I + I_1 = \iiint\limits_V [\cos^2(1+z)+\sin^2(1+z)+4]dv = 5\iiint\limits_V dv = 5\cdot\dfrac{1}{2}\cdot\dfrac{4}{3}\pi = \dfrac{10}{3}\pi$

$I_1=\iint\limits_{S_1}4dxdy=4\pi \qquad \therefore I=\frac{2}{3}\pi$

13. （1） $\forall x\in[\delta,+\infty)$，$\forall n\in N_+$， 恒有 $ne^{-nx}\le ne^{-n\delta}$

$\because \lambda=\lim\limits_{n\to\infty}\frac{(n+1)e^{-(n+1)\delta}}{ne^{-n\delta}}=e^{-\delta}<1 \qquad \therefore \sum\limits_{n=1}^{\infty}ne^{-nx}$ 收敛

由 M 判别法知 $\therefore \sum\limits_{n=1}^{\infty}ne^{-nx}$ 在 $[\delta,+\infty)$ 上一致收敛，但 $u(\frac{1}{n})=e^{-1}\not\to 0$ $\qquad \therefore$ 在 $(0,+\infty)$ 内不一致收敛

（2） $\lambda=\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{(2n+3)}{(n+1)!}x^{2n+2}/\frac{(2n+1)}{n!}x^{2n}=\lim\limits_{n\to\infty}\frac{2n+3}{(2n+1)(n+1)}x^2=0 \qquad \therefore$ 收敛域： $x\in(-\infty,+\infty)$

14. $\lim\limits_{\substack{x\to0\\y\to0}}f(x,y)=\lim\limits_{\substack{x\to0\\y\to0}}\frac{xy}{\sqrt{x^2+y^2}}\le\lim\limits_{x\to0}|x|=0 \qquad \therefore$ 在 $(0,0)$ 处连续； $f_x(0,0)=\lim\limits_{\Delta x\to0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0$

$f_y(0,0)=\lim\limits_{\Delta y\to0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=0 \qquad \therefore$ 在 $(0,0)$ 处可偏导

则： $\lim\limits_{\substack{\Delta x\to0\\\Delta y\to0}}\frac{f(\Delta x,\Delta y)-f_x(0,0)\Delta x-f_y(0,0)\Delta y-f(0,0)}{\sqrt{\Delta x^2+\Delta y^2}}=\lim\limits_{\substack{\Delta x\to0\\\Delta y\to0}}\frac{\Delta x\Delta y}{\Delta x^2+\Delta y^2}$ 不存在 $\qquad \therefore$ 在 $(0,0)$ 处不可微

15. 设 $C=\frac{1}{\pi}\iint\limits_D f(x,y)dxdy$， $f(x,y)=\sqrt{1-x^2-y^2}-C$

$C=\frac{1}{\pi}\iint\limits_D f(x,y)dxdy=\frac{1}{\pi}\iint\limits_D(\sqrt{1-x^2-y^2}-C)dxdy=\frac{1}{\pi}(\int_0^{2\pi}d\theta\int_0^1\sqrt{1-\rho^2}\cdot\rho d\rho-C\pi)=\frac{2}{3}-C$

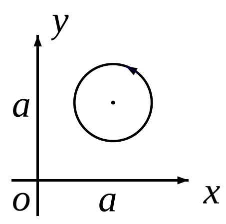
$\therefore C=\frac{1}{3} \quad f(x,y)=\sqrt{1-x^2-y^2}-\frac{1}{3}$

16. $I=\iiint\limits_V[(y+1)f''(x)+(1-2y)f(x)+yf'(x)-2e^x]dv=0 \quad \therefore (y+1)f''(x)+yf'(x)+(1-2y)f(x)=2e^x$

$y[f''(x)+f'(x)-2f(x)]+[f''(x)+f(x)]=2e^x$

$$\begin{cases}f''(x)+f'(x)-2f(x)=0\\f''(x)+f(x)=2e^x\end{cases}\Rightarrow f(x)=e^x$$

17. $I=\iint\limits_D[f(y)+2x+\frac{1}{f(x)}+4y]d\sigma=\iint\limits_D[f(y)+\frac{1}{f(x)}]d\sigma+\iint\limits_D(2x+4y)d\sigma$



$\because \iint\limits_D f(y)d\sigma=\iint\limits_D f(x)d\sigma \quad \iint\limits_D xd\sigma=\iint\limits_D ad\sigma \qquad \therefore I=\iint\limits_D[f(x)+\frac{1}{f(x)}]d\sigma+\iint\limits_D 6ad\sigma$

