3 Differentiation

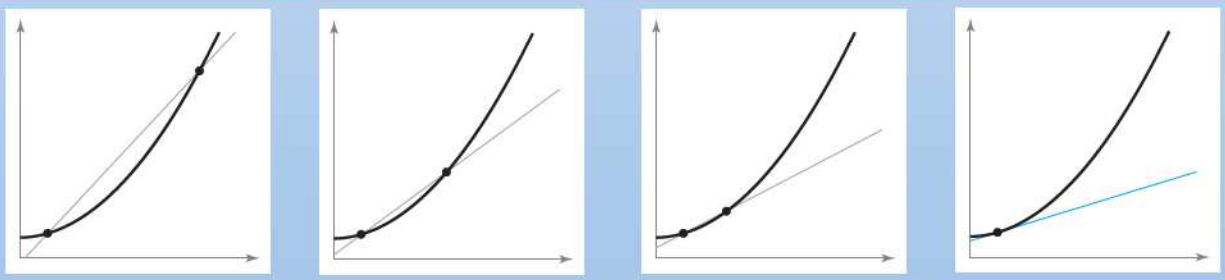
In this chapter you will study one of the most important processes of calculus–*differentiation*. In each section, you will learn new methods and rules for finding derivatives of functions. Then you will apply these rules to find such things as velocity, acceleration, and the rates of change of two or more related variables.

In this chapter, you should learn the following.

- How to find the derivative of a function using the limit definition and understand the relationship between differentiability and continuity. (3.1)
- How to find the derivative of a function using basic differentiation rules. (3.2)
- How to find the derivative of a function using the Product Rule and the Quotient Rule. (3.3)
- How to find the derivative of a function using the Chain Rule and the General Power Rule. (3.4)
- How to find the derivative of a function using implicit differentiation. (3.5)
- How to find the derivative of an inverse function. (3.6)
- How to find a related rate. (3.7)
- How to approximate a zero of a function using Newton's Method. (3.8)

Al Bello/Getty Images

When jumping from a platform, a diver's velocity is briefly positive because of the upward movement, but then becomes negative when falling. How can you use calculus to determine the velocity of a diver at impact? (See Section 3.2, Example 11.)



To approximate the slope of a tangent line to a graph at a given point, find the slope of the secant line through the given point and a second point on the graph. As the second point approaches the given point, the approximation tends to become more accurate. (See Section 3.1.)

3.1 The Derivative and the Tangent Line Problem

- Find the slope of the tangent line to a curve at a point.
- Use the limit definition to find the derivative of a function.
- Understand the relationship between differentiability and continuity.

The Tangent Line Problem

Calculus grew out of four major problems that European mathematicians were working on during the seventeenth century.

1. The tangent line problem (Section 2.1 and this section)
2. The velocity and acceleration problem (Sections 3.2 and 3.3)
3. The minimum and maximum problem (Section 4.1)
4. The area problem (Sections 2.1 and 5.2)

Each problem involves the notion of a limit, and calculus can be introduced with any of the four problems.

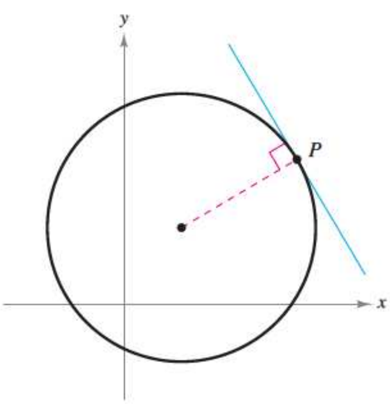
A brief introduction to the tangent line problem is given in Section 2.1. Although partial solutions to this problem were given by Pierre de Fermat (1601–1665), René Descartes (1596–1650), Christian Huygens (1629–1695), and Isaac Barrow (1630–1677), credit for the first general solution is usually given to Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716). Newton's work on this problem stemmed from his interest in optics and light refraction.

Mary Evans Picture Library/Alamy

ISAAC NEWTON (1642–1727)

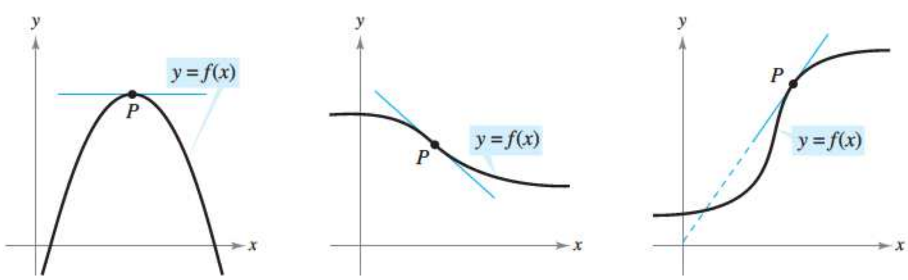
In addition to his work in calculus, Newton made revolutionary contributions to physics, including the Law of Universal Gravitation and his three laws of motion.

What does it mean to say that a line is tangent to a curve at a point? For a circle, the tangent line at a point P is the line that is perpendicular to the radial line at point P, as shown in Figure 3.1.



Tangent line to a circle
Figure 3.1

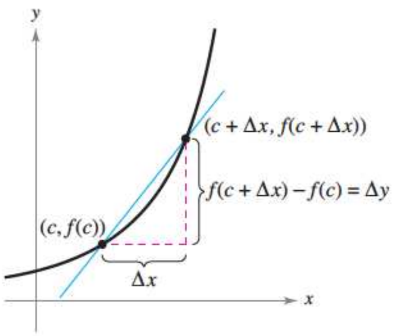
For a general curve, however, the problem is more difficult. For example, how would you define the tangent lines shown in Figure 3.2? You might say that a line is tangent to a curve at a point P if it touches, but does not cross, the curve at point P. This definition would work for the first curve shown in Figure 3.2, but not for the second. *Or* you might say that a line is tangent to a curve if the line touches or intersects the curve at exactly one point. This definition would work for a circle but not for more general curves, as the third curve in Figure 3.2 shows.



Tangent line to a curve at a point
Figure 3.2

EXPLORATION

Identifying a Tangent Line Use a graphing utility to graph the function $f(x) = 2x^3 - 4x^2 + 3x - 5$. On the same screen, graph $y = x - 5$, $y = 2x - 5$, and $y = 3x - 5$. Which of these lines, if any, appears to be tangent to the graph of f at the point $(0, -5)$? Explain your reasoning.



The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$
Figure 3.3

Essentially, the problem of finding the tangent line at a point P boils down to the problem of finding the *slope* of the tangent line at point P. You can approximate this slope using a **secant line*** through the point of tangency and a second point on the curve, as shown in Figure 3.3. If $(c, f(c))$ is the point of tangency and $(c + \Delta x, f(c + \Delta x))$ is a second point on the graph of f, the slope of the secant line through the two points is given by substitution into the slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c} \qquad \frac{\text{Change in } y}{\text{Change in } x}$$

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{\Delta x}. \qquad \text{Slope of secant line}$$

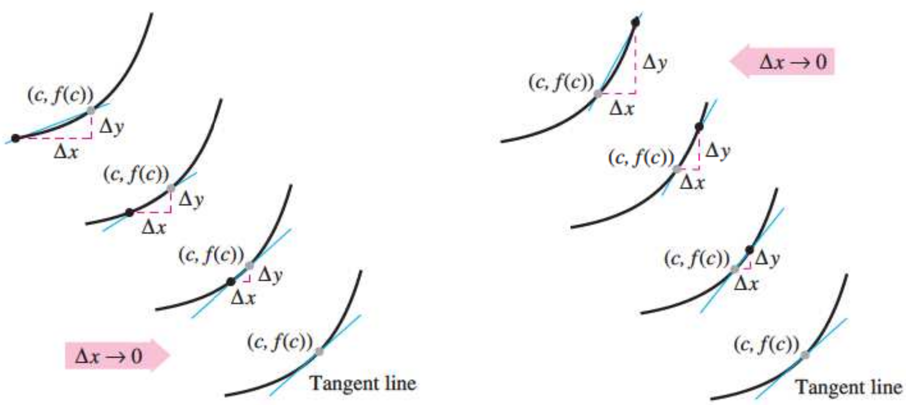
The right-hand side of this equation is a **difference quotient.** The denominator Δx is the **change in** x, and the numerator $\Delta y = f(c + \Delta x) - f(c)$ is the **change in** y.

The beauty of this procedure is that you can obtain more and more accurate approximations of the slope of the tangent line by choosing points closer and closer to the point of tangency, as shown in Figure 3.4.

THE TANGENT LINE PROBLEM

In 1637, mathematician René Descartes stated this about the tangent line problem:

"And I dare say that this is not only the most useful and general problem in geometry that I know, but even that I ever desire to know."



Tangent line approximations
Figure 3.4

DEFINITION OF TANGENT LINE WITH SLOPE m

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

The slope of the tangent line to the graph of f at the point $(c, f(c))$ is also called the **slope of the graph of f at $x = c$.**

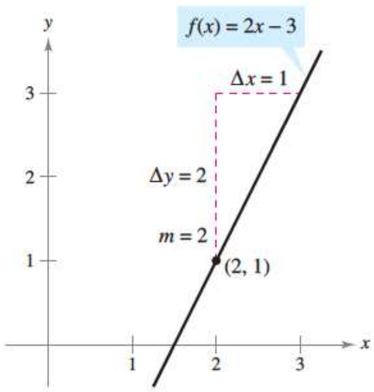
* *This use of the word* secant *comes from the Latin* secare, *meaning to cut, and is not a reference to the trigonometric function of the same name.*

EXAMPLE 1 The Slope of the Graph of a Linear Function

Find the slope of the graph of

$$f(x) = 2x - 3$$

at the point (2, 1).



The slope of f at (2, 1) is $m = 2$.
Figure 3.5

Solution To find the slope of the graph of f when $c = 2$, you can apply the definition of the slope of a tangent line, as shown.

$$\begin{aligned}\lim_{\Delta x \to 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{[2(2 + \Delta x) - 3] - [2(2) - 3]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{4 + 2\Delta x - 3 - 4 + 3}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2\cancel{\Delta x}}{\cancel{\Delta x}} \\ &= \lim_{\Delta x \to 0} 2 \\ &= 2\end{aligned}$$

The slope of f at $(c, f(c)) = (2, 1)$ is $m = 2$, as shown in Figure 3.5. ■

NOTE In Example 1, the limit definition of the slope of f agrees with the definition of the slope of a line as discussed in Section 1.2. ■

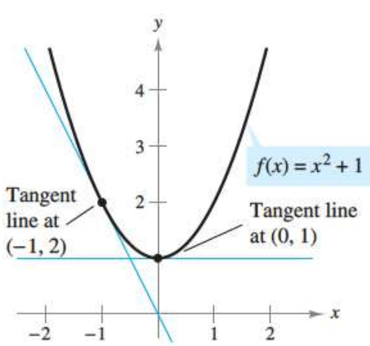
The graph of a linear function has the same slope at any point. This is not true of nonlinear functions, as shown in the following example.

EXAMPLE 2 Tangent Lines to the Graph of a Nonlinear Function

Find the slopes of the tangent lines to the graph of

$$f(x) = x^2 + 1$$

at the points (0, 1) and $(-1, 2)$, as shown in Figure 3.6.



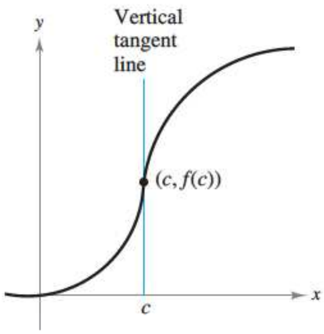
The slope of f at any point $(c, f(c))$ is $m = 2c$.
Figure 3.6

Solution Let $(c, f(c))$ represent an arbitrary point on the graph of f. Then the slope of the tangent line at $(c, f(c))$ is given by

$$\begin{aligned}\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{[(c + \Delta x)^2 + 1] - (c^2 + 1)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{c^2 + 2c(\Delta x) + (\Delta x)^2 + 1 - c^2 - 1}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2c(\Delta x) + (\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} (2c + \Delta x) \\ &= 2c.\end{aligned}$$

So, the slope at *any* point $(c, f(c))$ on the graph of f is $m = 2c$. At the point (0, 1), the slope is $m = 2(0) = 0$, and at $(-1, 2)$, the slope is $m = 2(-1) = -2$. ■

NOTE In Example 2, note that c is held constant in the limit process (as $\Delta x \to 0$). ■



The graph of f has a vertical tangent line at $(c, f(c))$.
Figure 3.7

The definition of a tangent line to a curve does not cover the possibility of a vertical tangent line. For vertical tangent lines, you can use the following definition. If f is continuous at c and

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = -\infty$$

the vertical line $x = c$ passing through $(c, f(c))$ is a **vertical tangent line** to the graph of f. For example, the function shown in Figure 3.7 has a vertical tangent line at $(c, f(c))$. If the domain of f is the closed interval $[a, b]$, you can extend the definition of a vertical tangent line to include the endpoints by considering continuity and limits from the right (for $x = a$) and from the left (for $x = b$).

The Derivative of a Function

You have now arrived at a crucial point in the study of calculus. The limit used to define the slope of a tangent line is also used to define one of the two fundamental operations of calculus—**differentiation.**

DEFINITION OF THE DERIVATIVE OF A FUNCTION

The **derivative** of f at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x.

Be sure you see that the derivative of a function of x is also a function of x. This "new" function gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, provided that the graph has a tangent line at this point.

The process of finding the derivative of a function is called **differentiation.** A function is **differentiable** at x if its derivative exists at x and is **differentiable on an open interval (a, b)** if it is differentiable at every point in the interval.

In addition to $f'(x)$, which is read as "f prime of x," other notations are used to denote the derivative of $y = f(x)$. The most common are

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad D_x[y].$$

Notation for derivatives

■ **FOR FURTHER INFORMATION** For more information on the crediting of mathematical discoveries to the first "discoverers," see the article "Mathematical Firsts—Who Done It?" by Richard H. Williams and Roy D. Mazzagatti in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

The notation dy/dx is read as "the derivative of y *with respect to* x" or simply "dy, dx." Using limit notation, you can write

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= f'(x). \end{aligned}$$

EXAMPLE 3 Finding the Derivative by the Limit Process

Find the derivative of $f(x) = x^3 + 2x$.

Solution

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} && \text{Definition of derivative} \\
&= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 + 2(x + \Delta x) - (x^3 + 2x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x + 2\Delta x - x^3 - 2x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}[3x^2 + 3x\Delta x + (\Delta x)^2 + 2]}{\cancel{\Delta x}} \\
&= \lim_{\Delta x \to 0} [3x^2 + 3x\Delta x + (\Delta x)^2 + 2] \\
&= 3x^2 + 2
\end{aligned}
$$

STUDY TIP When using the definition to find a derivative of a function, the key is to rewrite the difference quotient so that Δx does not occur as a factor of the denominator.

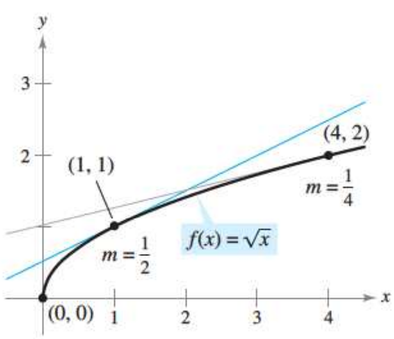
Remember that the derivative of a function f is itself a function, which can be used to find the slope of the tangent line at the point $(x, f(x))$ on the graph of f.

EXAMPLE 4 Using the Derivative to Find the Slope at a Point

Find $f'(x)$ for $f(x) = \sqrt{x}$. Then find the slopes of the graph of f at the points $(1, 1)$ and $(4, 2)$. Discuss the behavior of f at $(0, 0)$.

Solution Use the procedure for rationalizing numerators, as discussed in Section 2.3.

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} && \text{Definition of derivative} \\
&= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(\frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}\right)\left(\frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}\right) \\
&= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} \\
&= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x}\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} \\
&= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}, \quad x > 0
\end{aligned}
$$



The slope of f at $(x, f(x))$, $x > 0$, is $m = 1/\left(2\sqrt{x}\right)$.
Figure 3.8

At the point $(1, 1)$, the slope is $f'(1) = \frac{1}{2}$. At the point $(4, 2)$, the slope is $f'(4) = \frac{1}{4}$. See Figure 3.8. At the point $(0, 0)$, the slope is undefined. Moreover, the graph of f has a vertical tangent line at $(0, 0)$.

The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

In many applications, it is convenient to use a variable other than x as the independent variable, as shown in Example 5.

EXAMPLE 5 Finding the Derivative of a Function

Find the derivative with respect to t for the function $y = 2/t$.

Solution Considering $y = f(t)$, you obtain

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \qquad \text{Definition of derivative}$$

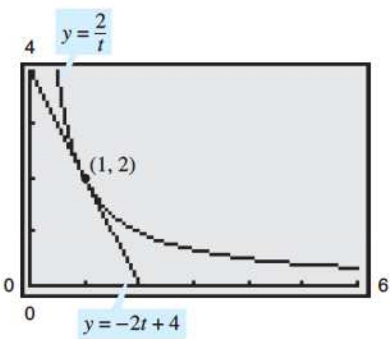
$$= \lim_{\Delta t \to 0} \frac{\dfrac{2}{t + \Delta t} - \dfrac{2}{t}}{\Delta t} \qquad f(t + \Delta t) = 2/(t + \Delta t) \text{ and } f(t) = 2/t$$

$$= \lim_{\Delta t \to 0} \frac{\dfrac{2t - 2(t + \Delta t)}{t(t + \Delta t)}}{\Delta t} \qquad \text{Combine fractions in numerator.}$$

$$= \lim_{\Delta t \to 0} \frac{-2\cancel{\Delta t}}{\cancel{\Delta t}(t)(t + \Delta t)} \qquad \text{Divide out common factor of } \Delta t.$$

$$= \lim_{\Delta t \to 0} \frac{-2}{t(t + \Delta t)} \qquad \text{Simplify.}$$

$$= -\frac{2}{t^2}. \qquad \text{Evaluate limit as } \Delta t \to 0.$$



At the point $(1, 2)$, the line $y = -2t + 4$ is tangent to the graph of $y = 2/t$.
Figure 3.9

TECHNOLOGY A graphing utility can be used to reinforce the result given in Example 5. For instance, using the formula $dy/dt = -2/t^2$, you know that the slope of the graph of $y = 2/t$ at the point $(1, 2)$ is $m = -2$. Using the point-slope form, you can find that the equation of the tangent line to the graph at $(1, 2)$ is

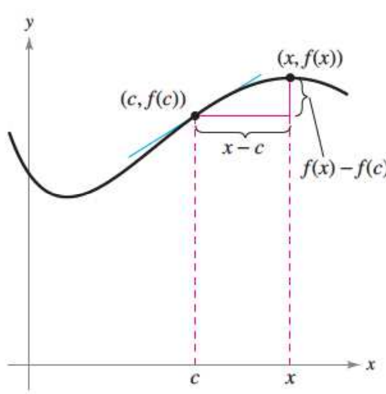
$$y - 2 = -2(t - 1) \quad \text{or} \quad y = -2t + 4$$

as shown in Figure 3.9.

Differentiability and Continuity

The following alternative limit form of the derivative is useful in investigating the relationship between differentiability and continuity. The derivative of f at c is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \text{Alternative form of derivative}$$

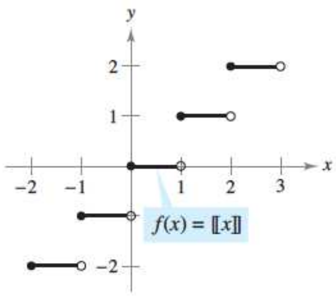


As x approaches c, the secant line approaches the tangent line.
Figure 3.10

provided this limit exists (see Figure 3.10). (A proof of the equivalence of this form is given in Appendix A.) Note that the existence of the limit in this alternative form requires that the one-sided limits

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

exist and are equal. These one-sided limits are called the **derivatives from the left and from the right,** respectively. It follows that f is **differentiable on the closed interval** $[a, b]$ if it is differentiable on (a, b) and if the derivative from the right at a and the derivative from the left at b both exist.



The greatest integer function is not differentiable at $x = 0$, because it is not continuous at $x = 0$.
Figure 3.11

If a function is not continuous at $x = c$, it is also not differentiable at $x = c$. For instance, the greatest integer function

$$f(x) = [\![x]\!]$$

is not continuous at $x = 0$, and so it is not differentiable at $x = 0$ (see Figure 3.11). You can verify this by observing that

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{[\![x]\!] - 0}{x} = \infty$$ Derivative from the left

and

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{[\![x]\!] - 0}{x} = 0.$$ Derivative from the right

(See Exercise 99.) Although it is true that differentiability implies continuity (as we will show in Theorem 3.1), the converse is not true. That is, it is possible for a function to be continuous at $x = c$ and *not* differentiable at $x = c$. Examples 6 and 7 illustrate this possibility.

EXAMPLE 6 A Graph with a Sharp Turn

The function

$$f(x) = |x - 2|$$

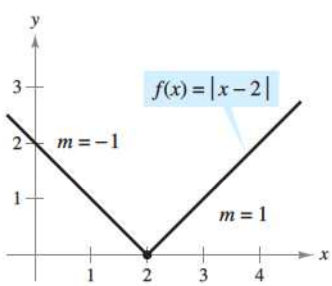
shown in Figure 3.12 is continuous at $x = 2$. However, the one-sided limits

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{|x - 2| - 0}{x - 2} = -1$$ Derivative from the left

and

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{|x - 2| - 0}{x - 2} = 1$$ Derivative from the right

are not equal. So, f is not differentiable at $x = 2$ and the graph of f does not have a tangent line at the point $(2, 0)$.



f is not differentiable at $x = 2$, because the derivatives from the left and from the right are not equal.
Figure 3.12

EXAMPLE 7 A Graph with a Vertical Tangent Line

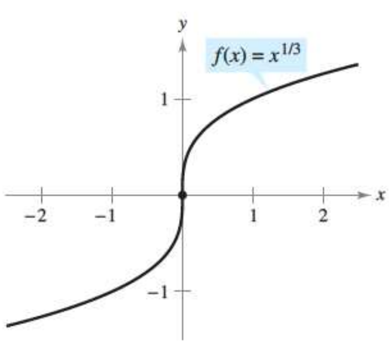
The function

$$f(x) = x^{1/3}$$

is continuous at $x = 0$, as shown in Figure 3.13. However, because the limit

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^{1/3} - 0}{x}$$
$$= \lim_{x \to 0} \frac{1}{x^{2/3}}$$
$$= \infty$$

is infinite, you can conclude that the tangent line is vertical at $x = 0$. So, f is not differentiable at $x = 0$. ■



f is not differentiable at $x = 0$, because f has a vertical tangent line at $x = 0$.
Figure 3.13

From Examples 6 and 7, you can see that a function is not differentiable at a point at which its graph has a sharp turn *or* a vertical tangent line.

TECHNOLOGY Some graphing utilities, such as *Maple*, *Mathematica*, and the *TI-89*, perform symbolic differentiation. Others perform *numerical differentiation* by finding values of derivatives using the formula

$$f'(x) \approx \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

where Δx is a small number such as 0.001. Can you see any problems with this definition? For instance, using this definition, what is the value of the derivative of $f(x) = |x|$ when $x = 0$?

THEOREM 3.1 DIFFERENTIABILITY IMPLIES CONTINUITY

If f is differentiable at $x = c$, then f is continuous at $x = c$.

PROOF You can prove that f is continuous at $x = c$ by showing that $f(x)$ approaches $f(c)$ as $x \to c$. To do this, use the differentiability of f at $x = c$ and consider the following limit.

$$\begin{aligned}\lim_{x\to c}[f(x) - f(c)] &= \lim_{x\to c}\left[(x - c)\left(\frac{f(x) - f(c)}{x - c}\right)\right]\\ &= \left[\lim_{x\to c}(x - c)\right]\left[\lim_{x\to c}\frac{f(x) - f(c)}{x - c}\right]\\ &= (0)[f'(c)]\\ &= 0\end{aligned}$$

Because the difference $f(x) - f(c)$ approaches zero as $x \to c$, you can conclude that $\lim_{x\to c} f(x) = f(c)$. So, f is continuous at $x = c$. ■

You can summarize the relationship between continuity and differentiability as follows.

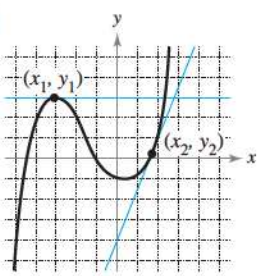
1. If a function is differentiable at $x = c$, then it is continuous at $x = c$. So, differentiability implies continuity.
2. It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, continuity does not imply differentiability (see Example 6).

3.1 Exercises

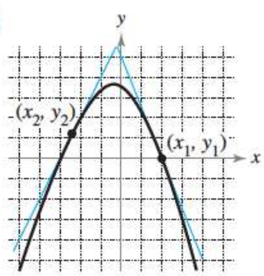
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, estimate the slope of the graph at the points (x_1, y_1) and (x_2, y_2).

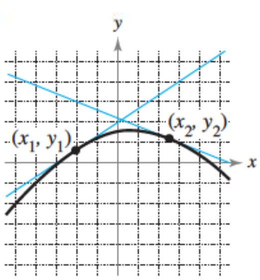
1. (a)



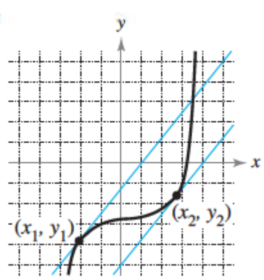
(b)



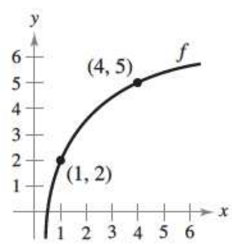
2. (a)



(b)



In Exercises 3 and 4, use the graph shown in the figure. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



3. Identify or sketch each of the quantities on the figure.

(a) $f(1)$ and $f(4)$ (b) $f(4) - f(1)$

(c) $y = \dfrac{f(4) - f(1)}{4 - 1}(x - 1) + f(1)$

4. Insert the proper inequality symbol ($<$ or $>$) between the given quantities.

(a) $\dfrac{f(4) - f(1)}{4 - 1} \quad \square \quad \dfrac{f(4) - f(3)}{4 - 3}$

(b) $\dfrac{f(4) - f(1)}{4 - 1} \quad \square \quad f'(1)$

In Exercises 5–10, find the slope of the tangent line to the graph of the function at the given point.

5. $f(x) = 3 - 5x, \quad (-1, 8)$

6. $g(x) = \frac{3}{2}x + 1, \quad (-2, -2)$

7. $g(x) = x^2 - 9, \quad (2, -5)$

8. $g(x) = 6 - x^2, \quad (1, 5)$

9. $f(t) = 3t - t^2, \quad (0, 0)$

10. $h(t) = t^2 + 3, \quad (-2, 7)$

In Exercises 11–24, find the derivative by the limit process.

11. $f(x) = 7$

12. $g(x) = -3$

13. $f(x) = -5x$

14. $f(x) = 3x + 2$

15. $h(s) = 3 + \frac{2}{3}s$

16. $f(x) = 8 - \frac{1}{5}x$

17. $f(x) = x^2 + x - 3$

18. $f(x) = 2 - x^2$

19. $f(x) = x^3 - 12x$

20. $f(x) = x^3 + x^2$

21. $f(x) = \dfrac{1}{x - 1}$

22. $f(x) = \dfrac{1}{x^2}$

23. $f(x) = \sqrt{x + 4}$

24. $f(x) = \dfrac{4}{\sqrt{x}}$

In Exercises 25–32, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

25. $f(x) = x^2 + 3, \quad (1, 4)$

26. $f(x) = x^2 + 3x + 4, \quad (-2, 2)$

27. $f(x) = x^3, \quad (2, 8)$

28. $f(x) = x^3 + 1, \quad (1, 2)$

29. $f(x) = \sqrt{x}, \quad (1, 1)$

30. $f(x) = \sqrt{x - 1}, \quad (5, 2)$

31. $f(x) = x + \dfrac{4}{x}, \quad (4, 5)$

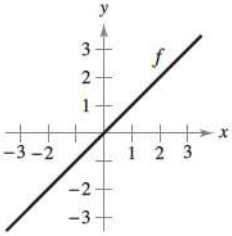
32. $f(x) = \dfrac{1}{x + 1}, \quad (0, 1)$

In Exercises 33–38, find an equation of the line that is tangent to the graph of f *and* parallel to the given line.

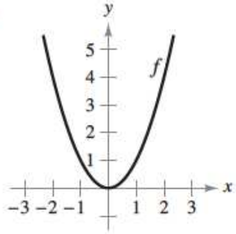
	Function	*Line*
33.	$f(x) = x^2$	$2x - y + 1 = 0$
34.	$f(x) = 2x^2$	$4x + y + 3 = 0$
35.	$f(x) = x^3$	$3x - y + 1 = 0$
36.	$f(x) = x^3 + 2$	$3x - y - 4 = 0$
37.	$f(x) = \dfrac{1}{\sqrt{x}}$	$x + 2y - 6 = 0$
38.	$f(x) = \dfrac{1}{\sqrt{x - 1}}$	$x + 2y + 7 = 0$

In Exercises 39–42, the graph of f is given. Select the graph of f'.

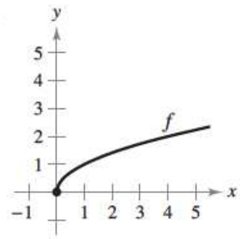
39.



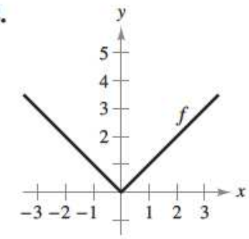
40.



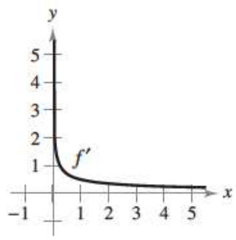
41.



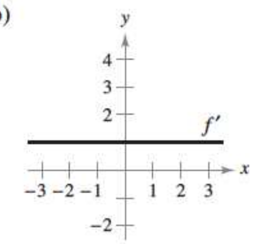
42.



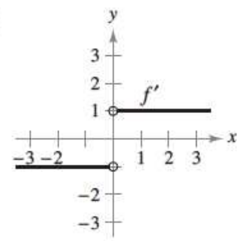
(a)



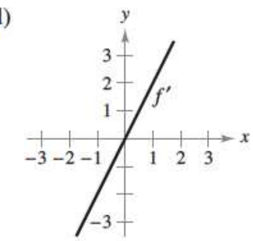
(b)



(c)



(d)



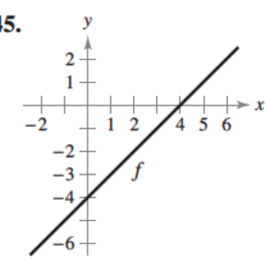
43. The tangent line to the graph of $y = g(x)$ at the point $(4, 5)$ passes through the point $(7, 0)$. Find $g(4)$ and $g'(4)$.

44. The tangent line to the graph of $y = h(x)$ at the point $(-1, 4)$ passes through the point $(3, 6)$. Find $h(-1)$ and $h'(-1)$.

WRITING ABOUT CONCEPTS

In Exercises 45–50, sketch the graph of f'. Explain how you found your answer.

45.



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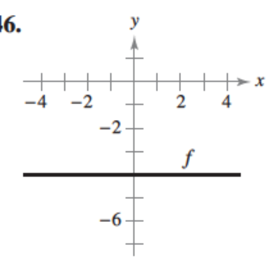
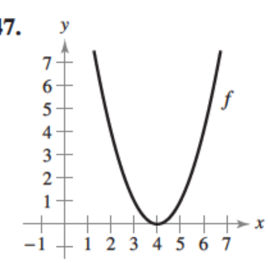
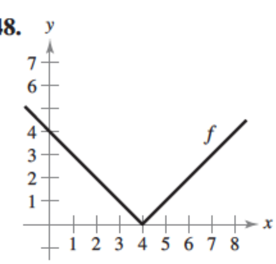


47.

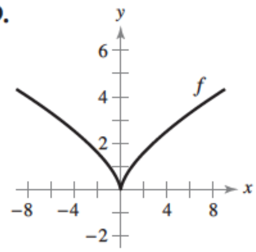


48.

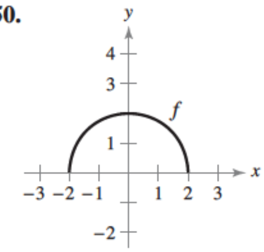


WRITING ABOUT CONCEPTS (continued)

49.



50.



51. Sketch a graph of a function whose derivative is always negative.

52. Sketch a graph of a function whose derivative is always positive.

In Exercises 53–56, the limit represents $f'(c)$ for a function f and a number c. Find f and c.

53. $\lim_{\Delta x \to 0} \dfrac{[5 - 3(1 + \Delta x)] - 2}{\Delta x}$

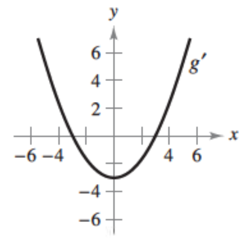
54. $\lim_{\Delta x \to 0} \dfrac{(-2 + \Delta x)^3 + 8}{\Delta x}$

55. $\lim_{x \to 6} \dfrac{-x^2 + 36}{x - 6}$

56. $\lim_{x \to 9} \dfrac{2\sqrt{x} - 6}{x - 9}$

CAPSTONE

64. The figure shows the graph of g'.



(a) $g'(0) = \square$ (b) $g'(3) = \square$

(c) What can you conclude about the graph of g knowing that $g'(1) = -\frac{8}{3}$?

(d) What can you conclude about the graph of g knowing that $g'(-4) = \frac{7}{3}$?

(e) Is $g(6) - g(4)$ positive or negative? Explain.

(f) Is it possible to find $g(2)$ from the graph? Explain.

In Exercises 57–59, identify a function f that has the given characteristics. Then sketch the function.

57. $f(0) = 2;$ $f'(x) = -3, -\infty < x < \infty$

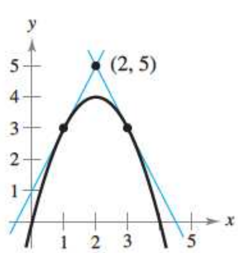
58. $f(0) = 4; f'(0) = 0;$ $f'(x) < 0$ for $x < 0;$ $f'(x) > 0$ for $x > 0$

59. $f(0) = 0; f'(0) = 0; f'(x) > 0$ for $x \neq 0$

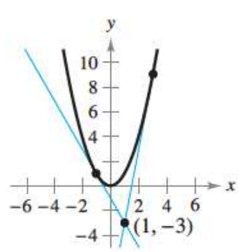
60. Assume that $f'(c) = 3$. Find $f'(-c)$ if (a) f is an odd function and (b) f is an even function.

In Exercises 61 and 62, find equations of the two tangent lines to the graph of f that pass through the given point.

61. $f(x) = 4x - x^2$



62. $f(x) = x^2$



63. *Graphical Reasoning* Use a graphing utility to graph each function and its tangent lines at $x = -1$, $x = 0$, and $x = 1$. Based on the results, determine whether the slopes of tangent lines to the graph of a function at different values of x are always distinct.

(a) $f(x) = x^2$ (b) $g(x) = x^3$

65. *Graphical Analysis* Consider the function $f(x) = \frac{1}{2}x^2$.

(a) Use a graphing utility to graph the function and estimate the values of $f'(0), f'\left(\frac{1}{2}\right), f'(1)$, and $f'(2)$.

(b) Use your results from part (a) to determine the values of $f'\left(-\frac{1}{2}\right), f'(-1)$, and $f'(-2)$.

(c) Sketch a possible graph of f'.

(d) Use the definition of derivative to find $f'(x)$.

66. *Graphical Analysis* Consider the function $f(x) = \frac{1}{3}x^3$.

(a) Use a graphing utility to graph the function and estimate the values of $f'(0), f'\left(\frac{1}{2}\right), f'(1), f'(2)$, and $f'(3)$.

(b) Use your results from part (a) to determine the values of $f'\left(-\frac{1}{2}\right), f'(-1), f'(-2)$, and $f'(-3)$.

(c) Sketch a possible graph of f'.

(d) Use the definition of derivative to find $f'(x)$.

Graphical Reasoning **In Exercises 67 and 68, use a graphing utility to graph the functions f and g in the same viewing window where**

$$g(x) = \frac{f(x + 0.01) - f(x)}{0.01}.$$

Label the graphs and describe the relationship between them.

67. $f(x) = 2x - x^2$

68. $f(x) = 3\sqrt{x}$

In Exercises 69 and 70, evaluate $f(2)$ and $f(2.1)$ and use the results to approximate $f'(2)$.

69. $f(x) = x(4 - x)$

70. $f(x) = \frac{1}{4}x^3$

Graphical Reasoning **In Exercises 71 and 72, use a graphing utility to graph the function and its derivative in the same viewing window. Label the graphs and describe the relationship between them.**

71. $f(x) = \dfrac{1}{\sqrt{x}}$

72. $f(x) = \dfrac{x^3}{4} - 3x$

In Exercises 73–82, use the alternative form of the derivative to find the derivative at $x = c$ (if it exists).

73. $f(x) = x^2 - 5, \quad c = 3$

74. $g(x) = x(x - 1), \quad c = 1$

75. $f(x) = x^3 + 2x^2 + 1, \quad c = -2$

76. $f(x) = x^3 + 6x, \quad c = 2$

77. $g(x) = \sqrt{|x|}, \quad c = 0$

78. $f(x) = 2/x, \quad c = 5$

79. $f(x) = (x - 6)^{2/3}, \quad c = 6$

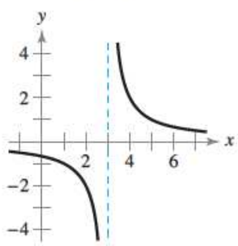
80. $g(x) = (x + 3)^{1/3}, \quad c = -3$

81. $h(x) = |x + 7|, \quad c = -7$

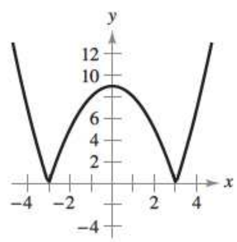
82. $f(x) = |x - 6|, \quad c = 6$

In Exercises 83–88, describe the x-values at which f is differentiable.

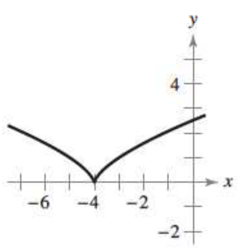
83. $f(x) = \dfrac{2}{x - 3}$



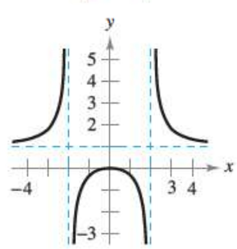
84. $f(x) = |x^2 - 9|$



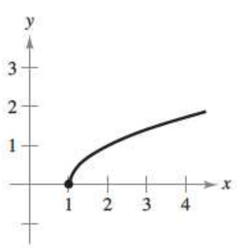
85. $f(x) = (x + 4)^{2/3}$



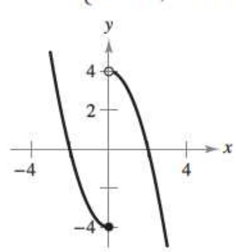
86. $f(x) = \dfrac{x^2}{x^2 - 4}$



87. $f(x) = \sqrt{x - 1}$



88. $f(x) = \begin{cases} x^2 - 4, & x \le 0 \\ 4 - x^2, & x > 0 \end{cases}$



Graphical Analysis **In Exercises 89–92, use a graphing utility to graph the function and find the x-values at which f is differentiable.**

89. $f(x) = |x - 5|$

90. $f(x) = \dfrac{4x}{x - 3}$

91. $f(x) = x^{2/5}$

92. $f(x) = \begin{cases} x^3 - 3x^2 + 3x, & x \le 1 \\ x^2 - 2x, & x > 1 \end{cases}$

In Exercises 93–96, find the derivatives from the left and from the right at $x = 1$ (if they exist). Is the function differentiable at $x = 1$?

93. $f(x) = |x - 1|$

94. $f(x) = \sqrt{1 - x^2}$

95. $f(x) = \begin{cases} (x - 1)^3, & x \le 1 \\ (x - 1)^2, & x > 1 \end{cases}$

96. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

In Exercises 97 and 98, determine whether the function is differentiable at $x = 2$.

