(2) area under curve

(1) Slope @ a point

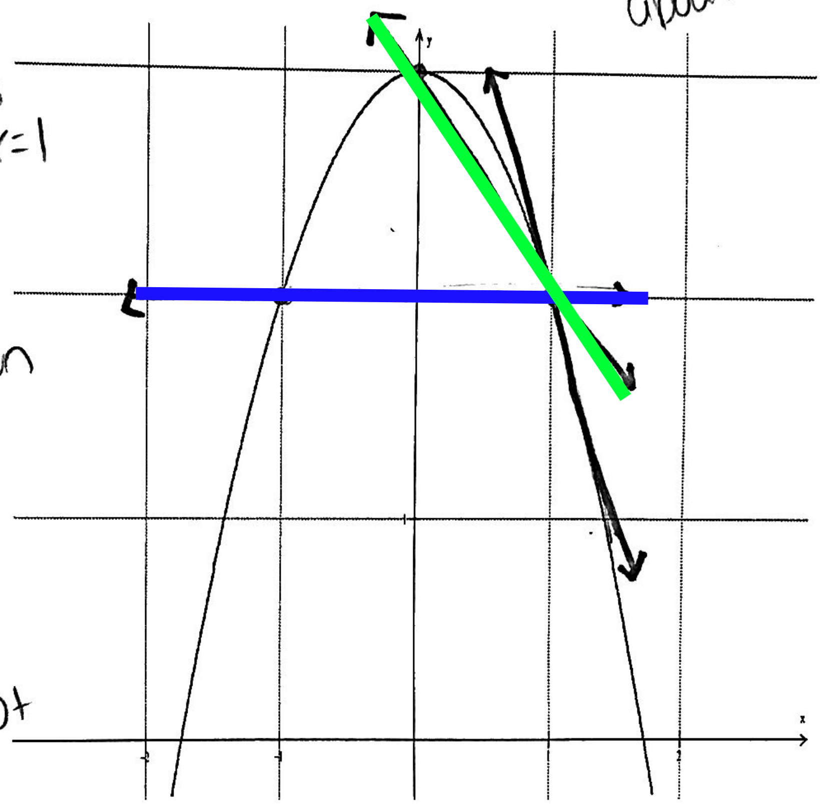
AP Calculus I
Notes 3.1
The Derivative and the Tangent Line Problem

The study of calculus originated, in part, from trying to find how a curve is changing at a *single* point. The measure of the change of a graph can be expressed as **slope**. We can find the slope between two points using the slope formula:

thus ex about $x=1$

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$ → average rate of change

a) Find: $\dfrac{f(1)-f(-1)}{1-{}^{-}1}$ Slope of f between $x=-1$ & $x=1$



$m = \dfrac{2-2}{1+1} = \dfrac{0}{2} = \boxed{0}$

not the best approximation

b) Find: $\dfrac{f(1)-f(0)}{1-0}$

$m = \dfrac{2-3}{1-0} = \dfrac{-1}{1} = \boxed{-1}$

negative so better, but not great

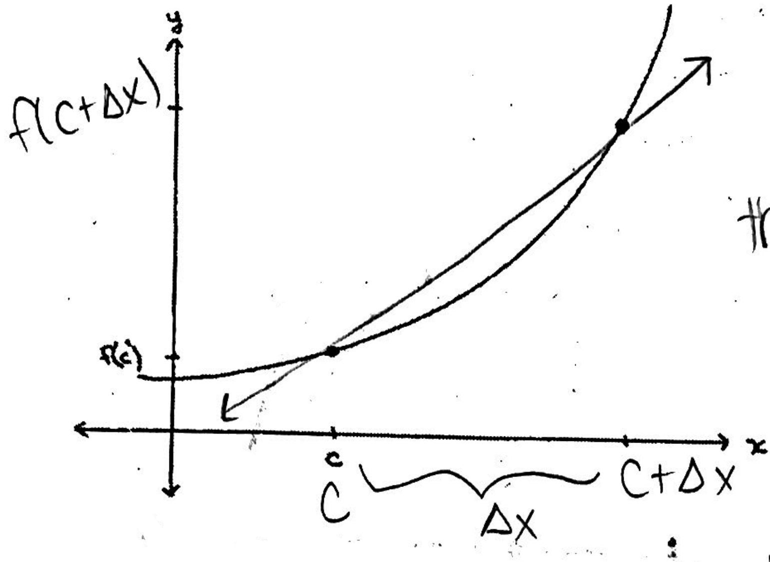
c) Find: $\dfrac{f(1)-f(1)}{1-1}$

$\dfrac{0}{0}$ indeterminate

$\dfrac{y_2 - y_1}{x_2 - x_1}$ (blue & green lines)

This expression for slope is known as both <u>average rate of change</u> and <u>Slope of the secant line</u>

The slope formula directly will not give the slope at a single point. So, let's try a different method:



Slope of the Secant line:

$m = \frac{y_2 - y_1}{x_2 - x}$ lets find the slope between $(c, f(c))$ \$ $(c+\Delta x, f(c+\Delta x))$ is ...

$$m_{sec} = \frac{f(c+\Delta x) - f(c)}{c+\Delta x - c}$$

$$m_{sec} = \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

To get a better representation of the slope at c lets bring the 2nd point closer, meaning Δx gets smaller

can $\Delta x = 0$?

No, ex 3 from front → instead $\Delta x \to 0$ (limit!!)

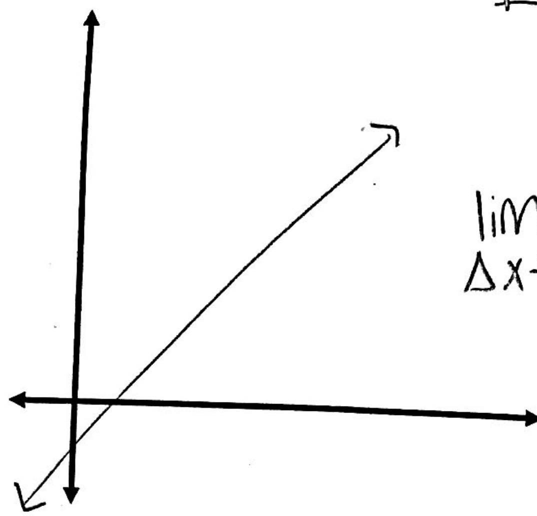
Definition of Tangent Line with Slope *m*:

$$\boxed{m_{tan} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}}$$

If f is defined on an open interval containing c, and if the limit exists, then the line passing through the point $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

* The slope of the tangent line to the graph of f at the point $(c, f(c))$ is also called the **slope of the graph of *f* at $x = c$.** This is also known as the instantaneous rate of change.

Ex. 1: Find the slope of the graph of $\boxed{f(x) = 3x - 1}$ at the $\boxed{\text{point } (2, 5).}$ $c = 2$



$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}, \quad f(2+\Delta x)$$

$$\lim_{\Delta x \to 0} \frac{f(2+\Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3(2+\Delta x) - 1 - 5}{\Delta x}$$

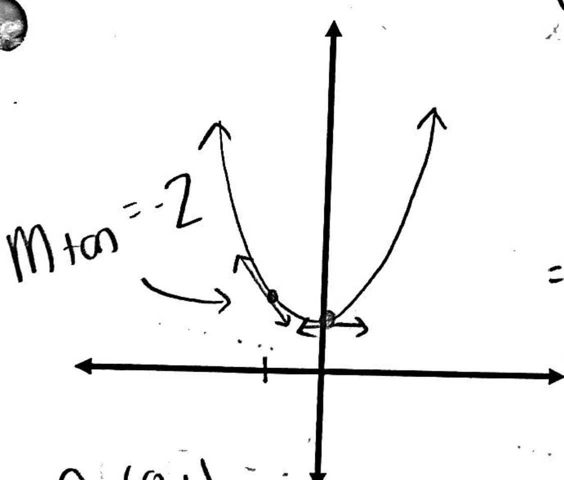
$$\lim_{\Delta x \to 0} \frac{6 + 3\Delta x - 1 - 5}{\Delta x} = \lim_{\Delta x \to 0} \frac{3\cancel{\Delta x}}{\cancel{\Delta x}} =$$

$$\lim_{\Delta x \to 0} 3 = \boxed{3}$$

Why does this answer make sense?

Slope of the Linear function!

Ex. 2: Find the slopes of the tangent lines to the graph of $f(x) = x^2 + 1$ at the points $(-1,2)$ and $(0,1)$. Then, find the slope of the secant line between the two points.



@ $(-1,2)$ $\displaystyle\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m_{tan}$

$$\lim_{\Delta x \to 0} \frac{f(-1+\Delta x) - f(-1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(-1+\Delta x)^2 + 1 - 2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{1} - 2\Delta x + \Delta x^2 \cancel{+1} \cancel{-2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(-2+\Delta x)}{\cancel{\Delta x}}$$

$$\lim_{\Delta x \to 0} -2 + \Delta x = \boxed{-2}$$

m_{tan} @ $(0,1)$

$$\lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{f(\Delta x) - 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x^{\cancel{2}} + \cancel{1} \cancel{-1}}{\cancel{\Delta x}} = \lim_{\Delta x \to 0} \Delta x \boxed{= 0}$$

Slope of tan line is 0

Slope of Secant

$(-1,2)(0,1)$

$$\frac{2-1}{-1-0} = \frac{1}{-1} = \boxed{-1}$$

The limit used to define the slope of a tangent line is also used to define one of the two fundamental operations of calculus – **DIFFERENTIATION**

not given
the x-value

Definition of the Derivative of a Function

The **derivative** of f at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$ provided the limit exists

The process of finding the derivative of a function is called **differentiation**. A function is **differentiable** x if its derivative exists at x and **differentiable on an open interval** (a, b) if it is differentiable at every point in that interval.

Notations used to Denote the Derivative of $y = f(x)$

1. $f'(x)$
2. $\frac{dy}{dx}$ Leibniz
3. y'
4. $\frac{d}{dx}[f(x)]$

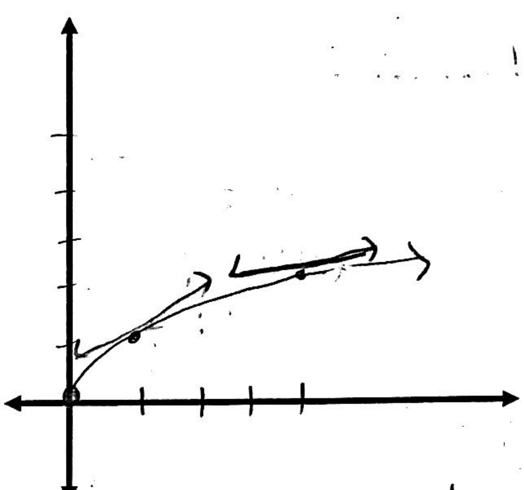
The notation $\frac{dy}{dx}$ is read as "the derivative of y with respect to x."

Ex. 3: Find the derivative of $f(x) = \sqrt{x}$. Then find the slope of the graph of $f(x)$ at the points $(1,1)$ and $(4,2)$. Discuss the behavior of f at $(0,0)$.

Derivative $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ $\frac{0}{0}$ conjugate!

$\lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}}$

$\lim_{\Delta x \to 0} \frac{\cancel{x}+\cancel{\Delta x}-\cancel{x}}{\cancel{\Delta x}(\sqrt{x+\Delta x}+\sqrt{x})} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta}}$



$f'(x) = \frac{1}{\sqrt{x}+\sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$

represents the slope of the graph at any given x-value

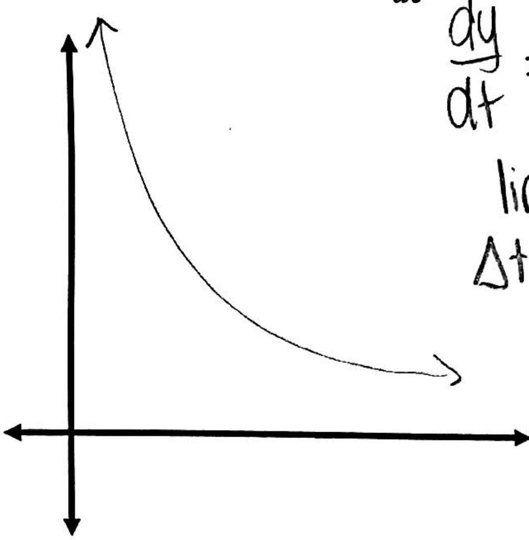
$(1,1)$ $m_{tan} = f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$

$(4,2)$ $m_{tan} = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$(0,0)$ $m_{tan} = f'(0) = \frac{1}{2\sqrt{0}}$ undefined

think of what lines had an undefined slope....

Ex. 4: Find the derivative $\frac{dy}{dt}$ for the function $y = \frac{2}{t}$.



$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\frac{2}{t+\Delta t} - \frac{2}{t}}{\Delta t} \cdot \frac{t(t+\Delta t)}{t(t+\Delta t)}$$

$$\lim_{\Delta t \to 0} \frac{2t - 2(t+\Delta t)}{\Delta t \cdot t \cdot (t+\Delta t)} = \lim_{\Delta t \to 0} \frac{\cancel{2t} - \cancel{2t} - 2\cancel{\Delta t}}{\cancel{\Delta t} \cdot t \cdot (t+\Delta t)}$$

$$\lim_{\Delta t \to 0} \frac{-2}{t(t+\Delta t)} = \frac{-2}{t^2}$$

$$\boxed{\frac{dy}{dt} = \frac{-2}{t^2}}$$

What is the equation of the tangent line at (1,2)? $y - y_1 = m(x - x_1)$

we need a point & slope

pt : (1,2)

means evaluated @ t=1

Slope of the tangent line (derivative) $\quad \left.\frac{dy}{dt} = \frac{-2}{t^2}\right|_{t=1} = -2 \quad \boxed{y - 2 = -2(t-1)}$

Another alternate limit form of the derivative is: $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ provided this limit exists

Yesterday $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

The existence of the limit in this alternative form requires that the one-sided limits of:

$$\lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} \text{ and } \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c}$$

are equal. These one-sided limits are called the **derivatives from the left and from the right.**

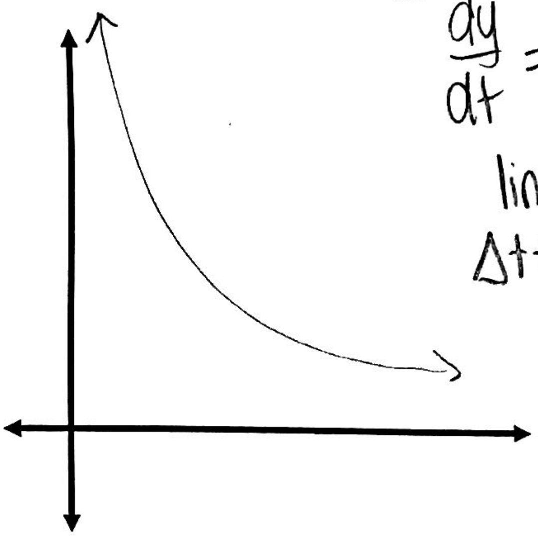
So the Example 2 could be seen as… $\quad f(x) = x^2+1$ @ $(-1,2)$

$$f'(-1) = \lim_{x \to -1} \frac{f(x)-f(-1)}{x - -1} = \lim_{x \to -1} \frac{x^2+1-2}{x+1} = \lim_{x \to -1} \frac{x^2-1}{x+1}$$

$$= \lim_{x \to -1} \frac{\cancel{(x+1)}(x-1)}{\cancel{x+1}} = \lim_{x \to -1} x-1 = \boxed{-2}$$

slope of f(x) @ x=-1

Ex. 4: Find the derivative $\frac{dy}{dt}$ for the function $y = \frac{2}{t}$.



