AP Calculus
Notes 5.1
Antiderivatives and Indefinite Integration

Exploration:

For each of the following derivatives, find the original function $F(x)$.

1) $F'(x) = 2x$

2) $F'(x) = 6x^2$

3) $F'(x) = \cos x$

4) $F'(x) = e^x - 3$

5) $F'(x) = \frac{1}{1+x^2}$

6) $F'(x) = \frac{-5}{2-5x}$

The function $F(x)$ is known as an *antiderivative* of $f(x)$ if $F'(x) = f(x)$ for all x. F is **an** antiderivative rather than **the** antiderivative because adding any constant C would work.

Explanation:
Find the derivative of $F(x) = x^2$, $F(x) = x^2 + 5$, and $F(x) = x^2 - \pi^e$.

Because of this, you can represent *all* antiderivatives of $f(x) = 2x$ by :___________________

The constant C is called the **constant of integration**. The functions represented by F is the **general antiderivative** of f, and $F(x) = x^2 + C$ is the **general solution** of the *differential equation* $F'(x) = 2x$.

Notation for Antiderivatives:

The operation of finding all solutions of this equation is called **antidifferentiation** or **indefinite integration** and is denoted by the integral sign $\int$. The general solution is:

$$y = \int f(x)dx = F(x) + C$$

The expression $\int f(x)dx$ is read as the *antiderivative of f with respect to x*. So, dx serves to identify x as the variable of integration. The term **indefinite integral** is a synonym for antiderivative.

Basic Integration Rules:

The inverse nature of integration and differentiation can be used to obtain: $\int F'(x)dx = F(x) + C$

Additionally, $\frac{d}{dx}[\int f(x)dx] = f(x)$

Exploration:

What is the antiderivative of each of the following? Try to develop the basic power rule for integration:

a) $f(x) = x^2$ b) $f(x) = x^3$ c) $f(x) = x^4$

So, the Power Rule for Integration is ___________________ $\int x^n dx =$ ___________

Ex. 1: Integrate each of the following polynomial functions:

a) $\int (x + 2)dx$ b) $\int dy$ c) $\int (3t^4 - 5t^2 + t)dt$

Differentiation Formula	Integration Formula
$\frac{d}{dx}[kx] =$	
$\frac{d}{dx}[kf(x)] =$	
$\frac{d}{dx}[f(x) \pm g(x)] =$	
$\frac{d}{dx}[x^n] =$	
$\frac{d}{dx}[\sin x] =$	
$\frac{d}{dx}[\cos x] =$	
$\frac{d}{dx}[\tan x] =$	
$\frac{d}{dx}[\sec x] =$	
$\frac{d}{dx}[\csc x] =$	
$\frac{d}{dx}[\cot x] =$	
$\frac{d}{dx}[e^x] =$	
$\frac{d}{dx}[a^x] =$	
$\frac{d}{dx}[\ln x] =$	

The most important step in integration is ***rewriting the integral*** in a form that fits the basic integration rules.

Ex. 2: Rewrite each of the following before integrating:

Original Integral	Rewrite	Integrate	Simplify
a) $\int \frac{1}{x^3} dx$			
b) $\int \frac{1}{2\sqrt{x}} dx$			
c) $\int 2t\left(3t^2 - 9\sqrt{t} + 1\right)dt$			
d) $\int (y^2 + 1)^2 dy$			
e) $\int \frac{x^3+3}{x^2} dx$			
f) $\int \left(sec^2\,\theta + \frac{2}{\theta} + csc^2\,\theta\right) d\theta$			

Initial Conditions and Particular Solutions:

There are several antiderivatives for a function, depending on C. In many applications of integration, you are given enough information to determine a **particular solution.** To do this you need only know the value of $y = f(x)$ for one value of x. (This information is called an **initial condition**).

Ex. 3: The solution to the differential equation $f'(x) = 3x^2 + 1$ has only one curve passing through the point $(2,4)$. Find the particular solution that satisfies this condition. Then, approach this graphically.

Ex. 4: Find the general solution of $F'(t) = 3^t$ and find the particular solution that satisfies the initial condition $F(0) = 3$.

Ex. 5: Find the particular solution $y = f(x)$ given $f''(x) = \sin x + 2$, the slope of $f(x)$ at $x = 0$ is 3 and the curve of $y = f(x)$ passes through $(\pi, 1)$.

<u>Ex. 6:</u> A ball is thrown upward with an initial velocity of 64 feet per second from an initial height of 80 feet.

Remember: To go from position to velocity to acceleration –

To go from acceleration to velocity to position –

a) Find the position function giving the height s as a function of time t.

b) What is the speed of the ball when it hits the ground?

c) After how many seconds after launch is the ball back at its initial height?

Ex. 7: A particle, starting at the origin, moves along the x −axis and its' velocity is modeled by the equation $v(t) = 6t^2 - 30t + 24$ where t is in seconds and $v(t)$ is meters per second.

a) How is the velocity changing at any time t? Be sure to include units of measure.

b) What is the particle's speed at 3 seconds? Be sure to include units of measure.

c) What is the particle's position when the acceleration is $6\ m/s^2$?

d) When is the particle changing directions?

e) When is the particle furthest to the left?

Ex. 8: A missile is accelerating at a rate of $4t\ m/s^2$ from a position at rest (velocity is zero) in a silo 750m below ground. How long will it take the missile to reach ground level?

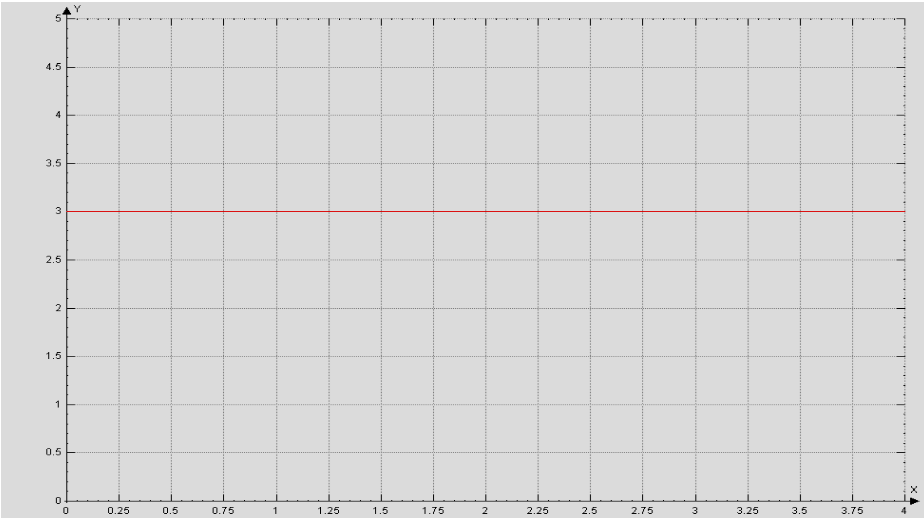
Ex. 9: The motion of a bear stalking its prey, walking left and right of a fixed point, can be modeled by the motion of a particle moving left and right along the x −axis, according to the acceleration equation $a(t) = -\sin t + \frac{1}{3}$. If the bear's velocity is 1 ft/s when $t = 0$…

a) Find the velocity equation.

b) What is the speed of the bear when $t = 7$?

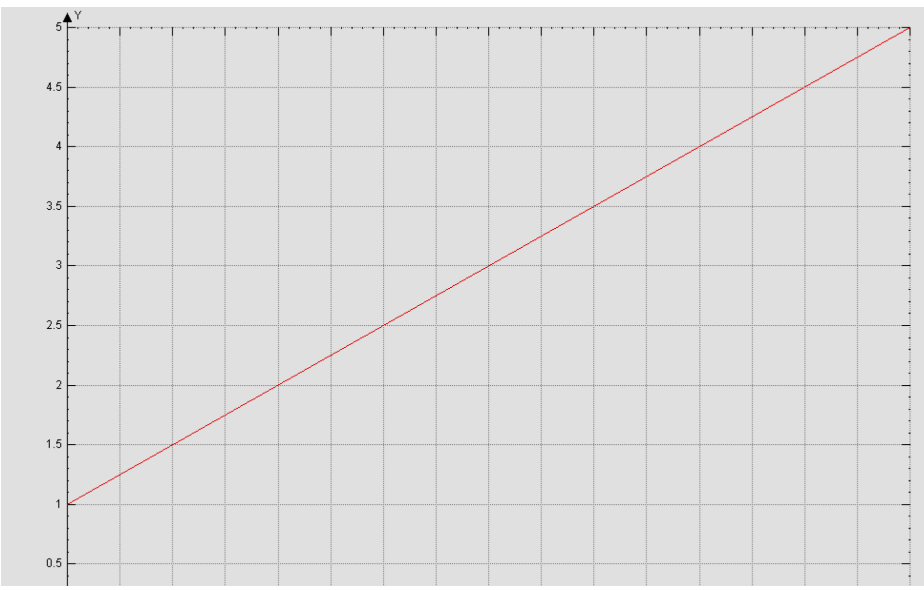
c) In what direction is the bear traveling when $t = 5$?