97. $f(x) = \begin{cases} x^2 + 1, & x \le 2 \\ 4x - 3, & x > 2 \end{cases}$

98. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \ge 2 \end{cases}$

99. Use a graphing utility to graph $g(x) = [\![x]\!]/x$. Then let $f(x) = [\![x]\!]$ and show that

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \infty \text{ and } \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = 0.$$

Is f differentiable? Explain.

100. *Conjecture* Consider the functions $f(x) = x^2$ and $g(x) = x^3$.

(a) Graph f and f' on the same set of axes.

(b) Graph g and g' on the same set of axes.

(c) Identify a pattern between f and g and their respective derivatives. Use the pattern to make a conjecture about $h'(x)$ if $h(x) = x^n$, where n is an integer and $n \ge 2$.

(d) Find $f'(x)$ if $f(x) = x^4$. Compare the result with the conjecture in part (c). Is this a proof of your conjecture? Explain.

True or False? **In Exercises 101–104, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

101. The slope of the tangent line to the differentiable function f at the point $(2, f(2))$ is $\dfrac{f(2 + \Delta x) - f(2)}{\Delta x}$.

102. If a function is continuous at a point, then it is differentiable at that point.

103. If a function has derivatives from both the right and the left at a point, then it is differentiable at that point.

104. If a function is differentiable at a point, then it is continuous at that point.

105. Let $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$ and $g(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$.

Show that f is continuous, but not differentiable, at $x = 0$. Show that g is differentiable at 0, and find $g'(0)$.

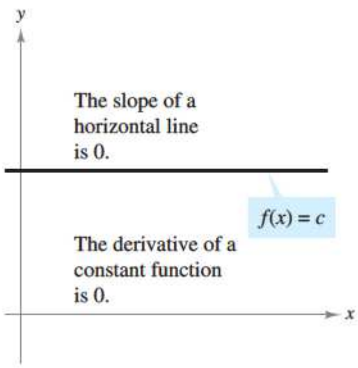
106. *Writing* Use a graphing utility to graph the two functions $f(x) = x^2 + 1$ and $g(x) = |x| + 1$ in the same viewing window. Use the *zoom* and *trace* features to analyze the graphs near the point $(0, 1)$. What do you observe? Which function is differentiable at this point? Write a short paragraph describing the geometric significance of differentiability at a point.

3.2 Basic Differentiation Rules and Rates of Change

- Find the derivative of a function using the Constant Rule.
- Find the derivative of a function using the Power Rule.
- Find the derivative of a function using the Constant Multiple Rule.
- Find the derivative of a function using the Sum and Difference Rules.
- Find the derivatives of the sine, cosine, and exponential functions.
- Use derivatives to find rates of change.

The Constant Rule

In Section 3.1 you used the limit definition to find derivatives. In this and the next two sections you will be introduced to several "differentiation rules" that allow you to find derivatives without the *direct* use of the limit definition.



Notice that the Constant Rule is equivalent to saying that the slope of a horizontal line is 0. This demonstrates the relationship between slope and derivative.
Figure 3.14

THEOREM 3.2 THE CONSTANT RULE

The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0.$$

(See Figure 3.14.)

PROOF Let $f(x) = c$. Then, by the limit definition of the derivative,

$$\begin{aligned}\frac{d}{dx}[c] &= f'(x)\\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x}\\ &= \lim_{\Delta x \to 0} 0\\ &= 0.\end{aligned}$$ ■

EXAMPLE 1 Using the Constant Rule

	Function	*Derivative*
a.	$y = 7$	$dy/dx = 0$
b.	$f(x) = 0$	$f'(x) = 0$
c.	$s(t) = -3$	$s'(t) = 0$
d.	$y = k\pi^2$, k is constant	$y' = 0$

■

EXPLORATION

Writing a Conjecture Use the definition of the derivative given in Section 3.1 to find the derivative of each of the following. What patterns do you see? Use your results to write a conjecture about the derivative of $f(x) = x^n$.

a. $f(x) = x^1$ **b.** $f(x) = x^2$ **c.** $f(x) = x^3$
d. $f(x) = x^4$ **e.** $f(x) = x^{1/2}$ **f.** $f(x) = x^{-1}$

The Power Rule

Before proving the next rule, review the procedure for expanding a binomial.

$$(x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$$
$$(x + \Delta x)^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$
$$(x + \Delta x)^4 = x^4 + 4x^3\Delta x + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4$$

The general binomial expansion for a positive integer n is

$$(x + \Delta x)^n = x^n + nx^{n-1}(\Delta x) + \underbrace{\frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n}_{(\Delta x)^2 \text{ is a factor of these terms.}}.$$

This binomial expansion is used in proving a special case of the Power Rule.

NOTE From Example 7 in Section 3.1, you know that the function $f(x) = x^{1/3}$ is defined at $x = 0$, but is not differentiable at $x = 0$. This is because $x^{-2/3}$ is not defined on an interval containing 0.

THEOREM 3.3 THE POWER RULE

If n is a real number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing 0.

PROOF If n is a positive integer greater than 1, then the binomial expansion produces the following.

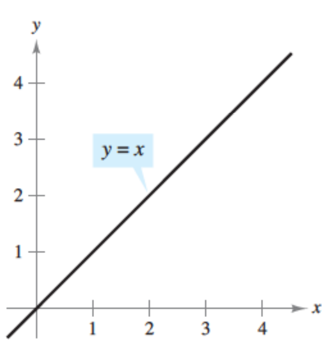
$$\begin{aligned}
\frac{d}{dx}[x^n] &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}(\Delta x) + \frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[nx^{n-1} + \frac{n(n-1)x^{n-2}}{2}(\Delta x) + \cdots + (\Delta x)^{n-1} \right] \\
&= nx^{n-1} + 0 + \cdots + 0 \\
&= nx^{n-1}.
\end{aligned}$$

This proves the case for which n is a positive integer greater than 1. It is left to you to prove the case for $n = 1$. Example 7 in Section 3.3 proves the case for which n is a negative integer. The cases for which n is rational and n is irrational are left as an exercise (see Section 3.5, Exercise 100). ■

When using the Power Rule, the case for which $n = 1$ is best thought of as a separate differentiation rule. That is,

$$\boxed{\frac{d}{dx}[x] = 1.}$$

Power Rule when $n = 1$



The slope of the line $y = x$ is 1.
Figure 3.15

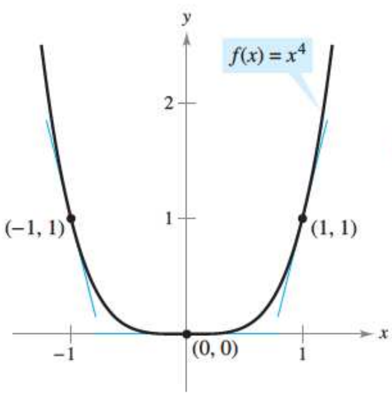
This rule is consistent with the fact that the slope of the line $y = x$ is 1, as shown in Figure 3.15.

EXAMPLE 2 Using the Power Rule

Function	*Derivative*
a. $f(x) = x^3$	$f'(x) = 3x^2$
b. $g(x) = \sqrt[3]{x}$	$g'(x) = \dfrac{d}{dx}[x^{1/3}] = \dfrac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$
c. $y = \dfrac{1}{x^2}$	$\dfrac{dy}{dx} = \dfrac{d}{dx}[x^{-2}] = (-2)x^{-3} = -\dfrac{2}{x^3}$ ■

In Example 2(c), note that *before* differentiating, $1/x^2$ was rewritten as x^{-2}. Rewriting is the first step in *many* differentiation problems.

Given:		Rewrite:		Differentiate:		Simplify:
$y = \dfrac{1}{x^2}$	⇨	$y = x^{-2}$	⇨	$\dfrac{dy}{dx} = (-2)x^{-3}$	⇨	$\dfrac{dy}{dx} = -\dfrac{2}{x^3}$



The slope of a graph at a point is the value of the derivative at that point.
Figure 3.16

EXAMPLE 3 Finding the Slope of a Graph

Find the slope of the graph of $f(x) = x^4$ when

a. $x = -1$ **b.** $x = 0$ **c.** $x = 1$.

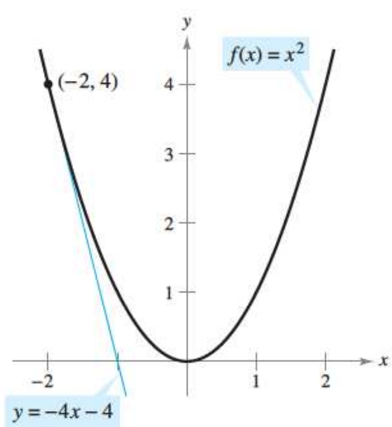
Solution The derivative of f is $f'(x) = 4x^3$.

a. When $x = -1$, the slope is $f'(-1) = 4(-1)^3 = -4$. Slope is negative.

b. When $x = 0$, the slope is $f'(0) = 4(0)^3 = 0$. Slope is zero.

c. When $x = 1$, the slope is $f'(1) = 4(1)^3 = 4$. Slope is positive.

In Figure 3.16, note that the slope of the graph is negative at the point $(-1, 1)$, the slope is zero at the point $(0, 0)$, and the slope is positive at the point $(1, 1)$.



The line $y = -4x - 4$ is tangent to the graph of $f(x) = x^2$ at the point $(-2, 4)$.
Figure 3.17

EXAMPLE 4 Finding an Equation of a Tangent Line

Find an equation of the tangent line to the graph of $f(x) = x^2$ when $x = -2$.

Solution To find the *point* on the graph of f, evaluate the original function at $x = -2$.

$$(-2, f(-2)) = (-2, 4) \qquad \text{Point on graph}$$

To find the *slope* of the graph when $x = -2$, evaluate the derivative, $f'(x) = 2x$, at $x = -2$.

$$m = f'(-2) = -4 \qquad \text{Slope of graph at } (-2, 4)$$

Now, using the point-slope form of the equation of a line, you can write

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point-slope form} \\ y - 4 &= -4[x - (-2)] && \text{Substitute for } y_1, m, \text{ and } x_1. \\ y &= -4x - 4. && \text{Simplify.} \end{aligned}$$

(See Figure 3.17.) ■

The Constant Multiple Rule

THEOREM 3.4 THE CONSTANT MULTIPLE RULE

If f is a differentiable function and c is a real number, then cf is also differentiable and $\frac{d}{dx}[cf(x)] = cf'(x)$.

PROOF

$$\begin{aligned}
\frac{d}{dx}[cf(x)] &= \lim_{\Delta x \to 0} \frac{cf(x+\Delta x) - cf(x)}{\Delta x} && \text{Definition of derivative} \\
&= \lim_{\Delta x \to 0} c\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right] \\
&= c\left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right] && \text{Apply Theorem 2.2.} \\
&= cf'(x)
\end{aligned}$$

Informally, the Constant Multiple Rule states that constants can be factored out of the differentiation process, even if the constants appear in the denominator.

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)] = cf'(x)$$

$$\begin{aligned}
\frac{d}{dx}\left[\frac{f(x)}{c}\right] &= \frac{d}{dx}\left[\left(\frac{1}{c}\right)f(x)\right] \\
&= \left(\frac{1}{c}\right)\frac{d}{dx}[f(x)] = \left(\frac{1}{c}\right)f'(x)
\end{aligned}$$

EXAMPLE 5 Using the Constant Multiple Rule

Function	*Derivative*
a. $y = \frac{2}{x}$	$\frac{dy}{dx} = \frac{d}{dx}[2x^{-1}] = 2\frac{d}{dx}[x^{-1}] = 2(-1)x^{-2} = -\frac{2}{x^2}$
b. $f(t) = \frac{4t^2}{5}$	$f'(t) = \frac{d}{dt}\left[\frac{4}{5}t^2\right] = \frac{4}{5}\frac{d}{dt}[t^2] = \frac{4}{5}(2t) = \frac{8}{5}t$
c. $y = 2\sqrt{x}$	$\frac{dy}{dx} = \frac{d}{dx}[2x^{1/2}] = 2\left(\frac{1}{2}x^{-1/2}\right) = x^{-1/2} = \frac{1}{\sqrt{x}}$
d. $y = \frac{1}{2\sqrt[3]{x^2}}$	$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{2}x^{-2/3}\right] = \frac{1}{2}\left(-\frac{2}{3}\right)x^{-5/3} = -\frac{1}{3x^{5/3}}$
e. $y = -\frac{3x}{2}$	$y' = \frac{d}{dx}\left[-\frac{3}{2}x\right] = -\frac{3}{2}(1) = -\frac{3}{2}$

The Constant Multiple Rule and the Power Rule can be combined into one rule. The combination rule is

$$\frac{d}{dx}[cx^n] = cnx^{n-1}.$$

EXAMPLE 6 Using Parentheses When Differentiating

	Original Function	*Rewrite*	*Differentiate*	*Simplify*
a.	$y = \dfrac{5}{2x^3}$	$y = \dfrac{5}{2}(x^{-3})$	$y' = \dfrac{5}{2}(-3x^{-4})$	$y' = -\dfrac{15}{2x^4}$
b.	$y = \dfrac{5}{(2x)^3}$	$y = \dfrac{5}{8}(x^{-3})$	$y' = \dfrac{5}{8}(-3x^{-4})$	$y' = -\dfrac{15}{8x^4}$
c.	$y = \dfrac{7}{3x^{-2}}$	$y = \dfrac{7}{3}(x^2)$	$y' = \dfrac{7}{3}(2x)$	$y' = \dfrac{14x}{3}$
d.	$y = \dfrac{7}{(3x)^{-2}}$	$y = 63(x^2)$	$y' = 63(2x)$	$y' = 126x$

The Sum and Difference Rules

THEOREM 3.5 THE SUM AND DIFFERENCE RULES

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \qquad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \qquad \text{Difference Rule}$$

PROOF A proof of the Sum Rule follows from Theorem 2.2. (The Difference Rule can be proved in a similar way.)

$$\begin{aligned}
\frac{d}{dx}[f(x) + g(x)] &= \lim_{\Delta x \to 0} \frac{[f(x + \Delta x) + g(x + \Delta x)] - [f(x) + g(x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) + g(x + \Delta x) - f(x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x}\right] \\
&= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\
&= f'(x) + g'(x)
\end{aligned}$$

The Sum and Difference Rules can be extended to any finite number of functions. For instance, if $F(x) = f(x) + g(x) - h(x)$, then $F'(x) = f'(x) + g'(x) - h'(x)$.

EXPLORATION

Use a graphing utility to graph the function

$$f(x) = \frac{\sin(x + \Delta x) - \sin x}{\Delta x}$$

for $\Delta x = 0.01$. What does this function represent? Compare this graph with that of the cosine function. What do you think the derivative of the sine function equals?

EXAMPLE 7 Using the Sum and Difference Rules

	Function	*Derivative*
a.	$f(x) = x^3 - 4x + 5$	$f'(x) = 3x^2 - 4$
b.	$g(x) = -\dfrac{x^4}{2} + 3x^3 - 2x$	$g'(x) = -2x^3 + 9x^2 - 2$

FOR FURTHER INFORMATION For the outline of a geometric proof of the derivatives of the sine and cosine functions, see the article "The Spider's Spacewalk Derivation of sin′ and cos′" by Tim Hesterberg in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

Derivatives of Sine and Cosine Functions

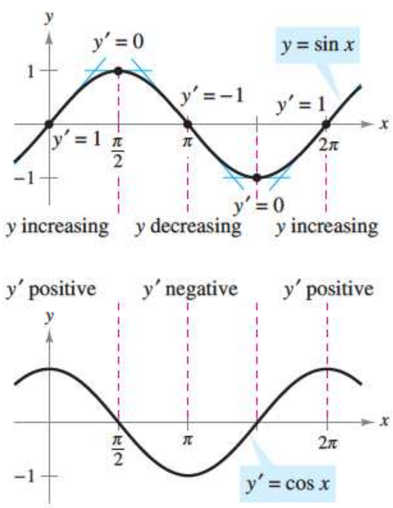
In Section 2.3, you studied the following limits.

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1 \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0$$

These two limits can be used to prove differentiation rules for the sine and cosine functions. (The derivatives of the other four trigonometric functions are discussed in Section 3.3.)

THEOREM 3.6 DERIVATIVES OF SINE AND COSINE FUNCTIONS

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$



The derivative of the sine function is the cosine function.
Figure 3.18

PROOF

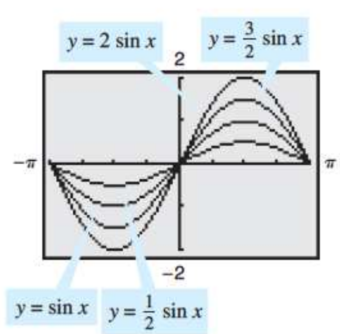
$$\begin{aligned}
\frac{d}{dx}[\sin x] &= \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} && \text{Definition of derivative} \\
&= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cos x \sin \Delta x - (\sin x)(1 - \cos \Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[(\cos x)\left(\frac{\sin \Delta x}{\Delta x}\right) - (\sin x)\left(\frac{1 - \cos \Delta x}{\Delta x}\right) \right] \\
&= \cos x \left(\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \right) - \sin x \left(\lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} \right) \\
&= (\cos x)(1) - (\sin x)(0) \\
&= \cos x
\end{aligned}$$

This differentiation rule is shown graphically in Figure 3.18. Note that for each x, the *slope* of the sine curve is equal to the value of the cosine. The proof of the second rule is left as an exercise (see Exercise 124). ■

EXAMPLE 8 Derivatives Involving Sines and Cosines

Function	*Derivative*
a. $y = 2 \sin x$	$y' = 2 \cos x$
b. $y = \dfrac{\sin x}{2} = \dfrac{1}{2} \sin x$	$y' = \dfrac{1}{2} \cos x = \dfrac{\cos x}{2}$
c. $y = x + \cos x$	$y' = 1 - \sin x$

■



$\frac{d}{dx}[a \sin x] = a \cos x$
Figure 3.19

TECHNOLOGY A graphing utility can provide insight into the interpretation of a derivative. For instance, Figure 3.19 shows the graphs of

$$y = a \sin x$$

for $a = \frac{1}{2}, 1, \frac{3}{2}$, and 2. Estimate the slope of each graph at the point $(0, 0)$. Then verify your estimates analytically by evaluating the derivative of each function when $x = 0$.

EXPLORATION

Use a graphing utility to graph the function

$$f(x) = \frac{e^{x+\Delta x} - e^x}{\Delta x}$$

for $\Delta x = 0.01$. What does this function represent? Compare this graph with that of the exponential function. What do you think the derivative of the exponential function equals?

Derivatives of Exponential Functions

One of the most intriguing (and useful) characteristics of the natural exponential function is that *it is its own derivative*. Consider the following.

Let $f(x) = e^x$.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{e^x(e^{\Delta x} - 1)}{\Delta x} \end{aligned}$$

The definition of e

$$\lim_{\Delta x \to 0} (1 + \Delta x)^{1/\Delta x} = e$$

tells you that for small values of Δx, you have $e \approx (1 + \Delta x)^{1/\Delta x}$, which implies that $e^{\Delta x} \approx 1 + \Delta x$. Replacing $e^{\Delta x}$ by this approximation produces the following.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{e^x[e^{\Delta x} - 1]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{e^x[(1+\Delta x) - 1]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{e^x \Delta x}{\Delta x} \\ &= e^x \end{aligned}$$

This result is stated in the next theorem.

STUDY TIP The key to the formula for the derivative of $f(x) = e^x$ is the limit

$$\lim_{x \to 0} (1 + x)^{1/x} = e.$$

This important limit was introduced on page 51 and formalized later on page 85. It is used to conclude that for $\Delta x \approx 0$,

$$(1 + \Delta x)^{1/\Delta x} \approx e.$$

THEOREM 3.7 DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

$$\frac{d}{dx}[e^x] = e^x$$

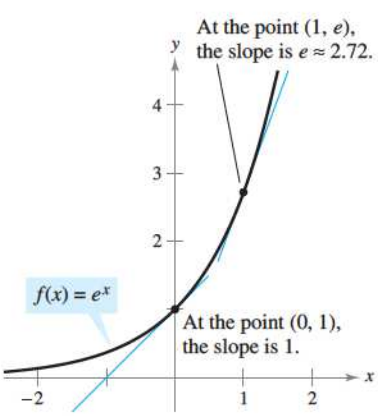


Figure 3.20

You can interpret Theorem 3.7 graphically by saying that the slope of the graph of $f(x) = e^x$ at any point (x, e^x) is equal to the y-coordinate of the point, as shown in Figure 3.20.

EXAMPLE 9 Derivatives of Exponential Functions

Find the derivative of each function.

a. $f(x) = 3e^x$ **b.** $f(x) = x^2 + e^x$ **c.** $f(x) = \sin x - e^x$

Solution

a. $f'(x) = 3\dfrac{d}{dx}[e^x] = 3e^x$

b. $f'(x) = \dfrac{d}{dx}[x^2] + \dfrac{d}{dx}[e^x] = 2x + e^x$

c. $f'(x) = \dfrac{d}{dx}[\sin x] - \dfrac{d}{dx}[e^x] = \cos x - e^x$ ■

Rates of Change

You have seen how the derivative is used to determine slope. The derivative can also be used to determine the rate of change of one variable with respect to another. Applications involving rates of change occur in a wide variety of fields. A few examples are population growth rates, production rates, water flow rates, velocity, and acceleration.

A common use for rate of change is to describe the motion of an object moving in a straight line. In such problems, it is customary to use either a horizontal or a vertical line with a designated origin to represent the line of motion. On such lines, movement to the right (or upward) is considered to be in the positive direction, and movement to the left (or downward) is considered to be in the negative direction.

The function s that gives the position (relative to the origin) of an object as a function of time t is called a **position function.** If, over a period of time Δt, the object changes its position by the amount $\Delta s = s(t + \Delta t) - s(t)$, then, by the familiar formula

$$\text{Rate} = \frac{\text{distance}}{\text{time}}$$

the **average velocity** is

$$\frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta s}{\Delta t}. \qquad \text{Average velocity}$$

EXAMPLE 10 Finding Average Velocity of a Falling Object

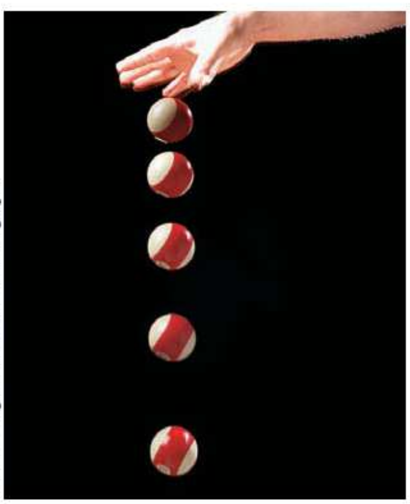
If a billiard ball is dropped from a height of 100 feet, its height s at time t is given by the position function

$$s = -16t^2 + 100 \qquad \text{Position function}$$

where s is measured in feet and t is measured in seconds. Find the average velocity over each of the following time intervals.

a. $[1, 2]$ **b.** $[1, 1.5]$ **c.** $[1, 1.1]$

Solution



Richard Megna/Fundamental Photographs

Time-lapse photograph of a free-falling billiard ball

a. For the interval $[1, 2]$, the object falls from a height of $s(1) = -16(1)^2 + 100 = 84$ feet to a height of $s(2) = -16(2)^2 + 100 = 36$ feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{36 - 84}{2 - 1} = \frac{-48}{1} = -48 \text{ feet per second.}$$

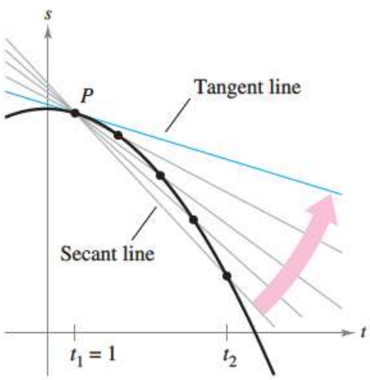
b. For the interval $[1, 1.5]$, the object falls from a height of 84 feet to a height of 64 feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{64 - 84}{1.5 - 1} = \frac{-20}{0.5} = -40 \text{ feet per second.}$$

c. For the interval $[1, 1.1]$, the object falls from a height of 84 feet to a height of 80.64 feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{80.64 - 84}{1.1 - 1} = \frac{-3.36}{0.1} = -33.6 \text{ feet per second.}$$

Note that the average velocities are *negative*, indicating that the object is moving downward. ■



The average velocity between t_1 and t_2 is the slope of the secant line, and the instantaneous velocity at t_1 is the slope of the tangent line.
Figure 3.21

Suppose that in Example 10 you wanted to find the *instantaneous* velocity (or simply the velocity) of the object when $t = 1$. Just as you can approximate the slope of the tangent line by calculating the slope of the secant line, you can approximate the velocity at $t = 1$ by calculating the average velocity over a small interval $[1, 1 + \Delta t]$ (see Figure 3.21). By taking the limit as Δt approaches zero, you obtain the velocity when $t = 1$. Try doing this—you will find that the velocity when $t = 1$ is -32 feet per second.

In general, if $s = s(t)$ is the position function for an object moving along a straight line, the **velocity** of the object at time t is

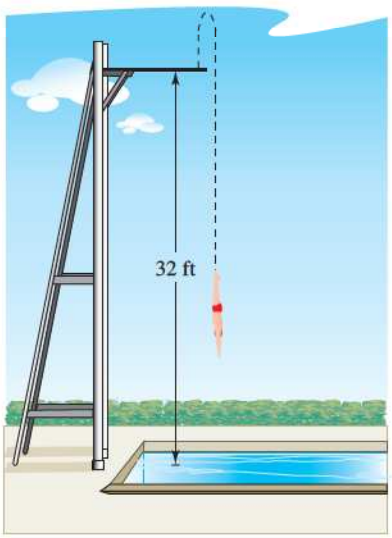
$$v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t).$$ Velocity function

In other words, the velocity function is the derivative of the position function. Velocity can be negative, zero, or positive. The **speed** of an object is the absolute value of its velocity. Speed cannot be negative.

The position of a free-falling object (neglecting air resistance) under the influence of gravity can be represented by the equation

$$s(t) = \frac{1}{2}gt^2 + v_0t + s_0$$ Position function

where s_0 is the initial height of the object, v_0 is the initial velocity of the object, and g is the acceleration due to gravity. On Earth, the value of g is approximately -32 feet per second per second or -9.8 meters per second per second.



Velocity is positive when an object is rising, and is negative when an object is falling. Notice that the diver moves upward for the first half-second because the velocity is positive for $0 < t < \frac{1}{2}$. When the velocity is 0, the diver has reached the maximum height of the dive.
Figure 3.22

EXAMPLE 11 Using the Derivative to Find Velocity

At time $t = 0$, a diver jumps from a platform diving board that is 32 feet above the water (see Figure 3.22). The position of the diver is given by

$$s(t) = -16t^2 + 16t + 32$$ Position function

where s is measured in feet and t is measured in seconds.

a. When does the diver hit the water?

b. What is the diver's velocity at impact?

Solution

a. To find the time t when the diver hits the water, let $s = 0$ and solve for t.

$$-16t^2 + 16t + 32 = 0$$ Set position function equal to 0.

$$-16(t + 1)(t - 2) = 0$$ Factor.

$$t = -1 \text{ or } 2$$ Solve for t.

Because $t \geq 0$, choose the positive value to conclude that the diver hits the water at $t = 2$ seconds.

b. The velocity at time t is given by the derivative $s'(t) = -32t + 16$. So, the velocity at time $t = 2$ is

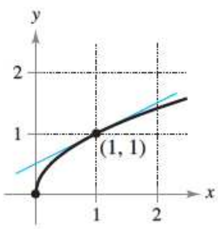
$$s'(2) = -32(2) + 16 = -48 \text{ feet per second.}$$ ■

3.2 Exercises

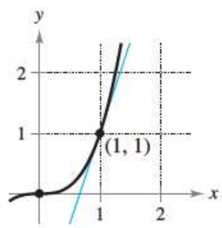
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use the graph to estimate the slope of the tangent line to $y = x^n$ at the point $(1, 1)$. Verify your answer analytically. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

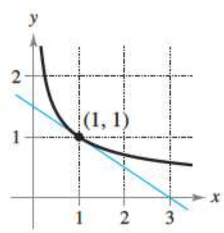
1. (a) $y = x^{1/2}$



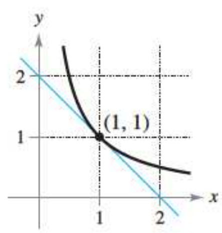
(b) $y = x^3$



2. (a) $y = x^{-1/2}$



(b) $y = x^{-1}$



In Exercises 3–26, use the rules of differentiation to find the derivative of the function.

3. $y = 12$

4. $f(x) = -9$

5. $y = x^7$

6. $y = x^{16}$

7. $y = \dfrac{1}{x^5}$

8. $y = \dfrac{1}{x^8}$

9. $f(x) = \sqrt[5]{x}$

10. $g(x) = \sqrt[6]{x}$

11. $f(x) = x + 11$

12. $g(x) = 3x - 1$

13. $f(t) = -2t^2 + 3t - 6$

14. $y = t^2 + 2t - 3$

15. $g(x) = x^2 + 4x^3$

16. $y = 8 - x^3$

17. $s(t) = t^3 + 5t^2 - 3t + 8$

18. $f(x) = 2x^3 - 4x^2 + 3x$

19. $f(x) = 6x - 5e^x$

20. $h(t) = t^3 + 2e^t$

21. $y = \dfrac{\pi}{2}\sin\theta - \cos\theta$

22. $g(t) = \pi\cos t$

23. $y = x^2 - \frac{1}{2}\cos x$

24. $y = 7 + \sin x$

25. $y = \frac{1}{2}e^x - 3\sin x$

26. $y = \frac{3}{4}e^x + 2\cos x$

In Exercises 27–32, complete the table.

	Original Function	*Rewrite*	*Differentiate*	*Simplify*
27.	$y = \dfrac{5}{2x^2}$			
28.	$y = \dfrac{4}{3x^2}$			
29.	$y = \dfrac{6}{(5x)^3}$			
30.	$y = \dfrac{\pi}{(5x)^2}$			
31.	$y = \dfrac{\sqrt{x}}{x}$			
32.	$y = \dfrac{4}{x^{-3}}$			

In Exercises 33–40, find the slope of the graph of the function at the given point. Use the *derivative* feature of a graphing utility to confirm your results.

	Function	*Point*
33.	$f(x) = \dfrac{8}{x^2}$	$(2, 2)$
34.	$f(t) = 3 - \dfrac{3}{5t}$	$\left(\frac{3}{5}, 2\right)$
35.	$f(x) = -\frac{1}{2} + \frac{7}{5}x^3$	$\left(0, -\frac{1}{2}\right)$
36.	$f(x) = 3(5 - x)^2$	$(5, 0)$
37.	$f(\theta) = 4\sin\theta - \theta$	$(0, 0)$
38.	$g(t) = -2\cos t + 5$	$(\pi, 7)$
39.	$f(t) = \frac{3}{4}e^t$	$\left(0, \frac{3}{4}\right)$
40.	$g(x) = -4e^x$	$(1, -4e)$

In Exercises 41–56, find the derivative of the function.

41. $g(t) = t^2 - \dfrac{4}{t^3}$

42. $f(x) = x + \dfrac{1}{x^2}$

43. $f(x) = \dfrac{4x^3 + 3x^2}{x}$

44. $f(x) = \dfrac{x^3 - 6}{x^2}$

45. $f(x) = \dfrac{x^3 - 3x^2 + 4}{x^2}$

46. $h(x) = \dfrac{2x^2 - 3x + 1}{x}$

47. $y = x(x^2 + 1)$

48. $y = 3x(6x - 5x^2)$

49. $f(x) = \sqrt{x} - 6\sqrt[3]{x}$

50. $f(x) = \sqrt[3]{x} + \sqrt[5]{x}$

51. $h(s) = s^{4/5} - s^{2/3}$

52. $f(t) = t^{2/3} - t^{1/3} + 4$

53. $f(x) = 6\sqrt{x} + 5\cos x$

54. $f(x) = \dfrac{2}{\sqrt[3]{x}} + 5\cos x$

55. $f(x) = x^{-2} - 2e^x$

56. $g(x) = \sqrt{x} - 3e^x$

In Exercises 57–60, (a) find an equation of the tangent line to the graph of *f* at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

	Function	*Point*
57.	$y = x^4 - x$	$(-1, 2)$
58.	$f(x) = \dfrac{2}{\sqrt[4]{x^3}}$	$(1, 2)$
59.	$g(x) = x + e^x$	$(0, 1)$
60.	$h(t) = \sin t + \frac{1}{2}e^t$	$\left(\pi, \frac{1}{2}e^{\pi}\right)$

In Exercises 61–68, determine the point(s) (if any) at which the graph of the function has a horizontal tangent line.

61. $y = x^4 - 2x^2 + 3$

62. $y = x^3 + x$

63. $y = \dfrac{1}{x^2}$

64. $y = x^2 + 9$

65. $y = x + \sin x, \quad 0 \le x < 2\pi$

66. $y = \sqrt{3}x + 2\cos x, \quad 0 \le x < 2\pi$

67. $y = -4x + e^x$

68. $y = x + 4e^x$

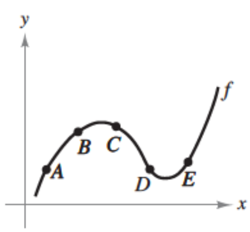
In Exercises 69–74, find k such that the line is tangent to the graph of the function.

	Function	*Line*
69.	$f(x) = x^2 - kx$	$y = 5x - 4$
70.	$f(x) = k - x^2$	$y = -6x + 1$
71.	$f(x) = \dfrac{k}{x}$	$y = -\dfrac{3}{4}x + 3$
72.	$f(x) = k\sqrt{x}$	$y = x + 4$
73.	$f(x) = kx^3$	$y = x + 1$
74.	$f(x) = kx^4$	$y = 4x - 1$

75. Sketch the graph of a function f such that $f' > 0$ for all x and the rate of change of the function is decreasing.

CAPSTONE

76. Use the graph of f to answer each question. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(a) Between which two consecutive points is the average rate of change of the function greatest?

(b) Is the average rate of change of the function between A and B greater than or less than the instantaneous rate of change at B?

(c) Sketch a tangent line to the graph between C and D such that the slope of the tangent line is the same as the average rate of change of the function between C and D.

WRITING ABOUT CONCEPTS

In Exercises 77 and 78, the relationship between f and g is given. Explain the relationship between f' and g'.

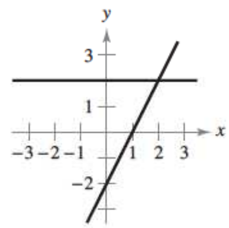
77. $g(x) = f(x) + 6$

78. $g(x) = -5f(x)$

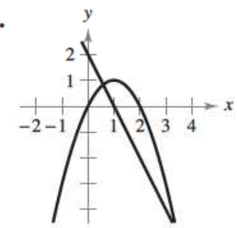
WRITING ABOUT CONCEPTS (continued)

In Exercises 79 and 80, the graphs of a function f and its derivative f' are shown on the same set of coordinate axes. Label the graphs as f or f' and write a short paragraph stating the criteria you used in making your selection. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

79.



80.



81. Sketch the graphs of $y = x^2$ and $y = -x^2 + 6x - 5$, and sketch the two lines that are tangent to both graphs. Find equations of these lines.

82. Show that the graphs of the two equations $y = x$ and $y = 1/x$ have tangent lines that are perpendicular to each other at their point of intersection.

83. Show that the graph of the function

$$f(x) = 3x + \sin x + 2$$

does not have a horizontal tangent line.

84. Show that the graph of the function

$$f(x) = x^5 + 3x^3 + 5x$$

does not have a tangent line with a slope of 3.

In Exercises 85 and 86, find an equation of the tangent line to the graph of the function f through the point (x_0, y_0) not on the graph. To find the point of tangency (x, y) on the graph of f, solve the equation

$$f'(x) = \frac{y_0 - y}{x_0 - x}.$$

85. $f(x) = \sqrt{x}$

$(x_0, y_0) = (-4, 0)$

86. $f(x) = \dfrac{2}{x}$

$(x_0, y_0) = (5, 0)$

87. ***Linear Approximation*** Use a graphing utility (in square mode) to zoom in on the graph of

$$f(x) = 4 - \tfrac{1}{2}x^2$$

to approximate $f'(1)$. Use the derivative to find $f'(1)$.

88. ***Linear Approximation*** Use a graphing utility (in square mode) to zoom in on the graph of

$$f(x) = 4\sqrt{x} + 1$$

to approximate $f'(4)$. Use the derivative to find $f'(4)$.

89. ***Linear Approximation*** Consider the function $f(x) = x^{3/2}$ with the solution point $(4, 8)$.

(a) Use a graphing utility to obtain the graph of f. Use the *zoom* feature to obtain successive magnifications of the graph in the neighborhood of the point $(4, 8)$. After zooming in a few times, the graph should appear nearly linear. Use the *trace* feature to determine the coordinates of a point near $(4, 8)$. Find an equation of the secant line $S(x)$ through the two points.

(b) Find the equation of the line

$$T(x) = f'(4)(x - 4) + f(4)$$

tangent to the graph of f passing through the given point. Why are the linear functions S and T nearly the same?

(c) Use a graphing utility to graph f and T in the same viewing window. Note that T is a good approximation of f when x is close to 4. What happens to the accuracy of the approximation as you move farther away from the point of tangency?

(d) Demonstrate the conclusion in part (c) by completing the table.

Δx	-3	-2	-1	-0.5	-0.1	0
$f(4 + \Delta x)$						
$T(4 + \Delta x)$						

Δx	0.1	0.5	1	2	3
$f(4 + \Delta x)$					
$T(4 + \Delta x)$					

90. ***Linear Approximation*** Repeat Exercise 89 for the function $f(x) = x^3$, where $T(x)$ is the line tangent to the graph at the point $(1, 1)$. Explain why the accuracy of the linear approximation decreases more rapidly than in Exercise 89.

True or False? **In Exercises 91–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

91. If $f'(x) = g'(x)$, then $f(x) = g(x)$.

92. If $f(x) = g(x) + c$, then $f'(x) = g'(x)$.

93. If $y = \pi^2$, then $dy/dx = 2\pi$.

94. If $y = x/\pi$, then $dy/dx = 1/\pi$.

95. If $g(x) = 3f(x)$, then $g'(x) = 3f'(x)$.

96. If $f(x) = 1/x^n$, then $f'(x) = 1/(nx^{n-1})$.

In Exercises 97–100, find the average rate of change of the function over the given interval. Compare this average rate of change with the instantaneous rates of change at the endpoints of the interval.

97. $f(x) = \dfrac{-1}{x}$, $[1, 2]$

98. $f(x) = \cos x$, $\left[0, \dfrac{\pi}{3}\right]$

99. $g(x) = x^2 + e^x$, $[0, 1]$

100. $h(x) = x^3 - \frac{1}{2}e^x$, $[0, 2]$

Vertical Motion **In Exercises 101 and 102, use the position function $s(t) = -16t^2 + v_0 t + s_0$ for free-falling objects.**

101. A silver dollar is dropped from the top of a building that is 1362 feet tall.

(a) Determine the position and velocity functions for the coin.

(b) Determine the average velocity on the interval $[1, 2]$.

(c) Find the instantaneous velocities when $t = 1$ and $t = 2$.

(d) Find the time required for the coin to reach ground level.

(e) Find the velocity of the coin at impact.

102. A ball is thrown straight down from the top of a 220-foot building with an initial velocity of -22 feet per second. What is its velocity after 3 seconds? What is its velocity after falling 108 feet?

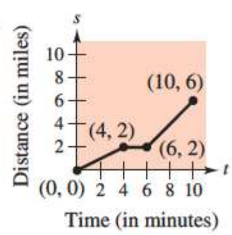
Vertical Motion **In Exercises 103 and 104, use the position function $s(t) = -4.9t^2 + v_0 t + s_0$ for free-falling objects.**

103. A projectile is shot upward from the surface of Earth with an initial velocity of 120 meters per second. What is its velocity after 5 seconds? After 10 seconds?

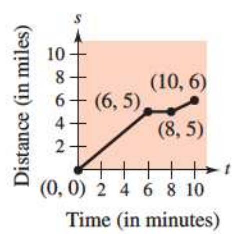
104. To estimate the height of a building, a stone is dropped from the top of the building into a pool of water at ground level. How high is the building if the splash is seen 5.6 seconds after the stone is dropped?

