A book dedicated to lovers of mathematics

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0. Problems

id aa. Consider the following stochastic process. First, a probability p is drawn from a uniform distribution on $[0, 1]$. Then 100 independent coins are drawn, each with probability p to show heads. Without integrals, determine the probability of seeing precisely k heads.

id ab. A group of n mathematicians will face the following challenge. Each will be given a hat showing an integer from $1, \dots, n$ (repetitions and misses allowed), and will be able to see the numbers written on all hats but their own. Once they see each other, and without any communication between them, they will all have to simultaneously guess the number written on their hat. They will win if at least one of them makes a correct guess. They have today to form a strategy that will guarantee a win. What do you suggest?

id ad. Let $f : \mathbf{Z}^{\mathbf{N}} \to \mathbf{Z}$ be an additive mapping (namely $f(x+y) = f(x) + f(y)$) from infinite integer sequences to integers. Show that $f(x_1, x_2, x_3, \dots) = \alpha_1 x_1 + \cdots + \alpha_N x_N$ for some finite integer sequence $\alpha_1, \dots, \alpha_N$.

id ah. Let $f : [0, 1] \to \mathbf{R}$ be a continuous function for which $\lim_{h \to 0} \dfrac{f(t_0 + h) + f(t_0 - h) - 2f(t_0)}{h^2} = 0$ for all $t_0 \in (0, 1)$. Then $f(t) = \alpha t + \beta$ is linear.

id ai. Smooth (or even just continuous) homomorphisms $\phi : \mathbf{R} \to \mathrm{GL}_d(\mathbf{C})$ are in $1 : 1$ correspondence with $A \in \mathbf{C}^{d \times d}$ via

$$\phi(t) = \exp(At)$$
$$A = \left.\frac{\mathrm{d}\phi(t)}{\mathrm{d}t}\right|_{t=0}$$

id aj. Let a_n be a sequence for which $\sum |a_n b_n| < \infty$ for all $b_n \in \ell^2$. Then $a_n \in \ell^2$.

id ak. A two player game goes as follows. Each player is given a number, drawn at random, independently, uniformly from $[0, 1]$. A player may keep their number, or ask for another - in which case they must keep the second number. Then the two players reveal their numbers, and the player with the bigger number wins. Find the optimal playing strategy.

id al. n ants are placed inside a circle of radius R, each is initially heading east,west,north, or south, and all ants have constant speed 1. When two ants facing opposite directions collide, they both immediately turn 90 degrees. Find the minimal length of time to guarantee all ants will leave the circle.

id am. Customers arrive to a store via a Poisson process (with some constant). For every arrival, the employee calculates the probability this will be the last customer of his shift. At the end of their shift, they write down the probability calculated for the last customer. Over many shifts, what is the distribution of probabilities?

id an. Two players take turns coloring the vertices of a graph. In the beginning, all vertices are white. Player 1 picks the first vertex and colors it black. Afterwards, the next player must color black a vertex that is currently white, and adjacent to the the last colored vertex. A player unable to do so losses the game. What property of the graph is equivalent to the first/second player having a winning strategy, and what is their strategy?

id ao. A very long road contains n cars, each having a constant speed drawn at random from 20 kmh to 200 kmh. The road is narrow so there's no overtaking, so fast cars may have to slow down to match the speed of the car in front. Eventually, how many meshes of connected cars are to be expected?

id ap. Alice and Bob play the following game. There is an outer circle, fixed in position, containing n lamps, and an inner circle, which may be rotated, containing the numbers $1..n$. Alice wins when all the lights are off, and Bob wins if she gives up/the game goes forever. Bob chooses the initial on/off states of the lamps. Every turn, Alice specifies a set of numbers in $1..n$ and tells Bob to switch the state (on $\leftrightarrow$ off) of the lamps corresponding to those positions, but Bob may spin the inner circle of numbers before doing so. For which values of n can Alice/Bob win, and how?

id aq. A rectangular grid R is painted using 10 colors. A rectangle is called *special* if its four vertices have the same color. Find dimensions for R guaranteeing the existence of a special rectangle.

id ar. There exists an uncountable family $\mathscr{F}$ of infinite subsets of $\mathbf{N}$ where the intersection of any pair of sets in $\mathscr{F}$ is finite.

id as. You find yourself in a huge circular train (say about the size of the equator) in which you can walk freely in both directions. You'll be free once you determine the exact number of cars, but you only have one guess. All the cars look exactly the same, and you can't leave any mark, except each car has a lamp with a switch which you may freely use. However, the initial state of the lamps is random. What do you do?

id at. An invisible frog lives on the integer number line, and each day it hops by a fixed unknown amount d. Every day you get one guess as to where the frog is, and if you hit it you win a prize. Find a strategy that guarantees you'll win the prize, eventually.