AP Calculus I
Notes 5.2
Area Under a Curve



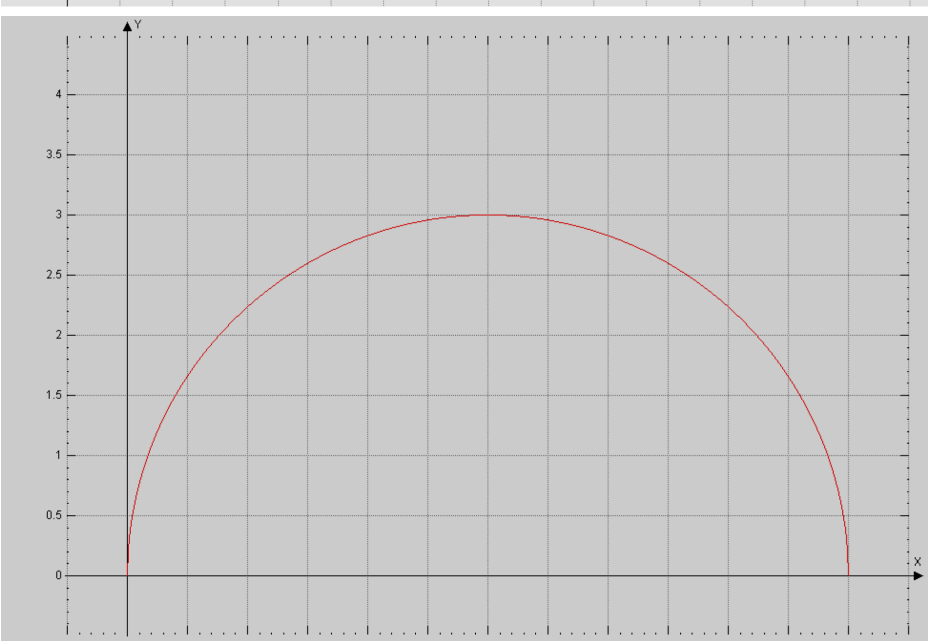
Area under the curve from $[0,4]$

= ______________________________



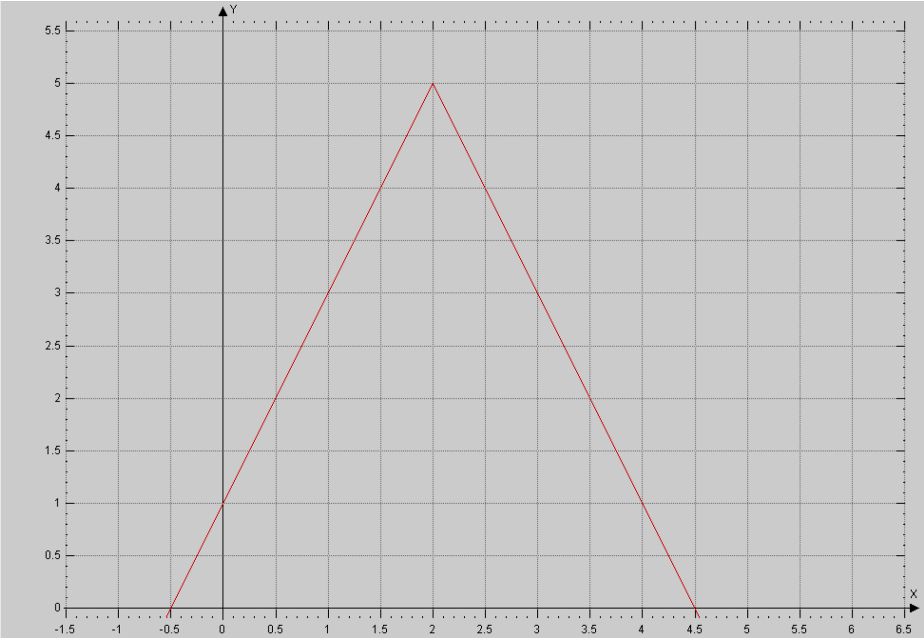
Area under the curve from $[1,3]$

= ______________________________



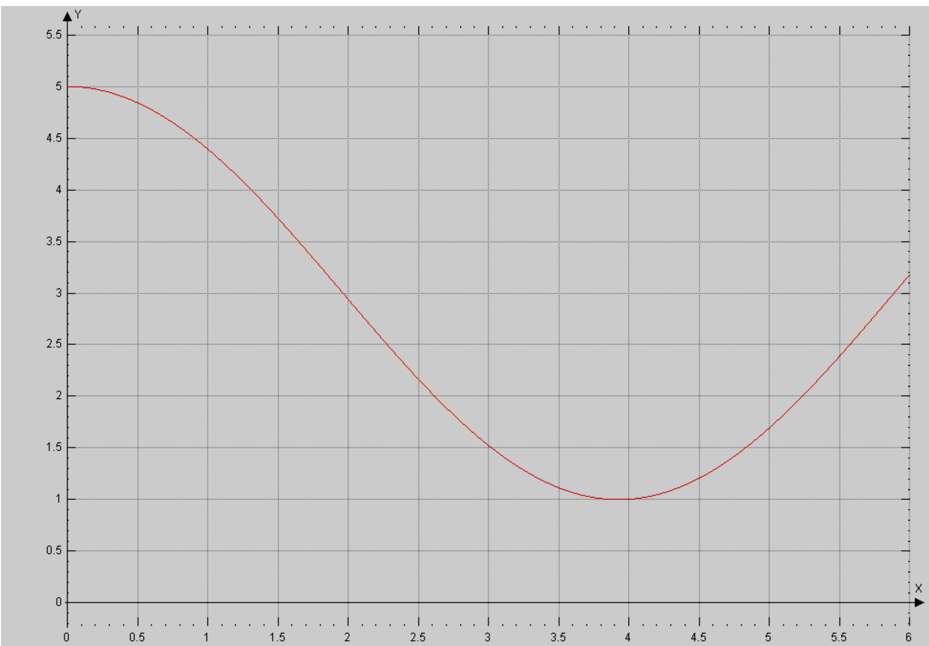
Area under the curve from $[0,6]$

= ______________________________



Area under the curve from $[-0.5, 4.5]$

= ______________________________



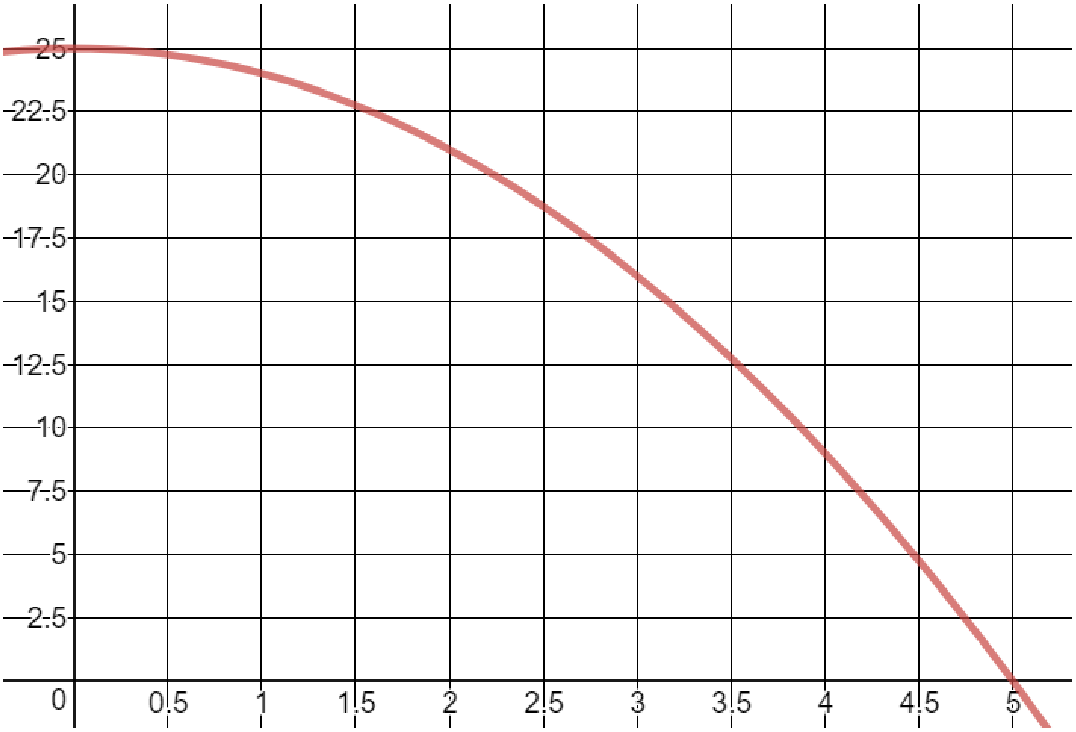
Area under the curve from $[1, 5]$

= ______________________________

In this section we will examine the problem of finding the area of a region in a plane.

Suppose you have to find the area under the curve $y = 25 - x^2$ from $x = 0$ to $x = 4$.