Think About It **In Exercises 105 and 106, the graph of a position function is shown. It represents the distance in miles that a person drives during a 10-minute trip to work. Make a sketch of the corresponding velocity function.**

105.

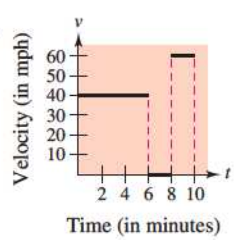


106.

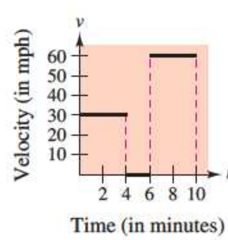


Think About It **In Exercises 107 and 108, the graph of a velocity function is shown. It represents the velocity in miles per hour during a 10-minute drive to work. Make a sketch of the corresponding position function.**

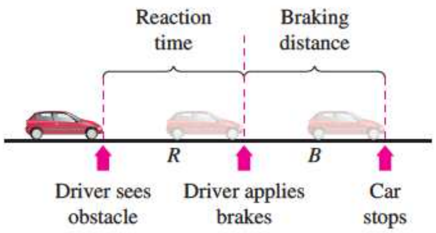
107.



108.



109. ***Modeling Data*** The stopping distance of an automobile, on dry, level pavement, traveling at a speed v (kilometers per hour) is the distance R (meters) the car travels during the reaction time of the driver plus the distance B (meters) the car travels after the brakes are applied (see figure). The table shows the results of an experiment.



Speed, v	20	40	60	80	100
Reaction Time Distance, R	8.3	16.7	25.0	33.3	41.7
Braking Time Distance, B	2.3	9.0	20.2	35.8	55.9

(a) Use the regression capabilities of a graphing utility to find a linear model for reaction time distance.

(b) Use the regression capabilities of a graphing utility to find a quadratic model for braking distance.

(c) Determine the polynomial giving the total stopping distance T.

(d) Use a graphing utility to graph the functions R, B, and T in the same viewing window.

(e) Find the derivative of T and the rates of change of the total stopping distance for $v = 40$, $v = 80$, and $v = 100$.

(f) Use the results of this exercise to draw conclusions about the total stopping distance as speed increases.

110. ***Fuel Cost*** A car is driven 15,000 miles a year and gets x miles per gallon. Assume that the average fuel cost is \$2.76 per gallon. Find the annual cost of fuel C as a function of x, and use this function to complete the table.

x	10	15	20	25	30	35	40
C							
dC/dx							

Who would benefit more from a one-mile-per-gallon increase in fuel efficiency—the driver of a car that gets 15 miles per gallon or the driver of a car that gets 35 miles per gallon? Explain.

111. ***Volume*** The volume of a cube with sides of length s is given by $V = s^3$. Find the rate of change of the volume with respect to s when $s = 6$ centimeters.

112. ***Area*** The area of a square with sides of length s is given by $A = s^2$. Find the rate of change of the area with respect to s when $s = 6$ meters.

113. ***Velocity*** Verify that the average velocity over the time interval $[t_0 - \Delta t, t_0 + \Delta t]$ is the same as the instantaneous velocity at $t = t_0$ for the position function

$$s(t) = -\tfrac{1}{2}at^2 + c.$$

114. ***Inventory Management*** The annual inventory cost C for a manufacturer is

$$C = \frac{1{,}008{,}000}{Q} + 6.3Q$$

where Q is the order size when the inventory is replenished. Find the change in annual cost when Q is increased from 350 to 351, and compare this with the instantaneous rate of change when $Q = 350$.

115. ***Writing*** The number of gallons N of regular unleaded gasoline sold by a gasoline station at a price of p dollars per gallon is given by $N = f(p)$.

(a) Describe the meaning of $f'(2.979)$.

(b) Is $f'(2.979)$ usually positive or negative? Explain.

116. ***Newton's Law of Cooling*** This law states that the rate of change of the temperature of an object is proportional to the difference between the object's temperature T and the temperature T_a of the surrounding medium. Write an equation for this law.

117. Find an equation of the parabola $y = ax^2 + bx + c$ that passes through $(0, 1)$ and is tangent to the line $y = x - 1$ at $(1, 0)$.

118. Let (a, b) be an arbitrary point on the graph of $y = 1/x$, $x > 0$. Prove that the area of the triangle formed by the tangent line through (a, b) and the coordinate axes is 2.

119. Find the tangent line(s) to the curve $y = x^3 - 9x$ through the point $(1, -9)$.

120. Find the equation(s) of the tangent line(s) to the parabola $y = x^2$ through the given point.

(a) $(0, a)$ (b) $(a, 0)$

Are there any restrictions on the constant a?

In Exercises 121 and 122, find *a* and *b* such that *f* is differentiable everywhere.

121. $f(x) = \begin{cases} ax^3, & x \le 2 \\ x^2 + b, & x > 2 \end{cases}$

122. $f(x) = \begin{cases} \cos x, & x < 0 \\ ax + b, & x \ge 0 \end{cases}$

123. Where are the functions $f_1(x) = |\sin x|$ and $f_2(x) = \sin |x|$ differentiable?

124. Prove that $\dfrac{d}{dx}[\cos x] = -\sin x$.

FOR FURTHER INFORMATION For a geometric interpretation of the derivatives of trigonometric functions, see the article "Sines and Cosines of the Times" by Victor J. Katz in *Math Horizons*. To view this article, go to the website *www.matharticles.com*.

3.3 Product and Quotient Rules and Higher-Order Derivatives

- Find the derivative of a function using the Product Rule.
- Find the derivative of a function using the Quotient Rule.
- Find the derivative of a trigonometric function.
- Find a higher-order derivative of a function.

The Product Rule

In Section 3.2 you learned that the derivative of the sum of two functions is simply the sum of their derivatives. The rules for the derivatives of the product and quotient of two functions are not as simple.

THEOREM 3.8 THE PRODUCT RULE

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

NOTE A version of the Product Rule that some people prefer is

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

The advantage of this form is that it generalizes easily to products of three or more factors.

PROOF Some mathematical proofs, such as the proof of the Sum Rule, are straightforward. Others involve clever steps that may appear unmotivated to a reader. This proof involves such a step—subtracting and adding the same quantity—which is shown in color.

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[f(x+\Delta x)\frac{g(x+\Delta x) - g(x)}{\Delta x} + g(x)\frac{f(x+\Delta x) - f(x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \left[f(x+\Delta x)\frac{g(x+\Delta x) - g(x)}{\Delta x} \right] + \lim_{\Delta x \to 0} \left[g(x)\frac{f(x+\Delta x) - f(x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} f(x+\Delta x) \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} g(x) \cdot \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

Note that $\lim_{\Delta x \to 0} f(x + \Delta x) = f(x)$ because f is given to be differentiable and therefore is continuous.

The Product Rule can be extended to cover products involving more than two factors. For example, if f, g, and h are differentiable functions of x, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

For instance, the derivative of $y = x^2 \sin x \cos x$ is

$$\begin{aligned}
\frac{dy}{dx} &= 2x \sin x \cos x + x^2 \cos x \cos x + x^2 \sin x(-\sin x) \\
&= 2x \sin x \cos x + x^2(\cos^2 x - \sin^2 x).
\end{aligned}$$

NOTE The proof of the Product Rule for products of more than two factors is left as an exercise (see Exercise 141).

THE PRODUCT RULE

When Leibniz originally wrote a formula for the Product Rule, he was motivated by the expression

$$(x + dx)(y + dy) - xy$$

from which he subtracted $dx\,dy$ (as being negligible) and obtained the differential form $x\,dy + y\,dx$. This derivation resulted in the traditional form of the Product Rule. (*Source: The History of Mathematics by David M. Burton*)

The derivative of a product of two functions is not (in general) given by the product of the derivatives of the two functions. To see this, try comparing the product of the derivatives of $f(x) = 3x - 2x^2$ and $g(x) = 5 + 4x$ with the derivative in Example 1.

EXAMPLE 1 Using the Product Rule

Find the derivative of $h(x) = (3x - 2x^2)(5 + 4x)$.

Solution

$$\begin{aligned} h'(x) &= \overbrace{(3x - 2x^2)}^{\text{First}}\overbrace{\frac{d}{dx}[5 + 4x]}^{\text{Derivative of second}} + \overbrace{(5 + 4x)}^{\text{Second}}\overbrace{\frac{d}{dx}[3x - 2x^2]}^{\text{Derivative of first}} && \text{Apply Product Rule.} \\ &= (3x - 2x^2)(4) + (5 + 4x)(3 - 4x) \\ &= (12x - 8x^2) + (15 - 8x - 16x^2) \\ &= -24x^2 + 4x + 15 \end{aligned}$$

In Example 1, you have the option of finding the derivative with or without the Product Rule. To find the derivative without the Product Rule, you can write

$$\begin{aligned} D_x[(3x - 2x^2)(5 + 4x)] &= D_x[-8x^3 + 2x^2 + 15x] \\ &= -24x^2 + 4x + 15. \end{aligned}$$

In the next example, you must use the Product Rule.

EXAMPLE 2 Using the Product Rule

Find the derivative of $y = xe^x$.

Solution

$$\begin{aligned} \frac{d}{dx}[xe^x] &= x\frac{d}{dx}[e^x] + e^x\frac{d}{dx}[x] && \text{Apply Product Rule.} \\ &= xe^x + e^x(1) \\ &= e^x(x + 1) \end{aligned}$$

EXAMPLE 3 Using the Product Rule

Find the derivative of $y = 2x \cos x - 2 \sin x$.

Solution

$$\begin{aligned} \frac{dy}{dx} &= \overbrace{(2x)\left(\frac{d}{dx}[\cos x]\right) + (\cos x)\left(\frac{d}{dx}[2x]\right)}^{\text{Product Rule}} - \overbrace{2\frac{d}{dx}[\sin x]}^{\text{Constant Multiple Rule}} \\ &= (2x)(-\sin x) + (\cos x)(2) - 2(\cos x) \\ &= -2x \sin x \end{aligned}$$

NOTE In Example 3, notice that you use the Product Rule when both factors of the product are variable, and you use the Constant Multiple Rule when one of the factors is a constant.

The Quotient Rule

THEOREM 3.9 THE QUOTIENT RULE

The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

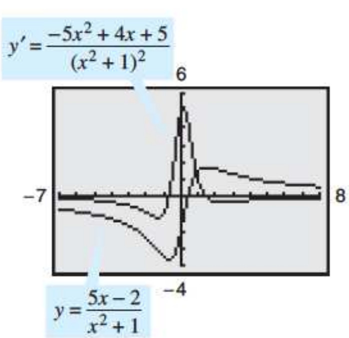
PROOF As with the proof of Theorem 3.8, the key to this proof is subtracting and adding the same quantity.

$$\begin{aligned}
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \lim_{\Delta x \to 0} \frac{\dfrac{f(x+\Delta x)}{g(x+\Delta x)} - \dfrac{f(x)}{g(x)}}{\Delta x} && \text{Definition of derivative} \\
&= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x+\Delta x)}{\Delta x g(x) g(x+\Delta x)} \\
&= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x) + f(x)g(x) - f(x)g(x+\Delta x)}{\Delta x g(x) g(x+\Delta x)} \\
&= \frac{\displaystyle\lim_{\Delta x \to 0} \frac{g(x)[f(x+\Delta x) - f(x)]}{\Delta x} - \lim_{\Delta x \to 0} \frac{f(x)[g(x+\Delta x) - g(x)]}{\Delta x}}{\displaystyle\lim_{\Delta x \to 0} [g(x)g(x+\Delta x)]} \\
&= \frac{g(x)\left[\displaystyle\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right] - f(x)\left[\displaystyle\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}\right]}{\displaystyle\lim_{\Delta x \to 0} [g(x)g(x+\Delta x)]} \\
&= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}
\end{aligned}$$

■

Note that $\lim_{\Delta x \to 0} g(x + \Delta x) = g(x)$ because g is given to be differentiable and therefore is continuous.

TECHNOLOGY Graphing utilities can be used to compare the graph of a function with the graph of its derivative. For instance, in Figure 3.23, the graph of the function in Example 4 appears to have two points that have horizontal tangent lines. What are the values of y' at these two points?



Graphical comparison of a function and its derivative
Figure 3.23

EXAMPLE 4 Using the Quotient Rule

Find the derivative of $y = \dfrac{5x - 2}{x^2 + 1}$.

Solution

$$\begin{aligned}
\frac{d}{dx}\left[\frac{5x-2}{x^2+1}\right] &= \frac{(x^2+1)\dfrac{d}{dx}[5x-2] - (5x-2)\dfrac{d}{dx}[x^2+1]}{(x^2+1)^2} && \text{Apply Quotient Rule.} \\
&= \frac{(x^2+1)(5) - (5x-2)(2x)}{(x^2+1)^2} \\
&= \frac{(5x^2+5) - (10x^2-4x)}{(x^2+1)^2} \\
&= \frac{-5x^2+4x+5}{(x^2+1)^2}
\end{aligned}$$

■

Note the use of parentheses in Example 4. A liberal use of parentheses is recommended for *all* types of differentiation problems. For instance, with the Quotient Rule, it is a good idea to enclose all factors and derivatives in parentheses, and to pay special attention to the subtraction required in the numerator.

When differentiation rules were introduced in the preceding section, the need for rewriting *before* differentiating was emphasized. The next example illustrates this point with the Quotient Rule.

EXAMPLE 5 Rewriting Before Differentiating

Find an equation of the tangent line to the graph of $f(x) = \dfrac{3 - (1/x)}{x + 5}$ at $(-1, 1)$.

Solution Begin by rewriting the function.

$$f(x) = \frac{3 - (1/x)}{x + 5} \qquad \text{Write original function.}$$

$$= \frac{x\left(3 - \frac{1}{x}\right)}{x(x + 5)} \qquad \text{Multiply numerator and denominator by } x.$$

$$= \frac{3x - 1}{x^2 + 5x} \qquad \text{Rewrite.}$$

$$f'(x) = \frac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2} \qquad \text{Apply Quotient Rule.}$$

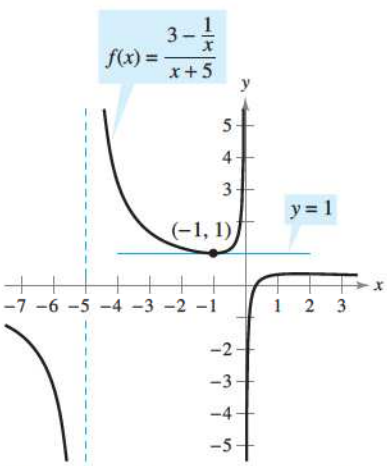
$$= \frac{(3x^2 + 15x) - (6x^2 + 13x - 5)}{(x^2 + 5x)^2}$$

$$= \frac{-3x^2 + 2x + 5}{(x^2 + 5x)^2} \qquad \text{Simplify.}$$

To find the slope at $(-1, 1)$, evaluate $f'(-1)$.

$$f'(-1) = 0 \qquad \text{Slope of graph at } (-1, 1)$$

Then, using the point-slope form of the equation of a line, you can determine that the equation of the tangent line at $(-1, 1)$ is $y = 1$. See Figure 3.24. ■



The line $y = 1$ is tangent to the graph of $f(x)$ at the point $(-1, 1)$.
Figure 3.24

Not every quotient needs to be differentiated by the Quotient Rule. For example, each quotient in the next example can be considered as the product of a constant times a function of x. In such cases it is more convenient to use the Constant Multiple Rule.

EXAMPLE 6 Using the Constant Multiple Rule

	Original Function	*Rewrite*	*Differentiate*	*Simplify*
a.	$y = \dfrac{x^2 + 3x}{6}$	$y = \dfrac{1}{6}(x^2 + 3x)$	$y' = \dfrac{1}{6}(2x + 3)$	$y' = \dfrac{2x + 3}{6}$
b.	$y = \dfrac{5x^4}{8}$	$y = \dfrac{5}{8}x^4$	$y' = \dfrac{5}{8}(4x^3)$	$y' = \dfrac{5}{2}x^3$
c.	$y = \dfrac{-3(3x - 2x^2)}{7x}$	$y = -\dfrac{3}{7}(3 - 2x)$	$y' = -\dfrac{3}{7}(-2)$	$y' = \dfrac{6}{7}$
d.	$y = \dfrac{9}{5x^2}$	$y = \dfrac{9}{5}(x^{-2})$	$y' = \dfrac{9}{5}(-2x^{-3})$	$y' = -\dfrac{18}{5x^3}$

■

NOTE To see the benefit of using the Constant Multiple Rule for some quotients, try using the Quotient Rule to differentiate the functions in Example 6—you should obtain the same results, but with more work.

In Section 3.2, the Power Rule was proved only for the case in which the exponent n is a positive integer greater than 1. The next example extends the proof to include negative integer exponents.

EXAMPLE 7 Proof of the Power Rule (Negative Integer Exponents)

If n is a negative integer, there exists a positive integer k such that $n = -k$. So, by the Quotient Rule, you can write

$$
\begin{aligned}
\frac{d}{dx}[x^n] &= \frac{d}{dx}\left[\frac{1}{x^k}\right] \\
&= \frac{x^k(0) - (1)(kx^{k-1})}{(x^k)^2} && \text{Quotient Rule and Power Rule} \\
&= \frac{0 - kx^{k-1}}{x^{2k}} \\
&= -kx^{-k-1} \\
&= nx^{n-1}. && n = -k
\end{aligned}
$$

So, the Power Rule

$$\frac{d}{dx}[x^n] = nx^{n-1} \qquad \text{Power Rule}$$

is valid for any integer. The cases for which n is rational and n is irrational are left as an exercise (see Section 3.5, Exercise 100). ■

Derivatives of Trigonometric Functions

Knowing the derivatives of the sine and cosine functions, you can use the Quotient Rule to find the derivatives of the four remaining trigonometric functions.

THEOREM 3.10 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

PROOF Considering $\tan x = (\sin x)/(\cos x)$ and applying the Quotient Rule, you obtain

$$
\begin{aligned}
\frac{d}{dx}[\tan x] &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} && \text{Apply Quotient Rule.} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x.
\end{aligned}
$$

The proofs of the other three parts of the theorem are left as an exercise (see Exercise 91). ■

EXAMPLE 8 Differentiating Trigonometric Functions

NOTE Because of trigonometric identities, the derivative of a trigonometric function can take many forms. This presents a challenge when you are trying to match your answers to those given in the back of the text.

Function	*Derivative*
a. $y = x - \tan x$	$\dfrac{dy}{dx} = 1 - \sec^2 x$
b. $y = x \sec x$	$y' = x(\sec x \tan x) + (\sec x)(1)$ $= (\sec x)(1 + x \tan x)$

EXAMPLE 9 Different Forms of a Derivative

Differentiate both forms of $y = \dfrac{1 - \cos x}{\sin x} = \csc x - \cot x.$

Solution

First form: $y = \dfrac{1 - \cos x}{\sin x}$

$$y' = \frac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{\sin^2 x + \cos^2 x - \cos x}{\sin^2 x}$$

$$= \frac{1 - \cos x}{\sin^2 x}$$

Second form: $y = \csc x - \cot x$

$$y' = -\csc x \cot x + \csc^2 x$$

To verify that the two derivatives are equal, you can write

$$\frac{1 - \cos x}{\sin^2 x} = \frac{1}{\sin^2 x} - \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)$$

$$= \csc^2 x - \csc x \cot x.$$

The summary below shows that much of the work in obtaining a simplified form of a derivative occurs *after* differentiating. Note that two characteristics of a simplified form are the absence of negative exponents and the combining of like terms.

	$f'(x)$ After Differentiating	$f'(x)$ After Simplifying
Example 1	$(3x - 2x^2)(4) + (5 + 4x)(3 - 4x)$	$-24x^2 + 4x + 15$
Example 3	$(2x)(-\sin x) + (\cos x)(2) - 2(\cos x)$	$-2x \sin x$
Example 4	$\dfrac{(x^2 + 1)(5) - (5x - 2)(2x)}{(x^2 + 1)^2}$	$\dfrac{-5x^2 + 4x + 5}{(x^2 + 1)^2}$
Example 5	$\dfrac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2}$	$\dfrac{-3x^2 + 2x + 5}{(x^2 + 5x)^2}$
Example 9	$\dfrac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x}$	$\dfrac{1 - \cos x}{\sin^2 x}$

EXPLORATION

For which of the functions

$y = e^x, \quad y = \dfrac{1}{e^x}$

$y = \sin x, \quad y = \cos x$

are the following equations true?

a. $y = y'$ **b.** $y = y''$
c. $y = y'''$ **d.** $y = y^{(4)}$

Without determining the actual derivative, is $y = y^{(8)}$ for $y = \sin x$ true? What conclusion can you draw from this?

Higher-Order Derivatives

Just as you can obtain a velocity function by differentiating a position function, you can obtain an **acceleration** function by differentiating a velocity function. Another way of looking at this is that you can obtain an acceleration function by differentiating a position function *twice*.

$$s(t) \qquad \text{Position function}$$
$$v(t) = s'(t) \qquad \text{Velocity function}$$
$$a(t) = v'(t) = s''(t) \qquad \text{Acceleration function}$$

The function given by $a(t)$ is the **second derivative** of $s(t)$ and is denoted by $s''(t)$.

The second derivative is an example of a **higher-order derivative.** You can define derivatives of any positive integer order. For instance, the **third derivative** is the derivative of the second derivative. Higher-order derivatives are denoted as follows.

First derivative:	y',	$f'(x)$,	$\dfrac{dy}{dx}$,	$\dfrac{d}{dx}[f(x)]$,	$D_x[y]$
Second derivative:	y'',	$f''(x)$,	$\dfrac{d^2y}{dx^2}$,	$\dfrac{d^2}{dx^2}[f(x)]$,	$D_x^2[y]$
Third derivative:	y''',	$f'''(x)$,	$\dfrac{d^3y}{dx^3}$,	$\dfrac{d^3}{dx^3}[f(x)]$,	$D_x^3[y]$
Fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\dfrac{d^4y}{dx^4}$,	$\dfrac{d^4}{dx^4}[f(x)]$,	$D_x^4[y]$
$\vdots$					
nth derivative:	$y^{(n)}$,	$f^{(n)}(x)$,	$\dfrac{d^ny}{dx^n}$,	$\dfrac{d^n}{dx^n}[f(x)]$,	$D_x^n[y]$

EXAMPLE 10 Finding the Acceleration Due to Gravity

Because the moon has no atmosphere, a falling object on the moon encounters no air resistance. In 1971, astronaut David Scott demonstrated that a feather and a hammer fall at the same rate on the moon. The position function for each of these falling objects is given by

$$s(t) = -0.81t^2 + 2$$

where $s(t)$ is the height in meters and t is the time in seconds. What is the ratio of Earth's gravitational force to the moon's?

Solution To find the acceleration, differentiate the position function twice.

$$s(t) = -0.81t^2 + 2 \qquad \text{Position function}$$
$$s'(t) = -1.62t \qquad \text{Velocity function}$$
$$s''(t) = -1.62 \qquad \text{Acceleration function}$$

So, the acceleration due to gravity on the moon is -1.62 meters per second per second. Because the acceleration due to gravity on Earth is -9.8 meters per second per second, the ratio of Earth's gravitational force to the moon's is

$$\frac{\text{Earth's gravitational force}}{\text{Moon's gravitational force}} = \frac{-9.8}{-1.62} \approx 6.0.$$


NASA

THE MOON

The moon's mass is 7.349×10^{22} kilograms, and Earth's mass is 5.976×10^{24} kilograms. The moon's radius is 1737 kilometers, and Earth's radius is 6378 kilometers. Because the gravitational force on the surface of a planet is directly proportional to its mass and inversely proportional to the square of its radius, the ratio of the gravitational force on Earth to the gravitational force on the moon is

$$\frac{(5.976 \times 10^{24})/6378^2}{(7.349 \times 10^{22})/1737^2} \approx 6.0.$$

3.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, use the Product Rule to differentiate the function.

1. $g(x) = (x^2 + 3)(x^2 - 4x)$
2. $f(x) = (6x + 5)(x^3 - 2)$
3. $h(t) = \sqrt{t}(1 - t^2)$
4. $g(s) = \sqrt{s}(s^2 + 8)$
5. $f(x) = e^x \cos x$
6. $g(x) = \sqrt{x} \sin x$

In Exercises 7–12, use the Quotient Rule to differentiate the function.

7. $f(x) = \dfrac{x}{x^2 + 1}$
8. $g(t) = \dfrac{t^2 + 4}{5t - 3}$
9. $h(x) = \dfrac{\sqrt{x}}{x^3 + 1}$
10. $h(s) = \dfrac{s}{\sqrt{s} - 1}$
11. $g(x) = \dfrac{\sin x}{e^x}$
12. $f(t) = \dfrac{\cos t}{t^3}$

In Exercises 13–18, find $f'(x)$ and $f'(c)$.

Function	*Value of c*
13. $f(x) = (x^3 + 4x)(3x^2 + 2x - 5)$	$c = 0$
14. $f(x) = \dfrac{x + 5}{x - 5}$	$c = 4$
15. $f(x) = x \cos x$	$c = \dfrac{\pi}{4}$
16. $f(x) = \dfrac{\sin x}{x}$	$c = \dfrac{\pi}{6}$
17. $f(x) = e^x \sin x$	$c = 0$
18. $f(x) = \dfrac{\cos x}{e^x}$	$c = 0$

In Exercises 19–24, complete the table without using the Quotient Rule.

Function	*Rewrite*	*Differentiate*	*Simplify*
19. $y = \dfrac{x^2 + 3x}{7}$			
20. $y = \dfrac{5x^2 - 3}{4}$			
21. $y = \dfrac{6}{7x^2}$			
22. $y = \dfrac{10}{3x^3}$			
23. $y = \dfrac{4x^{3/2}}{x}$			
24. $y = \dfrac{5x^2 - 8}{11}$			

In Exercises 25–38, find the derivative of the algebraic function.

25. $f(x) = \dfrac{4 - 3x - x^2}{x^2 - 1}$
26. $f(x) = \dfrac{x^3 + 5x + 3}{x^2 - 1}$
27. $f(x) = x\left(1 - \dfrac{4}{x + 3}\right)$
28. $f(x) = x^4\left(1 - \dfrac{2}{x + 1}\right)$
29. $f(x) = \dfrac{3x - 1}{\sqrt{x}}$
30. $f(x) = \sqrt[3]{x}(\sqrt{x} + 3)$
31. $h(s) = (s^3 - 2)^2$
32. $h(x) = (x^2 - 1)^2$
33. $f(x) = \dfrac{2 - \dfrac{1}{x}}{x - 3}$
34. $g(x) = x^2\left(\dfrac{2}{x} - \dfrac{1}{x + 1}\right)$
35. $f(x) = (2x^3 + 5x)(x - 3)(x + 2)$
36. $f(x) = (x^3 - x)(x^2 + 2)(x^2 + x - 1)$
37. $f(x) = \dfrac{x^2 + c^2}{x^2 - c^2}$, c is a constant
38. $f(x) = \dfrac{c^2 - x^2}{c^2 + x^2}$, c is a constant

In Exercises 39–56, find the derivative of the transcendental function.

39. $f(t) = t^2 \sin t$
40. $f(\theta) = (\theta + 1) \cos \theta$
41. $f(t) = \dfrac{\cos t}{t}$
42. $f(x) = \dfrac{\sin x}{x^3}$
43. $f(x) = -e^x + \tan x$
44. $y = e^x - \cot x$
45. $g(t) = \sqrt[4]{t} + 6 \csc t$
46. $h(x) = \dfrac{1}{x} - 12 \sec x$
47. $y = \dfrac{3(1 - \sin x)}{2 \cos x}$
48. $y = \dfrac{\sec x}{x}$
49. $y = -\csc x - \sin x$
50. $y = x \cos x + \sin x$
51. $f(x) = x^2 \tan x$
52. $f(x) = 2 \sin x \cos x$
53. $y = 2x \sin x + x^2 e^x$
54. $h(x) = 2e^x \cos x$
55. $y = \dfrac{e^x}{4\sqrt{x}}$
56. $y = \dfrac{2e^x}{x^2 + 1}$

CAS In Exercises 57–60, use a computer algebra system to differentiate the function.

57. $g(x) = \left(\dfrac{x + 1}{x + 2}\right)(2x - 5)$
58. $f(x) = \left(\dfrac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$
59. $g(\theta) = \dfrac{\theta}{1 - \sin \theta}$
60. $f(\theta) = \dfrac{\sin \theta}{1 - \cos \theta}$

The symbol **CAS** *indicates an exercise in which you are instructed to specifically use a computer algebra system.*

In Exercises 61–64, evaluate the derivative of the function at the given point. Use a graphing utility to verify your result.

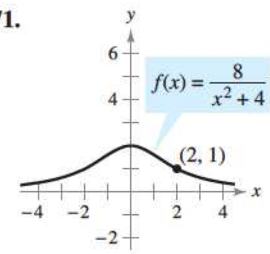
Function	*Point*
61. $y = \dfrac{1 + \csc x}{1 - \csc x}$	$\left(\dfrac{\pi}{6}, -3\right)$
62. $f(x) = \tan x \cot x$	$(1, 1)$
63. $h(t) = \dfrac{\sec t}{t}$	$\left(\pi, -\dfrac{1}{\pi}\right)$
64. $f(x) = \sin x(\sin x + \cos x)$	$\left(\dfrac{\pi}{4}, 1\right)$

In Exercises 65–70, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

Function	*Point*
65. $f(x) = (x^3 + 4x - 1)(x - 2)$	$(1, -4)$
66. $f(x) = \dfrac{x - 1}{x + 1}$	$\left(2, \dfrac{1}{3}\right)$
67. $f(x) = \tan x$	$\left(\dfrac{\pi}{4}, 1\right)$
68. $f(x) = \sec x$	$\left(\dfrac{\pi}{3}, 2\right)$
69. $f(x) = (x - 1)e^x$	$(1, 0)$
70. $f(x) = \dfrac{e^x}{x + 4}$	$\left(0, \dfrac{1}{4}\right)$

Famous Curves **In Exercises 71–74, find an equation of the tangent line to the graph at the given point. (The graphs in Exercises 71 and 72 are called *witches of Agnesi*. The graphs in Exercises 73 and 74 are called *serpentines*.)**

71.



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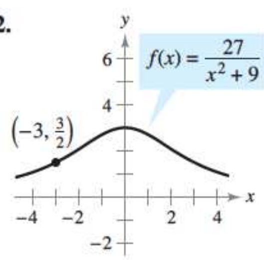

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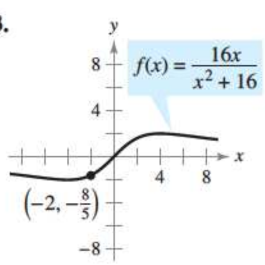

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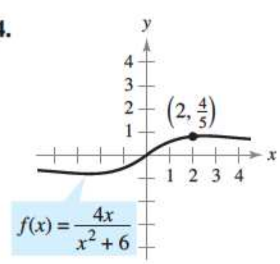


In Exercises 75–78, determine the point(s) at which the graph of the function has a horizontal tangent line.

75. $f(x) = \dfrac{2x - 1}{x^2}$

76. $f(x) = \dfrac{x^2}{x^2 + 1}$

77. $g(x) = \dfrac{8(x - 2)}{e^x}$

78. $f(x) = e^x \sin x, \ [0, \pi]$

79. ***Tangent Lines*** Find equations of the tangent lines to the graph of $f(x) = (x + 1)/(x - 1)$ that are parallel to the line $2y + x = 6$. Then graph the function and the tangent lines.

80. ***Tangent Lines*** Find equations of the tangent lines to the graph of $f(x) = x/(x - 1)$ that pass through the point $(-1, 5)$. Then graph the function and the tangent lines.

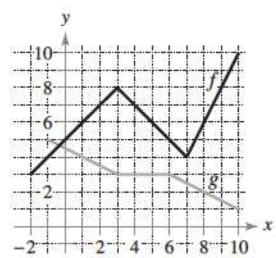
In Exercises 81 and 82, verify that $f'(x) = g'(x)$, and explain the relationship between f and g.

81. $f(x) = \dfrac{3x}{x + 2}, \ g(x) = \dfrac{5x + 4}{x + 2}$

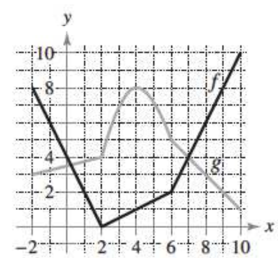
82. $f(x) = \dfrac{\sin x - 3x}{x}, \ g(x) = \dfrac{\sin x + 2x}{x}$

In Exercises 83 and 84, use the graphs of f and g. Let $p(x) = f(x)g(x)$ and $q(x) = f(x)/g(x)$.

83. (a) Find $p'(1)$.
(b) Find $q'(4)$.



84. (a) Find $p'(4)$.
(b) Find $q'(7)$.



85. ***Area*** The length of a rectangle is given by $6t + 5$ and its height is $\sqrt{t}$, where t is time in seconds and the dimensions are in centimeters. Find the rate of change of the area with respect to time.

86. ***Volume*** The radius of a right circular cylinder is given by $\sqrt{t + 2}$ and its height is $\frac{1}{2}\sqrt{t}$, where t is time in seconds and the dimensions are in inches. Find the rate of change of the volume with respect to time.

87. ***Inventory Replenishment*** The ordering and transportation cost C for the components used in manufacturing a product is

$$C = \frac{375{,}000 + 6x^2}{x}, \quad x \geq 1$$

where C is measured in dollars and x is the order size. Find the rate of change of C with respect to x when (a) $x = 200$, (b) $x = 250$, and (c) $x = 300$. Interpret the meanings of these values.

88. ***Boyle's Law*** This law states that if the temperature of a gas remains constant, its pressure is inversely proportional to its volume. Use the derivative to show that the rate of change of the pressure is inversely proportional to the square of the volume.

89. ***Population Growth*** A population of 500 bacteria is introduced into a culture and grows in number according to the equation

$$P(t) = 500\left(1 + \frac{4t}{50 + t^2}\right)$$

where t is measured in hours. Find the rate at which the population is growing when $t = 2$.

90. ***Gravitational Force*** Newton's Law of Universal Gravitation states that the force F between two masses, m_1 and m_2, is

$$F = \frac{Gm_1m_2}{d^2}$$

where G is a constant and d is the distance between the masses. Find an equation that gives an instantaneous rate of change of F with respect to d. (Assume that m_1 and m_2 represent moving points.)

91. Prove the following differentiation rules.

(a) $\frac{d}{dx}[\sec x] = \sec x \tan x$ (b) $\frac{d}{dx}[\csc x] = -\csc x \cot x$

(c) $\frac{d}{dx}[\cot x] = -\csc^2 x$

92. ***Rate of Change*** Determine whether there exist any values of x in the interval $[0, 2\pi)$ such that the rate of change of $f(x) = \sec x$ and the rate of change of $g(x) = \csc x$ are equal.

93. ***Modeling Data*** The table shows the quantities q (in millions) of personal computers shipped in the United States and the values v (in billions of dollars) of these shipments for the years 1999 through 2004. The year is represented by t, with $t = 9$ corresponding to 1999. *(Source: U.S. Census Bureau)*

Year, t	9	10	11	12	13	14
q	19.6	15.9	14.6	12.9	15.0	15.8
v	26.8	22.6	18.9	16.2	14.7	15.3

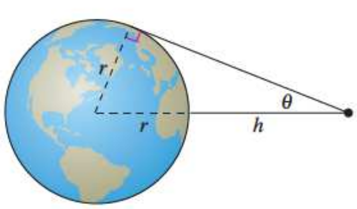
(a) Use a graphing utility to find cubic models for the quantity of personal computers shipped $q(t)$ and the value $v(t)$ of the personal computers.

(b) Graph each model found in part (a).

(c) Find $A = v(t)/q(t)$, then graph A. What does this function represent?

(d) Interpret $A'(t)$ in the context of these data.

94. ***Satellites*** When satellites observe Earth, they can scan only part of Earth's surface. Some satellites have sensors that can measure the angle θ shown in the figure. Let h represent the satellite's distance from Earth's surface and let r represent Earth's radius.



(a) Show that $h = r(\csc\theta - 1)$.

(b) Find the rate at which h is changing with respect to θ when $\theta = 30^\circ$. (Assume $r = 3960$ miles.)

In Exercises 95–102, find the second derivative of the function.

95. $f(x) = 4x^{3/2}$ **96.** $f(x) = x + 32x^{-2}$

97. $f(x) = \frac{x}{x-1}$ **98.** $f(x) = \frac{x^2 + 2x - 1}{x}$

99. $f(x) = x \sin x$ **100.** $f(x) = \sec x$

101. $g(x) = \frac{e^x}{x}$ **102.** $h(t) = e^t \sin t$

In Exercises 103–106, find the given higher-order derivative.

	Given	*Find*
103.	$f'(x) = x^2$	$f''(x)$
104.	$f''(x) = 2 - \frac{2}{x}$	$f'''(x)$
105.	$f'''(x) = 2\sqrt{x}$	$f^{(4)}(x)$
106.	$f^{(4)}(x) = 2x + 1$	$f^{(6)}(x)$

WRITING ABOUT CONCEPTS

107. Sketch the graph of a differentiable function f such that $f(2) = 0$, $f' < 0$ for $-\infty < x < 2$, and $f' > 0$ for $2 < x < \infty$.

108. Sketch the graph of a differentiable function f such that $f > 0$ and $f' < 0$ for all real numbers x.

In Exercises 109–112, use the given information to find $f'(2)$.

$g(2) = 3$ and $g'(2) = -2$

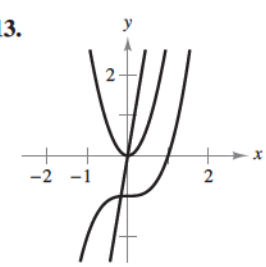
$h(2) = -1$ and $h'(2) = 4$

109. $f(x) = 2g(x) + h(x)$ **110.** $f(x) = 4 - h(x)$

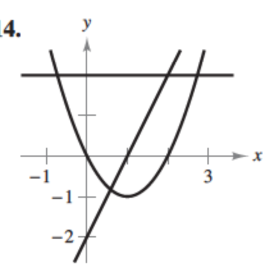
111. $f(x) = \frac{g(x)}{h(x)}$ **112.** $f(x) = g(x)h(x)$

In Exercises 113 and 114, the graphs of f, f', and f'' are shown on the same set of coordinate axes. Identify each graph. Explain your reasoning. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

113.

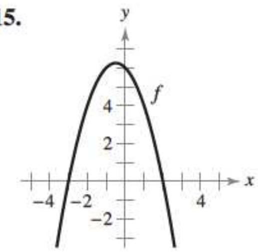


114.

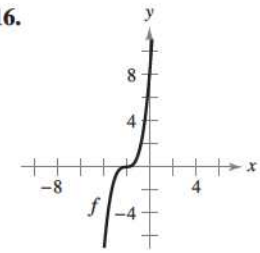


In Exercises 115–118, the graph of f is shown. Sketch the graphs of f' and f''. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

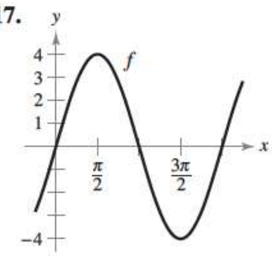
115.



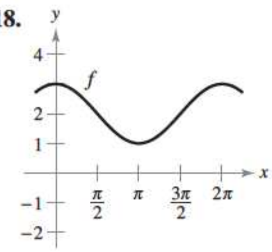
116.



117.



118.



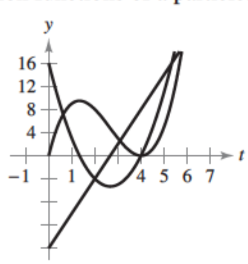
119. ***Acceleration*** The velocity of an object in meters per second is $v(t) = 36 - t^2, 0 \le t \le 6$. Find the velocity and acceleration of the object when $t = 3$. What can be said about the speed of the object when the velocity and acceleration have opposite signs?

CAPSTONE

120. ***Particle Motion*** The figure shows the graphs of the position, velocity, and acceleration functions of a particle.

(a) Copy the graphs of the functions shown. Identify each graph. Explain your reasoning. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(b) On your sketch, identify when the particle speeds up and when it slows down. Explain your reasoning.