$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\frac{2}{t+\Delta t} - \frac{2}{t}}{\Delta t} \cdot \frac{t(t+\Delta t)}{t(t+\Delta t)}$$

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$$\lim_{\Delta t \to 0} \frac{-2}{t(t+\Delta t)} = \frac{-2}{t^2}$$

$$\boxed{\frac{dy}{dt} = \frac{-2}{t^2}}$$

What is the equation of the tangent line at (1,2)? $y - y_1 = m(x - x_1)$

we need a point & slope

pt : (1,2)

means evaluated @ t=1

Slope of the tangent line (derivative) $\frac{dy}{dt} = \frac{-2}{t^2}\Big|_{t=1} = -2$ $\boxed{y - 2 = -2(t-1)}$

Another alternate limit form of the derivative is: $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ provided this limit exists

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$$= \lim_{x \to -1} \frac{\cancel{(x+1)}(x-1)}{\cancel{x+1}} = \lim_{x \to -1} (x-1) = \boxed{-2}$$

slope of f(x) @ x=-1

Differentiability and Continuity

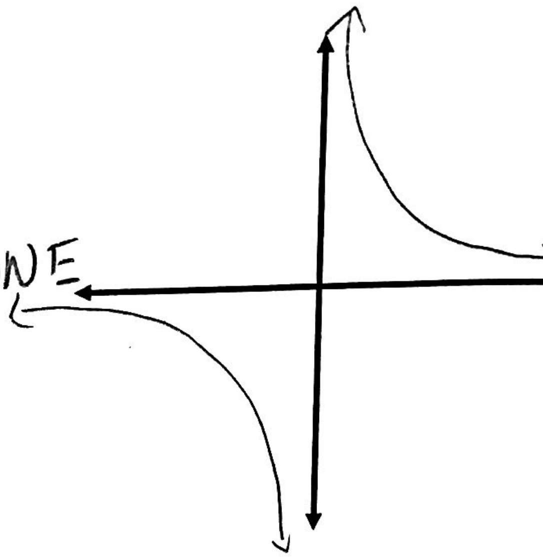
Sketch a graph of each function. Determine the intervals in which the function is continuous and the intervals in which the functions are differentiable.

1. $f(x) = \frac{1}{x}$

$\lim_{x \to c} f(x) = f(c)$

Continuous $(-\infty, 0), (0, \infty)$

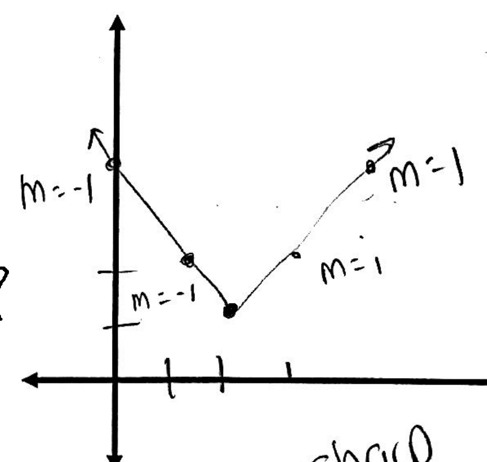
not cont @ $x = 0$ b/c $\lim_{x \to 0} f(x)$ DNE



Differentiable: $(-\infty, 0), (0, \infty)$

2. $f(x) = |x - 2| + 1$

Continuous: $(-\infty, \infty)$



Differentiable: what happens at $x = 2$?

$\lim_{x \to 2^-} f'(x) = -1$

$\lim_{x \to 2^+} f'(x) = 1$

$\lim_{x \to 2} f'(x) =$ DNE

(not differentiable @ $x = 2$)

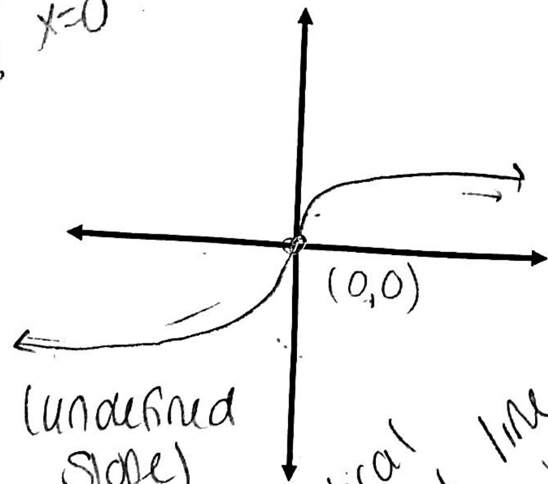
sharp point → abrupt change in slope

$(-\infty, 2), (2, \infty)$

3. $f(x) = x^{\frac{1}{3}}$

Continuous: $(-\infty, \infty)$

Differentiable: when $x = 0$? → can we find slope @ $x=0$?



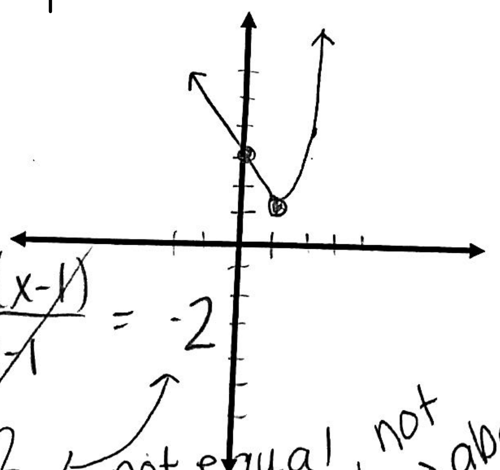
$f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$

$= \lim\limits_{x \to 0} \dfrac{x^{1/3} - 0}{x} = \lim\limits_{x \to 0} \dfrac{1}{x^{2/3}} = \dfrac{1}{0}$ (undefined slope)

(Vertical Tangent line undefined)

Diff: $(-\infty, 0), (0, \infty)$

4. $f(x) = \begin{cases} 3 - 2x & x < 1 \\ x^2 & x \geq 1 \end{cases}$ linear parabola → $\lim\limits_{x \to 1^-} f(x) = 1$ $\lim\limits_{x \to 1^+} f(x) = 1$ and $f(1) = 1$



Continuous? $\mathbb{R}$

Differentiable? @ $x = 1$)

$\lim\limits_{x \to 1^-} \dfrac{3 - 2x - 1}{x - 1} = \lim\limits_{x \to 1^-} \dfrac{2 - 2x}{x - 1} = \lim\limits_{x \to 1^-} \dfrac{-2\cancel{(x-1)}}{\cancel{x-1}} = -2$

$\lim\limits_{x \to 1^+} \dfrac{x^2 - 1}{x - 1} = \lim\limits_{x \to 1} \dfrac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim\limits_{x \to 1} x + 1 = 2$ ← not equal, not differentiable → abrupt change in slope (sharp pt)

$(-\infty, 1), (1, \infty)$ ←

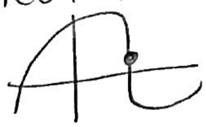
Theorem – If f is differentiable at $x = c$, then f is continuous at $x = c$

Summary: Same

1) If a function is __________ at $x = c$, then it is __________ at $x = c$.
Thus differentiability implies continuity.

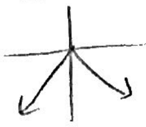
2) It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. Thus continuity does not imply differentiability.

Vertical tangent line



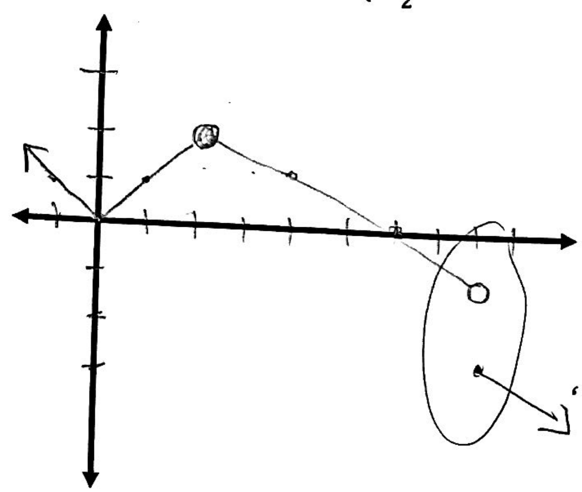
Abrupt slope change

$y = -|x|$



$y = x^{2/3}$

Ex. 5: Graph the function $f(x) = \begin{cases} |x| & x < 2 \\ -\frac{1}{2}x + 3 & 2 \le x < 6 \\ -\frac{1}{2}x & x \ge 6 \end{cases}$ and then determine the following:



If f is not cont @ 6, its not differentiable at x=6

a) Find all values of x such that f is continuous. $\boxed{(-\infty,6),(6,\infty)}$

b) $\lim\limits_{h\to 0} \frac{f(1+h)-f(1)}{h} = \boxed{1}$ formula means slope @ x=1 (+ side of abs value)

c) $f'(4) =$ Slope of f x=4 or slope of the tangent line @ x=

$\boxed{= -\frac{1}{2}}$ (Linear)

d) $\lim\limits_{h\to 0^-} \frac{f(6+h)-f(6)}{h} =$ slope approaching 6 from left

$= -\frac{1}{2}$ (linear)

e) $\lim\limits_{h\to 0^+} \frac{f(6+h)-f(6)}{h} =$ slope approaching 6 from right

$-\frac{1}{2}$

f) Find all values of x such that f is differentiable.

$(-\infty,0),(0,2),(2,6),(6,\infty)$

x=0 abrupt slope change

x=2 " "

x=6 discontinuity

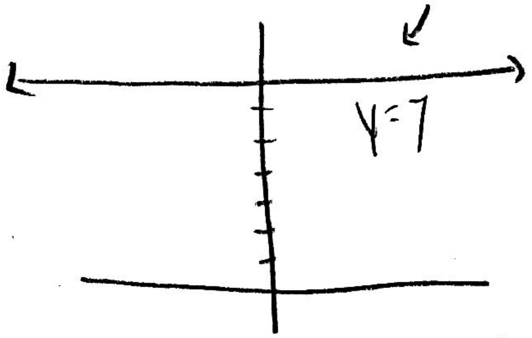
AP Calculus I

Notes 3.2

Basic Differentiation Rules and Rates of Change

The Constant Rule

Ex. 1: Use the definition of a derivative or knowledge of the graph to find the derivative of $y = 7$.



m=0

So, **the Constant Rule** is the derivative of a constant function is 0 $\frac{d}{dx}[c] = 0$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The Power Rule

Ex. 2: Use the definition of a derivative to find the derivative of each of the following:

a) $f(x) = x^3$

$$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\cancel{x^3} + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - \cancel{x^3}}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(3x^2 + 3x\Delta x + \Delta x^2)}{\cancel{\Delta x}}$$

$$\lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + \Delta x^2 =$$

$$\boxed{f' = 3x^2}$$

b) $g(x) = x^2$

$$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x\Delta x + \Delta x^2 - \cancel{x^2}}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(2x+\Delta x)}{\cancel{\Delta x}}$$

$$\lim_{\Delta x \to 0} 2x + \Delta x$$

$$\boxed{= 2x}$$

c) $h(x) = \sqrt{x} = x^{\frac{1}{2}}$

$$\lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$\lim_{\Delta x \to 0} \frac{\cancel{x} + \cancel{\Delta x} - \cancel{x}}{\cancel{\Delta x}(\sqrt{x+\Delta x} + \sqrt{x})}$$

$$\lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$$

So, **the Power Rule** is if n is a rational number, then the derivative of a function $f(x) = x^n$ is nx^{n-1}

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Ex. 3: Use the derivative rules to find the derivative of each of the following:

a) $f(x) = \pi^2$ b) $g(t) = \sqrt[3]{t} = t^{1/3}$ c) $h(x) = \frac{1}{x^2} = x^{-2}$

$f'(x) = 0$

(Constant)

$g'(t) = \frac{1}{3}t^{-2/3}$

$= \frac{1}{3t^{2/3}}$

$h'(x) = -2x^{-3}$

$= \frac{-2}{x^3}$

Ex. 4: Find the equation of the tangent line to the graph of $y = x^2$ when $x = -2$.

**What do we need to write the equation of a line? **

point & slope

$y(-2) = 4$

$\frac{dy}{dx} = 2x\Big|_{x=-2} = -4$

$\boxed{y - 4 = -4(x+2)}$

The Constant Multiple Rule

Ex. 5: Use the definition of a derivative to find the derivative of $f(x) = 3x^2$.

$\lim_{h\to 0} \frac{3(x+h)^2 - 3x^2}{h} = \lim_{h\to 0} \frac{3(x^2+2xh+h^2) - 3x^2}{h} = \lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}(6x+3h)}{\cancel{h}} = \lim_{h\to 0} 6x + 3h = \boxed{6x = f'(x)}$

So, **the Constant Multiple (Coefficient) Rule** is if f is a differentiable function and c is a real number, the derivative of a function $cf(x)$ is $c \cdot f'(x)$

$\frac{d}{dx}[cf(x)] = c \cdot f'(x)$

Ex. 6: Find the derivative of each of the following:

rewrite!

a) $f(t) = \frac{4t^3}{5}$

$f(t) = \frac{4}{5}t^3$

$f'(t) = \frac{12}{5}t^2$

b) $a(k) = \frac{3\pi}{2k}$

$a(k) = \frac{3\pi}{2} \cdot k^{-1}$

$a'(k) = \frac{-3\pi}{2} k^{-2}$

$= \frac{-3\pi}{2k^2}$

The Sum and Difference Rules

The derivative of the sum (or difference) of two differentiable functions is differentiable and is the sum (or difference) of their derivatives.

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

Ex. 7: Find the derivative of each of the following:

rewrite!

