Understand. The following are equivalent for a finite group G.

- The composition $G \xrightarrow{\text{Cay}} \mathrm{Perm}(G) \xrightarrow{\text{sgn}} \{\pm 1\}$ is non-trivial.
- $2 \mid \mathrm{ord}(G)$ and G has an element of order $2^{\nu_2(\mathrm{ord}(G))}$.
- $2 \mid \mathrm{ord}(G)$ and G's Sylow_2 subgroups are cyclic.

Corollary. Let G be a finite group. Then the set elements of odd order O forms a subgroup, under the assumption that G has an element of order $2^{\nu_2(\mathrm{ord}(G))}$. Moreover, $[G:O] = 2^{\nu_2(\mathrm{ord}(G))}$.

Proof. Write $\mathrm{ord}(G) = 2^k(2\ell - 1)$. The case $k = 0$ is trivial. If $k \geq 1$, then the composition $G \longrightarrow \mathrm{Perm}(G) \longrightarrow \{\pm 1\}$ is non-trivial. Thus G has a normal subgroup N of index 2. We have $\mathrm{ord}(N) = 2^{k-1}(2\ell - 1)$, $O_G = O_N$, and N has an element of order 2^{k-1}. q.e.d.

Understand. Let M be a semi-group of n elements. Then $m^{\mathrm{lcm}(1,\dots,n)} = m^{2\mathrm{lcm}(1,\dots,n)}$ for all $m \in M$.

Corollary. Let $A \in \mathbf{Z}^{n\times n}$ be a k-th power of an integral matrix for all $k \geq 2$. Then $A = A^2 = A^3 = \dots$

Proof. It suffices to show $A = A^2$ when reduced mod p for each prime p. However, working in $M = \mathbf{F}_p^{n\times n}$ we have $\bar{A} = X^{\mathrm{lcm}(1,\dots,p^{n^2})}$ which implies $\bar{A} = \bar{A}^2$.

Understand. For a linear mapping $T : V \longrightarrow W$ of vector spaces over a finite field we have $\#\{v \in V : Tv \in S\} = \#\{\mathrm{img}\, T \cap S\} \cdot \#\ker T$.

Exercise. Given a random coloring of the 30 edges of the icosahedron red green and blue, what is the probability that each of the 20 triangular faces have two edges of one color and one edge of another color?

A bashing proof of theorema egregium. Let $X = X(u,v) : D \subseteq \mathbf{R}^2 \xrightarrow{\sim} S \subset \mathbf{R}^3$ be a surface parametrization. Let

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|}$$

be the normal, and

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} X_u \cdot X_u & X_u \cdot X_v \\ X_u \cdot X_v & X_v \cdot X_v \end{pmatrix}$$

be the first fundamental form, namely the surface isometry invariant. To compute the partials of the first fundamental form entries we write

$$X_{uu} = \alpha X_u + \beta X_v + eN$$

$$X_{uv} = \varepsilon X_u + \zeta X_v + fN$$

$$X_{vv} = \gamma X_u + \delta X_v + gN$$

Then we have

$$E_u = 2(\alpha E + \beta F)$$

$$E_v = 2(\varepsilon E + \zeta F)$$

$$G_u = 2(\varepsilon F + \zeta G)$$

$$G_v = 2(\gamma F + \delta G)$$

$$F_u = \alpha F + \beta G + \varepsilon E + \zeta F$$

$$F_v = \gamma E + \delta F + \varepsilon F + \zeta G$$

Which mean

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \varepsilon \\ \zeta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} E_v \\ G_u \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} E_u \\ 2F_u - E_v \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2F_v - G_u \\ G_v \end{pmatrix}$$

And so the first fundamental form determines $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$. Gauss's theorem, which we shall now prove, is that $eg - f^2$ is also determined by the first fundamental form. Firstly, since $N \perp X_u$ and $N \perp X_v$ we have

$$\begin{aligned} e &= X_{uu} \cdot N = -N_u \cdot X_u \\ f &= X_{uv} \cdot N = -N_v \cdot X_u = -N_u \cdot X_v \\ g &= X_{vv} \cdot N = -N_v \cdot X_v \end{aligned}$$

Moreover, since $\|N\| \equiv 1$ we may write

$$\begin{aligned} N_u &= a^{11}X_u + a^{12}X_v \\ N_v &= a^{21}X_u + a^{22}X_v \end{aligned}$$

which yields

$$\begin{pmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

We write $K = a^{11}a^{22} - a^{12}a^{21} = \dfrac{eg - f^2}{EG - F^2}$ for the Gaussian curvature. Now, we have

$$\begin{aligned} X_{uuv} &= \alpha_v X_u + \alpha X_{uv} + \beta_v X_v + \beta X_{vv} + e_v N + e N_v \\ X_{uvu} &= \varepsilon_u X_u + \varepsilon X_{uu} + \zeta_u X_v + \zeta X_{uv} + f_u N + f N_u \end{aligned}$$

Comparing the X_v part we get

$$\alpha\zeta + \beta_v + \beta\delta + e a^{22} = \varepsilon\beta + \zeta_u + \zeta^2 + f a^{12}$$

Now, by the matrix equation we have that

$$\varepsilon\beta + \zeta_u + \zeta^2 - \alpha\zeta - \beta_v - \beta\delta = e a^{22} - f a^{12} = -EK$$

is determined by the first fundamental form. Since E is positive, K is determined by the first fundamental form. Gauss's theorem, for example, implies that no part of a sphere can be isometrically embedded in the plane. Any map of the globe, no matter of how small a region, will have distorsions.

In orthogonal coordinates $E = A^2, F = 0, G = B^2$ we get

$$K = -\frac{1}{AB}\left(\partial_v\left(\frac{A_v}{B}\right) + \partial_u\left(\frac{B_u}{A}\right)\right)$$

In particular, in isothermal coordinates $E = G = \lambda, F = 0$ we have

$$K = -\frac{\Delta \log \lambda}{2\lambda}$$

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To me, this finally gives a definition of the curvature of the hyperbolic plane and shows it is -1. Indeed, the hyperbolic plane is nothing but $H = \{y > 0\}$ with the metric given by $\dfrac{\sqrt{dx^2 + dy^2}}{y}$. Namely, the first fundamental form is $y^{-2}\mathrm{id}$. This immediately yields $K = -1$.

Theorem. $\nu_p(n!) = \lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \dots$

Theorem. $(p-1)\nu_p(n!) = n - s_p(n)$ where $s_p(n)$ is the digit sum of n in base p.

Theorem. $\nu_p\binom{p^k}{r} = k - \nu_p(r)$