

# 10

# Conics, Parametric Equations, and Polar Coordinates

In this chapter, you will analyze and write equations of conics using their properties. You will also learn how to write and graph parametric equations and polar equations, and see how calculus can be used to study these graphs. In addition to the rectangular equations of conics, you will also study polar equations of conics.

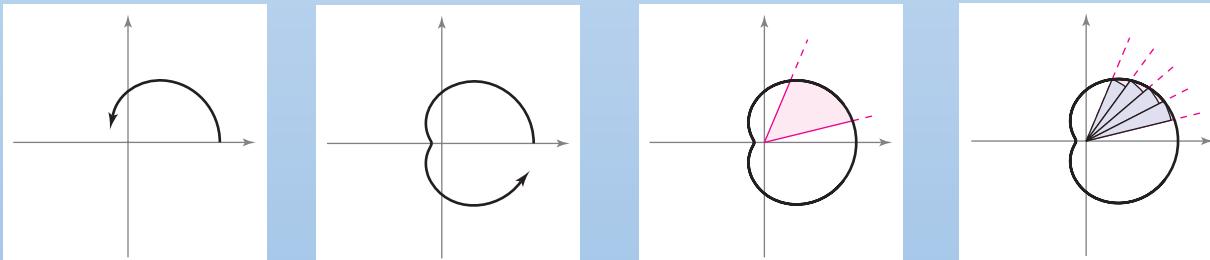
In this chapter, you should learn the following.

- How to analyze and write equations of a parabola, an ellipse, and a hyperbola. (10.1)
- How to sketch a curve represented by parametric equations. (10.2)
- How to use a set of parametric equations to find the slope of a tangent line to a curve and the arc length of a curve. (10.3)
- How to sketch the graph of an equation in polar form, find the slope of a tangent line to a polar graph, and identify special polar graphs. (10.4)
- How to find the area of a region bounded by a polar graph and find the arc length of a polar graph. (10.5)
- How to analyze and write a polar equation of a conic. (10.6)



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The path of a baseball hit at a particular height at an angle with the horizontal can be modeled using parametric equations. How can a set of parametric equations be used to find the minimum angle at which the ball must leave the bat in order for the hit to be a home run? (See Section 10.2, Exercise 75.)



In the polar coordinate system, graphing an equation involves tracing a curve about a fixed point called the pole. Consider a region bounded by a curve and by the rays that contain the endpoints of an interval on the curve. You can use sectors of circles to approximate the area of such a region. In Section 10.5, you will see how the limit process can be used to find this area.

## 10.1 Conics and Calculus

- Understand the definition of a conic section.
- Analyze and write equations of parabolas using properties of parabolas.
- Analyze and write equations of ellipses using properties of ellipses.
- Analyze and write equations of hyperbolas using properties of hyperbolas.

### Conic Sections

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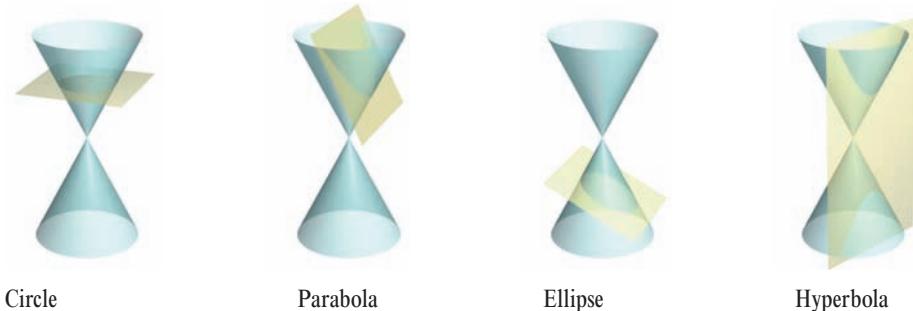


HYPATIA (370–415 A.D.)

The Greeks discovered conic sections sometime between 600 and 300 B.C. By the beginning of the Alexandrian period, enough was known about conics for Apollonius (262–190 B.C.) to produce an eight-volume work on the subject. Later, toward the end of the Alexandrian period, Hypatia wrote a textbook entitled *On the Conics of Apollonius*. Her death marked the end of major mathematical discoveries in Europe for several hundred years.

The early Greeks were largely concerned with the geometric properties of conics. It was not until 1900 years later, in the early seventeenth century, that the broader applicability of conics became apparent. Conics then played a prominent role in the development of calculus.

Each **conic section** (or simply **conic**) can be described as the intersection of a plane and a double-napped cone. Notice in Figure 10.1 that for the four basic conics, the intersecting plane does not pass through the vertex of the cone. When the plane passes through the vertex, the resulting figure is a **degenerate conic**, as shown in Figure 10.2.



Circle  
Conic sections  
**Figure 10.1**



Point  
Degenerate conics

**Figure 10.2**

There are several ways to study conics. You could begin as the Greeks did by defining the conics in terms of the intersections of planes and cones, or you could define them algebraically in terms of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

General second-degree equation

**FOR FURTHER INFORMATION** To learn more about the mathematical activities of Hypatia, see the article “Hypatia and Her Mathematics” by Michael A. B. Deakin in *The American Mathematical Monthly*. To view this article, go to the website [www.matharticles.com](http://www.matharticles.com).

However, a third approach, in which each of the conics is defined as a **locus** (collection) of points satisfying a certain geometric property, works best. For example, a circle can be defined as the collection of all points  $(x, y)$  that are equidistant from a fixed point  $(h, k)$ . This locus definition easily produces the standard equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2.$$

Standard equation of a circle

For information about rotating second-degree equations in two variables, see Appendix D.

## Parabolas

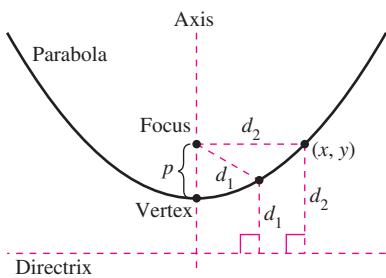


Figure 10.3

A **parabola** is the set of all points  $(x, y)$  that are equidistant from a fixed line called the **directrix** and a fixed point called the **focus** not on the line. The midpoint between the focus and the directrix is the **vertex**, and the line passing through the focus and the vertex is the **axis** of the parabola. Note in Figure 10.3 that a parabola is symmetric with respect to its axis.

### THEOREM 10.1 STANDARD EQUATION OF A PARABOLA

The **standard form** of the equation of a parabola with vertex  $(h, k)$  and directrix  $y = k - p$  is

$$(x - h)^2 = 4p(y - k). \quad \text{Vertical axis}$$

For directrix  $x = h - p$ , the equation is

$$(y - k)^2 = 4p(x - h). \quad \text{Horizontal axis}$$

The focus lies on the axis  $p$  units (*directed distance*) from the vertex. The coordinates of the focus are as follows.

$$(h, k + p) \quad \text{Vertical axis}$$

$$(h + p, k) \quad \text{Horizontal axis}$$

### EXAMPLE 1 Finding the Focus of a Parabola

Find the focus of the parabola given by  $y = \frac{1}{2} - x - \frac{1}{2}x^2$ .

**Solution** To find the focus, convert to standard form by completing the square.

$$y = \frac{1}{2} - x - \frac{1}{2}x^2 \quad \text{Write original equation.}$$

$$y = \frac{1}{2}(1 - 2x - x^2) \quad \text{Factor out } \frac{1}{2}.$$

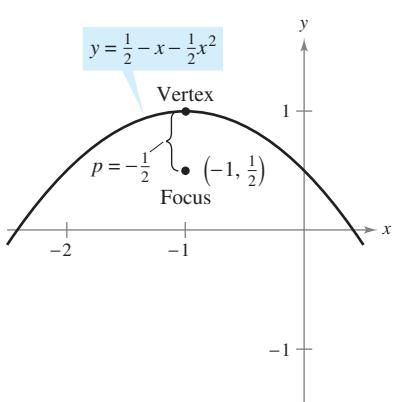
$$2y = 1 - 2x - x^2 \quad \text{Multiply each side by 2.}$$

$$2y = 1 - (x^2 + 2x) \quad \text{Group terms.}$$

$$2y = 2 - (x^2 + 2x + 1) \quad \text{Add and subtract 1 on right side.}$$

$$x^2 + 2x + 1 = -2y + 2$$

$$(x + 1)^2 = -2(y - 1) \quad \text{Write in standard form.}$$



Parabola with a vertical axis,  $p < 0$

Figure 10.4

Comparing this equation with  $(x - h)^2 = 4p(y - k)$ , you can conclude that

$$h = -1, \quad k = 1, \quad \text{and} \quad p = -\frac{1}{2}.$$

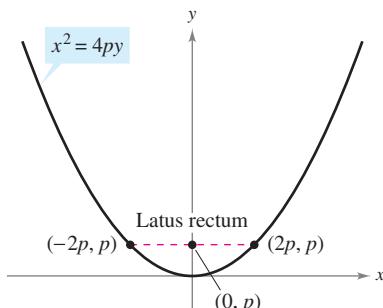
Because  $p$  is negative, the parabola opens downward, as shown in Figure 10.4. So, the focus of the parabola is  $p$  units from the vertex, or

$$(h, k + p) = \left(-1, \frac{1}{2}\right). \quad \text{Focus} \quad \blacksquare$$

A line segment that passes through the focus of a parabola and has endpoints on the parabola is called a **focal chord**. The specific focal chord perpendicular to the axis of the parabola is the **latus rectum**. The next example shows how to determine the length of the latus rectum and the length of the corresponding intercepted arc.

**EXAMPLE 2** Focal Chord Length and Arc Length

Find the length of the latus rectum of the parabola given by  $x^2 = 4py$ . Then find the length of the parabolic arc intercepted by the latus rectum.



Length of latus rectum:  $4p$

**Figure 10.5**

**Solution** Because the latus rectum passes through the focus  $(0, p)$  and is perpendicular to the  $y$ -axis, the coordinates of its endpoints are  $(-x, p)$  and  $(x, p)$ . Substituting  $p$  for  $y$  in the equation of the parabola produces

$$x^2 = 4p(p) \quad \Rightarrow \quad x = \pm 2p.$$

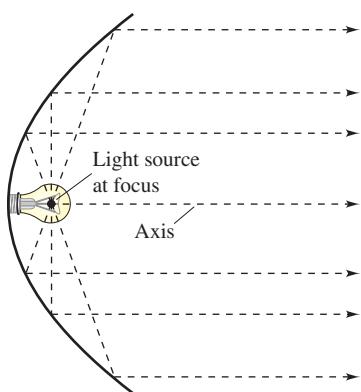
So, the endpoints of the latus rectum are  $(-2p, p)$  and  $(2p, p)$ , and you can conclude that its length is  $4p$ , as shown in Figure 10.5. In contrast, the length of the intercepted arc is

$$\begin{aligned} s &= \int_{-2p}^{2p} \sqrt{1 + (y')^2} dx && \text{Use arc length formula.} \\ &= 2 \int_0^{2p} \sqrt{1 + \left(\frac{x}{2p}\right)^2} dx && y = \frac{x^2}{4p} \quad \Rightarrow \quad y' = \frac{x}{2p} \\ &= \frac{1}{p} \int_0^{2p} \sqrt{4p^2 + x^2} dx && \text{Simplify.} \\ &= \frac{1}{2p} \left[ x \sqrt{4p^2 + x^2} + 4p^2 \ln|x + \sqrt{4p^2 + x^2}| \right]_0^{2p} && \text{Theorem 8.2} \\ &= \frac{1}{2p} \left[ 2p\sqrt{8p^2} + 4p^2 \ln(2p + \sqrt{8p^2}) - 4p^2 \ln(2p) \right] \\ &= 2p[\sqrt{2} + \ln(1 + \sqrt{2})] \\ &\approx 4.59p. \end{aligned}$$

■

One widely used property of a parabola is its reflective property. In physics, a surface is called **reflective** if the tangent line at any point on the surface makes equal angles with an incoming ray and the resulting outgoing ray. The angle corresponding to the incoming ray is the **angle of incidence**, and the angle corresponding to the outgoing ray is the **angle of reflection**. One example of a reflective surface is a flat mirror.

Another type of reflective surface is that formed by revolving a parabola about its axis. A special property of parabolic reflectors is that they allow us to direct all incoming rays parallel to the axis through the focus of the parabola—this is the principle behind the design of the parabolic mirrors used in reflecting telescopes. Conversely, all light rays emanating from the focus of a parabolic reflector used in a flashlight are parallel, as shown in Figure 10.6.



Parabolic reflector: light is reflected in parallel rays.

**Figure 10.6**

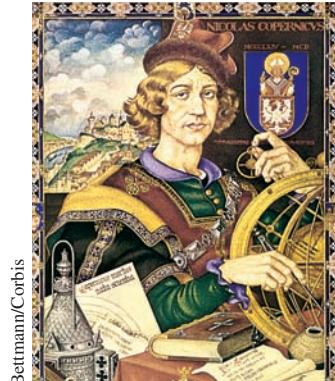
**THEOREM 10.2 REFLECTIVE PROPERTY OF A PARABOLA**

Let  $P$  be a point on a parabola. The tangent line to the parabola at the point  $P$  makes equal angles with the following two lines.

1. The line passing through  $P$  and the focus
2. The line passing through  $P$  parallel to the axis of the parabola

The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple and Mathematica.

## Ellipses



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NICOLAUS COPERNICUS (1473–1543)

Copernicus began to study planetary motion when asked to revise the calendar. At that time, the exact length of the year could not be accurately predicted using the theory that Earth was the center of the universe.

More than a thousand years after the close of the Alexandrian period of Greek mathematics, Western civilization finally began a Renaissance of mathematical and scientific discovery. One of the principal figures in this rebirth was the Polish astronomer Nicolaus Copernicus. In his work *On the Revolutions of the Heavenly Spheres*, Copernicus claimed that all of the planets, including Earth, revolved about the sun in circular orbits. Although some of Copernicus's claims were invalid, the controversy set off by his heliocentric theory motivated astronomers to search for a mathematical model to explain the observed movements of the sun and planets. The first to find an accurate model was the German astronomer Johannes Kepler (1571–1630). Kepler discovered that the planets move about the sun in elliptical orbits, with the sun not as the center but as a focal point of the orbit.

The use of ellipses to explain the movements of the planets is only one of many practical and aesthetic uses. As with parabolas, you will begin your study of this second type of conic by defining it as a locus of points. Now, however, *two* focal points are used rather than one.

An **ellipse** is the set of all points  $(x, y)$  the sum of whose distances from two distinct fixed points called **foci** is constant. (See Figure 10.7.) The line through the foci intersects the ellipse at two points, called the **vertices**. The chord joining the vertices is the **major axis**, and its midpoint is the **center** of the ellipse. The chord perpendicular to the major axis at the center is the **minor axis** of the ellipse. (See Figure 10.8.)

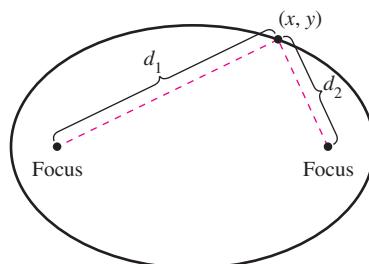


Figure 10.7

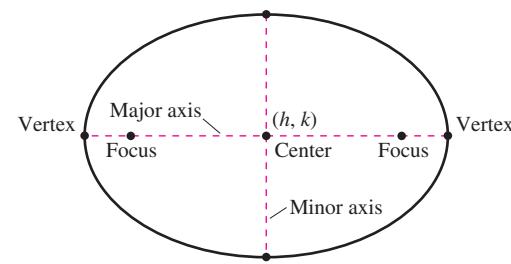
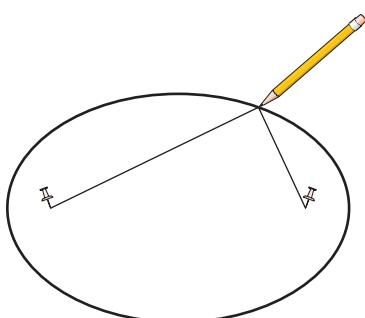


Figure 10.8

**FOR FURTHER INFORMATION** To learn about how an ellipse may be “exploded” into a parabola, see the article “Exploding the Ellipse” by Arnold Good in *Mathematics Teacher*. To view this article, go to the website [www.matharticles.com](http://www.matharticles.com).



If the ends of a fixed length of string are fastened to the thumbtacks and the string is drawn taut with a pencil, the path traced by the pencil will be an ellipse.

Figure 10.9

### THEOREM 10.3 STANDARD EQUATION OF AN ELLIPSE

The standard form of the equation of an ellipse with center  $(h, k)$  and major and minor axes of lengths  $2a$  and  $2b$ , where  $a > b$ , is

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \quad \text{Major axis is horizontal.}$$

or

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1. \quad \text{Major axis is vertical.}$$

The foci lie on the major axis,  $c$  units from the center, with  $c^2 = a^2 - b^2$ .

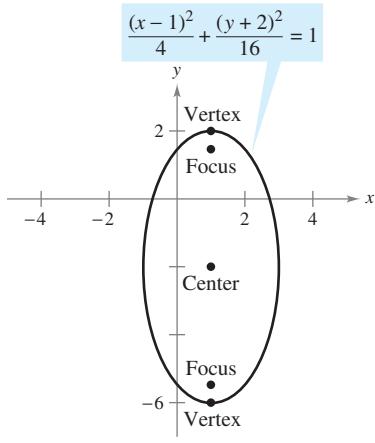
You can visualize the definition of an ellipse by imagining two thumbtacks placed at the foci, as shown in Figure 10.9.

**EXAMPLE 3** Analyzing an Ellipse

Find the center, vertices, and foci of the ellipse given by

$$4x^2 + y^2 - 8x + 4y - 8 = 0.$$

**Solution** By completing the square, you can write the original equation in standard form.



Ellipse with a vertical major axis

Figure 10.10

$$4x^2 + y^2 - 8x + 4y - 8 = 0$$

Write original equation.

$$4x^2 - 8x + y^2 + 4y = 8$$

$$4(x^2 - 2x + 1) + (y^2 + 4y + 4) = 8 + 4 + 4$$

$$4(x - 1)^2 + (y + 2)^2 = 16$$

$$\frac{(x - 1)^2}{4} + \frac{(y + 2)^2}{16} = 1$$

Write in standard form.

So, the major axis is parallel to the  $y$ -axis, where  $h = 1$ ,  $k = -2$ ,  $a = 4$ ,  $b = 2$ , and  $c = \sqrt{16 - 4} = 2\sqrt{3}$ . So, you obtain the following.

$$\text{Center: } (1, -2)$$

( $h, k$ )

$$\text{Vertices: } (1, -6) \text{ and } (1, 2)$$

( $h, k \pm a$ )

$$\text{Foci: } (1, -2 - 2\sqrt{3}) \text{ and } (1, -2 + 2\sqrt{3})$$

( $h, k \pm c$ )

The graph of the ellipse is shown in Figure 10.10. ■

**NOTE** If the constant term  $F = -8$  in the equation in Example 3 had been greater than or equal to 8, you would have obtained one of the following degenerate cases.

$$1. F = 8, \text{ single point, } (1, -2): \frac{(x - 1)^2}{4} + \frac{(y + 2)^2}{16} = 0$$

$$2. F > 8, \text{ no solution points: } \frac{(x - 1)^2}{4} + \frac{(y + 2)^2}{16} < 0$$

**EXAMPLE 4** The Orbit of the Moon

The moon orbits Earth in an elliptical path with the center of Earth at one focus, as shown in Figure 10.11. The major and minor axes of the orbit have lengths of 768,800 kilometers and 767,640 kilometers, respectively. Find the greatest and least distances (the apogee and perigee) from Earth's center to the moon's center.

**Solution** Begin by solving for  $a$  and  $b$ .

$$2a = 768,800 \quad \text{Length of major axis}$$

$$a = 384,400 \quad \text{Solve for } a.$$

$$2b = 767,640 \quad \text{Length of minor axis}$$

$$b = 383,820 \quad \text{Solve for } b.$$

Now, using these values, you can solve for  $c$  as follows.

$$c = \sqrt{a^2 - b^2} \approx 21,108$$

The greatest distance between the center of Earth and the center of the moon is  $a + c \approx 405,508$  kilometers, and the least distance is  $a - c \approx 363,292$  kilometers. ■

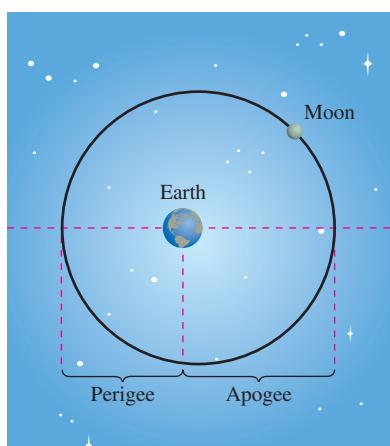


Figure 10.11

### FOR FURTHER INFORMATION

For more information on some uses of the reflective properties of conics, see the article “Parabolic Mirrors, Elliptic and Hyperbolic Lenses” by Mohsen Maesumi in *The American Mathematical Monthly*. Also see the article “The Geometry of Microwave Antennas” by William R. Parzynski in *Mathematics Teacher*.

Theorem 10.2 presented a reflective property of parabolas. Ellipses have a similar reflective property. You are asked to prove the following theorem in Exercise 112.

### THEOREM 10.4 REFLECTIVE PROPERTY OF AN ELLIPSE

Let  $P$  be a point on an ellipse. The tangent line to the ellipse at point  $P$  makes equal angles with the lines through  $P$  and the foci.

One of the reasons that astronomers had difficulty in detecting that the orbits of the planets are ellipses is that the foci of the planetary orbits are relatively close to the center of the sun, making the orbits nearly circular. To measure the ovalness of an ellipse, you can use the concept of **eccentricity**.

### DEFINITION OF ECCENTRICITY OF AN ELLIPSE

The **eccentricity**  $e$  of an ellipse is given by the ratio

$$e = \frac{c}{a}.$$

To see how this ratio is used to describe the shape of an ellipse, note that because the foci of an ellipse are located along the major axis between the vertices and the center, it follows that

$$0 < c < a.$$

For an ellipse that is nearly circular, the foci are close to the center and the ratio  $c/a$  is small, and for an elongated ellipse, the foci are close to the vertices and the ratio  $c/a$  is close to 1, as shown in Figure 10.12. Note that  $0 < e < 1$  for every ellipse.

The orbit of the moon has an eccentricity of  $e = 0.0549$ , and the eccentricities of the eight planetary orbits are as follows.