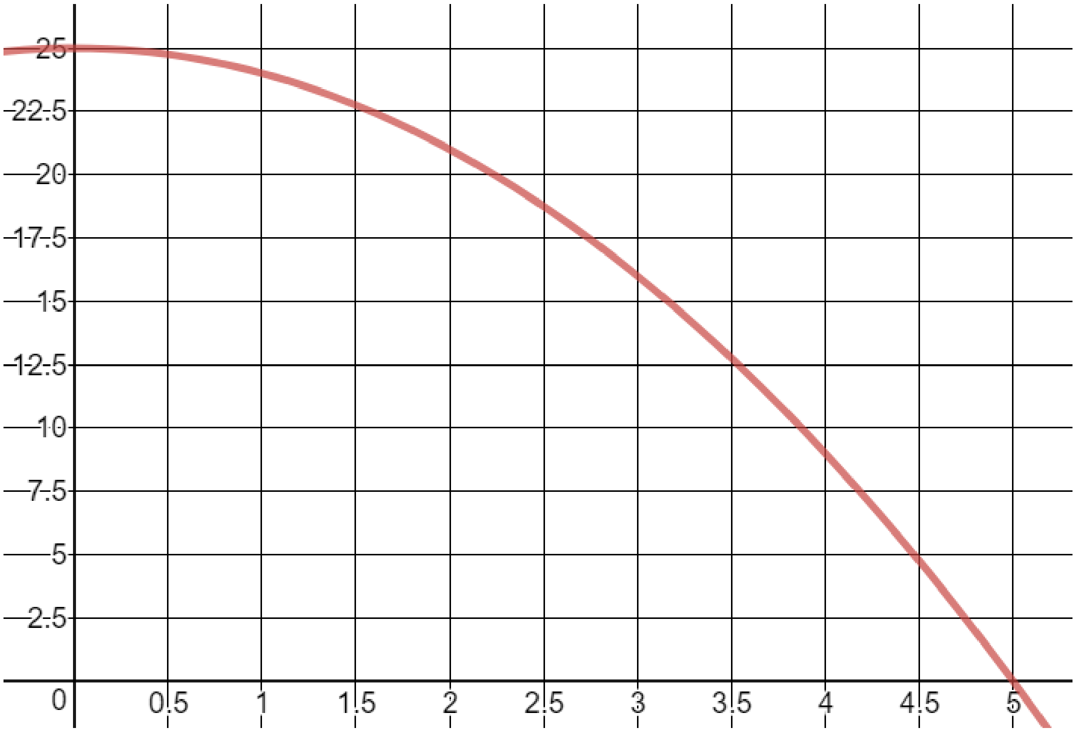
Graph:



Method 1: Divide the region into four rectangles, where the left endpoint of each rectangle comes just under the curve, and find the area.

Is this an over- or under-approximation of the actual?

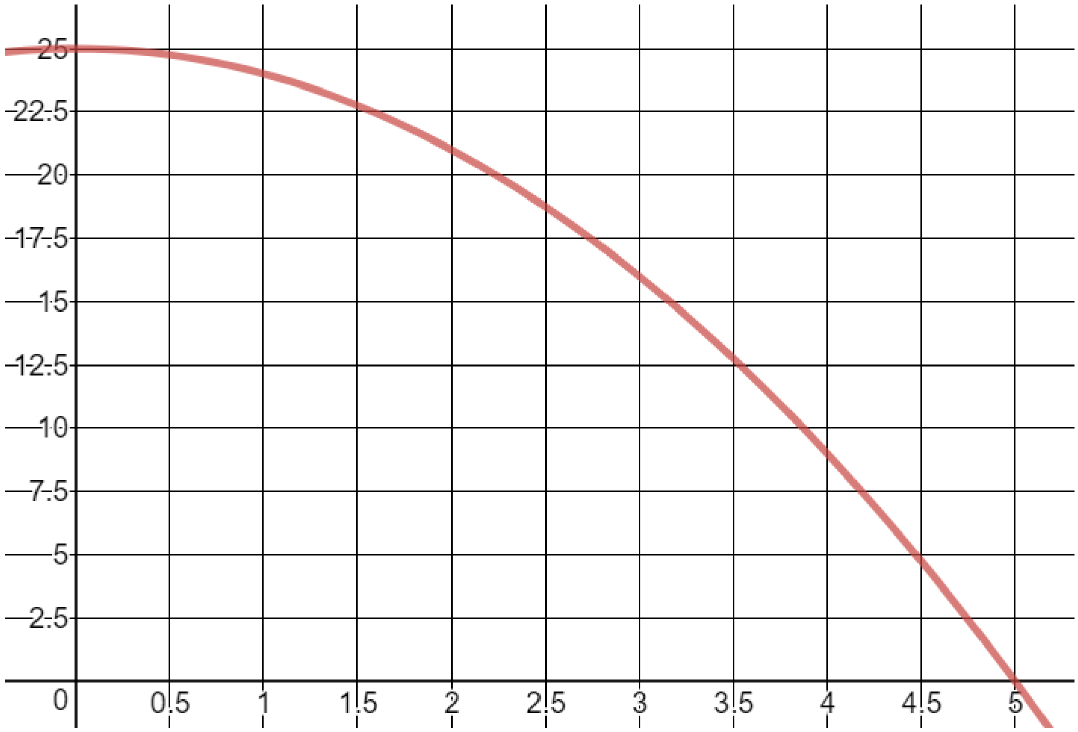
Graph:



Method 2: Divide the region into four rectangles, where the right endpoint of each rectangle comes just under the curve, and find the area.

Is this an over- or under-approximation of the actual?

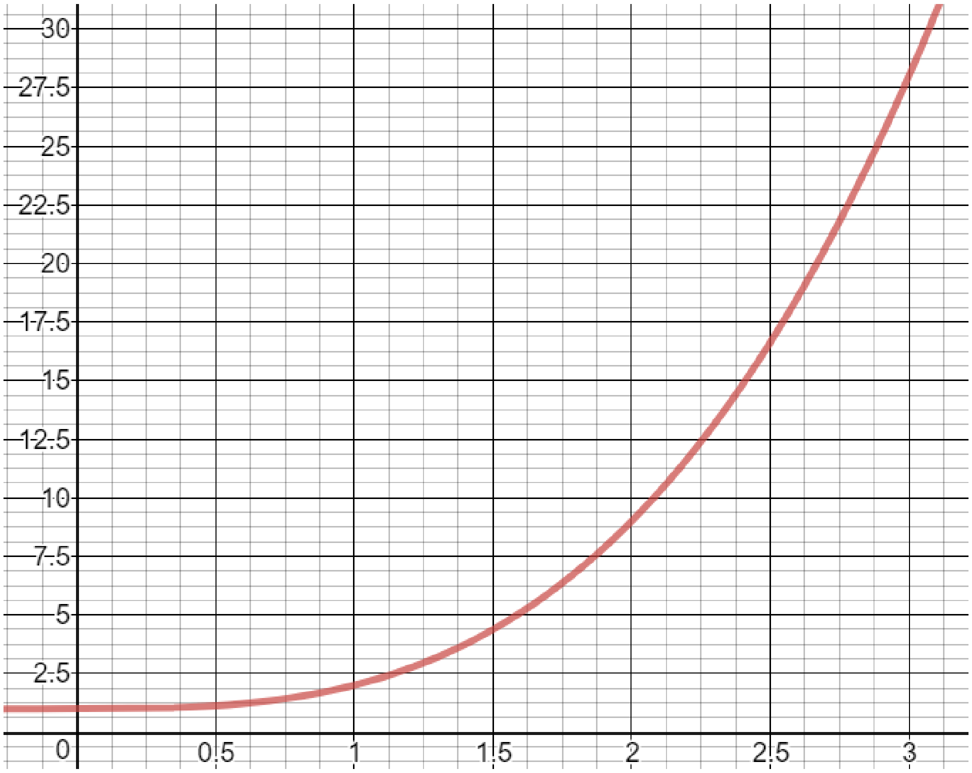
Graph:



Method 3: Divide the region into four rectangles, where the midpoint of each rectangle comes just under the curve, and find the area.

Is this an over- or under-approximation of the actual?

Suppose you have to find the area under the curve $y = x^3 + 1$ from $x = 1.5$ to $x = 3$ using 5 rectangles.



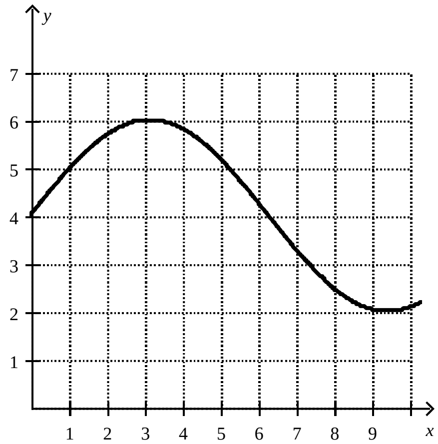
Left Endpoint Approximation: Over/Under?

Right Endpoint Approximation: Over/Under?

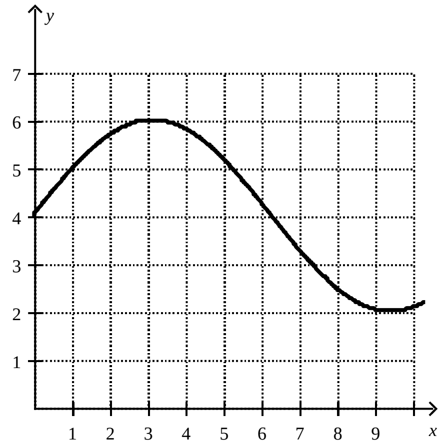
Midpoint Approximation: Over/Under?