$f(x)+\frac{1}{f(x)}\ge2 \qquad I\ge\iint\limits_D 2d\sigma+\iint\limits_D 6ad\sigma=2\pi+6a\pi$

2012 年高数下期末答案

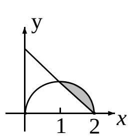
一、计算题

1. $\dot{r}(t)=(-\sin t,\cos t,\frac{1}{2}\cdot\frac{1}{\cos^2\frac{t}{2}})=(-1,0,1) \qquad \therefore \frac{x}{-1}=\frac{y-1}{0}=\frac{z-1}{1}$

2. 令 $F=z-e^z+2xy-3=0 \qquad F_x=2y=4 \qquad F_y=2x=2 \qquad F_z=1-e^z=0$

$\therefore 4(x-1)+2(y-2)=0 \qquad \therefore 2x+y-4=0$

3. 如图： $\int_0^1dy\int_{2-y}^{1+\sqrt{1-y^2}}f(x,y)dx$



4. $\lim\limits_{x\to\infty}\dfrac{(1-\cos\frac{1}{n})\sqrt{n}}{\frac{1}{n^{\frac{3}{2}}}}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{2}\cdot\frac{1}{n^2}}{\frac{1}{n^2}}=\dfrac{1}{2}$ $\qquad\because \dfrac{1}{n^{\frac{3}{2}}}$ 收敛 $\qquad\therefore$ 原级数收敛

5. 采用奇延拓：$a_n=0$ $\qquad f(x)=\sum\limits_{n=1}^{\infty}b_n\sin nx$ $\qquad$ 令 $f(0)=0$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}(x+\frac{\pi}{2})\sin nxdx=\frac{1}{n}-\frac{2}{n}\cos\frac{n\pi}{2}+\frac{2}{n^2\pi}\sin\frac{n\pi}{2}$$

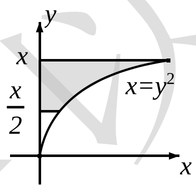
$$\therefore f(x)=\sum_{k=1}^{\infty}\left[\frac{1}{2k}-\frac{(-1)^k}{k}\right]\sin 2kx+\sum_{k=0}^{\infty}\left[\frac{1}{2k+1}-\frac{2}{(2k+1)^2\pi}\cdot(-1)^k\right]\sin(2k+1)x$$

6. $(x-\dfrac{a}{2})^2+y^2=\dfrac{a^2}{4}$ $\qquad\therefore\begin{cases}x=\dfrac{a}{2}+\dfrac{a}{2}\cos t\\ y=\dfrac{a}{2}\sin t\end{cases}$ $\qquad\dot{x}(t)=-\dfrac{a}{2}\sin t$ $\qquad\dot{y}(t)=\dfrac{a}{2}\cos t$

$$\therefore\int_L\sqrt{x^2+y^2}ds=\int_0^{2\pi}\sqrt{ax}\cdot\sqrt{\dot{x}^2(t)+\dot{y}^2(t)}dt=\int_0^{\pi}\frac{a^2}{2}\sqrt{\frac{1+\cos t}{2}}dt=a^2\int_0^{\pi}\cos\frac{t}{2}dt=2a^2$$

7. $z_x=2f_1+yf_2\cos x$ $\qquad z_{xy}=-2f_{11}+2f_{12}\sin x+y\cos x(-f_{21}+f_{22}\sin x)+f_2\cos x$

8. $\iint\limits_D\sin\dfrac{x}{y}dxdy=\int_{\frac{\pi}{2}}^{\pi}dy\int_0^{y^2}\sin\dfrac{x}{y}dx=\dfrac{3\pi^2}{8}+\dfrac{\pi}{2}+1$

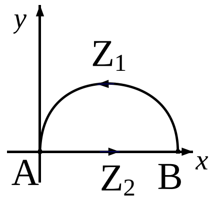


9. $\begin{cases}z=\sqrt{4-x^2-y^2}\\ z=\dfrac{1}{3}(x^2+y^2)\end{cases}$ $\qquad\begin{cases}x^2+y^2=3\\ z=1\end{cases}$

$$\therefore m=\iiint\limits_V zdv=\int_0^{2\pi}d\theta\int_0^{\sqrt{3}}\rho d\rho\int_{\frac{1}{3}\rho^2}^{\sqrt{4-\rho^2}}zdz=\frac{13}{4}\pi$$

10. $I_1+I_2=\iint ye^x d\sigma=\int_0^{\pi}dx\int_0^{\sin x}ye^x dy=\dfrac{e^{\pi}-1}{5}$