Finding a Pattern **In Exercises 121 and 122, develop a general rule for $f^{(n)}(x)$ given $f(x)$.**

121. $f(x) = x^n$

122. $f(x) = \dfrac{1}{x}$

123. ***Finding a Pattern*** Consider the function $f(x) = g(x)h(x)$.

(a) Use the Product Rule to generate rules for finding $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$.

(b) Use the results of part (a) to write a general rule for $f^{(n)}(x)$.

124. ***Finding a Pattern*** Develop a general rule for $[xf(x)]^{(n)}$, where f is a differentiable function of x.

In Exercises 125 and 126, find the derivatives of the function f for $n = 1, 2, 3,$ and 4. Use the results to write a general rule for $f'(x)$ in terms of n.

125. $f(x) = x^n \sin x$

126. $f(x) = \dfrac{\cos x}{x^n}$

Differential Equations **In Exercises 127–130, verify that the function satisfies the differential equation.**

	Function	*Differential Equation*
127.	$y = \dfrac{1}{x}, x > 0$	$x^3 y'' + 2x^2 y' = 0$
128.	$y = 2x^3 - 6x + 10$	$-y''' - xy'' - 2y' = -24x^2$
129.	$y = 2 \sin x + 3$	$y'' + y = 3$
130.	$y = 3 \cos x + \sin x$	$y'' + y = 0$

True or False? **In Exercises 131–136, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

131. If $y = f(x)g(x)$, then $dy/dx = f'(x)g'(x)$.

132. If $y = (x + 1)(x + 2)(x + 3)(x + 4)$, then $d^5y/dx^5 = 0$.

133. If $f'(c)$ and $g'(c)$ are zero and $h(x) = f(x)g(x)$, then $h'(c) = 0$.

134. If $f(x)$ is an nth-degree polynomial, then $f^{(n+1)}(x) = 0$.

135. The second derivative represents the rate of change of the first derivative.

136. If the velocity of an object is constant, then its acceleration is zero.

137. Find a second-degree polynomial $f(x) = ax^2 + bx + c$ such that its graph has a tangent line with slope 10 at the point $(2, 7)$ and an x-intercept at $(1, 0)$.

138. Consider the third-degree polynomial

$$f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0.$$

Determine conditions for a, b, c, and d if the graph of f has (a) no horizontal tangent lines, (b) exactly one horizontal tangent line, and (c) exactly two horizontal tangent lines. Give an example for each case.

139. Find the derivative of $f(x) = x|x|$. Does $f''(0)$ exist?

140. ***Think About It*** Let f and g be functions whose first and second derivatives exist on an interval I. Which of the following formulas is (are) true?

(a) $fg'' - f''g = (fg' - f'g)'$

(b) $fg'' + f''g = (fg)''$

141. Use the Product Rule twice to prove that if f, g, and h are differentiable functions of x, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

3.4 The Chain Rule

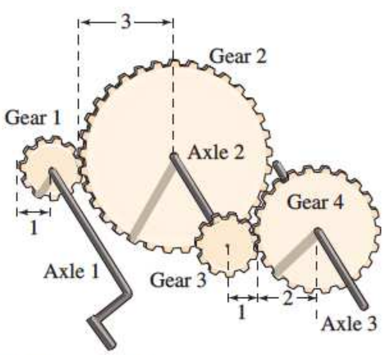
- **Find the derivative of a composite function using the Chain Rule.**
- **Find the derivative of a function using the General Power Rule.**
- **Simplify the derivative of a function using algebra.**
- **Find the derivative of a transcendental function using the Chain Rule.**
- **Find the derivative of a function involving the natural logarithmic function.**
- **Define and differentiate exponential functions that have bases other than *e*.**

The Chain Rule

This text has yet to discuss one of the most powerful differentiation rules—the **Chain Rule.** This rule deals with composite functions and adds a surprising versatility to the rules discussed in the two previous sections. For example, compare the following functions. Those on the left can be differentiated without the Chain Rule, and those on the right are best differentiated with the Chain Rule.

Without the Chain Rule	*With the Chain Rule*
$y = x^2 + 1$	$y = \sqrt{x^2 + 1}$
$y = \sin x$	$y = \sin 6x$
$y = 3x + 2$	$y = (3x + 2)^5$
$y = e^x + \tan x$	$y = e^{5x} + \tan x^2$

Basically, the Chain Rule states that if y changes dy/du times as fast as u, and u changes du/dx times as fast as x, then y changes $(dy/du)(du/dx)$ times as fast as x.



Axle 1: y revolutions per minute
Axle 2: u revolutions per minute
Axle 3: x revolutions per minute
Figure 3.25

EXAMPLE 1 The Derivative of a Composite Function

A set of gears is constructed, as shown in Figure 3.25, such that the second and third gears are on the same axle. As the first axle revolves, it drives the second axle, which in turn drives the third axle. Let y, u, and x represent the numbers of revolutions per minute of the first, second, and third axles, respectively. Find dy/du, du/dx, and dy/dx, and show that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Solution Because the circumference of the second gear is three times that of the first, the first axle must make three revolutions to turn the second axle once. Similarly, the second axle must make two revolutions to turn the third axle once, and you can write

$$\frac{dy}{du} = 3 \quad \text{and} \quad \frac{du}{dx} = 2.$$

Combining these two results, you know that the first axle must make six revolutions to turn the third axle once. So, you can write

$$\frac{dy}{dx} = \boxed{\begin{array}{l}\text{Rate of change of first axle}\\ \text{with respect to second axle}\end{array}} \cdot \boxed{\begin{array}{l}\text{Rate of change of second axle}\\ \text{with respect to third axle}\end{array}}$$

$$= \frac{dy}{du} \cdot \frac{du}{dx} = 3 \cdot 2 = 6 = \boxed{\begin{array}{l}\text{Rate of change of first axle}\\ \text{with respect to third axle}\end{array}}$$

In other words, the rate of change of y with respect to x is the product of the rate of change of y with respect to u and the rate of change of u with respect to x. ■

EXPLORATION

Using the Chain Rule Each of the following functions can be differentiated using rules that you studied in Sections 3.2 and 3.3. For each function, find the derivative using those rules. Then find the derivative using the Chain Rule. Compare your results. Which method is simpler?

a. $\dfrac{2}{3x + 1}$

b. $(x + 2)^3$

c. $\sin 2x$

Example 1 illustrates a simple case of the Chain Rule. The general rule is stated below.

THEOREM 3.11 THE CHAIN RULE

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

PROOF Let $h(x) = f(g(x))$. Then, using the alternative form of the derivative, you need to show that, for $x = c$,

$$h'(c) = f'(g(c))g'(c).$$

An important consideration in this proof is the behavior of g as x approaches c. A problem occurs if there are values of x, other than c, such that $g(x) = g(c)$. Appendix A shows how to use the differentiability of f and g to overcome this problem. For now, assume that $g(x) \neq g(c)$ for values of x other than c. In the proofs of the Product Rule and the Quotient Rule, the same quantity was added and subtracted to obtain the desired form. This proof uses a similar technique—multiplying and dividing by the same (nonzero) quantity. Note that because g is differentiable, it is also continuous, and it follows that $g(x) \to g(c)$ as $x \to c$.

$$\begin{aligned} h'(c) &= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} \\ &= \lim_{x \to c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c}\right], \quad g(x) \neq g(c) \\ &= \left[\lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)}\right]\left[\lim_{x \to c} \frac{g(x) - g(c)}{x - c}\right] \\ &= f'(g(c))g'(c) \end{aligned}$$

■

When applying the Chain Rule, it is helpful to think of the composite function $f \circ g$ as having two parts—an inner part and an outer part.

Outer function

$$y = f(g(x)) = f(u)$$

Inner function

The derivative of $y = f(u)$ is the derivative of the outer function (at the inner function u) *times* the derivative of the inner function.

$$y' = f'(u) \cdot u'$$

Derivative of outer function

Derivative of inner function

EXAMPLE 2 Decomposition of a Composite Function

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
a. $y = \dfrac{1}{x+1}$	$u = x + 1$	$y = \dfrac{1}{u}$
b. $y = \sin 2x$	$u = 2x$	$y = \sin u$
c. $y = \sqrt{3x^2 - x + 1}$	$u = 3x^2 - x + 1$	$y = \sqrt{u}$
d. $y = \tan^2 x$	$u = \tan x$	$y = u^2$

EXAMPLE 3 Using the Chain Rule

Find dy/dx for $y = (x^2 + 1)^3$.

STUDY TIP You could also solve the problem in Example 3 without using the Chain Rule by observing that

$$y = x^6 + 3x^4 + 3x^2 + 1$$

and

$$y' = 6x^5 + 12x^3 + 6x.$$

Verify that this is the same result as the derivative in Example 3. Which method would you use to find

$$\frac{d}{dx}(x^2 + 1)^{50}?$$

Solution For this function, you can consider the inside function to be $u = x^2 + 1$. By the Chain Rule, you obtain

$$\frac{dy}{dx} = \underbrace{3(x^2 + 1)^2}_{\frac{dy}{du}}\underbrace{(2x)}_{\frac{du}{dx}} = 6x(x^2 + 1)^2.$$

■

The General Power Rule

The function in Example 3 is an example of one of the most common types of composite functions, $y = [u(x)]^n$. The rule for differentiating such functions is called the **General Power Rule,** and it is a special case of the Chain Rule.

THEOREM 3.12 THE GENERAL POWER RULE

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a real number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = n u^{n-1} u'.$$

PROOF Because $y = u^n$, you apply the Chain Rule to obtain

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$

$$= \frac{d}{du}[u^n]\frac{du}{dx}.$$

By the (Simple) Power Rule in Section 3.2, you have $D_u[u^n] = n u^{n-1}$, and it follows that

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}.$$

■

EXAMPLE 4 Applying the General Power Rule

Find the derivative of $f(x) = (3x - 2x^2)^3$.

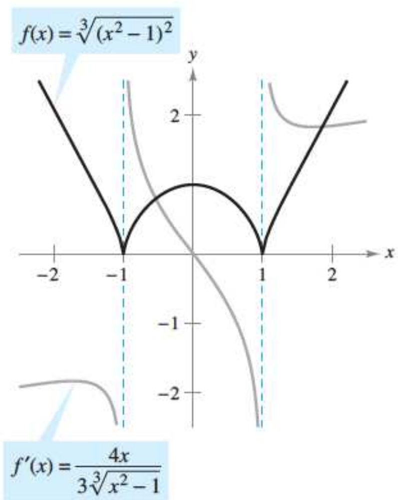
Solution Let $u = 3x - 2x^2$. Then

$$f(x) = (3x - 2x^2)^3 = u^3$$

and, by the General Power Rule, the derivative is

$$f'(x) = \underbrace{3}_{n}\underbrace{(3x - 2x^2)^2}_{u^{n-1}}\underbrace{\frac{d}{dx}[3x - 2x^2]}_{u'} \qquad \text{Apply General Power Rule.}$$

$$= 3(3x - 2x^2)^2(3 - 4x). \qquad \text{Differentiate } 3x - 2x^2.$$



The derivative of f is 0 at $x = 0$ and is undefined at $x = \pm 1$.
Figure 3.26

EXAMPLE 5 Differentiating Functions Involving Radicals

Find all points on the graph of $f(x) = \sqrt[3]{(x^2 - 1)^2}$ for which $f'(x) = 0$ and those for which $f'(x)$ does not exist.

Solution Begin by rewriting the function as

$$f(x) = (x^2 - 1)^{2/3}.$$

Then, applying the General Power Rule (with $u = x^2 - 1$) produces

$$f'(x) = \underbrace{\frac{2}{3}}_{n}\underbrace{(x^2 - 1)^{-1/3}}_{u^{n-1}}\underbrace{(2x)}_{u'} \qquad \text{Apply General Power Rule.}$$

$$= \frac{4x}{3\sqrt[3]{x^2 - 1}}. \qquad \text{Write in radical form.}$$

So, $f'(x) = 0$ when $x = 0$ and $f'(x)$ does not exist when $x = \pm 1$, as shown in Figure 3.26.

EXAMPLE 6 Differentiating Quotients with Constant Numerators

Differentiate $g(t) = \dfrac{-7}{(2t - 3)^2}$.

Solution Begin by rewriting the function as

$$g(t) = -7(2t - 3)^{-2}.$$

NOTE Try differentiating the function in Example 6 using the Quotient Rule. You should obtain the same result, but using the Quotient Rule is less efficient than using the General Power Rule.

Then, applying the General Power Rule produces

$$g'(t) = \underbrace{(-7)}_{\substack{\text{Constant}\\ \text{Multiple Rule}}}\overbrace{(-2)}^{n}\overbrace{(2t - 3)^{-3}}^{u^{n-1}}\overbrace{(2)}^{u'} \qquad \text{Apply General Power Rule.}$$

$$= 28(2t - 3)^{-3} \qquad \text{Simplify.}$$

$$= \frac{28}{(2t - 3)^3}. \qquad \text{Write with positive exponent.}$$

Simplifying Derivatives

The next three examples illustrate some techniques for simplifying the "raw derivatives" of functions involving products, quotients, and composites.

EXAMPLE 7 Simplifying by Factoring Out the Least Powers

$$\begin{aligned}
f(x) &= x^2\sqrt{1-x^2} && \text{Original function}\\
&= x^2(1-x^2)^{1/2} && \text{Rewrite.}\\
f'(x) &= x^2\frac{d}{dx}[(1-x^2)^{1/2}] + (1-x^2)^{1/2}\frac{d}{dx}[x^2] && \text{Product Rule}\\
&= x^2\left[\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right] + (1-x^2)^{1/2}(2x) && \text{General Power Rule}\\
&= -x^3(1-x^2)^{-1/2} + 2x(1-x^2)^{1/2} && \text{Simplify.}\\
&= x(1-x^2)^{-1/2}[-x^2(1) + 2(1-x^2)] && \text{Factor.}\\
&= \frac{x(2-3x^2)}{\sqrt{1-x^2}} && \text{Simplify.}
\end{aligned}$$

EXAMPLE 8 Simplifying the Derivative of a Quotient

TECHNOLOGY Symbolic differentiation utilities are capable of differentiating very complicated functions. Often, however, the result is given in unsimplified form. If you have access to such a utility, use it to find the derivatives of the functions given in Examples 7, 8, and 9. Then compare the results with those given on this page.

$$\begin{aligned}
f(x) &= \frac{x}{\sqrt[3]{x^2+4}} && \text{Original function}\\
&= \frac{x}{(x^2+4)^{1/3}} && \text{Rewrite.}\\
f'(x) &= \frac{(x^2+4)^{1/3}(1) - x(1/3)(x^2+4)^{-2/3}(2x)}{(x^2+4)^{2/3}} && \text{Quotient Rule}\\
&= \frac{1}{3}(x^2+4)^{-2/3}\left[\frac{3(x^2+4) - (2x^2)(1)}{(x^2+4)^{2/3}}\right] && \text{Factor.}\\
&= \frac{x^2+12}{3(x^2+4)^{4/3}} && \text{Simplify.}
\end{aligned}$$

EXAMPLE 9 Simplifying the Derivative of a Power

$$\begin{aligned}
y &= \left(\frac{3x-1}{x^2+3}\right)^2 && \text{Original function}\\
y' &= \underset{n}{2}\underbrace{\left(\frac{3x-1}{x^2+3}\right)}_{u^{n-1}}\underbrace{\frac{d}{dx}\left[\frac{3x-1}{x^2+3}\right]}_{u'} && \text{General Power Rule}\\
&= \left[\frac{2(3x-1)}{x^2+3}\right]\left[\frac{(x^2+3)(3)-(3x-1)(2x)}{(x^2+3)^2}\right] && \text{Quotient Rule}\\
&= \frac{2(3x-1)(3x^2+9-6x^2+2x)}{(x^2+3)^3} && \text{Multiply.}\\
&= \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3} && \text{Simplify.}
\end{aligned}$$

Transcendental Functions and the Chain Rule

The "Chain Rule versions" of the derivatives of the six trigonometric functions and the natural exponential function are as follows.

$$\frac{d}{dx}[\sin u] = (\cos u)\,u' \qquad \frac{d}{dx}[\cos u] = -(\sin u)\,u'$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u)\,u' \qquad \frac{d}{dx}[\cot u] = -(\csc^2 u)\,u'$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u)\,u' \qquad \frac{d}{dx}[\csc u] = -(\csc u \cot u)\,u'$$

$$\frac{d}{dx}[e^u] = e^u u'$$

EXAMPLE 10 Applying the Chain Rule to Transcendental Functions

NOTE Be sure that you understand the mathematical conventions regarding parentheses and trigonometric functions. For instance, in Example 10(a), $\sin 2x$ is written to mean $\sin(2x)$.

a. $y = \sin \overbrace{2x}^{u}$ $\qquad y' = \overbrace{\cos 2x}^{\cos u}\overbrace{\frac{d}{dx}[2x]}^{u'} = (\cos 2x)(2) = 2\cos 2x$

b. $y = \cos\overbrace{(x-1)}^{u}$ $\qquad y' = \overbrace{-\sin(x-1)}^{-\sin u}\overbrace{\frac{d}{dx}[x-1]}^{u'} = -\sin(x-1)$

c. $y = e^{\overbrace{3x}^{u}}$ $\qquad y' = \overbrace{e^{3x}}^{e^u}\overbrace{\frac{d}{dx}[3x]}^{u'} = 3e^{3x}$

EXAMPLE 11 Parentheses and Trigonometric Functions

a. $y = \cos 3x^2 = \cos(3x^2)$ $\qquad y' = (-\sin 3x^2)(6x) = -6x \sin 3x^2$

b. $y = (\cos 3)x^2$ $\qquad y' = (\cos 3)(2x) = 2x \cos 3$

c. $y = \cos(3x)^2 = \cos(9x^2)$ $\qquad y' = (-\sin 9x^2)(18x) = -18x \sin 9x^2$

d. $y = \cos^2 x = (\cos x)^2$ $\qquad y' = 2(\cos x)(-\sin x) = -2\cos x \sin x$ ■

To find the derivative of a function of the form $k(x) = f(g(h(x)))$, you need to apply the Chain Rule twice, as shown in Example 12.

EXAMPLE 12 Repeated Application of the Chain Rule

$f(t)$	$= \sin^3 4t$	Original function
	$= (\sin 4t)^3$	Rewrite.
$f'(t)$	$= 3(\sin 4t)^2 \frac{d}{dt}[\sin 4t]$	Apply Chain Rule once.
	$= 3(\sin 4t)^2(\cos 4t)\frac{d}{dt}[4t]$	Apply Chain Rule a second time.
	$= 3(\sin 4t)^2(\cos 4t)(4)$	
	$= 12 \sin^2 4t \cos 4t$	Simplify. ■

The Derivative of the Natural Logarithmic Function

Up to this point in the text, derivatives of algebraic functions have been algebraic and derivatives of transcendental functions have been transcendental. The next theorem looks at an unusual situation in which the derivative of a transcendental function is algebraic. Specifically, the derivative of the natural logarithmic function is the algebraic function $1/x$.

THEOREM 3.13 DERIVATIVE OF THE NATURAL LOGARITHMIC FUNCTION

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}, \quad x > 0$

2. $\dfrac{d}{dx}[\ln u] = \dfrac{1}{u}\dfrac{du}{dx} = \dfrac{u'}{u}, \quad u > 0$

EXPLORATION

Use the *table* feature of a graphing utility to display the values of $f(x) = \ln x$ and its derivative for $x = 0, 1, 2, 3, \ldots$. What do these values tell you about the derivative of the natural logarithmic function?

PROOF To prove the first part, let $y = \ln x$, which implies that $e^y = x$. Differentiating both sides of this equation produces the following.

$$y = \ln x$$
$$e^y = x$$
$$\frac{d}{dx}[e^y] = \frac{d}{dx}[x]$$
$$e^y\frac{dy}{dx} = 1 \qquad \text{Chain Rule}$$
$$\frac{dy}{dx} = \frac{1}{e^y}$$
$$\frac{dy}{dx} = \frac{1}{x}$$

The second part of the theorem can be obtained by applying the Chain Rule to the first part. ■

EXAMPLE 13 Differentiation of Logarithmic Functions

a. $\dfrac{d}{dx}[\ln(2x)] = \dfrac{u'}{u} = \dfrac{2}{2x} = \dfrac{1}{x}$ $\qquad u = 2x$

b. $\dfrac{d}{dx}[\ln(x^2 + 1)] = \dfrac{u'}{u} = \dfrac{2x}{x^2 + 1}$ $\qquad u = x^2 + 1$

c. $\dfrac{d}{dx}[x\ln x] = x\left(\dfrac{d}{dx}[\ln x]\right) + (\ln x)\left(\dfrac{d}{dx}[x]\right)$ $\qquad$ Product Rule

$$= x\left(\frac{1}{x}\right) + (\ln x)(1)$$
$$= 1 + \ln x$$

d. $\dfrac{d}{dx}[(\ln x)^3] = 3(\ln x)^2\dfrac{d}{dx}[\ln x]$ $\qquad$ Chain Rule

$$= 3(\ln x)^2\frac{1}{x}$$ ■

The Granger Collection

JOHN NAPIER (1550–1617)

Logarithms were invented by the Scottish mathematician John Napier. Although he did not introduce the *natural* logarithmic function, it is sometimes called the *Napierian* logarithm.

John Napier used logarithmic properties to simplify *calculations* involving products, quotients, and powers. Of course, given the availability of calculators, there is now little need for this particular application of logarithms. However, there is great value in using logarithmic properties to simplify *differentiation* involving products, quotients, and powers.

EXAMPLE 14 Logarithmic Properties as Aids to Differentiation

Differentiate $f(x) = \ln\sqrt{x+1}$.

Solution Because

$$f(x) = \ln\sqrt{x+1} = \ln(x+1)^{1/2} = \frac{1}{2}\ln(x+1) \qquad \text{Rewrite before differentiating.}$$

you can write

$$f'(x) = \frac{1}{2}\left(\frac{1}{x+1}\right) = \frac{1}{2(x+1)}. \qquad \text{Differentiate.}$$

EXAMPLE 15 Logarithmic Properties as Aids to Differentiation

Differentiate $f(x) = \ln\dfrac{x(x^2+1)^2}{\sqrt{2x^3-1}}$.

NOTE In Examples 14 and 15, be sure that you see the benefit of applying logarithmic properties *before* differentiation. Consider, for instance, the difficulty of direct differentiation of the function given in Example 15.

Solution

$$f(x) = \ln\frac{x(x^2+1)^2}{\sqrt{2x^3-1}} \qquad \text{Write original function.}$$

$$= \ln x + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^3-1) \qquad \text{Rewrite before differentiating.}$$

$$f'(x) = \frac{1}{x} + 2\left(\frac{2x}{x^2+1}\right) - \frac{1}{2}\left(\frac{6x^2}{2x^3-1}\right) \qquad \text{Differentiate.}$$

$$= \frac{1}{x} + \frac{4x}{x^2+1} - \frac{3x^2}{2x^3-1} \qquad \text{Simplify.}$$

Because the natural logarithm is undefined for negative numbers, you will often encounter expressions of the form $\ln|u|$. Theorem 3.14 states that you can differentiate functions of the form $y = \ln|u|$ as if the absolute value notation was not present.

THEOREM 3.14 DERIVATIVE INVOLVING ABSOLUTE VALUE

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u}.$$

PROOF If $u > 0$, then $|u| = u$, and the result follows from Theorem 3.13. If $u < 0$, then $|u| = -u$, and you have

$$\frac{d}{dx}[\ln|u|] = \frac{d}{dx}[\ln(-u)] = \frac{-u'}{-u} = \frac{u'}{u}.$$

Bases Other than e

The **base** of the natural exponential function is e. This "natural" base can be used to assign a meaning to a general base a.

DEFINITION OF EXPONENTIAL FUNCTION TO BASE a

If a is a positive real number $(a \neq 1)$ and x is any real number, then the **exponential function to the base a** is denoted by a^x and is defined by

$$a^x = e^{(\ln a)x}.$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

Logarithmic functions to bases other than e can be defined in much the same way as exponential functions to other bases are defined.

DEFINITION OF LOGARITHMIC FUNCTION TO BASE a

If a is a positive real number $(a \neq 1)$ and x is any positive real number, then the **logarithmic function to the base a** is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x.$$

To differentiate exponential and logarithmic functions to other bases, you have two options: (1) use the definitions of a^x and $\log_a x$ and differentiate using the rules for the natural exponential and logarithmic functions, or (2) use the following differentiation rules for bases other than e.

NOTE These differentiation rules are similar to those for the natural exponential function and the natural logarithmic function. In fact, they differ only by the constant factors $\ln a$ and $1/\ln a$. This points out one reason why, for calculus, e is the most convenient base.

THEOREM 3.15 DERIVATIVES FOR BASES OTHER THAN e

Let a be a positive real number $(a \neq 1)$ and let u be a differentiable function of x.

1. $\dfrac{d}{dx}[a^x] = (\ln a)a^x$ **2.** $\dfrac{d}{dx}[a^u] = (\ln a)a^u \dfrac{du}{dx}$

3. $\dfrac{d}{dx}[\log_a x] = \dfrac{1}{(\ln a)x}$ **4.** $\dfrac{d}{dx}[\log_a u] = \dfrac{1}{(\ln a)u} \dfrac{du}{dx}$

PROOF By definition, $a^x = e^{(\ln a)x}$. Therefore, you can prove the first rule by letting $u = (\ln a)x$ and differentiating with base e to obtain

$$\frac{d}{dx}[a^x] = \frac{d}{dx}[e^{(\ln a)x}] = e^u \frac{du}{dx} = e^{(\ln a)x}(\ln a) = (\ln a)a^x.$$

To prove the third rule, you can write

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\left[\frac{1}{\ln a} \ln x\right] = \frac{1}{\ln a}\left(\frac{1}{x}\right) = \frac{1}{(\ln a)x}.$$

The second and fourth rules are simply the Chain Rule versions of the first and third rules. ■

EXAMPLE 16 Differentiating Functions to Other Bases

Find the derivative of each function.

a. $y = 2^x$ **b.** $y = 2^{3x}$ **c.** $y = \log_{10} \cos x$

Solution

a. $y' = \dfrac{d}{dx}[2^x] = (\ln 2)2^x$

b. $y' = \dfrac{d}{dx}[2^{3x}] = (\ln 2)2^{3x}(3) = (3 \ln 2)2^{3x}$

Try writing 2^{3x} as 8^x and differentiating to see that you obtain the same result.

c. $y' = \dfrac{d}{dx}[\log_{10} \cos x] = \dfrac{-\sin x}{(\ln 10) \cos x} = -\dfrac{1}{\ln 10} \tan x$ ■

STUDY TIP To become skilled at differentiation, you should memorize each rule in words, not symbols. As an aid to memorization, note that the cofunctions (cosine, cotangent, and cosecant) require a negative sign as part of their derivatives.

This section concludes with a summary of the differentiation rules studied so far.

SUMMARY OF DIFFERENTIATION RULES

General Differentiation Rules Let u and v be differentiable functions of x.

Constant Rule: $\dfrac{d}{dx}[c] = 0$

(Simple) Power Rule: $\dfrac{d}{dx}[x^n] = nx^{n-1}, \quad \dfrac{d}{dx}[x] = 1$

Constant Multiple Rule: $\dfrac{d}{dx}[cu] = cu'$

Sum or Difference Rule: $\dfrac{d}{dx}[u \pm v] = u' \pm v'$

Product Rule: $\dfrac{d}{dx}[uv] = uv' + vu'$

Quotient Rule: $\dfrac{d}{dx}\left[\dfrac{u}{v}\right] = \dfrac{vu' - uv'}{v^2}$

Chain Rule: $\dfrac{d}{dx}[f(u)] = f'(u)\, u'$

General Power Rule: $\dfrac{d}{dx}[u^n] = nu^{n-1} u'$

Derivatives of Trigonometric Functions

$\dfrac{d}{dx}[\sin x] = \cos x$ $\quad \dfrac{d}{dx}[\tan x] = \sec^2 x$ $\quad \dfrac{d}{dx}[\sec x] = \sec x \tan x$

$\dfrac{d}{dx}[\cos x] = -\sin x$ $\quad \dfrac{d}{dx}[\cot x] = -\csc^2 x$ $\quad \dfrac{d}{dx}[\csc x] = -\csc x \cot x$

Derivatives of Exponential and Logarithmic Functions

$\dfrac{d}{dx}[e^x] = e^x$ $\quad \dfrac{d}{dx}[\ln x] = \dfrac{1}{x}$

$\dfrac{d}{dx}[a^x] = (\ln a)a^x$ $\quad \dfrac{d}{dx}[\log_a x] = \dfrac{1}{(\ln a)x}$

3.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, complete the table.

	$y = f(g(x))$	$u = g(x)$	$y = f(u)$
1.	$y = (5x - 8)^4$		
2.	$y = \dfrac{1}{\sqrt{x+1}}$		
3.	$y = \sqrt{x^3 - 7}$		
4.	$y = 3\tan(\pi x^2)$		
5.	$y = \csc^3 x$		
6.	$y = \sin \dfrac{5x}{2}$		
7.	$y = e^{-2x}$		
8.	$y = (\ln x)^3$		

In Exercises 9–38, find the derivative of the function.

9. $y = (4x - 1)^3$

10. $y = 2(6 - x^2)^5$

11. $g(x) = 3(4 - 9x)^4$

12. $y = 3(5 - x^2)^5$

13. $f(x) = (9 - x^2)^{2/3}$

14. $f(t) = (9t + 7)^{2/3}$

15. $f(t) = \sqrt{5 - t}$

16. $g(x) = \sqrt{9 - 4x}$

17. $y = \sqrt[3]{6x^2 + 1}$

18. $g(x) = \sqrt{x^2 - 2x + 1}$

19. $y = 2\sqrt[4]{9 - x^2}$

20. $f(x) = -3\sqrt[4]{2 - 9x}$

21. $y = \dfrac{1}{x - 2}$

22. $s(t) = \dfrac{1}{t^2 + 3t - 1}$

23. $f(t) = \left(\dfrac{1}{t - 3}\right)^2$

24. $y = -\dfrac{8}{(t + 3)^3}$

25. $y = \dfrac{1}{\sqrt{x + 2}}$

26. $g(t) = \sqrt{\dfrac{1}{t^2 - 2}}$

27. $f(x) = x^2(x - 2)^4$

28. $f(x) = x(3x - 7)^3$

29. $y = x\sqrt{1 - x^2}$

30. $y = \frac{1}{2}x^2\sqrt{16 - x^2}$

31. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

32. $y = \dfrac{x}{\sqrt{x^4 + 2}}$

33. $g(x) = \left(\dfrac{x + 5}{x^2 + 2}\right)^2$

34. $g(x) = \left(\dfrac{3x^2 - 1}{2x + 5}\right)^3$

35. $f(x) = ((x^2 + 3)^5 + x)^2$

36. $g(x) = (2 + (x^2 + 1)^4)^3$

37. $f(x) = \sqrt{2 + \sqrt{2 + \sqrt{x}}}$

38. $g(t) = \sqrt{\sqrt{t + 1} + 1}$

CAS **In Exercises 39–48, use a computer algebra system to find the derivative of the function. Then use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.**

39. $y = \dfrac{\sqrt{x} + 1}{x^2 + 1}$

40. $y = \sqrt{\dfrac{2x}{x + 1}}$

41. $g(t) = \dfrac{3t^2}{\sqrt{t^2 + 2t - 1}}$

42. $f(x) = \sqrt{x}(2 - x)^2$

43. $y = \sqrt{\dfrac{x + 1}{x}}$

44. $y = (t^2 - 9)\sqrt{t + 2}$

45. $s(t) = \dfrac{-2(2 - t)\sqrt{1 + t}}{3}$

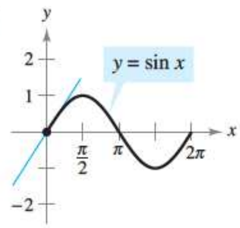
46. $g(x) = \sqrt{x - 1} + \sqrt{x + 1}$

47. $y = \dfrac{\cos \pi x + 1}{x}$

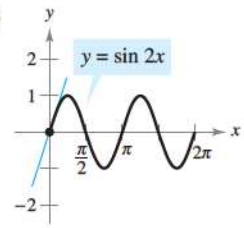
48. $y = x^2 \tan\dfrac{1}{x}$

In Exercises 49 and 50, find the slope of the tangent line to the sine function at the origin. Compare this value with the number of complete cycles in the interval $[0, 2\pi]$. What can you conclude about the slope of the sine function $\sin ax$ at the origin?

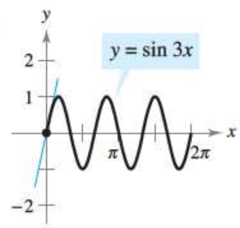
49. (a)



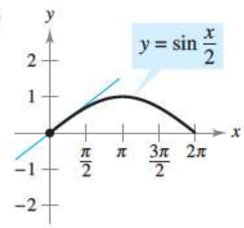
(b)



50. (a)

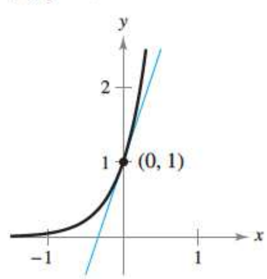


(b)

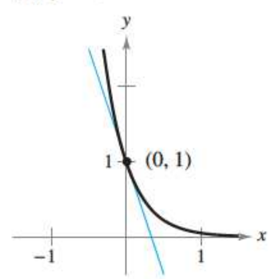


In Exercises 51 and 52, find the slope of the tangent line to the graph of the function at the point (0, 1).

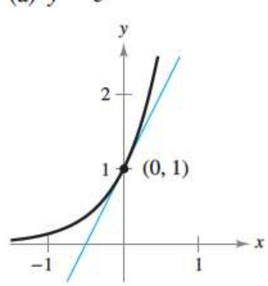
51. (a) $y = e^{3x}$



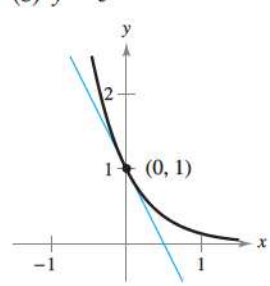
(b) $y = e^{-3x}$



52. (a) $y = e^{2x}$

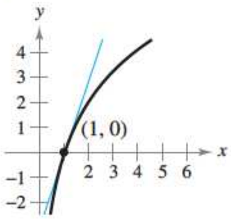


(b) $y = e^{-2x}$

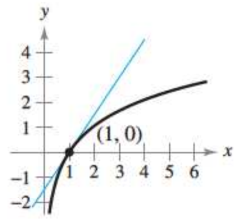


In Exercises 53–56, find the slope of the tangent line to the graph of the logarithmic function at the point (1, 0).

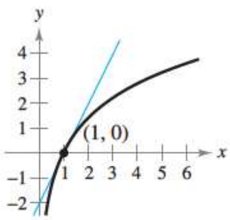
53. $y = \ln x^3$



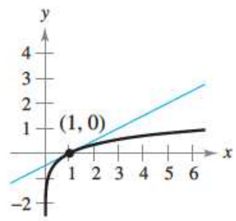
54. $y = \ln x^{3/2}$



55. $y = \ln x^2$



56. $y = \ln x^{1/2}$



In Exercises 57–106, find the derivative of the function.

57. $y = \cos 4x$

58. $y = \sin \pi x$

59. $g(x) = 5 \tan 3x$

60. $h(x) = \sec x^3$

61. $f(\theta) = \tan^2 5\theta$

62. $g(\theta) = \cos^2 8\theta$

63. $f(\theta) = \frac{1}{4}\sin^2 2\theta$

64. $g(t) = 5\cos^3 \pi t$

65. $y = \sqrt{x} + \frac{1}{4}\sin(2x)^2$

66. $y = 3x - 5\cos(\pi x)^2$

67. $y = \sin(\cos x)$

68. $y = \sin \sqrt[3]{x} + \sqrt[3]{\sin x}$

69. $y = \sin(\tan 2x)$

70. $y = \cos\sqrt{\sin(\tan \pi x)}$

71. $f(x) = e^{2x}$

72. $y = e^{-x^2}$

73. $y = e^{\sqrt{x}}$

74. $y = x^2 e^{-x}$

75. $g(t) = (e^{-t} + e^t)^3$

76. $g(t) = e^{-3/t^2}$

77. $y = \ln(e^{x^2})$

78. $y = \ln\left(\dfrac{1+e^x}{1-e^x}\right)$

79. $y = \dfrac{2}{e^x + e^{-x}}$

80. $y = \dfrac{e^x - e^{-x}}{2}$

81. $y = x^2e^x - 2xe^x + 2e^x$

82. $y = xe^x - e^x$

83. $f(x) = e^{-x}\ln x$

84. $f(x) = e^3 \ln x$

85. $y = e^x(\sin x + \cos x)$

86. $y = \ln e^x$

87. $g(x) = \ln x^2$

88. $h(x) = \ln(2x^2 + 3)$

89. $y = (\ln x)^4$

90. $y = x \ln x$

91. $y = \ln\left(x\sqrt{x^2 - 1}\right)$

92. $y = \ln\sqrt{x^2 - 9}$

93. $f(x) = \ln\left(\dfrac{x}{x^2+1}\right)$

94. $f(x) = \ln\left(\dfrac{2x}{x+3}\right)$

95. $g(t) = \dfrac{\ln t}{t^2}$

96. $h(t) = \dfrac{\ln t}{t}$

97. $y = \ln\sqrt{\dfrac{x+1}{x-1}}$

98. $y = \ln\sqrt[3]{\dfrac{x-2}{x+2}}$

99. $y = \dfrac{-\sqrt{x^2+1}}{x} + \ln\left(x + \sqrt{x^2+1}\right)$

100. $y = \dfrac{-\sqrt{x^2+4}}{2x^2} - \dfrac{1}{4}\ln\left(\dfrac{2+\sqrt{x^2+4}}{x}\right)$

101. $y = \ln|\sin x|$

102. $y = \ln|\csc x|$

103. $y = \ln\left|\dfrac{\cos x}{\cos x - 1}\right|$

104. $y = \ln|\sec x + \tan x|$

105. $y = \ln\left|\dfrac{-1 + \sin x}{2 + \sin x}\right|$

106. $y = \ln\sqrt{1 + \sin^2 x}$

In Exercises 107–114, find the second derivative of the function.

107. $f(x) = 5(2 - 7x)^4$

108. $f(x) = 4(x^2 - 2)^3$

109. $f(x) = \dfrac{1}{x - 6}$

110. $f(x) = \dfrac{4}{(x+2)^3}$

111. $f(x) = \sin x^2$

112. $f(x) = \sec^2 \pi x$

113. $f(x) = (3 + 2x)e^{-3x}$

114. $g(x) = \sqrt{x} + e^x \ln x$

In Exercises 115–122, evaluate the derivative of the function at the given point. Use a graphing utility to verify your result.

	Function	*Point*
115.	$s(t) = \sqrt{t^2 + 6t - 2}$	$(3, 5)$
116.	$y = \sqrt[5]{3x^3 + 4x}$	$(2, 2)$
117.	$f(x) = \dfrac{5}{x^3 - 2}$	$\left(-2, -\dfrac{1}{2}\right)$
118.	$f(x) = \dfrac{1}{(x^2 - 3x)^2}$	$\left(4, \dfrac{1}{16}\right)$
119.	$f(t) = \dfrac{3t + 2}{t - 1}$	$(0, -2)$
120.	$f(x) = \dfrac{x + 1}{2x - 3}$	$(2, 3)$
121.	$y = 26 - \sec^3 4x$	$(0, 25)$
122.	$y = \dfrac{1}{x} + \sqrt{\cos x}$	$\left(\dfrac{\pi}{2}, \dfrac{2}{\pi}\right)$

In Exercises 123–130, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

	Function	*Point*
123.	$f(x) = \sqrt{2x^2 - 7}$	$(4, 5)$
124.	$f(x) = \frac{1}{3}x\sqrt{x^2 + 5}$	$(2, 2)$
125.	$f(x) = \sin 2x$	$(\pi, 0)$
126.	$y = \cos 3x$	$\left(\dfrac{\pi}{4}, -\dfrac{\sqrt{2}}{2}\right)$
127.	$y = 2\tan^3 x$	$\left(\dfrac{\pi}{4}, 2\right)$
128.	$f(x) = \tan^2 x$	$\left(\dfrac{\pi}{4}, 1\right)$
129.	$y = 4 - x^2 - \ln\left(\frac{1}{2}x + 1\right)$	$(0, 4)$
130.	$y = 2e^{1-x^2}$	$(1, 2)$

In Exercises 131–146, find the derivative of the function.