Problems – Calculator Active

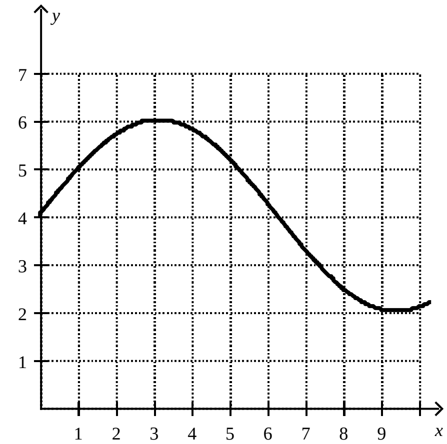
1. Approximate the area under the curve $y = 2\sin(0.5x) + 4$ from $x = 0$ to $x = 8$ using:



4 left Riemann rectangles



4 right Riemann rectangles



4 midpoint Riemann rectangles

2. Approximate the area under the curve $\boldsymbol{y = x^3}$ from $\boldsymbol{x = 2.1}$ to $\boldsymbol{x = 3.7}$ using:

a) five left-endpoint rectangles

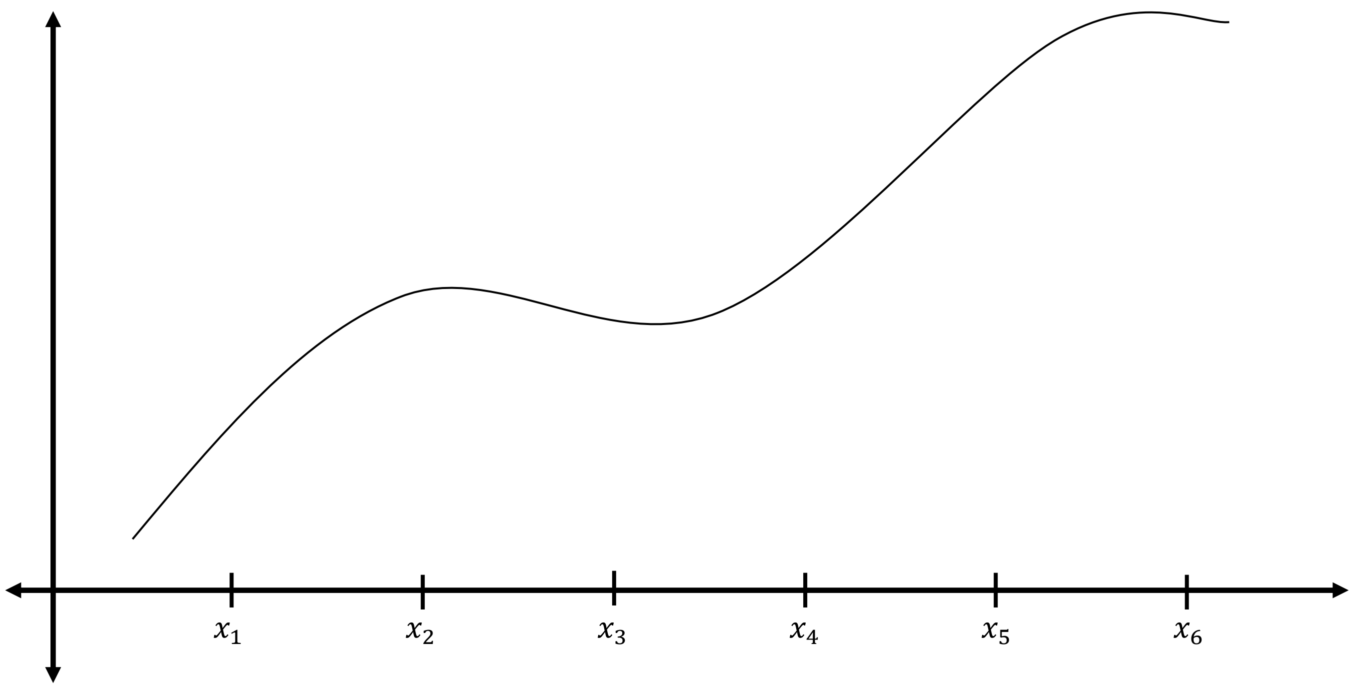
b) five right-endpoint rectangles

AP Calculus I
Notes 5.6
The Trapezoidal Rule

In addition to the three approximation techniques, there is a fourth technique that changes the geometric shape of the approximation. The trapezoidal rule approximates the area using a certain number of trapezoids.

Remember the area of a trapezoid is:

So, to use the trapezoidal rule to approximate the area under the curve of a function…



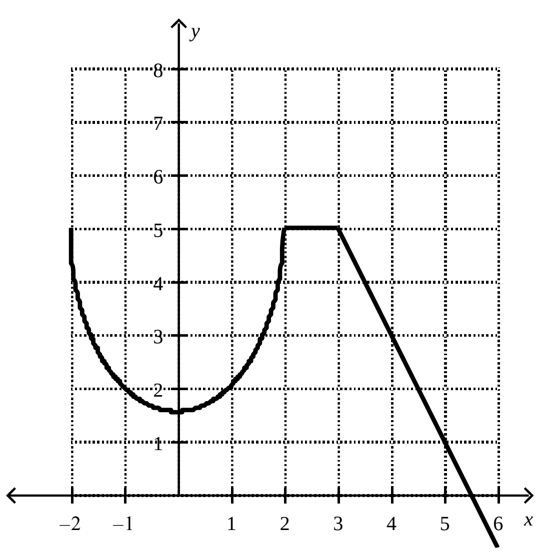
Area of trapezoids:

What feature about the curve made the trapezoidal approximation an over or underestimate?

Ex. 1: Use the trapezoidal rule to approximate the area under $f(x) = \sin x$ from $[0, \pi]$ with $n = 4$.

Ex. 2: Approximate the area between the x −axis and $f(x) = x^2 - 2x$ from $[1,3]$ using 5 trapezoids.

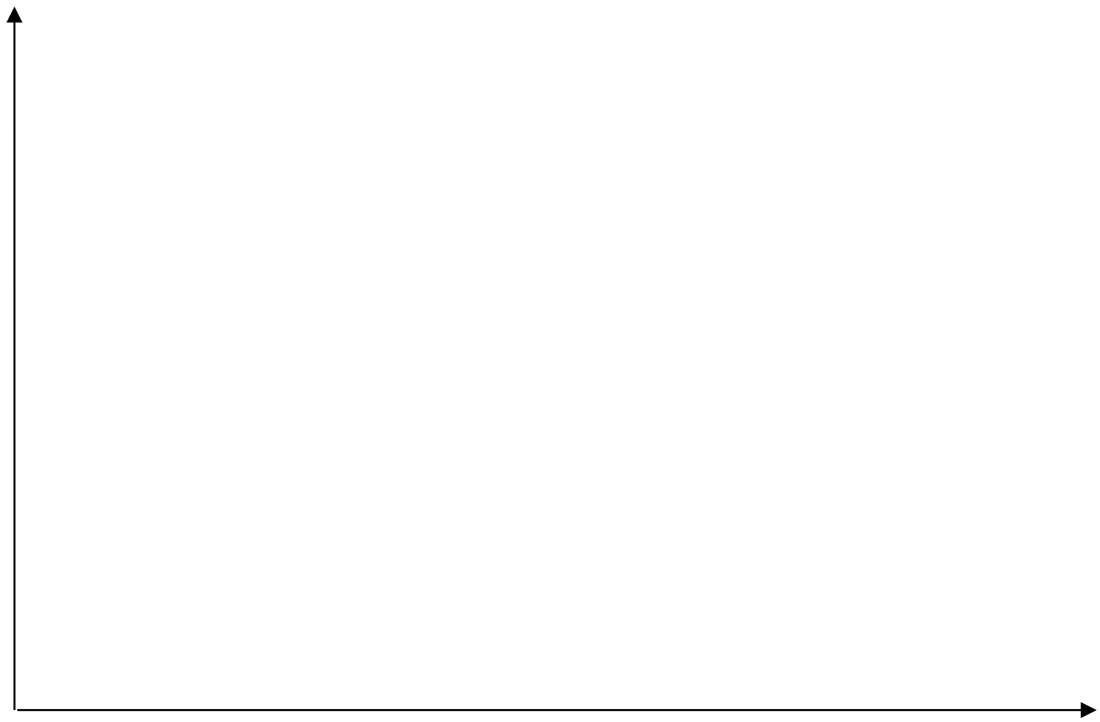
Ex. 3: Approximate the area between the x −axis and $f(x)$, found below, from $[-1,5]$ using 3 trapezoids.



Ex. 4: Readings from a car's speedometer at 10-minute intervals during a 1 – hour period are given in the table: t =hours, v =speed in miles per hour:

t	$0/60$	$10/60$	$20/60$	$30/60$	$40/60$	$50/60$	$60/60$
v	26	40	55	10	60	32	45

a) Draw a graph that could represent the car's velocity during the hour.



b) Find the area under the curve after 40 minutes of driving using the Right Riemann Sum with 4 equal intervals. What is the meaning of the area underneath the graph?

t	$0/60$	$10/60$	$20/60$	$30/60$	$40/60$	$50/60$	$60/60$
v	26	40	55	10	60	32	45

c) Find the area under the curve after 60 minutes using a Midpoint Riemann Sum with 3 equal intervals.

d) Approximate the distance traveled by the car for the hour using the Trapezoid Rule with 6 equal intervals.

