A book dedicated to lovers of mathematics

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0. Problems

id aa. Consider the following stochastic process. First, a probability p is drawn from a uniform distribution on $[0, 1]$. Then 100 independent coins are drawn, each with probability p to show heads. Without integrals, determine the probability of seeing precisely k heads.

id ab. A group of n mathematicians will face the following challenge. Each will be given a hat showing an integer from $1, \ldots, n$ (repetitions and misses allowed), and will be able to see the numbers written on all hats but their own. Once they see each other, and without any communication between them, they will all have to simultaneously guess the number written on their hat. They will win if at least one of them makes a correct guess. They have today to form a strategy that will guarantee a win. What do you suggest?

id ad. Let $f : \mathbf{Z}^{\mathbf{N}} \to \mathbf{Z}$ be an additive mapping (namely $f(x + y) = f(x) + f(y)$) from infinite integer sequences to integers. Show that $f(x_1, x_2, x_3, \ldots) = \alpha_1 x_1 + \cdots + \alpha_N x_N$ for some finite integer sequence $\alpha_1, \ldots, \alpha_N$.

id ah. Let $f : [0, 1] \to \mathbf{R}$ be a continuous function for which $\lim_{h \to 0} \dfrac{f(t_0 + h) + f(t_0 - h) - 2f(t_0)}{h^2} = 0$ for all $t_0 \in (0, 1)$. Then $f(t) = \alpha t + \beta$ is linear.

id ai. Smooth (or even just continuous) homomorphisms $\phi : \mathbf{R} \to \mathrm{GL}_d(\mathbf{C})$ are in $1 : 1$ correspondence with $A \in \mathbf{C}^{d \times d}$ via
$$\phi(t) = \exp(At)$$
$$A = \left. \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} \right|_{t=0}$$

id aj. Let a_n be a sequence for which $\sum |a_n b_n| < \infty$ for all $b_n \in \ell^2$. Then $a_n \in \ell^2$.

id ak. A two player game goes as follows. Each player is given a number, drawn at random, independently, uniformly from $[0, 1]$. A player may keep their number, or ask for another - in which case they must keep the second number. Then the two players reveal their numbers, and the player with the bigger number wins. Find the optimal playing strategy.

id al. n ants are placed inside a circle of radius R, each is initially heading east,west,north, or south, and all ants have constant speed 1. When two ants facing opposite directions collide, they both immediately turn 90 degrees. Find the minimal length of time to guarantee all ants will leave the circle.

id am. Customers arrive to a store via a Poisson process (with some constant). For every arrival, the employee calculates the probability this will be the last customer of his shift. At the end of their shift, they write down the probability calculated for the last customer. Over many shifts, what is the distribution of probabilities?

id an. Two players take turns coloring the vertices of a graph. In the beginning, all vertices are white. Player 1 picks the first vertex and colors it black. Afterwards, the next player must color black a vertex that is currently white, and adjacent to the the last colored vertex. A player unable to do so losses the game. What property of

the graph is equivalent to the first/second player having a winning strategy, and what is their strategy?

id ao. A very long road contains n cars, each having a constant speed drawn at random from 20 kmh to 200 kmh. The road is narrow so there's no overtaking, so fast cars may have to slow down to match the speed of the car in front. Eventually, how many meshes of connected cars are to be expected?

id ap. Alice and Bob play the following game. There is an outer circle, fixed in position, containing n lamps, and an inner circle, which may be rotated, containing the numbers $1..n$. Alice wins when all the lights are off, and Bob wins if she gives up/the game goes forever. Bob chooses the initial on/off states of the lamps. Every turn, Alice specifies a set of numbers in $1..n$ and tells Bob to switch the state (on $\leftrightarrow$ off) of the lamps corresponding to those positions, but Bob may spin the inner circle of numbers before doing so. For which values of n can Alice/Bob win, and how?

id aq. A rectangular grid R is painted using 10 colors. A rectangle is called *special* if its four vertices have the same color. Find dimensions for R guaranteeing the existence of a special rectangle.

id ar. There exists an uncountable family $\mathscr{F}$ of infinite subsets of $\mathbf{N}$ where the intersection of any pair of sets in $\mathscr{F}$ is finite.

id as. You find yourself in a huge circular train (say about the size of the equator) in which you can walk freely in both directions. You'll be free once you determine the exact number of cars, but you only have one guess. All the cars look exactly the same, and you can't leave any mark, except each car has a lamp with a switch which you may freely use. However, the initial state of the lamps is random. What do you do?

id at. An invisible frog lives on the integer number line, and each day it hops by a fixed unknown amount d. Every day you get one guess as to where the frog is, and if you hit it you win a prize. Find a strategy that guarantees you'll win the prize, eventually.