131. $f(x) = 4^x$

132. $g(x) = 5^{-x}$

133. $y = 5^{x-2}$

134. $y = x(6^{-2x})$

135. $g(t) = t^2 2^t$

136. $f(t) = \dfrac{3^{2t}}{t}$

137. $h(\theta) = 2^{-\theta} \cos \pi\theta$

138. $g(\alpha) = 5^{-\alpha/2} \sin 2\alpha$

139. $y = \log_3 x$

140. $y = \log_{10} 2x$

141. $f(x) = \log_2 \dfrac{x^2}{x-1}$

142. $h(x) = \log_3 \dfrac{x\sqrt{x-1}}{2}$

143. $y = \log_5 \sqrt{x^2-1}$

144. $y = \log_{10} \dfrac{x^2-1}{x}$

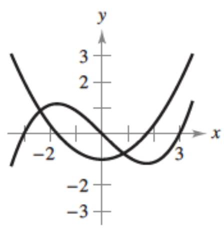
145. $g(t) = \dfrac{10 \log_4 t}{t}$

146. $f(t) = t^{3/2} \log_2 \sqrt{t+1}$

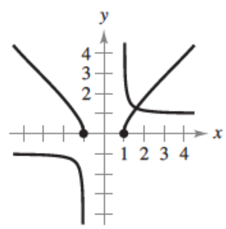
WRITING ABOUT CONCEPTS

In Exercises 147–150, the graphs of a function f and its derivative f' are shown. Label the graphs as f or f' and write a short paragraph stating the criteria you used in making your selection. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

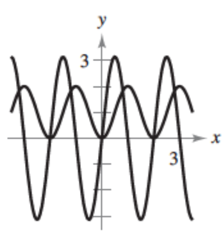
147.



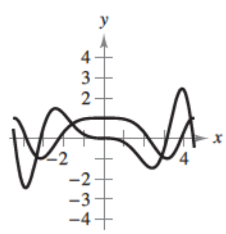
148.



149.



150.



In Exercises 151 and 152, the relationship between f and g is given. Explain the relationship between f' and g'.

151. $g(x) = f(3x)$

152. $g(x) = f(x^2)$

153. (a) Find the derivative of the function $g(x) = \sin^2 x + \cos^2 x$ in two ways.

(b) For $f(x) = \sec^2 x$ and $g(x) = \tan^2 x$, show that

$f'(x) = g'(x)$.

CAPSTONE

154. Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$, find $f'(5)$ (if possible) for each of the following. If it is not possible, state what additional information is required.

(a) $f(x) = g(x)h(x)$

(b) $f(x) = g(h(x))$

(c) $f(x) = \dfrac{g(x)}{h(x)}$

(d) $f(x) = [g(x)]^3$

In Exercises 155–158, (a) use a graphing utility to find the derivative of the function at the given point, (b) find an equation of the tangent line to the graph of the function at the given point, and (c) use the utility to graph the function and its tangent line in the same viewing window.

155. $g(t) = \dfrac{3t^2}{\sqrt{t^2+2t-1}}$, $\left(\dfrac{1}{2}, \dfrac{3}{2}\right)$

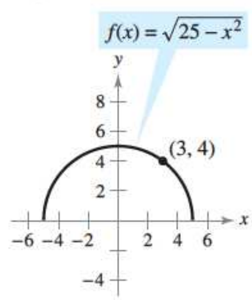
156. $f(x) = \sqrt{x}(2-x)^2$, $(4, 8)$

157. $s(t) = \dfrac{(4-2t)\sqrt{1+t}}{3}$, $\left(0, \dfrac{4}{3}\right)$

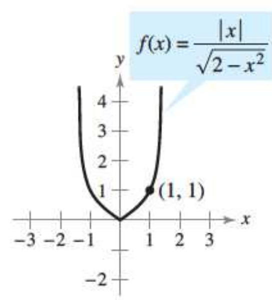
158. $y = (t^2-9)\sqrt{t+2}$, $(2, -10)$

Famous Curves **In Exercises 159 and 160, find an equation of the tangent line to the graph at the given point. Then use a graphing utility to graph the function and its tangent line in the same viewing window.**

159. Top half of circle



160. Bullet-nose curve



161. *Horizontal Tangent Line* Determine the point(s) in the interval $(0, 2\pi)$ at which the graph of $f(x) = 2\cos x + \sin 2x$ has a horizontal tangent line.

162. *Horizontal Tangent Line* Determine the point(s) at which the graph of $f(x) = \dfrac{x}{\sqrt{2x-1}}$ has a horizontal tangent line.

In Exercises 163–166, evaluate the second derivative of the function at the given point. Use a computer algebra system to verify your result.

163. $h(x) = \frac{1}{9}(3x+1)^3$, $\left(1, \frac{64}{9}\right)$

164. $f(x) = \dfrac{1}{\sqrt{x+4}}$, $\left(0, \dfrac{1}{2}\right)$

165. $f(x) = \cos x^2$, $(0, 1)$

166. $g(t) = \tan 2t$, $\left(\dfrac{\pi}{6}, \sqrt{3}\right)$

167. ***Doppler Effect*** The frequency F of a fire truck siren heard by a stationary observer is

$$F = \frac{132{,}400}{331 \pm v}$$

where $\pm v$ represents the velocity of the accelerating fire truck in meters per second. Find the rate of change of F with respect to v when

(a) the fire truck is approaching at a velocity of 30 meters per second (use $-v$).

(b) the fire truck is moving away at a velocity of 30 meters per second (use $+v$).

168. ***Harmonic Motion*** The displacement from equilibrium of an object in harmonic motion on the end of a spring is

$$y = \tfrac{1}{3}\cos 12t - \tfrac{1}{4}\sin 12t$$

where y is measured in feet and t is the time in seconds. Determine the position and velocity of the object when $t = \pi/8$.

169. ***Pendulum*** A 15-centimeter pendulum moves according to the equation $\theta = 0.2 \cos 8t$, where θ is the angular displacement from the vertical in radians and t is the time in seconds. Determine the maximum angular displacement and the rate of change of θ when $t = 3$ seconds.

170. ***Wave Motion*** A buoy oscillates in simple harmonic motion $y = A\cos \omega t$ as waves move past it. The buoy moves a total of 3.5 feet (vertically) from its low point to its high point. It returns to its high point every 10 seconds.

(a) Write an equation describing the motion of the buoy if it is at its high point at $t = 0$.

(b) Determine the velocity of the buoy as a function of t.

171. ***Circulatory System*** The speed S of blood that is r centimeters from the center of an artery is

$$S = C(R^2 - r^2)$$

where C is a constant, R is the radius of the artery, and S is measured in centimeters per second. Suppose a drug is administered and the artery begins to dilate at a rate of dR/dt. At a constant distance r, find the rate at which S changes with respect to t for $C = 1.76 \times 10^5$, $R = 1.2 \times 10^{-2}$, and $dR/dt = 10^{-5}$.

172. ***Modeling Data*** The normal daily maximum temperatures T (in degrees Fahrenheit) for Chicago, Illinois are shown in the table. *(Source: National Oceanic and Atmospheric Administration)*

Month	Jan	Feb	Mar	Apr	May	Jun
Temperature	29.6	34.7	46.1	58.0	69.9	79.2

Month	Jul	Aug	Sep	Oct	Nov	Dec
Temperature	83.5	81.2	73.9	62.1	47.1	34.4

(a) Use a graphing utility to plot the data and find a model for the data of the form

$$T(t) = a + b\sin(ct - d)$$

where T is the temperature and t is the time in months, with $t = 1$ corresponding to January.

(b) Use a graphing utility to graph the model. How well does the model fit the data?

(c) Find T' and use a graphing utility to graph the derivative.

(d) Based on the graph of the derivative, during what times does the temperature change most rapidly? Most slowly? Do your answers agree with your observations of the temperature changes? Explain.

173. ***Volume*** Air is being pumped into a spherical balloon so that the radius is increasing at the rate of $dr/dt = 3$ inches per second. What is the rate of change of the volume of the balloon, in cubic inches per second, when $r = 8$ inches? $\left[\textit{Hint: } V = \frac{4}{3}\pi r^3\right]$

174. ***Think About It*** The table shows some values of the derivative of an unknown function f. Complete the table by finding (if possible) the derivative of each transformation of f.

(a) $g(x) = f(x) - 2$ (b) $h(x) = 2f(x)$

(c) $r(x) = f(-3x)$ (d) $s(x) = f(x + 2)$

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$						
$h'(x)$						
$r'(x)$						
$s'(x)$						

175. ***Modeling Data*** The table shows the temperatures T (°F) at which water boils at selected pressures p (pounds per square inch). *(Source: Standard Handbook of Mechanical Engineers)*

p	5	10	14.696 (1 atm)	20
T	162.24°	193.21°	212.00°	227.96°

p	30	40	60	80	100
T	250.33°	267.25°	292.71°	312.03°	327.81°

A model that approximates the data is

$$T = 87.97 + 34.96 \ln p + 7.91\sqrt{p}.$$

(a) Use a graphing utility to plot the data and graph the model.

(b) Find the rates of change of T with respect to p when $p = 10$ and $p = 70$.

176. *Depreciation* After t years, the value of a car purchased for \$25,000 is

$$V(t) = 25{,}000\left(\tfrac{3}{4}\right)^t.$$

(a) Use a graphing utility to graph the function and determine the value of the car 2 years after it was purchased.

(b) Find the rates of change of V with respect to t when $t = 1$ and $t = 4$.

177. *Inflation* If the annual rate of inflation averages 5% over the next 10 years, the approximate cost C of goods or services during any year in that decade is $C(t) = P(1.05)^t$, where t is the time in years and P is the present cost.

(a) If the price of an oil change for your car is presently \$29.95, estimate the price 10 years from now.

(b) Find the rates of change of C with respect to t when $t = 1$ and $t = 8$.

(c) Verify that the rate of change of C is proportional to C. What is the constant of proportionality?

178. *Finding a Pattern* Consider the function $f(x) = \sin \beta x$, where β is a constant.

(a) Find the first-, second-, third-, and fourth-order derivatives of the function.

(b) Verify that the function and its second derivative satisfy the equation $f''(x) + \beta^2 f(x) = 0$.

(c) Use the results of part (a) to write general rules for the even- and odd-order derivatives

$$f^{(2k)}(x) \quad \text{and} \quad f^{(2k-1)}(x).$$

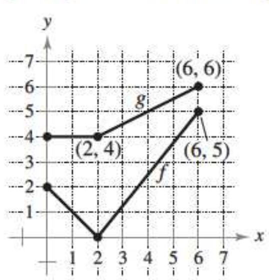
[*Hint:* $(-1)^k$ is positive if k is even and negative if k is odd.]

179. *Conjecture* Let f be a differentiable function of period p.

(a) Is the function f' periodic? Verify your answer.

(b) Consider the function $g(x) = f(2x)$. Is the function $g'(x)$ periodic? Verify your answer.

180. *Think About It* Let $r(x) = f(g(x))$ and $s(x) = g(f(x))$, where f and g are shown in the figure. Find (a) $r'(1)$ and (b) $s'(4)$.



181. (a) Show that the derivative of an odd function is even. That is, if $f(-x) = -f(x)$, then $f'(-x) = f'(x)$.

(b) Show that the derivative of an even function is odd. That is, if $f(-x) = f(x)$, then $f'(-x) = -f'(x)$.

182. Let u be a differentiable function of x. Use the fact that $|u| = \sqrt{u^2}$ to prove that

$$\frac{d}{dx}[|u|] = u'\frac{u}{|u|}, \quad u \neq 0.$$

In Exercises 183–186, use the result of Exercise 182 to find the derivative of the function.

183. $g(x) = |3x - 5|$

184. $f(x) = |x^2 - 9|$

185. $h(x) = |x| \cos x$

186. $f(x) = |\sin x|$

Linear and Quadratic Approximations **The linear and quadratic approximations of a function f at $x = a$ are**

$$P_1(x) = f'(a)(x - a) + f(a) \text{ and}$$

$$P_2(x) = \tfrac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a).$$

In Exercises 187–190, (a) find the specified linear and quadratic approximations of f, (b) use a graphing utility to graph f and the approximations, (c) determine whether P_1 or P_2 is the better approximation, and (d) state how the accuracy changes as you move farther from $x = a$.

187. $f(x) = \tan x$, $a = \dfrac{\pi}{4}$

188. $f(x) = \sec x$, $a = \dfrac{\pi}{6}$

189. $f(x) = e^x$, $a = 0$

190. $f(x) = \ln x$, $a = 1$

True or False? **In Exercises 191–194, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

191. If $y = (1 - x)^{1/2}$, then $y' = \frac{1}{2}(1 - x)^{-1/2}$.

192. If $f(x) = \sin^2(2x)$, then $f'(x) = 2(\sin 2x)(\cos 2x)$.

193. If y is a differentiable function of u, u is a differentiable function of v, and v is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$

194. If f and g are differentiable functions of x and $h(x) = f(g(x))$, then $h'(x) = f'(g(x))g'(x)$.

PUTNAM EXAM CHALLENGE

195. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$, where $a_1, a_2, \ldots, a_n$ are real numbers and where n is a positive integer. Given that $|f(x)| \leq |\sin x|$ for all real x, prove that $|a_1 + 2a_2 + \cdots + na_n| \leq 1$.

196. Let k be a fixed positive integer. The nth derivative of $\dfrac{1}{x^k - 1}$ has the form

$$\frac{P_n(x)}{(x^k - 1)^{n+1}}$$

where $P_n(x)$ is a polynomial. Find $P_n(1)$.

3.5 Implicit Differentiation

- **Distinguish between functions written in implicit form and explicit form.**
- **Use implicit differentiation to find the derivative of a function.**
- **Find derivatives of functions using logarithmic differentiation.**

EXPLORATION

Graphing an Implicit Equation How could you use a graphing utility to sketch the graph of the equation

$$x^2 - 2y^3 + 4y = 2?$$

Here are two possible approaches.

a. Solve the equation for x. Switch the roles of x and y and graph the two resulting equations. The combined graphs will show a 90° rotation of the graph of the original equation.

b. Set the graphing utility to *parametric* mode and graph the equations

$$x = -\sqrt{2t^3 - 4t + 2}$$
$$y = t$$

and

$$x = \sqrt{2t^3 - 4t + 2}$$
$$y = t.$$

From either of these two approaches, can you decide whether the graph has a tangent line at the point $(0, 1)$? Explain your reasoning.

Implicit and Explicit Functions

Up to this point in the text, most functions have been expressed in **explicit form.** For example, in the equation

$$y = 3x^2 - 5 \qquad \text{Explicit form}$$

the variable y is explicitly written as a function of x. Some functions, however, are only *implied* by an equation. For instance, the function $y = 1/x$ is defined **implicitly** by the equation $xy = 1$. Suppose you were asked to find dy/dx for this equation. You could begin by writing y explicitly as a function of x and then differentiating.

Implicit Form	*Explicit Form*	*Derivative*
$xy = 1$	$y = \dfrac{1}{x} = x^{-1}$	$\dfrac{dy}{dx} = -x^{-2} = -\dfrac{1}{x^2}$

This strategy works whenever you can solve for the function explicitly. You cannot, however, use this procedure when you are unable to solve for y as a function of x. For instance, how would you find dy/dx for the equation $x^2 - 2y^3 + 4y = 2$, where it is very difficult to express y as a function of x explicitly? To do this, you can use **implicit differentiation.**

To understand how to find dy/dx implicitly, you must realize that the differentiation is taking place *with respect to x*. This means that when you differentiate terms involving x alone, you can differentiate as usual. However, when you differentiate terms involving y, you must apply the Chain Rule, because you are assuming that y is defined implicitly as a differentiable function of x.

EXAMPLE 1 Differentiating with Respect to *x*

a. $\dfrac{d}{dx}[x^3] = 3x^2$ — Variables agree: use Simple Power Rule.

Variables agree

b. $\dfrac{d}{dx}[\overbrace{y^3}^{u^n}] = \overbrace{3y^2}^{nu^{n-1}}\overbrace{\dfrac{dy}{dx}}^{u'}$ — Variables disagree: use Chain Rule.

Variables disagree

c. $\dfrac{d}{dx}[x + 3y] = 1 + 3\dfrac{dy}{dx}$ — Chain Rule: $\dfrac{d}{dx}[3y] = 3y'$

d.
$$\begin{aligned}\frac{d}{dx}[xy^2] &= x\frac{d}{dx}[y^2] + y^2\frac{d}{dx}[x] && \text{Product Rule}\\ &= x\left(2y\frac{dy}{dx}\right) + y^2(1) && \text{Chain Rule}\\ &= 2xy\frac{dy}{dx} + y^2 && \text{Simplify.}\end{aligned}$$

Implicit Differentiation

GUIDELINES FOR IMPLICIT DIFFERENTIATION

1. Differentiate both sides of the equation *with respect to x*.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx by dividing both sides of the equation by the left-hand factor that does not contain dy/dx.

In Example 2, note that implicit differentiation can produce an expression for dy/dx that contains both x and y.

EXAMPLE 2 Implicit Differentiation

Find dy/dx given that $y^3 + y^2 - 5y - x^2 = -4$.

Solution

1. Differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}[y^3 + y^2 - 5y - x^2] = \frac{d}{dx}[-4]$$

$$\frac{d}{dx}[y^3] + \frac{d}{dx}[y^2] - \frac{d}{dx}[5y] - \frac{d}{dx}[x^2] = \frac{d}{dx}[-4]$$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

2. Collect the dy/dx terms on the left side of the equation.

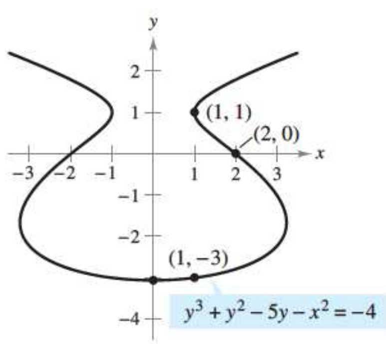
$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} = 2x$$

3. Factor dy/dx out of the left side of the equation.

$$\frac{dy}{dx}(3y^2 + 2y - 5) = 2x$$

4. Solve for dy/dx by dividing by $(3y^2 + 2y - 5)$.

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$$



Point on Graph	Slope of Graph
$(2, 0)$	$-\frac{4}{5}$
$(1, -3)$	$\frac{1}{8}$
$x = 0$	0
$(1, 1)$	Undefined

The implicit equation

$$y^3 + y^2 - 5y - x^2 = -4$$

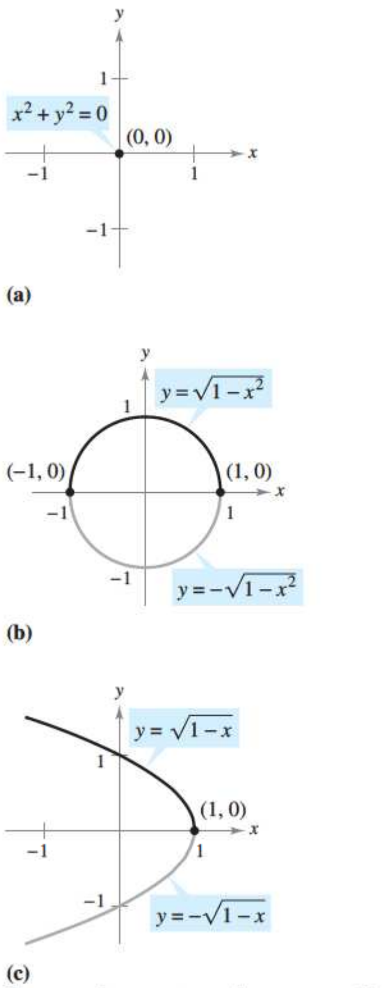
has the derivative

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}.$$

Figure 3.27

To see how you can use an *implicit derivative*, consider the graph shown in Figure 3.27. From the graph, you can see that y is not a function of x. Even so, the derivative found in Example 2 gives a formula for the slope of the tangent line at a point on this graph. The slopes at several points on the graph are shown below the graph.

TECHNOLOGY With most graphing utilities, it is easy to graph an equation that explicitly represents y as a function of x. Graphing other equations, however, can require some ingenuity. For instance, to graph the equation given in Example 2, use a graphing utility, set in *parametric* mode, to graph the parametric representations $x = \sqrt{t^3 + t^2 - 5t + 4}$, $y = t$, and $x = -\sqrt{t^3 + t^2 - 5t + 4}$, $y = t$, for $-5 \le t \le 5$. How does the result compare with the graph shown in Figure 3.27?



Some graph segments can be represented by differentiable functions.
Figure 3.28

It is meaningless to solve for dy/dx in an equation that has no solution points. (For example, $x^2 + y^2 = -4$ has no solution points.) If, however, a segment of a graph can be represented by a differentiable function, dy/dx will have meaning as the slope at each point on the segment. Recall that a function is not differentiable at (1) points with vertical tangents and (2) points at which the function is not continuous.

EXAMPLE 3 Representing a Graph by Differentiable Functions

If possible, represent y as a differentiable function of x (see Figure 3.28).

a. $x^2 + y^2 = 0$ **b.** $x^2 + y^2 = 1$ **c.** $x + y^2 = 1$

Solution

a. The graph of this equation is a single point. So, the equation does not define y as a differentiable function of x.

b. The graph of this equation is the unit circle, centered at $(0, 0)$. The upper semicircle is given by the differentiable function

$$y = \sqrt{1 - x^2}, \quad -1 < x < 1$$

and the lower semicircle is given by the differentiable function

$$y = -\sqrt{1 - x^2}, \quad -1 < x < 1.$$

At the points $(-1, 0)$ and $(1, 0)$, the slope of the graph is undefined.

c. The upper half of this parabola is given by the differentiable function

$$y = \sqrt{1 - x}, \quad x < 1$$

and the lower half of this parabola is given by the differentiable function

$$y = -\sqrt{1 - x}, \quad x < 1.$$

At the point $(1, 0)$, the slope of the graph is undefined.

EXAMPLE 4 Finding the Slope of a Graph Implicitly

Determine the slope of the tangent line to the graph of

$$x^2 + 4y^2 = 4$$

at the point $(\sqrt{2}, -1/\sqrt{2})$. See Figure 3.29.

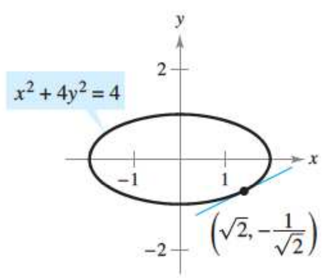


Figure 3.29

Solution

$$x^2 + 4y^2 = 4 \qquad \text{Write original equation.}$$

$$2x + 8y\frac{dy}{dx} = 0 \qquad \text{Differentiate with respect to } x.$$

$$\frac{dy}{dx} = \frac{-2x}{8y} = \frac{-x}{4y} \qquad \text{Solve for } \frac{dy}{dx}.$$

So, at $(\sqrt{2}, -1/\sqrt{2})$, the slope is

$$\frac{dy}{dx} = \frac{-\sqrt{2}}{-4/\sqrt{2}} = \frac{1}{2}. \qquad \text{Evaluate } \frac{dy}{dx} \text{ when } x = \sqrt{2} \text{ and } y = -\frac{1}{\sqrt{2}}.$$

NOTE To see the benefit of implicit differentiation, try doing Example 4 using the explicit function $y = -\frac{1}{2}\sqrt{4 - x^2}$.

EXAMPLE 5 Finding the Slope of a Graph Implicitly

Determine the slope of the graph of $3(x^2 + y^2)^2 = 100xy$ at the point $(3, 1)$.

Solution

$$\frac{d}{dx}[3(x^2 + y^2)^2] = \frac{d}{dx}[100xy]$$

$$3(2)(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 100\left[x\frac{dy}{dx} + y(1)\right]$$

$$12y(x^2 + y^2)\frac{dy}{dx} - 100x\frac{dy}{dx} = 100y - 12x(x^2 + y^2)$$

$$[12y(x^2 + y^2) - 100x]\frac{dy}{dx} = 100y - 12x(x^2 + y^2)$$

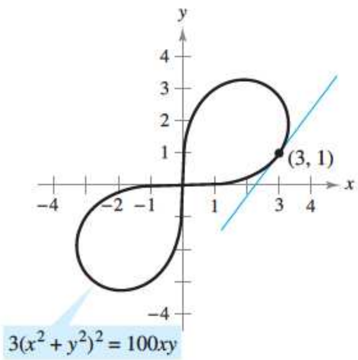
$$\frac{dy}{dx} = \frac{100y - 12x(x^2 + y^2)}{-100x + 12y(x^2 + y^2)}$$

$$= \frac{25y - 3x(x^2 + y^2)}{-25x + 3y(x^2 + y^2)}$$

At the point $(3, 1)$, the slope of the graph is

$$\frac{dy}{dx} = \frac{25(1) - 3(3)(3^2 + 1^2)}{-25(3) + 3(1)(3^2 + 1^2)} = \frac{25 - 90}{-75 + 30} = \frac{-65}{-45} = \frac{13}{9}$$

as shown in Figure 3.30. This graph is called a **lemniscate.**



Lemniscate
Figure 3.30

EXAMPLE 6 Determining a Differentiable Function

Find dy/dx implicitly for the equation $\sin y = x$. Then find the largest interval of the form $-a < y < a$ on which y is a differentiable function of x (see Figure 3.31).

Solution

$$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$$

$$\cos y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

The largest interval about the origin for which y is a differentiable function of x is $-\pi/2 < y < \pi/2$. To see this, note that $\cos y$ is positive for all y in this interval and is 0 at the endpoints. If you restrict y to the interval $-\pi/2 < y < \pi/2$, you should be able to write dy/dx explicitly as a function of x. To do this, you can use

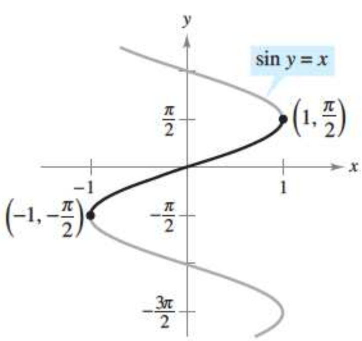
$$\cos y = \sqrt{1 - \sin^2 y}$$

$$= \sqrt{1 - x^2}, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

and conclude that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

You will study this example further when derivatives of inverse trigonometric functions are defined in Section 3.6. ■



The derivative is $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$.

Figure 3.31

The Granger Collection

ISAAC BARROW (1630–1677)

The graph in Example 8 is called the **kappa curve** because it resembles the Greek letter kappa, κ. The general solution for the tangent line to this curve was discovered by the English mathematician Isaac Barrow. Newton was Barrow's student, and they corresponded frequently regarding their work in the early development of calculus.

With implicit differentiation, the form of the derivative often can be simplified (as in Example 6) by an appropriate use of the *original* equation. A similar technique can be used to find and simplify higher-order derivatives obtained implicitly.

EXAMPLE 7 Finding the Second Derivative Implicitly

Given $x^2 + y^2 = 25$, find $\dfrac{d^2y}{dx^2}$.

Solution Differentiating each term with respect to x produces

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}.$$

Differentiating a second time with respect to x yields

$$\frac{d^2y}{dx^2} = -\frac{(y)(1) - (x)(dy/dx)}{y^2} \qquad \text{Quotient Rule}$$

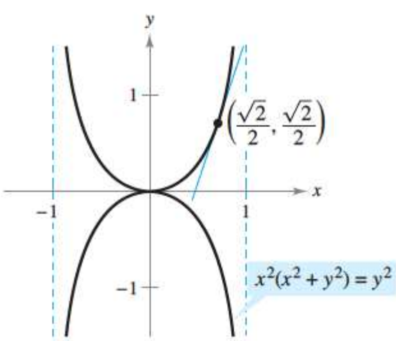
$$= -\frac{y - (x)(-x/y)}{y^2} \qquad \text{Substitute } -x/y \text{ for } \frac{dy}{dx}.$$

$$= -\frac{y^2 + x^2}{y^3} \qquad \text{Simplify.}$$

$$= -\frac{25}{y^3}. \qquad \text{Substitute 25 for } x^2 + y^2.$$

EXAMPLE 8 Finding a Tangent Line to a Graph

Find the tangent line to the graph given by $x^2(x^2 + y^2) = y^2$ at the point $(\sqrt{2}/2, \sqrt{2}/2)$, as shown in Figure 3.32.



Kappa curve
Figure 3.32

Solution By rewriting and differentiating implicitly, you obtain

$$x^4 + x^2y^2 - y^2 = 0$$

$$4x^3 + x^2\left(2y\frac{dy}{dx}\right) + 2xy^2 - 2y\frac{dy}{dx} = 0$$

$$2y(x^2 - 1)\frac{dy}{dx} = -2x(2x^2 + y^2)$$

$$\frac{dy}{dx} = \frac{x(2x^2 + y^2)}{y(1 - x^2)}.$$

At the point $(\sqrt{2}/2, \sqrt{2}/2)$, the slope is

$$\frac{dy}{dx} = \frac{(\sqrt{2}/2)[2(1/2) + (1/2)]}{(\sqrt{2}/2)[1 - (1/2)]} = \frac{3/2}{1/2} = 3$$

and the equation of the tangent line at this point is

$$y - \frac{\sqrt{2}}{2} = 3\left(x - \frac{\sqrt{2}}{2}\right)$$

$$y = 3x - \sqrt{2}.$$

Logarithmic Differentiation

On occasion, it is convenient to use logarithms as aids in differentiating nonlogarithmic functions. This procedure is called **logarithmic differentiation.**

EXAMPLE 9 Logarithmic Differentiation

Find the derivative of $y = \dfrac{(x-2)^2}{\sqrt{x^2+1}}, x \neq 2.$

Solution Note that $y > 0$ and so $\ln y$ is defined. Begin by taking the natural logarithms of both sides of the equation. Then apply logarithmic properties and differentiate implicitly. Finally, solve for y'.

$$\ln y = \ln \frac{(x-2)^2}{\sqrt{x^2+1}} \qquad \text{Take ln of both sides.}$$

$$\ln y = 2\ln(x-2) - \frac{1}{2}\ln(x^2+1) \qquad \text{Logarithmic properties}$$

$$\frac{y'}{y} = 2\left(\frac{1}{x-2}\right) - \frac{1}{2}\left(\frac{2x}{x^2+1}\right) \qquad \text{Differentiate.}$$

$$= \frac{x^2+2x+2}{(x-2)(x^2+1)} \qquad \text{Simplify.}$$

$$y' = y\left[\frac{x^2+2x+2}{(x-2)(x^2+1)}\right] \qquad \text{Solve for } y'.$$

$$= \frac{(x-2)^2}{\sqrt{x^2+1}}\left[\frac{x^2+2x+2}{(x-2)(x^2+1)}\right] \qquad \text{Substitute for } y.$$

$$= \frac{(x-2)(x^2+2x+2)}{(x^2+1)^{3/2}} \qquad \text{Simplify.}$$

3.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–22, find dy/dx by implicit differentiation.

1. $x^2 + y^2 = 9$
2. $x^2 - y^2 = 25$
3. $x^{1/2} + y^{1/2} = 16$
4. $x^3 + y^3 = 64$
5. $x^3 - xy + y^2 = 7$
6. $x^2y + y^2x = -3$
7. $xe^y - 10x + 3y = 0$
8. $e^{xy} + x^2 - y^2 = 10$
9. $x^3y^3 - y = x$
10. $\sqrt{xy} = x^2y + 1$
11. $x^3 - 3x^2y + 2xy^2 = 12$
12. $4\cos x \sin y = 1$
13. $\sin x + 2\cos 2y = 1$
14. $(\sin \pi x + \cos \pi y)^2 = 2$
15. $\sin x = x(1 + \tan y)$
16. $\cot y = x - y$
17. $y = \sin(xy)$
18. $x = \sec \dfrac{1}{y}$
19. $x^2 - 3\ln y + y^2 = 10$
20. $\ln xy + 5x = 30$
21. $4x^3 + \ln y^2 + 2y = 2x$
22. $4xy + \ln x^2y = 7$

In Exercises 23–26, (a) find two explicit functions by solving the equation for y in terms of x, (b) sketch the graph of the equation and label the parts given by the corresponding explicit functions, (c) differentiate the explicit functions, and (d) find dy/dx implicitly and show that the result is equivalent to that of part (c).

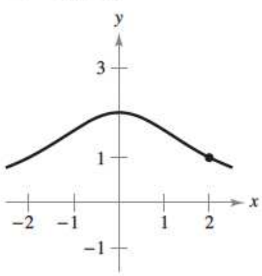
23. $x^2 + y^2 = 64$
24. $x^2 + y^2 - 4x + 6y + 9 = 0$
25. $16x^2 + 25y^2 = 400$
26. $16y^2 - x^2 = 16$

In Exercises 27–36, find dy/dx by implicit differentiation and evaluate the derivative at the given point.

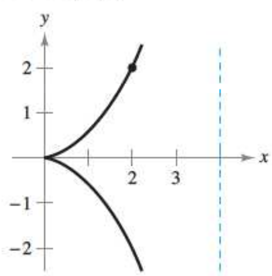
Equation	Point
27. $xy = 6$	$(-6, -1)$
28. $x^3 - y^2 = 0$	$(1, 1)$
29. $y^2 = \dfrac{x^2 - 49}{x^2 + 49}$	$(7, 0)$
30. $(x + y)^3 = x^3 + y^3$	$(-1, 1)$
31. $x^{2/3} + y^{2/3} = 5$	$(8, 1)$
32. $x^3 + y^3 = 6xy - 1$	$(2, 3)$
33. $\tan(x + y) = x$	$(0, 0)$
34. $x \cos y = 1$	$\left(2, \dfrac{\pi}{3}\right)$
35. $3e^{xy} - x = 0$	$(3, 0)$
36. $y^2 = \ln x$	$(e, 1)$

Famous Curves **In Exercises 37–40, find the slope of the tangent line to the graph at the given point.**

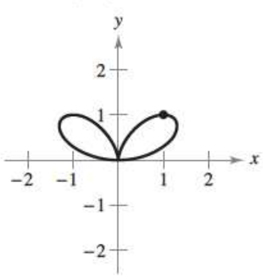
37. Witch of Agnesi:
$(x^2 + 4)y = 8$
Point: $(2, 1)$



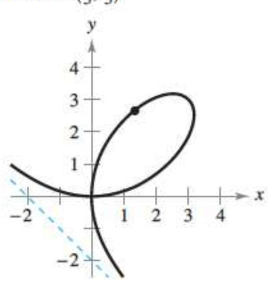
38. Cissoid:
$(4 - x)y^2 = x^3$
Point: $(2, 2)$



39. Bifolium:
$(x^2 + y^2)^2 = 4x^2y$
Point: $(1, 1)$



40. Folium of Descartes:
$x^3 + y^3 - 6xy = 0$
Point: $\left(\frac{4}{3}, \frac{8}{3}\right)$



In Exercises 41–44, use implicit differentiation to find an equation of the tangent line to the graph at the given point.

41. $4xy = 9$, $\left(1, \frac{9}{4}\right)$

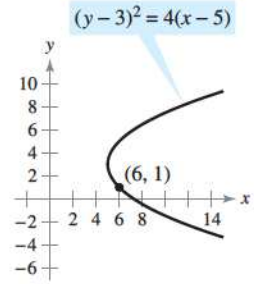
42. $x^2 + xy + y^2 = 4$, $(2, 0)$

43. $x + y - 1 = \ln(x^2 + y^2)$, $(1, 0)$

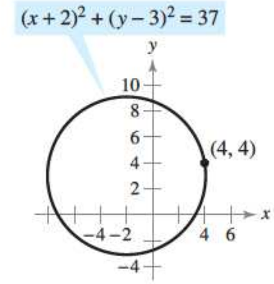
44. $y^2 + \ln xy = 2$, $(e, 1)$

Famous Curves **In Exercises 45–52, find an equation of the tangent line to the graph at the given point. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

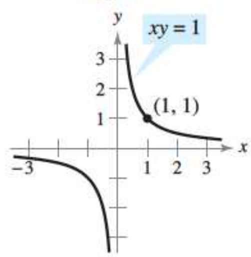
45. Parabola



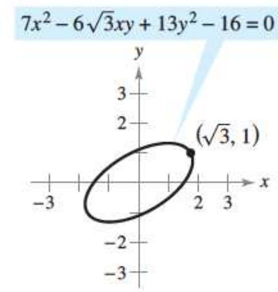
46. Circle



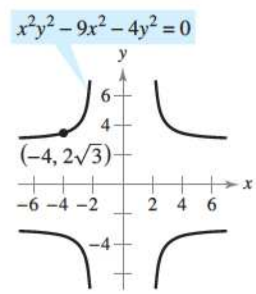
47. Rotated hyperbola



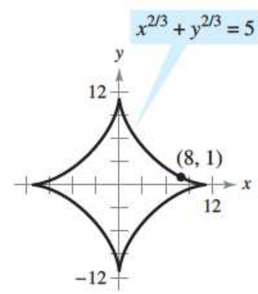
48. Rotated ellipse



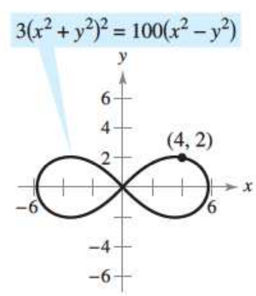
49. Cruciform



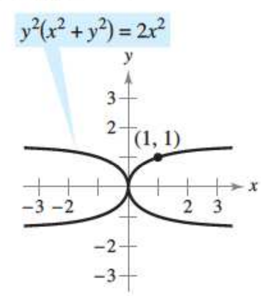
50. Astroid



51. Lemniscate



52. Kappa curve



53. (a) Use implicit differentiation to find an equation of the tangent line to the ellipse $\dfrac{x^2}{2} + \dfrac{y^2}{8} = 1$ at $(1, 2)$.

(b) Show that the equation of the tangent line to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at (x_0, y_0) is $\dfrac{x_0 x}{a^2} + \dfrac{y_0 y}{b^2} = 1$.

54. (a) Use implicit differentiation to find an equation of the tangent line to the hyperbola $\dfrac{x^2}{6} - \dfrac{y^2}{8} = 1$ at $(3, -2)$.

(b) Show that the equation of the tangent line to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at (x_0, y_0) is $\dfrac{x_0 x}{a^2} - \dfrac{y_0 y}{b^2} = 1$.

In Exercises 55 and 56, find dy/dx implicitly and find the largest interval of the form $-a < y < a$ or $0 < y < a$ such that y is a differentiable function of x. Write dy/dx as a function of x.

55. $\tan y = x$ **56.** $\cos y = x$

In Exercises 57–62, find d^2y/dx^2 in terms of x and y.

57. $x^2 + y^2 = 4$ **58.** $x^2y^2 - 2x = 3$

59. $x^2 - y^2 = 36$ **60.** $1 - xy = x - y$

61. $y^2 = x^3$ **62.** $y^2 = 10x$

In Exercises 63 and 64, use a graphing utility to graph the equation. Find an equation of the tangent line to the graph at the given point and graph the tangent line in the same viewing window.

63. $\sqrt{x} + \sqrt{y} = 5, \quad (9, 4)$ **64.** $y^2 = \dfrac{x-1}{x^2+1}, \quad \left(2, \dfrac{\sqrt{5}}{5}\right)$

In Exercises 65 and 66, find equations of the tangent line and normal line to the circle at each given point. (The *normal line* at a point is perpendicular to the tangent line at the point.) Use a graphing utility to graph the equation, tangent line, and normal line.

65. $x^2 + y^2 = 25$
$(4, 3), (-3, 4)$

66. $x^2 + y^2 = 36$
$(6, 0), (5, \sqrt{11})$

67. Show that the normal line at any point on the circle $x^2 + y^2 = r^2$ passes through the origin.

68. Two circles of radius 4 are tangent to the graph of $y^2 = 4x$ at the point $(1, 2)$. Find equations of these two circles.

In Exercises 69 and 70, find the points at which the graph of the equation has a vertical or horizontal tangent line.

69. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

70. $4x^2 + y^2 - 8x + 4y + 4 = 0$

In Exercises 71–82, find dy/dx using logarithmic differentiation.