Uneven Interval Widths

<u>Ex. 5:</u> The table below shows the rate at which water is coming out of a faucet in (mL/sec.) over different periods of time. $t =$ seconds. $R =$ rate of volume of water in mL/sec.

t	0	3	4	8	10
R	8	12	15	11	5

a) Is there guaranteed to be a time t, where $0 < t < 10$, such that $R'(t) = 0$? Explain your reasoning.

b) Approximate the area under the curve after 10 seconds using the Left Riemann Sum with 4 subintervals. What is the meaning of the approximate area underneath this graph?

t	0	3	4	8	10
R	8	12	15	11	5

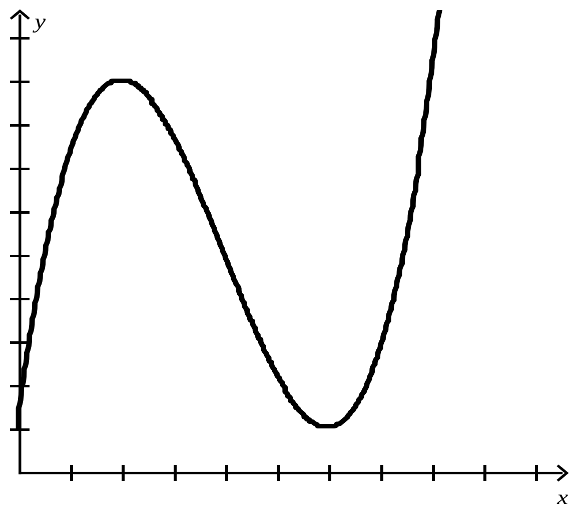
c) Approximate the volume of water that has come out of the faucet after 10 seconds using a Right Riemann Sum with 4 subintervals.

d) Approximate the volume of water that has come out of the faucet after 10 seconds using the Trapezoid Rule with 4 subintervals.

AP Calculus I
Notes 5.3
Definite Integrals

In the previous section, we estimated the area under a curve using a finite number of rectangles. Using the sum of those gave us an estimate for the area. Let's see if there is a way to find the **exact** area under a curve.

Let f be defined on the closed interval $[a, b]$ and let Δx_k be the width of the kth subinterval of $[a, b]$. Let's allow for equal widths for each subinterval, meaning that each subinterval has a width represented by:



Width =

Since the method of finding the exact area is similar, regardless of if we use LEA, REA or MPA, we will use a Right Endpoint Riemann sum here. As a reminder, a REA is the sum of the area of n rectangles, where the area of each rectangle is found by:

Area =

As you add up each of the areas of the n rectangles, the width remains the same, but the heights change as the function changes. So, we must develop a way to express the changing heights. Since we are using a REA from $[a, b]$ with n rectangles, the area = height * width are:

First subinterval $(k = 1)$

Second subinterval $(k = 2)$

Third subinterval $(k = 3)$

*k*th subinterval

Now that we can express the area of any rectangle, we can take the sum of the areas to get a **Riemann sum**:

Now, if we want this summation of the finite number of rectangles to become closer and closer to the actual area under the curve, we should allow two (related) things happen:

Therefore, we can formally state that the definite (exact) area under a curve is written as:

The notation changed from the mathematical Greek to common Roman language gives us our final answer.

Definite Integrals

If f is defined on the closed interval $[a, b]$ and the limit exists, then f is **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(a + k\Delta x)\Delta x = \int_{a}^{b} f(x)\,dx$$

The limit is called the **definite integral** of f from $x = a$ to $x = b$. The number $x = a$ is the **lower limit** of integration, and the number $x = b$ is the **upper limit** of integration.

** A definite integral is a *number*, whereas an indefinite integral is a *family of functions*.**

Theorem – The Definite Integral as the Area of a Region

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x −axis, and the vertical lines $x = a$ and $x = b$ is given by:

$$\text{Area} = \int_{a}^{b} f(x)dx$$

Ex. 1: Given the graph of $f(x) = x^2$ from $x = 2$ to $x = 5$, find:

a) An approximation of the area using a REA with 4 equal subintervals.

b) The limit definition of, and the definite integral representing, the exact area.

Ex. 2: Write the limit definition of the Riemann sum that is equivalent to the following definite integrals:

a) $\int_2^6 3 \ln x \, dx$

b) $\int_0^3 \sin(5x) \, dx$

Ex. 3: Which of the following limits is equal to $\int_3^5 x^4 \, dx$?

(A) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{k}{n}\right)^4 \frac{1}{n}$

(B) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{k}{n}\right)^4 \frac{2}{n}$

(C) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{2k}{n}\right)^4 \frac{1}{n}$

(D) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{2k}{n}\right)^4 \frac{2}{n}$

Ex. 4: Sketch the region corresponding to each definite integral. Then, evaluate each integral using a geometric formula:

a) $\int_{1}^{3} 4\,dx$

b) $\int_{-2}^{2} \sqrt{4-y^2}\,dy$

Ex. 5: Evaluate. $\int_{0}^{6} (|x-4|-2)dx$

Features and Theorems of Definite Integrals

1. If f is defined at $x = a$, then $\int_a^a f(x)dx = 0$.

2. If f is integrable on $[a, b]$, then $\int_b^a f(x)dx = -\int_a^b f(x)dx$.

3. If f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$.

4. If f is integrable on closed intervals containing a, b, and c, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

Ex. 6: Given $\int_{-1}^1 f(x)dx = 3$ and $\int_0^1 f(x)dx = 5$, find $\int_{-1}^0 f(x)dx$.

Ex. 7: Estimate $\int_0^8 (2x - 4)dx$ using a Left Endpoint Approximation with 4 equal intervals. Then, estimate using a Trapezoidal sum with 4 equal intervals. Finally, find the exact value of $\int_0^8 (2x - 4)dx$.

AP Calculus I
Notes 5.4
Fundamental Theorem of Calculus

The two branches of Calculus, differential calculus and integral calculus, seemed unrelated. However, the connection between the two was discovered independently by Sir Isaac Newton and Gottfried Leibniz. This connection is stated in a theorem appropriately named **The Fundamental Theorem of Calculus**.

Theorem – The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and G is the antiderivative of f on $[a, b]$, then:

$$\int_a^b f(x)dx = G(b) - G(a)$$

Guidelines for Using the Fundamental Theorem of Calculus

Provided you can find an antiderivative, you now have a way to evaluate a definite integral without having to use the limit of a sum or geometric means.

When using the FTC, you use the following steps with the given notation:

$$\int_a^b f(x)dx = G(x)\Big|_a^b = G(b) - G(a)$$

1) Find the antiderivative of f.
2) Evaluate the antiderivative function at the two bounds.
3) Find the difference of the upper bound and the lower bound.

****It is not necessary for use a constant of integration C in this process.****

Reason:

Ex. 1: Evaluate the definite integral $\int_0^3 4x\,dx$ using the Fundamental Theorem of Calculus and then verify by finding the area under the curve graphically.

Ex. 2: Evaluate the following definite integrals:

a) $\int_1^4 \left(t^2 - e^t + \frac{2}{t}\right) dt$

b) Find k where $\int_0^k (2y - 12)dy = -36$

c) $\int_1^4 \left(\frac{3x^4 + 2x^2 - x^{\frac{3}{2}}}{x^2}\right) dx$

d) $\int_{\frac{\pi}{4}}^{x} (\sec^2\theta + \sin\theta)d\theta$

Ex. 3: Find the area under the curve of each of the following functions from $[1,4]$:

a) $f(x) = \sqrt{\frac{4}{x}}$

b) $g(x) = \begin{cases} 2x^2 - 5 & x < 2 \\ 2x - 1 & x \geq 2 \end{cases}$

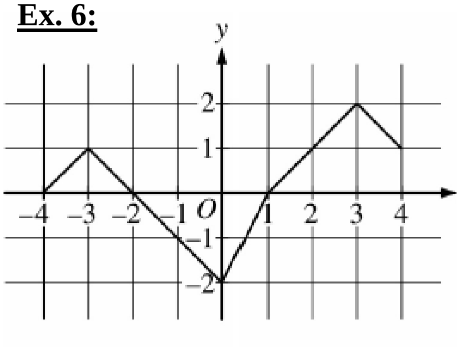
Definition of the Average Value of a Function on an Interval

If f is integrable on the closed interval $[a, b]$, then the average value of f is: $\frac{1}{b-a}\int_a^b f(x)dx$

Ex. 4: Find the average value of $f(x) = 3x^2 - 2x$ over the interval $[1,4]$.

Ex. 5: A ball is fired from a cannon and its position is modeled by the function, $s(t) = -16t^2 + 48t + 10$. Find the average velocity of the projectile over the interval $[1,3]$ using 2 different methods.

Ex. 6:



Graph of f

The function f is continuous for $-4 \le x \le 4$. The graph of f shown above consists of five line segments. What is the average value of f on the interval $-4 \le x \le 4$?

(A) $\frac{1}{8}$

(B) $\frac{3}{16}$

(C) $\frac{15}{16}$

(D) $\frac{3}{2}$

Ex. 7: **(CA)** OakMart creates a model to fit the monthly sales data for their top selling product Out of Sight, Out of Mime, a street performer repellant. The model is $S(t) = \frac{t^2}{4} + 1.8\sqrt{2t+1} + 0.74\sin\left(\frac{\pi t}{6}\right)$ over the interval $0 \le t \le 24$, where S is sales (in thousands) and t is time in months. Find the average sales of Out of Sight, Out of Mime over the 2 years.