Mercury:  $e = 0.2056$

Jupiter:  $e = 0.0484$

Venus:  $e = 0.0068$

Saturn:  $e = 0.0542$

Earth:  $e = 0.0167$

Uranus:  $e = 0.0472$

Mars:  $e = 0.0934$

Neptune:  $e = 0.0086$

You can use integration to show that the area of an ellipse is  $A = \pi ab$ . For instance, the area of the ellipse

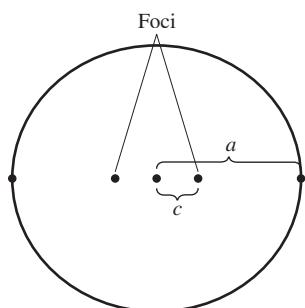
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is given by

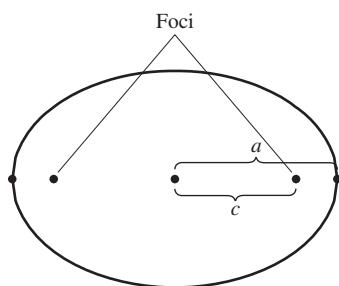
$$A = 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx \\ = \frac{4b}{a} \int_0^{\pi/2} a^2 \cos^2 \theta d\theta.$$

Trigonometric substitution  $x = a \sin \theta$ .

However, it is not so simple to find the *circumference* of an ellipse. The next example shows how to use eccentricity to set up an “elliptic integral” for the circumference of an ellipse.



(a)  $\frac{c}{a}$  is small.



(b)  $\frac{c}{a}$  is close to 1.

Eccentricity is the ratio  $\frac{c}{a}$ .

**Figure 10.12**

### EXAMPLE 5 Finding the Circumference of an Ellipse

Show that the circumference of the ellipse  $(x^2/a^2) + (y^2/b^2) = 1$  is

$$4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} d\theta. \quad e = \frac{c}{a}$$

**Solution** Because the given ellipse is symmetric with respect to both the  $x$ -axis and the  $y$ -axis, you know that its circumference  $C$  is four times the arc length of  $y = (b/a)\sqrt{a^2 - x^2}$  in the first quadrant. The function  $y$  is differentiable for all  $x$  in the interval  $[0, a]$  except at  $x = a$ . So, the circumference is given by the improper integral

$$C = \lim_{d \rightarrow a} 4 \int_0^d \sqrt{1 + (y')^2} dx = 4 \int_0^a \sqrt{1 + (y')^2} dx = 4 \int_0^a \sqrt{1 + \frac{b^2 x^2}{a^2(a^2 - x^2)}} dx.$$

Using the trigonometric substitution  $x = a \sin \theta$ , you obtain

$$\begin{aligned} C &= 4 \int_0^{\pi/2} \sqrt{1 + \frac{b^2 \sin^2 \theta}{a^2 \cos^2 \theta}} (a \cos \theta) d\theta \\ &= 4 \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta \\ &= 4 \int_0^{\pi/2} \sqrt{a^2(1 - \sin^2 \theta) + b^2 \sin^2 \theta} d\theta \\ &= 4 \int_0^{\pi/2} \sqrt{a^2 - (a^2 - b^2) \sin^2 \theta} d\theta. \end{aligned}$$

Because  $e^2 = c^2/a^2 = (a^2 - b^2)/a^2$ , you can rewrite this integral as

$$C = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} d\theta.$$

#### AREA AND CIRCUMFERENCE OF AN ELLIPSE

In his work with elliptic orbits in the early 1600's, Johannes Kepler successfully developed a formula for the area of an ellipse,  $A = \pi ab$ . He was less successful in developing a formula for the circumference of an ellipse, however; the best he could do was to give the approximate formula  $C = \pi(a + b)$ .

A great deal of time has been devoted to the study of elliptic integrals. Such integrals generally do not have elementary antiderivatives. To find the circumference of an ellipse, you must usually resort to an approximation technique.

### EXAMPLE 6 Approximating the Value of an Elliptic Integral

Use the elliptic integral in Example 5 to approximate the circumference of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

**Solution** Because  $e^2 = c^2/a^2 = (a^2 - b^2)/a^2 = 9/25$ , you have

$$C = (4)(5) \int_0^{\pi/2} \sqrt{1 - \frac{9 \sin^2 \theta}{25}} d\theta.$$

Applying Simpson's Rule with  $n = 4$  produces

$$\begin{aligned} C &\approx 20 \left(\frac{\pi}{6}\right) \left(\frac{1}{4}\right) [1 + 4(0.9733) + 2(0.9055) + 4(0.8323) + 0.8] \\ &\approx 28.36. \end{aligned}$$

So, the ellipse has a circumference of about 28.36 units, as shown in Figure 10.13.

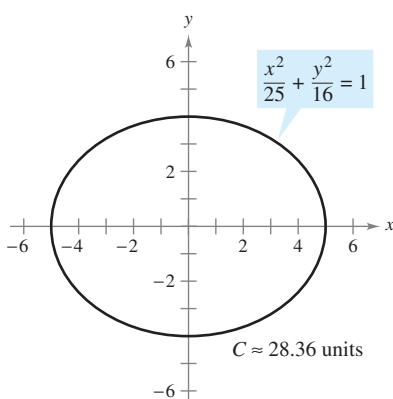


Figure 10.13

## Hyperbolas

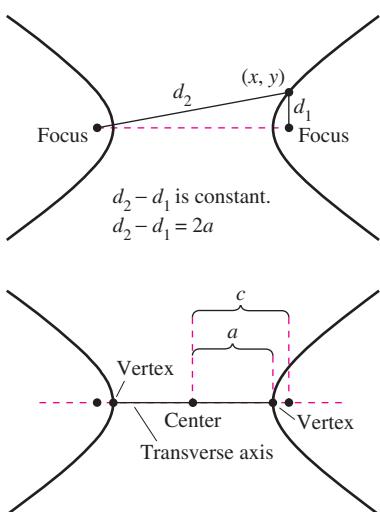


Figure 10.14

The definition of a hyperbola is similar to that of an ellipse. For an ellipse, the *sum* of the distances between the foci and a point on the ellipse is fixed, whereas for a hyperbola, the absolute value of the *difference* between these distances is fixed.

A **hyperbola** is the set of all points  $(x, y)$  for which the absolute value of the difference between the distances from two distinct fixed points called **foci** is constant. (See Figure 10.14.) The line through the two foci intersects a hyperbola at two points called the **vertices**. The line segment connecting the vertices is the **transverse axis**, and the midpoint of the transverse axis is the **center** of the hyperbola. One distinguishing feature of a hyperbola is that its graph has two separate *branches*.

### THEOREM 10.5 STANDARD EQUATION OF A HYPERBOLA

The standard form of the equation of a hyperbola with center at  $(h, k)$  is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \text{Transverse axis is horizontal.}$$

or

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \quad \text{Transverse axis is vertical.}$$

The vertices are  $a$  units from the center, and the foci are  $c$  units from the center, where,  $c^2 = a^2 + b^2$ .

**NOTE** The constants  $a$ ,  $b$ , and  $c$  do not have the same relationship for hyperbolas as they do for ellipses. For hyperbolas,  $c^2 = a^2 + b^2$ , but for ellipses,  $c^2 = a^2 - b^2$ . ■

An important aid in sketching the graph of a hyperbola is the determination of its **asymptotes**, as shown in Figure 10.15. Each hyperbola has two asymptotes that intersect at the center of the hyperbola. The asymptotes pass through the vertices of a rectangle of dimensions  $2a$  by  $2b$ , with its center at  $(h, k)$ . The line segment of length  $2b$  joining  $(h, k+b)$  and  $(h, k-b)$  is referred to as the **conjugate axis** of the hyperbola.

### THEOREM 10.6 ASYMPTOTES OF A HYPERBOLA

For a *horizontal* transverse axis, the equations of the asymptotes are

$$y = k + \frac{b}{a}(x-h) \quad \text{and} \quad y = k - \frac{b}{a}(x-h).$$

For a *vertical* transverse axis, the equations of the asymptotes are

$$y = k + \frac{a}{b}(x-h) \quad \text{and} \quad y = k - \frac{a}{b}(x-h).$$

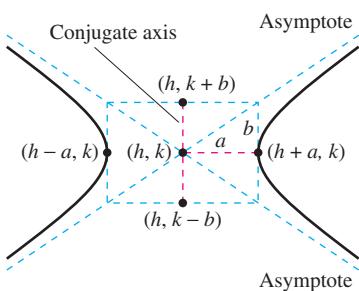


Figure 10.15

In Figure 10.15 you can see that the asymptotes coincide with the diagonals of the rectangle with dimensions  $2a$  and  $2b$ , centered at  $(h, k)$ . This provides you with a quick means of sketching the asymptotes, which in turn aids in sketching the hyperbola.



### EXAMPLE 7 Using Asymptotes to Sketch a Hyperbola

Sketch the graph of the hyperbola whose equation is  $4x^2 - y^2 = 16$ .

**TECHNOLOGY** You can use a graphing utility to verify the graph obtained in Example 7 by solving the original equation for  $y$  and graphing the following equations.

$$y_1 = \sqrt{4x^2 - 16}$$

$$y_2 = -\sqrt{4x^2 - 16}$$

**Solution** Begin by rewriting the equation in standard form.

$$\frac{x^2}{4} - \frac{y^2}{16} = 1$$

The transverse axis is horizontal and the vertices occur at  $(-2, 0)$  and  $(2, 0)$ . The ends of the conjugate axis occur at  $(0, -4)$  and  $(0, 4)$ . Using these four points, you can sketch the rectangle shown in Figure 10.16(a). By drawing the asymptotes through the corners of this rectangle, you can complete the sketch as shown in Figure 10.16(b).

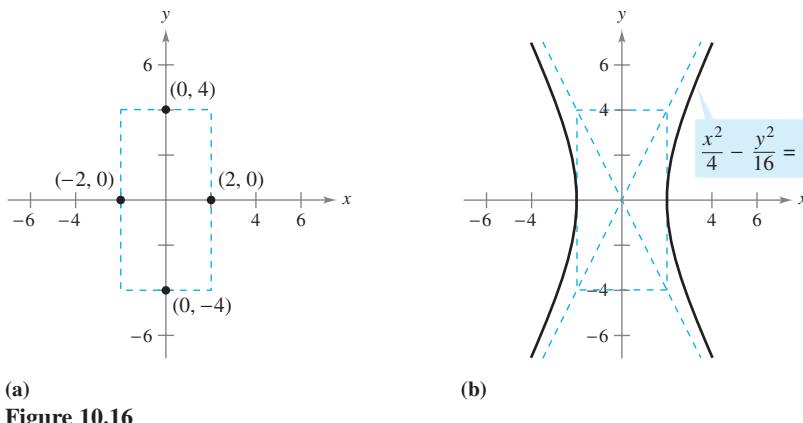


Figure 10.16

#### DEFINITION OF ECCENTRICITY OF A HYPERBOLA

The **eccentricity**  $e$  of a hyperbola is given by the ratio

$$e = \frac{c}{a}.$$

As with an ellipse, the **eccentricity** of a hyperbola is  $e = c/a$ . Because  $c > a$  for hyperbolas, it follows that  $e > 1$  for hyperbolas. If the eccentricity is large, the branches of the hyperbola are nearly flat. If the eccentricity is close to 1, the branches of the hyperbola are more pointed, as shown in Figure 10.17.

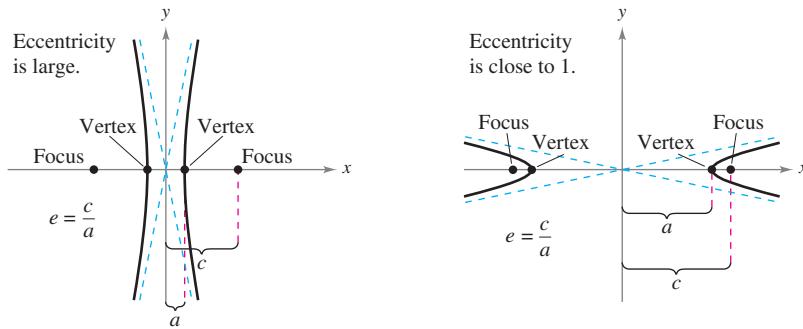


Figure 10.17

The following application was developed during World War II. It shows how the properties of hyperbolas can be used in radar and other detection systems.

### EXAMPLE 8 A Hyperbolic Detection System

Two microphones, 1 mile apart, record an explosion. Microphone  $A$  receives the sound 2 seconds before microphone  $B$ . Where was the explosion?

**Solution** Assuming that sound travels at 1100 feet per second, you know that the explosion took place 2200 feet farther from  $B$  than from  $A$ , as shown in Figure 10.18. The locus of all points that are 2200 feet closer to  $A$  than to  $B$  is one branch of the hyperbola  $(x^2/a^2) - (y^2/b^2) = 1$ , where

$$c = \frac{1 \text{ mile}}{2} = \frac{5280 \text{ ft}}{2} = 2640 \text{ feet}$$

and

$$a = \frac{2200 \text{ ft}}{2} = 1100 \text{ feet.}$$

Because  $c^2 = a^2 + b^2$ , it follows that

$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= 5,759,600 \end{aligned}$$

and you can conclude that the explosion occurred somewhere on the right branch of the hyperbola given by

$$\frac{x^2}{1,210,000} - \frac{y^2}{5,759,600} = 1. \quad \blacksquare$$

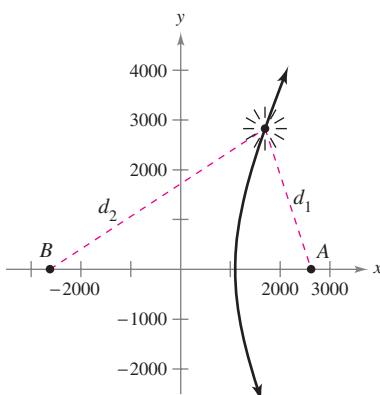
In Example 8, you were able to determine only the hyperbola on which the explosion occurred, but not the exact location of the explosion. If, however, you had received the sound at a third position  $C$ , then two other hyperbolas would be determined. The exact location of the explosion would be the point at which these three hyperbolas intersect.

Another interesting application of conics involves the orbits of comets in our solar system. Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits. The center of the sun is a focus of each orbit, and each orbit has a vertex at the point at which the comet is closest to the sun. Undoubtedly, many comets with parabolic or hyperbolic orbits have not been identified—such comets pass through our solar system only once. Only comets with elliptical orbits such as Halley's comet remain in our solar system.

The type of orbit for a comet can be determined as follows.

1. Ellipse:  $v < \sqrt{2GM/p}$
2. Parabola:  $v = \sqrt{2GM/p}$
3. Hyperbola:  $v > \sqrt{2GM/p}$

In these three formulas,  $p$  is the distance between one vertex and one focus of the comet's orbit (in meters),  $v$  is the velocity of the comet at the vertex (in meters per second),  $M \approx 1.989 \times 10^{30}$  kilograms is the mass of the sun, and  $G \approx 6.67 \times 10^{-11}$  cubic meters per kilogram-second squared is the gravitational constant.



$$2c = 5280$$

$$d_2 - d_1 = 2a = 2200$$

Figure 10.18

Mary Evans Picture Library/The Image Works



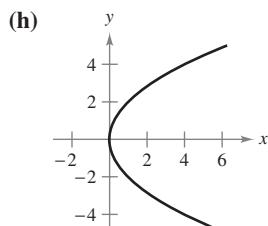
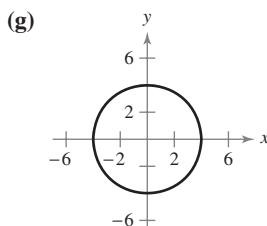
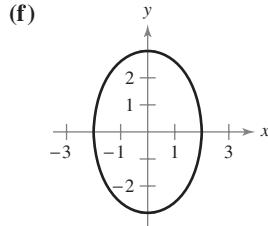
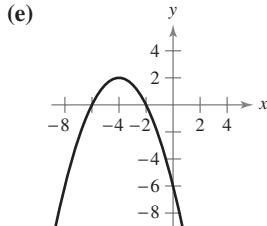
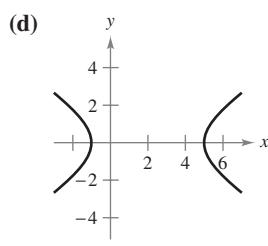
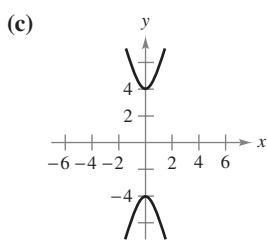
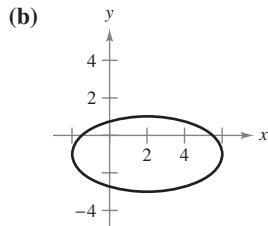
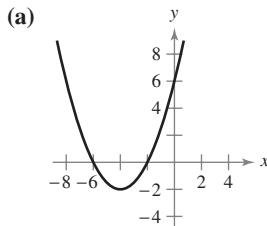
CAROLINE HERSCHEL (1750–1848)

The first woman to be credited with detecting a new comet was the English astronomer Caroline Herschel. During her life, Caroline Herschel discovered a total of eight new comets.

## 10.1 Exercises

See [www.CalcChat.com](http://www.CalcChat.com) for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]



1.  $y^2 = 4x$

2.  $(x + 4)^2 = 2(y + 2)$

3.  $(x + 4)^2 = -2(y - 2)$

4.  $\frac{(x - 2)^2}{16} + \frac{(y + 1)^2}{4} = 1$

5.  $\frac{x^2}{4} + \frac{y^2}{9} = 1$

6.  $\frac{x^2}{16} + \frac{y^2}{16} = 1$

7.  $\frac{y^2}{16} - \frac{x^2}{1} = 1$

8.  $\frac{(x - 2)^2}{9} - \frac{y^2}{4} = 1$

In Exercises 9–16, find the vertex, focus, and directrix of the parabola, and sketch its graph.

9.  $y^2 = -8x$

10.  $x^2 + 6y = 0$

11.  $(x + 5)^2 + (y - 3)^2 = 0$

12.  $(x - 6)^2 + 8(y + 7) = 0$

13.  $y^2 - 4y - 4x = 0$

14.  $y^2 + 6y + 8x + 25 = 0$

15.  $x^2 + 4x + 4y - 4 = 0$

16.  $y^2 + 4y + 8x - 12 = 0$



In Exercises 17–20, find the vertex, focus, and directrix of the parabola. Then use a graphing utility to graph the parabola.

17.  $y^2 + x + y = 0$

18.  $y = -\frac{1}{6}(x^2 - 8x + 6)$

19.  $y^2 - 4x - 4 = 0$

20.  $x^2 - 2x + 8y + 9 = 0$

In Exercises 21–28, find an equation of the parabola.

21. Vertex:  $(5, 4)$

22. Vertex:  $(-2, 1)$

Focus:  $(3, 4)$

Focus:  $(-2, -1)$

23. Vertex:  $(0, 5)$

Directrix:  $y = -3$

24. Focus:  $(2, 2)$

Directrix:  $x = -2$

25.

26.

27. Axis is parallel to  $y$ -axis; graph passes through  $(0, 3)$ ,  $(3, 4)$ , and  $(4, 11)$ .

28. Directrix:  $y = -2$ ; endpoints of latus rectum are  $(0, 2)$  and  $(8, 2)$ .

In Exercises 29–34, find the center, foci, vertices, and eccentricity of the ellipse, and sketch its graph.

29.  $16x^2 + y^2 = 16$

30.  $3x^2 + 7y^2 = 63$

31.  $\frac{(x - 3)^2}{16} + \frac{(y - 1)^2}{25} = 1$

32.  $(x + 4)^2 + \frac{(y + 6)^2}{1/4} = 1$

33.  $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

34.  $16x^2 + 25y^2 - 64x + 150y + 279 = 0$



In Exercises 35–38, find the center, foci, and vertices of the ellipse. Use a graphing utility to graph the ellipse.

35.  $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

36.  $36x^2 + 9y^2 + 48x - 36y + 43 = 0$

37.  $x^2 + 2y^2 - 3x + 4y + 0.25 = 0$

38.  $2x^2 + y^2 + 4.8x - 6.4y + 3.12 = 0$

In Exercises 39–44, find an equation of the ellipse.

39. Center:  $(0, 0)$

40. Vertices:  $(0, 3)$ ,  $(8, 3)$

Focus:  $(5, 0)$

Eccentricity:  $\frac{3}{4}$

Vertex:  $(6, 0)$

41. Vertices:  $(3, 1)$ ,  $(3, 9)$

42. Foci:  $(0, \pm 9)$

Minor axis length: 6

Major axis length: 22

43. Center:  $(0, 0)$

44. Center:  $(1, 2)$

Major axis: horizontal

Major axis: vertical

Points on the ellipse:

Points on the ellipse:

$(3, 1)$ ,  $(4, 0)$

$(1, 6)$ ,  $(3, 2)$

In Exercises 45–52, find the center, foci, and vertices of the hyperbola, and sketch its graph using asymptotes as an aid.

45.  $y^2 - \frac{x^2}{9} = 1$

46.  $\frac{x^2}{25} - \frac{y^2}{16} = 1$

47.  $\frac{(x-1)^2}{4} - \frac{(y+2)^2}{1} = 1$

48.  $\frac{(y+3)^2}{225} - \frac{(x-5)^2}{64} = 1$

49.  $9x^2 - y^2 - 36x - 6y + 18 = 0$

50.  $y^2 - 16x^2 + 64x - 208 = 0$

51.  $x^2 - 9y^2 + 2x - 54y - 80 = 0$

52.  $9x^2 - 4y^2 + 54x + 8y + 78 = 0$



In Exercises 53–56, find the center, foci, and vertices of the hyperbola. Use a graphing utility to graph the hyperbola and its asymptotes.

53.  $9y^2 - x^2 + 2x + 54y + 62 = 0$

54.  $9x^2 - y^2 + 54x + 10y + 55 = 0$

55.  $3x^2 - 2y^2 - 6x - 12y - 27 = 0$

56.  $3y^2 - x^2 + 6x - 12y = 0$

In Exercises 57–64, find an equation of the hyperbola.

57. Vertices:  $(\pm 1, 0)$

Asymptotes:  $y = \pm 5x$

58. Vertices:  $(0, \pm 4)$

Asymptotes:  $y = \pm 2x$

59. Vertices:  $(2, \pm 3)$

Point on graph:  $(0, 5)$

60. Vertices:  $(2, \pm 3)$

Foci:  $(2, \pm 5)$

61. Center:  $(0, 0)$

Vertex:  $(0, 2)$

Focus:  $(0, 4)$

62. Center:  $(0, 0)$

Vertex:  $(6, 0)$

Focus:  $(10, 0)$

63. Vertices:  $(0, 2), (6, 2)$

Asymptotes:  $y = \frac{2}{3}x$

$y = 4 - \frac{2}{3}x$

64. Focus:  $(20, 0)$

Asymptotes:  $y = \pm \frac{3}{4}x$

In Exercises 65 and 66, find equations for (a) the tangent lines and (b) the normal lines to the hyperbola for the given value of  $x$ .