71. $y = x\sqrt{x^2+1}, \quad x > 0$

72. $y = \sqrt{x^2(x+1)(x+2)}, \quad x > 0$

73. $y = \dfrac{x^2\sqrt{3x-2}}{(x+1)^2}, \quad x > \dfrac{2}{3}$ **74.** $y = \sqrt{\dfrac{x^2-1}{x^2+1}}, \quad x > 1$

75. $y = \dfrac{x(x-1)^{3/2}}{\sqrt{x+1}}, \quad x > 1$ **76.** $y = \dfrac{(x+1)(x-2)}{(x-1)(x+2)}, \quad x > 2$

77. $y = x^{2/x}, \quad x > 0$ **78.** $y = x^{x-1}, \quad x > 0$

79. $y = (x-2)^{x+1}, \quad x > 2$ **80.** $y = (1+x)^{1/x}, \quad x > 0$

81. $y = x^{\ln x}, \quad x > 0$ **82.** $y = (\ln x)^{\ln x}, \quad x > 1$

Orthogonal Trajectories **In Exercises 83–86, use a graphing utility to graph the intersecting graphs of the equations and show that they are orthogonal. [Two graphs are *orthogonal* if at their point(s) of intersection their tangent lines are perpendicular to each other.]**

83. $2x^2 + y^2 = 6$
$y^2 = 4x$

84. $y^2 = x^3$
$2x^2 + 3y^2 = 5$

85. $x + y = 0$
$x = \sin y$

86. $x^3 = 3(y - 1)$
$x(3y - 29) = 3$

Orthogonal Trajectories **In Exercises 87 and 88, verify that the two families of curves are orthogonal, where C and K are real numbers. Use a graphing utility to graph the two families for two values of C and two values of K.**

87. $xy = C, \quad x^2 - y^2 = K$

88. $x^2 + y^2 = C^2, \quad y = Kx$

In Exercises 89–92, differentiate (a) with respect to x (y is a function of x) and (b) with respect to t (x and y are functions of t).

89. $2y^2 - 3x^4 = 0$

90. $x^2 - 3xy^2 + y^3 = 10$

91. $\cos \pi y - 3 \sin \pi x = 1$

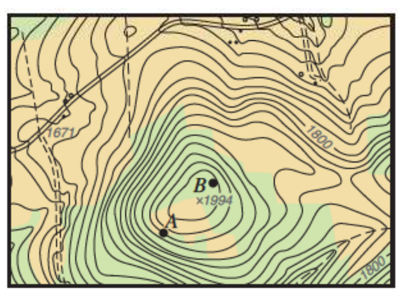
92. $4 \sin x \cos y = 1$

WRITING ABOUT CONCEPTS

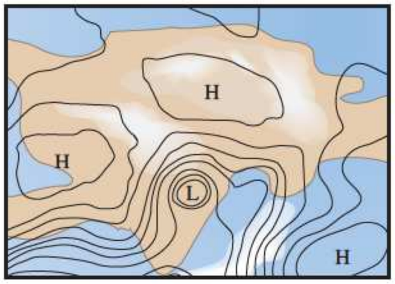
93. Describe the difference between the explicit form of a function and an implicit equation. Give an example of each.

94. In your own words, state the guidelines for implicit differentiation.

95. ***Orthogonal Trajectories*** The figure below shows the topographic map carried by a group of hikers. The hikers are in a wooded area on top of the hill shown on the map and they decide to follow a path of steepest descent (orthogonal trajectories to the contours on the map). Draw their routes if they start from point A and if they start from point B. If their goal is to reach the road along the top of the map, which starting point should they use? To print an enlarged copy of the map, go to the website *www.mathgraphs.com.*



96. ***Weather Map*** The weather map shows several *isobars*—curves that represent areas of constant air pressure. Three high pressures H and one low pressure L are shown on the map. Given that wind speed is greatest along the orthogonal trajectories of the isobars, use the map to determine the areas having high wind speed.



97. Consider the equation $x^4 = 4(4x^2 - y^2)$.

(a) Use a graphing utility to graph the equation.

(b) Find and graph the four tangent lines to the curve for $y = 3$.

(c) Find the exact coordinates of the point of intersection of the two tangent lines in the first quadrant.

CAPSTONE

98. Determine if the statement is true. If it is false, explain why and correct it. For each statement, assume y is a function of x.

(a) $\frac{d}{dx}\cos(x^2) = -2x\sin(x^2)$ (b) $\frac{d}{dy}\cos(y^2) = 2y\sin(y^2)$

(c) $\frac{d}{dx}\cos(y^2) = -2y\sin(y^2)$

99. Let L be any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$. Show that the sum of the x- and y-intercepts of L is c.

100. (a) Prove (Theorem 3.3) that $d/dx[x^n] = nx^{n-1}$ for the case in which n is a rational number. (*Hint:* Write $y = x^{p/q}$ in the form $y^q = x^p$ and differentiate implicitly. Assume that p and q are integers, where $q > 0$.)

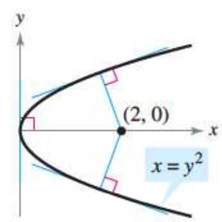
(b) Prove part (a) for the case in which n is an irrational number. (*Hint:* Let $y = x^r$, where r is a real number, and use logarithmic differentiation.)

101. ***Slope*** Find all points on the circle $x^2 + y^2 = 100$ where the slope is $\frac{3}{4}$.

102. ***Horizontal Tangent Line*** Determine the point(s) at which the graph of $y^4 = y^2 - x^2$ has a horizontal tangent line.

103. ***Tangent Lines*** Find equations of both tangent lines to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ that passes through the point $(4, 0)$.

104. ***Normals to a Parabola*** The graph shows the normal lines from the point $(2, 0)$ to the graph of the parabola $x = y^2$. How many normal lines are there from the point $(x_0, 0)$ to the graph of the parabola if (a) $x_0 = \frac{1}{4}$, (b) $x_0 = \frac{1}{2}$, and (c) $x_0 = 1$? For what value of x_0 are two of the normal lines perpendicular to each other?



105. ***Normal Lines*** (a) Find an equation of the normal line to the ellipse $\frac{x^2}{32} + \frac{y^2}{8} = 1$ at the point $(4, 2)$. (b) Use a graphing utility to graph the ellipse and the normal line. (c) At what other point does the normal line intersect the ellipse?

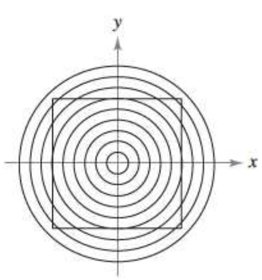
SECTION PROJECT

Optical Illusions

In each graph below, an optical illusion is created by having lines intersect a family of curves. In each case, the lines appear to be curved. Find the value of dy/dx for the given values of x and y.

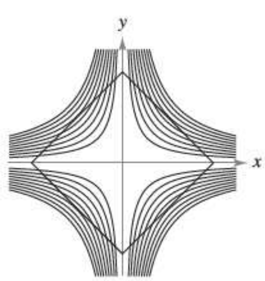
(a) Circles: $x^2 + y^2 = C^2$

$x = 3, y = 4, C = 5$



(b) Hyperbolas: $xy = C$

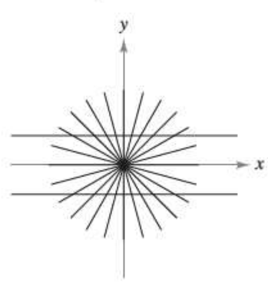
$x = 1, y = 4, C = 4$



(c) Lines: $ax = by$

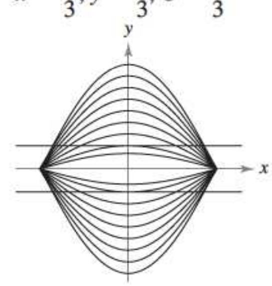
$x = \sqrt{3}, y = 3,$

$a = \sqrt{3}, b = 1$



(d) Cosine curves: $y = C\cos x$

$x = \frac{\pi}{3}, y = \frac{1}{3}, C = \frac{2}{3}$

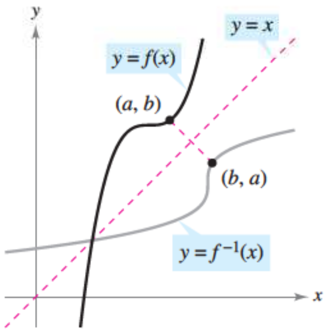


■ **FOR FURTHER INFORMATION** For more information on the mathematics of optical illusions, see the article "Descriptive Models for Perception of Optical Illusions" by David A. Smith in *The UMAP Journal*.

3.6 Derivatives of Inverse Functions

- Find the derivative of an inverse function.
- Differentiate an inverse trigonometric function.
- Review the basic differentiation rules for elementary functions.

Derivative of an Inverse Function



The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.
Figure 3.33

The next two theorems discuss the derivative of an inverse function. The reasonableness of Theorem 3.16 follows from the reflective property of inverse functions, as shown in Figure 3.33. Proofs of the two theorems are given in Appendix A.

THEOREM 3.16 CONTINUITY AND DIFFERENTIABILITY OF INVERSE FUNCTIONS

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

THEOREM 3.17 THE DERIVATIVE OF AN INVERSE FUNCTION

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

EXAMPLE 1 Evaluating the Derivative of an Inverse Function

Let $f(x) = \frac{1}{4}x^3 + x - 1$.

a. What is the value of $f^{-1}(x)$ when $x = 3$?

b. What is the value of $(f^{-1})'(x)$ when $x = 3$?

Solution Notice that f is one-to-one and therefore has an inverse function.

a. Because $f(2) = 3$, you know that $f^{-1}(3) = 2$.

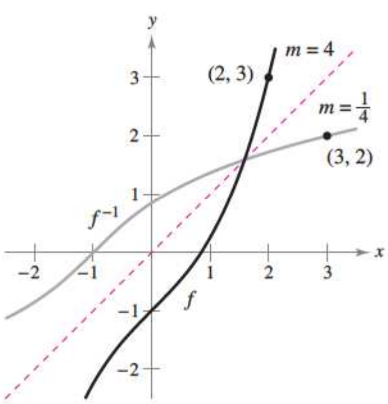
b. Because the function f is differentiable and has an inverse function, you can apply Theorem 3.17 (with $g = f^{-1}$) to write

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(2)}.$$

Moreover, using $f'(x) = \frac{3}{4}x^2 + 1$, you can conclude that

$$(f^{-1})'(3) = \frac{1}{f'(2)} = \frac{1}{\frac{3}{4}(2^2) + 1} = \frac{1}{4}.$$

(See Figure 3.34.)



The graphs of the inverse functions f and f^{-1} have reciprocal slopes at points (a, b) and (b, a).
Figure 3.34

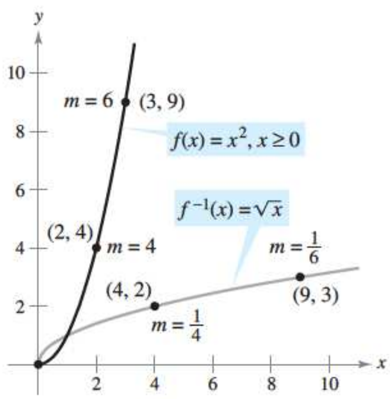
In Example 1, note that at the point $(2, 3)$ the slope of the graph of f is 4 and at the point $(3, 2)$ the slope of the graph of f^{-1} is $\frac{1}{4}$ (see Figure 3.34). This reciprocal relationship (which follows from Theorem 3.17) is sometimes written as

$$\frac{dy}{dx} = \frac{1}{dx/dy}.$$

EXAMPLE 2 Graphs of Inverse Functions Have Reciprocal Slopes

Let $f(x) = x^2$ (for $x \geq 0$) and let $f^{-1}(x) = \sqrt{x}$. Show that the slopes of the graphs of f and f^{-1} are reciprocals at each of the following points.

a. $(2, 4)$ and $(4, 2)$ **b.** $(3, 9)$ and $(9, 3)$



At $(0, 0)$, the derivative of f is 0 and the derivative of f^{-1} does not exist.
Figure 3.35

Solution The derivatives of f and f^{-1} are $f'(x) = 2x$ and $(f^{-1})'(x) = \dfrac{1}{2\sqrt{x}}$.

a. At $(2, 4)$, the slope of the graph of f is $f'(2) = 2(2) = 4$. At $(4, 2)$, the slope of the graph of f^{-1} is

$$(f^{-1})'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2(2)} = \frac{1}{4}.$$

b. At $(3, 9)$, the slope of the graph of f is $f'(3) = 2(3) = 6$. At $(9, 3)$, the slope of the graph of f^{-1} is

$$(f^{-1})'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}.$$

So, in both cases, the slopes are reciprocals, as shown in Figure 3.35. ■

When determining the derivative of an inverse function, you have two options: (1) you can apply Theorem 3.17, or (2) you can use implicit differentiation. The first approach is illustrated in Example 3, and the second in the proof of Theorem 3.18.

EXAMPLE 3 Finding the Derivative of an Inverse Function

Find the derivative of the inverse tangent function.

Solution Let $f(x) = \tan x$, $-\pi/2 < x < \pi/2$. Then let $g(x) = \arctan x$ be the inverse tangent function. To find the derivative of $g(x)$, use the fact that $f'(x) = \sec^2 x = \tan^2 x + 1$, and apply Theorem 3.17 as follows.

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f'(\arctan x)} = \frac{1}{[\tan(\arctan x)]^2 + 1} = \frac{1}{x^2 + 1}$$ ■

Derivatives of Inverse Trigonometric Functions

In Section 3.4, you saw that the derivative of the *transcendental* function $f(x) = \ln x$ is the *algebraic* function $f'(x) = 1/x$. You will now see that the derivatives of the inverse trigonometric functions also are algebraic (even though the inverse trigonometric functions are themselves transcendental).

The following theorem lists the derivatives of the six inverse trigonometric functions. Note that the derivatives of $\arccos u$, $\operatorname{arccot} u$, and $\operatorname{arccsc} u$ are the *negatives* of the derivatives of $\arcsin u$, $\arctan u$, and $\operatorname{arcsec} u$, respectively.

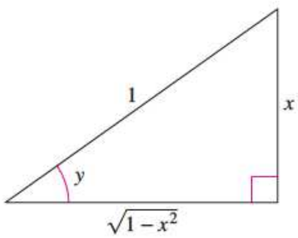
THEOREM 3.18 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\text{arccot } u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\text{arcsec } u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\text{arccsc } u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$



$y = \arcsin x$
Figure 3.36

PROOF Let $y = \arcsin x$, $-\pi/2 \le y \le \pi/2$ (see Figure 3.36). So, $\sin y = x$, and you can use implicit differentiation as follows.

$$\sin y = x$$

$$(\cos y)\left(\frac{dy}{dx}\right) = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

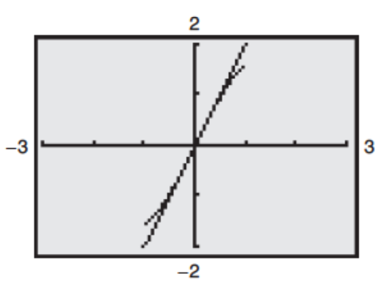
Because u is a differentiable function of x, you can use the Chain Rule to write

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}, \quad \text{where} \quad u' = \frac{du}{dx}.$$

Proofs of the other differentiation rules are left as an exercise (see Exercise 73). ■

EXPLORATION

Suppose that you want to find a linear approximation to the graph of the function in Example 4. You decide to use the tangent line at the origin, as shown below. Use a graphing utility to describe an interval about the origin where the tangent line is within 0.01 unit of the graph of the function. What might a person mean by saying that the original function is "locally linear"?



There is no common agreement on the definition of arcsec x (or arccsc x) for negative values of x. When we defined the range of the arcsecant, we chose to preserve the reciprocal identity arcsec $x = \arccos(1/x)$. For example, to evaluate arcsec(-2), you can write

$$\text{arcsec}(-2) = \arccos(-0.5)$$
$$\approx 2.09.$$

One of the consequences of the definition of the inverse secant function given in this text is that its graph has a positive slope at every x-value in its domain. This accounts for the absolute value sign in the formula for the derivative of arcsec x.

EXAMPLE 4 A Derivative That Can Be Simplified

Differentiate $y = \arcsin x + x\sqrt{1-x^2}$.

Solution

$$y' = \frac{1}{\sqrt{1-x^2}} + x\left(\frac{1}{2}\right)(-2x)(1-x^2)^{-1/2} + \sqrt{1-x^2}$$
$$= \frac{1}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}$$
$$= \sqrt{1-x^2} + \sqrt{1-x^2}$$
$$= 2\sqrt{1-x^2}$$ ■

EXAMPLE 5 Differentiating Inverse Trigonometric Functions

a. $\frac{d}{dx}[\arcsin(2x)] = \frac{2}{\sqrt{1-(2x)^2}}$ $\qquad u = 2x$

$= \frac{2}{\sqrt{1-4x^2}}$

b. $\frac{d}{dx}[\arctan(3x)] = \frac{3}{1+(3x)^2}$ $\qquad u = 3x$

$= \frac{3}{1+9x^2}$

c. $\frac{d}{dx}[\arcsin\sqrt{x}] = \frac{(1/2)x^{-1/2}}{\sqrt{1-x}}$ $\qquad u = \sqrt{x}$

$= \frac{1}{2\sqrt{x}\sqrt{1-x}}$

$= \frac{1}{2\sqrt{x-x^2}}$

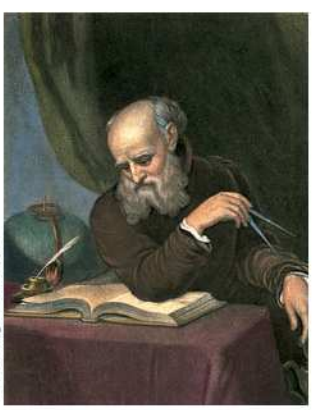
d. $\frac{d}{dx}[\operatorname{arcsec} e^{2x}] = \frac{2e^{2x}}{e^{2x}\sqrt{(e^{2x})^2-1}}$ $\qquad u = e^{2x}$

$= \frac{2e^{2x}}{e^{2x}\sqrt{e^{4x}-1}}$

$= \frac{2}{\sqrt{e^{4x}-1}}$

In part (d), the absolute value sign is not necessary because $e^{2x} > 0$. ■

Review of Basic Differentiation Rules



GALILEO GALILEI (1564–1642)

Galileo's approach to science departed from the accepted Aristotelian view that nature had describable *qualities,* such as "fluidity" and "potentiality." He chose to describe the physical world in terms of measurable *quantities,* such as time, distance, force, and mass.

In the 1600s, Europe was ushered into the scientific age by such great thinkers as Descartes, Galileo, Huygens, Newton, and Kepler. These men believed that nature is governed by basic laws—laws that can, for the most part, be written in terms of mathematical equations. One of the most influential publications of this period—*Dialogue on the Great World Systems*, by Galileo Galilei—has become a classic description of modern scientific thought.

As mathematics has developed during the past few hundred years, a small number of elementary functions has proven sufficient for modeling most* phenomena in physics, chemistry, biology, engineering, economics, and a variety of other fields. An **elementary function** is a function from the following list or one that can be formed as the sum, product, quotient, or composition of functions in the list.

Algebraic Functions	*Transcendental Functions*
Polynomial functions	Logarithmic functions
Rational functions	Exponential functions
Functions involving radicals	Trigonometric functions
	Inverse trigonometric functions

With the differentiation rules introduced so far in the text, you can differentiate any elementary function. For convenience, these differentiation rules are summarized on the next page.

* *Some important functions used in engineering and science (such as Bessel functions and gamma functions) are not elementary functions.*

BASIC DIFFERENTIATION RULES FOR ELEMENTARY FUNCTIONS

1. $\frac{d}{dx}[cu] = cu'$
2. $\frac{d}{dx}[u \pm v] = u' \pm v'$
3. $\frac{d}{dx}[uv] = uv' + vu'$
4. $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}[c] = 0$
6. $\frac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\frac{d}{dx}[x] = 1$
8. $\frac{d}{dx}[|u|] = \frac{u}{|u|}(u'), \quad u \neq 0$
9. $\frac{d}{dx}[\ln u] = \frac{u'}{u}$
10. $\frac{d}{dx}[e^u] = e^u u'$
11. $\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$
12. $\frac{d}{dx}[a^u] = (\ln a)a^u u'$
13. $\frac{d}{dx}[\sin u] = (\cos u)u'$
14. $\frac{d}{dx}[\cos u] = -(\sin u)u'$
15. $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$
16. $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$
17. $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$
18. $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
19. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$
20. $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$
21. $\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$
22. $\frac{d}{dx}[\text{arccot } u] = \frac{-u'}{1+u^2}$
23. $\frac{d}{dx}[\text{arcsec } u] = \frac{u'}{|u|\sqrt{u^2-1}}$
24. $\frac{d}{dx}[\text{arccsc } u] = \frac{-u'}{|u|\sqrt{u^2-1}}$

3.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, verify that f has an inverse. Then use the function f and the given real number a to find $(f^{-1})'(a)$. (*Hint:* See Example 1.)

Function	*Real Number*
1. $f(x) = x^3 - 1$	$a = 26$
2. $f(x) = 5 - 2x^3$	$a = 7$
3. $f(x) = x^3 + 2x - 1$	$a = 2$
4. $f(x) = \frac{1}{27}(x^5 + 2x^3)$	$a = -11$
5. $f(x) = \sin x, \ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$	$a = \frac{1}{2}$
6. $f(x) = \cos 2x, \ 0 \le x \le \frac{\pi}{2}$	$a = 1$
7. $f(x) = \frac{x+6}{x-2}, \ x > 2$	$a = 3$
8. $f(x) = \sqrt{x-4}$	$a = 2$

In Exercises 9–12, show that the slopes of the graphs of f and f^{-1} are reciprocals at the given points.

Function	*Point*
9. $f(x) = x^3$	$\left(\frac{1}{2}, \frac{1}{8}\right)$
$f^{-1}(x) = \sqrt[3]{x}$	$\left(\frac{1}{8}, \frac{1}{2}\right)$
10. $f(x) = 3 - 4x$	$(1, -1)$
$f^{-1}(x) = \frac{3-x}{4}$	$(-1, 1)$
11. $f(x) = \sqrt{x-4}$	$(5, 1)$
$f^{-1}(x) = x^2 + 4, \quad x \ge 0$	$(1, 5)$
12. $f(x) = \frac{4}{1+x^2}, \quad x \ge 0$	$(1, 2)$
$f^{-1}(x) = \sqrt{\frac{4-x}{x}}$	$(2, 1)$

In Exercises 13–16, (a) find an equation of the tangent line to the graph of f at the given point and (b) use a graphing utility to graph the function and its tangent line at the point.

Function	*Point*
13. $f(x) = \arccos x^2$	$\left(0, \frac{\pi}{2}\right)$
14. $f(x) = \arctan x$	$\left(-1, -\frac{\pi}{4}\right)$
15. $f(x) = \arcsin 3x$	$\left(\frac{\sqrt{2}}{6}, \frac{\pi}{4}\right)$
16. $f(x) = \text{arcsec } x$	$\left(\sqrt{2}, \frac{\pi}{4}\right)$

In Exercises 17–20, find dy/dx at the given point for the equation.

17. $x = y^3 - 7y^2 + 2, \ (-4, 1)$
18. $x = 2\ln(y^2 - 3), \ (0, 2)$
19. $x \arctan x = e^y, \ \left(1, \ln\frac{\pi}{4}\right)$
20. $\arcsin xy = \frac{2}{3}\arctan 2x, \ \left(\frac{1}{2}, 1\right)$

In Exercises 21–46, find the derivative of the function.

21. $f(x) = \arcsin(x + 1)$

22. $f(t) = \arcsin t^2$

23. $g(x) = 3 \arccos \dfrac{x}{2}$

24. $f(x) = \operatorname{arcsec} 4x$

25. $f(x) = \arctan e^x$

26. $f(x) = \operatorname{arccot} \sqrt{2x}$

27. $g(x) = \dfrac{\arcsin 3x}{x}$

28. $h(x) = x^2 \arctan 5x$

29. $g(x) = \dfrac{\arccos x}{x + 1}$

30. $g(x) = e^{2x} \arcsin x$

31. $h(x) = \operatorname{arccot} 6x$

32. $f(x) = \operatorname{arccsc} 3x$

33. $h(t) = \sin(\arccos t)$

34. $f(x) = \arcsin x + \arccos x$

35. $y = 2x \arccos x - 2\sqrt{1 - x^2}$

36. $y = \ln(t^2 + 4) - \dfrac{1}{2} \arctan \dfrac{t}{2}$

37. $y = \dfrac{1}{2}\left(\dfrac{1}{2} \ln \dfrac{x + 1}{x - 1} + \arctan x\right)$

38. $y = \dfrac{1}{2}\left[x\sqrt{4 - x^2} + 4 \arcsin\left(\dfrac{x}{2}\right)\right]$

39. $g(t) = \tan(\arcsin t)$

40. $f(x) = \operatorname{arcsec} x + \operatorname{arccsc} x$

41. $y = x \arcsin x + \sqrt{1 - x^2}$

42. $y = x \arctan 2x - \frac{1}{4} \ln(1 + 4x^2)$

43. $y = 8 \arcsin \dfrac{x}{4} - \dfrac{x\sqrt{16 - x^2}}{2}$

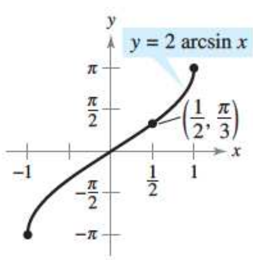
44. $y = 25 \arcsin \dfrac{x}{5} - x\sqrt{25 - x^2}$

45. $y = \arctan x + \dfrac{x}{1 + x^2}$

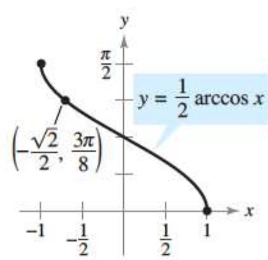
46. $y = \arctan \dfrac{x}{2} - \dfrac{1}{2(x^2 + 4)}$

In Exercises 47–52, find an equation of the tangent line to the graph of the function at the given point.

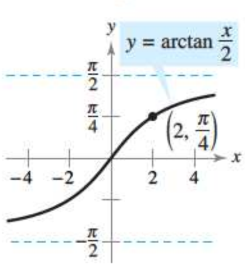
47. $y = 2 \arcsin x$



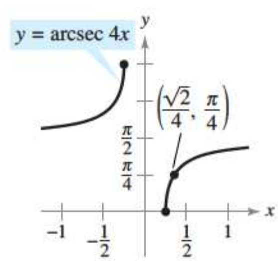
48. $y = \dfrac{1}{2} \arccos x$



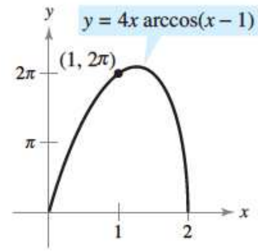
49. $y = \arctan \dfrac{x}{2}$



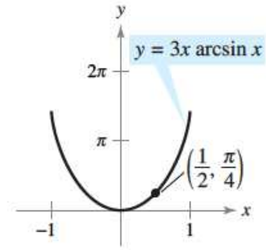
50. $y = \operatorname{arcsec} 4x$



51. $y = 4x \arccos(x - 1)$



52. $y = 3x \arcsin x$



53. Find equations of all tangent lines to the graph of $f(x) = \arccos x$ that have slope -2.

54. Find an equation of the tangent line to the graph of $g(x) = \arctan x$ when $x = 1$.

CAS *Linear and Quadratic Approximations* **In Exercises 55–58, use a computer algebra system to find the linear approximation**

$$P_1(x) = f(a) + f'(a)(x - a)$$

and the quadratic approximation

$$P_2(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2$$

to the function f at $x = a$. Sketch the graph of the function and its linear and quadratic approximations.

55. $f(x) = \arcsin x, \quad a = \frac{1}{2}$

56. $f(x) = \arctan x, \quad a = 1$

57. $f(x) = \arctan x, \quad a = 0$

58. $f(x) = \arccos x, \quad a = 0$

Implicit Differentiation **In Exercises 59–62, find an equation of the tangent line to the graph of the equation at the given point.**

59. $x^2 + x \arctan y = y - 1, \quad \left(-\dfrac{\pi}{4}, 1\right)$

60. $\arctan(xy) = \arcsin(x + y), \quad (0, 0)$

61. $\arcsin x + \arcsin y = \dfrac{\pi}{2}, \quad \left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$

62. $\arctan(x + y) = y^2 + \dfrac{\pi}{4}, \quad (1, 0)$

WRITING ABOUT CONCEPTS

In Exercises 63 and 64, the derivative of the function has the same sign for all x in its domain, but the function is not one-to-one. Explain.

63. $f(x) = \tan x$

64. $f(x) = \dfrac{x}{x^2 - 4}$

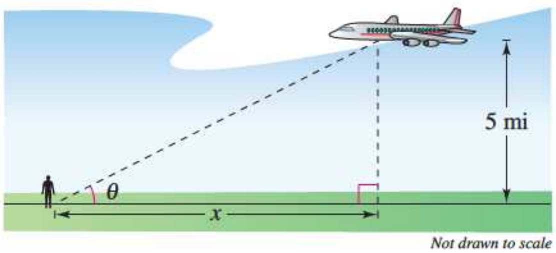
65. State the theorem that gives the method for finding the derivative of an inverse function.

66. Are the derivatives of the inverse trigonometric functions algebraic or transcendental functions? List the derivatives of the inverse trigonometric functions.

67. ***Angular Rate of Change*** An airplane flies at an altitude of 5 miles toward a point directly over an observer. Consider θ and x as shown in the figure.

(a) Write θ as a function of x.

(b) The speed of the plane is 400 miles per hour. Find $d\theta/dt$ when $x = 10$ miles and $x = 3$ miles.



68. ***Writing*** Repeat Exercise 67 if the altitude of the plane is 3 miles and describe how the altitude affects the rate of change of θ.

69. ***Angular Rate of Change*** In a free-fall experiment, an object is dropped from a height of 256 feet. A camera on the ground 500 feet from the point of impact records the fall of the object (see figure).

(a) Find the position function giving the height of the object at time t, assuming the object is released at time $t = 0$. At what time will the object reach ground level?

(b) Find the rates of change of the angle of elevation of the camera when $t = 1$ and $t = 2$.

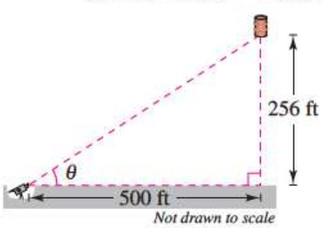


Figure for 69

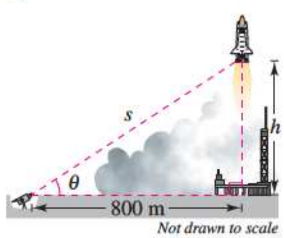


Figure for 70

70. ***Angular Rate of Change*** A television camera at ground level is filming the lift-off of a space shuttle at a point 800 meters from the launch pad. Let θ be the angle of elevation of the shuttle and let s be the distance between the camera and the shuttle (see figure). Write θ as a function of s for the period of time when the shuttle is moving vertically. Differentiate the result to find $d\theta/dt$ in terms of s and ds/dt.

71. ***Angular Rate of Change*** An observer is standing 300 feet from the point at which a balloon is released. The balloon rises at a rate of 5 feet per second. How fast is the angle of elevation of the observer's line of sight increasing when the balloon is 100 feet high?

72. ***Angular Speed*** A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of 30 revolutions per minute. Write θ as a function of x. How fast is the light beam moving along the wall when the beam makes an angle of $\theta = 45°$ with the line perpendicular from the light to the wall?

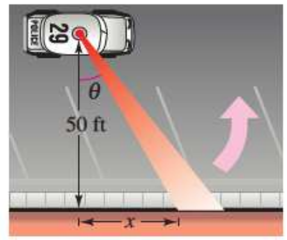


Figure for 72

73. Verify each differentiation formula.

(a) $\dfrac{d}{dx}[\arccos u] = \dfrac{-u'}{\sqrt{1-u^2}}$

(b) $\dfrac{d}{dx}[\arctan u] = \dfrac{u'}{1+u^2}$

(c) $\dfrac{d}{dx}[\operatorname{arcsec} u] = \dfrac{u'}{|u|\sqrt{u^2-1}}$

(d) $\dfrac{d}{dx}[\operatorname{arccot} u] = \dfrac{-u'}{1+u^2}$

(e) $\dfrac{d}{dx}[\operatorname{arccsc} u] = \dfrac{-u'}{|u|\sqrt{u^2-1}}$

74. ***Existence of an Inverse*** Determine the values of k such that the function $f(x) = kx + \sin x$ has an inverse function.

True or False? **In Exercises 75 and 76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

75. The slope of the graph of the inverse tangent function is positive for all x.

76. $\dfrac{d}{dx}[\arctan(\tan x)] = 1$ for all x in the domain.

77. Prove that $\arcsin x = \arctan\left(\dfrac{x}{\sqrt{1-x^2}}\right)$, $|x| < 1$.

78. Prove that $\arccos x = \dfrac{\pi}{2} - \arctan\left(\dfrac{x}{\sqrt{1-x^2}}\right)$, $|x| < 1$.

79. Some calculus textbooks define the inverse secant function using the range $[0, \pi/2) \cup [\pi, 3\pi/2)$.

(a) Sketch the graph of $y = \operatorname{arcsec} x$ using this range.

(b) Show that $y' = \dfrac{1}{x\sqrt{x^2-1}}$.

80. Compare the graphs of $y_1 = \sin(\arcsin x)$ and $y_2 = \arcsin(\sin x)$. What are the domains and ranges of y_1 and y_2?

81. Show that the function $f(x) = \arcsin \dfrac{x-2}{2} - 2\arcsin\left(\dfrac{\sqrt{x}}{2}\right)$ is constant for $0 \le x \le 4$.

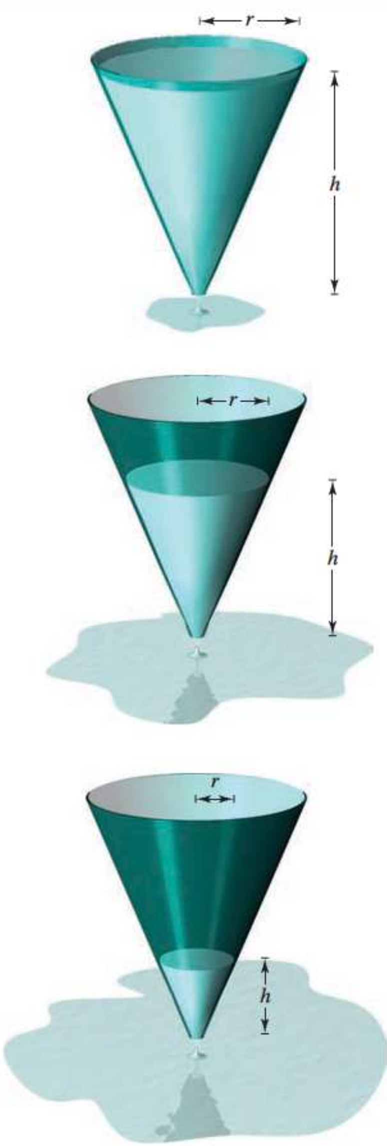
CAPSTONE

82. ***Think About It*** The point $(1, 3)$ lies on the graph of f, and the slope of the tangent line through this point is $m = 2$. Assume f^{-1} exists. What is the slope of the tangent line to the graph of f^{-1} at the point $(3, 1)$?

3.7 Related Rates

- **Find a related rate.**
- **Use related rates to solve real-life problems.**

Finding Related Rates



Volume is related to radius and height.
Figure 3.37

You have seen how the Chain Rule can be used to find dy/dx implicitly. Another important use of the Chain Rule is to find the rates of change of two or more related variables that are changing with respect to *time*.

For example, when water is drained out of a conical tank (see Figure 3.37), the volume V, the radius r, and the height h of the water level are all functions of time t. Knowing that these variables are related by the equation

$$V = \frac{\pi}{3} r^2 h \qquad \text{Original equation}$$

you can differentiate implicitly with respect to t to obtain the **related-rate** equation

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{\pi}{3} r^2 h\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2 \frac{dh}{dt} + h\left(2r\frac{dr}{dt}\right)\right] \qquad \text{Differentiate with respect to } t.$$

$$= \frac{\pi}{3}\left(r^2 \frac{dh}{dt} + 2rh\frac{dr}{dt}\right).$$

From this equation you can see that the rate of change of V is related to the rates of change of both h and r.

EXPLORATION

Finding a Related Rate In the conical tank shown in Figure 3.37, suppose that the height of the water level is changing at a rate of -0.2 foot per minute and the radius is changing at a rate of -0.1 foot per minute. What is the rate of change in the volume when the radius is $r = 1$ foot and the height is $h = 2$ feet? Does the rate of change of the volume depend on the values of r and h? Explain.

EXAMPLE 1 Two Rates That Are Related

Suppose x and y are both differentiable functions of t and are related by the equation $y = x^2 + 3$. Find dy/dt when $x = 1$, given that $dx/dt = 2$ when $x = 1$.

Solution Using the Chain Rule, you can differentiate both sides of the equation *with respect to t*.

$$y = x^2 + 3 \qquad \text{Write original equation.}$$

$$\frac{d}{dt}[y] = \frac{d}{dt}[x^2 + 3] \qquad \text{Differentiate with respect to } t.$$

$$\frac{dy}{dt} = 2x\frac{dx}{dt} \qquad \text{Chain Rule}$$

When $x = 1$ and $dx/dt = 2$, you have

$$\frac{dy}{dt} = 2(1)(2) = 4.$$

■ **FOR FURTHER INFORMATION** To learn more about the history of related-rate problems, see the article "The Lengthening Shadow: The Story of Related Rates" by Bill Austin, Don Barry, and David Berman in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*.

Problem Solving with Related Rates

In Example 1, you were *given* an equation that related the variables x and y and were asked to find the rate of change of y when $x = 1$.

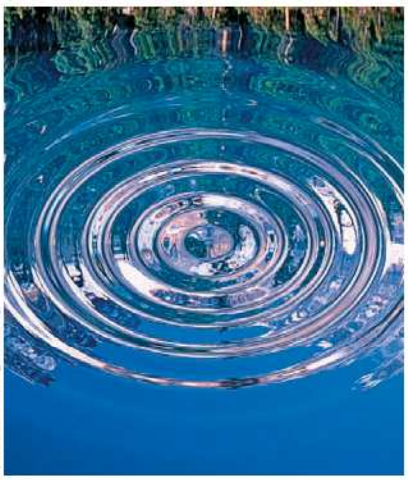
Equation: $y = x^2 + 3$

Given rate: $\frac{dx}{dt} = 2$ when $x = 1$

Find: $\frac{dy}{dt}$ when $x = 1$

In each of the remaining examples in this section, you must *create* a mathematical model from a verbal description.

EXAMPLE 2 Ripples in a Pond


© Russ Bishop/Alamy

Total area increases as the outer radius increases.
Figure 3.38

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles, as shown in Figure 3.38. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

Solution The variables r and A are related by $A = \pi r^2$. The rate of change of the radius r is $dr/dt = 1$.

Equation: $A = \pi r^2$

Given rate: $\frac{dr}{dt} = 1$

Find: $\frac{dA}{dt}$ when $r = 4$

With this information, you can proceed as in Example 1.

$$\frac{d}{dt}[A] = \frac{d}{dt}[\pi r^2] \qquad \text{Differentiate with respect to } t.$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \qquad \text{Chain Rule}$$

$$\frac{dA}{dt} = 2\pi(4)(1) = 8\pi \qquad \text{Substitute 4 for } r \text{ and 1 for } dr/dt.$$

When the radius is 4 feet, the area is changing at a rate of 8π square feet per second. ■

GUIDELINES FOR SOLVING RELATED-RATE PROBLEMS

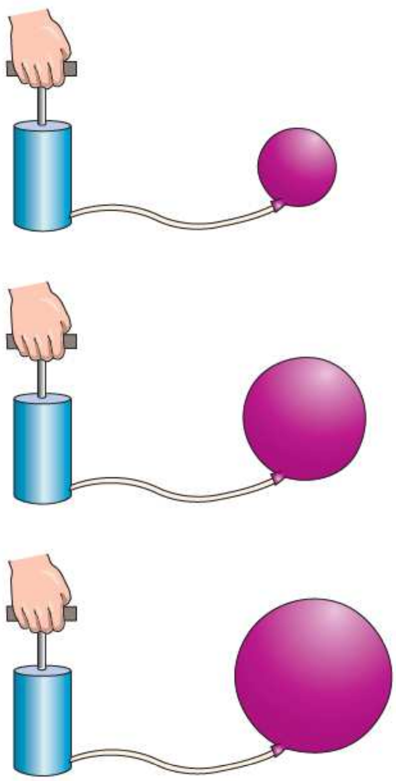
1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time t*.
4. *After* completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

NOTE When using these guidelines, be sure you perform Step 3 before Step 4. Substituting the known values of the variables before differentiating will produce an inappropriate derivative.