The Second Fundamental Theorem of Calculus Investigation

1. Let $f(t) = 2t + 3$

   a) Find $\int_1^x (2t+3)dt$

   b) Find $\frac{d}{dx}\left[\int_1^x (2t+3)dt\right]$

2. Let $g(t) = 3t^2 - 6t + 5$

   a) Find $\int_3^x g(t)dt$

   b) Find $\frac{d}{dx}\left[\int_3^x g(t)dt\right]$

Do you see a pattern? Use it to find $\frac{d}{dx}\left[\int_a^x j(t)dt\right]$ (where a is a constant)

3. Let $k(t) = \cos t$

   a) Find $\int_6^{2x^2} k(t)dt$

   b) Find $\frac{d}{dx}\left[\int_6^{2x^2} k(t)dt\right]$

4. Let $m(t) = e^t$

   a) Find $\int_3^{\cos x} m(t)dt$

   b) Find $\frac{d}{dx}\left[\int_3^{\cos x} m(t)dt\right]$

Do you see another pattern? Use it to find $\frac{d}{dx}\left[\int_a^{g(x)} f(t)dt\right]$ (where a is a constant)

Theorem – The Second Fundamental Theorem of Calculus

If f is continuous on an open interval, then for all x in the interval , $\frac{d}{dx}\left[\int_a^u f(t)dt\right] = f(u)\cdot u' - f(a)\cdot a'$

Ex. 8: Evaluate $\frac{d}{dx}\left[\int_3^{3x-2} t^2 dt\right]$

Ex. 9: Find the derivative of:

a) $F(x) = \int_x^{x^3} \cos t\, dt$

b) $F(x) = \int_{2x-1}^{6} \sin(3y+7)dy$

Ex. 10: If $f(x) = \int_1^{x^3} \frac{1}{1+\ln t}\, dt$ for $x \geq 1$, then $f'(2) =$

(A) $\frac{1}{1+\ln 2}$

(B) $\frac{12}{1+\ln 2}$

(C) $\frac{1}{1+\ln 8}$

(D) $\frac{12}{1+\ln 8}$

Ex. 11: $F(x) = \int_{2}^{2x} (3t^2 - 5)dt$

a) $F(1) =$

b) $F'(1) =$

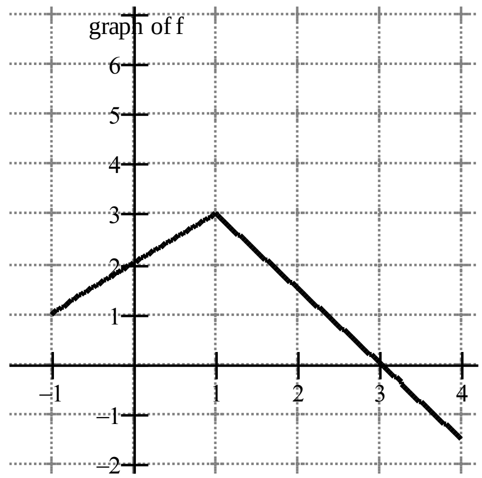
c) $F''(1) =$

Functions Defined by Integrals

Ex. 12: Let $F(x) = \int_{-1}^{x} f(t)dt$, $-1 \le x \le 4$, where f is the function graphed.

a) Complete the table of values for F.

x	-1	0	1	2	3	4
$F(x)$						

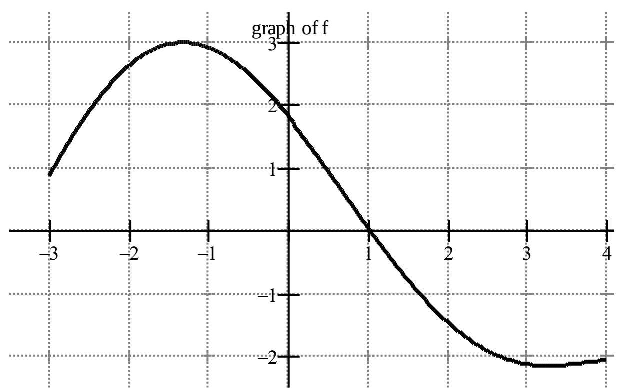


b) Sketch a graph of F.

c) Where is F increasing? Why?

Ex. 13: Let $A(x) = \int_{-3}^{x} f(t)dt$, $-3 \le x \le 4$ where f is graphed.

a) Which is larger, $A(-1)$ or $A(1)$? Justify.



b) Which is larger, $A(2)$ or $A(4)$? Justify.

c) Where is A increasing? Justify.

d) Does A have a relative minimum, relative maximum or neither at $x = 1$. Justify your answer.

Theorem – Net Change Theorem

The definite integral of the rate of change of a quantity $f(x)$, whose antiderivative is $G(x)$, gives the **total accumulation**, or **net change**, of that quantity on the interval $[a, b]$.

$$\int_a^b f(x)dx =$$

This can also be rewritten to represent a **final amount** **OR** an **initial amount**

$G(b) =$ $G(a) =$

Ex. 14: A chemical flows into a storage tank at a rate of $180 + 3t$ liters per minute at time t (in min), where $0 \leq t \leq 60$. Find the amount of the chemical that flows into the tank during the first 20 minutes, including units. Then, find the amount of chemical in the tank at $t = 20$, if possible.

Ex. 15: The tray of mystery meat at the Golden Trough, the local buffet restaurant, is being slopped onto customer's plates at a rate of $M(t) = 14 + 0.2t(3^t - t^2)$. Golden Trough employees refill the mystery meat tray at a rate of $R(t) = 3.7e^{2-\sin t}$. Both $M(t)$ and $R(t)$ are measured in pounds per hour at time t (in hours), where $0 \leq t \leq 4$.

a) At what rate is mystery meat being slopped onto customer's plates at $t = 4$?

b) How many pounds of mystery meat were slopped onto customer's plates by $t = 4$?

c) If 11 pounds were in the tray at $t = 0$, how many pounds of mystery meat are in the tray at $t = 4$?

Ex. 16: Given $f(x)$ is the antiderivative of $F(x)$, $f(2) = -3$ and $F(x) = 2\sqrt{\arctan x + 5x^3}$, find $f(5)$.

Ex. 17: Given that $g(x) = f'(x)$, $f(9) = 3$ and $g(x) = \frac{40e^{\sin x - 1}}{2+3^x}$, find $f(5)$.

Ex. 18: Mr. Gough is baking cookies for his favorite Calculus class at a temperature of 350 degrees Fahrenheit. He then takes out the cookies and turns off the oven ($t = 0$ minutes). The temperature of the oven is changing at a rate of $-110e^{-0.4t}$ degrees Fahrenheit per minute. To the nearest degree, what is the temperature of the oven at time $t = 5$ minutes?

Ex. 19: The rate at which people enter an amusement park on a day is modeled by the function $E(t)$ and the rate at which people leave the park on the same day is modeled by the function $L(t)$ shown below.

$$E(t) = \frac{15600}{t^2 - 24t + 160} \qquad\qquad L(t) = \frac{9890}{t^2 - 38t + 370}$$

Both $E(t)$ and $L(t)$ are measured in people per hour and t is in hours after midnight. These functions are valid for $9 \leq t \leq 23$, the hours when the park is open. At $t = 9$, there are no people in the park.

a) How many people have entered the park by 5:00 P.M. ($t = 17$)? Round your answer to the nearest whole number.

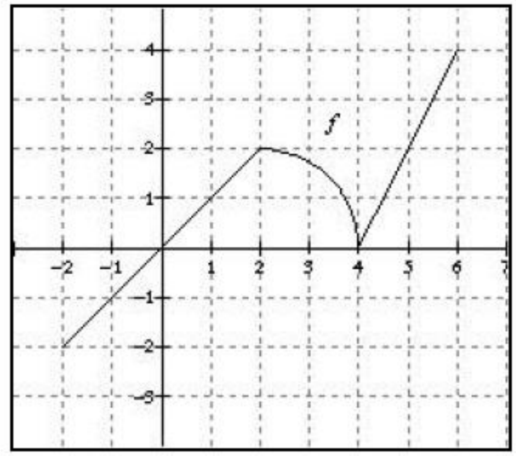
b) The price of admission to the park is \$15 until 5:00 P.M. ($t = 17$). After 5:00 P.M., the price of admission to the park is \$11. How many dollars are collected from admissions to the park on the given day? Round your answer to the nearest whole number.

c) Let $H(t) = \int_9^t (E(x) - L(x))dx$ for $[9, 23]$. The value of $H(17)$ to the nearest number is 3725.

Find the value of $H'(17)$ and explain the meaning of $H(17)$ and $H'(17)$ in the context of the park.

d) At what time t, for $9 \leq t \leq 23$, does the model predict that the number of people in the park is a maximum?

Ex. 20: Let $F(x) = \int_{2}^{x} f(t)dt$. The graph of f, shown to the right, consists of two line segments and a quarter circle.



a) Find the average rate of change of F from $x = 0$ to $x = 4$.

b) Determine the interval where $F(x)$ is increasing. Justify your answer.

c) Find the critical numbers of $F(x)$ and identify each as a relative min, max or neither.

d) Find the absolute extreme values of $F(x)$.

e) Find the intervals in which $F(x)$ is concave up. Explain your reasoning.

Ex. 21: A water tank at Camp Itsunderthere holds 1200 gallons of water at time $t = 0$. During the time interval $0 \le t \le 18$ hours, water is pumped into the tank at the rate $W(t) = 95\sqrt{t}\sin^2\left(\frac{t}{6}\right)$ gallons per hour. During the same time interval, water is removed from the tank, in gallons per hour, at the rate $R(t) = 275\sin^2\left(\frac{t}{3}\right)$.

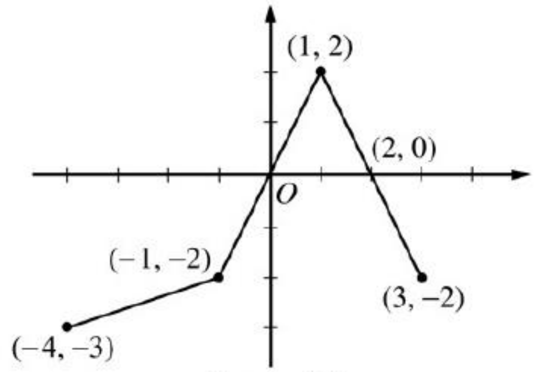
a) Is the amount of water in the tank increasing at time $t = 15$? Why or why not?

b) To the nearest whole number, how many gallons of water are in the tank at time $t = 18$?

c) At what time t, for $0 \le t \le 18$, is the amount of water in the tank at an absolute minimum?

d) For $t > 18$, no water is pumped into the tank, but water continues to be removed at the rate $R(t)$ until the tank becomes empty. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.