65.  $\frac{x^2}{9} - y^2 = 1, \quad x = 6$

66.  $\frac{y^2}{4} - \frac{x^2}{2} = 1, \quad x = 4$

In Exercises 67–76, classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

67.  $x^2 + 4y^2 - 6x + 16y + 21 = 0$

68.  $4x^2 - y^2 - 4x - 3 = 0$

69.  $y^2 - 8y - 8x = 0$

70.  $25x^2 - 10x - 200y - 119 = 0$

71.  $4x^2 + 4y^2 - 16y + 15 = 0$

72.  $y^2 - 4y = x + 5$

73.  $9x^2 + 9y^2 - 36x + 6y + 34 = 0$

74.  $2x(x - y) = y(3 - y - 2x)$

75.  $3(x - 1)^2 = 6 + 2(y + 1)^2$

76.  $9(x + 3)^2 = 36 - 4(y - 2)^2$

### WRITING ABOUT CONCEPTS

77. (a) Give the definition of a parabola.  
 (b) Give the standard forms of a parabola with vertex at  $(h, k)$ .  
 (c) In your own words, state the reflective property of a parabola.
78. (a) Give the definition of an ellipse.  
 (b) Give the standard forms of an ellipse with center at  $(h, k)$ .
79. (a) Give the definition of a hyperbola.  
 (b) Give the standard forms of a hyperbola with center at  $(h, k)$ .  
 (c) Write equations for the asymptotes of a hyperbola.
80. Define the eccentricity of an ellipse. In your own words, describe how changes in the eccentricity affect the ellipse.

81. **Solar Collector** A solar collector for heating water is constructed with a sheet of stainless steel that is formed into the shape of a parabola (see figure). The water will flow through a pipe that is located at the focus of the parabola. At what distance from the vertex is the pipe?

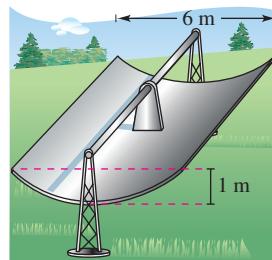


Figure for 81

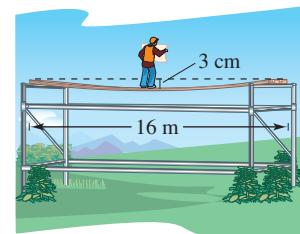


Figure for 82

82. **Beam Deflection** A simply supported beam that is 16 meters long has a load concentrated at the center (see figure). The deflection of the beam at its center is 3 centimeters. Assume that the shape of the deflected beam is parabolic.

- (a) Find an equation of the parabola. (Assume that the origin is at the center of the beam.)
- (b) How far from the center of the beam is the deflection 1 centimeter?

83. Find an equation of the tangent line to the parabola  $y = ax^2$  at  $x = x_0$ . Prove that the  $x$ -intercept of this tangent line is  $(x_0/2, 0)$ .

84. (a) Prove that any two distinct tangent lines to a parabola intersect.
- (b) Demonstrate the result of part (a) by finding the point of intersection of the tangent lines to the parabola  $x^2 - 4x - 4y = 0$  at the points  $(0, 0)$  and  $(6, 3)$ .
85. (a) Prove that if any two tangent lines to a parabola intersect at right angles, their point of intersection must lie on the directrix.
- (b) Demonstrate the result of part (a) by proving that the tangent lines to the parabola  $x^2 - 4x - 4y + 8 = 0$  at the points  $(-2, 5)$  and  $(3, \frac{5}{4})$  intersect at right angles, and that the point of intersection lies on the directrix.

86. Find the point on the graph of  $x^2 = 8y$  that is closest to the focus of the parabola.

87. **Radio and Television Reception** In mountainous areas, reception of radio and television is sometimes poor. Consider an idealized case where a hill is represented by the graph of the parabola  $y = x - x^2$ , a transmitter is located at the point  $(-1, 1)$ , and a receiver is located on the other side of the hill at the point  $(x_0, 0)$ . What is the closest the receiver can be to the hill while still maintaining unobstructed reception?



88. **Modeling Data** The table shows the average amounts of time  $A$  (in minutes) women spent watching television each day for the years 1999 through 2005. (Source: Nielsen Media Research)

Year	1999	2000	2001	2002	2003	2004	2005
$A$	280	286	291	298	305	307	317

- (a) Use the regression capabilities of a graphing utility to find a model of the form  $A = at^2 + bt + c$  for the data. Let  $t$  represent the year, with  $t = 9$  corresponding to 1999.  
(b) Use a graphing utility to plot the data and graph the model.  
(c) Find  $dA/dt$  and sketch its graph for  $9 \leq t \leq 15$ . What information about the average amount of time women spent watching television is given by the graph of the derivative?

89. **Architecture** A church window is bounded above by a parabola and below by the arc of a circle (see figure). Find the surface area of the window.

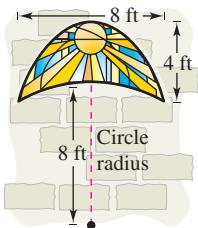


Figure for 89

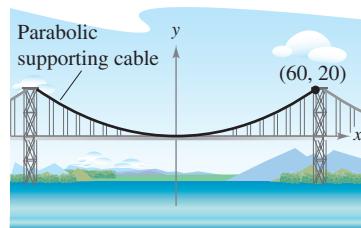


Figure for 91

90. **Arc Length** Find the arc length of the parabola  $4x - y^2 = 0$  over the interval  $0 \leq y \leq 4$ .

91. **Bridge Design** A cable of a suspension bridge is suspended (in the shape of a parabola) between two towers that are 120 meters apart and 20 meters above the roadway (see figure). The cables touch the roadway midway between the towers.

- (a) Find an equation for the parabolic shape of each cable.  
(b) Find the length of the parabolic supporting cable.

92. **Surface Area** A satellite signal receiving dish is formed by revolving the parabola given by  $x^2 = 20y$  about the  $y$ -axis. The radius of the dish is  $r$  feet. Verify that the surface area of the dish is given by

$$2\pi \int_0^r x \sqrt{1 + \left(\frac{x}{10}\right)^2} dx = \frac{\pi}{15}[(100 + r^2)^{3/2} - 1000].$$

93. **Investigation** Sketch the graphs of  $x^2 = 4py$  for  $p = \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}$ , and 2 on the same coordinate axes. Discuss the change in the graphs as  $p$  increases.

94. **Area** Find a formula for the area of the shaded region in the figure.

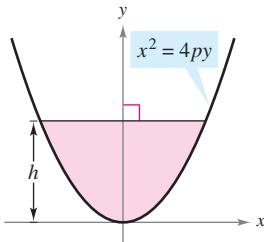


Figure for 94

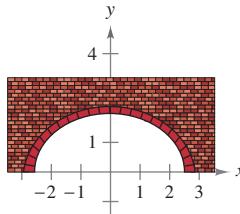


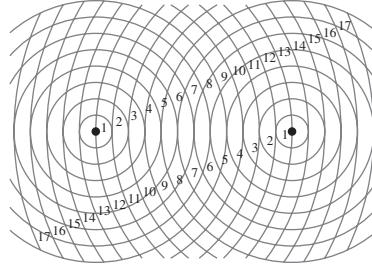
Figure for 96

95. **Writing** On page 699, it was noted that an ellipse can be drawn using two thumbtacks, a string of fixed length (greater than the distance between the tacks), and a pencil. If the ends of the string are fastened at the tacks and the string is drawn taut with a pencil, the path traced by the pencil will be an ellipse.

- (a) What is the length of the string in terms of  $a$ ?  
(b) Explain why the path is an ellipse.

96. **Construction of a Semielliptical Arch** A fireplace arch is to be constructed in the shape of a semiellipse. The opening is to have a height of 2 feet at the center and a width of 5 feet along the base (see figure). The contractor draws the outline of the ellipse by the method shown in Exercise 95. Where should the tacks be placed and what should be the length of the piece of string?

97. Sketch the ellipse that consists of all points  $(x, y)$  such that the sum of the distances between  $(x, y)$  and two fixed points is 16 units, and the foci are located at the centers of the two sets of concentric circles in the figure. To print an enlarged copy of the graph, go to the website [www.mathgraphs.com](http://www.mathgraphs.com).



98. **Orbit of Earth** Earth moves in an elliptical orbit with the sun at one of the foci. The length of half of the major axis is 149,598,000 kilometers, and the eccentricity is 0.0167. Find the minimum distance (*perihelion*) and the maximum distance (*aphelion*) of Earth from the sun.

99. **Satellite Orbit** The *apogee* (the point in orbit farthest from Earth) and the *perigee* (the point in orbit closest to Earth) of an elliptical orbit of an Earth satellite are given by  $A$  and  $P$ . Show that the eccentricity of the orbit is

$$e = \frac{A - P}{A + P}.$$

100. **Explorer 18** On November 27, 1963, the United States launched the research satellite Explorer 18. Its low and high points above the surface of Earth were 119 miles and 123,000 miles. Find the eccentricity of its elliptical orbit.

- 101. Explorer 55** On November 20, 1975, the United States launched the research satellite Explorer 55. Its low and high points above the surface of Earth were 96 miles and 1865 miles. Find the eccentricity of its elliptical orbit.

**CAPSTONE**

- 102.** Consider the equation

$$9x^2 + 4y^2 - 36x - 24y - 36 = 0.$$

- (a) Classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.
- (b) Change the  $4y^2$ -term in the equation to  $-4y^2$ . Classify the graph of the new equation.
- (c) Change the  $9x^2$ -term in the original equation to  $4x^2$ . Classify the graph of the new equation.
- (d) Describe one way you could change the original equation so that its graph is a parabola.

- 103. Halley's Comet** Probably the most famous of all comets, Halley's comet, has an elliptical orbit with the sun at one focus. Its maximum distance from the sun is approximately 35.29 AU (1 astronomical unit  $\approx 92.956 \times 10^6$  miles), and its minimum distance is approximately 0.59 AU. Find the eccentricity of the orbit.

- 104.** The equation of an ellipse with its center at the origin can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1.$$

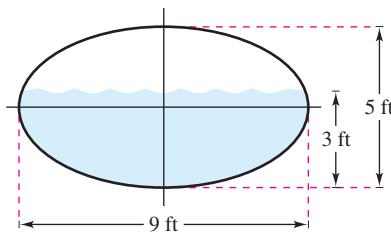
Show that as  $e \rightarrow 0$ , with  $a$  remaining fixed, the ellipse approaches a circle.

- 105.** Consider a particle traveling clockwise on the elliptical path

$$\frac{x^2}{100} + \frac{y^2}{25} = 1.$$

The particle leaves the orbit at the point  $(-8, 3)$  and travels in a straight line tangent to the ellipse. At what point will the particle cross the  $y$ -axis?

- 106. Volume** The water tank on a fire truck is 16 feet long, and its cross sections are ellipses. Find the volume of water in the partially filled tank as shown in the figure.



In Exercises 107 and 108, determine the points at which  $dy/dx$  is zero or does not exist to locate the endpoints of the major and minor axes of the ellipse.

**107.**  $16x^2 + 9y^2 + 96x + 36y + 36 = 0$

- 108.**  $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

**Area and Volume** In Exercises 109 and 110, find (a) the area of the region bounded by the ellipse, (b) the volume and surface area of the solid generated by revolving the region about its major axis (prolate spheroid), and (c) the volume and surface area of the solid generated by revolving the region about its minor axis (oblate spheroid).

**109.**  $\frac{x^2}{4} + \frac{y^2}{1} = 1$

**110.**  $\frac{x^2}{16} + \frac{y^2}{9} = 1$

- 111. Arc Length** Use the integration capabilities of a graphing utility to approximate to two-decimal-place accuracy the elliptical integral representing the circumference of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{49} = 1.$$

- 112.** Prove Theorem 10.4 by showing that the tangent line to an ellipse at a point  $P$  makes equal angles with lines through  $P$  and the foci (see figure). [Hint: (1) Find the slope of the tangent line at  $P$ , (2) find the slopes of the lines through  $P$  and each focus, and (3) use the formula for the tangent of the angle between two lines.]

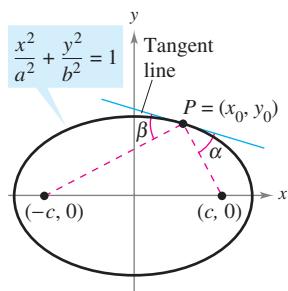


Figure for 112

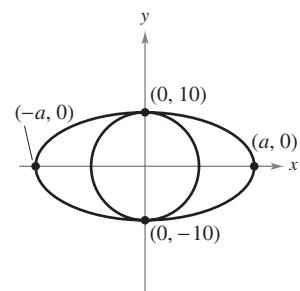


Figure for 113

- 113. Geometry** The area of the ellipse in the figure is twice the area of the circle. What is the length of the major axis?

- 114. Conjecture**

- (a) Show that the equation of an ellipse can be written as

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2(1-e^2)} = 1.$$

- (b) Use a graphing utility to graph the ellipse

$$\frac{(x-2)^2}{4} + \frac{(y-3)^2}{4(1-e^2)} = 1$$

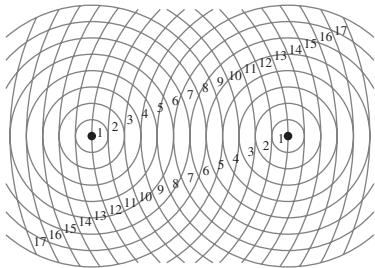
for  $e = 0.95$ ,  $e = 0.75$ ,  $e = 0.5$ ,  $e = 0.25$ , and  $e = 0$ .

- (c) Use the results of part (b) to make a conjecture about the change in the shape of the ellipse as  $e$  approaches 0.

- 115.** Find an equation of the hyperbola such that for any point on the hyperbola, the difference between its distances from the points  $(2, 2)$  and  $(10, 2)$  is 6.

- 116.** Find an equation of the hyperbola such that for any point on the hyperbola, the difference between its distances from the points  $(-3, 0)$  and  $(-3, 3)$  is 2.

- 117.** Sketch the hyperbola that consists of all points  $(x, y)$  such that the difference of the distances between  $(x, y)$  and two fixed points is 10 units, and the foci are located at the centers of the two sets of concentric circles in the figure. To print an enlarged copy of the graph, go to the website [www.mathgraphs.com](http://www.mathgraphs.com).



- 118.** Consider a hyperbola centered at the origin with a horizontal transverse axis. Use the definition of a hyperbola to derive its standard form:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

- 119. Sound Location** A rifle positioned at point  $(-c, 0)$  is fired at a target positioned at point  $(c, 0)$ . A person hears the sound of the rifle and the sound of the bullet hitting the target at the same time. Prove that the person is positioned on one branch of the hyperbola given by

$$\frac{x^2}{c^2 v_s^2/v_m^2} - \frac{y^2}{c^2(v_m^2 - v_s^2)/v_m^2} = 1$$

where  $v_m$  is the muzzle velocity of the rifle and  $v_s$  is the speed of sound, which is about 1100 feet per second.

- 120. Navigation** LORAN (long distance radio navigation) for aircraft and ships uses synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci. Assume that two stations, 300 miles apart, are positioned on a rectangular coordinate system at  $(-150, 0)$  and  $(150, 0)$  and that a ship is traveling on a path with coordinates  $(x, 75)$  (see figure). Find the  $x$ -coordinate of the position of the ship if the time difference between the pulses from the transmitting stations is 1000 microseconds (0.001 second).

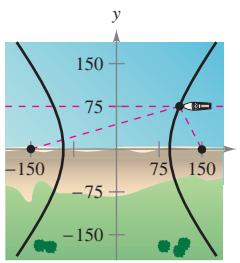


Figure for 120

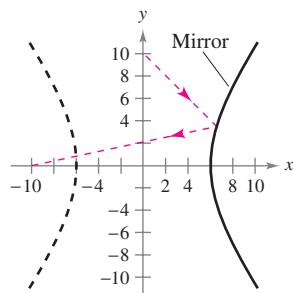


Figure for 121

- 121. Hyperbolic Mirror** A hyperbolic mirror (used in some telescopes) has the property that a light ray directed at the focus will be reflected to the other focus. The mirror in the figure has the equation  $(x^2/36) - (y^2/64) = 1$ . At which point on the mirror will light from the point  $(0, 10)$  be reflected to the other focus?

- 122.** Show that the equation of the tangent line to  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

at the point  $(x_0, y_0)$  is  $(x_0/a^2)x - (y_0/b^2)y = 1$ .

- 123.** Show that the graphs of the equations intersect at right angles:

$$\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \quad \text{and} \quad \frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1.$$

- 124.** Prove that the graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

is one of the following (except in degenerate cases).

Conic	Condition
(a) Circle	$A = C$
(b) Parabola	$A = 0$ or $C = 0$ (but not both)
(c) Ellipse	$AC > 0$
(d) Hyperbola	$AC < 0$

**True or False?** In Exercises 125–130, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

- 125.** It is possible for a parabola to intersect its directrix.

- 126.** The point on a parabola closest to its focus is its vertex.

- 127.** If  $C$  is the circumference of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b < a$$

then  $2\pi b \leq C \leq 2\pi a$ .

- 128.** If  $D \neq 0$  or  $E \neq 0$ , then the graph of  $y^2 - x^2 + Dx + Ey = 0$  is a hyperbola.

- 129.** If the asymptotes of the hyperbola  $(x^2/a^2) - (y^2/b^2) = 1$  intersect at right angles, then  $a = b$ .

- 130.** Every tangent line to a hyperbola intersects the hyperbola only at the point of tangency.

### PUTNAM EXAM CHALLENGE

- 131.** For a point  $P$  on an ellipse, let  $d$  be the distance from the center of the ellipse to the line tangent to the ellipse at  $P$ . Prove that  $(PF_1)(PF_2)d^2$  is constant as  $P$  varies on the ellipse, where  $PF_1$  and  $PF_2$  are the distances from  $P$  to the foci  $F_1$  and  $F_2$  of the ellipse.

- 132.** Find the minimum value of  $(u - v)^2 + \left(\sqrt{2 - u^2} - \frac{9}{v}\right)^2$  for  $0 < u < \sqrt{2}$  and  $v > 0$ .

These problems were composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.

## 10.2 Plane Curves and Parametric Equations

- Sketch the graph of a curve given by a set of parametric equations.
- Eliminate the parameter in a set of parametric equations.
- Find a set of parametric equations to represent a curve.
- Understand two classic calculus problems, the tautochrone and brachistochrone problems.

### Plane Curves and Parametric Equations

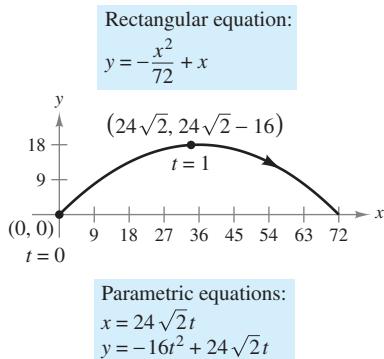


Figure 10.19

Until now, you have been representing a graph by a single equation involving *two* variables. In this section you will study situations in which *three* variables are used to represent a curve in the plane.

Consider the path followed by an object that is propelled into the air at an angle of  $45^\circ$ . If the initial velocity of the object is 48 feet per second, the object travels the parabolic path given by

$$y = -\frac{x^2}{72} + x \quad \text{Rectangular equation}$$

as shown in Figure 10.19. However, this equation does not tell the whole story. Although it does tell you *where* the object has been, it doesn't tell you *when* the object was at a given point  $(x, y)$ . To determine this time, you can introduce a third variable  $t$ , called a **parameter**. By writing both  $x$  and  $y$  as functions of  $t$ , you obtain the **parametric equations**

$$x = 24\sqrt{2}t \quad \text{Parametric equation for } x$$

and

$$y = -16t^2 + 24\sqrt{2}t. \quad \text{Parametric equation for } y$$

From this set of equations, you can determine that at time  $t = 0$ , the object is at the

point  $(0, 0)$ . Similarly, at time  $t = 1$ , the object is at the point  $(24\sqrt{2}, 24\sqrt{2} - 16)$ , and so on. (You will learn a method for determining this particular set of parametric equations—the equations of motion—later, in Section 12.3.)

For this particular motion problem,  $x$  and  $y$  are continuous functions of  $t$ , and the resulting path is called a **plane curve**.

#### DEFINITION OF A PLANE CURVE

If  $f$  and  $g$  are continuous functions of  $t$  on an interval  $I$ , then the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are called **parametric equations** and  $t$  is called the **parameter**. The set of points  $(x, y)$  obtained as  $t$  varies over the interval  $I$  is called the **graph** of the parametric equations. Taken together, the parametric equations and the graph are called a **plane curve**, denoted by  $C$ .

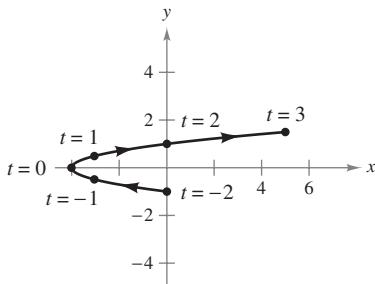
**NOTE** At times it is important to distinguish between a graph (the set of points) and a curve (the points together with their defining parametric equations). When it is important, we will make the distinction explicit. When it is not important, we will use  $C$  to represent the graph or the curve.

When sketching (by hand) a curve represented by a set of parametric equations, you can plot points in the  $xy$ -plane. Each set of coordinates  $(x, y)$  is determined from a value chosen for the parameter  $t$ . By plotting the resulting points in order of increasing values of  $t$ , the curve is traced out in a specific direction. This is called the **orientation** of the curve.

### EXAMPLE 1 Sketching a Curve

Sketch the curve described by the parametric equations

$$x = t^2 - 4 \quad \text{and} \quad y = \frac{t}{2}, \quad -2 \leq t \leq 3.$$



Parametric equations:

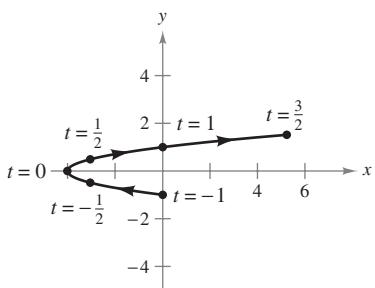
$$x = t^2 - 4 \quad \text{and} \quad y = \frac{t}{2}, \quad -2 \leq t \leq 3$$

Figure 10.20

**Solution** For values of  $t$  on the given interval, the parametric equations yield the points  $(x, y)$  shown in the table.

<b><i>t</i></b>	-2	-1	0	1	2	3
<b><i>x</i></b>	0	-3	-4	-3	0	5
<b><i>y</i></b>	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

By plotting these points in order of increasing  $t$  and using the continuity of  $f$  and  $g$ , you obtain the curve  $C$  shown in Figure 10.20. Note that the arrows on the curve indicate its orientation as  $t$  increases from  $-2$  to  $3$ . ■



Parametric equations:

$$x = 4t^2 - 4 \quad \text{and} \quad y = t, \quad -1 \leq t \leq \frac{3}{2}$$

Figure 10.21

**NOTE** From the Vertical Line Test, you can see that the graph shown in Figure 10.20 does not define  $y$  as a function of  $x$ . This points out one benefit of parametric equations—they can be used to represent graphs that are more general than graphs of functions. ■

It often happens that two different sets of parametric equations have the same graph. For example, the set of parametric equations

$$x = 4t^2 - 4 \quad \text{and} \quad y = t, \quad -1 \leq t \leq \frac{3}{2}$$

has the same graph as the set given in Example 1. (See Figure 10.21.) However, comparing the values of  $t$  in Figures 10.20 and 10.21, you can see that the second graph is traced out more *rapidly* (considering  $t$  as time) than the first graph. So, in applications, different parametric representations can be used to represent various speeds at which objects travel along a given path.