Part ii. Now the frog moved to the real line, and still hops every day by a fixed amount. It also gained an unknown small but positive length ε. What's your strategy?

id au. A standard 2-dimensional Gaussian is given in polar coordinates by $\sqrt{2\log(1/u)}(\cos 2\pi\theta, \sin 2\pi\theta)$ where u, θ are independent and uniform on $[0, 1]$.

id aw. There's a deck with 100 numbered $1..100$ in some order. In each step we peek at the top card, say numbered k, take the first k cards from the deck, flip their order, and return them at the top of the deck. In particular, after this move the k-th card now shows k. Show that eventually the top card becomes 1.

id ay. Let $f : [0, 1] \to \mathbf{R}$ be a continuous function satisfying $\int_0^1 f(x)x^k = 1$ for $k = 0, \ldots, n-1$. What is the best lower bound on $\int_0^1 f^2$?

id bd. From each vertex of a triangle, extend the two edges by the length of the opposite edge, then the six endpoints lie on a circle with center I and radius $\sqrt{r^2 + s^2}$.

id bq. If $a_n \to a$ then $\left(1 + \frac{a_n}{n}\right)^n \to e^a$.

id br. A betting game with odds $p > 1/2$ in your favour goes as follows. In each round, you may wager an amount x of your capital. If the odds were right you get back $2x$, and if not you lose the initial x. You start

with \$1 and the game goes for n rounds. What strategy should be played to maximize the expected value of your capital at the end of the game? What about maximizing the expected value of the log of your capital.

2. Calculus

This chapter emphasizes intuition over rigour. Proofs will later be given in the Analysis chapter.

id ac. Fact: For all $x \in \mathbf{R}$ we have

$$\exp(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

Nomenclature. The *Laplace transform* of a bounded continuous function $f : [0, \infty) \to \mathbf{R}$ is

$$(\mathcal{L}[f])(\lambda) = \int_0^{\infty} e^{-\lambda x} f(x) \mathrm{d}x$$

defined for $\lambda > 0$.

id bi. Fact: We have

$$\mathcal{L}[f'](\lambda) = \lambda \mathcal{L}[f] - f(0)$$

3. Number Theory

id ax. Fact: We have $\operatorname{lcm}(1, \ldots, n) \geq 2^{n-2}$ for all n.

id ax. Explanation: Consider $I = \int_0^1 t^m (1-t)^m \mathrm{d}t$. We have $I = \dfrac{r}{\operatorname{lcm}(1, \ldots, 2m+1)}$ for some positive integer r, as well as $I \leq 4^{-m}$. Thus $4^m \leq \operatorname{lcm}(1, \ldots, 2m+1)$.

4. Combinatorics and Probability

id az. Fact: A monkey is typing letters uniformly at random. The the expected number of letters until ABRACADABRA is typed equals $26 + 26^4 + 26^{11}$.

id az. Explanation. Imagine a casino where gamblers can bet on the next letter. There are 26 letters and the casino gives fair odds, meaning a gambler can pay the casino \$$x$ dollars and bet that the next letter will be y. If the bet was right he will get back \$$26x$. Now imagine that before every letter, a new gambler arrives with \$1 and bets all their money that the next 11 letters will be ABRACADABRA (of course this will take several turns, the first putting down \$1 on the first letter being A. If the bet works out, the next bet would be \$26 dollars on the next letter being B, and so on). Starting with \$0, let X_n be the cash of the casino after the n letter was typed. Let T be the stopping time - when ABRACADABRA is first typed. Then X_n is a martingale with bounded differences, $\mathbf{E}T < \infty$ and so $\mathbf{E}X_T = \mathbf{E}X_0 = \$0$. Let's consider the cash after time T. Every unsuccessful gambler lost \$1 to the casino. There are 3 successful gamblers, who put in an initial \$1 and took $26 + 26^4 + 26^{11}$ from the casino. In total, $X_T = T - 26 + 26^4 + 26^{11}$.

Nomenclature. In a *branching process* of microbes, the 0-th generation consists of a single microbe, and each microbe gives birth to a number of children, randomly generated from a fixed distribution X (taking non-negative integer values) independently of all other microbes. In formula, $Z_0 = 1$, and $Z_{n+1} = A_{1,n} + \cdots + A_{Z_n,n}$, where $A_{i,j} \sim X$ are independent.

id bb. Fact: Let $(X, Z_n)_{n=0}^\infty$ be a branching process, and set $f(\theta) = \mathbf{E}[\theta^X]$, $g_n(\theta) = \mathbf{E}[\theta^{Z_n}]$ for $\theta \in [0,1]$. Then
i. $g_n = f \circ f \cdots \circ f$ is the n-th composition of f.
ii. The *probability of extinction* $p = \mathbf{P}[Z_m = 0 \ \text{ for some } m]$ is the smallest non-negative fixed point of f.

id bg. Fact: If the edges of K_n are exactly covered by the edges of m complete bi-partite graphs, then $m \geq n-1$.

id bg. Explanation: Suppose $E(K_n) = \bigsqcup_{k=1}^m E(L_k, R_k)$ with $m + 1 < n$. Fix a later chosen N, and for each mapping $\sigma : [n] \to [N]$ let $P(\sigma) = (\sum_{L_1} \sigma, \ldots, \sum_{L_m} \sigma, \sum_{[n]} \sigma)$. Then P has domain of size N^n and range of size at most $(nN)^{m+1}$. We choose N guaranteeing P is not injective. Fix $\sigma_1 \neq \sigma_2$ with $P(\sigma_1) = P(\sigma_2)$, and set $\tau = \sigma_1 - \sigma_2$. We have $\sum_{i<j} \tau(i)\tau(j) = \sum_{k=1}^m \sum_{L_k} \tau \sum_{R_k} \tau = 0$, and $\sum \tau(i) = 0$. Thus $\sum \tau(i)^2 = [\sum \tau(i)]^2 - 2\sum_{i<j} \tau(i)\tau(j) = 0$, in contradiction to $\sigma_1 \neq \sigma_2$.