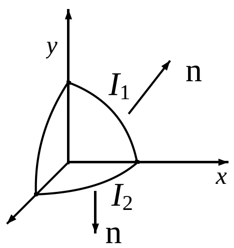
$I_2=\int_0^{\pi}e^x dx=e^{\pi}-1$ $\qquad\therefore$ 原式 $=-I_1=\dfrac{4}{5}(e^{\pi}-1)$



11. S_2 为平面 $x^2+y^2=1$，取 $z=0$ 下侧

$$I=I_1+I_2=\iiint(P_x+Q_y+R_z)dv=\iiint 3dv=3\int_0^{2\pi}d\theta\int_0^1\rho d\rho\int_0^{1-\rho^2}dz=\frac{3\pi}{2}$$

$I_2=-\iint d\sigma=-\pi$ $\qquad\therefore$ 原式 $=I-I_2=\dfrac{5\pi}{2}$



12. $\lambda=\lim\limits_{x\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim\limits_{x\to\infty}\left|\dfrac{x^n}{(n+1)2^{n+1}}/\dfrac{x^{n-1}}{n2^n}\right|=\dfrac{|x|}{2}<1\Rightarrow -2<x<2$

当 $x=2$ 时，原式 $=\sum\limits_{n=1}^{\infty}\dfrac{2^{n-1}}{n2^n}=\sum\limits_{n=1}^{\infty}\dfrac{1}{2n}$ 发散；当 $x=-2$ 时，原式 $=\sum\limits_{n=1}^{\infty}\dfrac{(-2)^{n-1}}{n2^n}=\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{2n}$ 收敛

$\therefore$ 收敛域：$[-2,2)$ $\qquad x\sum\limits_{n=1}^{\infty}\dfrac{x^{n-1}}{n2^n}=\sum\limits_{n=1}^{\infty}\dfrac{1}{n}\left(\dfrac{x}{2}\right)^n=S(\dfrac{x}{2})$ $\qquad$ 令 $\dfrac{x}{2}=t$ $\qquad S(t)=\sum\limits_{n=1}^{\infty}\dfrac{1}{n}t^n$ $\qquad S'(t)=\sum\limits_{n=1}^{\infty}t^{n-1}=\dfrac{1}{1-t}$

$\therefore S(t)-S(0)=\int_0^t\dfrac{1}{1-t}dt=-\ln(1-t)$ $\qquad S(t)=-\ln(1-t)$ $\qquad\sum\limits_{n=1}^{\infty}\dfrac{x^{n-1}}{n2^n}=-\dfrac{1}{x}\ln(1-\dfrac{x}{2})$

13.（1）$\forall x\in[\delta,+\infty),\forall n\in N^{+}$ 恒有 $\sqrt{n}\cdot 2^{-nx}\le\sqrt{n}\cdot 2^{-n\delta}$ $\qquad \lambda=\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}\le 2^{-\delta}<1$

$\because\sum\limits_{n=1}^{\infty}\sqrt{n}\cdot 2^{-n\delta}$ 收敛 $\qquad$ 由 M 判别法知 $\sum\limits_{n=1}^{\infty}\sqrt{n}\cdot 2^{-nx}$ 在 $[\delta,+\infty)$ 上一致收敛

$\because u(\dfrac{1}{n})=\dfrac{\sqrt{n}}{2}\not\to 0$ $\qquad\therefore$ 在 $(0,+\infty)$ 内不一致收敛

（2）$f(x)=\dfrac{1}{5}(\dfrac{2}{x+2}+\dfrac{1}{2x-1})$ $\qquad \dfrac{2}{x+2}=\dfrac{1}{2}\cdot\dfrac{1}{1+\dfrac{x-2}{4}}=\dfrac{1}{2}\sum\limits_{n=0}^{\infty}(-1)^n\left(\dfrac{x-2}{4}\right)^n$

$\dfrac{1}{2x-1}=\dfrac{1}{3}\cdot\dfrac{1}{1+\dfrac{2(x-2)}{3}}=\dfrac{1}{3}\sum\limits_{n=0}^{\infty}(-1)^n\left[\dfrac{2(x-2)}{3}\right]^n$ $\qquad\therefore f(x)=\dfrac{1}{10}\sum\limits_{n=0}^{\infty}(-1)^n\left(\dfrac{x-2}{4}\right)^n+\dfrac{1}{15}\sum\limits_{n=0}^{\infty}(-1)^n\left[\dfrac{2(x-2)}{3}\right]^n$

14. $z_x=\dfrac{x}{\sqrt{x^2+y^2}}$ $\qquad z_y=\dfrac{y}{\sqrt{x^2+y^2}}$ $\qquad \sqrt{1+z_x{}^2+z_y{}^2}=\sqrt{2}$