Part ii. Now the frog moved to the real line, and still hops every day by a fixed amount. It also gained an unknown small but positive length ε. What's your strategy?

id au. A standard 2-dimensional Gaussian is given in polar coordinates by $\sqrt{2\log(1/u)}(\cos 2\pi\theta, \sin 2\pi\theta)$ where u, θ are independent and uniform on $[0, 1]$.

id aw. There's a deck with 100 numbered 1..100 in some order. In each step we peek at the top card, say numbered k, take the first k cards from the deck, flip their order, and return them at the top of the deck. In particular, after this move the k-th card now shows k. Show that eventually the top card becomes 1.

id ay. Let $f : [0, 1] \to \mathbf{R}$ be a continuous function satisfying $\int_0^1 f(x)x^k = 1$ for $k = 0, \ldots, n-1$. What is the best lower bound on $\int_0^1 f^2$?

id bd. From each vertex of a triangle, extend the two edges by the length of the opposite edge, then the six endpoints lie on a circle with center I and radius $\sqrt{r^2 + s^2}$.

2. Calculus

This chapter emphasizes intuition over rigour. Proofs will later be given in the Analysis chapter.

id ac. Fact: For all $x \in \mathbf{R}$ we have

$$\exp(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

3. Number Theory

id ax. Fact: We have $\operatorname{lcm}(1, \ldots, n) \geq 2^{n-2}$ for all n.

id ax. Explanation: Consider $I = \int_0^1 t^m (1-t)^m \mathrm{d}t$. We have $I = \dfrac{r}{\operatorname{lcm}(1, \ldots, 2m+1)}$ for some positive integer r, as well as $I \leq 4^{-m}$. Thus $4^m \leq \operatorname{lcm}(1, \ldots, 2m+1)$.

4. Combinatorics and Probability

id az. Fact: A monkey is typing letters uniformly at random. The the expected number of letters until ABRACADABRA is typed equals $26 + 26^4 + 26^{11}$.

id az. Explanation. Imagine a casino where gamblers can bet on the next letter. There are 26 letters and the casino gives fair odds, meaning a gambler can pay the casino \$$x$ dollars and bet that the next letter will be y. If the bet was right he will get back \$$26x$. Now imagine that before every letter, a new gambler arrives with \$1 and bets all their money that the next 11 letters will be ABRACADABRA (of course this will take several turns, the first putting down \$1 on the first letter being A. If the bet works out, the next bet would be \$26 dollars on the next letter being B, and so on). Starting with \$0, let X_n be the cash of the casino after the n letter was typed. Let T be the stopping time - when ABRACADABRA is first typed. Then X_n is a martingale with bounded differences, $\mathbf{E}T < \infty$ and so $\mathbf{E}X_T = \mathbf{E}X_0 = \$0$. Let's consider the cash after time T. Every unsuccessful gambler lost \$1 to the casino. There are 3 successful gamblers, who put in an initial \$1 and took $26 + 26^4 + 26^{11}$ from the casino. In total, $X_T = T - 26 + 26^4 + 26^{11}$.

Nomenclature. In a *branching process* of microbes, the 0-th generation consists of a single microbe, and each microbe gives birth to a number of children, randomly generated from a fixed distribution X (taking non-negative integer values) independently of all other microbes. In formula, $Z_0 = 1$, and $Z_{n+1} = A_{1,n} + \cdots + A_{Z_n,n}$, where $A_{i,j} \sim X$ are independent.

id bb. Fact: Let $(X, Z_n)_{n=0}^{\infty}$ be a branching process, and set $f(\theta) = \mathbf{E}[\theta^X]$, $g_n(\theta) = \mathbf{E}[\theta^{Z_n}]$ for $\theta \in [0,1]$. Then
i. $g_n = f \circ f \cdots \circ f$ is the n-th composition of f.
ii. The *probability of extinction* $p = \mathbf{P}[Z_m = 0 \quad \text{for some } m]$ is the smallest non-negative fixed point of f.

