misc

Definition. A (mathematical) juggling pattern is a permutation $f : \mathbf{Z} \longrightarrow \mathbf{Z}$ for which $df(t) = f(t) - t \geq 0$.

We think of $t \in \mathbf{Z}$ as a discrete, infinite in both directions time parameter. If there is no "throw" at time t, we set $f(t) = t$, and otherwise we set $f(t)$ to be the next time the "ball" thrown at t will be thrown again. We think of $df(t)$ as the "height" of the throw. This model assumes only one ball may be thrown at once.

Observation. Each orbit is either a fixed point (no throw) or infinite in both directions.

The "balls" are in one to one correspondence with the infinite orbits.

Observation. If df is bounded by M then so is the number of balls.

What does $\sum_{t \in I} df(t)$ count for an interval I? well, it counts the "skips" of one ball over another. Approximately, if I is very large, this equals $|I|$ times the number of balls, since each time frame is "skipped over" once by each ball. Another way to see this is the following. The sum of df over a ball's orbit in an interval is the last after minus first appearence in that interval, which is $|I| \pm O(M)$.

combinatorics in finite vector spaces

In this chapter F is a finite field with q elements and all vector spaces are over F and finite dimensional.

Result. $|V| = q^{\dim V}$.

Result. Given a linear surjective mapping $T : V \longrightarrow W$, T is $|\ker T| : 1$.

Exercise. Given a random coloring of the 30 edges of the icosahedron red green and blue, what is the probability that each of the 20 triangular faces have two edges of one color and one edge of another color?

Result. The number of bases of V is $b_d = (q^d - 1)(q^d - q)\dots(q^d - q^{d-1})$, where $d = \dim V$.

Result. The number of subspaces of V of dimension k is $\dfrac{(q^d-1)(q^d-q)\dots(q^d-q^{k-1})}{(q^k-1)(q^k-q)\dots(q^k-q^{k-1})} = \begin{bmatrix} d \\ k \end{bmatrix}_q$.

Result. The number of pairs (U, W) with $U \oplus W = V$ and $\dim U = k$ is $\dfrac{b_d}{b_k b_{d-k}}$.

Result. The number of $T \in \operatorname{End} V$ such that $\ker T \oplus \operatorname{img} T = V$ and $\dim \ker T = k$ is $\dfrac{b_d}{b_k}$.

Result. Given $U \leq V$ of dimension r, the number of pairs (W_1, W_2) with $W_1 + W_2 = V$, $W_1 \cap W_2 = U$ and $\dim W_1 = r + j$ is $\dfrac{b_{d-r}}{b_j, b_{d-r-j}}$.

Result. The number of $T \in \operatorname{End} V$ such that $\dim(\ker T \cap \operatorname{img} T) = r$ and $\dim \operatorname{img} T = r + j$ is $\begin{bmatrix} d \\ d-r \end{bmatrix}_q \begin{bmatrix} d-r \\ r \end{bmatrix}_q \dfrac{b_{d-2r} b_{j+r}}{b_j b_{d-2r-j}}$.

Result. $q^{d^2} = \sum_{r=0}^{\lfloor d/2 \rfloor} \sum_{j=0}^{d-2r} \begin{bmatrix} d \\ d-r \end{bmatrix}_q \begin{bmatrix} d-r \\ r \end{bmatrix}_q \frac{b_{d-2r} b_{j+r}}{b_j b_{d-2r-j}}$

Result. The expected value of the number of nonzero fixed points of a random $g \in GL_d(F)$ in F^d is one.

random abstract algebra bits

Exercise. Given an operator $T \in \mathrm{Lin}(V)$ if $v_0, Tv_0, \ldots, T^{d-1}v_0$ is a basis of V for some v_0, then $ST = TS \iff S$ is a polynomial in T.

Observation. The following are equivalent for a finite group G.