原式$=\iint\sqrt{x^2+y^2}\cdot\sqrt{1+z_x{}^2+z_y{}^2}\,d\sigma=\sqrt{2}\int_0^{2\pi}d\theta\int_1^2\rho^2 d\rho=\dfrac{14\sqrt{2}\pi}{3}$

15. $f=(x-a)^2+(y-a)^2+(z-a)^2-a^2=0$ $\qquad$ 令 $F=u+\lambda f$ $\qquad F_x=1+2\lambda(x-a)=0$

$F_y=1+2\lambda(y-a)=0$ $\qquad F_z=1+2\lambda(z-a)=0$ $\qquad F_\lambda=(x-a)^2+(y-a)^2+(z-a)^2-a^2=0$

$\therefore x=y=z=(1-\dfrac{\sqrt{3}}{3})a$ 时 $u_{\min}=(3-\sqrt{3})a$ $\qquad \oiint\limits_{\Sigma}(x+y+z+\sqrt{3}a)^3ds\ge\oiint\limits_{\Sigma}(3a)^3ds=108\lambda a^5$

2011 年高数下期末答案

一、填空题

1. $\dfrac{x-1}{3}=\dfrac{y-2}{2}=\dfrac{z-1}{1}$

解析：$y=2t=2$ $\qquad\therefore t=1$ $\qquad x_t=3t^3=3$ $\qquad y_t=2$ $\qquad z_t=1$

2. $-\dfrac{2}{3}$

解析：$u_x=z\cdot\dfrac{-y/x^2}{1+y^2/x^2}=\dfrac{-zy}{x^2+y^2}=0$ $\qquad u_y=z\cdot\dfrac{1/x}{1+y^2/x^2}=1$ $\qquad u_z=\arctan\dfrac{y}{x}=0$

$\overrightarrow{AB}=(2,-2,1)$ $\qquad \vec{e_l}=(\dfrac{2}{3},-\dfrac{2}{3},\dfrac{1}{3})$ $\qquad\therefore\dfrac{\partial f}{\partial l}\Big|_{x_0}=f_x\cos\alpha+f_y\cos\beta+f_z\cos\gamma=-\dfrac{2}{3}$

3. π

解析：$\because I=\oint\limits_{x^2+y^2=1}x^2+y^2ds=2\pi$ $\qquad\therefore\oint x^2ds=\dfrac{1}{2}I=\pi$

4. 3

解析：$\lambda_1=\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}x\right|<1$ $\qquad R_1=\dfrac{a_n}{a_{n+1}}=3$

$\lambda_2=\lim\limits_{n\to\infty}\left|\dfrac{n+1}{n}\dfrac{a_{n+1}}{a_n}(x-1)\right|=\dfrac{|x-1|}{3}<1$ $\qquad\therefore|x-1|<3$ $\qquad\because R_2=3$

5. $\vec{0}$

解析：$u_x = yze^{xyz} + xy^2\sin xy$　　$u_y = xze^{xyz} + x^2 y\sin xy$　　$u_z = xye^{xyz}$

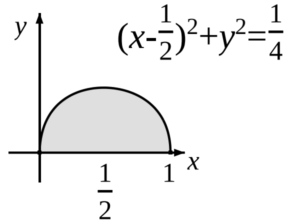
$\because \operatorname{grad} u = (u_x, u_y, u_z)$　　$\therefore \operatorname{rot}(\operatorname{grad} u) = \nabla \times (u_x, u_y, u_z) = \vec{0}$

二、选择题

1. B

解析：$du = Pdx + Qdy$　　$\therefore \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$　　$\therefore 3axy^2 - 2y\cos x = by\cos x + bxy^2$

$\therefore \begin{cases} 3a = 6 \\ -2 = b \end{cases}$　　$\begin{cases} a = 2 \\ b = -2 \end{cases}$



2. D

解析：如图：原式$= \int_0^1 dx \int_0^{\sqrt{x-x^2}} f(x, y)dy$

3. B

解析：$\lim\limits_{n\to\infty} \tan(\dfrac{1}{n} + \dfrac{k}{n^2}) = \dfrac{1}{n} + \dfrac{k}{n^2} > \dfrac{1}{n}$　　$\because \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ 发散　　$\therefore \sum\limits_{n=1}^{\infty} \tan(\dfrac{1}{n} + \dfrac{k}{n^2})$ 发散

对于交错级数 $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \tan(\dfrac{1}{n} + \dfrac{k}{n^2})$　　$a_n > a_{n+1}$　　$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \tan(\dfrac{1}{n} + \dfrac{k}{n^2}) = 0$