5. Abstract Algebra

id ba. Fact: If A is a Noetherian ring then $A[t]$ is a Noetherian ring.

id ba. Explanation: Fix an ideal $\mathscr{I} \trianglelefteq A[t]$. A best effort attempt to generate $\mathscr{I}$ is as follows: as long as $f_1(t), ..., f_{n-1}(t)$ don't generate $\mathscr{I}$, let $f_n(t)$ be the polynomial of minimal degree in $\mathscr{I}$ and not in $\langle f_1(t), \ldots, f_{n-1}(t) \rangle$. Note that $d_n = \deg f_n(t)$ is weakly increasing, and that the leading coefficient of $f_n(t)$ does not belong to the ideal of A generated by the leading coefficients of $f_1(t), .., f_{n-1}(t)$, for otherwise there are $a_1, \ldots, a_{n-1} \in A$ for which $g(t) = f_n(t) - a_1 f_1(t) t^{d_n - d_1} \cdots - a_{n-1} f_{n-1}(t) t^{d_n - d_{n-1}}$ has degree smaller than $f_n(t)$ and is also member of $\mathscr{I} \setminus \langle f_1(t), .., f_n(t) \rangle$, which can't be. It follows that the sequence of ideals of A generated by the leading coefficients of $f_1(t), \ldots, f_n(t)$ is strictly increasing, and A is not Noetherian.

6. Analysis

Nomenclature. A function f defined on the vertices of a graph will be called *harmonic* if for every vertex v the average value of $f(u)$ over the neighbours u of v equals $f(v)$.

id ae. Fact: Let f be a real valued harmonic function on the lattice $\mathbf{Z}^d$. If f is non-constant, then it is unbounded.

id ae. Explanation: If f is non-constant, then wlog the bounded harmonic function $g(x) = f(x+e_1) - f(x)$ attains a positive value. Let $S = \sup g > 0$. Given ε we may find x_ε for which $g(x_\varepsilon) > S - \varepsilon$. Since the average of g over the $2d$ neighbours of x_ε is larger than $S - \varepsilon$, and each value is at most S, it follows that g assigns each neighbour a value greater than $S - 2d\varepsilon$. Thus $g(x_\varepsilon + e_1) > S - 2d\varepsilon$, and generally $g(x_\varepsilon + ke_1) > S - (2d)^k\varepsilon$. However, if we pick k large enough and ε small enough, we can make $f(x+(k+1)e_1) - f(x) = g(x) + g(x+e_1) + \cdots + g(x+ke_1) > (k+1)(S-(2d)^k\varepsilon)$ arbitrarily large, meaning f is unbounded.

id af. Fact: Let T be a strict contraction of a complete metric space M. Then in M there exists a unique fixed point m^* of T. Moreover, for any $m \in M$ we have $\lim\limits_{n\to\infty} T^n(m) = m^*$.

id af. Explanation: A strict contraction clearly cannot have more than one fixed point. Let m be an arbitrary point, and let $m_n = T^n(m)$ be its orbit under T. We have $d(m_n, m_{n+1}) \leq q^n d(m, Tm)$, where q is the contraction constant for T. Thus $\sum d(m_n, m_{n+1}) \leq \frac{d(m,Tm)}{1-q}$ is convergent, implying m_n is Cauchy, and hence convergent. By continuity, the limit of the orbit is a fixed point.

id av. Fact: The Fourier transform is injective on $\mathcal{L}^1(\mathbf{R})$.

id av. Explanation: Let $f \in \mathcal{L}^1, \hat{f} \equiv 0$. For all $a, \xi \in \mathbf{R}$ we have

$$\int_{-\infty}^{a} f(t)e^{i\xi(t-a)}\mathrm{d}t = -\int_{a}^{\infty} f(t)e^{i\xi(t-a)}\mathrm{d}t$$

and we denote this value by $F_a(\xi)$. The right hand side may be analytically continued to arguments in the closed upper half plane, and the left hand side may be continued to the closed lower half plane. It follows that $F_a(\xi)$ is an entire function. However, it is everywhere bounded by $\|f\|_1$, and so must be constant. By dominated convergence we have $\lim_{r\to\infty} F_a(ir) = 0$, and so $F_a(\xi) = 0$. In particular $\int_{-\infty}^{a} f(t)\mathrm{d}t = F_a(0) = 0$ for all a and so $f = 0$ almost everywhere.