id bh. Fact: Let f be a bounded continuous function on $[0, \infty)$, $\mathcal{L}[f]$ its Laplace transform, let $X_1, \ldots, X_n$ be a sequence of iid variables $\sim \text{Exp}(\lambda)$ and $S_n = X_1 + \cdots + X_n$. We have
i. $\mathbf{E}[f(S_n)] = \dfrac{(-1)^{n-1}\lambda^n}{(n-1)!} \dfrac{\mathrm{d}^{n-1}\mathcal{L}[f](\lambda)}{\mathrm{d}\lambda^{n-1}}$.
ii. For all $y > 0$ we have $f(y) = \lim\limits_{n\to\infty} \left(\dfrac{(-1)^{n-1}\lambda^n}{(n-1)!} \dfrac{\mathrm{d}^{n-1}\mathcal{L}[f](\lambda)}{\mathrm{d}\lambda^{n-1}} \right)\Big|_{\lambda=n/y}$, a formula for inverting $\mathcal{L}[f]$ back to f.

id bn. Fact: Let S_n be a symmetric random walk on $\mathbf{Z}$, $T = \inf\{n : S_n = 1\}$. Then $P(T = 2m - 1) = \binom{\frac{1}{2}}{m}(-1)^{m+1}$.

id bn. Explanation: We know that $T < \infty$ almost surely (though $\mathbf{E}[T] = \infty$). Let $f(\alpha) = \mathbf{E}[\alpha^T]$. Then $f(\alpha) = \frac{1}{2}\alpha + \frac{1}{2}\alpha\mathbf{E}[\alpha^{T'}]$ where T' is the time to get to 1, starting at -1. Now T' is a sum of two independent variables, both distributed as T. Thus $\mathbf{E}[\alpha^{T'}] = f(\alpha)^2$. It follows that $f(\alpha) = \dfrac{1 - \sqrt{1 - \alpha^2}}{\alpha}$. It remains to compare coefficients.

id bo. Fact: Let Z_n be a Markov chain on (E, P). Then the following are equivalent.
i. For any $i, j \in E$ we have $\mathbf{P}\left(\exists n \geq 1 : j = Z_n \mid Z_0 = i\right) = 1$.
ii. All P-super-harmonic non-negative functions h on E are constant.

id bo. Explanation: i. $\implies$ ii. Clearly $h(Z_n)$ is a non-negative super-martingale. Starting at $Z_0 = i$, let T be the stopping time of reaching j, so T is almost-surely finite, and thus $h(j)\mathbf{E}[h(Z_T)] \leq h(Z_0) = h(i)$.
ii. $\implies$ i. For fixed j the function $h_j(i) = \mathbf{P}[\text{hitting } j \mid Z_0 = i]$ is P-super-harmonic. Indeed, $\sum_k h_j(k)p_{ik} \leq \sum_{k \neq j} h_j(k)p_{ik} + p_{ij} = h_j(i)$. Therefore $h_j \equiv c_j$ is constant and $p_{ij} = p_{ij}c_j$. If $p_{ij} = 0$ for all i then the function δ_j would be P-super-harmonic and non-constant, and so $c_j = 1$.

5. Abstract Algebra

id bm. Fact: For a subset $S \subseteq \mathbf{F}_2^d$ let $\Gamma_S(x,y) = \sum_{s \in S} x^{d-W(s)} y^{W(s)}$ where $W(s) = |i \in [d] : s_i = 1|$ is the Hamming weight. If U is a subspace of $\mathbf{F}_2^d$ and $W = \{v \in \mathbf{F}_2^d : u^t v = 0 \ \ \forall u \in U\}$ is the "orthogonal" subspace, then

$$\Gamma_W(x,y) = \frac{1}{|U|} \Gamma_U(x+y, x-y)$$

In particular, the multiset of Hamming weights of U is sufficient information for that of W.

id bm. Explanation: Let $f(s) = x^{d-W(s)} y^{W(s)}$ be a function on $\mathbf{F}_2^d$ and ψ be a linear character of $\mathbf{F}_2^d$. We have $\psi = \psi_v : x \mapsto (-1)^{v^t x}$ for some $v \in \mathbf{F}_2^d$. Thus $2^d \hat{f}(\psi) = \sum_{s \in \mathbf{F}_2^d} x^{d-W(s)} y^{W(s)} (-1)^{v^t s} = \prod_{i=1}^d \sum_{s_i=0}^1 x^{1-s_i} y^{s_i} (-1)^{v_i s_i} = \prod_{i=1}^d (x+y)^{1-v_i} (x-y)^{v_i} = (x+y)^{d-W(v)} (x-y)^{W(v)}$. It remains to utilize Poisson summation formula.

id ba. Fact: If A is a Noetherian ring then $A[t]$ is a Noetherian ring.

id ba. Explanation: Fix an ideal $\mathscr{I} \trianglelefteq A[t]$. A best effort attempt to generate $\mathscr{I}$ is as follows: as long as $f_1(t), ..., f_{n-1}(t)$ don't generate $\mathscr{I}$, let $f_n(t)$ be the polynomial of minimal degree in $\mathscr{I}$ and not in $\langle f_1(t), \ldots, f_{n-1}(t) \rangle$. Note that $d_n = \deg f_n(t)$ is weakly increasing, and that the leading coefficient of $f_n(t)$ does not belong to the ideal of A generated by the leading coefficients of $f_1(t), .., f_{n-1}(t)$, for otherwise there are $a_1, \ldots, a_{n-1} \in A$ for which $g(t) = f_n(t) - a_1 f_1(t) t^{d_n - d_1} \cdots - a_{n-1} f_{n-1}(t) t^{d_n - d_{n-1}}$ has degree smaller than $f_n(t)$ and is also member of $\mathscr{I} \setminus \langle f_1(t), .., f_n(t) \rangle$, which can't be. It follows that the sequence of ideals of A generated by the leading coefficients of $f_1(t), \ldots, f_n(t)$ is strictly increasing, and A is not Noetherian.