$\sum\limits_{n=1}^{\infty} (-1)^{n-1} \tan(\dfrac{1}{n} + \dfrac{k}{n^2})$ 收敛

4. A

解析：$\because 1 \le 1 + \cos^2 x + \sin^2 y \le 3$　　又 $\iint\limits_{|x|+|y|\le 1} dxdy = 2$　　$\therefore \dfrac{2}{3} \le I \le 2$

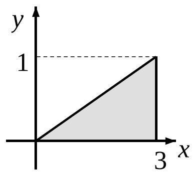
5. C

解析：原式$= \iint y^2 + x^2 d\sigma = \int_0^{2\pi} d\theta \int_0^R \rho^3 d\rho = \dfrac{\pi R^4}{2}$

三、计算题

1. $z_y = f_1 \cdot e^x \cos y + f_2$

$z_{yx} = e^x \cos y \cdot f_{11} \cdot e^x \sin y + f_1 \cdot e^x \cos y + f_{21} e^x \sin y$



2. 如图，交换积分次序，原式$= \int_0^3 dx \int_0^{\frac{x}{3}} e^{x^2} dy = \dfrac{1}{6} e^{x^2} \Big|_0^3 = \dfrac{1}{6}(e^9 - 1)$

3. $\rho = r$　　$V = x^2 + y^2 + (z-1)^2 \le 1$　　$m = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \sin\theta d\theta \int_0^{2\cos\theta} r^3 dr = \dfrac{8}{5}\pi$

4. $d = \sqrt{x^2 + y^2 + z^2}$　　令 $L(x, y, z, \lambda) = \sqrt{x^2 + y^2 + z^2} + \lambda(xy - z^2 + 1)$

$L_x = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}} + \lambda y = 0$　　$L_y = \dfrac{y}{\sqrt{x^2 + y^2 + z^2}} + \lambda x = 0$

$L_z = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}} - 2\lambda z = 0$　　$L_\lambda = xy - z^2 + 1 = 0$

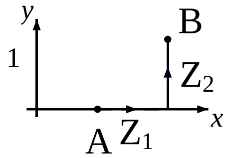
得 $P_1(1, -1, 0)$，$P_2(-1, 1, 0)$，$P_3(0, 0, 1)$，$P_4(0, 0, -1)$

而 $f(1, -1, 0) = f(-1, 1, 0) = 2$　　$f(0, 0, 1) = f(0, 0, -1) = 1$　　$\therefore$最近点为$(0, 0, \pm 1)$

5. $z = -y + 5$　　$z_x = 0$　　$z_y = -1$

$I = \iint 5 + x\sqrt{1 + z_x^2 + z_y^2} d\sigma = \sqrt{2} \int_0^{2\pi} d\theta \int_0^5 (5 + \rho\cos\theta)\rho d\theta = 125\sqrt{2}\pi$

6. $\dfrac{\partial P}{\partial y}=\dfrac{x-\varphi(x)}{x}=\dfrac{1}{2}+\dfrac{1}{2x^2}$ $\qquad\dfrac{\partial Q}{\partial y}=\varphi'(x)=\dfrac{1}{2}+\dfrac{1}{2x^2}$ $\qquad\because\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial y}$



$\therefore$曲线积分 I 与路径无关 $\qquad$ 如图：$I=\int_0^{\pi}\varphi(\pi)dy=\dfrac{\pi^2-1}{2}$

7. $\because f(x,y)$在$(0,0)$点哟定义，又$\lim\limits_{\substack{x\to0\\y\to0}}f(x,y)=0=f(0,0)$ $\qquad\therefore$连续

$f_x(0,0)=\lim\limits_{x\to0}\dfrac{f(x,0)-f(0,0)}{x}=0$ $\qquad f_y(0,0)=\lim\limits_{x\to0}\dfrac{f(0,y)-f(0,0)}{y}=0$ $\qquad\therefore f(x,y)$偏导数存在

$\lim\limits_{\rho\to0}\dfrac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}}=\dfrac{\sqrt[3]{x^2y}}{\sqrt{x^2+y^2}}$不存在 $\qquad\therefore$不可微

8. 原式$=\iiint\left[3x^2+yf'(xy)+3y^2+xf'(xy)+3z^2\right]dv=\iiint\left[3(x^2+y^2+z^2)+(x+y)f'(xy)\right]dv$

$\because f(u)$为奇函数 $\qquad\therefore f'(u)$为偶函数 $\qquad\therefore(x+y)f'(xy)$为奇函数 $\qquad$ 又区间对称

