Notes on the longest run of heads

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These notes provide additional explanation on computing the probabilities of events associated with the longest run of consecutive heads in repeated coin tossing. A reference for all of the ideas here is Schilling (1990).

The specific problem I asked was to find the probability that the longest run of consecutive heads is exactly three from ten coin tosses of a fair coin. The general approach I described was to solve a more general problem: what is the probability that the longest run of heads has length x from n coin tosses. Each of these probabilities can be found by breaking the event into a partition of events involving shorter sequences by conditioning on how the sequences begin. The desired probability would then be a sum of these probabilities events on shorter sequences. We can then find the answer to probabilities of events on large sequences by building up from shorter ones.

I also mentioned that in this case, it was simpler to consider events where the longest run was less than or equal to some amount rather than exactly equal to an amount.

The event for which we want to find the probability can be defined as

$$B = \{\text{the longest run of heads in ten coin tosses has length 3}\} .$$

If we define a value X_n to be the length of the longest run of consecutive heads in n coin tosses, it will be easier to specify events with fewer words. For example, $B = \{X_{10} = 3\}$. (In chapter 2 we will call X_n a random variable.) If we also define

$$R_3 = \{\text{the longest run of heads is less than or equal to 3}\} = \{X \leq 3\}$$

and

$$R_2 = \{\text{the longest run of heads is less than or equal to 2}\} = \{X \leq 2\}$$

we can note that $R_2 \subset R_3$ and $B = R_3 \cap R_2^c$ since this is the set of sequences where the longest run of heads is three or less but not two or less; hence it is exactly three. Thus, $\mathsf{P}(B) = \mathsf{P}(R_3) - \mathsf{P}(R_2)$ as B contains all elements in R_3 that are not in R_2.

Next, we can find $\mathsf{P}(R_3)$. The elements of R_3 can be partitioned by how the sequences begin. In particular, we can define these four subsets of R_3.

$$\begin{aligned}
R_{30} &= \{\text{sequences that begin with } T \text{ where } X_{10} \leq 3\} \\
R_{31} &= \{\text{sequences that begin with } HT \text{ where } X_{10} \leq 3\} \\
R_{32} &= \{\text{sequences that begin with } HHT \text{ where } X_{10} \leq 3\} \\
R_{33} &= \{\text{sequences that begin with } HHHT \text{ where } X_{10} \leq 3\}
\end{aligned}$$

In other words, sequences in R_3 begin with 0, 1, 2, or 3 heads. They cannot begin with four or more heads since the longest run of consecutive heads is three or less. Note also that the four subsets of R_3 are disjoint.

Therefore,

$$R_3 = R_{30} \cup R_{31} \cup R_{32} \cup R_{33}$$

and

$$\mathsf{P}(R_3) = \mathsf{P}(R_{30}) + \mathsf{P}(R_{31}) + \mathsf{P}(R_{32}) + \mathsf{P}(R_{33}) \; .$$

To find all of these probabilities, it is useful to answer the more general question about how many sequences of n coin tosses there are where the longest run of heads is x or less. If we define this as $A_n(x)$, we see that $\mathsf{P}(R_3) = A_{10}(3)/2^{10}$ and from the set equation above, that

$$A_{10}(3) = A_9(3) + A_8(3) + A_7(3) + A_6(3) \; .$$

since $\mathsf{P}(R_{30}(3)) = A_9(3)/2^{10}$ and so on. In general, for large enough n,

$$A_n(3) = A_{n-1}(3) + A_{n-2}(3) + A_{n-3}(3) + A_{n-4}(3),$$

but for $n \le 3$, all sequences will have the longest run of heads of length 3 or less. So, we define $A_n(3)$ as follows:

$$A_n(3) = \begin{cases} 2^n & \text{for } n = 0,1,2,3 \\ A_{n-1}(3) + A_{n-2}(3) + A_{n-3}(3) + A_{n-4}(3) & \text{for } n = 4,5,\ldots \end{cases}$$

Thus we can build up a table for $A_n(3)$ for small n as large as necessary. (Defining $A_0(3) = 2^0 = 1$ makes the recursion work for $A_4(3)$.)

There is a similar relationship for $A_n(2)$, but the recursion only involves summing the previous three terms as the number of leading heads is 0, 1, or 2.

$$A_n(2) = \begin{cases} 2^n & \text{for } n = 0,1,2 \\ A_{n-1}(2) + A_{n-2}(2) + A_{n-3}(2) & \text{for } n = 3,4,\ldots \end{cases}$$

Putting this all together, here is a partial table of $A_n(x)$.

$A_n(x)$	x			
n	0	1	2	3
0	1	1	1	1
1	1	2	2	2
2	1	3	4	4
3	1	5	7	8
4	1	8	13	15
5	1	13	24	29
6	1	21	44	56
7	1	34	81	108
8	1	55	149	208
9	1	89	274	401
10	1	144	504	773

The final probability, then, is

$$\begin{aligned}
\mathsf{P}(B) &= \mathsf{P}(R_3) - \mathsf{P}(R_2) \\
&= \frac{773}{1024} - \frac{504}{1024} \\
&= \frac{269}{1024} \\
&\doteq 0.263
\end{aligned}$$

LITERATURE CITED

Schilling, M. F. (1990) The longest run of heads. *The College Mathematics Journal*, **21**, 196–207.