id be. Fact: In a vector space, if $(u_1, \ldots, u_i)$ is linearly independent and $(w_1, \ldots, w_s)$ is spanning, then $i \leq s$. Moreover, we may replace some i members of the latter with the members of the former to produce a spanning set.

id be. Explanation: By induction on i, the base $i = 0$ being trivial. For the step, we have wlog $(u_j)_{j<i} \cup (w_k)_{k \geq i}$ spanning. Writing u_i as a linear combination, a coefficient of one of the w's must be non-zero, wlog say that of w_i. Thus w_i is spanned by $(u_j)_{j \leq i} \cup (w_k)_{k > i}$. This set is spanning, since it contains the remaining members of the previously spanning set.

We also give specific explanations for when $\mathbf{F}$ is a finite field and when $\mathbf{F} = \mathbf{Q}$.

id be. Explanation for finite fields: If $|\mathbf{F}| = q$ is finite, then the span of $u_1, \ldots, u_i$ has precisely q^i elements, where as the span of $w_1, \ldots, w_s$ has at most q^s elements.

id be. Explanation for $\mathbf{F} = \mathbf{Q}$: Write $u_t = \sum_j \alpha_{tj} w_j$ for $\alpha_{tj} \in \mathbf{Q}$, $t \in [i], j \in [s]$. Let M be a common multiple of the denominators, $f_t = M u_t$ and $\beta_{tj} = M \alpha_{tj} \in \mathbf{Z}$. Let R be a positive integer variable, $A_R = \{r_1 f_1 + \cdots + r_i f_i : r_t \in \mathbf{Z}, |r_t| \leq R\}$ and $B_R = \{m_1 w_1 + \cdots + m_s w_s : m_j \in \mathbf{Z}, |m_j| \leq CR\}$ where $C = \max_j \sum_t |\beta_{tj}|$. We have $A_R \subseteq B_R$, and with the polynomial inequality $(2R+1)^i = |A_R| \leq |B_R| \leq (2CR+1)^s$ valid for all values of R we deduce $i \leq s$.

id bf. Fact: If $f : \mathbf{F}_2^d \to \pm 1$ satisfies $f(x+y) = f(x)f(y)$ in probability $\geq 1 - \varepsilon$, then there exists a character $\phi : \mathbf{F}_2^d \to \pm 1$ with $f = \phi$ in probability $\geq 1 - \varepsilon$.

id bf. Explanation: Write $f = \sum \hat{f}(\psi)\psi$. We have

$$4^d(1 - 2\varepsilon) \leq \sum_{x,y} f(x+y)f(x)f(y) = \sum_{x,y,\psi_1,\psi_2,\psi_3} \hat{f}(\psi_1)\hat{f}(\psi_2)\hat{f}(\psi_3)\psi_1(x+y)\psi_2(x)\psi_3(y) = 4^d \sum_\psi [\hat{f}(\psi)]^3$$

by character orthogonality. Moreover $1 = \langle f, f \rangle = \sum_\psi [\hat{f}(\psi)]^2$. Noting that the $\hat{f}(\psi)$ are real, we have $\max_\psi \hat{f}(\psi) \geq \dfrac{\sum_\psi [\hat{f}(\psi)]^3}{\sum_\psi [\hat{f}(\psi)]^2} \geq 1 - 2\varepsilon$ and so for $\phi = \arg\max \hat{f}$ we have $f = \phi$ in probability $\geq 1 - \varepsilon$.

id bl. Fact: Let G be a finite abelian group. For a subgroup H let $H^\perp = \{\phi \in \hat{G} : \phi|_H \equiv 1\}$. Then
i. $H^\perp$ is a subgroup naturally isomorphic to $\widehat{G/H}$.
ii. $\hat{G}/H^\perp$ is naturally isomorphic to $\hat{H}$.
iii. $H^{\perp\perp}$ is naturally isomorphic to H.
iv. We have $\displaystyle\sum_{\psi \in H^\perp} \psi(g) = \begin{cases} [G : H] & g \in H \\ 0 & g \notin H \end{cases}$
v. *Poisson summation formula*: For $f : G \to \mathbf{C}$ we have

$$\sum_{h \in H} f(h) = |H| \sum_{\psi \in H^\perp} \hat{f}(\psi)$$

id bl. Explanation:
iv. This follows from i. and character orthogonality.
v. We may assume f is a character of G. If $f \in H^\perp$ the formula is clear. If $f \notin H^\perp$ then $f|_H$ is a non-trivial character on H so that the lhs vanishes, as does the right.