id ag. Fact: Let $f \in \mathscr{C}(\mathbf{T})$ and let $F_m(t) = \sum\limits_{j=-m}^{m} \hat{f}(j)e^{2\pi ijt}$ be the Fourier approximations of f. Then the sequence of averages $\sigma_n(t) = \dfrac{F_0(t) + \cdots + F_n(t)}{n+1}$ converges uniformly to f.

id aq. Explanation: We have $\hat{f}(j)e^{2\pi ijt} = \int_0^1 f(x)e^{2\pi ij(t-x)}\mathrm{d}x = \int_{-t}^{1-t} f(u+t)e^{-2\pi iju}\mathrm{d}u = \int_0^1 f(u+t)e^{-2\pi iju}\mathrm{d}u$

and therefore

$$F_m(t) = \int_0^1 f(u+t)P_m(u)\mathrm{d}u \qquad \sigma_n(t) = \int_0^1 f(u+t)D_n(u)\mathrm{d}u$$

For

$$P_m(u) = \sum_{j=-m}^{m} e^{2\pi i j u} \qquad D_n(u) = \frac{P_0(u)+\cdots+P_n(u)}{n+1}$$

We have the identities $\int_0^1 D_n(u) = 1$ and $(n+1)D_n(u) = \left(\sum_{k=0}^{n} e^{2\pi i(k-\frac{n}{2})u}\right)^2 = \left(\frac{\sin(\pi(n+1)u)}{\sin(\pi u)}\right)^2 \geq 0$. (To clarify, $D_n(u) = n+1$ for $u \in \mathbf{Z}$). We continue

$$\sigma_n(t) - f(t) = \int_0^1 [f(u+t)-f(t)]D_n(u)\mathrm{d}u \implies |\sigma_n(t) - f(t)| \leq \int_0^1 |f(u+t)-f(t)|\, D_n(u)\mathrm{d}u$$

Given ε we find δ such that $|x-y| \leq \delta \implies |f(x)-f(y)| \leq \varepsilon$. The first part of the integral is bounded independent of n or t: $\int_{-\delta}^{\delta} |f(u+t)-f(t)|\, D_n(u)\mathrm{d}u \leq \varepsilon \int_0^1 D_n(u)\mathrm{d}u = \varepsilon$. Finally, on $[\delta, 1-\delta]$ we have $D_n(u) \leq \frac{1}{(n+1)\sin(\pi\delta)^2}$ and so if n is large enough then $D_n(u) \leq \varepsilon$ on this segment, and in total $|\sigma_n(t) - f(t)| \leq \varepsilon + 2\left\|f\right\|_\infty \varepsilon$ independent of t.

id bc. Fact: Let b, m, n be elements of a C^* algebra with m, n normal and $bn = mb$. Then $bn^* = m^*b$.

id bc. Explanation: We have $bn^k = m^k b$, and so $bf(n) = f(m)b$ for analytic functions f. In particular $b = \exp(\xi m) b \exp(-\xi n)$ for all $\xi \in \mathbf{C}$. Normality yields

$$\exp(i\zeta m^*) b \exp(-i\zeta n^*) = \exp(i(\zeta m^* + \zeta^* m)) b \exp(-i(\zeta^* n - \zeta n^*))$$

The rhs is an entire function of ζ, and the lhs is bounded by $\|b\|$, and so the expression is a constant, namely b. Finally, the lhs coefficient of ζ is $im^*b - ibn^*$.

-1. Solutions

id aa. The probability is $1/101$, independent of $k = 0, \ldots, 100$. To show this, note that we may generate a biased coin toss by generating a uniform random number t from $[0,1]$ and then show heads iff $t < p$. Therefore the process can be described by generating $1 + 100$ independent uniform random numbers $p = t_0, t_1, \ldots, t_{100}$ from $[0,1]$ and the number of heads is the index of t_0 after sorting, which is distributed uniformly in $0, \ldots, 100$.

id ab. First consider the case $n = 2$. The two players have numbers in $0, 1$, and see the other person's number. To guarantee a correct guess, one will guess they have the same number, and the other will guess they have different numbers. We generalize this to arbitrary n by assigning the players id's $0, 1, \ldots, n-1$, and the player with id i guessing that the sum of all their numbers is congruent to i modulo n. This strategy guarantees exactly one of them will be right, namely the one with id equal to the sum of their numbers modulo n.