$\therefore$原式$=3\iiint(x^2+y^2+z^2)dv=3\int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{4}}\sin\theta d\theta\int_1^{\sqrt{2}}r^4dr=\dfrac{6}{5}\pi(4\sqrt{2}-1)(1-\dfrac{\sqrt{2}}{2})=\dfrac{3}{5}\pi(9\sqrt{2}-10)$

9. 设$f(x)=\dfrac{a_0}{2}+\sum\limits_{n=1}^{\infty}a_n\cos nx$ $\qquad a_0=\dfrac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\dfrac{2}{\pi}\int_0^{\pi}xdx=\pi$

$a_n=\dfrac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\dfrac{2}{\pi}\int_0^{\pi}x\cos nxdx=\dfrac{1}{n^2}\left[(-1)^n-1\right]$ $\qquad\therefore f(x)=\dfrac{\pi}{2}+\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}\left[(-1)^n-1\right]\cos nx$

10. $f(x)=\dfrac{1}{x-3}-\dfrac{1}{2x+1}$ $\qquad\dfrac{1}{x-3}=-\dfrac{1}{2}\cdot\dfrac{1}{1-\dfrac{x-1}{2}}=-\dfrac{1}{2}\sum\limits_{n=0}^{\infty}\left(\dfrac{x-1}{2}\right)^n$ $\qquad\left|\dfrac{x-1}{2}\right|<1\Rightarrow-1<x<3$

$\dfrac{1}{2x+1}=\dfrac{1}{3}\cdot\dfrac{1}{1+\dfrac{2(x-1)}{3}}=\dfrac{1}{3}\sum\limits_{n=0}^{\infty}(-1)^n\left[\dfrac{2}{3}(x-1)\right]^n$ $\qquad\dfrac{2}{3}|x-1|<1\Rightarrow-\dfrac{1}{2}<x<\dfrac{5}{2}$

$\therefore f(x)=-\dfrac{1}{2}\sum\limits_{n=0}^{\infty}\left(\dfrac{x-1}{2}\right)^n-\dfrac{1}{3}\sum\limits_{n=0}^{\infty}(-1)^n\left[\dfrac{2(x-1)}{3}\right]^n$ $\qquad$ 收敛域：$x\in(-\dfrac{1}{2},\dfrac{5}{2})$

11. 同 2012 年计算题第 12 题

12. 令$\dfrac{x}{a}=u$，$\dfrac{u}{b}=v$ $\qquad\dfrac{\partial(x,u)}{\partial(u,v)}=ab$

左边$=\iint\limits_{u^2+v^2\le1}\dfrac{(b+1)f(u)+(a-1)f(u)}{f(u)+f(v)}abdudv=ab(a+b)\iint\limits_{u^2+v^2\le1}\dfrac{f(u)}{f(u)+f(v)}dudv$

$\because\iint\limits_{u^2+v^2\le1}dudv=\pi$ $\qquad\because$互换u,v，积分域$u^2+v^2\le1$不变

$\therefore\iint\limits_{u^2+v^2\le1}\dfrac{f(u)}{f(u)+f(v)}dudv=\iint\limits_{u^2+v^2\le1}\dfrac{f(v)}{f(u)+f(v)}dudv=\dfrac{1}{2}\iint\limits_{u^2+v^2\le1}dudv=\dfrac{1}{2}\pi$ $\qquad\therefore$右边$=\dfrac{\pi}{2}ab(a+b)$ $\qquad$ 得证

2010 年高数下期末答案

一、填空题

1. -5

解析：$\because$取得极值 $\therefore$该点为驻点 $\qquad\nabla f(x_0,y_0)=(f_x,f_y)\big|_{(x_0,y_0)}=0$

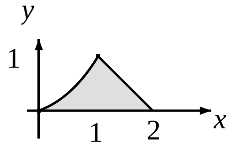
2. $-\dfrac{6}{11}$

解析：$(x_t,y_t,z_t)=(2t,3t^2,\frac{2}{3}t-\frac{1}{3})=(2,3,\frac{2}{3})$　　∵该切向量与 OZ 轴正向成钝角

∴该切向量为$(-2,-3,-\frac{2}{3})$　　$\cos\alpha=\dfrac{(-2,-3,-\frac{2}{3})\cdot(1,0,0)}{\sqrt{2^2+3^2+(\frac{2}{3})^2}}=-\dfrac{6}{11}$