Applications

Nomenclature. A society has voters and politicians. A *voting system* is a mapping, taking in a linear preference of the politicians from each voter, and returning a linear preference of the politicians. The *majority* system works if there's an odd number of voters and two politicians, and ranks on top the politician the majority of voters ranked on top. The majority system is pretty fair, and on the other spectrum we have a *dictatorship*: a system which always returns the preferences of one specific voter, the dictator.

id bp. Fact: Suppose a voting system on three politicians satisfies the following:
i. If all the voters rank one politician higher than another, so will the system.
ii. For any pair of politicians, the preference between the two returned by the system depends only on the preferences of the voters between the two.
iii. If the voters choose their preferences (uniformly) at random, then any one of politician has even odds at ranking higher than any other.
Then the system is a dictatorship.

id bp. Explanation: We first need a model. If for example there's four voters, the three politicians are named a, b, c and the preferences are $a > b > c$, $\quad c > a > b$, $\quad b > a > c$, $\quad b > c > a$ then the preferences of the voters between any pair of pair of politicians are as follows:

$$\begin{bmatrix} a : b & a > b & a > b & b > a & b > a \\ b : c & b > c & c > b & b > c & b > c \\ c : a & a > c & c > a & a > c & c > a \end{bmatrix}$$

For each row, the voting system returns a preference of the two politicians. In other words, the system is composed of three functions taking in n signs (where n is the number of voters) and returning one sign. In our example, the preferences are described as the sign vectors

$$\begin{bmatrix} x: & +1 & +1 & -1 & -1 \\ y: & +1 & -1 & +1 & +1 \\ z: & -1 & +1 & -1 & +1 \end{bmatrix}$$

and assumption *ii.* implies that the voting system returns a triple $(f(x), g(y), h(z))$ describing the final preferences $(a : b, b : c, c : a)$. It is important to see that each of f, g, h is defined on the full space $\{\pm 1\}^n$ but out of the 8^n triples x, y, z, only 6^n describe potential preferences of voters, namely for each $i \in [n]$ the three signs x_i, y_i, z_i cannot all be the same. We draw at random preferences for the voters (out of a space of 6^n possibilities), described as sign vectors x, y, z. The three signs $f(x), g(y), h(z)$ are not all the same, meaning

$$f(x)g(y) + g(y)h(z) + h(z)f(x) = -1$$

is a constant random variable. On the other hand

$$\mathbf{E}[f(x)g(y)] = \sum_{\psi_1, \psi_2 \in \widehat{\{\pm 1\}^n}} \hat{f}(\psi_1)\hat{g}(\psi_2)\mathbf{E}[\psi_1(x)\psi_2(y)]$$

Now a character ψ corresponds to a set S via $\psi(x) = \prod_{s \in S} x_s$. If $\psi_1 \neq \psi_2$ then from $\mathbf{E}[\psi_1(x)\psi_2(y)]$ we may factor out a zero in the form of $\mathbf{E}x_j$ or $\mathbf{E}y_j$. Since $\mathbf{E}[x_j y_j] = -\frac{1}{3}$ we have $\mathbf{E}[f(x)g(y)] = \sum_{S \subseteq [n]} \hat{f}(S)\hat{g}(S)\left(-\frac{1}{3}\right)^{|S|}$. In total we have

$$1 = \sum_{S \subseteq [n]} -\left(-\frac{1}{3}\right)^{|S|} \left(\hat{f}\hat{g} + \hat{g}\hat{h} + \hat{h}\hat{f}\right)(S)$$

Now $\hat{f}(\varnothing) = \hat{g}(\varnothing) = \hat{h}(\varnothing) = 0$ by assumption *iii.* Moreover $|r_1 r_2 + r_2 r_3 + r_3 r_1| \leq r_1^2 + r_2^2 + r_3^2$ for real r_i so

$$1 \leq \sum_{\varnothing \neq S \subseteq [n]} \frac{\hat{f}(S)^2 + \hat{g}(S)^2 + \hat{h}(S)^2}{3^S} \leq \frac{\langle f, f \rangle + \langle g, g \rangle + \langle h, h \rangle}{3} = 1$$

implying the only non-trivial coefficients are singletons, so f, g, h are linear functions. Since they only take two values, they're of the form $f(x) = \pm x_j$, $g(y) = \pm y_k$, $h(z) = \pm z_\ell$. Assumption *i.* yields positive signs, and it's easy to see $j = k = \ell$ is a dictator.

6. Analysis

Nomenclature. A function f defined on the vertices of a graph will be called *harmonic* if for every vertex v the average value of $f(u)$ over the neighbours u of v equals $f(v)$.

id ae. Fact: Let f be a real valued harmonic function on the lattice $\mathbf{Z}^d$. If f is non-constant, then it is unbounded.

id ae. Explanation: If f is non-constant, then wlog the bounded harmonic function $g(x) = f(x + e_1) - f(x)$ attains a positive value. Let $S = \sup g > 0$. Given ε we may find x_ε for which $g(x_\varepsilon) > S - \varepsilon$. Since the average of g over the $2d$ neighbours of x_ε is larger than $S - \varepsilon$, and each value is at most S, it follows that g assigns each neighbour a value greater than $S - 2d\varepsilon$. Thus $g(x_\varepsilon + e_1) > S - 2d\varepsilon$, and generally $g(x_\varepsilon + ke_1) > S - (2d)^k\varepsilon$. However, if we pick k large enough and ε small enough, we can make $f(x + (k+1)e_1) - f(x) = g(x) + g(x + e_1) + \cdots + g(x + ke_1) > (k+1)(S - (2d)^k\varepsilon)$ arbitrarily large, meaning f is unbounded.

id af. Fact: Let T be a strict contraction of a complete metric space M. Then in M there exists a unique fixed point m^* of T. Moreover, for any $m \in M$ we have $\lim_{n \to \infty} T^n(m) = m^*$.