2 rewrite steps!

a) $f(x) = x^3 - 4x + 5$

$f'(x) = 3x^2 - 4$

b) $r = \sqrt[4]{\theta^3} - \frac{5}{2\theta^3} - \frac{\theta^4}{2}$

$r = \theta^{3/4} - \frac{5}{2}\theta^{-3} - \frac{\theta^4}{2}$

$r' = \frac{3}{4}\theta^{-1/4} + \frac{15}{2}\theta^{-4} - \frac{4\theta^3}{2}$

$r' = \frac{3}{4\theta^{1/4}} + \frac{15}{2\theta^4} - 2\theta^3$

c) $h(t) = \frac{3t^3+5t^2-\sqrt{t}}{t}$

$h(t) = \frac{3t^3}{t} + \frac{5t^2}{t} - \frac{t^{1/2}}{t}$

$h(t) = 3t^2 + 5t - t^{-1/2}$

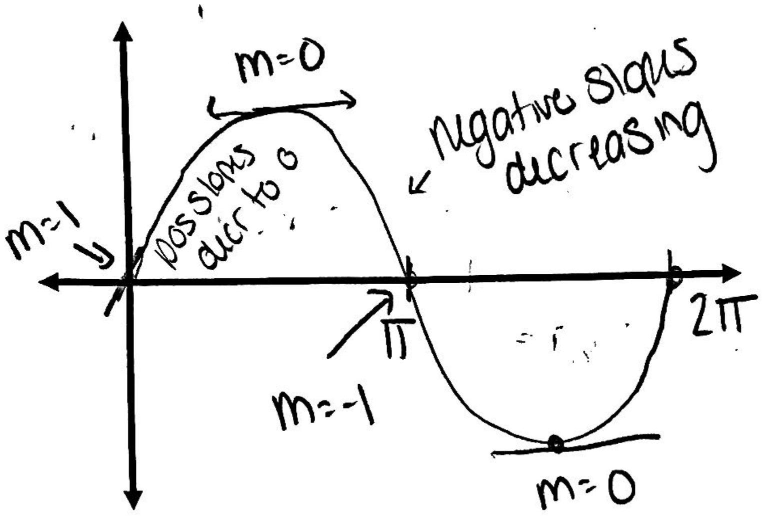
$h'(t) = 6t + 5 + \frac{1}{2}t^{-3/2}$

$h'(t) = 6t + 5 + \frac{1}{2t^{3/2}}$

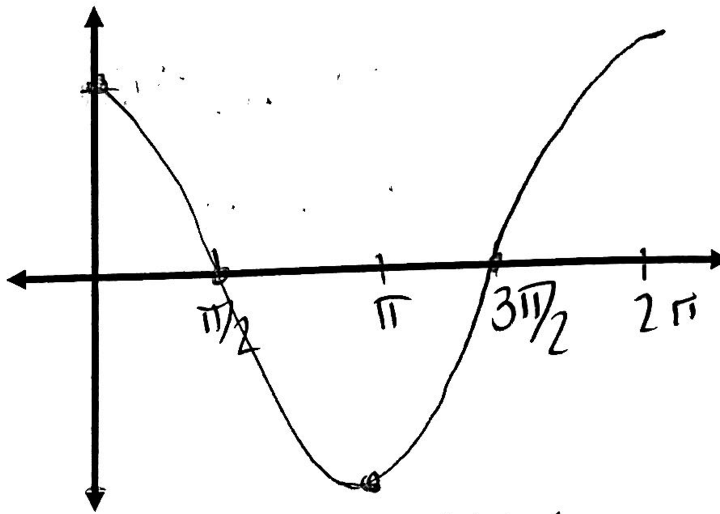
Derivatives of Sine and Cosine Functions

Proofs: Graphically:

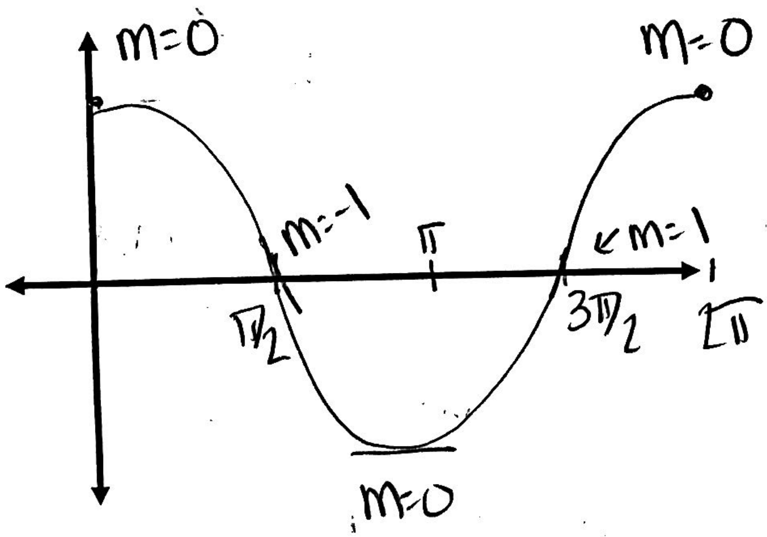
f'



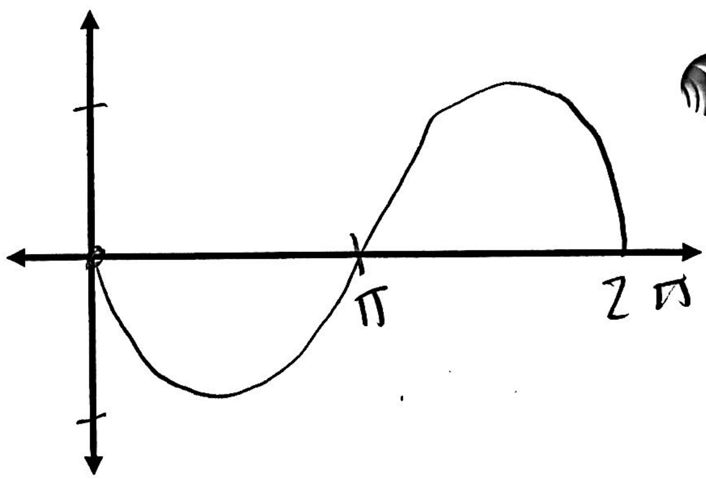
Graph of $f(x) = \sin x$



Graph of $f'(x) =$ cos x



Graph of $f(x) = \cos x$



Graph of $f'(x) =$ -sin x

Derivative Rules of Sine and Cosine Functions

$\frac{d}{dx}[\sin x] =$ cos x

$\frac{d}{dx}[\cos x] =$ -sin x

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Ex. 8: Evaluate the following:

a) $\underbrace{\lim_{\Delta x \to 0} \frac{2\sin(x+\Delta x)-2\sin x}{\Delta x}}$

means find the deriv of $y = 2\sin x$

$$\boxed{y' = 2\cos x}$$

b) $\lim_{\Delta x \to 0} \frac{\overbrace{(x+\Delta x)^2-\cos(x+\Delta x)}^{f(x+\Delta x)}-\left(x^2+\cos x\right)}{\Delta x}$

give f(

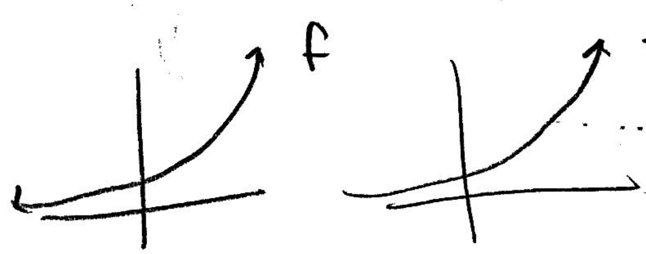
Find deriv. of

$f(x) = x^2 - \cos x$

$f'(x) = 2x + \sin x$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}[e^x] = e^x$$



Ex. 9: Find the derivative of the following functions:

a) $f(x) = 3\cos x - e^x$

$f'(x) = -3\sin x - e^x$

↑ represents slope @ any x-value

b) $g(n) = \frac{3}{\sqrt{n}} + \frac{4\pi}{e^2}e^n$

coeffi

$g(n) = 3n^{-1/2} + \frac{4\pi}{e^2}e^n$

$g'(n) = \frac{-3}{2}n^{-3/2} + \frac{4\pi}{e^2}e^n$

$= \frac{-3}{2n^{3/2}} + \frac{4\pi}{e^2}e^n$

Ex. 10: Determine the points on the curve in which the tangent line is horizontal or vertical.

slope is 0   slope undefind

a) $f(x) = 3x + 3\sin x \qquad 0 \le x \le 2\pi$

$f'(x) = 3 + 3\cos x$

Horiz

$3 + 3\cos x = 0$

$3\cos x = -3$

$\cos x = -1$

$\boxed{x = \pi \quad (\pi, 3\pi)}$

Vertical Tangent

$f' = 3 + 3\cos x$ undefined?

$\boxed{\text{never}}$

b) $g(x) = 6\sqrt[3]{x} - 5 \rightarrow 6x^{1/3} - 5$

$g'(x) = 2x^{-2/3}$

Horizontal

$\frac{2}{x^{2/3}} = 0$

$\boxed{\text{Never}}$

vert

$g' = \frac{2}{x^{2/3}}$ undefind?

$x = 0$

$g(0) = 6\sqrt[3]{0} - 5 \quad \boxed{(0, -5)}$

Ex. 11: Find the average rate of change from [1,3] of $h(x) = x^2 + e^x$ and then find the instantaneous rate of change at each of the endpoints. Give the graphical interpretation of the results.

average rate of change = slope of the secant line $\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$

$$\frac{h(3) - h(1)}{3 - 1} = \frac{9 + e^3 - (1 + e)}{3 - 1} = \boxed{\frac{8 + e^3 - e}{2}}$$

Instantanous rate of change = slope of tangent line ; derivative

@ end points

$h'(x) = 2x + e^x \qquad h'(1) = 2 + e$

$h'(3) = 6 + e^3$

AP Calculus I
Notes 3.3
The Product/Quotient Rules and Higher-Order Derivatives

Need for More Rules

$f(x) = g(x) \cdot h(x)$

1) Given the function $f(x) = (x^2)(x^4)$, find the derivative of $f(x)$ by taking the derivative of both terms, then simplify by multiplying the two answers.

g h

$$(2x)(4x^3) = 8x^4 \;??$$

Pro

2) Next, go back to $f(x)$ and start off by multiplying the terms and then differentiate.

confident this answ. is correct

$f(x) = x^6$

$f'(x) = 6x^5$

$6x^5 \neq 8x^4$

$(2x)(x^4) + (4x^3)(x^2)$

$2x^5 + 4x^5$

$= 6x^5$ ✓

3) Is it true that $\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$? No!

$$\frac{d}{dx}\left[f(x) \cdot g(x)\right] = ??$$

The Product Rule:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

+ B − some quantity

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)(g(x)) + f(x+\Delta x) \cdot g(x) - f(x)g(x)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x)}{\Delta x} + \frac{f(x+\Delta x)g(x) - f(x) \cdot g(x)}{\Delta x} \right]$$

$$\lim_{\Delta x \to 0} f(x+\Delta x)\left(\frac{g(x+\Delta x) - g(x)}{\Delta x}\right) + g(x)\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)$$

$f \quad g' \quad + \quad g \cdot f'$

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x)$$

product

coefficient

Ex. 1: Find the derivative of each of the following:

a) $h(x) = (3x - 2x^2)(5 + 4x)$

$(x) = (3-4x)(5+4x) + (4)(3x-2x^2)$

$= 15 - 20x + 12x - 16x^2 + 12x - 8x^2$

$= -24x^2 + 4x + 15$

Expand 1st $h(x) = 15x \cdot 10x^2 + 12x^2 - 8x^3$

$h(x) = -8x^3 + 2x^2 + 15$

$h'(x) = -24x^2 + 4x + 15$ ✓

b) $y = 3t^2e^t + 2\sin t - 3$

$\frac{dy}{dt} = 6te^t + 3t^2e^t + 2\cos t$

Need for MORE Rules

1) Given the function $f(x) = \frac{x^6}{x^2}$, find the derivative taking the derivative of both functions, then simplify by dividing the two answers.

$= \frac{6x^5}{2x} \rightarrow 3x^4$

2) Next, simplify the expression, then differentiate.

$f(x) = x^4$

$f'(x) = 4x^3$

3) Is it true that $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)}{g'(x)}$?

NO

The Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)^2]}$ $\frac{lo\,d(hi) - hi\,d(lo)}{lo^2}$

hi

lo

Ex. 2: Find each derivative:

a) $f(x) = \frac{5x-2}{x^2+1}$

$= \frac{(x^2+1)5 - (5x-2)(2x)}{(x^2+1)^2}$

$= \frac{5x^2+5-10x^2+4x}{(x^2+1)^2}$

$= \frac{-5x^2+4x+5}{(x^2+1)^2}$

b) $\lim_{h\to 0} \frac{\frac{e^{x+h}}{x+h} - \frac{e^x}{x}}{h}$

f(x+h) → ← f(x)

means deriv of

$f(x) = \frac{e^x}{x}$

$f'(x) = \frac{xe^x - e^x(1)}{x^2}$

$f'(x) = \frac{xe^x - e^x}{x^2}$

c) $\lim_{h\to 0} \frac{\frac{\pi+h}{\cos(\pi+h)} - \frac{\pi}{\cos\pi}}{h}$

f(π+h) ← f(π)

What is deriv. of

$f(x) = \frac{x}{\cos x}$ @ $x=\pi$?

$f'(x) = \frac{\cos x(1) - x(-\sin x)}{\cos^2 x}$

$f'(\pi) = \frac{\cos\pi + \pi\sin\pi}{\cos^2\pi} = \frac{-1+0}{1}$

$\boxed{=-1}$

Ex. 3: Assume that $f(x)$ and $g(x)$ are differentiable functions about which we know information about a few discrete data points. The information we know is summarized in the table below:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
−2	4	−1	5	6
1	1	6	2	3
2	−1	5	1	0

find p' first, then plug in 2. If you do it the wrong way you derive a constant!

Use your differentiation rules to determine each of the following.

a) If $p(x) = x^2 f(x)$, find $p'(2)$.

$p'(x) = 2xf(x) + f'(x)\cdot x^2$

$p'(2) = 2(2)f(2) + f'(2)\cdot 2^2$

$= 4(-1) + 5\cdot 4$

$-4 + 20 = \boxed{16}$

b) If $q(x) = \frac{3f(x)}{g(x)}$, find $q'(-2)$.

$q'(x) = \frac{g(x)\cdot 3f'(x) - 3f(x)\,g'(x)}{g^2(x)}$

$q'(-2) = \frac{g(-2)\cdot 3f'(-2) - 3f(-2)\,g'(-2)}{g^2(-2)}$

$= \frac{5\cdot 3(-1) - 3(4)(6)}{5^2} = \frac{-87}{25}$

$\frac{d}{dx}(\sin x) = \cos x$ $\frac{d}{dx}(\cos x) = \sin x$

Derivatives of Trigonometric Functions/Proofs Using Quotient Rule

$\frac{d}{dx}[\tan x] = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$

$= \frac{\cos x \cdot \cos x - \sin x \cdot -\sin x}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \boxed{= \sec^2 x}$

$\frac{d}{dx}[\cot x] = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$

$= \frac{\sin x \cdot -\sin x - \cos x \cdot \cos x}{\sin^2 x}$

$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$

$\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right]$

$= \frac{\cos x \cdot 0 - 1 \cdot (-\sin x)}{\cos^2 x}$

two x's!

$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \boxed{\sec x \tan x}$

$\frac{d}{dx}[\csc x] = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$

$= \frac{\sin x \cdot 0 - 1 \cos x}{\sin^2 x}$

$= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$

Ex. 4: Find the instantaneous rate of change of $y = \frac{\cot x}{6x^2}$.

deriv.