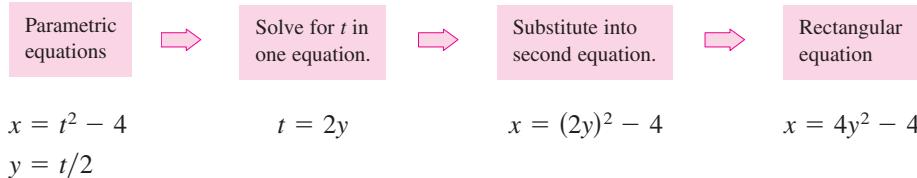
**TECHNOLOGY** Most graphing utilities have a *parametric* graphing mode. If you have access to such a utility, use it to confirm the graphs shown in Figures 10.20 and 10.21. Does the curve given by the parametric equations

$$x = 4t^2 - 8t \quad \text{and} \quad y = 1 - t, \quad -\frac{1}{2} \leq t \leq 2$$

represent the same graph as that shown in Figures 10.20 and 10.21? What do you notice about the *orientation* of this curve?

## Eliminating the Parameter

Finding a rectangular equation that represents the graph of a set of parametric equations is called **eliminating the parameter**. For instance, you can eliminate the parameter from the set of parametric equations in Example 1 as follows.



Once you have eliminated the parameter, you can recognize that the equation  $x = 4y^2 - 4$  represents a parabola with a horizontal axis and vertex at  $(-4, 0)$ , as shown in Figure 10.20.

The range of  $x$  and  $y$  implied by the parametric equations may be altered by the change to rectangular form. In such instances the domain of the rectangular equation must be adjusted so that its graph matches the graph of the parametric equations. Such a situation is demonstrated in the next example.

### EXAMPLE 2 Adjusting the Domain After Eliminating the Parameter

Sketch the curve represented by the equations

$$x = \frac{1}{\sqrt{t+1}} \quad \text{and} \quad y = \frac{t}{t+1}, \quad t > -1$$

by eliminating the parameter and adjusting the domain of the resulting rectangular equation.

**Solution** Begin by solving one of the parametric equations for  $t$ . For instance, you can solve the first equation for  $t$  as follows.

$$\begin{aligned} x &= \frac{1}{\sqrt{t+1}} && \text{Parametric equation for } x \\ x^2 &= \frac{1}{t+1} && \text{Square each side.} \\ t+1 &= \frac{1}{x^2} \\ t &= \frac{1}{x^2} - 1 = \frac{1-x^2}{x^2} && \text{Solve for } t. \end{aligned}$$

Now, substituting into the parametric equation for  $y$  produces

$$\begin{aligned} y &= \frac{t}{t+1} && \text{Parametric equation for } y \\ y &= \frac{(1-x^2)/x^2}{[(1-x^2)/x^2] + 1} && \text{Substitute } (1-x^2)/x^2 \text{ for } t. \\ y &= 1-x^2. && \text{Simplify.} \end{aligned}$$

The rectangular equation,  $y = 1 - x^2$ , is defined for all values of  $x$ , but from the parametric equation for  $x$  you can see that the curve is defined only when  $t > -1$ . This implies that you should restrict the domain of  $x$  to positive values, as shown in Figure 10.22. ■

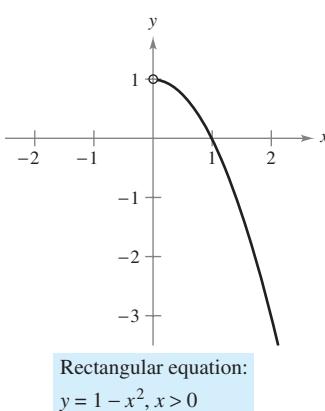
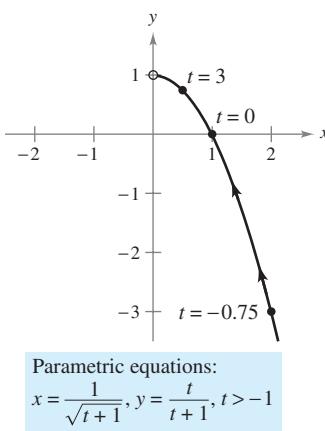


Figure 10.22

It is not necessary for the parameter in a set of parametric equations to represent time. The next example uses an *angle* as the parameter.

### EXAMPLE 3 Using Trigonometry to Eliminate a Parameter

Sketch the curve represented by

$$x = 3 \cos \theta \quad \text{and} \quad y = 4 \sin \theta, \quad 0 \leq \theta \leq 2\pi$$

by eliminating the parameter and finding the corresponding rectangular equation.

**Solution** Begin by solving for  $\cos \theta$  and  $\sin \theta$  in the given equations.

$$\cos \theta = \frac{x}{3} \quad \text{and} \quad \sin \theta = \frac{y}{4} \quad \text{Solve for } \cos \theta \text{ and } \sin \theta.$$

Next, make use of the identity  $\sin^2 \theta + \cos^2 \theta = 1$  to form an equation involving only  $x$  and  $y$ .

$$\cos^2 \theta + \sin^2 \theta = 1 \quad \text{Trigonometric identity}$$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1 \quad \text{Substitute.}$$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1 \quad \text{Rectangular equation}$$

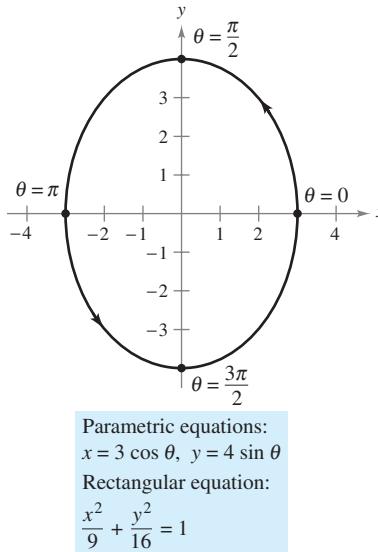


Figure 10.23

From this rectangular equation you can see that the graph is an ellipse centered at  $(0, 0)$ , with vertices at  $(0, 4)$  and  $(0, -4)$  and minor axis of length  $2b = 6$ , as shown in Figure 10.23. Note that the ellipse is traced out *counterclockwise* as  $\theta$  varies from  $0$  to  $2\pi$ . ■

Using the technique shown in Example 3, you can conclude that the graph of the parametric equations

$$x = h + a \cos \theta \quad \text{and} \quad y = k + b \sin \theta, \quad 0 \leq \theta \leq 2\pi$$

is the ellipse (traced counterclockwise) given by

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.$$

The graph of the parametric equations

$$x = h + a \sin \theta \quad \text{and} \quad y = k + b \cos \theta, \quad 0 \leq \theta \leq 2\pi$$

is also the ellipse (traced clockwise) given by

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.$$

Use a graphing utility in *parametric* mode to graph several ellipses.

In Examples 2 and 3, it is important to realize that eliminating the parameter is primarily an *aid to curve sketching*. If the parametric equations represent the path of a moving object, the graph alone is not sufficient to describe the object's motion. You still need the parametric equations to tell you the *position, direction, and speed* at a given time.

## Finding Parametric Equations

The first three examples in this section illustrate techniques for sketching the graph represented by a set of parametric equations. You will now investigate the reverse problem. How can you determine a set of parametric equations for a given graph or a given physical description? From the discussion following Example 1, you know that such a representation is not unique. This is demonstrated further in the following example, which finds two different parametric representations for a given graph.

### EXAMPLE 4 Finding Parametric Equations for a Given Graph

Find a set of parametric equations that represents the graph of  $y = 1 - x^2$ , using each of the following parameters.

- a.  $t = x$       b. The slope  $m = \frac{dy}{dx}$  at the point  $(x, y)$

#### Solution

- a. Letting  $x = t$  produces the parametric equations

$$x = t \quad \text{and} \quad y = 1 - x^2 = 1 - t^2.$$

- b. To write  $x$  and  $y$  in terms of the parameter  $m$ , you can proceed as follows.

$$m = \frac{dy}{dx} = -2x \quad \text{Differentiate } y = 1 - x^2.$$

$$x = -\frac{m}{2} \quad \text{Solve for } x.$$

This produces a parametric equation for  $x$ . To obtain a parametric equation for  $y$ , substitute  $-m/2$  for  $x$  in the original equation.

$$y = 1 - x^2 \quad \text{Write original rectangular equation.}$$

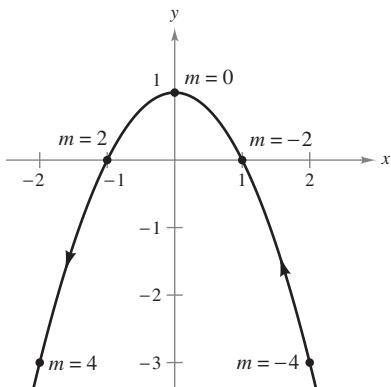
$$y = 1 - \left(-\frac{m}{2}\right)^2 \quad \text{Substitute } -m/2 \text{ for } x.$$

$$y = 1 - \frac{m^2}{4} \quad \text{Simplify.}$$

So, the parametric equations are

$$x = -\frac{m}{2} \quad \text{and} \quad y = 1 - \frac{m^2}{4}.$$

In Figure 10.24, note that the resulting curve has a right-to-left orientation as determined by the direction of increasing values of slope  $m$ . For part (a), the curve would have the opposite orientation. ■



Rectangular equation:  $y = 1 - x^2$

Parametric equations:

$$x = -\frac{m}{2}, y = 1 - \frac{m^2}{4}$$

**Figure 10.24**

**TECHNOLOGY** To be efficient at using a graphing utility, it is important that you develop skill in representing a graph by a set of parametric equations. The reason for this is that many graphing utilities have only three graphing modes—(1) functions, (2) parametric equations, and (3) polar equations. Most graphing utilities are not programmed to graph a general equation. For instance, suppose you want to graph the hyperbola  $x^2 - y^2 = 1$ . To graph the hyperbola in *function* mode, you need two equations:  $y = \sqrt{x^2 - 1}$  and  $y = -\sqrt{x^2 - 1}$ . In *parametric* mode, you can represent the graph by  $x = \sec t$  and  $y = \tan t$ .

**CYCLOIDS**

Galileo first called attention to the cycloid, once recommending that it be used for the arches of bridges. Pascal once spent 8 days attempting to solve many of the problems of cycloids, such as finding the area under one arch, and the volume of the solid of revolution formed by revolving the curve about a line. The cycloid has so many interesting properties and has caused so many quarrels among mathematicians that it has been called “the Helen of geometry” and “the apple of discord.”

**FOR FURTHER INFORMATION** For more information on cycloids, see the article “The Geometry of Rolling Curves” by John Bloom and Lee Whitt in *The American Mathematical Monthly*. To view this article, go to the website [www.matharticles.com](http://www.matharticles.com).

**EXAMPLE 5** **Parametric Equations for a Cycloid**

Determine the curve traced by a point  $P$  on the circumference of a circle of radius  $a$  rolling along a straight line in a plane. Such a curve is called a **cycloid**.

**Solution** Let the parameter  $\theta$  be the measure of the circle’s rotation, and let the point  $P = (x, y)$  begin at the origin. When  $\theta = 0$ ,  $P$  is at the origin. When  $\theta = \pi$ ,  $P$  is at a maximum point  $(\pi a, 2a)$ . When  $\theta = 2\pi$ ,  $P$  is back on the  $x$ -axis at  $(2\pi a, 0)$ . From Figure 10.25, you can see that  $\angle APC = 180^\circ - \theta$ . So,

$$\sin \theta = \sin(180^\circ - \theta) = \sin(\angle APC) = \frac{AC}{a} = \frac{BD}{a}$$

$$\cos \theta = -\cos(180^\circ - \theta) = -\cos(\angle APC) = \frac{AP}{-a}$$

which implies that  $AP = -a \cos \theta$  and  $BD = a \sin \theta$ .

Because the circle rolls along the  $x$ -axis, you know that  $OD = PD = a\theta$ . Furthermore, because  $BA = DC = a$ , you have

$$x = OD - BD = a\theta - a \sin \theta$$

$$y = BA + AP = a - a \cos \theta.$$

So, the parametric equations are  $x = a(\theta - \sin \theta)$  and  $y = a(1 - \cos \theta)$ .

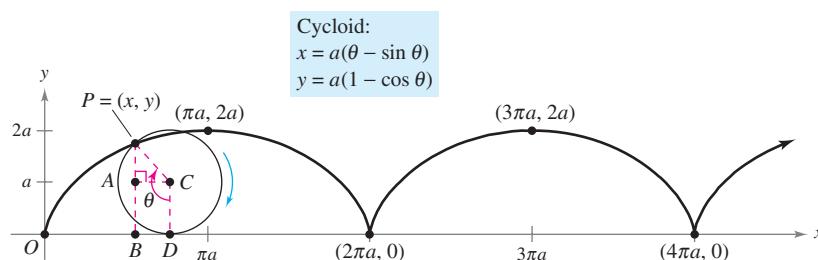


Figure 10.25

**TECHNOLOGY** Some graphing utilities allow you to simulate the motion of an object that is moving in the plane or in space. If you have access to such a utility, use it to trace out the path of the cycloid shown in Figure 10.25.

The cycloid in Figure 10.25 has sharp corners at the values  $x = 2n\pi a$ . Notice that the derivatives  $x'(\theta)$  and  $y'(\theta)$  are both zero at the points for which  $\theta = 2n\pi$ .

$$x(\theta) = a(\theta - \sin \theta) \quad y(\theta) = a(1 - \cos \theta)$$

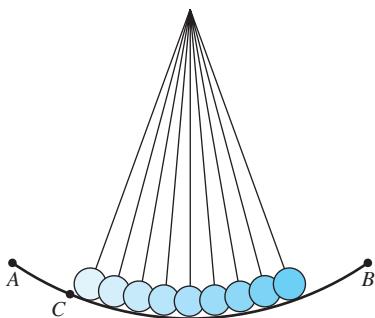
$$x'(\theta) = a - a \cos \theta \quad y'(\theta) = a \sin \theta$$

$$x'(2n\pi) = 0 \quad y'(2n\pi) = 0$$

Between these points, the cycloid is called **smooth**.

**DEFINITION OF A SMOOTH CURVE**

A curve  $C$  represented by  $x = f(t)$  and  $y = g(t)$  on an interval  $I$  is called **smooth** if  $f'$  and  $g'$  are continuous on  $I$  and not simultaneously 0, except possibly at the endpoints of  $I$ . The curve  $C$  is called **piecewise smooth** if it is smooth on each subinterval of some partition of  $I$ .



The time required to complete a full swing of the pendulum when starting from point  $C$  is only approximately the same as when starting from point  $A$ .

**Figure 10.26**

## The Tautochrone and Brachistochrone Problems

The type of curve described in Example 5 is related to one of the most famous pairs of problems in the history of calculus. The first problem (called the **tautochrone problem**) began with Galileo's discovery that the time required to complete a full swing of a given pendulum is *approximately* the same whether it makes a large movement at high speed or a small movement at lower speed (see Figure 10.26). Late in his life, Galileo (1564–1642) realized that he could use this principle to construct a clock. However, he was not able to conquer the mechanics of actual construction. Christian Huygens (1629–1695) was the first to design and construct a working model. In his work with pendulums, Huygens realized that a pendulum does not take exactly the same time to complete swings of varying lengths. (This doesn't affect a pendulum clock, because the length of the circular arc is kept constant by giving the pendulum a slight boost each time it passes its lowest point.) But, in studying the problem, Huygens discovered that a ball rolling back and forth on an inverted cycloid does complete each cycle in exactly the same time.



An inverted cycloid is the path down which a ball will roll in the shortest time.

**Figure 10.27**

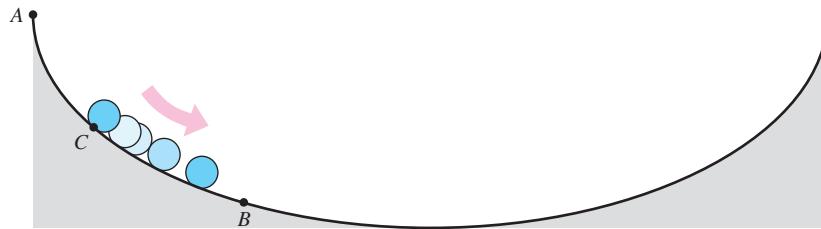
The Granger Collection



**JAMES BERNOULLI (1654–1705)**

James Bernoulli, also called Jacques, was the older brother of John. He was one of several accomplished mathematicians of the Swiss Bernoulli family. James's mathematical accomplishments have given him a prominent place in the early development of calculus.

The second problem, which was posed by John Bernoulli in 1696, is called the **brachistochrone problem**—in Greek, *brachys* means short and *chronos* means time. The problem was to determine the path down which a particle will slide from point  $A$  to point  $B$  in the *shortest time*. Several mathematicians took up the challenge, and the following year the problem was solved by Newton, Leibniz, L'Hôpital, John Bernoulli, and James Bernoulli. As it turns out, the solution is not a straight line from  $A$  to  $B$ , but an inverted cycloid passing through the points  $A$  and  $B$ , as shown in Figure 10.27. The amazing part of the solution is that a particle starting at rest at *any* other point  $C$  of the cycloid between  $A$  and  $B$  will take exactly the same time to reach  $B$ , as shown in Figure 10.28.



A ball starting at point  $C$  takes the same time to reach point  $B$  as one that starts at point  $A$ .

**Figure 10.28**

■ **FOR FURTHER INFORMATION** To see a proof of the famous brachistochrone problem, see the article “A New Minimization Proof for the Brachistochrone” by Gary Lawlor in *The American Mathematical Monthly*. To view this article, go to the website [www.matharticles.com](http://www.matharticles.com).

## 10.2 Exercises

See [www.CalcChat.com](http://www.CalcChat.com) for worked-out solutions to odd-numbered exercises.

- Consider the parametric equations  $x = \sqrt{t}$  and  $y = 3 - t$ .
  - Construct a table of values for  $t = 0, 1, 2, 3$ , and  $4$ .
  - Plot the points  $(x, y)$  generated in the table, and sketch a graph of the parametric equations. Indicate the orientation of the graph.
  - Use a graphing utility to confirm your graph in part (b).
  - Find the rectangular equation by eliminating the parameter, and sketch its graph. Compare the graph in part (b) with the graph of the rectangular equation.
- Consider the parametric equations  $x = 4 \cos^2 \theta$  and  $y = 2 \sin \theta$ .
  - Construct a table of values for  $\theta = -\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}$ , and  $\frac{\pi}{2}$ .
  - Plot the points  $(x, y)$  generated in the table, and sketch a graph of the parametric equations. Indicate the orientation of the graph.
  - Use a graphing utility to confirm your graph in part (b).
  - Find the rectangular equation by eliminating the parameter, and sketch its graph. Compare the graph in part (b) with the graph of the rectangular equation.
  - If values of  $\theta$  were selected from the interval  $[\pi/2, 3\pi/2]$  for the table in part (a), would the graph in part (b) be different? Explain.

In Exercises 3–20, sketch the curve represented by the parametric equations (indicate the orientation of the curve), and write the corresponding rectangular equation by eliminating the parameter.

- $x = 2t - 3$ ,  $y = 3t + 1$
- $x = 5 - 4t$ ,  $y = 2 + 5t$
- $x = t + 1$ ,  $y = t^2$
- $x = 2t^2$ ,  $y = t^4 + 1$
- $x = t^3$ ,  $y = \frac{t^2}{2}$
- $x = t^2 + t$ ,  $y = t^2 - t$
- $x = \sqrt{t}$ ,  $y = t - 5$
- $x = \sqrt[4]{t}$ ,  $y = 8 - t$
- $x = t - 3$ ,  $y = \frac{t}{t - 3}$
- $x = 1 + \frac{1}{t}$ ,  $y = t - 1$
- $x = 2t$ ,  $y = |t - 2|$
- $x = |t - 1|$ ,  $y = t + 2$
- $x = e^t$ ,  $y = e^{3t} + 1$
- $x = e^{-t}$ ,  $y = e^{2t} - 1$
- $x = \sec \theta$ ,  $y = \cos \theta$ ,  $0 \leq \theta < \pi/2$ ,  $\pi/2 < \theta \leq \pi$
- $x = \tan^2 \theta$ ,  $y = \sec^2 \theta$
- $x = 8 \cos \theta$ ,  $y = 8 \sin \theta$
- $x = 3 \cos \theta$ ,  $y = 7 \sin \theta$

 In Exercises 21–32, use a graphing utility to graph the curve represented by the parametric equations (indicate the orientation of the curve). Eliminate the parameter and write the corresponding rectangular equation.

- $x = 6 \sin 2\theta$ ,  $y = 4 \cos 2\theta$
- $x = \cos \theta$ ,  $y = 2 \sin 2\theta$
- $x = 4 + 2 \cos \theta$
- $y = -1 + \sin \theta$
- $x = -2 + 3 \cos \theta$
- $y = -5 + 3 \sin \theta$

- $x = -3 + 4 \cos \theta$
- $y = 2 + 5 \sin \theta$
- $x = 4 \sec \theta$ ,  $y = 3 \tan \theta$
- $x = t^3$ ,  $y = 3 \ln t$
- $x = e^{-t}$ ,  $y = e^{3t}$
- $x = \sec \theta$
- $y = \tan \theta$
- $x = \cos^3 \theta$ ,  $y = \sin^3 \theta$
- $x = \ln 2t$ ,  $y = t^2$
- $x = e^{2t}$ ,  $y = e^t$

**Comparing Plane Curves** In Exercises 33–36, determine any differences between the curves of the parametric equations. Are the graphs the same? Are the orientations the same? Are the curves smooth? Explain.

- (a)  $x = t$   
 $y = 2t + 1$
- (b)  $x = \cos \theta$   
 $y = 2 \cos \theta + 1$
- (c)  $x = e^{-t}$   
 $y = 2e^{-t} + 1$
- (d)  $x = e^t$   
 $y = 2e^t + 1$
- (a)  $x = 2 \cos \theta$   
 $y = 2 \sin \theta$
- (b)  $x = \sqrt{4t^2 - 1}/|t|$   
 $y = 1/t$
- (c)  $x = \sqrt{t}$   
 $y = \sqrt{4 - t}$
- (d)  $x = -\sqrt{4 - e^{2t}}$   
 $y = e^t$
- (a)  $x = \cos \theta$   
 $y = 2 \sin^2 \theta$   
 $0 < \theta < \pi$
- (b)  $x = \cos(-\theta)$   
 $y = 2 \sin^2(-\theta)$   
 $0 < \theta < \pi$
- (a)  $x = t + 1$ ,  $y = t^3$
- (b)  $x = -t + 1$ ,  $y = (-t)^3$

 37. *Conjecture*

- Use a graphing utility to graph the curves represented by the two sets of parametric equations.  
 $x = 4 \cos t$      $x = 4 \cos(-t)$   
 $y = 3 \sin t$      $y = 3 \sin(-t)$
- Describe the change in the graph when the sign of the parameter is changed.
- Make a conjecture about the change in the graph of parametric equations when the sign of the parameter is changed.
- Test your conjecture with another set of parametric equations.