id af. Explanation: A strict contraction clearly cannot have more than one fixed point. Let m be an arbitrary point, and let $m_n = T^n(m)$ be its orbit under T. We have $d(m_n, m_{n+1}) \leq q^n d(m, Tm)$, where q is the contraction constant for T. Thus $\sum d(m_n, m_{n+1}) \leq \frac{d(m, Tm)}{1-q}$ is convergent, implying m_n is Cauchy, and hence convergent. By continuity, the limit of the orbit is a fixed point.

id av. Fact: The Fourier transform is injective on $\mathcal{L}^1(\mathbf{R})$.

id av. Explanation: Let $f \in \mathcal{L}^1, \hat{f} \equiv 0$. For all $a, \xi \in \mathbf{R}$ we have

$$\int_{-\infty}^{a} f(t)e^{i\xi(t-a)}\mathrm{d}t = -\int_{a}^{\infty} f(t)e^{i\xi(t-a)}\mathrm{d}t$$

and we denote this value by $F_a(\xi)$. The right hand side may be analytically continued to arguments in the closed upper half plane, and the left hand side may be continued to the closed lower half plane. It follows that $F_a(\xi)$ is an entire function. However, it is everywhere bounded by $\|f\|_1$, and so must be constant. By dominated convergence we have $\lim_{r \to \infty} F_a(ir) = 0$, and so $F_a(\xi) = 0$. In particular $\int_{-\infty}^{a} f(t)\mathrm{d}t = F_a(0) = 0$ for all a and so $f = 0$ almost everywhere.

id ag. Fact: Let $f \in \mathscr{C}(\mathbf{T})$ and let $F_m(t) = \sum_{j=-m}^{m} \hat{f}(j)e^{2\pi ijt}$ be the Fourier approximations of f. Then the sequence of averages $\sigma_n(t) = \dfrac{F_0(t) + \cdots + F_n(t)}{n+1}$ converges uniformly to f.

id aq. Explanation: We have $\hat{f}(j)e^{2\pi ijt} = \int_0^1 f(x)e^{2\pi ij(t-x)}\mathrm{d}x = \int_{-t}^{1-t} f(u+t)e^{-2\pi iju}\mathrm{d}u = \int_0^1 f(u+t)e^{-2\pi iju}\mathrm{d}u$

and therefore
$$F_m(t) = \int_0^1 f(u+t)P_m(u)\mathrm{d}u \qquad \sigma_n(t) = \int_0^1 f(u+t)D_n(u)\mathrm{d}u$$

For
$$P_m(u) = \sum_{j=-m}^{m} e^{2\pi i j u} \qquad D_n(u) = \frac{P_0(u) + \cdots + P_n(u)}{n+1}$$

We have the identities $\int_0^1 D_n(u) = 1$ and $(n+1)D_n(u) = \left(\sum_{k=0}^{n} e^{2\pi i(k-\frac{n}{2})u}\right)^2 = \left(\frac{\sin(\pi(n+1)u)}{\sin(\pi u)}\right)^2 \geq 0$. (To clarify, $D_n(u) = n+1$ for $u \in \mathbf{Z}$). We continue

$$\sigma_n(t) - f(t) = \int_0^1 [f(u+t) - f(t)]D_n(u)\mathrm{d}u \implies |\sigma_n(t) - f(t)| \leq \int_0^1 |f(u+t) - f(t)|\, D_n(u)\mathrm{d}u$$

Given ε we find δ such that $|x - y| \leq \delta \implies |f(x) - f(y)| \leq \varepsilon$. The first part of the integral is bounded independent of n or t: $\int_{-\delta}^{\delta} |f(u+t) - f(t)|\, D_n(u)\mathrm{d}u \leq \varepsilon \int_0^1 D_n(u)\mathrm{d}u = \varepsilon$. Finally, on $[\delta, 1-\delta]$ we have $D_n(u) \leq \frac{1}{(n+1)\sin(\pi\delta)^2}$ and so if n is large enough then $D_n(u) \leq \varepsilon$ on this segment, and in total $|\sigma_n(t) - f(t)| \leq \varepsilon + 2\|f\|_\infty \varepsilon$ independent of t.

id bc. Fact: Let b, m, n be elements of a C^* algebra with m, n normal and $bn = mb$. Then $bn^* = m^*b$.

id bc. Explanation: We have $bn^k = m^k b$, and so $bf(n) = f(m)b$ for analytic functions f. In particular $b = \exp(\xi m)b\exp(-\xi n)$ for all $\xi \in \mathbf{C}$. Normality yields

$$\exp(i\zeta m^*)b\exp(-i\zeta n^*) = \exp(i(\zeta m^* + \zeta^* m))b\exp(-i(\zeta^* n - \zeta n^*))$$

The rhs is an entire function of ζ, and the lhs is bounded by $\|b\|$, and so the expression is a constant, namely b. Finally, the lhs coefficient of ζ is $im^*b - ibn^*$.

id bj. Fact: For a box $I \in \mathbf{R}^d$, not necessarily parallel to the standard axes, write $\rho(I)$ for the sum of the d edge lengths of I. If $I_1 \subseteq I_2$ then $\rho(I_1) \leq \rho(I_2)$.