$\frac{dy}{dt} = \frac{6x^2 \cdot -\csc^2 x - \cot x (12x)}{(6x^2)^2}$

$= \frac{-6x^2 \csc^2 x - 12x \cot x}{36x^4}$

Ex. 5: Find the local linear approximation (equation of the tangent line) of $y = x \sec x$ at $x = \pi$.

point $(\pi, \pi \sec \pi)$

$(\pi, -\pi)$

slope

$y' = \sec x + x \sec x \tan x$

$\sec \pi + \pi \sec \pi \tan \pi$

$-1 + \pi(-1)(0) = -1$

$\boxed{y + \pi = -1(x - \pi)}$

Higher Order Derivatives

The second derivative is the derivative of the first derivative.

$$f''(x) = \frac{d}{dx}[f'(x)] = \frac{d^2y}{dx^2} \qquad f'''(x) = \frac{d}{dx}[f''(x)] = \frac{d^2}{dx^2}[f'(x)] = \frac{d^3y}{dx^3}$$

Ex. 6: Find the first, second, third, fourth and eighth derivatives of $f(x) = x^4 - 3x^3 + 2e^x - 17$.

$f'(x) = 4x^3 - 9x^2 + 2e^x$

$f''(x) = 12x^2 - 18x + 2e^x$

$f'''(x) = 24x - 18 + 2e^x$

$f^4(x) = 24 + 2e^x \rightarrow f^5(x) = 2e^x \rightarrow f^8(x) = 2e^x$

Real-World Applications

Prior to Calculus, people could only measure rates as **average** rates of change. For example, if you drove 60 miles in 1 hour, you travelled an average of 60 miles per hour. That, however, does not tell you how fast you travelled at any moment. Introducing Calculus, we now can determine the **instantaneous** rate of change. For our example, you can now determine how fast you travelled at any moment along the hour trip.

Ex. 7: The height of a tree, in feet, is modeled by the function $h(t) = 8 + 0.55t^2 - 3^{2-t}\cos 4t$, where t is in years. Find the average growth rate from years 1 to 4 and the growth rate at $t = 3$, with units.

$\frac{h(4)-h(1)}{4-1} = \frac{8.8726 - 1.070907}{4-1}$ ft/yr $= 3.3145$ ft/yr

at $t=3$

$h'(3) \rightarrow$ Math 8 [f(x), x, 3] $= 2.984$ ft/yr

Ex. 8: The cost, in dollars, to shred the confidential documents of a company is modeled by C, a differentiable function of the weight of documents in pounds. Of the following, which is the best interpretation of $C'(500)=80$?

~~(A)~~ The cost to shred 500 pounds of documents is \$80. $C(500) = 80$?

~~(B)~~ The average cost to shred documents is $\frac{80}{500}$ dollar per pound.

(C) Increasing the weight of documents by 500 pounds will increase the cost to shred the documents by approximately \$80. ?

(D) The cost to shred documents is increasing at a rate of \$80 per pound when the weight of the documents is 500 pounds.

Particle Motion

In discussing motion, there are three closely related concepts that you need to keep straight. They are:

X' Position $x(t)$

Velocity $v(t) = x'(t)$ → speed w/ direction

acceleration $a(t) = v'(t) = x''(t)$ → rate of change of velocity

If $x(t)$ represents the position of a particle along the x –axis at any time t, then this means…

1) “Initially” means when t = 0. “At the origin” means $x(t)$ = 0.
“At rest” means $v(t)$ = 0.

2) If the velocity of the particle is positive, then the particle is moving to the right.
If the velocity of the particle is negative, then the particle is moving to the left.

$\frac{x(b)-x(a)}{b-a}$

3) To find the average velocity over a time interval, divide the change in position by the change in time. This is the average rate of change of the position function.

4) Instantaneous velocity is the velocity at a single moment (instant!) in time. This is the instantaneous rate of change (aka the derivative) of the position function.

5) If the acceleration of the particle is positive, then the velocity is increasing. If the acceleration of the particle is negative, then the velocity is decreasing.

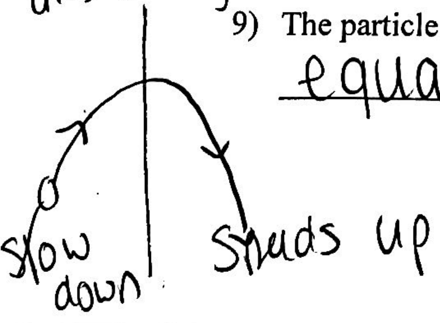
6) In order for a particle to change direction, the velocity must change signs.

7) Speed is the absolute value of velocity.

8) The amount of “ground” the particle covers is the distance. The change in the final position and the initial position is the displacement.

a(t) is neg

9) The particle’s speed is increasing when the signs of the velocity and acceleration are equal. The particle’s speed is decreasing when the signs are opposite.



Ex. 9: The data in the table below give selected values for the velocity, in meters/minute, of a particle moving along the x –axis. The velocity v is a differentiable function of time t.

Time t (min)	0	2	5	6	8	12
Velocity $v(t)$ (meters/min)	−3	2	3	5	7	5

a) At $t = 0$, is the particle moving to the right or to the left? Explain your answer.

velocity

$v(0) = -3$ $\quad -3 < 0$ so left

b) Is there a time during $0 \le t \le 12$ minutes when the particle is at rest? Explain your answer using correct mathematical language.

existence?

IVT!

$v(t)$ is differentiable therefore it $v(t)=0$ is continuous. Since $v(0) < 0 < v(2)$, then there must exist a time t on $0 < t < 2$ where $v(t) = 0$

c) What is the average acceleration of the particle over the interval $2 \le t \le 5$? Show the computations that lead to your answer and indicate units of measure.

avg rate of change of velocity

$$\frac{v(5)-v(2)}{5-2} = \frac{3-2}{5-2} = \frac{1}{3}\ m/min^2$$

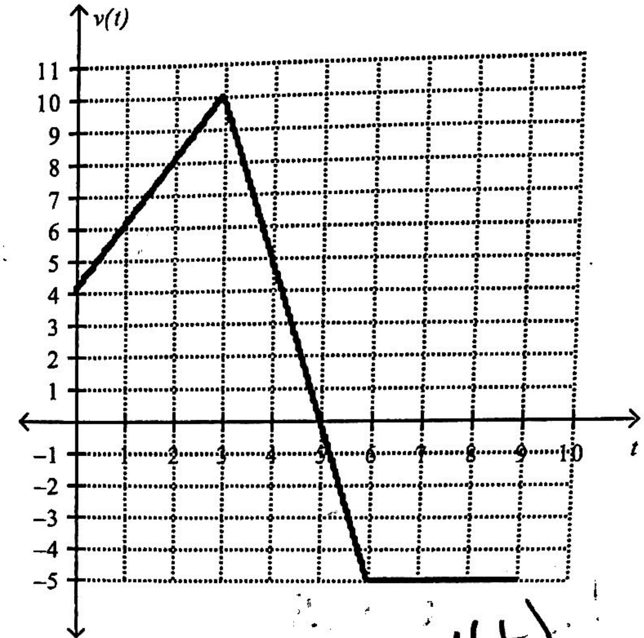
d) Use data from the table to find an approximation for $v'(10)$ and explain the meaning of $v'(10)$ in terms of the motion of the particle. Show your work and indicate units of measure.

$$v'(10) \approx \frac{v(12)-v(8)}{12-8} = \frac{5-7}{12-8} = \frac{-1}{2}\ m/min^2$$

e) Is the particle strictly increasing from $6 < t < 8$? Is the maximum velocity of the particle at $t = 8$? Explain your reasoning.

Not necessarily! Anything can happen between those integers of t

Ex. 10: The graph below represents the velocity $v(t)$, in feet per second, of a particle moving along the x –axis over the time interval $0 \leq t \leq 9$ seconds.



$v(t)$

a) At $t = 4$ seconds, is the particle moving to the right or the left? Explain your answer.

$v(4) > 0 \therefore$ right

b) Over what time interval is the particle moving to the left? Explain your answer.

$(5,9)$ because $v(t) < 0$ (below x axis)

c) Find the acceleration of the particle at $t = 2,3,4,7$ seconds or state that it does not exist.

$a(2) = v'(2) = 2$

$a(3) = v'(3) =$ Does not exist

$a(4) = v'(4) = -5$ $\qquad$ $a(7) = v'(7) = 0$

d) Is the particle's speed increasing or decreasing at $t = 4$ seconds? Explain your answer.

v & a same sign $\nearrow$ $\qquad$ v & a opp sign $\searrow$ $\qquad$ $v(4) = 5 > 0$

$a(4) = -5 < 0$

Speed decreasing (slows down)

e) What is the average acceleration of the particle over the interval $2 \leq t \leq 4$? Show the computations that lead to your answer and indicate units of measure.

$$\frac{v(4)-v(2)}{4-2} = \frac{5-8}{4-2} = \frac{-3}{2}\ ft/sec^2$$

Ex. 11: A particle moves along the y –axis such that its position, in meters, can be modeled by the equation $y(t) = 6.03t^2 + 9\sin t - e^t$, where t represents the time in seconds, $t \geq 0$.

a) Find the velocity and acceleration function.

calc

$y'(t) = v(t) = 12.06t + 9\cos t - e^t$

$y''(t) = v'(t) = a(t) = 12.06 - 9\sin t - e^t$

b) Find the initial position, velocity and acceleration of the particle. Indicate units of measure.

$t = 0$

$y(0) = -1$ m

$v(0) = 8$ m/s

$a(0) = 11.06$ m/s^2

c) What is the velocity of the particle when the acceleration is zero?

$a(t) = 0$

$12.06 - 9\sin t - e^t = 0$

$t = 1.25455$

$v(1.25455) = 14.422$

d) When is the particle moving up? Justify your answer.

$v(t) > 0$

$v(t)$ above x axis

$(0, 3.536)$

Ex. 12: A particle moves along the x –axis so that at time t, its position is given by:

$$x(t) = (t-1)(t^2 - 5t + 4)$$

a) Find the displacement of the particle over the time interval [1,3].

$x(3) - x(1)$

$(2)(-2) - 0(\ )$

$= -4$

b) At $t = 1$, is the velocity of the particle increasing or decreasing? Explain your answer.

Change in velocity ? acceleration

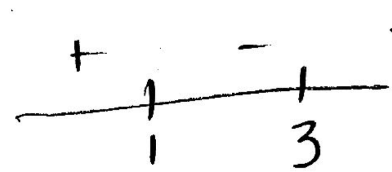
$V(t) = (t^2 - 5t + 4) + (t-1)(2t-5) = 3t^2 - 12t + 9$

$a(t) = 6t - 12$

$a(1) = -6$ so velocity is decreasing

c) Find all the values of t for which the particle is moving to the left.

$v(t) < 0$



$3(t^2 - 4t + 3) = 0$

$3(t-3)(t-1) = 0$

$t = 1, 3$

move left from $(1,3)$

b/c

d) Find the velocity of the particle when the acceleration is 0.

$a(t) = 0$

$6t - 12 = 0$

$t = 2$

$v(2) = -3$

AP Calculus I
Notes 3.4
The Chain Rule

Exploration

1) Use the basic power rule to find $f'(x)$ given $f(x) = (x+3)^2$. Next, multiply $f(x)$ out and then differentiate. Are they equivalent? If not, how can you go from one to the other?

Assume

$f' = 2(x+3)'$

$= 2x+6$

$f(x) = x^2+6x+9$

$f'(x) = 2x+6$

same!

2) Now, use the basic power rule to find $g'(x)$ given $g(x) = (2x-1)^2$. Next, multiply $g(x)$ out and then differentiate. Do you notice any similarities or differences?

Assume

$g'(x) = 2(2x-1)$

$g' = 4x-2$

$g(x) = 4x^2-4x+1$

$g'(x) = 8x-4$

not the same!

missing a factor of 2

confident

3) Next, use the basic power rule to find $h'(x)$ given $h(x) = (3x^2+1)^2$. Next, multiply $h(x)$ out and then differentiate. Do you notice any similarities or differences?

$h' = 2(3x^2+1)'$

$= 6x^2+2$

$h(x) = 9x^4+6x^2+1$

$h'(x) = 36x^3+12x$

not same

missing 6x

conf ✓

4) Next, use the basic power rule to find $m'(x)$ given $m(x) = (x^3+x)^2$. Next, multiply out $m(x)$ and then differentiate. Do you notice any similarities or differences?

$m' = 2(x^3+x)$

$= 2x^3+2x$

$m(x) = x^6+2x^4+x^2$

$m'(x) = 6x^5+8x^3+2x$

how?

multiply by $(3x^2+1)$

When differentiating functions with inside parts and outside parts, we must use **The Chain Rule**.

The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) \quad \text{or} \quad y' = f'(u)\cdot u'$$

Ex. 1: Rewrite the following composite functions in terms of u and x. Then, use the Chain Rule to find the derivative:

u = inside function

a) $f(x) = (3x - 2x^2)^3$

$u = (3x - 2x^2)$

$u' = 3 - 4x$

$f = u^3$

$f' = 3u^2 \cdot u'$

$f'(x) = 3(3x-2x^2)^2(3-4x)$

$f'(g(x)) \cdot g'(x)$

b) $y = \sqrt{x^2+1} = (x^2+1)^{1/2}$

$$\frac{dy}{dx} = \underbrace{\cancel{\tfrac{1}{2}}(x^2+1)^{-1/2}}_{f'(g(x))} \cdot \underbrace{\cancel{2}x}_{g'(x)}$$

$$= \frac{x}{\sqrt{x^2+1}}$$

c) $h(t) = \dfrac{-7}{(4t-1)^2} = -7(4t-1)^{-2}$

$h'(t) = 14(4t-1)^{-3} \cdot 4$

$$= \frac{56}{(4t-1)^3}$$

d) $w = (5-n)^2(3n+1)^3$ Product Rule

$$w' = \underbrace{2(5-n)(-1)}_{f'\ \text{chain}}\underbrace{(3n+1)^3}_{g} + \underbrace{(5-n)^2}_{f}\underbrace{3(3n+1)^2 \cdot 3}_{g'\ \text{chain}}$$

$w' = -2(5-n)(3n+1)^3 + 9(5-n)^2(3n+1)^2$

if you want to simplify...

$w' = (5-n)(3n+1)^2[-2(3n+1) + 9(5-n)]$

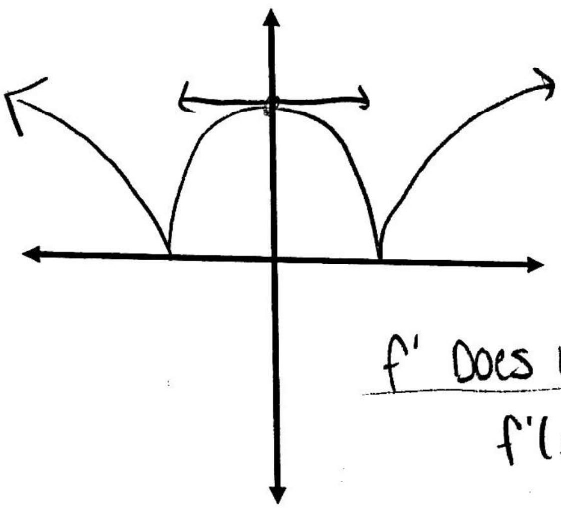
$= (5-n)(3n+1)^2[-6n - 2 + 45 - 9n]$

$= (5-n)(3n+1)^2(-15n + 43)$

Ex. 2: Find all points on the graph of $f(x) = \sqrt[3]{(x^2-1)^2}$ for which $f'(x) = 0$ and those for which $f'(x)$ does not exist.