- The composition $G \xrightarrow{\mathrm{Cay}} \mathrm{Perm}(G) \xrightarrow{\mathrm{sgn}} \{\pm 1\}$ is non-trivial.
- $2 \mid \mathrm{ord}(G)$ and G has an element of order $2^{\nu_2(\mathrm{ord}(G))}$.
- $2 \mid \mathrm{ord}(G)$ and G's Sylow_2 subgroups are cyclic.

Corollary. Let G be a finite group. Then the set elements of odd order O forms a subgroup, under the assumption that G has an element of order $2^{\nu_2(\mathrm{ord}(G))}$. Moreover, $[G : O] = 2^{\nu_2(\mathrm{ord}(G))}$.

Proof. Write $\mathrm{ord}(G) = 2^k(2\ell - 1)$. The case $k = 0$ is trivial. If $k \geq 1$, then the composition $G \longrightarrow \mathrm{Perm}(G) \longrightarrow \{\pm 1\}$ is non-trivial. Thus G has a normal subgroup N of index 2. We have $\mathrm{ord}(N) = 2^{k-1}(2\ell - 1)$, $O_G = O_N$, and N has an element of order 2^{k-1}. q.e.d.

Observation. Let M be a semi-group of n elements. Then $m^{\mathrm{lcm}(1,\ldots,n)} = m^{2\mathrm{lcm}(1,\ldots,n)}$ for all $m \in M$.

Corollary. Let $A \in \mathbf{Z}^{n \times n}$ be a k-th power of an integral matrix for all $k \geq 2$. Then $A = A^2 = A^3 = \ldots$

Proof. It suffices to show $A = A^2$ when reduced mod p for each prime p. However, working in $M = \mathbf{F}_p^{n \times n}$ we have $\bar{A} = X^{\mathrm{lcm}(1,\ldots,p^{n^2})}$ which implies $\bar{A} = \bar{A}^2$.

A bashing proof of theorema egregium. Let $X = X(u,v) : D \subseteq \mathbf{R}^2 \xrightarrow{\sim} S \subset \mathbf{R}^3$ be a surface parametrization. Let

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|}$$

be the normal, and

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} X_u \cdot X_u & X_u \cdot X_v \\ X_u \cdot X_v & X_v \cdot X_v \end{pmatrix}$$

be the first fundamental form, namely the surface isometry invariant. To compute the partials of the first fundamental form entries we write

$$X_{uu} = \alpha X_u + \beta X_v + eN$$

$$X_{uv} = \varepsilon X_u + \zeta X_v + fN$$

$$X_{vv} = \gamma X_u + \delta X_v + gN$$

Then we have

$$E_u = 2(\alpha E + \beta F)$$

$$E_v = 2(\varepsilon E + \zeta F)$$

$$G_u = 2(\varepsilon F + \zeta G)$$

$$G_v = 2(\gamma F + \delta G)$$

$$F_u = \alpha F + \beta G + \varepsilon E + \zeta F$$

$$F_v = \gamma E + \delta F + \varepsilon F + \zeta G$$

Which mean

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \varepsilon \\ \zeta \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} E_v \\ G_u \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} E_u \\ 2F_u - E_v \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 2F_v - G_u \\ G_v \end{pmatrix}$$

And so the first fundamental form determines $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$. Gauss's theorem, which we shall now prove, is that $eg - f^2$ is also determined by the first fundamental form. Firstly, since $N \perp X_u$ and $N \perp X_v$ we have

$$\begin{gathered} e = X_{uu} \cdot N = -N_u \cdot X_u \\ f = X_{uv} \cdot N = -N_v \cdot X_u = -N_u \cdot X_v \\ g = X_{vv} \cdot N = -N_v \cdot X_v \end{gathered}$$