- Writing** Review Exercises 33–36 and write a short paragraph describing how the graphs of curves represented by different sets of parametric equations can differ even though eliminating the parameter from each yields the same rectangular equation.

In Exercises 39–42, eliminate the parameter and obtain the standard form of the rectangular equation.

- Line through  $(x_1, y_1)$  and  $(x_2, y_2)$ :  
 $x = x_1 + t(x_2 - x_1)$ ,  $y = y_1 + t(y_2 - y_1)$
- Circle:  $x = h + r \cos \theta$ ,  $y = k + r \sin \theta$
- Ellipse:  $x = h + a \cos \theta$ ,  $y = k + b \sin \theta$
- Hyperbola:  $x = h + a \sec \theta$ ,  $y = k + b \tan \theta$

In Exercises 43–50, use the results of Exercises 39–42 to find a set of parametric equations for the line or conic.

43. Line: passes through  $(0, 0)$  and  $(4, -7)$
44. Line: passes through  $(1, 4)$  and  $(5, -2)$
45. Circle: center:  $(3, 1)$ ; radius: 2
46. Circle: center:  $(-6, 2)$ ; radius: 4
47. Ellipse: vertices:  $(\pm 10, 0)$ ; foci:  $(\pm 8, 0)$
48. Ellipse: vertices:  $(4, 7), (4, -3)$ ; foci:  $(4, 5), (4, -1)$
49. Hyperbola: vertices:  $(\pm 4, 0)$ ; foci:  $(\pm 5, 0)$
50. Hyperbola: vertices:  $(0, \pm 1)$ ; foci:  $(0, \pm 2)$

In Exercises 51–54, find two different sets of parametric equations for the rectangular equation.

51.  $y = 6x - 5$
52.  $y = 4/(x - 1)$
53.  $y = x^3$
54.  $y = x^2$

In Exercises 55–58, find a set of parametric equations for the rectangular equation that satisfies the given condition.

55.  $y = 2x - 5$ ,  $t = 0$  at the point  $(3, 1)$
56.  $y = 4x + 1$ ,  $t = -1$  at the point  $(-2, -7)$
57.  $y = x^2$ ,  $t = 4$  at the point  $(4, 16)$
58.  $y = 4 - x^2$ ,  $t = 1$  at the point  $(1, 3)$

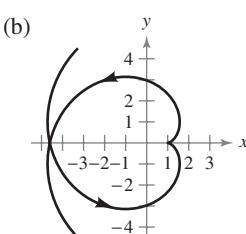
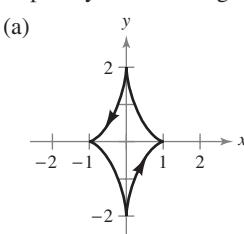


In Exercises 59–66, use a graphing utility to graph the curve represented by the parametric equations. Indicate the direction of the curve. Identify any points at which the curve is not smooth.

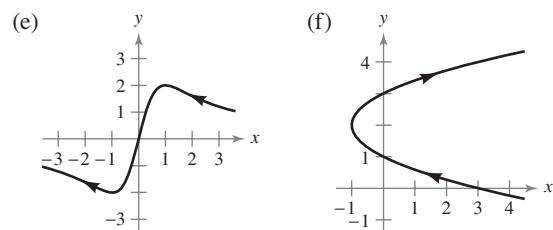
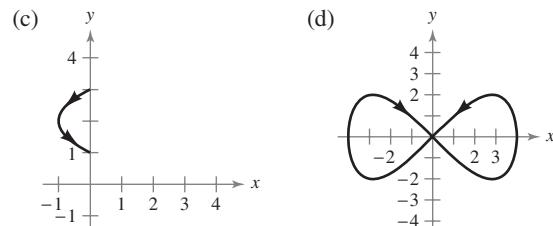
59. Cycloid:  $x = 2(\theta - \sin \theta)$ ,  $y = 2(1 - \cos \theta)$
60. Cycloid:  $x = \theta + \sin \theta$ ,  $y = 1 - \cos \theta$
61. Prolate cycloid:  $x = \theta - \frac{3}{2} \sin \theta$ ,  $y = 1 - \frac{3}{2} \cos \theta$
62. Prolate cycloid:  $x = 2\theta - 4 \sin \theta$ ,  $y = 2 - 4 \cos \theta$
63. Hypocycloid:  $x = 3 \cos^3 \theta$ ,  $y = 3 \sin^3 \theta$
64. Curtate cycloid:  $x = 2\theta - \sin \theta$ ,  $y = 2 - \cos \theta$
65. Witch of Agnesi:  $x = 2 \cot \theta$ ,  $y = 2 \sin^2 \theta$
66. Folium of Descartes:  $x = 3t/(1 + t^3)$ ,  $y = 3t^2/(1 + t^3)$

### WRITING ABOUT CONCEPTS

67. Explain the process of sketching a plane curve given by parametric equations. What is meant by the orientation of the curve?
68. Match each set of parametric equations with the correct graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).] Explain your reasoning.



### WRITING ABOUT CONCEPTS (continued)



- (e)  $x = t^2 - 1$ ,  $y = t + 2$
- (ii)  $x = \sin^2 \theta - 1$ ,  $y = \sin \theta + 2$
- (iii) Lissajous curve:  $x = 4 \cos \theta$ ,  $y = 2 \sin 2\theta$
- (iv) Evolute of ellipse:  $x = \cos^3 \theta$ ,  $y = 2 \sin^3 \theta$
- (v) Involute of circle:  $x = \cos \theta + \theta \sin \theta$ ,  $y = \sin \theta - \theta \cos \theta$
- (vi) Serpentine curve:  $x = \cot \theta$ ,  $y = 4 \sin \theta \cos \theta$

69. **Curtate Cycloid** A wheel of radius  $a$  rolls along a line without slipping. The curve traced by a point  $P$  that is  $b$  units from the center ( $b < a$ ) is called a **curtate cycloid** (see figure). Use the angle  $\theta$  to find a set of parametric equations for this curve.

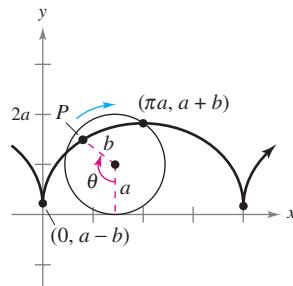


Figure for 69

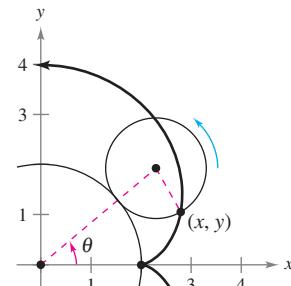


Figure for 70

70. **Epicycloid** A circle of radius 1 rolls around the outside of a circle of radius 2 without slipping. The curve traced by a point on the circumference of the smaller circle is called an **epicycloid** (see figure). Use the angle  $\theta$  to find a set of parametric equations for this curve.

**True or False?** In Exercises 71–73, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

71. The graph of the parametric equations  $x = t^2$  and  $y = t^2$  is the line  $y = x$ .
72. If  $y$  is a function of  $t$  and  $x$  is a function of  $t$ , then  $y$  is a function of  $x$ .

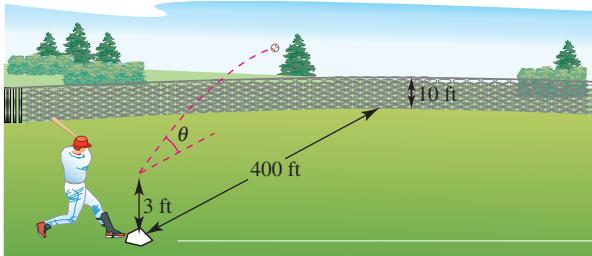
73. The curve represented by the parametric equations  $x = t$  and  $y = \cos t$  can be written as an equation of the form  $y = f(x)$ .

**CAPSTONE**

74. Consider the parametric equations  $x = 8 \cos t$  and  $y = 8 \sin t$ .
- Describe the curve represented by the parametric equations.
  - How does the curve represented by the parametric equations  $x = 8 \cos t + 3$  and  $y = 8 \sin t + 6$  compare to the curve described in part (a)?
  - How does the original curve change when cosine and sine are interchanged?

**Projectile Motion** In Exercises 75 and 76, consider a projectile launched at a height  $h$  feet above the ground and at an angle  $\theta$  with the horizontal. If the initial velocity is  $v_0$  feet per second, the path of the projectile is modeled by the parametric equations  $x = (v_0 \cos \theta)t$  and  $y = h + (v_0 \sin \theta)t - 16t^2$ .

-  75. The center field fence in a ballpark is 10 feet high and 400 feet from home plate. The ball is hit 3 feet above the ground. It leaves the bat at an angle of  $\theta$  degrees with the horizontal at a speed of 100 miles per hour (see figure).



- Write a set of parametric equations for the path of the ball.
- Use a graphing utility to graph the path of the ball when  $\theta = 15^\circ$ . Is the hit a home run?
- Use a graphing utility to graph the path of the ball when  $\theta = 23^\circ$ . Is the hit a home run?
- Find the minimum angle at which the ball must leave the bat in order for the hit to be a home run.

-  76. A rectangular equation for the path of a projectile is  $y = 5 + x - 0.005x^2$ .

- Eliminate the parameter  $t$  from the position function for the motion of a projectile to show that the rectangular equation is

$$y = -\frac{16 \sec^2 \theta}{v_0^2} x^2 + (\tan \theta) x + h.$$

- Use the result of part (a) to find  $h$ ,  $v_0$ , and  $\theta$ . Find the parametric equations of the path.
- Use a graphing utility to graph the rectangular equation for the path of the projectile. Confirm your answer in part (b) by sketching the curve represented by the parametric equations.
- Use a graphing utility to approximate the maximum height of the projectile and its range.

**SECTION PROJECT****Cycloids**

In Greek, the word *cycloid* means *wheel*, the word *hypocycloid* means *under the wheel*, and the word *epicycloid* means *upon the wheel*. Match the hypocycloid or epicycloid with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

**Hypocycloid,  $H(A, B)$** 

The path traced by a fixed point on a circle of radius  $B$  as it rolls around the inside of a circle of radius  $A$

$$x = (A - B) \cos t + B \cos\left(\frac{A - B}{B}\right)t$$

$$y = (A - B) \sin t - B \sin\left(\frac{A - B}{B}\right)t$$

**Epicycloid,  $E(A, B)$** 

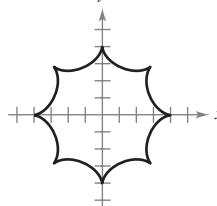
The path traced by a fixed point on a circle of radius  $B$  as it rolls around the outside of a circle of radius  $A$

$$x = (A + B) \cos t - B \cos\left(\frac{A + B}{B}\right)t$$

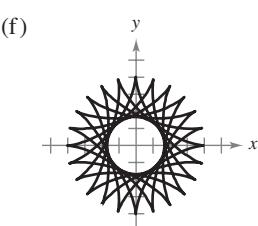
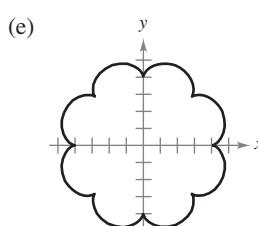
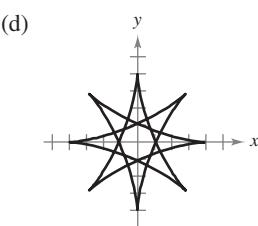
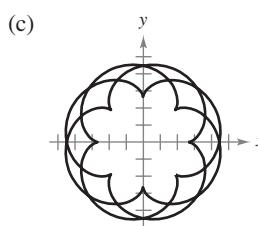
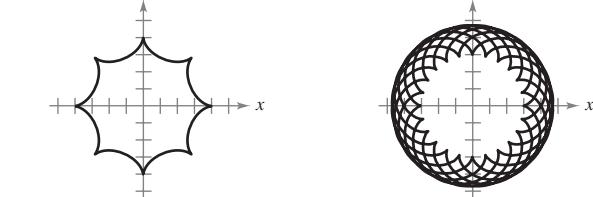
$$y = (A + B) \sin t - B \sin\left(\frac{A + B}{B}\right)t$$

- I.  $H(8, 3)$       II.  $E(8, 3)$

- IV.  $E(24, 3)$       V.  $H(24, 7)$

- (a) 

- III.  $H(8, 7)$       VI.  $E(24, 7)$

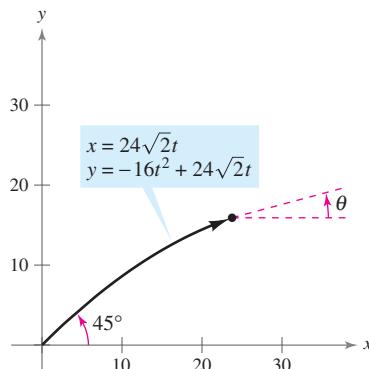


Exercises based on “Mathematical Discovery via Computer Graphics: Hypocycloids and Epicycloids” by Florence S. Gordon and Sheldon P. Gordon, *College Mathematics Journal*, November 1984, p.441. Used by permission of the authors.

## 10.3 Parametric Equations and Calculus

- Find the slope of a tangent line to a curve given by a set of parametric equations.
- Find the arc length of a curve given by a set of parametric equations.
- Find the area of a surface of revolution (parametric form).

### Slope and Tangent Lines



At time  $t$ , the angle of elevation of the projectile is  $\theta$ , the slope of the tangent line at that point.

**Figure 10.29**

Now that you can represent a graph in the plane by a set of parametric equations, it is natural to ask how to use calculus to study plane curves. To begin, let's take another look at the projectile represented by the parametric equations

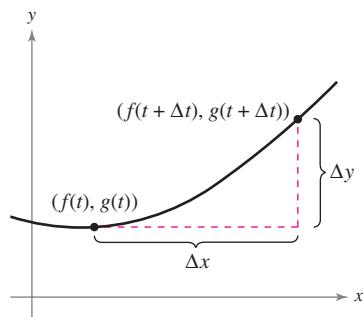
$$x = 24\sqrt{2}t \quad \text{and} \quad y = -16t^2 + 24\sqrt{2}t$$

as shown in Figure 10.29. From the discussion at the beginning of Section 10.2, you know that these equations enable you to locate the position of the projectile at a given time. You also know that the object is initially projected at an angle of  $45^\circ$ . But how can you find the angle  $\theta$  representing the object's direction at some other time  $t$ ? The following theorem answers this question by giving a formula for the slope of the tangent line as a function of  $t$ .

#### THEOREM 10.7 PARAMETRIC FORM OF THE DERIVATIVE

If a smooth curve  $C$  is given by the equations  $x = f(t)$  and  $y = g(t)$ , then the slope of  $C$  at  $(x, y)$  is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0.$$



The slope of the secant line through the points  $(f(t), g(t))$  and  $(f(t + \Delta t), g(t + \Delta t))$  is  $\Delta y/\Delta x$ .

**Figure 10.30**

**PROOF** In Figure 10.30, consider  $\Delta t > 0$  and let

$$\Delta y = g(t + \Delta t) - g(t) \quad \text{and} \quad \Delta x = f(t + \Delta t) - f(t).$$

Because  $\Delta x \rightarrow 0$  as  $\Delta t \rightarrow 0$ , you can write

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta t \rightarrow 0} \frac{g(t + \Delta t) - g(t)}{f(t + \Delta t) - f(t)}. \end{aligned}$$

Dividing both the numerator and denominator by  $\Delta t$ , you can use the differentiability of  $f$  and  $g$  to conclude that

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta t \rightarrow 0} \frac{[g(t + \Delta t) - g(t)]/\Delta t}{[f(t + \Delta t) - f(t)]/\Delta t} \\ &= \frac{\lim_{\Delta t \rightarrow 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}}{\lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}} \\ &= \frac{g'(t)}{f'(t)} \\ &= \frac{dy/dt}{dx/dt}. \end{aligned}$$

**EXAMPLE 1** Differentiation and Parametric Form

Find  $dy/dx$  for the curve given by  $x = \sin t$  and  $y = \cos t$ .

**STUDY TIP** The curve traced out in Example 1 is a circle. Use the formula

$$\frac{dy}{dx} = -\tan t$$

to find the slopes at the points  $(1, 0)$  and  $(0, 1)$ .

**Solution**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{\cos t} = -\tan t$$

Because  $dy/dx$  is a function of  $t$ , you can use Theorem 10.7 repeatedly to find higher-order derivatives. For instance,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx} \left[ \frac{dy}{dx} \right] = \frac{d}{dt} \left[ \frac{dy}{dx} \right] \Big|_{dx/dt} \\ \frac{d^3y}{dx^3} &= \frac{d}{dx} \left[ \frac{d^2y}{dx^2} \right] = \frac{d}{dt} \left[ \frac{d^2y}{dx^2} \right] \Big|_{dx/dt}.\end{aligned}$$

Second derivative

Third derivative

**EXAMPLE 2** Finding Slope and Concavity

For the curve given by

$$x = \sqrt{t} \quad \text{and} \quad y = \frac{1}{4}(t^2 - 4), \quad t \geq 0$$

find the slope and concavity at the point  $(2, 3)$ .

**Solution** Because

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(1/2)t}{(1/2)t^{-1/2}} = t^{3/2}$$

Parametric form of first derivative

you can find the second derivative to be

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[ \frac{dy}{dx} \right] = \frac{d}{dt} \left[ t^{3/2} \right] \Big|_{dx/dt} = \frac{(3/2)t^{1/2}}{(1/2)t^{-1/2}} = 3t.$$

Parametric form of second derivative

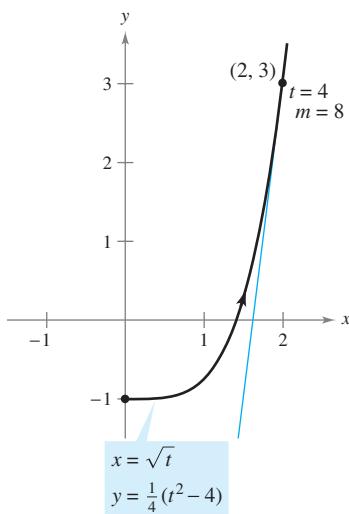
At  $(x, y) = (2, 3)$ , it follows that  $t = 4$ , and the slope is

$$\frac{dy}{dx} = (4)^{3/2} = 8.$$

Moreover, when  $t = 4$ , the second derivative is

$$\frac{d^2y}{dx^2} = 3(4) = 12 > 0$$

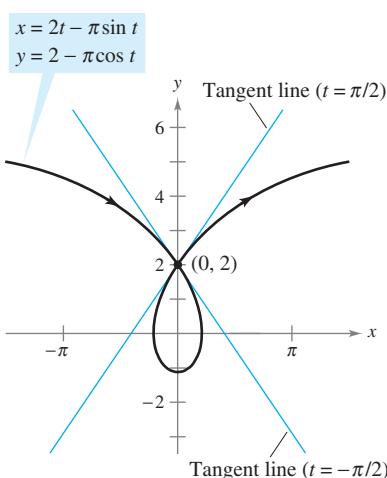
and you can conclude that the graph is concave upward at  $(2, 3)$ , as shown in Figure 10.31. ■



The graph is concave upward at  $(2, 3)$ , when  $t = 4$ .

Figure 10.31

Because the parametric equations  $x = f(t)$  and  $y = g(t)$  need not define  $y$  as a function of  $x$ , it is possible for a plane curve to loop around and cross itself. At such points the curve may have more than one tangent line, as shown in the next example.



This prolate cycloid has two tangent lines at the point  $(0, 2)$ .

Figure 10.32

### EXAMPLE 3 A Curve with Two Tangent Lines at a Point

The **prolate cycloid** given by

$$x = 2t - \pi \sin t \quad \text{and} \quad y = 2 - \pi \cos t$$

crosses itself at the point  $(0, 2)$ , as shown in Figure 10.32. Find the equations of both tangent lines at this point.

**Solution** Because  $x = 0$  and  $y = 2$  when  $t = \pm\pi/2$ , and

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\pi \sin t}{2 - \pi \cos t}$$

you have  $dy/dx = -\pi/2$  when  $t = -\pi/2$  and  $dy/dx = \pi/2$  when  $t = \pi/2$ . So, the two tangent lines at  $(0, 2)$  are

$$y - 2 = -\left(\frac{\pi}{2}\right)x \quad \text{Tangent line when } t = -\frac{\pi}{2}$$

and

$$y - 2 = \left(\frac{\pi}{2}\right)x \quad \text{Tangent line when } t = \frac{\pi}{2}$$

If  $dy/dt = 0$  and  $dx/dt \neq 0$  when  $t = t_0$ , the curve represented by  $x = f(t)$  and  $y = g(t)$  has a horizontal tangent at  $(f(t_0), g(t_0))$ . For instance, in Example 3, the given curve has a horizontal tangent at the point  $(0, 2 - \pi)$  (when  $t = 0$ ). Similarly, if  $dx/dt = 0$  and  $dy/dt \neq 0$  when  $t = t_0$ , the curve represented by  $x = f(t)$  and  $y = g(t)$  has a vertical tangent at  $(f(t_0), g(t_0))$ .

### Arc Length

You have seen how parametric equations can be used to describe the path of a particle moving in the plane. You will now develop a formula for determining the *distance* traveled by the particle along its path.

Recall from Section 7.4 that the formula for the arc length of a curve  $C$  given by  $y = h(x)$  over the interval  $[x_0, x_1]$  is

$$\begin{aligned} s &= \int_{x_0}^{x_1} \sqrt{1 + [h'(x)]^2} dx \\ &= \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \end{aligned}$$

If  $C$  is represented by the parametric equations  $x = f(t)$  and  $y = g(t)$ ,  $a \leq t \leq b$ , and if  $dx/dt = f'(t) > 0$ , you can write

$$\begin{aligned} s &= \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} dx \\ &= \int_a^b \sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}} \frac{dx}{dt} dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt. \end{aligned}$$

**THEOREM 10.8 ARC LENGTH IN PARAMETRIC FORM**

If a smooth curve  $C$  is given by  $x = f(t)$  and  $y = g(t)$  such that  $C$  does not intersect itself on the interval  $a \leq t \leq b$  (except possibly at the endpoints), then the arc length of  $C$  over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt.$$

**NOTE** When applying the arc length formula to a curve, be sure that the curve is traced out only once on the interval of integration. For instance, the circle given by  $x = \cos t$  and  $y = \sin t$  is traced out once on the interval  $0 \leq t \leq 2\pi$ , but is traced out twice on the interval  $0 \leq t \leq 4\pi$ . ■

In the preceding section you saw that if a circle rolls along a line, a point on its circumference will trace a path called a cycloid. If the circle rolls around the circumference of another circle, the path of the point is an **epicycloid**. The next example shows how to find the arc length of an epicycloid.

**ARCH OF A CYCLOID**

The arc length of an arch of a cycloid was first calculated in 1658 by British architect and mathematician Christopher Wren, famous for rebuilding many buildings and churches in London, including St. Paul's Cathedral.