id bj. Explanation: This is easy for $d = 1, 2$. For $d \geq 3$, we consider the volume of the set of points of distance $\leq r$ from I, denoted $f(r; I)$, where we take I as a solid. $f(r; I) = c_0 r^d + c_1 \rho(I)r^{d-1} + \cdots + \mathrm{Vol}(I)$ is a polynomial in I of degree d, and clearly $f(r; I_1) \leq f(r; I_2)$ for all r.

id bk. Fact: Let $I, I_1, I_2, \ldots$ be intervals with $I \subseteq \bigcup_k I_k$. Then $\mathrm{len}(I) \leq \sum_k \mathrm{len}(I_k)$.

id bk. Explanation: We first assume I is compact and all the I_k are open. Thus $I \subseteq I_1 \cup \cdots \cup I_N$ for some N, and we'll show $\mathrm{len}(I) \leq \mathrm{len}(I_1) + \cdots + \mathrm{len}(I_N)$ by induction on N. The base case is trivial. For the step, assume wlog $m = \min I \in I_1 = (\alpha, \beta)$. Then $[\beta, \max I] \subseteq I_2 \cup \cdots \cup I_N \implies \mathrm{len}(I) < \mathrm{len}(I_1) + \mathrm{len}([\beta, \max I]) \leq \mathrm{len}(I_1) + \cdots + \mathrm{len}(I_N)$. A second step is to remove the openness assumption on the I_k, but keep the compactness assumption on I. Let J_k be the interior of I_k, and for some fixed ε let L_k, R_k be open intervals of length $\varepsilon/2^k$ around the left and right endpoints of I_k (if one of the I_k is infinite there's nothing to prove). We get that $\mathrm{len}(I) \leq \sum \mathrm{len}(I_k) + 2\varepsilon$, and the desired inequality follows in the limit. The final step of removing the compactness assumption on I is by an approximation of I using compact intervals.

-1. Solutions

id aa. The probability is $1/101$, independent of $k = 0, \ldots, 100$. To show this, note that we may generate a biased coin toss by generating a uniform random number t from $[0, 1]$ and then show heads iff $t < p$. Therefore the process can be described by generating $1 + 100$ independent uniform random numbers $p = t_0, t_1, \ldots, t_{100}$ from $[0, 1]$ and the number of heads is the index of t_0 after sorting, which is distributed uniformly in $0, \ldots, 100$.

id ab. First consider the case $n = 2$. The two players have numbers in $0, 1$, and see the other person's number. To guarantee a correct guess, one will guess they have the same number, and the other will guess they have different numbers. We generalize this to arbitrary n by assigning the players id's $0, 1, \ldots, n-1$, and the player with id i guessing that the sum of all their numbers is congruent to i modulo n. This strategy guarantees exactly one of them will be right, namely the one with id equal to the sum of their numbers modulo n.

id ad. Let e_n denote the n-th coordinate sequence, set $\alpha_n = f(e_n)$. We'll show that $\alpha_{N+1} = \alpha_{N+2} = \cdots = 0$ after some value of N. To do this, we form a sequence $b = (b_1, b_2, b_3, \ldots)$ as follows: $b_1 = 1$, and b_{k+1} is a power of 2 greater than both b_k and $2\,|f(b_1 e_1 + \cdots + b_k e_k)|$. Note that $b_k \mid b_{k+1}$ and $\lim b = \infty$. Set $B = f(b)$. We have $B = f(b_1 e_1 + \cdots + b_k e_k) + b_{k+1} f(0..0, 1, b_{k+2}/b_{k+1}, b_{k+3}/b_{k+1}, \ldots)$. If k is sufficiently large, we must have $B = f(b_1 e_1 + \cdots + b_k e_k)$, or otherwise $|B| \geq b_{k+1} - |f(b_1 e_1 + \cdots + b_k e_k)| \geq b_{k+1}/2$. Therefore, if k is sufficiently large, $\alpha_k = f(e_k) = 0$. Now we may consider $g(x) = f(x) - \sum \alpha_i x_i$ as an additive function satisfying $g(e_n) = 0$ for all n, and we wish to show $g(x) \equiv 0$. Write $x_n = 2^n d_n + 3^n r_n$ for some integer sequences d, r. We have $g(x) = g(2d_1, 4d_2, 8d_3, \ldots) + g(3r_1, 9r_2, 27r_3, \ldots)$, but since $g(e_n) = 0$, the left summand is divisible by all powers of two, and the right summand is divisible by all powers of three, so they are both zero, and $g(x) \equiv 0$.

id ah. Subtracting a linear factor, we may assume $f(0) = f(1) = 0$, with the intention of showing $f \equiv 0$. Let $f_\varepsilon(t) = f(t) - \varepsilon t(1 - t)$. Then $\lim\limits_{h \to 0} \dfrac{f_\varepsilon(t_0 + h) - 2f_\varepsilon(t_0) + f_\varepsilon(t_0 - h)}{h^2} = \varepsilon$, implying f_ε does not attain its maximum at any inner point $t_0 \in (0, 1)$. Therefore $f_\varepsilon \leq 0$, and in the limit $f \leq 0$. Applying the same to $-f$ we have $f \equiv 0$.

id ai. What demands proof is that all smooth/continuous homomorphisms are exponentials. Indeed, for ϕ smooth

$$\phi(s + t) = \phi(s)\phi(t) \implies \phi'(s + t) = \phi'(s)\phi(t) \implies \phi'(t) = \phi'(0)\phi(t) \implies \phi(t) = \exp(t\phi'(0))$$