The table below lists examples of mathematical models involving rates of change. For instance, the rate of change in the first example is the velocity of a car.

Verbal Statement	Mathematical Model
The velocity of a car after traveling for 1 hour is 50 miles per hour.	$x =$ distance traveled $\dfrac{dx}{dt} = 50$ when $t = 1$
Water is being pumped into a swimming pool at a rate of 10 cubic meters per hour.	$V =$ volume of water in pool $\dfrac{dV}{dt} = 10 \text{ m}^3/\text{hr}$
A gear is revolving at a rate of 25 revolutions per minute (1 revolution $= 2\pi$ radians).	$\theta =$ angle of revolution $\dfrac{d\theta}{dt} = 25(2\pi)$ rad/min

EXAMPLE 3 An Inflating Balloon



Inflating a balloon
Figure 3.39

Air is being pumped into a spherical balloon (see Figure 3.39) at a rate of 4.5 cubic feet per minute. Find the rate of change of the radius when the radius is 2 feet.

Solution Let V be the volume of the balloon and let r be its radius. Because the volume is increasing at a rate of 4.5 cubic feet per minute, you know that at time t the rate of change of the volume is $dV/dt = \frac{9}{2}$. So, the problem can be stated as shown.

Given rate: $\dfrac{dV}{dt} = \dfrac{9}{2}$ (constant rate)

Find: $\dfrac{dr}{dt}$ when $r = 2$

To find the rate of change of the radius, you must find an equation that relates the radius r to the volume V.

Equation: $V = \dfrac{4}{3}\pi r^3$ Volume of a sphere

Differentiating both sides of the equation with respect to t produces

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad \text{Differentiate with respect to } t.$$

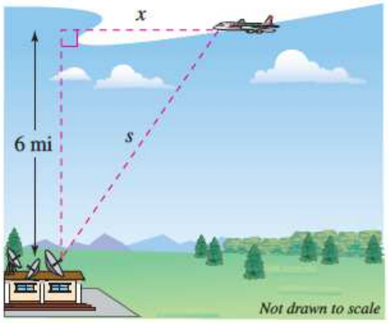
$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\left(\frac{dV}{dt}\right). \qquad \text{Solve for } dr/dt.$$

Finally, when $r = 2$, the rate of change of the radius is

$$\frac{dr}{dt} = \frac{1}{16\pi}\left(\frac{9}{2}\right) \approx 0.09 \text{ foot per minute.}$$

In Example 3, note that the volume is increasing at a *constant* rate but the radius is increasing at a *variable* rate. Just because two rates are related does not mean that they are proportional. In this particular case, the radius is growing more and more slowly as t increases. Do you see why?

EXAMPLE 4 The Speed of an Airplane Tracked by Radar



An airplane is flying at an altitude of 6 miles, s miles from the station.
Figure 3.40

An airplane is flying on a flight path that will take it directly over a radar tracking station, as shown in Figure 3.40. If s is decreasing at a rate of 400 miles per hour when $s = 10$ miles, what is the speed of the plane?

Solution Let x be the horizontal distance from the station, as shown in Figure 3.40. Notice that when $s = 10$, $x = \sqrt{10^2 - 36} = 8$.

Given rate: $ds/dt = -400$ when $s = 10$

Find: dx/dt when $s = 10$ and $x = 8$

You can find the velocity of the plane as shown.

Equation:
$$x^2 + 6^2 = s^2 \qquad \text{Pythagorean Theorem}$$
$$2x\frac{dx}{dt} = 2s\frac{ds}{dt} \qquad \text{Differentiate with respect to } t.$$
$$\frac{dx}{dt} = \frac{s}{x}\left(\frac{ds}{dt}\right) \qquad \text{Solve for } dx/dt.$$
$$\frac{dx}{dt} = \frac{10}{8}(-400) \qquad \text{Substitute for } s, x, \text{ and } ds/dt.$$
$$= -500 \text{ miles per hour} \qquad \text{Simplify.}$$

Because the velocity is -500 miles per hour, the *speed* is 500 miles per hour.

NOTE Note that the velocity in Example 4 is negative because x represents a distance that is decreasing.

EXAMPLE 5 A Changing Angle of Elevation

Find the rate of change in the angle of elevation of the camera shown in Figure 3.41 at 10 seconds after lift-off.

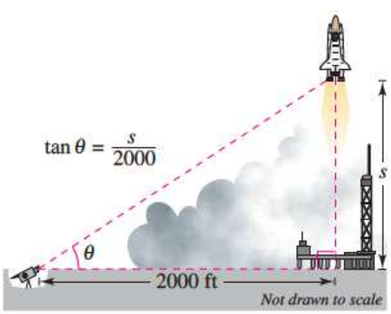
Solution Let θ be the angle of elevation, as shown in Figure 3.41. When $t = 10$, the height s of the rocket is $s = 50t^2 = 50(10)^2 = 5000$ feet.

Given rate: $ds/dt = 100t =$ velocity of rocket

Find: $d\theta/dt$ when $t = 10$ and $s = 5000$

Using Figure 3.41, you can relate s and θ by the equation $\tan\theta = s/2000$.

Equation:
$$\tan\theta = \frac{s}{2000} \qquad \text{See Figure 3.41.}$$
$$(\sec^2\theta)\frac{d\theta}{dt} = \frac{1}{2000}\left(\frac{ds}{dt}\right) \qquad \text{Differentiate with respect to } t.$$
$$\frac{d\theta}{dt} = \cos^2\theta\,\frac{100t}{2000} \qquad \text{Substitute } 100t \text{ for } ds/dt.$$
$$= \left(\frac{2000}{\sqrt{s^2 + 2000^2}}\right)^2 \frac{100t}{2000} \qquad \cos\theta = 2000/\sqrt{s^2 + 2000^2}$$



A television camera at ground level is filming the lift-off of a space shuttle that is rising vertically according to the position equation $s = 50t^2$, where s is measured in feet and t is measured in seconds. The camera is 2000 feet from the launch pad.
Figure 3.41

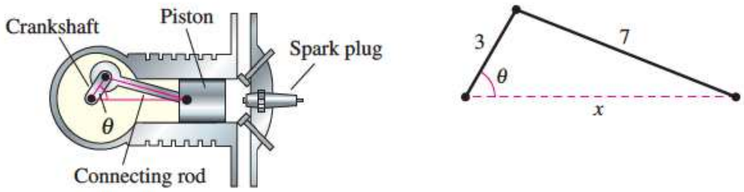
When $t = 10$ and $s = 5000$, you have

$$\frac{d\theta}{dt} = \frac{2000(100)(10)}{5000^2 + 2000^2} = \frac{2}{29} \text{ radian per second.}$$

So, when $t = 10$, θ is changing at a rate of $\frac{2}{29}$ radian per second.

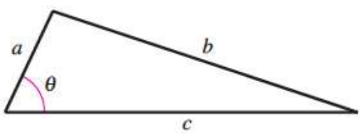
EXAMPLE 6 The Velocity of a Piston

In the engine shown in Figure 3.42, a 7-inch connecting rod is fastened to a crank of radius 3 inches. The crankshaft rotates counterclockwise at a constant rate of 200 revolutions per minute. Find the velocity of the piston when $\theta = \pi/3$.



The velocity of a piston is related to the angle of the crankshaft.
Figure 3.42

Solution Label the distances as shown in Figure 3.42. Because a complete revolution corresponds to 2π radians, it follows that $d\theta/dt = 200(2\pi) = 400\pi$ radians per minute.



Law of Cosines:
$b^2 = a^2 + c^2 - 2ac\cos\theta$
Figure 3.43

Given rate: $\dfrac{d\theta}{dt} = 400\pi$ (constant rate)

Find: $\dfrac{dx}{dt}$ when $\theta = \dfrac{\pi}{3}$

You can use the Law of Cosines (Figure 3.43) to find an equation that relates x and θ.

Equation:

$$7^2 = 3^2 + x^2 - 2(3)(x)\cos\theta$$

$$0 = 2x\frac{dx}{dt} - 6\left(-x\sin\theta\frac{d\theta}{dt} + \cos\theta\frac{dx}{dt}\right)$$

$$(6\cos\theta - 2x)\frac{dx}{dt} = 6x\sin\theta\frac{d\theta}{dt}$$

$$\frac{dx}{dt} = \frac{6x\sin\theta}{6\cos\theta - 2x}\left(\frac{d\theta}{dt}\right)$$

When $\theta = \pi/3$, you can solve for x as shown.

$$7^2 = 3^2 + x^2 - 2(3)(x)\cos\frac{\pi}{3}$$

$$49 = 9 + x^2 - 6x\left(\frac{1}{2}\right)$$

$$0 = x^2 - 3x - 40$$

$$0 = (x - 8)(x + 5)$$

$$x = 8 \qquad \text{Choose positive solution.}$$

So, when $x = 8$ and $\theta = \pi/3$, the velocity of the piston is

$$\frac{dx}{dt} = \frac{6(8)(\sqrt{3}/2)}{6(1/2) - 16}(400\pi)$$

$$= \frac{9600\pi\sqrt{3}}{-13}$$

$$\approx -4018 \text{ inches per minute.}$$

NOTE Note that the velocity in Example 6 is negative because x represents a distance that is decreasing.

3.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, assume that x and y are both differentiable functions of t and find the required values of dy/dt and dx/dt.

Equation	*Find*	*Given*
1. $y = \sqrt{x}$	(a) $\dfrac{dy}{dt}$ when $x = 4$	$\dfrac{dx}{dt} = 3$
	(b) $\dfrac{dx}{dt}$ when $x = 25$	$\dfrac{dy}{dt} = 2$
2. $y = 4(x^2 - 5x)$	(a) $\dfrac{dy}{dt}$ when $x = 3$	$\dfrac{dx}{dt} = 2$
	(b) $\dfrac{dx}{dt}$ when $x = 1$	$\dfrac{dy}{dt} = 5$
3. $xy = 4$	(a) $\dfrac{dy}{dt}$ when $x = 8$	$\dfrac{dx}{dt} = 10$
	(b) $\dfrac{dx}{dt}$ when $x = 1$	$\dfrac{dy}{dt} = -6$
4. $x^2 + y^2 = 25$	(a) $\dfrac{dy}{dt}$ when $x = 3, y = 4$	$\dfrac{dx}{dt} = 8$
	(b) $\dfrac{dx}{dt}$ when $x = 4, y = 3$	$\dfrac{dy}{dt} = -2$

In Exercises 5–8, a point is moving along the graph of the given function such that dx/dt is 2 centimeters per second. Find dy/dt for the given values of x.

5. $y = 2x^2 + 1$	(a) $x = -1$	(b) $x = 0$	(c) $x = 1$
6. $y = \dfrac{1}{1 + x^2}$	(a) $x = -2$	(b) $x = 0$	(c) $x = 2$
7. $y = \tan x$	(a) $x = -\dfrac{\pi}{3}$	(b) $x = -\dfrac{\pi}{4}$	(c) $x = 0$
8. $y = \cos x$	(a) $x = \dfrac{\pi}{6}$	(b) $x = \dfrac{\pi}{4}$	(c) $x = \dfrac{\pi}{3}$

WRITING ABOUT CONCEPTS

9. Consider the linear function $y = ax + b$. If x changes at a constant rate, does y change at a constant rate? If so, does it change at the same rate as x? Explain.

10. In your own words, state the guidelines for solving related-rate problems.

11. Find the rate of change of the distance between the origin and a moving point on the graph of $y = x^2 + 1$ if $dx/dt = 2$ centimeters per second.

12. Find the rate of change of the distance between the origin and a moving point on the graph of $y = \sin x$ if $dx/dt = 2$ centimeters per second.

13. ***Area*** The radius r of a circle is increasing at a rate of 4 centimeters per minute. Find the rates of change of the area when (a) $r = 8$ centimeters and (b) $r = 32$ centimeters.

14. ***Area*** Let A be the area of a circle of radius r that is changing with respect to time. If dr/dt is constant, is dA/dt constant? Explain.

15. ***Area*** The included angle of the two sides of constant equal length s of an isosceles triangle is θ.

(a) Show that the area of the triangle is given by $A = \frac{1}{2}s^2 \sin \theta$.

(b) If θ is increasing at the rate of $\frac{1}{2}$ radian per minute, find the rates of change of the area when $\theta = \pi/6$ and $\theta = \pi/3$.

(c) Explain why the rate of change of the area of the triangle is not constant even though $d\theta/dt$ is constant.

16. ***Volume*** The radius r of a sphere is increasing at a rate of 3 inches per minute.

(a) Find the rates of change of the volume when $r = 9$ inches and $r = 36$ inches.

(b) Explain why the rate of change of the volume of the sphere is not constant even though dr/dt is constant.

17. ***Volume*** A hemispherical water tank with radius 6 meters is filled to a depth of h meters. The volume of water in the tank is given by $V = \frac{1}{3}\pi h(108 - h^2)$, $0 < h < 6$. If water is being pumped into the tank at the rate of 3 cubic meters per minute, find the rate of change of the depth of the water when $h = 2$ meters.

18. ***Volume*** All edges of a cube are expanding at a rate of 6 centimeters per second. How fast is the volume changing when each edge is (a) 2 centimeters and (b) 10 centimeters?

19. ***Surface Area*** The conditions are the same as in Exercise 18. Determine how fast the *surface area* is changing when each edge is (a) 2 centimeters and (b) 10 centimeters.

20. ***Volume*** The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Find the rates of change of the volume if dr/dt is 2 inches per minute and $h = 3r$ when (a) $r = 6$ inches and (b) $r = 24$ inches.

21. ***Volume*** At a sand and gravel plant, sand is falling off a conveyor and onto a conical pile at a rate of 10 cubic feet per minute. The diameter of the base of the cone is approximately three times the altitude. At what rate is the height of the pile changing when the pile is 15 feet high?

22. ***Depth*** A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.

23. ***Depth*** A swimming pool is 12 meters long, 6 meters wide, 1 meter deep at the shallow end, and 3 meters deep at the deep end (see figure on next page). Water is being pumped into the pool at $\frac{1}{4}$ cubic meter per minute, and there is 1 meter of water at the deep end.

(a) What percent of the pool is filled?

(b) At what rate is the water level rising?

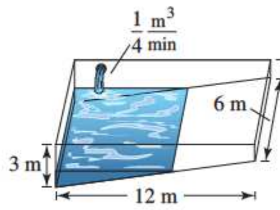


Figure for 23

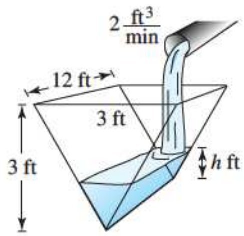


Figure for 24

24. *Depth* A trough is 12 feet long and 3 feet across the top (see figure). Its ends are isosceles triangles with altitudes of 3 feet.

(a) If water is being pumped into the trough at 2 cubic feet per minute, how fast is the water level rising when the depth h is 1 foot?

(b) If the water is rising at a rate of $\frac{3}{8}$ inch per minute when $h = 2$, determine the rate at which water is being pumped into the trough.

25. *Moving Ladder* A ladder 25 feet long is leaning against the wall of a house (see figure). The base of the ladder is pulled away from the wall at a rate of 2 feet per second.

(a) How fast is the top of the ladder moving down the wall when its base is 7 feet, 15 feet, and 24 feet from the wall?

(b) Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 feet from the wall.

(c) Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 feet from the wall.

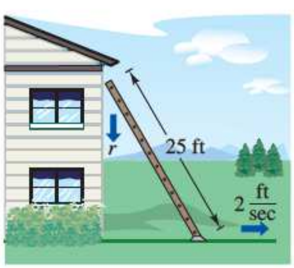


Figure for 25

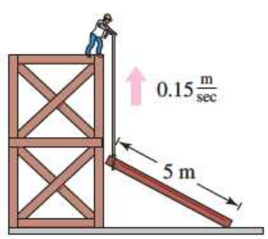


Figure for 26

FOR FURTHER INFORMATION For more information on the mathematics of moving ladders, see the article "The Falling Ladder Paradox" by Paul Scholten and Andrew Simoson in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

26. *Construction* A construction worker pulls a five-meter plank up the side of a building under construction by means of a rope tied to one end of the plank (see figure). Assume the opposite end of the plank follows a path perpendicular to the wall of the building and the worker pulls the rope at a rate of 0.15 meter per second. How fast is the end of the plank sliding along the ground when it is 2.5 meters from the wall of the building?

27. *Construction* A winch at the top of a 12-meter building pulls a pipe of the same length to a vertical position, as shown in the figure. The winch pulls in rope at a rate of -0.2 meter per second. Find the rate of vertical change and the rate of horizontal change at the end of the pipe when $y = 6$.

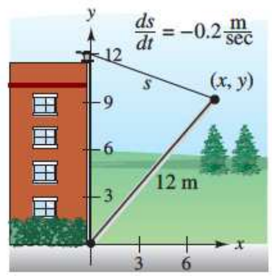


Figure for 27

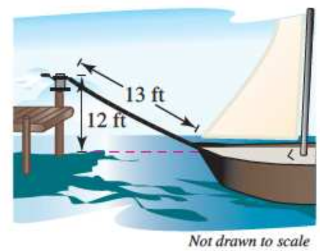


Figure for 28

28. *Boating* A boat is pulled into a dock by means of a winch 12 feet above the deck of the boat (see figure).

(a) The winch pulls in rope at a rate of 4 feet per second. Determine the speed of the boat when there is 13 feet of rope out. What happens to the speed of the boat as it gets closer to the dock?

(b) Suppose the boat is moving at a constant rate of 4 feet per second. Determine the speed at which the winch pulls in rope when there is a total of 13 feet of rope out. What happens to the speed at which the winch pulls in rope as the boat gets closer to the dock?

29. *Air Traffic Control* An air traffic controller spots two planes at the same altitude converging on a point as they fly at right angles to each other (see figure). One plane is 225 miles from the point moving at 450 miles per hour. The other plane is 300 miles from the point moving at 600 miles per hour.

(a) At what rate is the distance between the planes decreasing?

(b) How much time does the air traffic controller have to get one of the planes on a different flight path?

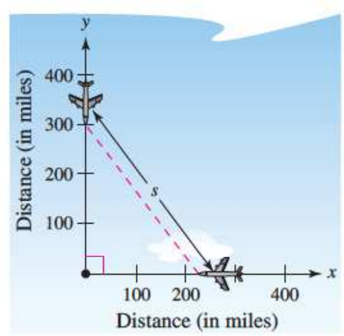


Figure for 29

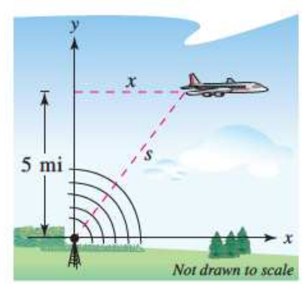


Figure for 30

30. *Air Traffic Control* An airplane is flying at an altitude of 5 miles and passes directly over a radar antenna (see figure). When the plane is 10 miles away ($s = 10$), the radar detects that the distance s is changing at a rate of 240 miles per hour. What is the speed of the plane?

31. ***Sports*** A baseball diamond has the shape of a square with sides 90 feet long (see figure). A player running from second base to third base at a speed of 25 feet per second is 20 feet from third base. At what rate is the player's distance s from home plate changing?

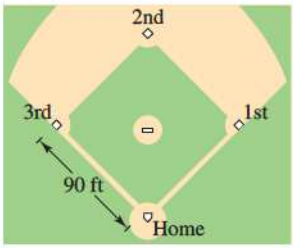


Figure for 31 and 32

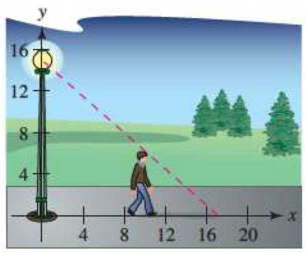


Figure for 33

32. ***Sports*** For the baseball diamond in Exercise 31, suppose the player is running from first base to second base at a speed of 25 feet per second. Find the rate at which the distance from home plate is changing when the player is 20 feet from second base.

33. ***Shadow Length*** A man 6 feet tall walks at a rate of 5 feet per second away from a light that is 15 feet above the ground (see figure). When he is 10 feet from the base of the light,

(a) at what rate is the tip of his shadow moving?

(b) at what rate is the length of his shadow changing?

34. ***Shadow Length*** Repeat Exercise 33 for a man 6 feet tall walking at a rate of 5 feet per second *toward* a light that is 20 feet above the ground (see figure).

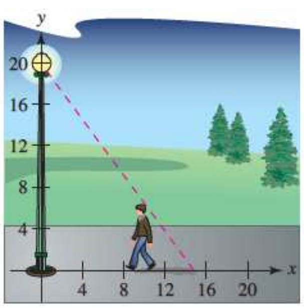


Figure for 34

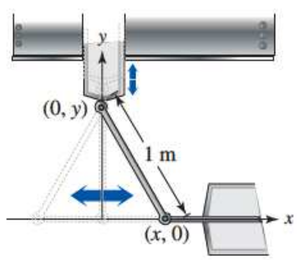


Figure for 35

35. ***Machine Design*** The endpoints of a movable rod of length 1 meter have coordinates $(x, 0)$ and $(0, y)$ (see figure). The position of the end on the x-axis is

$$x(t) = \frac{1}{2} \sin \frac{\pi t}{6}$$

where t is the time in seconds.

(a) Find the time of one complete cycle of the rod.

(b) What is the lowest point reached by the end of the rod on the y-axis?

(c) Find the speed of the y-axis endpoint when the x-axis endpoint is $\left(\frac{1}{4}, 0\right)$.

36. ***Machine Design*** Repeat Exercise 35 for a position function of $x(t) = \frac{3}{5} \sin \pi t$. Use the point $\left(\frac{3}{10}, 0\right)$ for part (c).

37. ***Evaporation*** As a spherical raindrop falls, it reaches a layer of dry air and begins to evaporate at a rate that is proportional to its surface area $(S = 4\pi r^2)$. Show that the radius of the raindrop decreases at a constant rate.

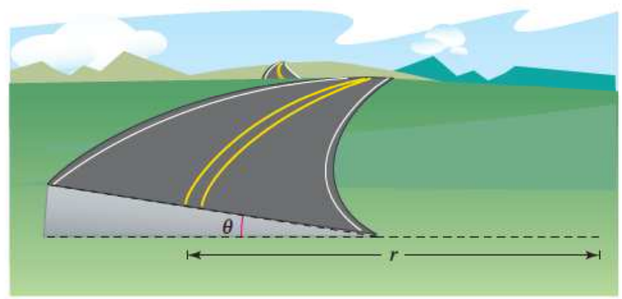
38. ***Electricity*** The combined electrical resistance R of R_1 and R_2, connected in parallel, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

where R, R_1, and R_2 are measured in ohms. R_1 and R_2 are increasing at rates of 1 and 1.5 ohms per second, respectively. At what rate is R changing when $R_1 = 50$ ohms and $R_2 = 75$ ohms?

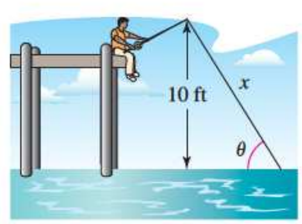
39. ***Adiabatic Expansion*** When a certain polyatomic gas undergoes adiabatic expansion, its pressure p and volume V satisfy the equation $pV^{1.3} = k$, where k is a constant. Find the relationship between the related rates dp/dt and dV/dt.

40. ***Roadway Design*** Cars on a certain roadway travel on a circular arc of radius r. In order not to rely on friction alone to overcome the centrifugal force, the road is banked at an angle of magnitude θ from the horizontal (see figure). The banking angle must satisfy the equation $rg \tan \theta = v^2$, where v is the velocity of the cars and $g = 32$ feet per second per second is the acceleration due to gravity. Find the relationship between the related rates dv/dt and $d\theta/dt$.



41. ***Angle of Elevation*** A balloon rises at a rate of 4 meters per second from a point on the ground 50 meters from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 50 meters above the ground.

42. ***Angle of Elevation*** A fish is reeled in at a rate of 1 foot per second from a point 10 feet above the water (see figure). At what rate is the angle θ between the line and the water changing when there is a total of 25 feet of line from the end of the rod to the water?



43. ***Relative Humidity*** When the dewpoint is 65° Fahrenheit, the relative humidity H is

$$H = \frac{4347}{400{,}000{,}000}e^{369{,}444/(50t + 19{,}793)}$$

where t is the temperature in degrees Fahrenheit.

(a) Determine the relative humidity when $t = 65°$ and $t = 80°$.

(b) At 10 A.M., the temperature is 75° and increasing at the rate of 2° per hour. Find the rate at which the relative humidity is changing.

44. ***Linear vs. Angular Speed*** A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of 30 revolutions per minute. How fast is the light beam moving along the wall when the beam makes angles of (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 70°$ with the line perpendicular from the light to the wall?

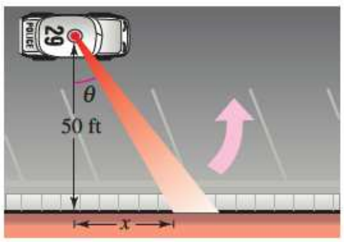


Figure for 44

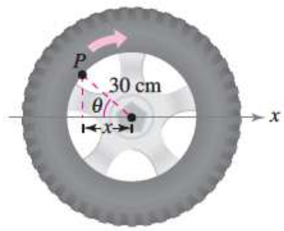


Figure for 45

45. ***Linear vs. Angular Speed*** A wheel of radius 30 centimeters revolves at a rate of 10 revolutions per second. A dot is painted at a point P on the rim of the wheel (see figure).

(a) Find dx/dt as a function of θ.

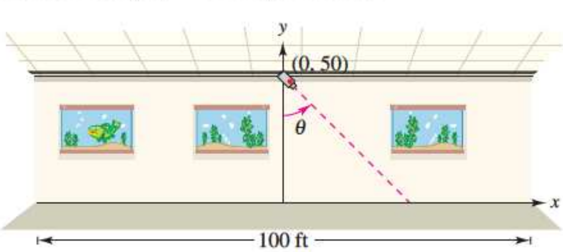
(b) Use a graphing utility to graph the function in part (a).

(c) When is the absolute value of the rate of change of x greatest? When is it least?

(d) Find dx/dt when $\theta = 30°$ and $\theta = 60°$.

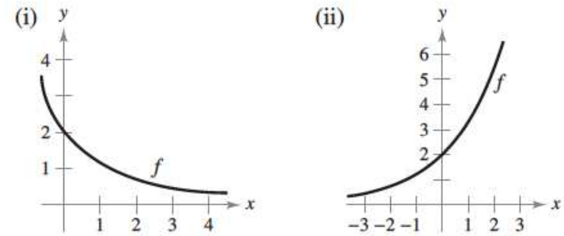
46. ***Flight Control*** An airplane is flying in still air with an airspeed of 275 miles per hour. If it is climbing at an angle of 18°, find the rate at which it is gaining altitude.

47. ***Security Camera*** A security camera is centered 50 feet above a 100-foot hallway (see figure). It is easiest to design the camera with a constant angular rate of rotation, but this results in a variable rate at which the images of the surveillance area are recorded. So, it is desirable to design a system with a variable rate of rotation and a constant rate of movement of the scanning beam along the hallway. Find a model for the variable rate of rotation if $|dx/dt| = 2$ feet per second.



CAPSTONE

48. Using the graph of f, (a) determine whether dy/dt is positive or negative given that dx/dt is negative, and (b) determine whether dx/dt is positive or negative given that dy/dt is positive.



49. ***Angle of Elevation*** An airplane flies at an altitude of 5 miles toward a point directly over an observer (see figure). The speed of the plane is 600 miles per hour. Find the rates at which the angle of elevation θ is changing when the angle is (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 75°$.

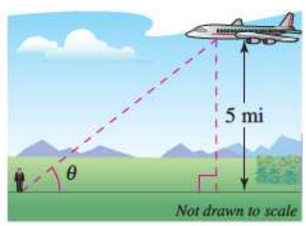


Figure for 49

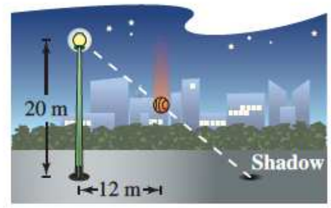


Figure for 50

50. ***Moving Shadow*** A ball is dropped from a height of 20 meters, 12 meters away from the top of a 20-meter lamppost (see figure). The ball's shadow, caused by the light at the top of the lamppost, is moving along the level ground. How fast is the shadow moving 1 second after the ball is released? *(Submitted by Dennis Gittinger, St. Philips College, San Antonio, TX)*

Acceleration **In Exercises 51 and 52, find the acceleration of the specified object. (*Hint:* Recall that if a variable is changing at a constant rate, its acceleration is zero.)**

51. Find the acceleration of the top of the ladder described in Exercise 25 when the base of the ladder is 7 feet from the wall.

52. Find the acceleration of the boat in Exercise 28(a) when there is a total of 13 feet of rope out.

53. ***Think About It*** Describe the relationship between the rate of change of y and the rate of change of x in each expression. Assume all variables and derivatives are positive.

(a) $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$

(b) $\dfrac{dy}{dt} = x(L - x)\dfrac{dx}{dt}, \quad 0 \le x \le L$

3.8 Newton's Method

■ **Approximate a zero of a function using Newton's Method.**

Newton's Method

In this section you will study a technique for approximating the real zeros of a function. The technique is called **Newton's Method,** and it uses tangent lines to approximate the graph of the function near its x-intercepts.

To see how Newton's Method works, consider a function f that is continuous on the interval $[a, b]$ and differentiable on the interval (a, b). If $f(a)$ and $f(b)$ differ in sign, then, by the Intermediate Value Theorem, f must have at least one zero in the interval (a, b). Suppose you estimate this zero to occur at

$$x = x_1 \qquad \text{First estimate}$$

as shown in Figure 3.44(a). Newton's Method is based on the assumption that the graph of f and the tangent line at $(x_1, f(x_1))$ both cross the x-axis at *about* the same point. Because you can easily calculate the x-intercept for this tangent line, you can use it as a second (and, usually, better) estimate of the zero of f. The tangent line passes through the point $(x_1, f(x_1))$ with a slope of $f'(x_1)$. In point-slope form, the equation of the tangent line is therefore

$$y - f(x_1) = f'(x_1)(x - x_1)$$
$$y = f'(x_1)(x - x_1) + f(x_1).$$

Letting $y = 0$ and solving for x produces

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

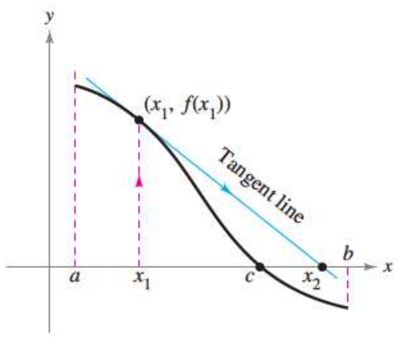
So, from the initial estimate x_1 you obtain a new estimate

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}. \qquad \text{Second estimate [see Figure 3.44(b)]}$$

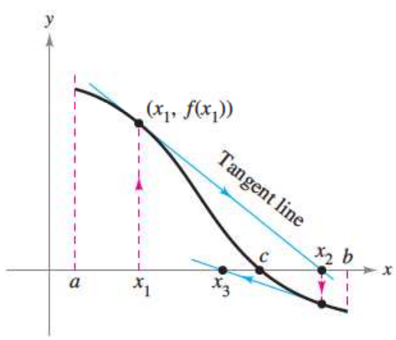
You can improve on x_2 and calculate yet a third estimate

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}. \qquad \text{Third estimate}$$

Repeated application of this process is called Newton's Method.



(a)



(b)
The x-intercept of the tangent line approximates the zero of f.
Figure 3.44

NEWTON'S METHOD FOR APPROXIMATING THE ZEROS OF A FUNCTION

Let $f(c) = 0$, where f is differentiable on an open interval containing c. Then, to approximate c, use the following steps.

1. Make an initial estimate x_1 that is close to c. (A graph is helpful.)
2. Determine a new approximation
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$
3. If $|x_n - x_{n+1}|$ is within the desired accuracy, let x_{n+1} serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.

Each successive application of this procedure is called an **iteration.**

NEWTON'S METHOD

Isaac Newton first described the method for approximating the real zeros of a function in his text *Method of Fluxions.* Although the book was written in 1671, it was not published until 1736. Meanwhile, in 1690, Joseph Raphson (1648–1715) published a paper describing a method for approximating the real zeros of a function that was very similar to Newton's. For this reason, the method is often referred to as the Newton-Raphson method.

NOTE For many functions, just a few iterations of Newton's Method will produce approximations having very small errors, as shown in Example 1.

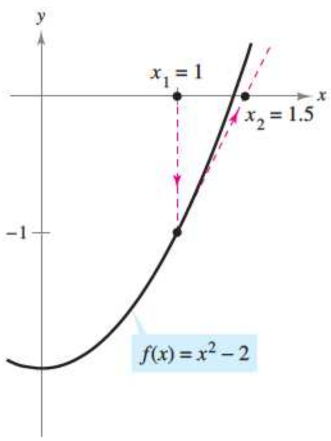
EXAMPLE 1 Using Newton's Method

Calculate three iterations of Newton's Method to approximate a zero of $f(x) = x^2 - 2$. Use $x_1 = 1$ as the initial guess.

Solution Because $f(x) = x^2 - 2$, you have $f'(x) = 2x$, and the iterative process is given by the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n}.$$

The calculations for three iterations are shown in the table.



The first iteration of Newton's Method
Figure 3.45

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.000000	−1.000000	2.000000	−0.500000	1.500000
2	1.500000	0.250000	3.000000	0.083333	1.416667
3	1.416667	0.006945	2.833334	0.002451	1.414216
4	1.414216				

Of course, in this case you know that the two zeros of the function are $\pm\sqrt{2}$. To six decimal places, $\sqrt{2} = 1.414214$. So, after only three iterations of Newton's Method, you have obtained an approximation that is within 0.000002 of an actual root. The first iteration of this process is shown in Figure 3.45.

EXAMPLE 2 Using Newton's Method

Use Newton's Method to approximate the zero(s) of

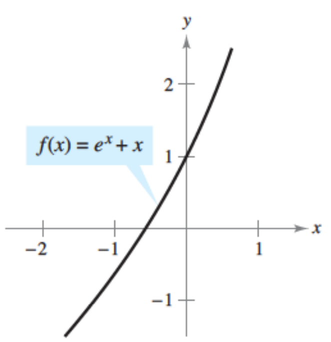
$$f(x) = e^x + x.$$

Continue the iterations until two successive approximations differ by less than 0.0001.

Solution Begin by sketching a graph of f, as shown in Figure 3.46. From the graph, you can observe that the function has only one zero, which occurs near $x = -0.6$. Next, differentiate f and form the iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{x_n} + x_n}{e^{x_n} + 1}.$$

The calculations are shown in the table.



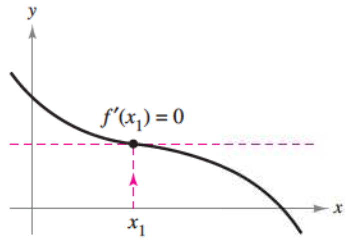
After three iterations of Newton's Method, the zero of f is approximated to the desired accuracy.
Figure 3.46

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	−0.60000	−0.05119	1.54881	−0.03305	−0.56695
2	−0.56695	0.00030	1.56725	0.00019	−0.56714
3	−0.56714	0.00000	1.56714	0.00000	−0.56714
4	−0.56714				

Because two successive approximations differ by less than the required 0.0001, you can estimate the zero of f to be -0.56714. ■

When, as in Examples 1 and 2, the approximations approach a limit, the sequence $x_1, x_2, x_3, \ldots, x_n, \ldots$ is said to **converge.** Moreover, if the limit is c, it can be shown that c must be a zero of f.

Newton's Method does not always yield a convergent sequence. One way it can fail to do so is shown in Figure 3.47. Because Newton's Method involves division by $f'(x_n)$, it is clear that the method will fail if the derivative is zero for any x_n in the sequence. When you encounter this problem, you can usually overcome it by choosing a different value for x_1. Another way Newton's Method can fail is shown in the next example.



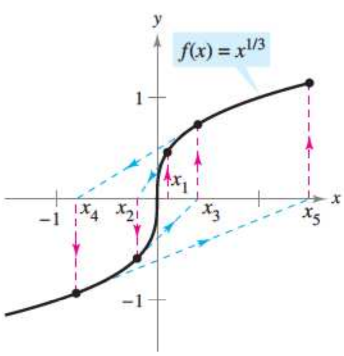
Newton's Method fails to converge if $f'(x_n) = 0$.
Figure 3.47

EXAMPLE 3 An Example in Which Newton's Method Fails

The function $f(x) = x^{1/3}$ is not differentiable at $x = 0$. Show that Newton's Method fails to converge using $x_1 = 0.1$.

Solution Because $f'(x) = \frac{1}{3}x^{-2/3}$, the iterative formula is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} \\ &= x_n - 3x_n \\ &= -2x_n. \end{aligned}$$



Newton's Method fails to converge for every x-value other than the actual zero of f.
Figure 3.48

The calculations are shown in the table. This table and Figure 3.48 indicate that x_n continues to increase in magnitude as $n \to \infty$, and so the limit of the sequence does not exist.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.10000	0.46416	1.54720	0.30000	−0.20000
2	−0.20000	−0.58480	0.97467	−0.60000	0.40000
3	0.40000	0.73681	0.61401	1.20000	−0.80000
4	−0.80000	−0.92832	0.38680	−2.40000	1.60000

NOTE In Example 3, the initial estimate $x_1 = 0.1$ fails to produce a convergent sequence. Try showing that Newton's Method also fails for every other choice of x_1 (other than the actual zero).

It can be shown that a condition sufficient to produce convergence of Newton's Method to a zero of f is that

$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| < 1 \qquad \text{Condition for convergence}$$

on an open interval containing the zero. For instance, in Example 1 this test would yield $f(x) = x^2 - 2$, $f'(x) = 2x$, $f''(x) = 2$, and

$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| = \left|\frac{(x^2-2)(2)}{4x^2}\right| = \left|\frac{1}{2} - \frac{1}{x^2}\right|. \qquad \text{Example 1}$$

On the interval $(1, 3)$, this quantity is less than 1 and therefore the convergence of Newton's Method is guaranteed. On the other hand, in Example 3, you have $f(x) = x^{1/3}$, $f'(x) = \frac{1}{3}x^{-2/3}$, $f''(x) = -\frac{2}{9}x^{-5/3}$, and

$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| = \left|\frac{x^{1/3}(-2/9)(x^{-5/3})}{(1/9)(x^{-4/3})}\right| = 2 \qquad \text{Example 3}$$

which is not less than 1 for any value of x, so you cannot conclude that Newton's Method will converge.

NIELS HENRIK ABEL (1802–1829)

EVARISTE GALOIS (1811–1832)

Although the lives of both Abel and Galois were brief, their work in the fields of analysis and abstract algebra was far-reaching.

Algebraic Solutions of Polynomial Equations

The zeros of some functions, such as

$$f(x) = x^3 - 2x^2 - x + 2$$

can be found by simple algebraic techniques, such as factoring. The zeros of other functions, such as

$$f(x) = x^3 - x + 1$$

cannot be found by *elementary* algebraic methods. This particular function has only one real zero, and by using more advanced algebraic techniques you can determine the zero to be

$$x = -\sqrt[3]{\frac{3 - \sqrt{23/3}}{6}} - \sqrt[3]{\frac{3 + \sqrt{23/3}}{6}}.$$

Because the *exact* solution is written in terms of square roots and cube roots, it is called a **solution by radicals.**

NOTE Try approximating the real zero of $f(x) = x^3 - x + 1$ and compare your result with the exact solution shown above.

The determination of radical solutions of a polynomial equation is one of the fundamental problems of algebra. The earliest such result is the Quadratic Formula, which dates back at least to Babylonian times. The general formula for the zeros of a cubic function was developed much later. In the sixteenth century an Italian mathematician, Jerome Cardan, published a method for finding radical solutions to cubic and quartic equations. Then, for 300 years, the problem of finding a general quintic formula remained open. Finally, in the nineteenth century, the problem was answered independently by two young mathematicians. Niels Henrik Abel, a Norwegian mathematician, and Evariste Galois, a French mathematician, proved that it is not possible to solve a *general* fifth- (or higher-) degree polynomial equation by radicals. Of course, you can solve particular fifth-degree equations such as $x^5 - 1 = 0$, but Abel and Galois were able to show that no general *radical* solution exists.