**EXAMPLE 4 Finding Arc Length**

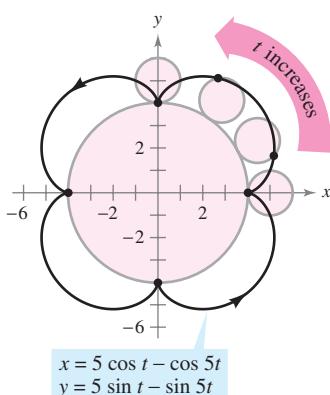
A circle of radius 1 rolls around the circumference of a larger circle of radius 4, as shown in Figure 10.33. The epicycloid traced by a point on the circumference of the smaller circle is given by

$$x = 5 \cos t - \cos 5t \quad \text{and} \quad y = 5 \sin t - \sin 5t.$$

Find the distance traveled by the point in one complete trip about the larger circle.

**Solution** Before applying Theorem 10.8, note in Figure 10.33 that the curve has sharp points when  $t = 0$  and  $t = \pi/2$ . Between these two points,  $dx/dt$  and  $dy/dt$  are not simultaneously 0. So, the portion of the curve generated from  $t = 0$  to  $t = \pi/2$  is smooth. To find the total distance traveled by the point, you can find the arc length of that portion lying in the first quadrant and multiply by 4.

$$\begin{aligned} s &= 4 \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt && \text{Parametric form for arc length} \\ &= 4 \int_0^{\pi/2} \sqrt{(-5 \sin t + 5 \sin 5t)^2 + (5 \cos t - 5 \cos 5t)^2} dt \\ &= 20 \int_0^{\pi/2} \sqrt{2 - 2 \sin t \sin 5t - 2 \cos t \cos 5t} dt \\ &= 20 \int_0^{\pi/2} \sqrt{2 - 2 \cos 4t} dt \\ &= 20 \int_0^{\pi/2} \sqrt{4 \sin^2 2t} dt && \text{Trigonometric identity} \\ &= 40 \int_0^{\pi/2} \sin 2t dt \\ &= -20 \left[ \cos 2t \right]_0^{\pi/2} \\ &= 40 \end{aligned}$$



An epicycloid is traced by a point on the smaller circle as it rolls around the larger circle.

**Figure 10.33**

For the epicycloid shown in Figure 10.33, an arc length of 40 seems about right because the circumference of a circle of radius 6 is  $2\pi r = 12\pi \approx 37.7$ . ■

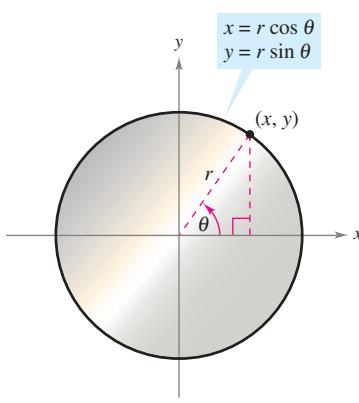
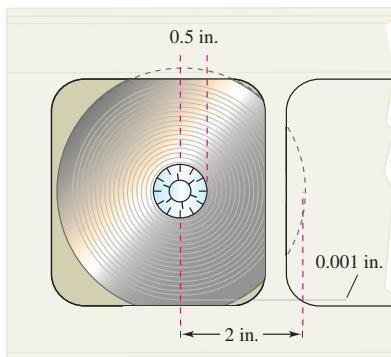


Figure 10.34

**EXAMPLE 5 Length of a Recording Tape**

A recording tape 0.001 inch thick is wound around a reel whose inner radius is 0.5 inch and whose outer radius is 2 inches, as shown in Figure 10.34. How much tape is required to fill the reel?

**Solution** To create a model for this problem, assume that as the tape is wound around the reel, its distance  $r$  from the center increases linearly at a rate of 0.001 inch per revolution, or

$$r = (0.001) \frac{\theta}{2\pi} = \frac{\theta}{2000\pi}, \quad 1000\pi \leq \theta \leq 4000\pi$$

where  $\theta$  is measured in radians. You can determine that the coordinates of the point  $(x, y)$  corresponding to a given radius are

$$x = r \cos \theta$$

and

$$y = r \sin \theta.$$

Substituting for  $r$ , you obtain the parametric equations

$$x = \left( \frac{\theta}{2000\pi} \right) \cos \theta \quad \text{and} \quad y = \left( \frac{\theta}{2000\pi} \right) \sin \theta.$$

You can use the arc length formula to determine that the total length of the tape is

$$\begin{aligned} s &= \int_{1000\pi}^{4000\pi} \sqrt{\left( \frac{dx}{d\theta} \right)^2 + \left( \frac{dy}{d\theta} \right)^2} d\theta \\ &= \frac{1}{2000\pi} \int_{1000\pi}^{4000\pi} \sqrt{(-\theta \sin \theta + \cos \theta)^2 + (\theta \cos \theta + \sin \theta)^2} d\theta \\ &= \frac{1}{2000\pi} \int_{1000\pi}^{4000\pi} \sqrt{\theta^2 + 1} d\theta \\ &= \frac{1}{2000\pi} \left( \frac{1}{2} \right) \left[ \theta \sqrt{\theta^2 + 1} + \ln \left| \theta + \sqrt{\theta^2 + 1} \right| \right]_{1000\pi}^{4000\pi} \\ &\approx 11,781 \text{ inches} \\ &\approx 982 \text{ feet.} \end{aligned}$$

Integration tables  
(Appendix B), Formula 26

**NOTE** The graph of  $r = a\theta$  is called the **spiral of Archimedes**. The graph of  $r = \theta/2000\pi$  (in Example 5) is of this form.

■ **FOR FURTHER INFORMATION** For more information on the mathematics of recording tape, see “Tape Counters” by Richard L. Roth in *The American Mathematical Monthly*. To view this article, go to the website [www.matharticles.com](http://www.matharticles.com).

The length of the tape in Example 5 can be approximated by adding the circumferences of circular pieces of tape. The smallest circle has a radius of 0.501 and the largest has a radius of 2.

$$\begin{aligned} s &\approx 2\pi(0.501) + 2\pi(0.502) + 2\pi(0.503) + \dots + 2\pi(2.000) \\ &= \sum_{i=1}^{1500} 2\pi(0.5 + 0.001i) \\ &= 2\pi[1500(0.5) + 0.001(1500)(1501)/2] \\ &\approx 11,786 \text{ inches} \end{aligned}$$

## Area of a Surface of Revolution

You can use the formula for the area of a surface of revolution in rectangular form to develop a formula for surface area in parametric form.

### THEOREM 10.9 AREA OF A SURFACE OF REVOLUTION

If a smooth curve  $C$  given by  $x = f(t)$  and  $y = g(t)$  does not cross itself on an interval  $a \leq t \leq b$ , then the area  $S$  of the surface of revolution formed by revolving  $C$  about the coordinate axes is given by the following.

$$1. S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad \text{Revolution about the } x\text{-axis: } g(t) \geq 0$$

$$2. S = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad \text{Revolution about the } y\text{-axis: } f(t) \geq 0$$

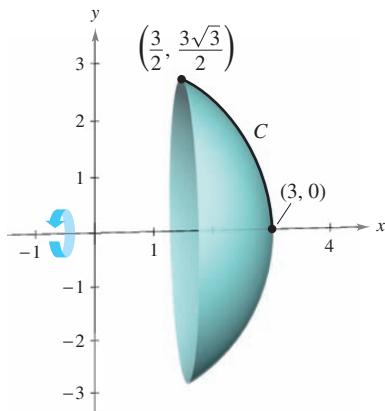
These formulas are easy to remember if you think of the differential of arc length as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Then the formulas are written as follows.

$$1. S = 2\pi \int_a^b g(t) ds \quad 2. S = 2\pi \int_a^b f(t) ds$$

### EXAMPLE 6 Finding the Area of a Surface of Revolution



The surface of revolution has a surface area of  $9\pi$ .

**Figure 10.35**

Let  $C$  be the arc of the circle

$$x^2 + y^2 = 9$$

from  $(3, 0)$  to  $(3/2, 3\sqrt{3}/2)$ , as shown in Figure 10.35. Find the area of the surface formed by revolving  $C$  about the  $x$ -axis.

**Solution** You can represent  $C$  parametrically by the equations

$$x = 3 \cos t \quad \text{and} \quad y = 3 \sin t, \quad 0 \leq t \leq \pi/3.$$

(Note that you can determine the interval for  $t$  by observing that  $t = 0$  when  $x = 3$  and  $t = \pi/3$  when  $x = 3/2$ .) On this interval,  $C$  is smooth and  $y$  is nonnegative, and you can apply Theorem 10.9 to obtain a surface area of

$$\begin{aligned} S &= 2\pi \int_0^{\pi/3} (3 \sin t) \sqrt{(-3 \sin t)^2 + (3 \cos t)^2} dt && \text{Formula for area of a surface of revolution} \\ &= 6\pi \int_0^{\pi/3} \sin t \sqrt{9(\sin^2 t + \cos^2 t)} dt \\ &= 6\pi \int_0^{\pi/3} 3 \sin t dt && \text{Trigonometric identity} \\ &= -18\pi \left[ \cos t \right]_0^{\pi/3} \\ &= -18\pi \left( \frac{1}{2} - 1 \right) \\ &= 9\pi. \end{aligned}$$

## 10.3 Exercises

See [www.CalcChat.com](http://www.CalcChat.com) for worked-out solutions to odd-numbered exercises.

**In Exercises 1–4, find  $dy/dx$ .**

1.  $x = t^2$ ,  $y = 7 - 6t$

3.  $x = \sin^2 \theta$ ,  $y = \cos^2 \theta$

2.  $x = \sqrt[3]{t}$ ,  $y = 4 - t$

4.  $x = 2e^\theta$ ,  $y = e^{-\theta}/2$

**In Exercises 5–14, find  $dy/dx$  and  $d^2y/dx^2$ , and find the slope and concavity (if possible) at the given value of the parameter.**

*Parametric Equations*

5.  $x = 4t$ ,  $y = 3t - 2$

6.  $x = \sqrt{t}$ ,  $y = 3t - 1$

7.  $x = t + 1$ ,  $y = t^2 + 3t$

8.  $x = t^2 + 5t + 4$ ,  $y = 4t$

9.  $x = 4 \cos \theta$ ,  $y = 4 \sin \theta$

10.  $x = \cos \theta$ ,  $y = 3 \sin \theta$

11.  $x = 2 + \sec \theta$ ,  $y = 1 + 2 \tan \theta$

12.  $x = \sqrt{t}$ ,  $y = \sqrt{t-1}$

13.  $x = \cos^3 \theta$ ,  $y = \sin^3 \theta$

14.  $x = \theta - \sin \theta$ ,  $y = 1 - \cos \theta$

*Point*

t = 3

t = 1

t = -1

t = 0

$\theta = \frac{\pi}{4}$

$\theta = 0$

$\theta = \frac{\pi}{6}$

t = 2

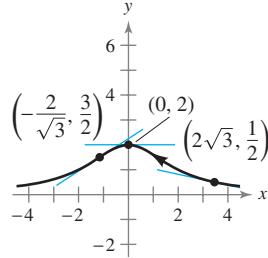
$\theta = \frac{\pi}{4}$

$\theta = \pi$

**In Exercises 15–18, find an equation of the tangent line at each given point on the curve.**

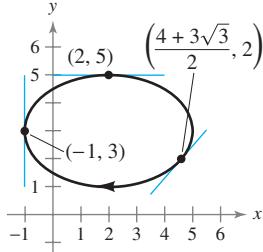
15.  $x = 2 \cot \theta$

$y = 2 \sin^2 \theta$



16.  $x = 2 - 3 \cos \theta$

$y = 3 + 2 \sin \theta$



17.  $x = t^2 - 4$

$y = t^2 - 2t$

(0, 0), (-3, -1), and (-3, 3)

18.  $x = t^4 + 2$

$y = t^3 + t$

(2, 0), (3, -2), and (18, 10)



**In Exercises 19–22, (a) use a graphing utility to graph the curve represented by the parametric equations, (b) use a graphing utility to find  $dx/dt$ ,  $dy/dt$ , and  $dy/dx$  at the given value of the parameter, (c) find an equation of the tangent line to the curve at the given value of the parameter, and (d) use a graphing utility to graph the curve and the tangent line from part (c).**

*Parametric Equations*

19.  $x = 6t$ ,  $y = t^2 + 4$

20.  $x = t - 2$ ,  $y = \frac{1}{t} + 3$

*Parameter*

t = 1

t = 1

*Parametric Equations*

21.  $x = t^2 - t + 2$ ,  $y = t^3 - 3t$

22.  $x = 4 \cos \theta$ ,  $y = 3 \sin \theta$

*Parameter*

$t = -1$

$\theta = \frac{3\pi}{4}$

**In Exercises 23–26, find the equations of the tangent lines at the point where the curve crosses itself.**

23.  $x = 2 \sin 2t$ ,  $y = 3 \sin t$

24.  $x = 2 - \pi \cos t$ ,  $y = 2t - \pi \sin t$

25.  $x = t^2 - t$ ,  $y = t^3 - 3t - 1$

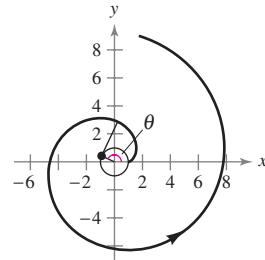
26.  $x = t^3 - 6t$ ,  $y = t^2$

**In Exercises 27 and 28, find all points (if any) of horizontal and vertical tangency to the portion of the curve shown.**

27. Involute of a circle:

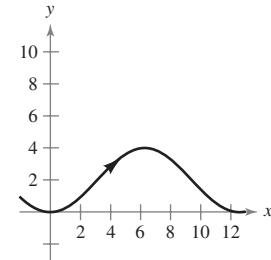
$x = \cos \theta + \theta \sin \theta$

$y = \sin \theta - \theta \cos \theta$



28.  $x = 2\theta$

$y = 2(1 - \cos \theta)$



**In Exercises 29–38, find all points (if any) of horizontal and vertical tangency to the curve. Use a graphing utility to confirm your results.**

29.  $x = 4 - t$ ,  $y = t^2$

30.  $x = t + 1$ ,  $y = t^2 + 3t$

31.  $x = t + 4$ ,  $y = t^3 - 3t$

32.  $x = t^2 - t + 2$ ,  $y = t^3 - 3t$

33.  $x = 3 \cos \theta$ ,  $y = 3 \sin \theta$

34.  $x = \cos \theta$ ,  $y = 2 \sin 2\theta$

35.  $x = 5 + 3 \cos \theta$ ,  $y = -2 + \sin \theta$

36.  $x = 4 \cos^2 \theta$ ,  $y = 2 \sin \theta$

37.  $x = \sec \theta$ ,  $y = \tan \theta$

38.  $x = \cos^2 \theta$ ,  $y = \cos \theta$

**In Exercises 39–44, determine the  $t$  intervals on which the curve is concave downward or concave upward.**

39.  $x = 3t^2$ ,  $y = t^3 - t$

40.  $x = 2 + t^2$ ,  $y = t^2 + t^3$

41.  $x = 2t + \ln t$ ,  $y = 2t - \ln t$

42.  $x = t^2$ ,  $y = \ln t$

43.  $x = \sin t$ ,  $y = \cos t$ ,  $0 < t < \pi$

44.  $x = 4 \cos t$ ,  $y = 2 \sin t$ ,  $0 < t < 2\pi$

**Arc Length** In Exercises 45–48, write an integral that represents the arc length of the curve on the given interval. Do not evaluate the integral.

<i>Parametric Equations</i>	<i>Interval</i>
45. $x = 3t - t^2$ , $y = 2t^{3/2}$	$1 \leq t \leq 3$
46. $x = \ln t$ , $y = 4t - 3$	$1 \leq t \leq 5$
47. $x = e^t + 2$ , $y = 2t + 1$	$-2 \leq t \leq 2$
48. $x = t + \sin t$ , $y = t - \cos t$	$0 \leq t \leq \pi$

**Arc Length** In Exercises 49–56, find the arc length of the curve on the given interval.

<i>Parametric Equations</i>	<i>Interval</i>
49. $x = 3t + 5$ , $y = 7 - 2t$	$-1 \leq t \leq 3$
50. $x = t^2$ , $y = 2t$	$0 \leq t \leq 2$
51. $x = 6t^2$ , $y = 2t^3$	$1 \leq t \leq 4$
52. $x = t^2 + 1$ , $y = 4t^3 + 3$	$-1 \leq t \leq 0$
53. $x = e^{-t} \cos t$ , $y = e^{-t} \sin t$	$0 \leq t \leq \frac{\pi}{2}$
54. $x = \arcsin t$ , $y = \ln \sqrt{1 - t^2}$	$0 \leq t \leq \frac{1}{2}$
55. $x = \sqrt{t}$ , $y = 3t - 1$	$0 \leq t \leq 1$
56. $x = t$ , $y = \frac{t^5}{10} + \frac{1}{6t^3}$	$1 \leq t \leq 2$

**Arc Length** In Exercises 57–60, find the arc length of the curve on the interval  $[0, 2\pi]$ .

- 57. Hypocycloid perimeter:  $x = a \cos^3 \theta$ ,  $y = a \sin^3 \theta$
- 58. Circle circumference:  $x = a \cos \theta$ ,  $y = a \sin \theta$
- 59. Cycloid arch:  $x = a(\theta - \sin \theta)$ ,  $y = a(1 - \cos \theta)$
- 60. Involute of a circle:  $x = \cos \theta + \theta \sin \theta$ ,  $y = \sin \theta - \theta \cos \theta$

**61. Path of a Projectile** The path of a projectile is modeled by the parametric equations

$$x = (90 \cos 30^\circ)t \quad \text{and} \quad y = (90 \sin 30^\circ)t - 16t^2$$

where  $x$  and  $y$  are measured in feet.

- (a) Use a graphing utility to graph the path of the projectile.
- (b) Use a graphing utility to approximate the range of the projectile.
- (c) Use the integration capabilities of a graphing utility to approximate the arc length of the path. Compare this result with the range of the projectile.

**62. Path of a Projectile** If the projectile in Exercise 61 is launched at an angle  $\theta$  with the horizontal, its parametric equations are

$$x = (90 \cos \theta)t \quad \text{and} \quad y = (90 \sin \theta)t - 16t^2.$$

Use a graphing utility to find the angle that maximizes the range of the projectile. What angle maximizes the arc length of the trajectory?

**P** 63. *Folium of Descartes* Consider the parametric equations

$$x = \frac{4t}{1+t^3} \quad \text{and} \quad y = \frac{4t^2}{1+t^3}.$$

- (a) Use a graphing utility to graph the curve represented by the parametric equations.
- (b) Use a graphing utility to find the points of horizontal tangency to the curve.
- (c) Use the integration capabilities of a graphing utility to approximate the arc length of the closed loop. (*Hint:* Use symmetry and integrate over the interval  $0 \leq t \leq 1$ .)

**P** 64. *Witch of Agnesi* Consider the parametric equations

$$x = 4 \cot \theta \quad \text{and} \quad y = 4 \sin^2 \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

- (a) Use a graphing utility to graph the curve represented by the parametric equations.
- (b) Use a graphing utility to find the points of horizontal tangency to the curve.
- (c) Use the integration capabilities of a graphing utility to approximate the arc length over the interval  $\pi/4 \leq \theta \leq \pi/2$ .

**65. Writing**

- P** (a) Use a graphing utility to graph each set of parametric equations.

$$\begin{array}{ll} x = t - \sin t & x = 2t - \sin(2t) \\ y = 1 - \cos t & y = 1 - \cos(2t) \\ 0 \leq t \leq 2\pi & 0 \leq t \leq \pi \end{array}$$

- (b) Compare the graphs of the two sets of parametric equations in part (a). If the curve represents the motion of a particle and  $t$  is time, what can you infer about the average speeds of the particle on the paths represented by the two sets of parametric equations?
- (c) Without graphing the curve, determine the time required for a particle to traverse the same path as in parts (a) and (b) if the path is modeled by

$$x = \frac{1}{2}t - \sin\left(\frac{1}{2}t\right) \quad \text{and} \quad y = 1 - \cos\left(\frac{1}{2}t\right).$$

**66. Writing**

- P** (a) Each set of parametric equations represents the motion of a particle. Use a graphing utility to graph each set.

<i>First Particle</i>	<i>Second Particle</i>
$x = 3 \cos t$	$x = 4 \sin t$
$y = 4 \sin t$	$y = 3 \cos t$
$0 \leq t \leq 2\pi$	$0 \leq t \leq 2\pi$

- (b) Determine the number of points of intersection.
- (c) Will the particles ever be at the same place at the same time? If so, identify the point(s).
- (d) Explain what happens if the motion of the second particle is represented by

$$x = 2 + 3 \sin t, \quad y = 2 - 4 \cos t, \quad 0 \leq t \leq 2\pi.$$

 **Surface Area** In Exercises 67–70, write an integral that represents the area of the surface generated by revolving the curve about the  $x$ -axis. Use a graphing utility to approximate the integral.

<i>Parametric Equations</i>	<i>Interval</i>
67. $x = 3t$ , $y = t + 2$	$0 \leq t \leq 4$
68. $x = \frac{1}{4}t^2$ , $y = t + 3$	$0 \leq t \leq 3$
69. $x = \cos^2 \theta$ , $y = \cos \theta$	$0 \leq \theta \leq \frac{\pi}{2}$
70. $x = \theta + \sin \theta$ , $y = \theta + \cos \theta$	$0 \leq \theta \leq \frac{\pi}{2}$

**Surface Area** In Exercises 71–76, find the area of the surface generated by revolving the curve about each given axis.

71.  $x = 2t$ ,  $y = 3t$ ,  $0 \leq t \leq 3$ , (a)  $x$ -axis (b)  $y$ -axis  
 72.  $x = t$ ,  $y = 4 - 2t$ ,  $0 \leq t \leq 2$ , (a)  $x$ -axis (b)  $y$ -axis  
 73.  $x = 5 \cos \theta$ ,  $y = 5 \sin \theta$ ,  $0 \leq \theta \leq \frac{\pi}{2}$ ,  $y$ -axis  
 74.  $x = \frac{1}{3}t^3$ ,  $y = t + 1$ ,  $1 \leq t \leq 2$ ,  $y$ -axis  
 75.  $x = a \cos^3 \theta$ ,  $y = a \sin^3 \theta$ ,  $0 \leq \theta \leq \pi$ ,  $x$ -axis  
 76.  $x = a \cos \theta$ ,  $y = b \sin \theta$ ,  $0 \leq \theta \leq 2\pi$ ,  
     (a)  $x$ -axis (b)  $y$ -axis

### WRITING ABOUT CONCEPTS

77. Give the parametric form of the derivative.

In Exercises 78 and 79, mentally determine  $dy/dx$ .

78.  $x = t$ ,  $y = 3$       79.  $x = t$ ,  $y = 6t - 5$

80. Give the integral formula for arc length in parametric form.

81. Give the integral formulas for the areas of the surfaces of revolution formed when a smooth curve  $C$  is revolved about (a) the  $x$ -axis and (b) the  $y$ -axis.

### CAPSTONE

82. (a) Sketch a graph of a curve defined by the parametric equations  $x = g(t)$  and  $y = f(t)$  such that  $dx/dt > 0$  and  $dy/dt < 0$  for all real numbers  $t$ .  
 (b) Sketch a graph of a curve defined by the parametric equations  $x = g(t)$  and  $y = f(t)$  such that  $dx/dt < 0$  and  $dy/dt < 0$  for all real numbers  $t$ .