It remains to show that a continuous homomorphism is smooth. Indeed, we have $\int_x^{x+a} \phi(t)\mathrm{d}t = \phi(x) \int_0^a \phi(t)\mathrm{d}t$. We pick a small enough a so that $\frac{1}{a}\int_0^a \phi(t)\mathrm{d}t = \mathrm{id} + o(1)$ is invertible, yielding

$$\phi(x) = \left(\int_x^{x+a} \phi(t)\mathrm{d}t \right) \left(\int_0^a \phi(t)\mathrm{d}t \right)^{-1}$$

is smooth.

id aj. Suppose $\sum a_n^2$ diverges, and form a partition $0 = N_0 < N_1 < \ldots$ for which $s_k = \sum_{(N_{k-1}, N_k]} a_n^2 > 1$. The sequence $b_n = a_n/ks_k$ (where $n \in (N_{k-1}, N_k]$) is in ℓ^2, since $\sum b_n^2 = \sum_k \frac{1}{k^2 s_k}$. However, $\sum a_n b_n = \sum_k 1/k$ diverges.

id aw. The number 100 can only be the top card once, after that it will stay forever at the bottom. Between any two times the number 99 is on top, 100 has to be on top. Between any two times the number 98 is on top, either 100 or 99 has to be on top, ..., between any two times the number 2 is on top a number bigger than 2 has to be on top. Therefore each number except for 1 can only be on top finitely many times.

id aw. Solution 2. If 100 ever becomes the top card, it'll later be stuck at the bottom. If not, the bottom card will never change, and the game will be indifferent to switching the numbers of the bottom card with 100. So in either case we have an induction step.

id bh. Explanation:

i. we have $\mathbf{E}[f(S_n)] = \int_{x_i \geq 0} f(\sum x_i) \lambda^n e^{-\lambda \sum x_i} \mathrm{d}x_1...\mathrm{d}x_n$. Change variables to $y_i = x_i$ for $i < n$ and $t = \sum x_i$. The Jacobian is 1, the domain $0 \leq y_i, \sum y_i \leq t$, and we get $\lambda^n \int_{t=0}^{\infty} \mathrm{d}t f(t) e^{-\lambda t} \int_{y_i \geq 0, \sum y_i \leq t} \mathrm{d}y_1...\mathrm{d}y_{n-1}$. We recognise the inner volume as $\frac{t^{n-1}}{(n-1)!}$. Finally, $\frac{\mathrm{d}^{n-1}\mathcal{L}[f](\lambda)}{\mathrm{d}\lambda^{n-1}} = \int_0^\infty (-x)^{n-1} e^{-\lambda x} f(x) \mathrm{d}x$.

ii. We have $\mathbf{E}[X] = \lambda^{-1}$, $\mathrm{Var}X = \lambda^{-2}$, so for $\lambda = n/y$ we have $\mathbf{E}[S_n] = y$, $\mathrm{Var}(S_n) = y^2/n$. Chebyshev gives $\mathbf{P}[|S_n - y| > yn^{-1/4}] \leq n^{-1/2}$. If $\varepsilon_n = \max|f(\alpha) - f(\beta)|$ over $\{|\alpha - \beta| \leq yn^{-1/4}, 0 \leq \alpha, \beta \leq 2y\}$, then ε_n tends to 0 by uniform continuity, and $|f(y) - \mathbf{E}[f(S_n)]| \leq \mathbf{E}[|f(y) - f(S_n)|] \leq 2\|f\|_\infty n^{-1/2} + \varepsilon_n$ tends to 0, so $f(y) = \lim \mathbf{E}[f(S_n)]$.

id br. Let Z_k be our capital after the k-th bet. We have $\mathbf{E}[Z_{k+1}|Z_k] \leq 2pZ_k$, with equality iff we bet all our money on the $k+1$-st round. Thus the strategy maximizing $\mathbf{E}[Z_n]$ is to always bet everything, and has a value of $\$(2p)^n$. This strategy is extremely risky, since only in probability p^n we don't lose our initial investment. Replacing the utility Z by $\log(Z)$ has two effects: it greatly punishes losing all your capital, and returns are diminishing. Now if we wager a proportion α of our capital Z_k in the $k+1$-st round we'll have $\mathbf{E}[\log(Z_{k+1})|Z_k, \alpha] = p\log(Z_k(1+\alpha)) + q\log(Z_k(1-\alpha))$ where $q = 1 - p$. This is maximized for $\alpha = 2p - 1$. Thus the optimal strategy always wager a proportion of $2p-1$ of our capital (which could be guessed, since for $p = 1/2$ we'd not wager anything and for $p = 1$ we'd be all in). This strategy has $\mathbf{E}[\log(Z_n)] = n(p\log p + q\log q + \log 2)$.

id br. engineer's solution: Suppose we always wager a fixed proportion α of our capital. We'll have approximately pn wins and qn losses, meaning $Z_n \approx (1+\alpha)^{pn}(1-\alpha)^{qn}$. Thus $\log(Z_n) \approx n(p\log(1+\alpha) + q\log(1-\alpha))$ is to be maximized, as before.

-2. Notes

The following is a minimal attempt at giving due credit.

id av. The given explanation is attributed to D.J. Newman.

id ax. The given explanation is attributed to M. Nair.

id be. The given explanation for $\mathbf{F} = \mathbf{Q}$ is attributed to S. Vishwanathan.

id bg. The given explanation is attributed to S. Vishwanathan.

id bm. The Fact is attributed to F.J. MacWilliams.