3.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, complete two iterations of Newton's Method for the function using the given initial guess.

1. $f(x) = x^2 - 5,\quad x_1 = 2.2$

2. $f(x) = x^3 - 3,\quad x_1 = 1.4$

3. $f(x) = \cos x,\quad x_1 = 1.6$

4. $f(x) = \tan x,\quad x_1 = 0.1$

In Exercises 5–16, approximate the zero(s) of the function. Use Newton's Method and continue the process until two successive approximations differ by less than 0.001. Then find the zero(s) using a graphing utility and compare the results.

5. $f(x) = x^3 + 4$

6. $f(x) = 2 - x^3$

7. $f(x) = x^3 + x - 1$

8. $f(x) = x^5 + x - 1$

9. $f(x) = 5\sqrt{x-1} - 2x$

10. $f(x) = x - 2\sqrt{x+1}$

11. $f(x) = x - e^{-x}$

12. $f(x) = x - 3 + \ln x$

13. $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$

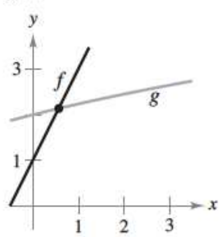
14. $f(x) = x^4 + x^3 - 1$

15. $f(x) = 1 - x + \sin x$

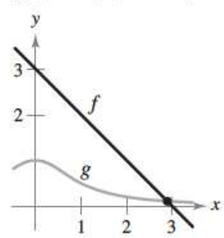
16. $f(x) = x^3 - \cos x$

In Exercises 17–24, apply Newton's Method to approximate the x-value(s) of the given point(s) of intersection of the two graphs. Continue the process until two successive approximations differ by less than 0.001. [*Hint:* Let $h(x) = f(x) - g(x)$.]

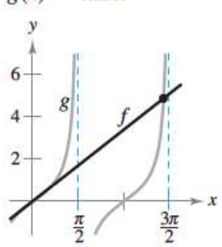
17. $f(x) = 2x + 1$
$g(x) = \sqrt{x+4}$



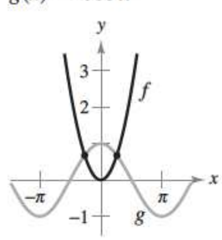
18. $f(x) = 3 - x$
$g(x) = 1/(x^2 + 1)$



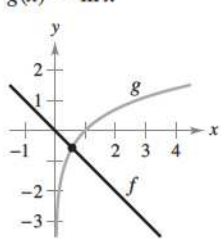
19. $f(x) = x$
$g(x) = \tan x$



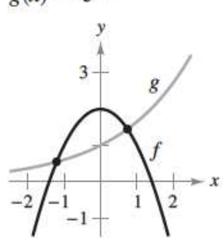
20. $f(x) = x^2$
$g(x) = \cos x$



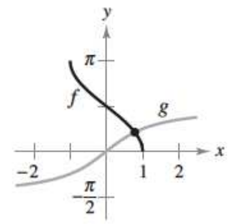
21. $f(x) = -x$
$g(x) = \ln x$



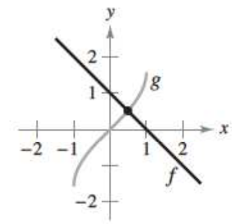
22. $f(x) = 2 - x^2$
$g(x) = e^{x/2}$



23. $f(x) = \arccos x$
$g(x) = \arctan x$



24. $f(x) = 1 - x$
$g(x) = \arcsin x$



25. ***Mechanic's Rule*** The Mechanic's Rule for approximating $\sqrt{a}$, $a > 0$, is

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \quad n = 1, 2, 3 \ldots$$

where x_1 is an approximation of $\sqrt{a}$.

(a) Use Newton's Method and the function $f(x) = x^2 - a$ to derive the Mechanic's Rule.

(b) Use the Mechanic's Rule to approximate $\sqrt{5}$ and $\sqrt{7}$ to three decimal places.

26. (a) Use Newton's Method and the function $f(x) = x^n - a$ to obtain a general rule for approximating $x = \sqrt[n]{a}$.

(b) Use the general rule found in part (a) to approximate $\sqrt[4]{6}$ and $\sqrt[3]{15}$ to three decimal places.

In Exercises 27–30, apply Newton's Method using the given initial guess, and explain why the method fails.

27. $y = 2x^3 - 6x^2 + 6x - 1,\quad x_1 = 1$

28. $y = x^3 - 2x - 2,\quad x_1 = 0$

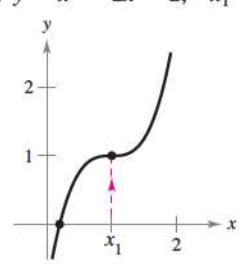


Figure for 27

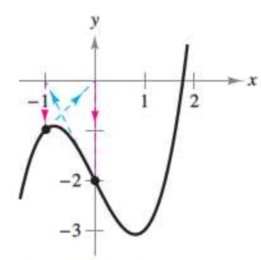


Figure for 28

29. $f(x) = -x^3 + 6x^2 - 10x + 6,\quad x_1 = 2$

30. $f(x) = 2\sin x + \cos 2x,\quad x_1 = \dfrac{3\pi}{2}$

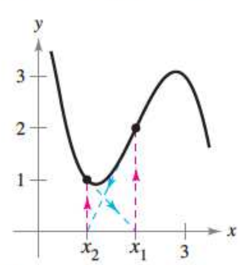


Figure for 29

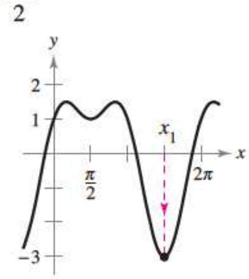


Figure for 30

Fixed Point **In Exercises 31–34, approximate the fixed point of the function to two decimal places. [A *fixed point* x_0 of a function f is a value of x such that $f(x_0) = x_0$.]**

31. $f(x) = \cos x$

32. $f(x) = \cot x, \quad 0 < x < \pi$

33. $f(x) = e^{x/10}$

34. $f(x) = -\ln x$

WRITING ABOUT CONCEPTS

35. Consider the function $f(x) = x^3 - 3x^2 + 3$.

(a) Use a graphing utility to graph f.

(b) Use Newton's Method with $x_1 = 1$ as an initial guess.

(c) Repeat part (b) using $x_1 = \frac{1}{4}$ as an initial guess and observe that the result is different.

(d) To understand why the results in parts (b) and (c) are different, sketch the tangent lines to the graph of f at the points $(1, f(1))$ and $\left(\frac{1}{4}, f\left(\frac{1}{4}\right)\right)$. Find the x-intercept of each tangent line and compare the intercepts with the first iteration of Newton's Method using the respective initial guesses.

(e) Write a short paragraph summarizing how Newton's Method works. Use the results of this exercise to describe why it is important to select the initial guess carefully.

36. Repeat the steps in Exercise 35 for the function $f(x) = \sin x$ with initial guesses of $x_1 = 1.8$ and $x_1 = 3$.

37. In your own words and using a sketch, describe Newton's Method for approximating the zeros of a function.

CAPSTONE

38. Under what conditions will Newton's Method fail?

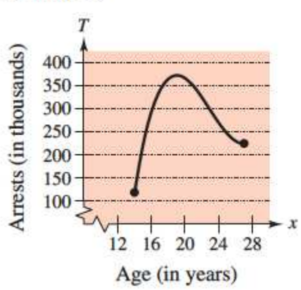
39. Use Newton's Method to show that the equation $x_{n+1} = x_n(2 - ax_n)$ can be used to approximate $1/a$ if x_1 is an initial guess of the reciprocal of a. Note that this method of approximating reciprocals uses only the operations of multiplication and subtraction. [*Hint:* Consider $f(x) = (1/x) - a$.]

40. Use the result of Exercise 39 to approximate (a) $\frac{1}{3}$ and (b) $\frac{1}{11}$ to three decimal places.

41. ***Crime*** The total number of arrests T (in thousands) for all males ages 14 to 27 in 2006 is approximated by the model

$$T = 0.602x^3 - 41.44x^2 + 922.8x - 6330, \quad 14 \le x \le 27$$

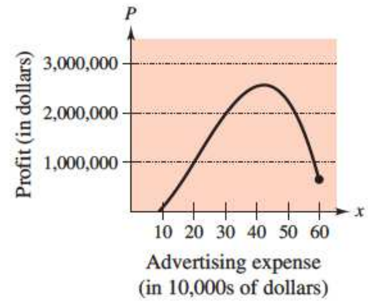
where x is the age in years (see figure). Approximate the two ages that had total arrests of 225 thousand. (*Source: U.S. Department of Justice*)



42. ***Advertising Costs*** A company that produces digital audio players estimates that the profit for selling a particular model is

$$P = -76x^3 + 4830x^2 - 320{,}000, \quad 0 \le x \le 60$$

where P is the profit in dollars and x is the advertising expense in 10,000s of dollars (see figure). According to this model, find the smaller of two advertising amounts that yield a profit P of \$2,500,000.



True or False? **In Exercises 43–46, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

43. The zeros of $f(x) = p(x)/q(x)$ coincide with the zeros of $p(x)$.

44. If the coefficients of a polynomial function are all positive, then the polynomial has no positive zeros.

45. If $f(x)$ is a cubic polynomial such that $f'(x)$ is never zero, then any initial guess will force Newton's Method to converge to the zero of f.

46. The roots of $\sqrt{f(x)} = 0$ coincide with the roots of $f(x) = 0$.

In Exercises 47 and 48, *write a computer program or use a spreadsheet* to find the zeros of a function using Newton's Method. Approximate the zeros of the function accurate to three decimal places. The output should be a table with the following headings.

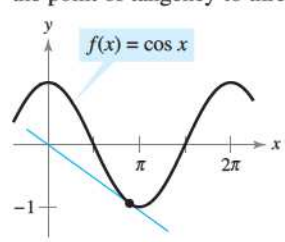
$$n, \quad x_n, \quad f(x_n), \quad f'(x_n), \quad \frac{f(x_n)}{f'(x_n)}, \quad x_n - \frac{f(x_n)}{f'(x_n)}$$

47. $f(x) = \frac{1}{4}x^3 - 3x^2 + \frac{3}{4}x - 2$

48. $f(x) = \sqrt{4 - x^2}\sin(x - 2)$

49. ***Tangent Lines*** The graph of $f(x) = -\sin x$ has infinitely many tangent lines that pass through the origin. Use Newton's Method to approximate the slope of the tangent line having the greatest slope to three decimal places.

50. ***Point of Tangency*** The graph of $f(x) = \cos x$ and a tangent line to f through the origin are shown. Find the coordinates of the point of tangency to three decimal places.



3 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, find the derivative of the function by using the definition of the derivative.

1. $f(x) = x^2 - 4x + 5$

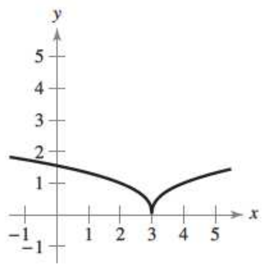
2. $f(x) = \dfrac{x + 1}{x - 1}$

3. $f(x) = \sqrt{x} + 1$

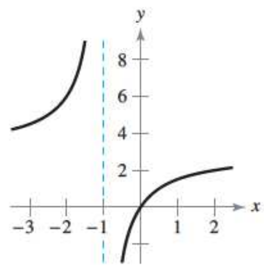
4. $f(x) = \dfrac{6}{x}$

In Exercises 5 and 6, describe the x-values at which f is differentiable.

5. $f(x) = (x - 3)^{2/5}$



6. $f(x) = \dfrac{3x}{x + 1}$



7. Sketch the graph of $f(x) = 4 - |x - 2|$.

(a) Is f continuous at $x = 2$?

(b) Is f differentiable at $x = 2$? Explain.

8. Sketch the graph of $f(x) = \begin{cases} x^2 + 4x + 2, & x < -2 \\ 1 - 4x - x^2, & x \geq -2. \end{cases}$

(a) Is f continuous at $x = -2$?

(b) Is f differentiable at $x = -2$? Explain.

In Exercises 9 and 10, find the slope of the tangent line to the graph of the function at the given point.

9. $g(x) = \dfrac{2}{3}x^2 - \dfrac{x}{6}, \quad \left(-1, \dfrac{5}{6}\right)$

10. $h(x) = \dfrac{3x}{8} - 2x^2, \quad \left(-2, -\dfrac{35}{4}\right)$

In Exercises 11 and 12, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of the graphing utility to confirm your results.

11. $f(x) = x^3 - 1, \quad (-1, -2)$

12. $f(x) = \dfrac{2}{x + 1}, \quad (0, 2)$

In Exercises 13 and 14, use the alternative form of the derivative to find the derivative at $x = c$ (if it exists).

13. $g(x) = x^2(x - 1), \quad c = 2$

14. $f(x) = \dfrac{1}{x + 4}, \quad c = 3$

In Exercises 15–30, use the rules of differentiation to find the derivative of the function.

15. $y = 25$

16. $y = -30$

17. $f(x) = x^8$

18. $g(x) = x^{20}$

19. $h(t) = 3t^4$

20. $f(t) = -8t^5$

21. $f(x) = x^3 - 3x^2$

22. $g(s) = 4s^4 - 5s^2$

23. $h(x) = 6\sqrt{x} + 3\sqrt[3]{x}$

24. $f(x) = x^{1/2} - x^{-1/2}$

25. $g(t) = \dfrac{2}{3t^2}$

26. $h(x) = \dfrac{10}{(7x)^2}$

27. $f(\theta) = 4\theta - 5 \sin \theta$

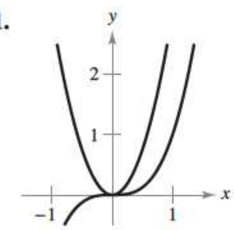
28. $g(\alpha) = 4 \cos \alpha + 6$

29. $f(t) = 3 \cos t - 4e^t$

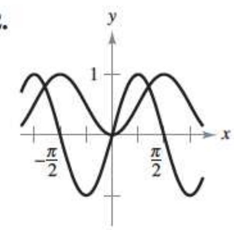
30. $g(s) = \frac{5}{3} \sin s - 2e^s$

Writing **In Exercises 31 and 32, the figure shows the graphs of a function and its derivative. Label the graphs as f or f' and write a short paragraph stating the criteria you used in making your selection. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

31.



32.



33. ***Vibrating String*** When a guitar string is plucked, it vibrates with a frequency of $F = 200\sqrt{T}$, where F is measured in vibrations per second and the tension T is measured in pounds. Find the rates of change of F when (a) $T = 4$ and (b) $T = 9$.

34. ***Vertical Motion*** A ball is dropped from a height of 100 feet. One second later, another ball is dropped from a height of 75 feet. Which ball hits the ground first?

35. ***Vertical Motion*** To estimate the height of a building, a weight is dropped from the top of the building into a pool at ground level. How high (in feet) is the building if the splash is seen 9.2 seconds after the weight is dropped?

36. ***Vertical Motion*** A bomb is dropped from an airplane at an altitude of 14,400 feet. How long will it take for the bomb to reach the ground? (Because of the motion of the plane, the fall will not be vertical, but the time will be the same as that for a vertical fall.) The plane is moving at 600 miles per hour. How far will the bomb move horizontally after it is released from the plane?

37. ***Projectile Motion*** A thrown ball follows a path described by $y = x - 0.02x^2$.

(a) Sketch a graph of the path.

(b) Find the total horizontal distance the ball is thrown.

(c) At what x-value does the ball reach its maximum height? (Use the symmetry of the path.)

(d) Find an equation that gives the instantaneous rate of change of the height of the ball with respect to the horizontal change. Evaluate the equation at $x = 0, 10, 25, 30,$ and 50.

(e) What is the instantaneous rate of change of the height when the ball reaches its maximum height?

38. *Projectile Motion* The path of a projectile thrown at an angle of 45° with level ground is

$$y = x - \frac{32}{v_0^2}(x^2)$$

where the initial velocity is v_0 feet per second.

(a) Find the x-coordinate of the point where the projectile strikes the ground. Use the symmetry of the path of the projectile to locate the x-coordinate of the point where the projectile reaches its maximum height.

(b) What is the instantaneous rate of change of the height when the projectile is at its maximum height?

(c) Show that doubling the initial velocity of the projectile multiplies both the maximum height and the range by a factor of 4.

(d) Find the maximum height and range of a projectile thrown with an initial velocity of 70 feet per second. Use a graphing utility to graph the path of the projectile.

39. *Horizontal Motion* The position function of a particle moving along the x-axis is

$$x(t) = t^2 - 3t + 2 \quad \text{for} \quad -\infty < t < \infty.$$

(a) Find the velocity of the particle.

(b) Find the open t-interval(s) in which the particle is moving to the left.

(c) Find the position of the particle when the velocity is 0.

(d) Find the speed of the particle when the position is 0.

40. *Modeling Data* The speed of a car in miles per hour and the stopping distance in feet are recorded in the table.

Speed, x	20	30	40	50	60
Stopping Distance, y	25	55	105	188	300

(a) Use the regression capabilities of a graphing utility to find a quadratic model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use a graphing utility to graph dy/dx.

(d) Use the model to approximate the stopping distance at a speed of 65 miles per hour.

(e) Use the graphs in parts (b) and (c) to explain the change in stopping distance as the speed increases.

In Exercises 41–56, find the derivative of the function.

41. $f(x) = (5x^2 + 8)(x^2 - 4x - 6)$

42. $g(x) = (x^3 + 7x)(x + 3)$

43. $h(x) = \sqrt{x} \sin x$

44. $f(t) = 2t^5 \cos t$

45. $f(x) = \dfrac{x^2 + x - 1}{x^2 - 1}$

46. $f(x) = \dfrac{6x - 5}{x^2 + 1}$

47. $f(x) = \dfrac{1}{9 - 4x^2}$

48. $f(x) = \dfrac{9}{3x^2 - 2x}$

49. $y = \dfrac{x^4}{\cos x}$

50. $y = \dfrac{\sin x}{x^4}$

51. $y = 3x^2 \sec x$

52. $y = 2x - x^2 \tan x$

53. $y = x \cos x - \sin x$

54. $g(x) = 3x \sin x + x^2 \cos x$

55. $y = 4xe^x$

56. $y = \dfrac{1 + \sin x}{1 - \sin x}$

In Exercises 57–60, find an equation of the tangent line to the graph of *f* at the given point.

57. $f(x) = \dfrac{2x^3 - 1}{x^2}$, $(1, 1)$

58. $f(x) = \dfrac{x + 1}{x - 1}$, $\left(\dfrac{1}{2}, -3\right)$

59. $f(x) = -x \tan x$, $(0, 0)$

60. $f(x) = \dfrac{1 + \cos x}{1 - \cos x}$, $\left(\dfrac{\pi}{2}, 1\right)$

61. *Acceleration* The velocity of an object in meters per second is $v(t) = 36 - t^2$, $0 \le t \le 6$. Find the velocity and acceleration of the object when $t = 4$.

62. *Acceleration* The velocity of an automobile starting from rest is

$$v(t) = \frac{90t}{4t + 10}$$

where v is measured in feet per second. Find the vehicle's velocity and acceleration at each of the following times.

(a) 1 second (b) 5 seconds (c) 10 seconds

In Exercises 63–68, find the second derivative of the function.

63. $g(t) = -8t^3 - 5t + 12$

64. $h(x) = 21x^{-3} + 3x$

65. $f(x) = 15x^{5/2}$

66. $f(x) = 20\sqrt[5]{x}$

67. $f(\theta) = 3 \tan \theta$

68. $h(t) = 10 \cos t - 15 \sin t$

In Exercises 69 and 70, show that the function satisfies the equation.

Function	*Equation*
69. $y = 2 \sin x + 3 \cos x$	$y'' + y = 0$
70. $y = \dfrac{10 - \cos x}{x}$	$xy' + y = \sin x$

71. *Rate of Change* Determine whether there exist any values of x in the interval $[0, 2\pi)$ such that the rate of change of $f(x) = \sec x$ and the rate of change of $g(x) = \csc x$ are equal.

72. *Volume* The radius of a right circular cylinder is given by $\sqrt{t + 2}$ and its height is $\frac{1}{2}\sqrt{t}$, where t is time in seconds and the dimensions are in inches. Find the rate of change of the volume with respect to time.

In Exercises 73–98, find the derivative of the function.

73. $h(x) = \left(\dfrac{x + 5}{x^2 + 3}\right)^2$

74. $f(x) = \left(x^2 + \dfrac{1}{x}\right)^5$

75. $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

76. $h(\theta) = \dfrac{\theta}{(1 - \theta)^3}$

77. $y = 5\cos(9x + 1)$

78. $y = 1 - \cos 2x + 2\cos^2 x$

79. $y = \dfrac{x}{2} - \dfrac{\sin 2x}{4}$

80. $y = \dfrac{\sec^7 x}{7} - \dfrac{\sec^5 x}{5}$

81. $y = \dfrac{2}{3}\sin^{3/2} x - \dfrac{2}{7}\sin^{7/2} x$

82. $f(x) = \dfrac{3x}{\sqrt{x^2 + 1}}$

83. $y = \dfrac{\sin \pi x}{x + 2}$

84. $y = \dfrac{\cos(x - 1)}{x - 1}$

85. $g(t) = t^2 e^{t/4}$

86. $h(z) = e^{-z^2/2}$

87. $y = \sqrt{e^{2x} + e^{-2x}}$

88. $y = 3e^{-3/t}$

89. $g(x) = \dfrac{x^2}{e^x}$

90. $f(\theta) = \dfrac{1}{2} e^{\sin 2\theta}$

91. $g(x) = \ln \sqrt{x}$

92. $h(x) = \ln \dfrac{x(x - 1)}{x - 2}$

93. $f(x) = x\sqrt{\ln x}$

94. $f(x) = \ln[x(x^2 - 2)^{2/3}]$

95. $y = \dfrac{1}{b^2}\left[\ln(a + bx) + \dfrac{a}{a + bx}\right]$

96. $y = \dfrac{1}{b^2}[a + bx - a\ln(a + bx)]$

97. $y = -\dfrac{1}{a}\ln\dfrac{a + bx}{x}$

98. $y = -\dfrac{1}{ax} + \dfrac{b}{a^2}\ln\dfrac{a + bx}{x}$

In Exercises 99–102, find the derivative of the function at the given point.

99. $f(x) = \sqrt{1 - x^3}$, $(-2, 3)$

100. $f(x) = \sqrt[3]{x^2 - 1}$, $(3, 2)$

101. $y = \dfrac{1}{2}\csc 2x$, $\left(\dfrac{\pi}{4}, \dfrac{1}{2}\right)$

102. $y = \csc 3x + \cot 3x$, $\left(\dfrac{\pi}{6}, 1\right)$

CAS **In Exercises 103–110, use a computer algebra system to find the derivative of the function. Use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.**

103. $f(t) = t^2(t - 1)^5$

104. $f(x) = [(x - 2)(x + 4)]^2$

105. $g(x) = \dfrac{2x}{\sqrt{x + 1}}$

106. $g(x) = x\sqrt{x^2 + 1}$

107. $f(t) = \sqrt{t + 1}\,\sqrt[3]{t + 1}$

108. $y = \sqrt{3x}(x + 2)^3$

109. $y = \tan\sqrt{1 - x}$

110. $y = 2\csc^3(\sqrt{x})$

In Exercises 111–114, find the second derivative of the function.

111. $y = 7x^2 + \cos 2x$

112. $y = \dfrac{1}{x} + \tan x$

113. $f(x) = \cot x$

114. $y = \sin^2 x$

CAS **In Exercises 115–120, use a computer algebra system to find the second derivative of the function.**

115. $f(t) = \dfrac{4t^2}{(1 - t)^2}$

116. $g(x) = \dfrac{6x - 5}{x^2 + 1}$

117. $g(\theta) = \tan 3\theta - \sin(\theta - 1)$

118. $h(x) = 5x\sqrt{x^2 - 16}$

119. $g(x) = x^3 \ln x$

120. $f(x) = 6x^2 e^{-x/3}$

121. ***Refrigeration*** The temperature T of food put in a freezer is

$$T = \frac{700}{t^2 + 4t + 10}$$

where t is the time in hours. Find the rate of change of T with respect to t at each of the following times.

(a) $t = 1$ (b) $t = 3$ (c) $t = 5$ (d) $t = 10$

122. ***Fluid Flow*** The emergent velocity v of a liquid flowing from a hole in the bottom of a tank is given by $v = \sqrt{2gh}$, where g is the acceleration due to gravity (32 feet per second per second) and h is the depth of the liquid in the tank. Find the rate of change of v with respect to h when (a) $h = 9$ and (b) $h = 4$. (Note that $g = +32$ feet per second per second. The sign of g depends on how a problem is modeled. In this case, letting g be negative would produce an imaginary value for v.)

123. ***Modeling Data*** The atmospheric pressure decreases with increasing altitude. At sea level, the average air pressure is one atmosphere (1.033227 kilograms per square centimeter). The table gives the pressures p (in atmospheres) at various altitudes h (in kilometers).

h	0	5	10	15	20	25
p	1	0.55	0.25	0.12	0.06	0.02

(a) Use a graphing utility to find a model of the form $p = a + b\ln h$ for the data. Explain why the result is an error message.

(b) Use a graphing utility to find the logarithmic model $h = a + b\ln p$ for the data.

(c) Use a graphing utility to plot the data and graph the logarithmic model.

(d) Use the model to estimate the altitude at which the pressure is 0.75 atmosphere.

(e) Use the model to estimate the pressure at an altitude of 13 kilometers.

(f) Find the rates of change of pressure when $h = 5$ and $h = 20$. Interpret the results in the context of the problem.

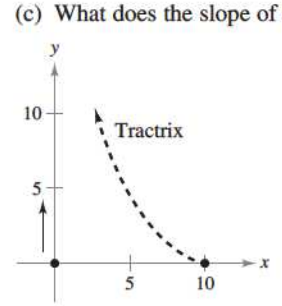
124. ***Tractrix*** A person walking along a dock drags a boat by a 10-meter rope. The boat travels along a path known as a *tractrix* (see figure). The equation of this path is

$$y = 10\ln\left(\frac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2}.$$

(a) Use a graphing utility to graph the function.

(b) What is the slope of the path when $x = 5$ and $x = 9$?

(c) What does the slope of the path approach as $x \to 10$?



In Exercises 125–132, find *dy/dx* by implicit differentiation.

125. $x^2 + 3xy + y^3 = 10$
126. $y^2 = (x - y)(x^2 + y)$
127. $\cos x^2 = xe^y$
128. $ye^x + xe^y = xy$
129. $\sqrt{xy} = x - 4y$
130. $y\sqrt{x} - x\sqrt{y} = 25$
131. $x \sin y = y \cos x$
132. $\cos(x + y) = x$

In Exercises 133–136, find the equations of the tangent line and the normal line to the graph of the equation at the given point. Use a graphing utility to graph the equation, the tangent line, and the normal line.

133. $x^2 + y^2 = 10, \quad (3, 1)$
134. $x^2 - y^2 = 20, \quad (6, 4)$
135. $y \ln x + y^2 = 0, \quad (e, -1)$
136. $\ln(x + y) = x, \quad (0, 1)$

In Exercises 137 and 138, use logarithmic differentiation to find *dy/dx*.

137. $y = \dfrac{x\sqrt{x^2 + 1}}{x + 4}$
138. $y = \dfrac{(2x + 1)^3(x^2 - 1)^2}{x + 3}$

In Exercises 139–142, verify that *f* has an inverse. Then use the function *f* and the given real number *a* to find $(f^{-1})'(a)$. (*Hint:* Use Theorem 3.17.)

Function	*Real number*
139. $f(x) = x^3 + 2$	$a = -1$
140. $f(x) = x\sqrt{x - 3}$	$a = 4$
141. $f(x) = \tan x, \ -\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}$	$a = \dfrac{\sqrt{3}}{3}$
142. $f(x) = \cos x, \ 0 \le x \le \pi$	$a = 0$

In Exercises 143–148, find the derivative of the function.

143. $y = \tan(\arcsin x)$
144. $y = \arctan(x^2 - 1)$
145. $y = x \operatorname{arcsec} x$
146. $y = \frac{1}{2} \arctan e^{2x}$
147. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1 - x^2} \arcsin x$
148. $y = \sqrt{x^2 - 4} - 2 \operatorname{arcsec} \dfrac{x}{2}, \quad 2 < x < 4$

149. A point moves along the curve $y = \sqrt{x}$ in such a way that the y-value is increasing at a rate of 2 units per second. At what rate is x changing for each of the following values?

(a) $x = \frac{1}{2}$ (b) $x = 1$ (c) $x = 4$

150. *Surface Area* The edges of a cube are expanding at a rate of 8 centimeters per second. How fast is the surface area changing when each edge is 6.5 centimeters?

151. *Changing Depth* The cross section of a 5-meter trough is an isosceles trapezoid with a 2-meter lower base, a 3-meter upper base, and an altitude of 2 meters. Water is running into the trough at a rate of 1 cubic meter per minute. How fast is the water level rising when the water is 1 meter deep?

152. *Linear and Angular Velocity* A rotating beacon is located 1 kilometer off a straight shoreline (see figure). If the beacon rotates at a rate of 3 revolutions per minute, how fast (in kilometers per hour) does the beam of light appear to be moving to a viewer who is $\frac{1}{2}$ kilometer down the shoreline?

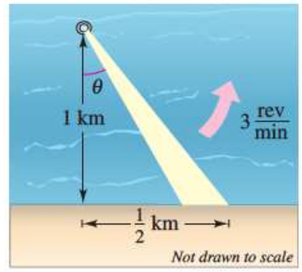
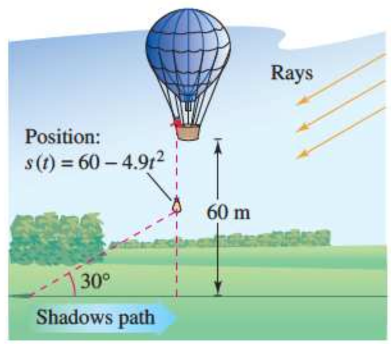


Figure for 152

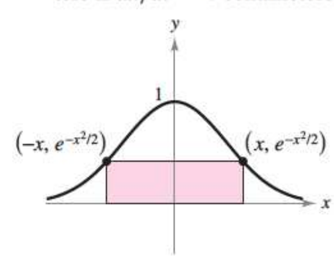
153. *Moving Shadow* A sandbag is dropped from a balloon at a height of 60 meters when the angle of elevation to the sun is 30° (see figure). Find the rate at which the shadow of the sandbag is traveling along the ground when the sandbag is at a height of 35 meters. [*Hint:* The position of the sandbag is given by $s(t) = 60 - 4.9t^2$.]



154. *Geometry* Consider the rectangle shown in the figure.

(a) Find the area of the rectangle as a function of x.

(b) Find the rate of change of the area when $x = 4$ centimeters if $dx/dt = 4$ centimeters per minute.



In Exercises 155–158, use Newton's Method to approximate any real zeros of the function accurate to three decimal places. Use the root-finding capabilities of a graphing utility to verify your results.

155. $f(x) = x^3 - 3x - 1$
156. $f(x) = x^3 + 2x + 1$
157. $g(x) = xe^x - 4$
158. $f(x) = 3 - x \ln x$

In Exercises 159 and 160, use Newton's Method to approximate, to three decimal places, the x-values of any points of intersection of the graphs of the equations. Use a graphing utility to verify your results.

159. $y = x^4$
$y = x + 3$

160. $y = \sin \pi x$
$y = 1 - x$

P.S. PROBLEM SOLVING

1. Consider the graph of the parabola $y = x^2$.

(a) Find the radius r of the largest possible circle centered on the y-axis that is tangent to the parabola at the origin, as indicated in the figure. This circle is called the **circle of curvature** (see Section 12.5). Use a graphing utility to graph the circle and parabola in the same viewing window.

(b) Find the center $(0, b)$ of the circle of radius 1 centered on the y-axis that is tangent to the parabola at two points, as indicated in the figure. Use a graphing utility to graph the circle and parabola in the same viewing window.

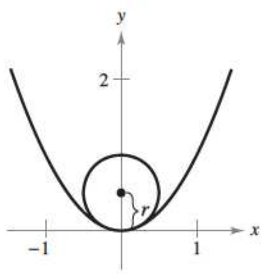


Figure for 1(a)

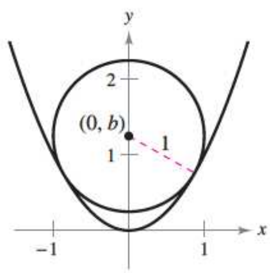


Figure for 1(b)

2. Graph the two parabolas $y = x^2$ and $y = -x^2 + 2x - 5$ in the same coordinate plane. Find equations of the two lines simultaneously tangent to both parabolas.

3. (a) Find the polynomial $P_1(x) = a_0 + a_1x$ whose value and slope agree with the value and slope of $f(x) = \cos x$ at the point $x = 0$.

(b) Find the polynomial $P_2(x) = a_0 + a_1x + a_2x^2$ whose value and first two derivatives agree with the value and first two derivatives of $f(x) = \cos x$ at the point $x = 0$. This polynomial is called the second-degree **Taylor polynomial** of $f(x) = \cos x$ at $x = 0$.

(c) Complete the table comparing the values of $f(x) = \cos x$ and $P_2(x)$. What do you observe?

x	-1.0	-0.1	-0.001	0	0.001	0.1	1.0
$\cos x$							
$P_2(x)$							

(d) Find the third-degree Taylor polynomial of $f(x) = \sin x$ at $x = 0$.

4. (a) Find an equation of the tangent line to the parabola $y = x^2$ at the point $(2, 4)$.

(b) Find an equation of the normal line to $y = x^2$ at the point $(2, 4)$. (The normal line is perpendicular to the tangent line.) Where does this line intersect the parabola a second time?

(c) Find equations of the tangent line and normal line to $y = x^2$ at the point $(0, 0)$.

(d) Prove that for any point $(a, b) \neq (0, 0)$ on the parabola $y = x^2$, the normal line intersects the graph a second time.

5. Find a third-degree polynomial $p(x)$ that is tangent to the line $y = 14x - 13$ at the point $(1, 1)$, and tangent to the line $y = -2x - 5$ at the point $(-1, -3)$.

6. Find a function of the form $f(x) = a + b\cos cx$ that is tangent to the line $y = 1$ at the point $(0, 1)$, and tangent to the line

$$y = x + \frac{3}{2} - \frac{\pi}{4}$$

at the point $\left(\frac{\pi}{4}, \frac{3}{2}\right)$.

7. The graph of the **eight curve**

$$x^4 = a^2(x^2 - y^2),\quad a \neq 0$$

is shown below.

(a) Explain how you could use a graphing utility to graph this curve.

(b) Use a graphing utility to graph the curve for various values of the constant a. Describe how a affects the shape of the curve.

(c) Determine the points on the curve at which the tangent line is horizontal.

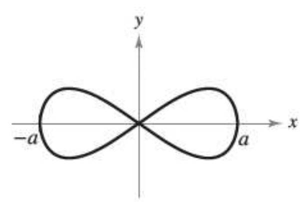


Figure for 7

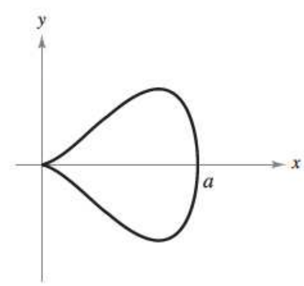


Figure for 8

8. The graph of the **pear-shaped quartic**

$$b^2y^2 = x^3(a - x),\quad a, b > 0$$

is shown above.

(a) Explain how you could use a graphing utility to graph this curve.

(b) Use a graphing utility to graph the curve for various values of the constants a and b. Describe how a and b affect the shape of the curve.

(c) Determine the points on the curve at which the tangent line is horizontal.

9. To approximate e^x, you can use a function of the form $f(x) = \dfrac{a + bx}{1 + cx}$. (This function is known as a **Padé approximation.**) The values of $f(0)$, $f'(0)$, and $f''(0)$ are equal to the corresponding values of e^x. Show that these values are equal to 1 and find the values of a, b, and c such that $f(0) = f'(0) = f''(0) = 1$. Then use a graphing utility to compare the graphs of f and e^x.

10. A man 6 feet tall walks at a rate of 5 feet per second toward a streetlight that is 30 feet high (see figure). The man's 3-foot-tall child follows at the same speed, but 10 feet behind the man. At times, the shadow behind the child is caused by the man, and at other times, by the child.

(a) Suppose the man is 90 feet from the streetlight. Show that the man's shadow extends beyond the child's shadow.

(b) Suppose the man is 60 feet from the streetlight. Show that the child's shadow extends beyond the man's shadow.

(c) Determine the distance d from the man to the streetlight at which the tips of the two shadows are exactly the same distance from the streetlight.

(d) Determine how fast the tip of the shadow is moving as a function of x, the distance between the man and the streetlight. Discuss the continuity of this shadow speed function.

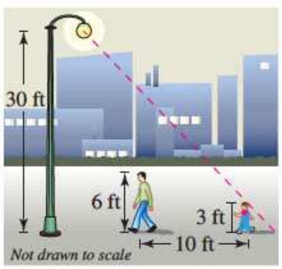


Figure for 10

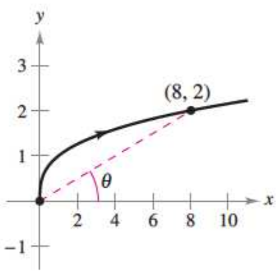


Figure for 11

11. A particle is moving along the graph of $y = \sqrt[3]{x}$ (see figure). When $x = 8$, the y-component of the position of the particle is increasing at the rate of 1 centimeter per second.

(a) How fast is the x-component changing at this moment?

(b) How fast is the distance from the origin changing at this moment?

(c) How fast is the angle of inclination θ changing at this moment?

12. The figure shows the graph of the function $y = \ln x$ and its tangent line L at the point (a, b). Show that the distance between b and c is always equal to 1.

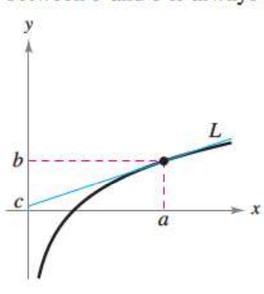


Figure for 12

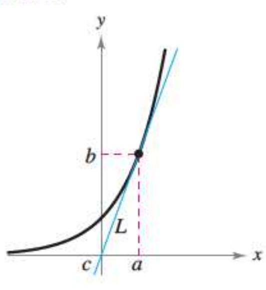


Figure for 13

13. The figure shows the graph of the function $y = e^x$ and its tangent line L at the point (a, b). Show that the distance between a and c is always equal to 1.

14. The fundamental limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

assumes that x is measured in radians. What happens if we assume that x is measured in degrees instead of radians?

(a) Set your calculator to degree mode and complete the table.

z (in degrees)	0.1	0.01	0.0001
$\dfrac{\sin z}{z}$			

(b) Use the table to estimate $\lim\limits_{z \to 0} \dfrac{\sin z}{z}$ for z in degrees. What is the exact value of this limit? (*Hint:* $180° = \pi$ radians)

(c) Use the limit definition of the derivative to find $\dfrac{d}{dz} \sin z$ for z in degrees.

(d) Define the new functions $S(z) = \sin(cz)$ and $C(z) = \cos(cz)$, where $c = \pi/180$. Find $S(90)$ and $C(180)$. Use the Chain Rule to calculate $\dfrac{d}{dz} S(z)$.

(e) Explain why calculus is made easier by using radians instead of degrees.

15. An astronaut standing on the moon throws a rock upward. The height of the rock is $s = -\frac{27}{10}t^2 + 27t + 6$, where s is measured in feet and t is measured in seconds.

(a) Find expressions for the velocity and acceleration of the rock.

(b) Find the time when the rock is at its highest point by finding the time when the velocity is zero. What is the rock's height at this time?

(c) How does the acceleration of the rock compare with the acceleration due to gravity on Earth?

16. If a is the acceleration of an object, the *jerk* j is defined by $j = a'(t)$.

(a) Use this definition to give a physical interpretation of j.

(b) The figure shows the graphs of the position, velocity, acceleration, and jerk functions of a vehicle. Identify each graph and explain your reasoning.