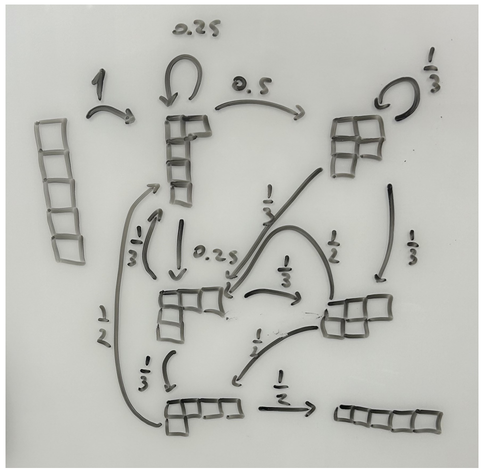
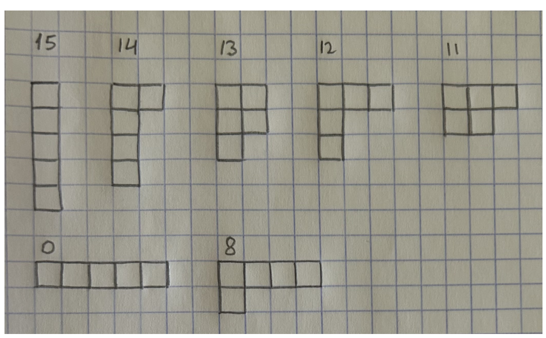
Problem. Consider the following game of stochastic dodgeball. There are n players, initially all active. As long as there's more than one, two active players A, B are chosen at random, then A captures B, leading to all the players captured by B to become active and rejoin the game. Find the expected number of turns for the game to finish (with only one active player, the rest are his captives).

Solution path. We first wrote python code to simulate the game, leading us to
Conjecture 1. The expected number of turns is $2^{n-1} - 1$.
We realized the state of the game can be modeled as a partition of n, namely if there are ℓ active players then we'll have ℓ rows, their lengths corresponding to the number of captives of each active player. We drew the Markov chain for the game with $n = 5$, leading to 7 states (partitions).



To answer what is the expected number of turns until the game finishes, starting from each given state is now an exercise in linear algebra. Solving it we reach the following expected values of turns.



Leading us to
Conjecture 2. The expected number of turns, from any starting state, is an integer.
Staring hard at these values, we noticed that they are an additive function of the rows! We have the additive values $\bullet = 3$, $\bullet\bullet = 5$, $\bullet\bullet\bullet = 6$, $\bullet\bullet\bullet\bullet = 5$, $\bullet\bullet\bullet\bullet\bullet = 0$. For this to work in general along with conjecture 1, we'd need a value $\bullet = \dfrac{2^{n-1}-1}{n}$, which isn't always an integer, but it is for odd primes, so to get a better feeling we wrote more python code to check
Conjecture 3. The expected number of turns, from any starting state, is an additive function of the lengths of the partitions.
The conjecture held for $n = 11$, with the additive values $(93, 185, 276, 365, 450, 527, 588, 617, 582, 419, 0)$. Not seeing a pattern, we plugged this sequence into OEIS, which immediately recognized that the second discrete derivative of this sequence is $-(1, 2, 4, 8, \ldots, 256)$. The second derivative of $(3, 5, 6, 5, 0)$ is $-(1, 2, 4)$ has this pattern too, so along with the two initial conditions of starting at $\dfrac{2^{n-1}-1}{n}$ and ending at 0, we arrive at
Conjecture 4. The additive function in conjecture 3 is $f(x) = x\dfrac{2^{n-1}-1}{n} - (2^{x-1} - 1)$, namely the expected number of turns starting at parition λ equals $2^{n-1} - 1 - \sum_j (2^{\lambda_j - 1} - 1)$

Proof (of conjecture 4, containing all other conjectures). Let $F(\lambda) = 2^{n-1} - 1 - \sum_j (2^{\lambda_j - 1} - 1)$. If two active players i, j are chosen, let λ_{ij} and λ_{ji} be the new states depending on whether i captures j or j captures i. Then $\operatorname{avg}(F(\lambda_{ij}), F(\lambda_{ji})) = F(\lambda) - 1$. In particular, $\mathbf{E}F(\lambda') = F(\lambda) - 1$ where λ' is the (random) state following λ. This means that $F(\lambda) = E(\lambda)$ is the expected number of turns starting at parition λ (to be fancy, $F - E$ is a martingale, and both vanish at $\lambda = (n)$ the end state) as we wanted.