Ex. 22:



Graph of f

The graph of the function f above consists of three line segments.

a) Let g be the function given by $g(x) = \int_{-4}^{x} f(t)dt$. For each $g(-1)$, $g'(-1)$, $g''(-1)$, find the value or state that it does not exist.

b) Find the x −coordinate of each point of inflection of the graph of g. Explain your reasoning.

c) Evaluate the following integral expression $\int_{0}^{3} [4f(t) - 2]dt$.

d) Let h be the function given by $h(x) = \int_{x}^{3} f(t)dt$. Find all intervals on which h is decreasing. Explain your reasoning.

Ex. 23: The penguin population on an island is modeled by a differentiable function P of time t, where $P(t)$ is the number of penguins and t is measured in years, for $0 \le t \le 40$. There are 100,000 penguins on the island at time $t = 0$. The birth rate for the penguins on the island is modeled by

$$B(t) = 1000e^{0.06t} \text{ penguins per year}$$

and the death rate for the penguins on the island is modeled by

$$D(t) = 250e^{0.1t} \text{ penguins per year.}$$

(a) What is the rate of change of the penguin population on the island at time $t = 0$?

(b) To the nearest whole number, what is the penguin population on the island at time $t = 40$?

(c) To the nearest whole number, what is the average rate of change of the penguin population on the island for $0 \le t \le 40$?

(d) To the nearest whole number, find the absolute minimum penguin population and the absolute maximum penguin population on the island for $0 \le t \le 40$. Show the analysis that leads to your answers.

Ex. 24: A particle is moving along a line such that its velocity can be modeled by $v(t) = 4t^3 + 6t^2 - 7e^t$.

a) Find the position of the particle at time $t = 3$ given the particle is at $s(0) = 13$.

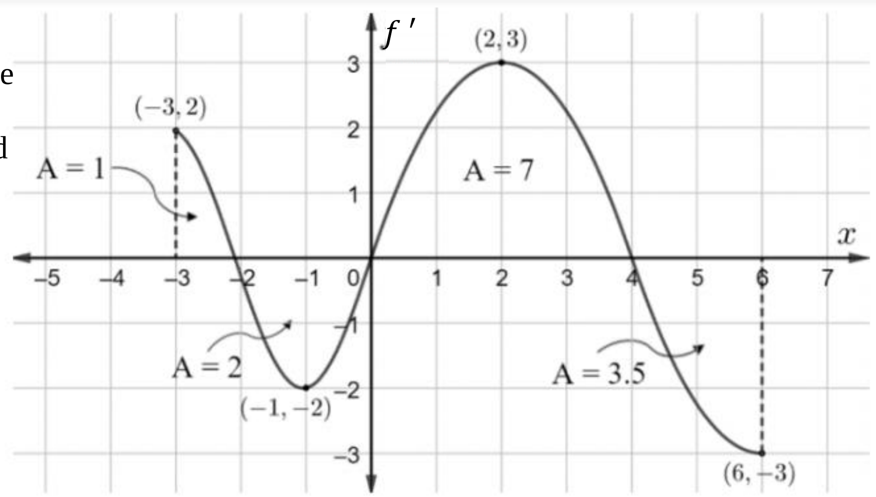
b) Find the total distance travelled by the particle over the first 3 seconds.

c) Find the displacement of the particle over the first 3 seconds.

d) Find the average acceleration from the time interval $[3,5]$.

e) Find the average velocity from the time interval $[3,5]$.

Ex. 25: The graph of $f'(x)$ is shown to the right. The graph has horizontal tangencies at $x = -3, -1, 2, 6$ and has areas provided as well.



a) Evaluate $\int_{-3}^{6} f'(x)dx$.

b) Evaluate $\int_{-1}^{2} [3f''(x) + 2f'''(x)]dx$.

The graph of $f'(x)$ represents the velocity $v(t)$ of a particle moving along the x −axis from $-3 \leq t \leq 6$. Its position at time t (sec) is $s(t)$ feet and it is known that $s(4) = 5$.

c) Is the particle further right at $t = -2$ or $t = 6$? Justify your answer.

d) When is the particle slowing down? Justify your answer.

e) At what time t, $-3 \leq t \leq 6$, does $s(t)$ have its smallest value? Justify your answer.

Ex. 26: Let $y(t)$ represent the temperature of a pie that has been removed from a $450°F$ oven and left to cool in a room with a temperature of $72°F$ where y is a differentiable function of t. The table below shows the temperature recorded every five minutes.

t (min)	0	5	10	15	20	25	30
$y(t)$	450	388	338	292	257	226	200

a) Use the data from the table to find an approximation for $y'(18)$ and explain the meaning of $y'(18)$ in terms of the temperature of the pie. Indicate units of measure.

b) Use data from the table to find the value of $\int_{10}^{25} y'(t)dt$ and explain the meaning of $\int_{10}^{25} y'(t)dt$ in terms of the temperature of the pie. Indicate units of measure.

c) A model for the temperature of the pie is given by the function $W(t) = 72 + 380e^{-0.037t}$ where t is measured in minutes and $W(t)$ is measured in degrees Fahrenheit $(°F)$. Use the model to find the value of $W'(18)$. Indicate units of measure.

d) Use the model given in part c) to find the time at which the temperature of the pie is $300°F$.

Ex. 27: Let $y(t)$ represent the population of the town of Sugar Mill over a 10 – year period, where y is a differentiable function of t. The table below shows the population recorded every two years.

t (yrs)	0	2	4	6	8	10
$y(t)$	2500	2912	3360	3815	4330	4875

a) Use the data from the table to find an approximation for $y'(7)$ and explain the meaning of $y'(7)$ in terms of the population of Sugar Mill. Indicate units of measure.

b) Use data from the table to approximate the average population of Sugar Mill over the time interval $0 \leq t \leq 10$ by using a left Riemann sum with five subintervals. Indicate units of measure.

c) A model for the population of another town, Pine Grove, over the same 10 – year period is given by the function $P(t) = (2t + 50)^2$ where t is measured in years and $P(t)$ is measured in people. Use the model to find the value of $P'(7)$. Indicate units of measure.

d) Use the model given in part c) to find the value of $\frac{1}{10}\int_0^{10} P(t)dt$. Explain the meaning of this integral expression in terms of the population of Pine Grove.

Ex. 28: The rate at which water is pumped into a tank is given by the differentiable, increasing function $R(t)$. A table of selected values of $R(t)$, for the time interval $0 \le t \le 20$ minutes is shown below.

t (min)	0	4	9	17	20
$R(t)$ (gal/min)	25	28	33	42	46

a) Use a right Riemann sum with four subintervals to approximate the value of $\int_0^{20} R(t)dt$.

Is your approximation greater or less than the true value? Give a reason for your answer.

b) Show that there is a time $t = c$, where $0 \le c \le 20$, such that $R'(c) = 1$.

c) A model for the rate at which water is being pumped into the tank is given by the function $G(m) = 25e^{0.03m}$ where m is measured in minutes and $G(m)$ is measured in gallons per minute. The tank contained 100 gallons of water at time $t = 0$. Find the amount of water in the tank at $t = 20$ minutes.

Ex. 29: A bowl of soup is placed on the kitchen counter to cool. Let $T(x)$ represent the temperature of the soup at time x, where T is a differentiable function of x. The temperature of the soup at selected times is given in the table below.

x (min)	0	4	7	12
$T(x)$ ($°F$)	108	101	99	95

a) Use data from the table to find $\int_0^{12} T'(x)dx$. Explain the meaning of this definite integral in terms of the temperature of the soup.

b) Use data from the table to find the average rate of change of $T(x)$, such that $x = 4$ to $x = 7$.

c) Explain the meaning of $\frac{1}{12}\int_0^{12} T(x)dx$ in terms of the temperature of the soup, and approximate the value of this integral expression by using a trapezoidal sum with three subintervals.

AP Calculus I
Notes 5.5
Integration by Substitution

In this section, we look at how to integrate composite functions. The major technique involved is called **u-substitution**. The objective is to know the few antiderivative rules from before and rewrite the integrand to fit those rules. The role of substitution is comparable to the role of **The Chain Rule** in differentiation.

<u>Theorem – U-Substitution Integration</u>

Let g and f be a function that is continuous and differentiable on an interval I. If F is an antiderivative of f on I, then,

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

If $u = g(x)$, then $du = g'(x)dx$ and $\int f(u)du = F(u) + C$.

<u>Ex. 1:</u> The integrand in each of the following integrals fits the pattern $f(g(x))g'(x)$. Identify the pattern and use the result to evaluate the integral. Then check through differentiation.

a) $\int 2x(x^2 + 1)^4 dx$

b) $\int \sec^2 x\,(\tan x + 3)dx$

c) $\int 3e^{3y} dy$

d) $\int 5\cos(5x)\,dx$

For Example 1, the integrands fit the $f(g(x))g'(x)$ pattern exactly – you only had to recognize the pattern. You can extend this technique (if it doesn't fit perfectly) with the Constant Multiple Rule.