$f(x) = (x^2-1)^{2/3}$

$f'(x) = \frac{2}{3}(x^2-1)^{-1/3}(2x) = \frac{4x}{3(x^2-1)^{1/3}}$



$f'(x)=0$ Horizont. Tangent

$\frac{4x}{3(x^2-1)^{1/3}} = 0 \rightarrow 4x=0$ so $\boxed{x=0}$

$\boxed{f(0)=1}$

f' Does not exist? Not Differentiable

$f'(x) = \frac{4x}{3(x^2-1)^{1/3}}$ is undefined when

$3(x^2-1)^{1/3} = 0$

$(x^2-1)^{1/3} = 0$

$x^2-1=0 \quad x=\pm 1$

$\boxed{\begin{array}{l} f(1)=0 \\ f(-1)=0 \end{array}}$

Ex. 3: Find $h'(0)$ given $h(x) = f(g(x))$ and $f(x) = \sqrt{x^2-6}$, $g(x) = 4\sin x + 3$.

create $h(x) = \sqrt{(4\sin x+3)^2-6} = ((4\sin x+3)^2-6)^{1/2}$

$h'(x) = \cancel{\tfrac{1}{2}}((4\sin x+3)^2-6)^{-1/2}(\cancel{2}(4\sin x+3)' \cdot 4\cos x)$

$h'(x) = \frac{4\cos x(4\sin x+3)}{\sqrt{(4\sin x+3)^2-6}}$ $\quad h'(0) = \frac{4(3)}{\sqrt{3^2-6}} = \frac{12}{\sqrt{3}}$

formula $h'(x) = f'(g(x)) \cdot g'(x)$

$f'(x) = \frac{1}{2}(x^2-6)^{-1/2}(\cancel{2}x)$

$f'(x) = \frac{x}{\sqrt{x^2-6}}$

$\frac{(4\sin x+3)}{\sqrt{(4\sin x+3)^2-6}} \cdot 4\cos x = \frac{12}{\sqrt{3}}$

Ex. 4: A particle is moving such that its position, in feet, is modeled by $s = (2t^2 + 1)^8$ where t is in seconds. Find the velocity and acceleration of the particle at $t = 0$, indicating units of measure.

$$v(t) = s'(t) = 8(2t^2+1)^7(4t) = \underbrace{32t}\,\underbrace{(2t^2+1)^7} \rightarrow v(0) = 0 \text{ ft/sec}$$

$$a(t) = v'(t) = s''(t) = 32(2t^2+1)^7 + 32t \cdot 7(2t^2+1)^6(4t)$$

product rule!

$$= 32(2t^2+1)^7 + 896t^2(2t^2+1)^6$$

$$a(0) = 32(1) + 0$$
$$= 32 \text{ ft/sec}^2$$

Ex. 5: Find the derivative of each of the following:

a) $y = \cos(3x^2 + 2)$

$$\frac{dy}{dt} = -\sin(3x^2+2)(6x)$$
$$= -6x\sin(3x^2+2)$$

b) $W = \sqrt{k^3 + \csc 5k} = (k^3 + \csc 5k)^{1/2}$

$$\frac{dW}{dk} = \frac{1}{2}(k^3 + \csc 5k)^{-1/2}(3k^2 + -\csc 5k \cot 5k \cdot 5)$$
$$= \frac{3k^2 - 5\csc k \cot k}{2\sqrt{k^3 + \csc 5k}}$$

c) $f(t) = \sin^3(4t) + \sin(4t^3)$

$$f(t) = (\sin 4t)^3 + \sin(4t^3)$$
$$f'(t) = 3(\sin 4t)^2 \cos 4t \cdot 4 + \cos(4t^3)(12t^2)$$
$$f'(t) = 12\sin^2(4t)\cos(4t) + 12t^2\cos(4t^3)$$

d) $y = \sec(\theta e^{f(2\theta)})$

chain

product

chain

$$\text{d) } \frac{dy}{dt} = \sec(\theta \cdot e^{f(2\theta)})\tan(\theta \cdot e^{f(2\theta)}) \cdot \left[\underbrace{1 \cdot e^{f(2\theta)}}_{f' \cdot g} + \underbrace{\theta e^{f(2\theta)} \cdot f'(2\theta) \cdot 2}_{g' \cdot f}\right]$$

$$= \boxed{\sec(\theta e^{f(2\theta)})\tan(\theta e^{f(2\theta)}) \cdot e^{f(2\theta)}\left[1 + 2\theta f'(2\theta)\right]}$$

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$$\frac{dw}{dk} = \frac{1}{2}(k^3+\csc 5k)^{-1/2}(3k^2 + -\csc 5k \cot 5k \cdot 5)$$
$$= \frac{3k^2 - 5\csc k \cot k}{2\sqrt{k^3+\csc 5k}}$$

c) $f(t) = \sin^3(4t) + \sin(4t^3)$

$$f(t) = (\sin 4t)^3 + \sin(4t^3)$$
$$f'(t) = 3(\sin 4t)^2 \cos 4t \cdot 4 + \cos(4t^3)(12t^2)$$
$$f'(t) = 12\sin^2(4t)\cos(4t) + 12t^2\cos(4t^3)$$

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chain

product

chain

$$\text{d) } \frac{dy}{dt} = \sec(\theta \cdot e^{f(2\theta)})\tan(\theta \cdot e^{f(2\theta)}) \cdot \Big[\underbrace{1 \cdot e^{f(2\theta)}}_{f' \cdot g} + \underbrace{\theta\, e^{f(2\theta)} \cdot f'(2\theta) \cdot 2}_{g' \cdot f}\Big]$$

$$= \boxed{\sec(\theta e^{f(2\theta)})\tan(\theta e^{f(2\theta)}) \cdot e^{f(2\theta)}[1 + 2\theta f'(2\theta)]}$$

Theorem – Derivative of the Natural Logarithmic Function

Let u be a differentiable function of x.

$$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx} = \frac{u'}{u}$$

Ex. 6: Differentiate each of the following logarithmic functions:

a) $\frac{d}{dx}[\ln\sqrt{x}]$ $\ln(x^{1/2})$

$$= \frac{1}{x^{1/2}} \cdot \frac{1}{2}x^{-1/2}$$

$$= \boxed{\frac{1}{2x}}$$

b) $\frac{d}{dx}[\ln(x^2+1)]$

$$\frac{1}{x^2+1} \cdot (2x) = \boxed{\frac{2x}{x^2+1}}$$

c) $\frac{d}{dx}[x\ln x]$

product

$$= 1 \cdot \ln x + \frac{1}{\cancel{x}} \cdot \cancel{x}$$

$$= \boxed{\ln x + 1} = \boxed{1 + \ln x}$$

d) $\frac{d}{dx}[(\ln(6x-7))^3]$ 2 3 1st

$$3(\ln(6x-7))^2 \cdot \frac{1}{6x-7} \cdot 6$$

$$\boxed{= \frac{18(\ln(6x-7))^2}{6x-7}}$$

Ex. 7: Use logarithmic properties to differentiate the following functions:

a) $y = \ln\sqrt{2g(x)+1}$

$$y = \ln(2g(x)+1)^{1/2}$$

$$y = \frac{1}{2}\ln(2g(x)+1)$$

$$y' = \frac{1}{\cancel{2}} \cdot \frac{1}{2g(x)+1} \cdot \cancel{2}g'(x)$$

$$\boxed{= \frac{g'(x)}{2g(x)+1}}$$

b) $f(x) = \ln\frac{x(x^2+1)^2}{\sqrt{2x^3-1}}$

$$f = \ln(x(x^2+1)^2) - \ln(2x^3-1)^{1/2}$$

$$f = \ln x + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^3-1)$$

$$f'(x) = \frac{1}{x} + \frac{2}{x^2+1} \cdot 2x - \frac{1}{2} \cdot \frac{1}{2x^3-1} \cdot 6x^2$$

$$f'(x) = \frac{1}{x} + \frac{4x}{x^2+1} - \frac{3x^2}{2x^3-1}$$

Because the natural logarithm is undefined for negative numbers, you will often encounter expressions of the form $\ln|u|$. When you differentiate functions in the form $y = \ln|u|$, do everything as usual.

★ **Ex. 8:** Find the equation of the tangent line for $f(x) = \ln|2\sin^2 x + 3|$ at $x = \frac{\pi}{4}$

point: $f(\pi/4) = \ln|2\sin^2\frac{\pi}{4} + 3| = \ln\left|2\left(\frac{\sqrt{2}}{2}\right)^2 + 3\right| = \ln 4$

slope $f'(x) = \frac{1}{2(\sin x)^2+3} \cdot 4\sin x \cdot \cos x$

$f'(\pi/4) = \frac{4\sin(\pi/4)\cos(\pi/4)}{2(\sin \pi/4)^2 + 3}$ $\quad \frac{4 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}}{2\left(\frac{\sqrt{2}}{2}\right)^2 + 3}$ $\quad \frac{2}{2(\frac{1}{2})+3}$

$\boxed{y - \ln 4 = \frac{1}{2}\left(x - \frac{\pi}{4}\right)}$

Ex. 9: If $f(x) = \sin(\ln(2x))$, then $f'(x) =$

(A) $\frac{\sin(\ln(2x))}{2x}$

(B) $\frac{\cos(\ln(2x))}{x}$

(C) $\frac{\cos(\ln(2x))}{2x}$

(D) $\cos\left(\frac{1}{2x}\right)$

$\cos(\ln(2x)) \cdot \frac{1}{2x} \cdot 2$

Theorem – Derivatives for Bases Other than *e*

Let a be a positive real number ($a \neq 1$) and let u be a differentiable function of x.

1) $\frac{d}{dx}[a^x] = a^x \cdot \ln a$

2) $\frac{d}{dx}[a^u] = a^u \cdot \ln a \cdot \frac{du}{dx}$

Ex. 10: Find the derivative of each of the following:

a) $y = 2^{3x}$

$\frac{dy}{dx} = \underbrace{2^{3x}\ln 2}_{a^u} \cdot \underbrace{3}_{u'}$

$= 3\ln 2 \cdot 2^{3x}$

b) $f(x) = x^4 4^x$ product rule!

$f'(x) = 4x^3 \cdot 4^x + 4^x \ln 4 \cdot x^4$

$f' = x^3 4^x [4 + x\ln 4]$

AP Calculus I
Notes 3.5
Implicit Differentiation

So far, most functions have been expressed in **explicit form**. Some, however, are defined **implicitly**.

For example:

Implicit Form	**Explicit Form**	**Derivative**
$xy = 1$	$y = \frac{1}{x}$	$y' = \frac{-1}{x^2}$

It is not always possible, however, to solve for y explicitly. For example, $x^2 - 2y^3 + 4y = 2$. In these cases, we must use **implicit differentiation**.

The key to finding $\frac{dy}{dx}$ implicitly is understanding that the differentiation is happening ***with respect to x***.

- When you differentiate terms involving x alone, you can differentiate as usual.
- When you differentiate terms involving another variable, you must apply the Chain Rule.

Ex. 1: Differentiate each of the following:

a) $\frac{d}{dx}[x^3]$

$3x^2 \frac{dx}{dx}$

$3x^2 \cdot 1 = 3x^2$

b) $\frac{d}{dx}[y^4 - 5x + 3]$

$4y^3\left(\frac{dy}{dx}\right) - 5$

c) $\frac{d}{dt}[\cos x + 3e^y]$

$-\sin x \cdot \frac{dx}{dt} + 3e^y \frac{dy}{dt}$

d) $\frac{d}{da}[ab^2]$

↑ product rule

$1 \cdot b^2 + a \cdot 2b \cdot \frac{db}{da}$

$= b^2 + 2ab\frac{db}{da}$

Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation *with respect to x*.
2. Collect all terms involving $\frac{dy}{dx}$ on the left side of the equation and move all other terms to the right side of the equation.
3. Factor $\frac{dy}{dx}$ out of the left side of the equation and divide to solve for $\frac{dy}{dx}$.

Ex. 2: Find $\frac{dy}{dx}$ given that $y^3 + y^2 - x^2 = -4 + 5y$. Then, find $\frac{dx}{dy}$.

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 2x = 5\frac{dy}{dx} \to \frac{dy}{dx}(3y^2+2y-5) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{3y^2+2y-5}$$

$$3y^2+2y-2x\frac{dx}{dy} = 5$$

$$-2x\frac{dx}{dy} = 5-3y^2-2x \to \frac{dx}{dy} = \frac{5-3y^2-2y}{-2x}$$

Ex. 3: Find all points of horizontal and vertical tangencies for the graph of $x^2 + y^2 = 1$.