83. Use integration by substitution to show that if  $y$  is a continuous function of  $x$  on the interval  $a \leq x \leq b$ , where  $x = f(t)$  and  $y = g(t)$ , then

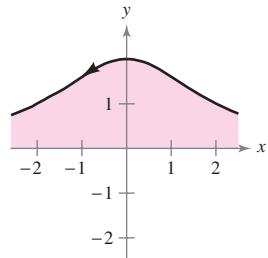
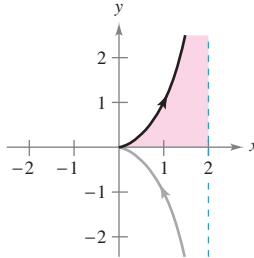
$$\int_a^b y \, dx = \int_{t_1}^{t_2} g(t)f'(t) \, dt$$

where  $f(t_1) = a$ ,  $f(t_2) = b$ , and both  $g$  and  $f'$  are continuous on  $[t_1, t_2]$ .

84. **Surface Area** A portion of a sphere of radius  $r$  is removed by cutting out a circular cone with its vertex at the center of the sphere. The vertex of the cone forms an angle of  $2\theta$ . Find the surface area removed from the sphere.

**Area** In Exercises 85 and 86, find the area of the region. (Use the result of Exercise 83.)

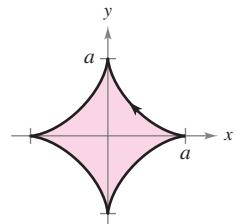
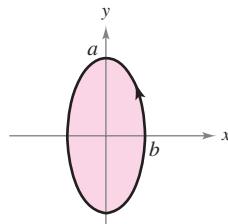
85.  $x = 2 \sin^2 \theta$   
 $y = 2 \sin^2 \theta \tan \theta$   
 $0 \leq \theta < \frac{\pi}{2}$
86.  $x = 2 \cot \theta$   
 $y = 2 \sin^2 \theta$   
 $0 < \theta < \pi$



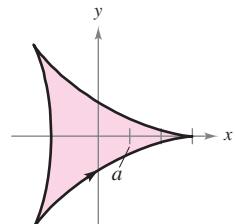
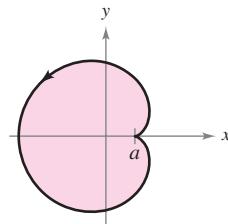
**CAS Areas of Simple Closed Curves** In Exercises 87–92, use a computer algebra system and the result of Exercise 83 to match the closed curve with its area. (These exercises were based on “The Surveyor’s Area Formula,” by Bart Braden, *College Mathematics Journal*, September 1986, pp. 335–337, by permission of the author.)

- (a)  $\frac{8}{3}ab$       (b)  $\frac{3}{8}\pi a^2$       (c)  $2\pi a^2$   
 (d)  $\pi ab$       (e)  $2\pi ab$       (f)  $6\pi a^2$

87. Ellipse:  $(0 \leq t \leq 2\pi)$   
 $x = b \cos t$   
 $y = a \sin t$
88. Astroid:  $(0 \leq t \leq 2\pi)$   
 $x = a \cos^3 t$   
 $y = a \sin^3 t$



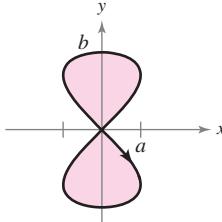
89. Cardioid:  $(0 \leq t \leq 2\pi)$   
 $x = 2a \cos t - a \cos 2t$   
 $y = 2a \sin t - a \sin 2t$
90. Deltoid:  $(0 \leq t \leq 2\pi)$   
 $x = 2a \cos t + a \cos 2t$   
 $y = 2a \sin t - a \sin 2t$



91. Hourglass:  $(0 \leq t \leq 2\pi)$ 

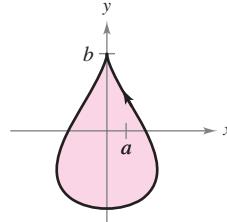
$$x = a \sin 2t$$

$$y = b \sin t$$

92. Teardrop:  $(0 \leq t \leq 2\pi)$ 

$$x = 2a \cos t - a \sin 2t$$

$$y = b \sin t$$



**Centroid** In Exercises 93 and 94, find the centroid of the region bounded by the graph of the parametric equations and the coordinate axes. (Use the result of Exercise 83.)

93.  $x = \sqrt{t}, \quad y = 4 - t$

94.  $x = \sqrt{4-t}, \quad y = \sqrt{t}$

**Volume** In Exercises 95 and 96, find the volume of the solid formed by revolving the region bounded by the graphs of the given equations about the  $x$ -axis. (Use the result of Exercise 83.)

95.  $x = 6 \cos \theta, \quad y = 6 \sin \theta$

96.  $x = \cos \theta, \quad y = 3 \sin \theta, \quad a > 0$

97. **Cycloid** Use the parametric equations

$$x = a(\theta - \sin \theta) \quad \text{and} \quad y = a(1 - \cos \theta), \quad a > 0$$

to answer the following.

(a) Find  $dy/dx$  and  $d^2y/dx^2$ .(b) Find the equation of the tangent line at the point where  $\theta = \pi/6$ .

(c) Find all points (if any) of horizontal tangency.

(d) Determine where the curve is concave upward or concave downward.

(e) Find the length of one arc of the curve.

98. Use the parametric equations

$$x = t^2 \sqrt{3} \quad \text{and} \quad y = 3t - \frac{1}{3}t^3$$

to answer the following.

(a) Use a graphing utility to graph the curve on the interval  $-3 \leq t \leq 3$ .(b) Find  $dy/dx$  and  $d^2y/dx^2$ .(c) Find the equation of the tangent line at the point  $(\sqrt{3}, \frac{8}{3})$ .

(d) Find the length of the curve.

(e) Find the surface area generated by revolving the curve about the  $x$ -axis.99. **Involute of a Circle** The involute of a circle is described by the endpoint  $P$  of a string that is held taut as it is unwound from a spool that does not turn (see figure). Show that a parametric representation of the involute is

$$x = r(\cos \theta + \theta \sin \theta) \quad \text{and} \quad y = r(\sin \theta - \theta \cos \theta).$$

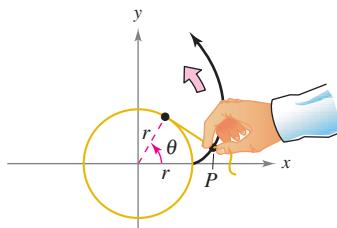


Figure for 99

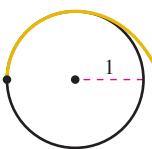


Figure for 100

100. **Involute of a Circle** The figure shows a piece of string tied to a circle with a radius of one unit. The string is just long enough to reach the opposite side of the circle. Find the area that is covered when the string is unwound counterclockwise.

101. (a) Use a graphing utility to graph the curve given by

$$x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2}, \quad -20 \leq t \leq 20.$$

- (b) Describe the graph and confirm your result analytically.

- (c) Discuss the speed at which the curve is traced as  $t$  increases from  $-20$  to  $20$ .

102. **Tractrix** A person moves from the origin along the positive  $y$ -axis pulling a weight at the end of a 12-meter rope. Initially, the weight is located at the point  $(12, 0)$ .

- (a) In Exercise 96 of Section 8.7, it was shown that the path of the weight is modeled by the rectangular equation

$$y = -12 \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$$

where  $0 < x \leq 12$ . Use a graphing utility to graph the rectangular equation.

- (b) Use a graphing utility to graph the parametric equations

$$x = 12 \operatorname{sech} \frac{t}{12} \quad \text{and} \quad y = t - 12 \tanh \frac{t}{12}$$

where  $t \geq 0$ . How does this graph compare with the graph in part (a)? Which graph (if either) do you think is a better representation of the path?

- (c) Use the parametric equations for the tractrix to verify that the distance from the  $y$ -intercept of the tangent line to the point of tangency is independent of the location of the point of tangency.

**True or False?** In Exercises 103 and 104, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

103. If  $x = f(t)$  and  $y = g(t)$ , then  $d^2y/dx^2 = g''(t)/f''(t)$ .

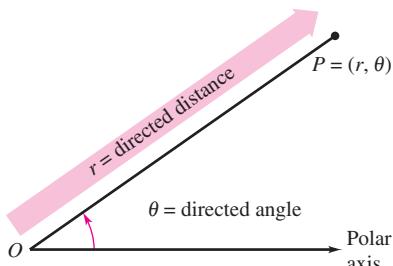
104. The curve given by  $x = t^3, y = t^2$  has a horizontal tangent at the origin because  $dy/dt = 0$  when  $t = 0$ .

105. **Recording Tape** Another method you could use to solve Example 5 is to find the area of the reel with an inner radius of 0.5 inch and an outer radius of 2 inches, and then use the formula for the area of the rectangle where the width is 0.001 inch. Use this method to determine how much tape is required to fill the reel.

## 10.4 Polar Coordinates and Polar Graphs

- Understand the polar coordinate system.
- Rewrite rectangular coordinates and equations in polar form and vice versa.
- Sketch the graph of an equation given in polar form.
- Find the slope of a tangent line to a polar graph.
- Identify several types of special polar graphs.

### Polar Coordinates



Polar coordinates

Figure 10.36

So far, you have been representing graphs as collections of points  $(x, y)$  on the rectangular coordinate system. The corresponding equations for these graphs have been in either rectangular or parametric form. In this section you will study a coordinate system called the **polar coordinate system**.

To form the polar coordinate system in the plane, fix a point  $O$ , called the **pole** (or **origin**), and construct from  $O$  an initial ray called the **polar axis**, as shown in Figure 10.36. Then each point  $P$  in the plane can be assigned **polar coordinates**  $(r, \theta)$ , as follows.

$r = \text{directed distance from } O \text{ to } P$

$\theta = \text{directed angle, counterclockwise from polar axis to segment } \overline{OP}$

Figure 10.37 shows three points on the polar coordinate system. Notice that in this system, it is convenient to locate points with respect to a grid of concentric circles intersected by **radial lines** through the pole.

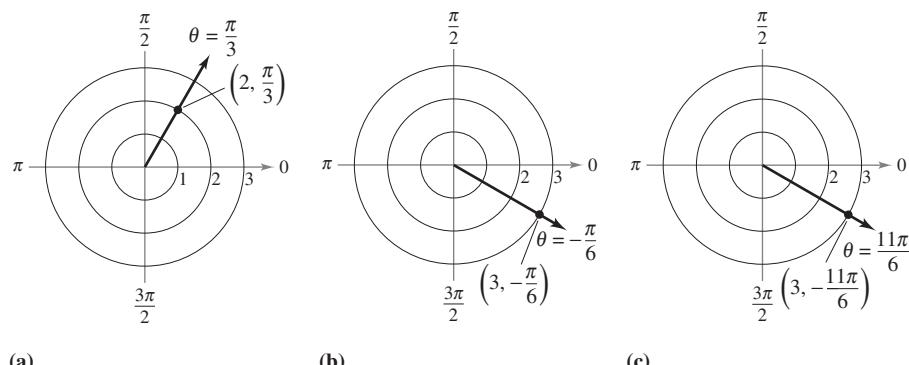


Figure 10.37

With rectangular coordinates, each point  $(x, y)$  has a unique representation. This is not true with polar coordinates. For instance, the coordinates  $(r, \theta)$  and  $(r, 2\pi + \theta)$  represent the same point [see parts (b) and (c) in Figure 10.37]. Also, because  $r$  is a *directed distance*, the coordinates  $(r, \theta)$  and  $(-r, \theta + \pi)$  represent the same point. In general, the point  $(r, \theta)$  can be written as

$$(r, \theta) = (r, \theta + 2n\pi)$$

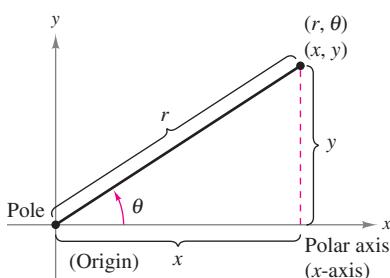
or

$$(r, \theta) = (-r, \theta + (2n + 1)\pi)$$

where  $n$  is any integer. Moreover, the pole is represented by  $(0, \theta)$ , where  $\theta$  is any angle.

#### POLAR COORDINATES

The mathematician credited with first using polar coordinates was James Bernoulli, who introduced them in 1691. However, there is some evidence that it may have been Isaac Newton who first used them.



Relating polar and rectangular coordinates  
Figure 10.38

## Coordinate Conversion

To establish the relationship between polar and rectangular coordinates, let the polar axis coincide with the positive  $x$ -axis and the pole with the origin, as shown in Figure 10.38. Because  $(x, y)$  lies on a circle of radius  $r$ , it follows that  $r^2 = x^2 + y^2$ . Moreover, for  $r > 0$ , the definitions of the trigonometric functions imply that

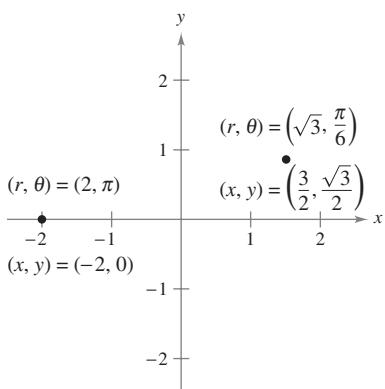
$$\tan \theta = \frac{y}{x}, \quad \cos \theta = \frac{x}{r}, \quad \text{and} \quad \sin \theta = \frac{y}{r}.$$

If  $r < 0$ , you can show that the same relationships hold.

### THEOREM 10.10 COORDINATE CONVERSION

The polar coordinates  $(r, \theta)$  of a point are related to the rectangular coordinates  $(x, y)$  of the point as follows.

1. $x = r \cos \theta$	2. $\tan \theta = \frac{y}{x}$
$y = r \sin \theta$	$r^2 = x^2 + y^2$



To convert from polar to rectangular coordinates, let  $x = r \cos \theta$  and  $y = r \sin \theta$ .  
Figure 10.39

### EXAMPLE 1 Polar-to-Rectangular Conversion

- a. For the point  $(r, \theta) = (2, \pi)$ ,

$$x = r \cos \theta = 2 \cos \pi = -2 \quad \text{and} \quad y = r \sin \theta = 2 \sin \pi = 0.$$

So, the rectangular coordinates are  $(x, y) = (-2, 0)$ .

- b. For the point  $(r, \theta) = (\sqrt{3}, \pi/6)$ ,

$$x = \sqrt{3} \cos \frac{\pi}{6} = \frac{3}{2} \quad \text{and} \quad y = \sqrt{3} \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

So, the rectangular coordinates are  $(x, y) = (3/2, \sqrt{3}/2)$ .

See Figure 10.39.

### EXAMPLE 2 Rectangular-to-Polar Conversion

- a. For the second quadrant point  $(x, y) = (-1, 1)$ ,

$$\tan \theta = \frac{y}{x} = -1 \quad \Rightarrow \quad \theta = \frac{3\pi}{4}.$$

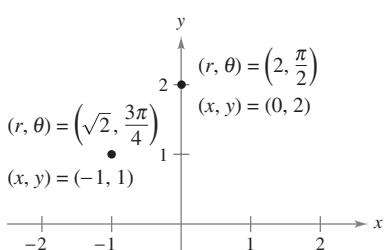
Because  $\theta$  was chosen to be in the same quadrant as  $(x, y)$ , you should use a positive value of  $r$ .

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-1)^2 + (1)^2} \\ &= \sqrt{2} \end{aligned}$$

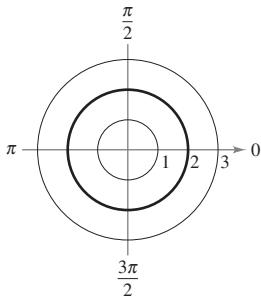
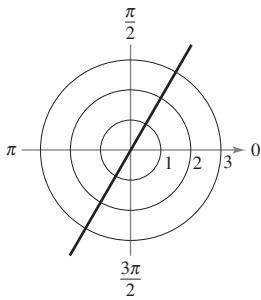
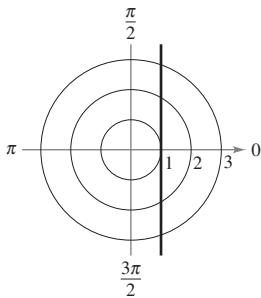
This implies that one set of polar coordinates is  $(r, \theta) = (\sqrt{2}, 3\pi/4)$ .

- b. Because the point  $(x, y) = (0, 2)$  lies on the positive  $y$ -axis, choose  $\theta = \pi/2$  and  $r = 2$ , and one set of polar coordinates is  $(r, \theta) = (2, \pi/2)$ .

See Figure 10.40. ■



To convert from rectangular to polar coordinates, let  $\tan \theta = y/x$  and  $r = \sqrt{x^2 + y^2}$ .  
Figure 10.40

(a) Circle:  $r = 2$ (b) Radial line:  $\theta = \frac{\pi}{3}$ (c) Vertical line:  $r = \sec \theta$ **Figure 10.41**

## Polar Graphs

One way to sketch the graph of a polar equation is to convert to rectangular coordinates and then sketch the graph of the rectangular equation.

### EXAMPLE 3 Graphing Polar Equations

Describe the graph of each polar equation. Confirm each description by converting to a rectangular equation.

a.  $r = 2$       b.  $\theta = \frac{\pi}{3}$       c.  $r = \sec \theta$

#### Solution

- a. The graph of the polar equation  $r = 2$  consists of all points that are two units from the pole. In other words, this graph is a circle centered at the origin with a radius of 2. [See Figure 10.41(a).] You can confirm this by using the relationship  $r^2 = x^2 + y^2$  to obtain the rectangular equation

$$x^2 + y^2 = 2^2. \quad \text{Rectangular equation}$$

- b. The graph of the polar equation  $\theta = \pi/3$  consists of all points on the line that makes an angle of  $\pi/3$  with the positive  $x$ -axis. [See Figure 10.41(b).] You can confirm this by using the relationship  $\tan \theta = y/x$  to obtain the rectangular equation

$$y = \sqrt{3}x. \quad \text{Rectangular equation}$$

- c. The graph of the polar equation  $r = \sec \theta$  is not evident by simple inspection, so you can begin by converting to rectangular form using the relationship  $r \cos \theta = x$ .

$$r = \sec \theta \quad \text{Polar equation}$$

$$r \cos \theta = 1$$

$$x = 1 \quad \text{Rectangular equation}$$

From the rectangular equation, you can see that the graph is a vertical line. [See Figure 10.41(c).]

**TECHNOLOGY** Sketching the graphs of complicated polar equations *by hand* can be tedious. With technology, however, the task is not difficult. If your graphing utility has a *polar* mode, use it to graph the equations in the exercise set. If your graphing utility doesn't have a *polar* mode, but does have a *parametric* mode, you can graph  $r = f(\theta)$  by writing the equation as

$$x = f(\theta) \cos \theta$$

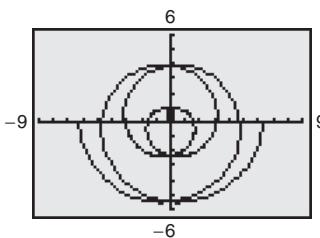
$$y = f(\theta) \sin \theta.$$

For instance, the graph of  $r = \frac{1}{2}\theta$  shown in Figure 10.42 was produced with a graphing calculator in *parametric* mode. This equation was graphed using the parametric equations

$$x = \frac{1}{2}\theta \cos \theta$$

$$y = \frac{1}{2}\theta \sin \theta$$

with the values of  $\theta$  varying from  $-4\pi$  to  $4\pi$ . This curve is of the form  $r = a\theta$  and is called a **spiral of Archimedes**.



Spiral of Archimedes

**Figure 10.42**

### EXAMPLE 4 Sketching a Polar Graph

**NOTE** One way to sketch the graph of  $r = 2 \cos 3\theta$  by hand is to make a table of values.

$\theta$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$r$	2	0	-2	0	2

By extending the table and plotting the points, you will obtain the curve shown in Example 4.

Sketch the graph of  $r = 2 \cos 3\theta$ .

**Solution** Begin by writing the polar equation in parametric form.

$$x = 2 \cos 3\theta \cos \theta \quad \text{and} \quad y = 2 \cos 3\theta \sin \theta$$

After some experimentation, you will find that the entire curve, which is called a **rose curve**, can be sketched by letting  $\theta$  vary from 0 to  $\pi$ , as shown in Figure 10.43. If you try duplicating this graph with a graphing utility, you will find that by letting  $\theta$  vary from 0 to  $2\pi$ , you will actually trace the entire curve *twice*.

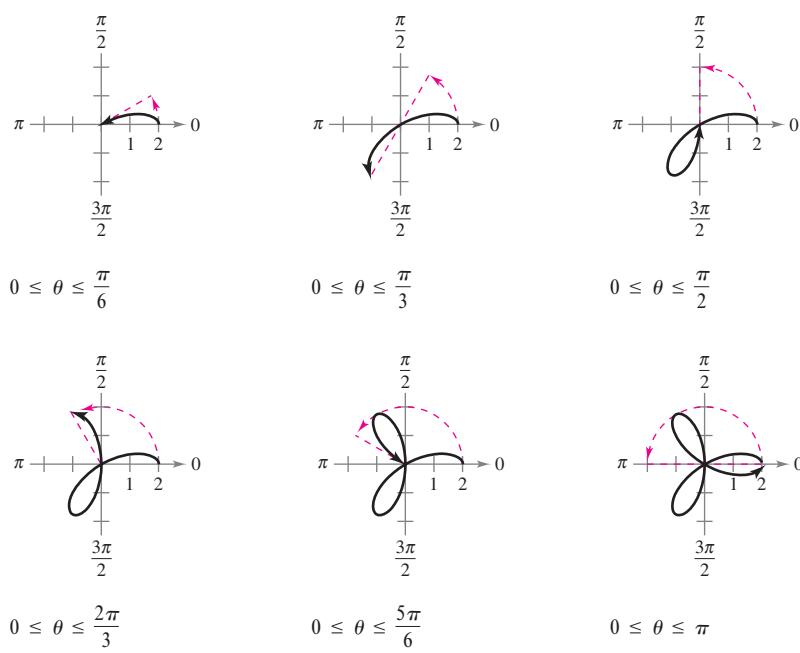
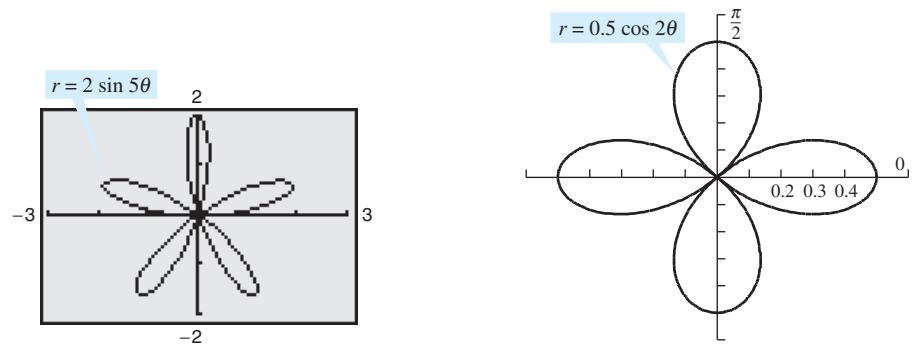


Figure 10.43

Use a graphing utility to experiment with other rose curves (they are of the form  $r = a \cos n\theta$  or  $r = a \sin n\theta$ ). For instance, Figure 10.44 shows the graphs of two other rose curves.