Ex. 2: Evaluate the following indefinite integrals:

a) $\displaystyle\int x(x^2-1)^2 dx$

b) $\displaystyle\int 7.1 \sec(4.2\theta) \tan(4.2\theta)\, d\theta$

c) $\displaystyle\int \frac{-5x}{(1-2x^2)^2} dx$

d) $\displaystyle\int (3e^x+3x)(e^x+1) dx$

e) $\int \frac{4}{x\left(\sqrt[5]{6+\ln x}\right)} dx$

f) $\int 3\cos(f(2t))\, f'(2t) dt$

g) $\int \left(\sin^2\left(\frac{x}{3}\right)\right) \cos\left(\frac{x}{3}\right) dx$

h) $\int \frac{4^{\ln x}}{5x} dx$

Definite Integrals

When it comes to definite integrals, you can still integrate and use the Fundamental Theorem of Calculus to evaluate the bounds. Another method is to rewrite the definite integral limits when you do the u-substitution.

Ex. 3: Evaluate the following:

a) $\displaystyle\int_1^5 \frac{dx}{\sqrt{10x-1}}$

b) $\displaystyle\int_1^3 \frac{e^{\frac{6}{y}}}{y^2}\,dy$

c) Find $f(\sqrt{\pi})$ given $f'(x) = x\sin(x^2+\pi)$ and $f(0) = 6$.

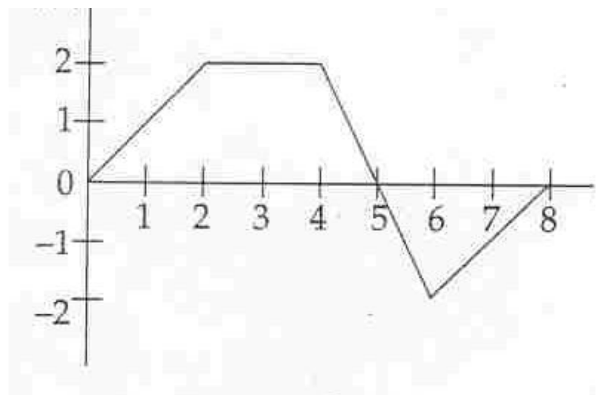
d) $\displaystyle\int_{\frac{1}{6}}^{\frac{1}{3}} (6t + \sec(\pi t)\tan(\pi t))\,dt$

Sometimes, when basic u-substitution doesn't work, you have to get a little creative and do some rewriting.

Ex. 4: Evaluate $\int x\sqrt{x-1}dx$

For Examples 5 and 6, use the graph of g given to the right:

Ex. 5: The function f is $f(t) = \int_1^t g(2x)dx$. Find $f(3)$.



Ex. 6: Evaluate the following integrals:

a) $\int_2^4 g(x+2)dx$

b) $\int_2^4 (g(x)+2)dx$

c) $\int_2^5 g'(\theta)\sin\big(g(\theta)\big)\,d\theta$

Ex. 7: A particle is moving along the x-axis for all $t \geq 0$ and has an acceleration modeled by the function $a(t) = 8(2t - 1)^3$. The particle has an initial velocity of $v(0) = -3$ and starts at the origin.

a) Find the velocity function $v(t)$ for any time $t \geq 0$.

b) Find the position function $x(t)$ for any time $t \geq 0$.

c) Find the average velocity of the particle from $[0,1]$.

d) Find the total distance travelled by the particle from $[0,2]$.

AP Calculus
Notes 5.7
The Natural Logarithmic Function and Integration

<u>Theorem – Log Rule for Integration</u>

Let $\boldsymbol{u}$ be a differentiable function of $\boldsymbol{x}$.

1) $\int \frac{1}{x} dx =$

2) $\int \frac{1}{u} du = \int \frac{du}{u} =$

<u>Ex. 1:</u> Evaluate the following integrals:

a) $\int \frac{2}{x} dx$

b) $\int \frac{1}{4x - 1} dx$

c) $\int \frac{\pi x + \pi}{x^2 + 2x} dx$

d) $\int \frac{1}{t \ln t} dt$

Ex. 2: Evaluate the following definite integrals:

a) $\displaystyle\int_0^1 \frac{e^x}{1+e^x}\,dx$

b) $\displaystyle\int_1^e \frac{4x^2+1}{2x}\,dx$

c) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{9\sec^2\theta}{7\tan\theta}\,d\theta$

d) $\displaystyle\int_1^2 \frac{3n}{\sqrt{5-n^2}}\,dn$

Ex. 3: Santa's favorite elves are tasked to create the best gifts: the Calculus ones. The 2nd most popular gift is the Yuletide Log (behind the Pumpkin Rolle), so the elves realize they must increase their production rate to $\boldsymbol{Y(t)} = \frac{\boldsymbol{417t}}{\boldsymbol{t^2+1}}$, where $\boldsymbol{Y(t)}$ is measure in millions of units per day and $\boldsymbol{t}$ is days since December 20th $(\boldsymbol{t = 0})$. Given the elves have already made 73 million by 12/20, how many would they have produced by the time Santa leaves on Christmas Eve 12/24?

As we continue our study of integration, we will devote much time to integration techniques. To master these techniques, you must recognize the "form-fitting" nature of integration.

Guidelines for Integration

1. Memorize a basic list of integration formulas (by the end of section 5.8, you will have 20 rules).

2. Find an integration formula that resembles all or part of the integrand, and, by ***trial and error***, find a choice of $\boldsymbol{u}$that will make the integrand conform to the formula.

3. If you cannot find a $\boldsymbol{u}$ - substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity or addition and subtraction of the same quantity. Be creative, but PATIENT most of all.

Ex. 4: Evaluate $\int \tan x \, dx$.

Integrals of the Trigonometric Functions

$\int \sin u \, du$

$\int \cos u \, du$

$\int \sec^2 u \, du$

$\int \cot u \, du$

$\int \sec u \tan u \, du$

$\int \csc u \cot u \, du$

$\int \tan u \, du$

$\int \csc^2 u \, du$

$\int \sec u \, du$

$\int \csc u \, du$

Ex. 5: Evaluate:

a) $\int 5x \csc 3x^2 dx$

b) $\int (\sec x \tan x - 2 \sec x + \tan 2x) dx$

Integrals to which the Log Rule can be applied often appear in disguised form. For instance, if a rational function has a numerator of degree greater than or equal to that of the denominator, divide!

Ex. 6: Evaluate the following integrals:

a) $\displaystyle\int \frac{x^2 + x + 1}{x^2 + 1} dx$

b) $\displaystyle\int_4^6 \frac{2y^2 + 7y - 3}{y - 2} dy$

Ex. 7: A population $\boldsymbol{P}$ of bacteria is changing at a rate of $\frac{\boldsymbol{dP}}{\boldsymbol{dt}} = \frac{\mathbf{3000}}{\mathbf{1+0.25}\boldsymbol{t}}$ where $\boldsymbol{t}$ is the time in days. The initial population is 1000. Find the population of bacteria after 3 days.

AP Calculus I
Notes 5.8
Inverse Trigonometric Functions and Integration

The derivatives of the six inverse trigonometric functions fall into three pairs. In each pair, the derivative of one function is the opposite of the other.

When listing the *antiderivative* that corresponds to each of the inverse trigonometric functions, you need to use only one member from each pair.

Theorem – Integrals Involving Inverse Trigonometric Functions

Let u be a differentiable function of x.

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\text{arcsec}\frac{|u|}{a} + C$$

Ex. 1: Integrate each of the following:

a) $\int \frac{dx}{\sqrt{4 - x^2}}$

b) $\int \frac{dx}{2 + 9x^2}$

c) $\int \frac{5}{x\sqrt{x^2 - 9}}dx$

Unfortunately, integration is not usually straightforward. The inverse trigonometric integration formulas can be disguised in many ways. Remember that **REWRITING** to a formula is the key to integration!

<u>Ex. 2:</u> Evaluate $\int \frac{2e^x dx}{\sqrt{1-e^{2x}}}$

<u>Ex. 3:</u> Find the area between the x −axis, the vertical lines $x = 0$, $x = 1$ and $f(x) = \frac{x+4}{\sqrt{4-x^2}}$.

Ex. 4: Evaluate $\int \frac{dx}{x^2 - 4x + 7}$

Ex. 5: Find the displacement of a particle whose velocity is modeled by $v(t) = \frac{3t^2+4t+12}{t^2+4}$ from $0 \leq t \leq 2$.

Guidelines for Integration

1. Memorize a basic list of integration formulas (you now have 20 rules).

2. Find an integration formula that resembles all or part of the integrand, and, by ***trial and error***, find a choice of $\boldsymbol{u}$ that will make the integrand conform to the formula.

3. If you cannot find a $\boldsymbol{u}$ −substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity or addition and subtraction of the same quantity. Be creative, but **PATIENT** most of all.

Ex. 6: Describe how to evaluate the following integrals using the formulas and techniques we have studied so far. (Hint: 2 of the following cannot be integrated as of yet)

a) $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$ b) $\displaystyle\int \frac{xdx}{\sqrt{x^2-1}}$ c) $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$

d) $\displaystyle\int \frac{e^x}{e^x+1}dx$ e) $\displaystyle\int \frac{e^x}{e^{2x}+1}dx$ f) $\displaystyle\int \frac{e^x+1}{e^x}dx$

g) $\displaystyle\int \frac{1}{x\ln x}dx$ h) $\displaystyle\int \ln x\,dx$ i) $\displaystyle\int \frac{\ln x}{x}dx$