Horizontal

Num = 0

$$\frac{-x}{y} = 0 \to -x = 0 \to x = 0$$

$$0^2 + y^2 = 1$$

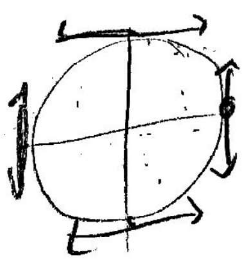
$$y = \pm 1$$

(0,1) & (0,-1)

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$



Vertical

Denom = 0

$\frac{dy}{dx} = \frac{-x}{y}$ is undefined (vertical line)

when $y = 0$

$$x^2 + 0 = 1$$

$$x = \pm 1$$

(-1,0) & (1,0)

Ex. 4: Determine the slope of the tangent line to the graph of $x^2 - 2xy + 4y^2 = 6$ at the point $\left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right)$.

$$2x - 2y - 2x\frac{dy}{dx} + 8y\frac{dy}{dx} = 0$$

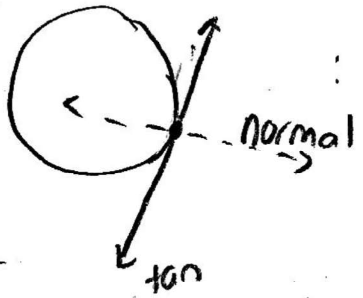
$$\frac{dy}{dx}(-2x+8y) = 2y-2x$$

$$\frac{dy}{dx} = \frac{2y-2x}{-2x+8y}\Bigg|_{\sqrt{2},\, -\frac{1}{\sqrt{2}}}$$

$$= \frac{y-x}{-x+4y}\Bigg|_{\sqrt{2},\, -\frac{1}{\sqrt{2}}} = \frac{-\frac{1}{\sqrt{2}} - \sqrt{2}}{-\sqrt{2} + 4\left(-\frac{1}{\sqrt{2}}\right)} \cdot \frac{(\sqrt{2})}{\sqrt{2}} = \frac{-1-2}{-2-4} = \frac{-3}{-6}$$

$$\boxed{= \frac{1}{2}}$$

Ex. 5: Determine the slope of the ***normal*** line of $3(x^2+y^2)^2 = 100xy$ at the point $(3,1)$.



lines are perpendicular → slopes are opp reciprocal

$$6(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 100y + 100x\frac{dy}{dx}$$

$$(6x^2+6y^2)\left(2x+2y\frac{dy}{dx}\right) = 100y + 100x\frac{dy}{dx}$$

$$12x^3 + 12xy^2 + 12x^2y\frac{dy}{dx} + 12y^3\frac{dy}{dx} = 100y + 100x\frac{dy}{dx}$$

$$\frac{dy}{dx}(12x^2y + 12y^3 - 100x) = 100y - 12x^3 - 12xy^2$$

$$\frac{dy}{dx} = \frac{25y - 3x^3 - 3xy^2}{3x^2y + 3y^3 - 25x}\Bigg|_{(3,1)} = \frac{25 - 3(27) - 3(3)}{3(1)(3)^2 + 3(1)}$$

$$m_{norm} = \frac{-9}{13} \qquad m_{tan} = \frac{13}{9}$$

Ex. 6: Find $\frac{dw}{dt}$: $2\sin w\cos t = \tan(2t^2) + \sec^3(2w)$

prod. chain

$$2\cos w\frac{dw}{dt}\cdot\cos t + -2\sin w\sin t\frac{dt}{dt} = \sec^2(2t^2)(4t)\frac{dt}{dt} + 3\sec^2(2w)\cdot\sec 2w\cdot\tan 2w\cdot 2\frac{dw}{dt}$$

$$2\cos w\cos t\frac{dw}{dt} - 6\sec^2(2w)\sec(2w)\tan(2w)\frac{dw}{dt} = \sec^2(2t^2)(4t) + 2\sin w\sin t$$

$$\frac{dw}{dt}(2\cos w\cos t - 6\sec^3 2w\tan 2w)$$

$$\frac{dw}{dt} = \frac{\sec^2(2t^2)(4t) + 2\sin w\sin t}{2\cos w\cos t - 6\sec^3 2w\tan 2w}$$

Ex. 7: Given $x^2 + y^2 = 25$, find $\frac{d^2y}{dx^2}$. ← 2nd!

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$\frac{d^2y}{dx^2} = \left[\frac{y(-1) - (-x)\cdot\frac{dy}{dx}}{y^2}\right] = \frac{-y + x\left(\frac{-x}{y}\right)}{y^2}\cdot\frac{y}{y}$$

$x^2+y^2=25$

$$\frac{d^2y}{dx^2} = \frac{-y^2 - x^2}{y^3} = \frac{-1(x^2+y^2)}{y^3} = \frac{-25}{y^3}$$

Ex. 8: Given $t^2y + \ln|t-1| = 6t$, find $\frac{d^2y}{dt^2}$ at (2,3).

$$2t\frac{dt}{dt}\cdot y + t^2\frac{dy}{dt} + \frac{1}{t-1}\frac{dt}{dt} = 6\frac{dt}{dt}$$

$$\cancel{t^2}\frac{dy}{dt} = \frac{6 - \frac{1}{t-1} - 2ty}{t^2}$$

$$\frac{dy}{dt}\Big|_{(2,3)} = \frac{6 - \frac{1}{1} - 2(2)(3)}{4} = \frac{6-1-12}{4} = \frac{-7}{4}$$

$$\frac{d^2y}{dx^2} = \frac{(t^2)\left(\frac{+1}{(t-1)^2} - 2y + -2t\frac{dy}{dt}\right) - \left(6 - \frac{1}{t-1} + 2ty\right)(2t)}{(t^2)^2}$$

$$\frac{d^2y}{dx^2}\Big|_{(2,3)} = \frac{4\left(1 - 6 - 4\frac{dy}{dt}\right) - (6-1-12)(4)}{(2^2)^2}$$

$$= \frac{4\left(-5 - 4\left(\frac{-7}{4}\right)\right) - (-7)(4)}{16}$$

$$= \frac{4(2) + 28}{16} = \boxed{\frac{36}{16}} \text{ or } \frac{9}{4}$$

Implicit Differentiation Worksheet

Find $\frac{dy}{dx}$ for each of the following. Show all work!

1) $x^2 + y^2 = 16$

$2x + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -2x$

$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$

2) $x^3 + y^3 = 8xy$

$3x^2 + 3y^2\frac{dy}{dx} = 8y + 8x\frac{dy}{dx}$

$\frac{dy}{dx}(3y^2 - 8x) = 8y - 3x^2$

$\boxed{\frac{dy}{dx} = \frac{8y-3x^2}{3y^2-8x}}$

3) $\frac{1}{x} + \frac{1}{y} = 1$

$x^{-1} + y^{-1} = 1$

$-1x^{-2} - 1y^{-2}\frac{dy}{dx} = 0 \quad + \frac{1}{x^2}$

$+\frac{1}{x^2}$

$\frac{-1}{y^2}\frac{dy}{dx} = \frac{1}{x^2}$

$\frac{dy}{dx} = \frac{-y^2}{x^2}$

4) $2\sqrt{x} + \sqrt{y} = e^{2x}$

$2x^{1/2} + y^{1/2} = e^{2x}$

$x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 2e^{2x}$

$\frac{1}{2y^{1/2}}\frac{dy}{dx} = 2e^{2x} - \frac{1}{\sqrt{x}}$

$\frac{dy}{dx} = 2\sqrt{y}\left(2e^{2x} - \frac{1}{\sqrt{x}}\right)$

5) $5 - 3y^2 = \csc(xy)$

$0 - 6y\frac{dy}{dx} = -\csc(xy)\cot(xy)\left(1y + x\frac{dy}{dx}\right)$

$-6y\frac{dy}{dx} = -y\csc(xy)\cot(xy) + -x\csc(xy)\cot(xy)\frac{dy}{dx}$

$\frac{dy}{dx}\left(x\csc(xy)\cot(xy) - 6y\right) = -y\csc(xy)\cot(xy)$

$\frac{dy}{dx} = \frac{-y\csc(xy)\cot(x,y)}{x\csc(xy)\cot(xy) - 6y}$

6) $x\sin y + y\cos x = 1$

$1\sin y + \cos y\frac{dy}{dx}\cdot x + \frac{dy}{dx}\cos x + -y\sin x = 0$

$\frac{dy}{dx}(x\cos y + \cos x) = y\sin x - \sin y$

$\boxed{\frac{dy}{dx} = \frac{y\sin x - \sin y}{x\cos y + \cos x}}$

$(2,-2) \quad (2,3)$

7) Find the equation(s) of each tangent line to $x^2 + xy + y^2 - 3y = 10$ at $x = 2$.

$2x + 1y + x\frac{dy}{dx} + 2y\frac{dy}{dx} - 3\frac{dy}{dx} = 0$

$2^2 + 2y + y^2 - 3y = 10$

$y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$\frac{dy}{dx}(x+2y-3) = -2x - 1y$

$\frac{dy}{dx} = \frac{-2x-y}{x+2y-3}$

$(2,-2)$: $= \frac{-4+2}{2-4-3} = \frac{-2}{-5} = \frac{2}{5}$ $\quad y+2 = \frac{2}{5}(x-2)$

$(2,3)$: $\frac{-4-3}{2+6-3} = \frac{-7}{5}$ $\quad y-3 = -\frac{7}{5}(x-2)$

8) Find $\frac{d^2y}{dt^2}$ of $t^3 - y^3 = 7t$

$3t^2 - 3y^2\frac{dy}{dt} = 7$

$-3y^2\frac{dy}{dt} = 7 - 3t^2$

$\frac{dy}{dt} = \frac{7-3t^2}{-3y^2}$

$\frac{d^2y}{dt^2} = \frac{(-3y^2)(-6t) - (7-3t^2)(-6y\frac{dy}{dt})}{(-3y^2)^2}$

$\frac{d^2y}{dt^2} = \frac{18y^2t - (7-3t^2)\left(-6y\left(\frac{7-3t^2}{-3y^2}\right)\right)}{(9y^4)} \cdot \frac{y}{y}$

$= \frac{18y^3t - (7-3t^2)(-2(7-3t^2))}{9y^5} = \boxed{\frac{18ty^3 - 2(7-3t^2)^2}{9y^5}}$

9) Find $\frac{d^2y}{dx^2}$ of $\ln|y(x^2+1)| = 0$ at $(0,-1)$

$\frac{1}{y(x^2+1)} \cdot \left(\frac{dy}{dx}(x^2+1) + 2xy\right) = 0$

$\frac{x^2+1}{y(x^2+1)}\frac{dy}{dx} + \frac{2xy}{y(x^2+1)} = 0$

$\frac{1}{y}\frac{dy}{dx} = \frac{-2x}{x^2+1}$

$\frac{dy}{dx} = \frac{-2xy}{x^2+1}$ $\quad \frac{dy}{dx}\Big|_{(0,-1)} = \frac{0}{2} = \boxed{0}$

$\frac{d^2y}{dx^2} = \frac{(x^2+1)(-2x\frac{dy}{dx} - 2y) - (-2xy)(2x)}{(x^2+1)^2}$

$\frac{d^2y}{dx^2}\Big|_{(0,-1)} = \frac{(1)(0+2) - (0)(0)}{1^2}$

$\boxed{= 2}$

Answers: 1) $\frac{dy}{dx} = -\frac{x}{y}$ 2) $\frac{dy}{dx} = \frac{3x^2-8y}{8x-3y^2}$ 3) $\frac{dy}{dx} = -\frac{y^2}{x^2}$ 4) $\frac{dy}{dx} = 4e^{2x}\sqrt{y} - \frac{2\sqrt{y}}{\sqrt{x}}$

5) $\frac{dy}{dx} = \frac{-y\csc xy\cot xy}{x\csc xy\cot xy - 6y}$ 6) $\frac{dy}{dx} = \frac{y\sin x - \sin y}{x\cos y + \cos x}$ 7) $y - 3 = -\frac{7}{5}(x-2)$, $y + 2 = \frac{2}{5}(x-2)$

8) $\frac{dy}{dt} = \frac{18ty^3 - 2(7-3t^2)^2}{9y^5}$ 9) 2

AP Calculus I
Notes 3.6
Inverse Trigonometric Functions and Differentiation

Ex. 1: Evaluate each of the following:

$\sin^{-1} x = \arcsin x$

a) $\arcsin\left(-\frac{1}{2}\right)$

$\sin\theta = -\frac{1}{2}$

$\theta = -\frac{\pi}{6}$

b) $\arccos(0)$

$\pi/2$

c) $\arctan(\sqrt{3})$

$\pi/3$

d) $\operatorname{arcsec} 2$

$\frac{1}{\cos\theta} = 2$

$\cos\theta = \frac{1}{2}$

$\theta = \pi/3$

e) $\operatorname{arccsc}\sqrt{2}$

$\csc\theta = \sqrt{2}$

$\sin\theta = \frac{1}{\sqrt{2}}$

$\theta = \pi/4$

f) $\operatorname{arccot}(-1)$

$\tan\theta = -1$

$\theta = -\pi/4$

Ex. 2: Use right triangles to evaluate the following expressions:

a) Given $y = \arcsin x$, find $\cos y$

$\sin y = x$

$\cos y = \sqrt{1-x^2}$

$a^2 + x^2 = 1^2$

$a^2 = 1 - x^2$

$a = \sqrt{1-x^2}$

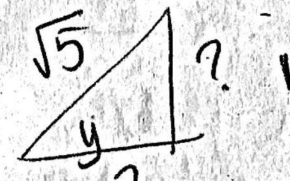
b) Given $y = \operatorname{arcsec}\left(\frac{\sqrt{5}}{2}\right)$, find $\tan y$

$\sec y = \frac{\sqrt{5}}{2}$

$\frac{1}{\cos y} = \frac{\sqrt{5}}{2}$

$\cos y = \frac{2}{\sqrt{5}}$

$\tan y = \frac{1}{2}$



$2^2 + a^2 = \sqrt{5}^2$

$a = 1$

Derivatives of Inverse Trigonometric Functions

Proof: $\frac{d}{dx}[\arcsin u] =$

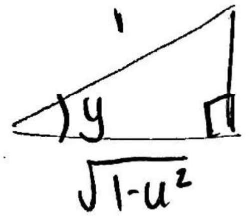
$$\frac{d}{dx}(y) = \frac{d}{dx}(\arcsin u)$$

$$\frac{d}{dx}(y) = \frac{dy}{dx}$$

$$\frac{d}{dx}[\sin y = u]$$

$$= \left[\cos y \frac{dy}{dx} = \frac{du}{dx}\right] \leftarrow u' \text{ chain!}$$

if $\sin y = u$



$$a^2 + u^2 = 1^2$$
$$a^2 = 1 - u^2$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \cdot u'$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot u'$$

Let u be a differentiable function of x.