id ad. Let e_n denote the n-th coordinate sequence, set $\alpha_n = f(e_n)$. We'll show that $\alpha_{N+1} = \alpha_{N+2} = \cdots = 0$ after some value of N. To do this, we form a sequence $b = (b_1, b_2, b_3, \ldots)$ as follows: $b_1 = 1$, and b_{k+1} is a power of 2 greater than both b_k and $2\,|f(b_1e_1 + \cdots + b_ke_k)|$. Note that $b_k \mid b_{k+1}$ and $\lim b = \infty$. Set $B = f(b)$. We have $B = f(b_1e_1 + \cdots + b_ke_k) + b_{k+1}f\left(0..0, 1, b_{k+2}/b_{k+1}, b_{k+3}/b_{k+1}, \ldots\right)$. If k is sufficiently large, we must have $B = f(b_1e_1 + \cdots + b_ke_k)$, or otherwise $|B| \geq b_{k+1} - |f(b_1e_1 + \cdots + b_ke_k)| \geq b_{k+1}/2$. Therefore, if k is sufficiently large, $\alpha_k = f(e_k) = 0$. Now we may consider $g(x) = f(x) - \sum \alpha_i x_i$ as an additive function satisfying $g(e_n) = 0$ for all n, and we wish to show $g(x) \equiv 0$. Write $x_n = 2^n d_n + 3^n r_n$ for some integer sequences d, r. We have $g(x) = g(2d_1, 4d_2, 8d_3, \ldots) + g(3r_1, 9r_2, 27r_3, \ldots)$, but since $g(e_n) = 0$, the left summand is divisible by all powers of two, and the right summand is divisible by all powers of three, so they are both zero, and $g(x) \equiv 0$.

id ah. Subtracting a linear factor, we may assume $f(0) = f(1) = 0$, with the intention of showing $f \equiv 0$. Let $f_\varepsilon(t) = f(t) - \varepsilon t(1-t)$. Then $\displaystyle\lim_{h\to 0} \frac{f_\varepsilon(t_0+h) - 2f_\varepsilon(t_0) + f_\varepsilon(t_0-h)}{h^2} = \varepsilon$, implying f_ε does not attain its maximum at any inner point $t_0 \in (0,1)$. Therefore $f_\varepsilon \leq 0$, and in the limit $f \leq 0$. Applying the same to $-f$ we have $f \equiv 0$.

id ai. What demands proof is that all smooth/continuous homomorphisms are exponentials. Indeed, for ϕ smooth

$$\phi(s+t) = \phi(s)\phi(t) \implies \phi'(s+t) = \phi'(s)\phi(t) \implies \phi'(t) = \phi'(0)\phi(t) \implies \phi(t) = \exp(t\phi'(0))$$

It remains to show that a continuous homomorphism is smooth. Indeed, we have $\int_x^{x+a} \phi(t)\mathrm{d}t = \phi(x)\int_0^a \phi(t)\mathrm{d}t$. We pick a small enough a so that $\frac{1}{a}\int_0^a \phi(t)\mathrm{d}t = \mathrm{id} + o(1)$ is invertible, yielding

$$\phi(x) = \left(\int_x^{x+a} \phi(t)\mathrm{d}t\right)\left(\int_0^a \phi(t)\mathrm{d}t\right)^{-1}$$

is smooth.

id aj. Suppose $\sum a_n^2$ diverges, and form a partition $0 = N_0 < N_1 < \ldots$ for which $s_k = \sum_{(N_{k-1}, N_k]} a_n^2 > 1$. The sequence $b_n = a_n/ks_k$ (where $n \in (N_{k-1}, N_k]$) is in ℓ^2, since $\sum b_n^2 = \sum_k \frac{1}{k^2 s_k}$. However, $\sum a_n b_n = \sum_k 1/k$ diverges.

id aw. The number 100 can only be the top card once, after that it will stay forever at the bottom. Between any two times the number 99 is on top, 100 has to be on top. Between any two times the number 98 is on top, either 100 or 99 has to be on top, ..., between any two times the number 2 is on top a number bigger than 2 has to be on top. Therefore each number except for 1 can only be on top finitely many times.

id aw. Solution 2. If 100 ever becomes the top card, it'll later be stuck at the bottom. If not, the bottom card will never change, and the game will be indifferent to switching the numbers of the bottom card with 100. So in either case we have an induction step.

-2. Notes

id av. The given explanation is attributed to D.J. Newman.

id ax. The given explanation is attributed to M. Nair.