3. $\int_0^1 dy\int_{\sqrt{y}}^{2-y} fdx$

解析：由图知：原式$=\int_0^1 dy\int_{\sqrt{y}}^{2-y} fdx$



4. 16π

解析：$\oint_L x^2+y^2ds=4\oint_L ds=16\pi$　　∵交换x,y，积分域$x^2+y^2=4$不变

$\therefore\oint_L x^2ds=\oint_L y^2ds=8\pi$　　$\therefore\oint_L 2y^2ds=16\pi$

二、选择题

1. C

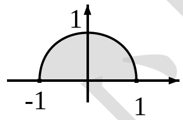
解析：$\because F(x)=e^x\cos 2x$　　$y''-2y'+5y\Rightarrow\lambda^2-2\lambda+5\lambda=0\Rightarrow\lambda=1\pm 2i$　　$\therefore y^*=xe^x(a\cos 2x+b\sin 2x)$

2. D

解析：原式$=\oiint xdy\wedge dz+ydx\wedge dz+zdx\wedge dy=3\underset{V}{\iiint} dv=4\pi R^3$

3. C

解析：如图：原式$=\int_0^{\pi}d\theta\int_0^1 f(\rho^2)\rho d\rho$



4. B

解析：$\vec{n}=(\frac{1}{2\sqrt{x}},\frac{1}{2\sqrt{y}},\frac{1}{2\sqrt{z}})$　　$\frac{1}{2\sqrt{x_0}}(x-x_0)+\frac{1}{2\sqrt{y_0}}(y-y_0)+\frac{1}{2\sqrt{z_0}}(z-z_0)=0$

各截距为$x=\sqrt{5x_0}$　　$y=\sqrt{5y_0}$　　$z=\sqrt{5z_0}$　　$\therefore x+y+z=5$

5. C

解析：偏导数连续可推出可微，可微不能推出偏导数连续

6. C

解析：$\sum\limits_{n=1}^{\infty}\frac{\sin na}{n^2}$收敛　　$\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{n}}$发散　　∴原级数发散

三、计算题

1. $\frac{\partial z}{\partial x}=\frac{1}{y}f_1+2xf_2$　　$\frac{\partial^2 z}{\partial x\partial y}=-\frac{1}{y^2}f_1+\frac{1}{y}(f_{11}-\frac{x}{y^2}+f_{12})+2x(f_{21}-\frac{x}{y^2}+f_{22})$

2. 设长方体在锥面上第一卦限的顶点为(x,y,z)　　$V=4xy(h-z)$　　$(x>0,y>0,z>0)$

设$L(x,y,z,\lambda)=4xy(h-z)+\lambda(h\sqrt{x^2+y^2}-Rz)$　　令$L_x=L_y=L_z=L_\lambda=0$

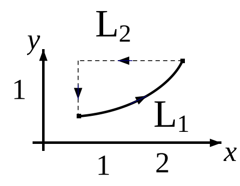
得$x=y=\frac{\sqrt{2}}{3}R,z=\frac{2}{3}h$　　$\therefore V=\frac{8}{27}R^2h$

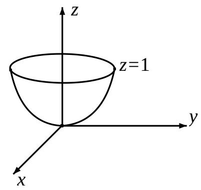
3. $W_1=\int_{L_1}\vec{F}\cdot d\vec{s}=\int_{L_1}[\varphi(y)\cos x-\pi y]dx+[\varphi'(y)\sin x-\pi]dy$

$W_1+W_2=\int_{L_1}\vec{F}\cdot d\vec{s}=\iint[\varphi'(y)\cos x-\varphi'(y)\cos x+\pi]d\sigma=\iint\pi d\sigma=(2+2\pi)\pi$

又$W_1=\int_{L_2}\vec{F}\cdot d\vec{s}=\int_{3\pi}^{\pi}[\varphi(4)\cos\pi-4\pi]dx+\int_4^2[\varphi'(4)\sin\pi-\pi]dy=8\pi^2+2\pi$

$\therefore W_1=(2+2\pi)\pi-(8\pi^2+2\pi)=-6\pi^2$





4. 取 $S_1:\begin{cases}x^2+y^2\le 1\\ z=1\end{cases}$ 的上侧　　如图，由高斯公式：

$$I+I_1=\iiint_V 3dv=3\int_0^1 \pi z dz=\frac{3}{2}\pi \quad \sum_{n=1}^{\infty}\frac{\sin na}{n^2} \qquad I_1=\iint_{S_1}(1+x)dxdy=\iint_{S_1}dxdy=\pi \qquad \therefore I=\frac{\pi}{2}$$

5. $P=\left[e^x-f'(x)\right]y$ 　　$Q=f'(x)$ 　　$P_y=Q_x\Rightarrow f''(x)=e^x-f'(x)\Rightarrow f(x)=C_1e^{-x}+C_2+\frac{1}{2}e^x$

6.（1）$\forall x\in[\delta,+\infty)$，$\forall n\in N^+$，恒有 $n2^{-nx}\le n2^{-n\delta}$ 　　$\because \lambda=\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{(n+1)2^{-(n+1)\delta}}{n2^{-n\delta}}=2^{-\delta}<1$