99% $\frac{d}{dx}[\arcsin u] = \frac{1}{\sqrt{1-u^2}} \cdot u'$

$\frac{d}{dx}[\arctan u] = \frac{1}{1+u^2} \cdot u'$

1% $\frac{d}{dx}[\operatorname{arcsec} u] = \frac{1}{|u|\sqrt{u^2-1}} \cdot u'$

$\frac{d}{dx}[\arccos u] = \frac{-1}{\sqrt{1-u^2}} \cdot u'$

$\frac{d}{dx}[\operatorname{arccot} u] = \frac{-1}{1+u^2} \cdot u'$

$\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-1}{|u|\sqrt{u^2-1}} \cdot u'$

Ex. 3: Differentiate each of the following:

a) $\frac{d}{dx}[\arctan 3x]$

$$= \frac{1}{1+(3x)^2} \cdot 3$$

$$\boxed{= \frac{3}{1+9x^2}}$$

b) $\frac{d}{dx}[\arcsin \sqrt{y}]$

$$= \frac{1}{\sqrt{1-(\sqrt{y})^2}} \cdot \frac{1}{2} y^{-1/2} \cdot \frac{dy}{dx}$$

$$\boxed{= \frac{1}{2\sqrt{1-y} \cdot \sqrt{y}} \frac{dy}{dx}}$$

c) $\frac{d}{dx}[\operatorname{arcsec} e^{2x}]$

$$\frac{1}{|e^{2x}|\sqrt{(e^{2x})^2-1}} \cdot e^{2x} \cdot 2$$

$$= \frac{2\cancel{e^{2x}}}{\cancel{|e^{2x}|}\sqrt{e^{4x}-1}} \rightarrow \text{always positive}$$

$$\boxed{= \frac{2}{\sqrt{e^{4x}-1}}}$$

Ex. 4: The velocity of a particle is moving according to the function $v(t) = \sin^{-1} t^2 + \sqrt{1-t}$. Is the particle's speed increasing or decreasing at $t = 0$?

increasing: V & a have same signs

decreasing: V & a have opp signs

$v(0) = \sin^{-1}(0) + \sqrt{1}$
$= 0 + 1$ which is > 0

$a(t) = \frac{1}{\sqrt{1-(t^2)^2}} \cdot 2t + \frac{1}{2}(1-t)^{-1/2}(-1)$

$a(t) = \frac{2t}{\sqrt{1-t^4}} - \frac{1}{2\sqrt{1-t}}$

$a(0) = \frac{0}{\sqrt{1}} - \frac{1}{2\sqrt{1}} = 0 - \frac{1}{2} \rightarrow < 0$

Speed is decreasing because $v(0)$ & $a(0)$ have opposite signs

Ex. 5: Write the equation of the tangent line to $y = 3^{\arctan x}$ at $y = 3^{\frac{\pi}{4}}$.

point: $3^{\arctan x} = 3^{\pi/4}$

$\arctan x = \frac{\pi}{4}$

$\tan \frac{\pi}{4} = x$

$x = 1$

$(1, 3^{\pi/4})$

slope: $\frac{dy}{dx} = 3^{\arctan x} \cdot \ln 3 \cdot \frac{1}{1+x^2} (1)$

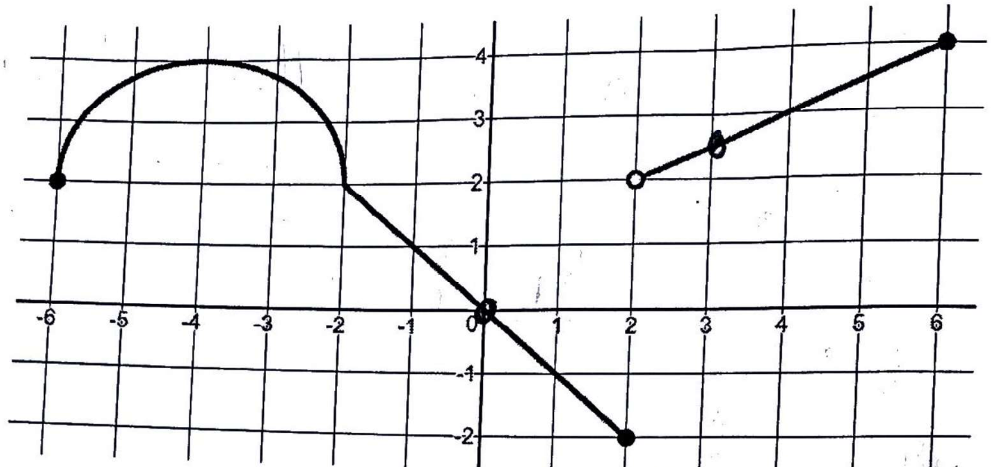
$\left.\frac{dy}{dx}\right|_{x=1} = \frac{3^{\arctan x} \ln 3}{1+x^2}$

$= \frac{3^{\arctan 1} \cdot \ln 3}{2}$

$= \frac{3^{\pi/4} \ln 3}{2}$

$$\boxed{y - 3^{\pi/4} = \frac{3^{\pi/4} \ln 3}{2}(x-1)}$$

Ex. 6: Answer the following questions given the graph of f, consisting of a semicircle and 2 line segments



a) For what value of x is f continuous but not differentiable on the interval $-6 < x < 6$? Be sure to explain your reasoning.

$\boxed{x=-2}$

$f(-2) = 2 = \lim_{x \to -2} f(x)$ so continuous

$\lim_{x \to -2^-} f'(x) \neq \lim_{x \to -2^+} f'(x)$

(abrupt change in slope)

b) Evaluate $\lim_{h \to 0} \frac{f(3+h)-f(3)}{h}$

wants the derivative (slope) of f @ $x=3$

$= \boxed{\frac{1}{2}}$

c) Find $p'(0)$ given $p(x) = \sin(f(2x))$.

$p'(x) = \cos(f(2x)) \cdot f'(2x) \cdot 2$

$p'(0) = \cos(f(0)) \cdot f'(0) \cdot 2 = \cos(0) \cdot (-1) \cdot 2$

$\boxed{=-2}$

d) Find the equation of the tangent line of g at $x = -1$ given $g(x) = \arctan[f(x)]^2$.

$g'(x) = \frac{1}{1+[(f(x))^2]^2} \cdot 2f(x) \cdot f'(x)$

$g(-1) = \arctan[f(-1)]^2$
$= \arctan 1$
$= (-1, \pi/4)$

$g'(x) = \frac{2f(x)f'(x)}{1+[f(x)]^4}$

$\boxed{y - \frac{\pi}{4} = -1(x+1)}$

$g'(-1) = \frac{2f(-1)f'(-1)}{1+[f(-1)]^4} = \frac{2(1)(-1)}{1+1^4} = \frac{-2}{2} = \boxed{-1}$

AP Calculus
Notes 3.6
Inverse Functions

A function can be written as a set of ordered pairs:

Function

$f = \{(1,4),(2,5),(3,6),(4,7)\}$

Inverse Function

$f^{-1} = \{(4,1)(5,2)(6,3)(7,4)\}$

Definition of Inverse Function

A function g is the inverse of the function f if:

$f(g(x)) = x$ for each x in the domain of g and $g(f(x)) = x$ for each x in the domain of f

Ex. 1: Given $f(x)$ and $g(x)$ are inverse functions, find 3 sets of inverse points.

$f(x) = 2x^3 - 1$ and $g(x) = \sqrt[3]{\frac{x+1}{2}}$

$f(0) = -1$

$f(1) = 1$

$f(2) = 15$

$g(-1) = 0 = f^{-1}(-1)$

$g(1) = 1 = f^{-1}(1)$

$g(15) = 2 = f^{-1}(15)$

Ex. 2: Determine the domain and range of $f(x) = e^{2x-3}$. Then, find the domain and range of $f^{-1}(x)$.

$f(x)$

D: $(-\infty,\infty)$ $\rightarrow$

R: $(0,\infty)$

$f^{-1}(x)$

D: $(0,\infty)$

R: $(-\infty,\infty)$

$x = e^{2y-3}$

$\ln x = 2y-3$

$3+\ln x = 2y$

$y = \frac{3+\ln x}{2}$

$f^{-1}(x) = \frac{3+\ln x}{2}$

Derivative of an Inverse Function Investigation:

1. Verify the functions $f(x) = x^2$ (for $x \geq 0$) and $g(x) = \sqrt{x}$ are inverses.

$f(g(x)) = (\sqrt{x})^2$ $g(f(x)) = \sqrt{x^2}$

$= x$ ✓ $= |x|$ ✓

2. Write the ordered pair for $f(x)$ at $x = 3$ and determine the inverse point on $g(x)$.

$f(3) = 9 \longrightarrow g(9) = 3$

3. Determine the slope at the points found above. Is there any relationship between the two slopes?

$f' = 2x$ $\frac{1}{2}x^{-1/2}$ $\frac{1}{2\sqrt{9}} =$

$m: f'(3) =$ 6 $m: g'(9) = (f^{-1})'(9) =$ $\frac{1}{6}$

4. Repeat the step above for $f(x)$ at $x = 2$ and $x = 4$. Is there any relationship between the slopes?

Reciprocals!

$m: f'(2) =$ 4 $m: g'(4) = (f^{-1})'(4) =$ $1/4$

$m: f'(4) =$ 8 $m: g'(16) = (f^{-1})'(16) =$ $1/8$

5. Make a conjecture about the derivative of inverse functions:

derivatives are reciprocals at their coorresponding x-value

$f(g(x)) = x$

$f'(g(x)) \cdot g'(x) = 1 \rightarrow g'(x) = \frac{1}{f'(g(x))}$

Derivative of an Inverse Function:

Let f^{-1} be the inverse of f, a differentiable function. The derivative of f^{-1} at any point $x = c$ is:

$$(f^{-1})'(c) = \frac{1}{f'(f^{-1}(c))}$$

← is the reciprocal of the derivative at the corresponding x-value

the deriv. of the inves

Ex. 3: Let $f(x) = x^3 - 2x^2 + 4x - 5$.

If 3 is an x for f^{-1}, its a y of f(x)

a) What is the value of $f^{-1}(3)$?

$3 = x^3 - 2x^2 + 4x - 5$

$0 = [x^3 - 2x^2] + [4x - 8]$

$0 = x^2(x-2) + 4(x-2)$

$0 = (x-2)(x^2+4)$

$x - 2 = 0$

$\boxed{x = 2}$

$x^2 + 4 \neq 0$

$f(2) = 3$

$f^{-1}(3) = 2$

b) What is the value of $(f^{-1})'(3)$?

slope / deriv!

3 is the y value for f when $x = 2$

$(f^{-1})'(3) = \frac{1}{f'(2)}$

$f'(x) = 3x^2 - 4x + 4$

$f'(2) = 3(4) - 8 + 4$

$12 - 8 + 4$

$= 8$

$(f^{-1})'(3) = \frac{1}{18}$

Ex. 4: Let $f(x) = \sin x$ where $0 \leq x \leq \frac{\pi}{2}$.

$x = \frac{1}{2}$ for f^{-1}

so $y = 1/2$ for f

a) What is the value of $f^{-1}\left(\frac{1}{2}\right)$?

$\frac{1}{2} = \sin x$

$x = \pi/6$

$f(\pi/6) = \frac{1}{2} \rightarrow f^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

b) What is the value of $(f^{-1})'\left(\frac{1}{2}\right)$? Verify this by identifying and differentiating $f^{-1}(x)$.

oof

$f(f^{-1}(x)) = x$

$f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$

$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

$f'(x) = \cos x$

$f'(\pi/6) = \frac{\sqrt{3}}{2}$

$(f^{-1})'\left(\frac{1}{2}\right) = \frac{1}{f'\left(f^{-1}\left(\frac{1}{2}\right)\right)}$

$= \frac{1}{f'\left(\frac{\pi}{6}\right)}$

$= \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

$f(x) = \sin x$

$f^{-1}(x) = \sin^{-1} x$

$(f^{-1})'(x) = \frac{1}{\sqrt{1-x^2}}$

$f^{-1'}\left(\frac{1}{2}\right) = \frac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}}$

$= \frac{1}{\sqrt{\frac{3}{4}}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

Ex. 5: Let f be a differentiable function such that $f(3) = 15$, $\boxed{f(6) = 3}$, $f'(3) = -8$, and $f'(6) = -2$. The function g is differentiable and $g(x) = f^{-1}(x)$ for all x. What is the value of $g'(3)$?

3 is an x for inverse so a y for f(x) so f(6) = 3

(A) $-\frac{1}{2}$

(B) $-\frac{1}{8}$

(C) $\frac{1}{6}$

(D) $\frac{1}{3}$

(E) The value of $g'(3)$ cannot be determined from the information given.

$g'(3) = (f^{-1})'(3)$

$(f^{-1})'(3) = \frac{1}{f'(6)} = \frac{1}{-2}$

Ex. 6:

x	0	1	2
$f(x)$	5	2	−7
$f'(x)$	−2	−5	−14

The table above gives selected values of a differentiable and decreasing function f and its derivative. If g is the inverse function of f, what is the value of $g'(2)$? → $f(1) = 2$

(A) $-\frac{1}{5}$

(B) $-\frac{1}{14}$

(C) $\frac{1}{5}$

(D) 5

$g'(2) = (f^{-1})'(2) = \frac{1}{f'(1)}$

$= \frac{1}{-5}$

Ex. 7: Let f be the function defined by $f(x) = 2x + e^x$. If $g(x) = f^{-1}(x)$ for all x and the point $(0, 1)$ is on the graph of f, what is the value of $g'(1)$?

A) $\frac{1}{2+e}$ B) $\frac{1}{3}$ C) 3 D) $2 + e$

$f(0) = 1$

$f'(x) = 2 + e^x$

$f'(0) = 2 + e^0 = 3$

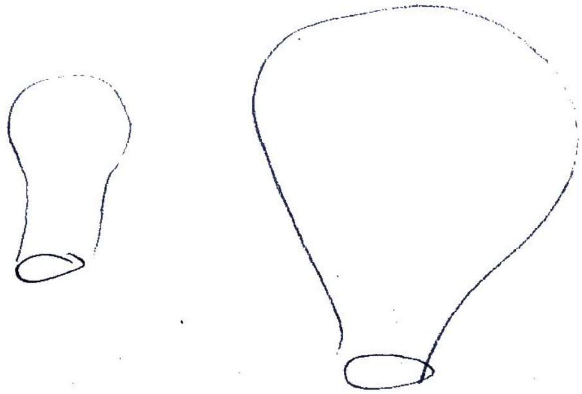
$(f^{-1})'(1) = \frac{1}{f'(0)}$

$= \boxed{\frac{1}{3}}$

AP Calculus I
Notes 3.7
Related Rates

Another important use of The Chain Rule is to find the rates of change of two or more related variables that are changing with respect to time.

For example, when a balloon is inflated, there are numerous characteristics (variables) of the balloon that are changing as time goes on. Here are a few:



- Volume
- Surface Area
- Radius
- pressure
- circumference
- Surface tension

Ex. 1: Suppose x and y are both differentiable functions of t and $x \leq 0$. Both functions are related by the equation $y^3 = 10 - x^2y$. Find $\boxed{\frac{dy}{dt}}$ when $y = 1$, given that $\boxed{\frac{dx}{dt}} = 2$ when $y = 1$.

need x

$1^3 = 10 - x^2(1)$

$x^2 = 9$

$x = \pm 3$

$x = -3$

$\frac{d}{dt}\left[y^3 = 10 - x^2y\right]$ ← product

$= 3y^2\frac{dy}{dt} = -2x \cdot \frac{dx}{dt} \cdot y + -x^2\frac{dy}{dt}$

$3(1)^2\frac{dy}{dt} = -2(-3)(2)(1) + -(-3)^2\frac{dy}{dt}$

$3\frac{dy}{dt} = 12 - 9\frac{dy}{dt}$

$12\frac{dy}{dt} = 12$

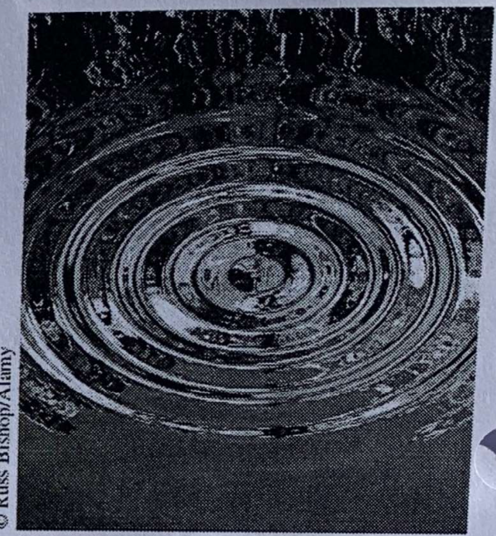
$$\boxed{\frac{dy}{dt} = 1}$$

Guidelines for Solving Related Rate Problems

1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label.
2. Write an equation, usually geometric, involving the variables that are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time t*.
4. *After* completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

positive rate

Ex. 2: A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?



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1

$\frac{dr}{dt}$ = rate of change of the radius = 1 ft/sec

$r = 4$

$\frac{dA}{dt}$ = rate of change of Area → ?