Moreover, since $\|N\| \equiv 1$ we may write

$$\begin{gathered} N_u = a^{11}X_u + a^{12}X_v \\ N_v = a^{21}X_u + a^{22}X_v \end{gathered}$$

which yields

$$\begin{pmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

We write $K = a^{11}a^{22} - a^{12}a^{21} = \dfrac{eg - f^2}{EG - F^2}$ for the Gaussian curvature. Now, we have

$$\begin{gathered} X_{uuv} = \alpha_v X_u + \alpha X_{uv} + \beta_v X_v + \beta X_{vv} + e_v N + e N_v \\ X_{uvu} = \varepsilon_u X_u + \varepsilon X_{uu} + \zeta_u X_v + \zeta X_{uv} + f_u N + f N_u \end{gathered}$$

Comparing the X_v part we get

$$\alpha\zeta + \beta_v + \beta\delta + e a^{22} = \varepsilon\beta + \zeta_u + \zeta^2 + f a^{12}$$

Now, by the matrix equation we have that

$$\varepsilon\beta + \zeta_u + \zeta^2 - \alpha\zeta - \beta_v - \beta\delta = e a^{22} - f a^{12} = -EK$$

is determined by the first fundamental form. Since E is positive, K is determined by the first fundamental form. Gauss's theorem, for example, implies that no part of a sphere can be isometrically embedded in the plane. Any map of the globe, no matter of how small a region, will have distorsions.

In orthogonal coordinates $E = A^2, F = 0, G = B^2$ we get

$$K = -\frac{1}{AB}\left(\partial_v\left(\frac{A_v}{B}\right) + \partial_u\left(\frac{B_u}{A}\right)\right)$$

In particular, in isothermal coordinates $E = G = \lambda$, $F = 0$ we have

$$K = -\frac{\Delta \log \lambda}{2\lambda}$$

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To me, this finally gives a definition of the curvature of the hyperbolic plane and shows it is -1. Indeed, the hyperbolic plane is nothing but $H = \{y > 0\}$ with the metric given by $\frac{\sqrt{dx^2 + dy^2}}{y}$. Namely, the first fundamental form is $y^{-2}\mathrm{id}$. This immediately yields $K = -1$.

Theorem. $\nu_p(n!) = \lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \dots$

Theorem. $(p-1)\nu_p(n!) = n - s_p(n)$ where $s_p(n)$ is the digit sum of n in base p.

Theorem. $\nu_p \binom{p^k}{r} = k - \nu_p(r)$

Given $v_1, \dots, v_{n-1} \in \mathbf{R}^n$ linearly independent, how can one find a vector w spanning $(v_1, \dots, v_{n-1})^\perp$?
Assuming these are row vectors, put them in an $n \times n$ determinant, where the first row is $x_1, \dots, x_n$.
This gives a nontrivial linear form in x, which vanishes for $x \in \{v_1, \dots, v_{n-1}\}$. It is of course of the form $x \mapsto x \cdot w$ for the w we are looking for. To get w explicitly, note that it's i-th coordinate is $e_i \cdot w = \det(e_i, v_1, \dots, v_{n-1})$. Moreover, the length of w is precisely the $n-1$ dimensional volume of the parallelepide spanned by $v_1, \dots, v_{n-1}$. Indeed, $w \cdot w = \det(w, v_1, \dots, v_{n-1}) = \mathrm{vol}_{n-1}(v_i) \cdot \|w\|$, since w is perpendicular to the v_i.

problems

A two player game with n lamps in a circle is played as follows. In the beginning, player 1 chooses which lamps should be on and which should be off. After that, in each turn, player 2 specifies which lamps should be switched (on↔off), but player 1 may rotate the cicle of lamps as they please before making the switches. Player 2 wins once all lamps are off. Find for which n player 2 can win, and how.

A two player game with a standard deck of cards is played as follows. In the beginning, player 1 randomly shuffles the deck. Then, at any point, player 2 can either bet that the next card will be red, or choose to see the next card. The game ends once the bet is made, and player 2 wins if they guessed right. (Player 2 must make a bet at some point, so if 51 cards were seen the last card has to be betted on). Show that no matter what strategy player 2 picks, they will win an expected 1/2 of the games they play.