$\because \sum\limits_{n=1}^{\infty}n2^{-n\delta}$ 收敛　　由 M 判别法知 $\sum\limits_{n=1}^{\infty}n2^{-nx}$ 在 $[\delta,+\infty)$ 上一致收敛　　$S(x)=\sum\limits_{n=1}^{\infty}n\left(2^{-x}\right)^n$

令 $y=2^{-x}$ 则 $S(y)=\sum\limits_{n=1}^{\infty}ny^n$ 　　$\frac{S(y)}{y}=\sum\limits_{n=1}^{\infty}ny^{n-1}\Rightarrow\int_0^y\frac{S(t)}{t}dt=\sum\limits_{n=1}^{\infty}y^n=\frac{1}{1-y}-1\Rightarrow S(y)=\frac{y}{(1-y)^2}$

$\therefore S(x)=\frac{2^{-x}}{(1-2^{-x})^2}$

（2）$\because \lambda=\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{x}{2}\right|<1\Rightarrow -2<x<2$ 　　当 $x=-2$ 时 $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n}$ 收敛

当 $x=2$ 时 $\sum\limits_{n=1}^{\infty}\frac{1}{2n}$ 发散　　$\therefore$ 收敛域：$[-2,2)$

$$S(x)=\sum_{n=1}^{\infty}\frac{1}{n2^n}x^{n-1}\Rightarrow\left[xS(x)\right]'=\sum_{n=1}^{\infty}\frac{x^{n-1}}{2^n}=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^{n-1}=\frac{1}{2}\cdot\frac{1}{1-\frac{x}{2}}=\frac{1}{2-x} \qquad \therefore S(x)=-\frac{1}{x}\ln(2-x)$$

故和函数 $S(x)=-\frac{1}{x}\ln(2-x)$ 　$x\in[-2,2)$ 且 $x\ne 0$ 　　$S(0)=\frac{1}{2}$

7. 奇延拓：设 $\therefore f(x)=\sum\limits_{n=1}^{\infty}b_n\sin\frac{n\pi x}{2}$

$$b_n=\frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}dx=\int_0^1\sin\frac{n\pi x}{2}dx+\int_1^2 2\sin\frac{n\pi x}{2}dx=\frac{2}{n\pi}\cos\frac{n\pi}{2}+\frac{2}{n\pi}-\frac{4\cos n\pi}{n\pi}$$

$$\therefore f(x)=\sum_{n=1}^{\infty}\left[\frac{2}{n\pi}\cos\frac{n\pi}{2}+\frac{2}{n\pi}-\frac{4\cdot(-1)^n}{n\pi}\right]\sin\frac{n\pi x}{2}$$

当 $x=n$，$(n=1,2,3\cdots)$ 时收敛于 $\frac{3}{2}$；当 $x=-n$，$(n=1,2,3\cdots)$ 时收敛于 $-\frac{3}{2}$

当 $x=0$ 时收敛于 0 　　$\therefore f(5)=\frac{3}{2}$

8. 取 C：$x^2+xy+y^2=\varepsilon^2$，$\varepsilon>0$ 足够

$$\because \int_{(+L)\cup(-C)}\frac{xdy-ydx}{x^2+xy+y^2}=\iint_D\left[\frac{-x^2+y^2}{\left(x^2+xy+y^2\right)^2}+\frac{x^2-y^2}{\left(x^2+xy+y^2\right)^2}\right]d\sigma=0 \qquad \therefore \oint_L\frac{xdy-ydx}{x^2+xy+y^2}=\oint_{+C}\frac{xdy-ydx}{x^2+xy+y^2}$$

化 C 为 $\frac{3}{4}x^2+(y+\frac{x}{2})^2=\varepsilon^2$ 　　令 $x=\frac{2}{\sqrt{3}}\varepsilon\cos t$，$y=(\sin t-\frac{1}{\sqrt{3}}\cos t)\varepsilon$

$$\therefore \oint_L\frac{xdy-ydx}{x^2+xy+y^2}=\int_0^{2\pi}\frac{\frac{2}{\sqrt{3}}\varepsilon\cos t\cdot\left(\cos t+\frac{1}{\sqrt{3}}\sin t\right)\varepsilon+\left(\sin t-\frac{1}{\sqrt{3}}\cos t\right)\varepsilon\cdot\frac{2}{\sqrt{3}}\varepsilon\sin t}{\varepsilon^2}dt=\int_0^{2\pi}\frac{2}{\sqrt{3}}dt=\frac{4}{\sqrt{3}}\pi$$

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