2

Equation $A_{circle} = \pi r^2$

3

$$\frac{dA}{dt} = 2 \cdot \pi r \cdot \frac{dr}{dt}$$

4

$$\frac{dA}{dt} = 2 \cdot \pi (4)(1)$$

$$dA/dt = 8\pi \text{ ft}^2/\text{sec}$$

$A(4) = 16\pi$ is the area at this time. Not needed for this problem

→ positive rate

Ex. 3: Air is being pumped into a spherical balloon at a rate of 4.5 cubic inches per minute. Find the rate of change of the diameter when the volume of the balloon is $36\pi\ in^3$. Indicate units of measure.

1 $\frac{dV}{dt} = 4.5$ $\frac{dD}{dt} = ?$ $V = 36\pi\ in^3$

2 $V_{sphere} = \frac{4}{3}\pi r^3$ $D = 2r$ $r = \frac{D}{2}$

$V = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3$

$V = \frac{4}{3} \cdot \pi \cdot \frac{D^3}{8}$

$V = \frac{\pi D^3}{6}$ $36\pi = \frac{\pi D^3}{6}$

$216 = D^3$

$\boxed{D = 6}$

3 $\frac{dV}{dt} = \frac{3\pi D^2}{6}\frac{dD}{dt}$

$4.5 = \frac{\pi}{2}(6)^2 \cdot \frac{dD}{dt}$

$\frac{9}{36\pi} = \frac{36\pi}{36\pi}\frac{dD}{dt}$

4 $\boxed{\frac{dD}{dt} = \frac{1}{4\pi}\ in/min}$

Ex. 4: The radius, r, of a circle is increasing at a rate of 2cm per minute. At the instant the radius of the circle is 6cm, find:

a) The rate of change of the area. Round your answer to the nearest whole number.

1 $\frac{dr}{dt} = 2$ $r = 6$ $\frac{da}{dt} = ?$

2 $A = \pi r^2$

$dA/dt = 2 \cdot \pi r \frac{dr}{dt}$

$= 2\pi(6)(2)$

$dA/dt = 24\pi$

$= 75\ cm^2/min$

b) The rate of change of the circumference.

$C = 2\pi r$

$\frac{dc}{dt} = 2\pi\frac{dr}{dt}$

$= 2(\pi)(2) = 4\pi\ cm/min$

Ex. 5: Liquid is dripping into a conical cup at the rate of 2.5 cubic inches per minute. The cup has a height that is always twice the radius. How fast is the liquid level rising in the cup when the liquid is 2 inches deep? Indicate units of measure.

$\frac{dV}{dt} = 2.5 \quad h = 2r \quad \frac{dh}{dt} = ?$

eliminate to one variable!!

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1\pi}{3}\left(\frac{h}{2}\right)^2 \cdot h$$

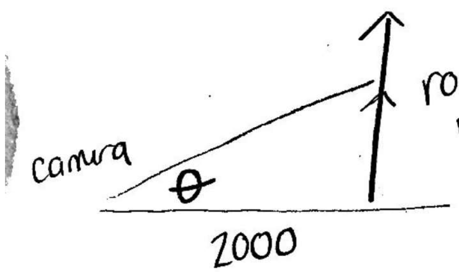
$$V = \frac{\pi}{3 \cdot 4} h^3$$

$$\frac{dV}{dt} = \frac{\cancel{3} \cdot \pi h^2}{\cancel{3} \cdot 4} \frac{dh}{dt}$$

$$2.5 = \frac{\pi}{\cancel{4}} \cancel{(2)^2} \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2.5}{\pi} \approx .7958 \text{ in/min}$$

Ex. 6: A television camera at ground level is filming the lift-off of a space shuttle that is rising according to the position equation $s = 50t^2$, where s is in feet and t is in seconds. The camera is 2000 feet from the launch. Find the rate of change in the angle of elevation of the camera 10 seconds after lift-off.



$$\left[\tan\theta = \frac{50t^2}{2000}\right] \frac{d}{dt}$$

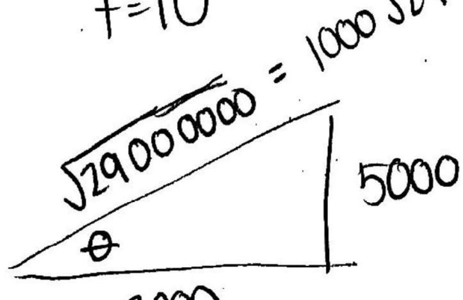
$$\sec^2\theta \frac{d\theta}{dt} = \frac{100t}{2000} \cdot \frac{\cancel{dt}}{\cancel{dt}}$$

$t = 10$

$\frac{d\theta}{dt} = ?$

$$\frac{d\theta}{dt} = \frac{1}{20}t \cdot \cos^2\theta \Big|_{t=10}$$

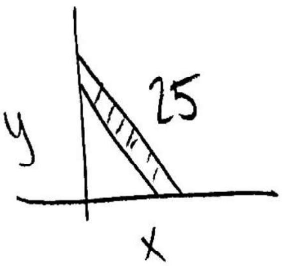
$t = 10$



$$\frac{d\theta}{dt} = \frac{1}{20}(10) \cdot \left(\frac{2000}{1000\sqrt{29}}\right)^2$$

$$= \frac{1}{2}\left(\frac{4}{29}\right) = \frac{2}{29} \text{ radians/sec}$$

Ex. 7: A 25-foot ladder leans against a vertical wall. If the bottom of the ladder is slipping away from the base of the wall at the rate of 1 foot per second, how fast is the top of the ladder moving down the wall when the bottom of the ladder is 7 feet from the base? Indicate units of measure.



$\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = ? \quad x = 7$

$x^2 + y^2 = 25^2$

$7^2 + y^2 = 25^2$

$y = 24$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

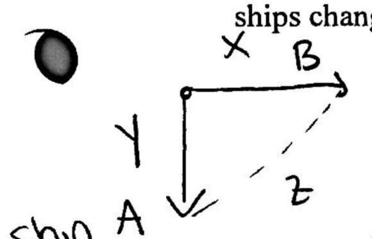
why is it neg? y is decreasing

$2(7)(1) + 2(24)\left(\frac{dy}{dt}\right) = 0$

$48\frac{dy}{dt} = -14$

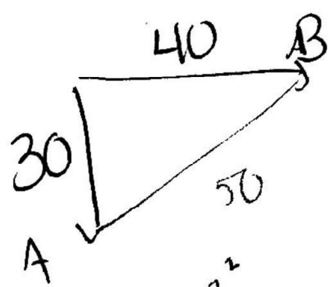
$\boxed{\frac{dy}{dt} = \frac{-7}{24} \text{ ft/sec}}$

Ex. 8: Ship A leaves from the same dock as Ship B. Ship A is sailing south at the rate of 15 km/h and after 1 hour, Ship B starts sailing east at the rate of 40 km/h. How fast is the distance between the two ships changing 2 hours after Ship A left the dock?



$\frac{dy}{dt} = 15 \quad \frac{dx}{dt} = 40 \quad \frac{dz}{dt} = ? \quad t = 2 \ldots ?$

$x^2 + y^2 = z^2$



$30^2 + 40^2 = z^2$

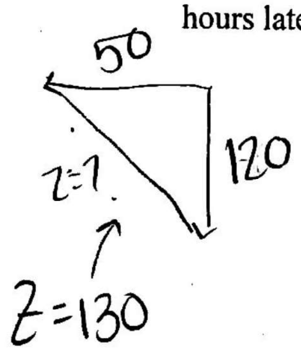
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$2(30)(15) + 2(40)(40) = 2(50)\frac{dz}{dt}$

$900 + 3200 = 100\frac{dz}{dt}$

$\frac{dz}{dt} = 41 \text{ km/hour}$

Ex. 9: Two cars start moving from the same point. One travels south at a velocity of 60 mi/h and the other travels west at a velocity of 25 mi/h. At what rate is the distance between the cars increasing two hours later?



$$\frac{dy}{dt} = 60 \quad \frac{dx}{dt} = 25 \quad \frac{dz}{dt} = ?$$

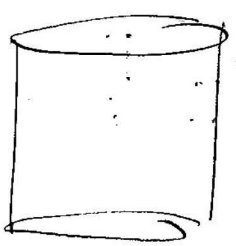
$$x^2 + y^2 = z^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$2(50)(25) + 2(120)(60) = 2(130)\frac{dz}{dt}$$

$$\frac{dz}{dt} = 65 \text{ mph}$$

Ex, 10: A cylindrical tank of radius 10 feet is being filled with wheat at the rate of 314 cubic feet per minute. How fast is the depth of the wheat increasing? (Think about what $\frac{dr}{dt}$ is for a cylinder)



r=10
dlwags
so $\frac{dr}{dt} = 0$

$$dV/dt = 314 \quad dh/dt = ?$$

$$V = \pi r^2 h$$

$$V = 100\pi h$$

$$\frac{dV}{dt} = 100\pi \frac{dh}{dt}$$

$$314 = 100\pi \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{314}{100\pi} \text{ ft/min}$$

1. → Volume is decreasing (rate is negative)

Air is leaking out of a spherical balloon at the rate of 3 cubic inches per minute. When the radius is 5 inches, how fast is the radius decreasing? Indicate units of measure.

$\frac{dv}{dt} = -3$

$r = 5$ $\quad dr/dt = ?$

$V = \frac{4}{3}\pi r^3$

$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$

$-3 = 4\pi(25)\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{-3}{100\pi}$ in/min

2. A cone-shaped paper cup is being filled with water at the rate of $3cm^3/s$. The height of the cup is always three times as big as the radius. How fast is the water level rising when the level is 4 cm?

$h = 3r$

$dV/dt = 3$

$dh/dt = ?$

$h = 4$

$V = \frac{1}{3}\pi r^2(h)$

$V = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h$

$V = \frac{1}{27}\pi h^3$

$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \cdot \frac{dh}{dt}$

$3 = \frac{1}{9}\pi(16)\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{27}{16\pi}$ cm/sec

3. An ice sculpture in the form of a sphere melts in such a way that it maintains its spherical shape. The volume of the sphere is decreasing at a constant rate of 2π cubic meters per hour. At what rate, in square meters per hour, is the surface area of the sphere decreasing at the moment when the radius is 5 meters? (Note: For a sphere of radius r, the surface area is $4\pi r^2$ and the volume is $\frac{4}{3}\pi r^3$.)

$\frac{dv}{dt} = -2\pi$ $\quad r = 5$ $\quad \frac{dSA}{dt} = ?$ $\quad V = \frac{4}{3}\pi r^3$

(A) $\frac{4\pi}{5}$

(B) 40π

(C) $80\pi^2$

(D) 100π

$SA = 4\pi r^2$

$\frac{dSA}{dt} = 8\pi r \frac{dr}{dt}$

$= 8 \cdot \pi \cdot 5\left(\frac{-1}{50}\right)$

$= \frac{-40\pi}{50}$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$-2\cancel{\pi} = 4\cancel{\pi}(5)^2\frac{dr}{dt}$

$\frac{-1}{50} = \frac{dr}{dt}$

4. Poiseuille's Law describes the relationship between the volume of a liquid travelling through a conduit and the conduit's radius. One application is blood travelling through blood vessels. For a 10cm long artery, the law states: $V = \frac{\pi}{0.24}R^4$ with R representing the artery's radius and V is the volume of blood. What is the change in the artery's radius if the heart is increasing blood flow at a rate of 10 cubic centimeters (cc) per second when the radius is 1.5cm? Indicate units of measure.

$\frac{dR}{dt} = ?$ $\quad r = 1.5$

$\frac{dV}{dt} = 10$

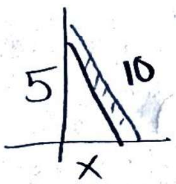
$V = \frac{\pi}{.24}R^4$

$V = \frac{4\pi}{.24}R^3\frac{dR}{dt}$

$10 = \frac{4\pi}{.24}(1.5)^3\frac{dR}{dt}$

$\frac{dR}{dt} = .056\ \frac{cm}{s}$

5. A ladder 10 m long leans against a vertical building. If the bottom of the ladder slides away from the building horizontally at a rate of 2 m/s, how fast is the ladder sliding down the building when the top of the ladder is 5 m above the ground?



$$x^2 + y^2 = 10^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -3.464 \text{ m/sec}$$

$10^2 = 5^2 + y^2$

$75 = x^2$

$x = \sqrt{75}$

$$2(\sqrt{75})(2) + 2(5)\left(\frac{dy}{dt}\right) = 0$$

6. A maple tree is tapped by drilling a hole into the truck and the sap is collected in a pail with base radius 5 inches as shown in the figure below. Let h be the height of the sap in the pail and let r be the upper radius of the sap collected, both measured in inches. The volume V of sap in the pail is changing at a rate of $\frac{1}{20}$ cubic inches per minute. The V of a pail with base radius 5, height h, and upper radius r is $V = \frac{\pi}{3}\cdot h\cdot(r^2 + 5r + 25)$.

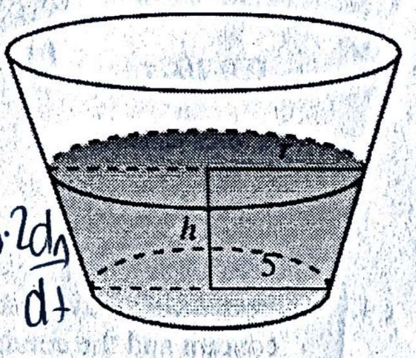
$\frac{dV}{dt} = \frac{1}{20}$

a) Find an expression for $\frac{dV}{dt}$ in terms of r, h and their derivatives.

b) Suppose when $h = 3$, $r = 8$ and $\frac{dr}{dt} = 2\frac{dh}{dt}$ Find $\frac{dh}{dt}$ and explain the meaning of this answer in the context of the problem.

c) Suppose the pail is replaced by a circular cylinder, so that $r = 5$ is constant. Find $\frac{dh}{dt}$.

a) $$\frac{dV}{dt} = \frac{\pi}{3}\cdot\frac{dh}{dt}(r^2+5r+25) + \frac{\pi}{3}h\left(2r\frac{dr}{dt} + 5\frac{dr}{dt}\right)$$



b) $$\frac{1}{20} = \frac{\pi}{3}\frac{dh}{dt}(8^2+40+25) + \frac{\pi}{\cancel{3}}(\cancel{3})\left(2(8)\left(2\frac{dh}{dt}\right) + 5\cdot 2\frac{dh}{dt}\right)$$

$$\frac{1}{20\pi} = 43\frac{dh}{dt} + 32\frac{dh}{dt} + 10\frac{dh}{dt} \rightarrow \frac{1}{20\pi} = 85\frac{dh}{dt} \rightarrow \frac{dh}{dt} = \frac{1}{1700\pi}$$

this is the rate of change of the height of sap in the pail in in/min when the height of the sap is 3 inches

c) $V = \frac{\pi}{3}h(75)$ $\qquad \frac{dV}{dt} = 25\pi\frac{dh}{dt}$ $\qquad \frac{dh}{dt} = \frac{1}{500\pi}$ in/min

$V = 25h\pi$ $\qquad \frac{1}{20} = 25\pi\frac{dh}{dt}$