Rose curves

Figure 10.44

## Slope and Tangent Lines

To find the slope of a tangent line to a polar graph, consider a differentiable function given by  $r = f(\theta)$ . To find the slope in polar form, use the parametric equations

$$x = r \cos \theta = f(\theta) \cos \theta \quad \text{and} \quad y = r \sin \theta = f(\theta) \sin \theta.$$

Using the parametric form of  $dy/dx$  given in Theorem 10.7, you have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\ &= \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{-f(\theta) \sin \theta + f'(\theta) \cos \theta}\end{aligned}$$

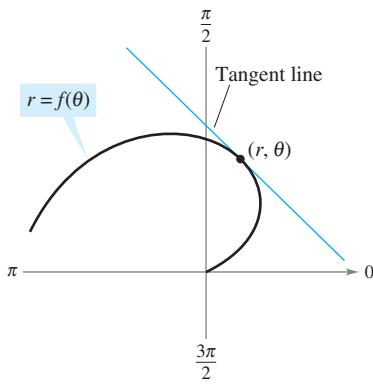
which establishes the following theorem.

### THEOREM 10.11 SLOPE IN POLAR FORM

If  $f$  is a differentiable function of  $\theta$ , then the *slope* of the tangent line to the graph of  $r = f(\theta)$  at the point  $(r, \theta)$  is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{-f(\theta) \sin \theta + f'(\theta) \cos \theta}$$

provided that  $dx/d\theta \neq 0$  at  $(r, \theta)$ . (See Figure 10.45.)



**Figure 10.45**

From Theorem 10.11, you can make the following observations.

1. Solutions to  $\frac{dy}{d\theta} = 0$  yield horizontal tangents, provided that  $\frac{dx}{d\theta} \neq 0$ .
2. Solutions to  $\frac{dx}{d\theta} = 0$  yield vertical tangents, provided that  $\frac{dy}{d\theta} \neq 0$ .

If  $dy/d\theta$  and  $dx/d\theta$  are simultaneously 0, no conclusion can be drawn about tangent lines.

### EXAMPLE 5 Finding Horizontal and Vertical Tangent Lines

Find the horizontal and vertical tangent lines of  $r = \sin \theta$ ,  $0 \leq \theta \leq \pi$ .

**Solution** Begin by writing the equation in parametric form.

$$x = r \cos \theta = \sin \theta \cos \theta$$

and

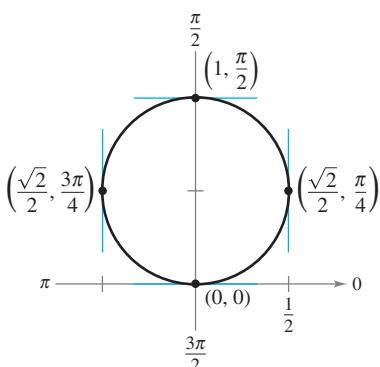
$$y = r \sin \theta = \sin \theta \sin \theta = \sin^2 \theta$$

Next, differentiate  $x$  and  $y$  with respect to  $\theta$  and set each derivative equal to 0.

$$\frac{dx}{d\theta} = \cos^2 \theta - \sin^2 \theta = \cos 2\theta = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\frac{dy}{d\theta} = 2 \sin \theta \cos \theta = \sin 2\theta = 0 \quad \Rightarrow \quad \theta = 0, \frac{\pi}{2}$$

So, the graph has vertical tangent lines at  $(\sqrt{2}/2, \pi/4)$  and  $(\sqrt{2}/2, 3\pi/4)$ , and it has horizontal tangent lines at  $(0, 0)$  and  $(1, \pi/2)$ , as shown in Figure 10.46. ■



Horizontal and vertical tangent lines of  $r = \sin \theta$

**Figure 10.46**

### EXAMPLE 6 Finding Horizontal and Vertical Tangent Lines

Find the horizontal and vertical tangents to the graph of  $r = 2(1 - \cos \theta)$ .

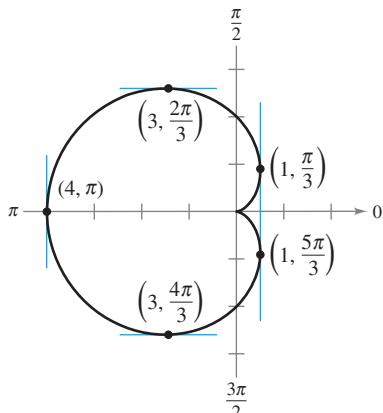
**Solution** Using  $y = r \sin \theta$ , differentiate and set  $dy/d\theta$  equal to 0.

$$\begin{aligned}y &= r \sin \theta = 2(1 - \cos \theta) \sin \theta \\ \frac{dy}{d\theta} &= 2[(1 - \cos \theta)(\cos \theta) + \sin \theta(\sin \theta)] \\ &= -2(2 \cos \theta + 1)(\cos \theta - 1) = 0\end{aligned}$$

So,  $\cos \theta = -\frac{1}{2}$  and  $\cos \theta = 1$ , and you can conclude that  $dy/d\theta = 0$  when  $\theta = 2\pi/3, 4\pi/3$ , and 0. Similarly, using  $x = r \cos \theta$ , you have

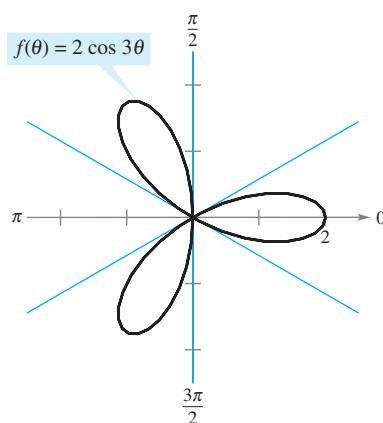
$$\begin{aligned}x &= r \cos \theta = 2 \cos \theta - 2 \cos^2 \theta \\ \frac{dx}{d\theta} &= -2 \sin \theta + 4 \cos \theta \sin \theta = 2 \sin \theta(2 \cos \theta - 1) = 0.\end{aligned}$$

So,  $\sin \theta = 0$  or  $\cos \theta = \frac{1}{2}$ , and you can conclude that  $dx/d\theta = 0$  when  $\theta = 0, \pi/3$ , and  $5\pi/3$ . From these results, and from the graph shown in Figure 10.47, you can conclude that the graph has horizontal tangents at  $(3, 2\pi/3)$  and  $(3, 4\pi/3)$ , and has vertical tangents at  $(1, \pi/3), (1, 5\pi/3)$ , and  $(4, \pi)$ . This graph is called a **cardioid**. Note that both derivatives ( $dy/d\theta$  and  $dx/d\theta$ ) are 0 when  $\theta = 0$ . Using this information alone, you don't know whether the graph has a horizontal or vertical tangent line at the pole. From Figure 10.47, however, you can see that the graph has a cusp at the pole. ■



Horizontal and vertical tangent lines of  $r = 2(1 - \cos \theta)$

Figure 10.47



This rose curve has three tangent lines ( $\theta = \pi/6, \theta = \pi/2$ , and  $\theta = 5\pi/6$ ) at the pole.

Figure 10.48

#### THEOREM 10.12 TANGENT LINES AT THE POLE

If  $f(\alpha) = 0$  and  $f'(\alpha) \neq 0$ , then the line  $\theta = \alpha$  is tangent at the pole to the graph of  $r = f(\theta)$ .

Theorem 10.12 is useful because it states that the zeros of  $r = f(\theta)$  can be used to find the tangent lines at the pole. Note that because a polar curve can cross the pole more than once, it can have more than one tangent line at the pole. For example, the rose curve

$$f(\theta) = 2 \cos 3\theta$$

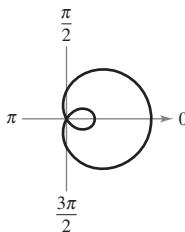
has three tangent lines at the pole, as shown in Figure 10.48. For this curve,  $f(\theta) = 2 \cos 3\theta$  is 0 when  $\theta$  is  $\pi/6, \pi/2$ , and  $5\pi/6$ . Moreover, the derivative  $f'(\theta) = -6 \sin 3\theta$  is not 0 for these values of  $\theta$ .

## Special Polar Graphs

Several important types of graphs have equations that are simpler in polar form than in rectangular form. For example, the polar equation of a circle having a radius of  $a$  and centered at the origin is simply  $r = a$ . Later in the text you will come to appreciate this benefit. For now, several other types of graphs that have simpler equations in polar form are shown below. (Conics are considered in Section 10.6.)

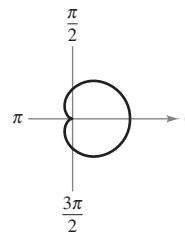
### Limaçons

$$\begin{aligned}r &= a \pm b \cos \theta \\r &= a \pm b \sin \theta \\(a > 0, b > 0)\end{aligned}$$



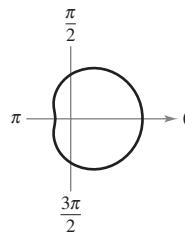
$$\frac{a}{b} < 1$$

Limaçon with inner loop



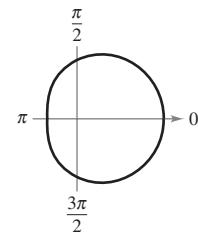
$$\frac{a}{b} = 1$$

Cardioid (heart-shaped)



$$1 < \frac{a}{b} < 2$$

Dimpled limaçon

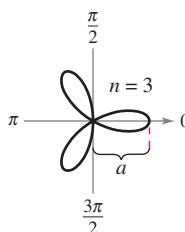


$$\frac{a}{b} \geq 2$$

Convex limaçon

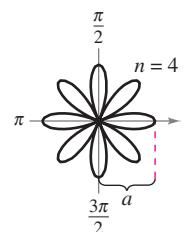
### Rose Curves

$n$  petals if  $n$  is odd  
 $2n$  petals if  $n$  is even  
( $n \geq 2$ )



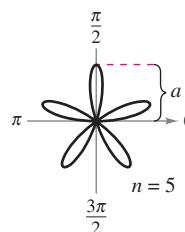
$$r = a \cos n\theta$$

Rose curve



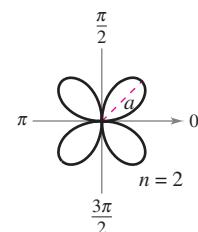
$$r = a \cos n\theta$$

Rose curve



$$r = a \sin n\theta$$

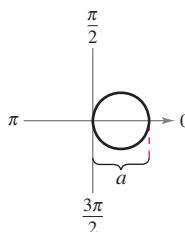
Rose curve



$$r = a \sin n\theta$$

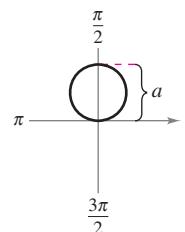
Rose curve

### Circles and Lemniscates



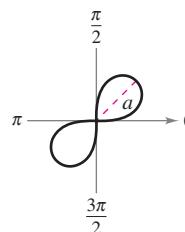
$$r = a \cos \theta$$

Circle



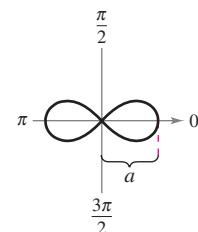
$$r = a \sin \theta$$

Circle



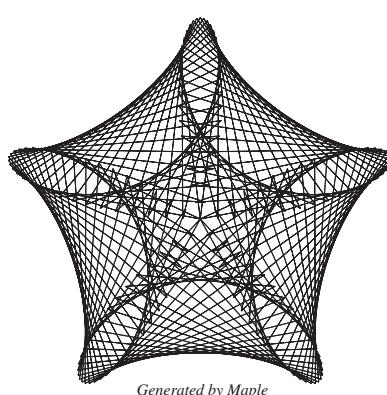
$$r^2 = a^2 \sin 2\theta$$

Lemniscate



$$r^2 = a^2 \cos 2\theta$$

Lemniscate



**TECHNOLOGY** The rose curves described above are of the form  $r = a \cos n\theta$  or  $r = a \sin n\theta$ , where  $n$  is a positive integer that is greater than or equal to 2. Use a graphing utility to graph  $r = a \cos n\theta$  or  $r = a \sin n\theta$  for some noninteger values of  $n$ . Are these graphs also rose curves? For example, try sketching the graph of  $r = \cos \frac{2}{3}\theta$ ,  $0 \leq \theta \leq 6\pi$ .

**FOR FURTHER INFORMATION** For more information on rose curves and related curves, see the article “A Rose is a Rose . . .” by Peter M. Maurer in *The American Mathematical Monthly*. The computer-generated graph at the left is the result of an algorithm that Maurer calls “The Rose.” To view this article, go to the website [www.matharticles.com](http://www.matharticles.com).

## 10.4 Exercises

See [www.CalcChat.com](http://www.CalcChat.com) for worked-out solutions to odd-numbered exercises.

**In Exercises 1–6, plot the point in polar coordinates and find the corresponding rectangular coordinates for the point.**

- |                       |                   |
|-----------------------|-------------------|
| 1. $(8, \pi/2)$       | 2. $(-2, 5\pi/3)$ |
| 3. $(-4, -3\pi/4)$    | 4. $(0, -7\pi/6)$ |
| 5. $(\sqrt{2}, 2.36)$ | 6. $(-3, -1.57)$  |

**In Exercises 7–10, use the angle feature of a graphing utility to find the rectangular coordinates for the point given in polar coordinates. Plot the point.**

- |                  |                    |
|------------------|--------------------|
| 7. $(7, 5\pi/4)$ | 8. $(-2, 11\pi/6)$ |
| 9. $(-4.5, 3.5)$ | 10. $(9.25, 1.2)$  |

**In Exercises 11–16, the rectangular coordinates of a point are given. Plot the point and find two sets of polar coordinates for the point for  $0 \leq \theta < 2\pi$ .**

- |                       |                      |
|-----------------------|----------------------|
| 11. $(2, 2)$          | 12. $(0, -6)$        |
| 13. $(-3, 4)$         | 14. $(4, -2)$        |
| 15. $(-1, -\sqrt{3})$ | 16. $(3, -\sqrt{3})$ |

**In Exercises 17–20, use the angle feature of a graphing utility to find one set of polar coordinates for the point given in rectangular coordinates.**

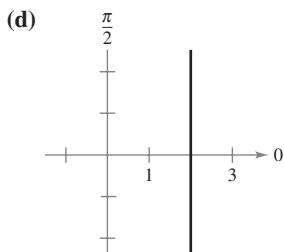
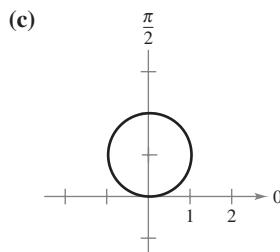
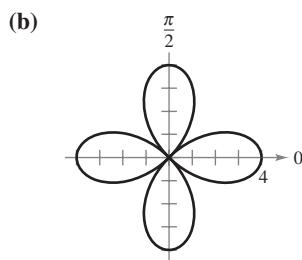
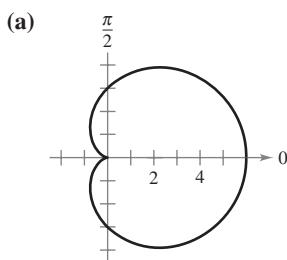
- |                                  |                              |
|----------------------------------|------------------------------|
| 17. $(3, -2)$                    | 18. $(3\sqrt{2}, 3\sqrt{2})$ |
| 19. $(\frac{7}{4}, \frac{5}{2})$ | 20. $(0, -5)$                |

21. Plot the point  $(4, 3.5)$  if the point is given in (a) rectangular coordinates and (b) polar coordinates.

**22. Graphical Reasoning**

- Set the window format of a graphing utility to rectangular coordinates and locate the cursor at any position off the axes. Move the cursor horizontally and vertically. Describe any changes in the displayed coordinates of the points.
- Set the window format of a graphing utility to polar coordinates and locate the cursor at any position off the axes. Move the cursor horizontally and vertically. Describe any changes in the displayed coordinates of the points.
- Why are the results in parts (a) and (b) different?

**In Exercises 23–26, match the graph with its polar equation. [The graphs are labeled (a), (b), (c), and (d).]**



23.  $r = 2 \sin \theta$       24.  $r = 4 \cos 2\theta$   
25.  $r = 3(1 + \cos \theta)$       26.  $r = 2 \sec \theta$

**In Exercises 27–36, convert the rectangular equation to polar form and sketch its graph.**

27.  $x^2 + y^2 = 9$       28.  $x^2 - y^2 = 9$   
29.  $x^2 + y^2 = a^2$       30.  $x^2 + y^2 - 2ax = 0$   
31.  $y = 8$       32.  $x = 12$   
33.  $3x - y + 2 = 0$       34.  $xy = 4$   
35.  $y^2 = 9x$   
36.  $(x^2 + y^2)^2 - 9(x^2 - y^2) = 0$

**In Exercises 37–46, convert the polar equation to rectangular form and sketch its graph.**

37.  $r = 4$       38.  $r = -5$   
39.  $r = 3 \sin \theta$       40.  $r = 5 \cos \theta$   
41.  $r = \theta$       42.  $\theta = \frac{5\pi}{6}$   
43.  $r = 3 \sec \theta$       44.  $r = 2 \csc \theta$   
45.  $r = \sec \theta \tan \theta$       46.  $r = \cot \theta \csc \theta$

**In Exercises 47–56, use a graphing utility to graph the polar equation. Find an interval for  $\theta$  over which the graph is traced only once.**

47.  $r = 2 - 5 \cos \theta$       48.  $r = 3(1 - 4 \cos \theta)$   
49.  $r = 2 + \sin \theta$       50.  $r = 4 + 3 \cos \theta$   
51.  $r = \frac{2}{1 + \cos \theta}$       52.  $r = \frac{2}{4 - 3 \sin \theta}$   
53.  $r = 2 \cos\left(\frac{3\theta}{2}\right)$       54.  $r = 3 \sin\left(\frac{5\theta}{2}\right)$   
55.  $r^2 = 4 \sin 2\theta$       56.  $r^2 = \frac{1}{\theta}$

57. Convert the equation

$$r = 2(h \cos \theta + k \sin \theta)$$

to rectangular form and verify that it is the equation of a circle. Find the radius and the rectangular coordinates of the center of the circle.

**58. Distance Formula**

- (a) Verify that the Distance Formula for the distance between the two points  $(r_1, \theta_1)$  and  $(r_2, \theta_2)$  in polar coordinates is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)}.$$

- (b) Describe the positions of the points relative to each other for  $\theta_1 = \theta_2$ . Simplify the Distance Formula for this case. Is the simplification what you expected? Explain.
- (c) Simplify the Distance Formula for  $\theta_1 - \theta_2 = 90^\circ$ . Is the simplification what you expected? Explain.
- (d) Choose two points on the polar coordinate system and find the distance between them. Then choose different polar representations of the same two points and apply the Distance Formula again. Discuss the result.

**In Exercises 59–62, use the result of Exercise 58 to approximate the distance between the two points in polar coordinates.**

59.  $\left(1, \frac{5\pi}{6}\right), \left(4, \frac{\pi}{3}\right)$

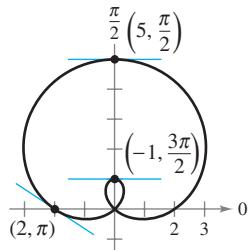
60.  $\left(8, \frac{7\pi}{4}\right), (5, \pi)$

61.  $(2, 0.5), (7, 1.2)$

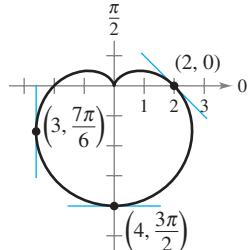
62.  $(4, 2.5), (12, 1)$

**In Exercises 63 and 64, find  $dy/dx$  and the slopes of the tangent lines shown on the graph of the polar equation.**

63.  $r = 2 + 3 \sin \theta$



64.  $r = 2(1 - \sin \theta)$



In Exercises 65–68, use a graphing utility to (a) graph the polar equation, (b) draw the tangent line at the given value of  $\theta$ , and (c) find  $dy/dx$  at the given value of  $\theta$ . (Hint: Let the increment between the values of  $\theta$  equal  $\pi/24$ .)

65.  $r = 3(1 - \cos \theta), \theta = \frac{\pi}{2}$

66.  $r = 3 - 2 \cos \theta, \theta = 0$

67.  $r = 3 \sin \theta, \theta = \frac{\pi}{3}$

68.  $r = 4, \theta = \frac{\pi}{4}$

**In Exercises 69 and 70, find the points of horizontal and vertical tangency (if any) to the polar curve.**

69.  $r = 1 - \sin \theta$

70.  $r = a \sin \theta$

**In Exercises 71 and 72, find the points of horizontal tangency (if any) to the polar curve.**

71.  $r = 2 \csc \theta + 3$

72.  $r = a \sin \theta \cos^2 \theta$

In Exercises 73–76, use a graphing utility to graph the polar equation and find all points of horizontal tangency.

73.  $r = 4 \sin \theta \cos^2 \theta$

74.  $r = 3 \cos 2\theta \sec \theta$

75.  $r = 2 \csc \theta + 5$

76.  $r = 2 \cos(3\theta - 2)$

**In Exercises 77–84, sketch a graph of the polar equation and find the tangents at the pole.**

77.  $r = 5 \sin \theta$

78.  $r = 5 \cos \theta$

79.  $r = 2(1 - \sin \theta)$

80.  $r = 3(1 - \cos \theta)$

81.  $r = 4 \cos 3\theta$

82.  $r = -\sin 5\theta$

83.  $r = 3 \sin 2\theta$

84.  $r = 3 \cos 2\theta$

**In Exercises 85–96, sketch a graph of the polar equation.**

85.  $r = 8$

86.  $r = 1$

87.  $r = 4(1 + \cos \theta)$

88.  $r = 1 + \sin \theta$

89.  $r = 3 - 2 \cos \theta$

90.  $r = 5 - 4 \sin \theta$

91.  $r = 3 \csc \theta$

92.  $r = \frac{6}{2 \sin \theta - 3 \cos \theta}$

93.  $r = 2\theta$

94.  $r = \frac{1}{\theta}$

95.  $r^2 = 4 \cos 2\theta$

96.  $r^2 = 4 \sin \theta$

In Exercises 97–100, use a graphing utility to graph the equation and show that the given line is an asymptote of the graph.

Name of Graph	Polar Equation	Asymptote
97. Conchoid	$r = 2 - \sec \theta$	$x = -1$
98. Conchoid	$r = 2 + \csc \theta$	$y = 1$
99. Hyperbolic spiral	$r = 2/\theta$	$y = 2$
100. Strophoid	$r = 2 \cos 2\theta \sec \theta$	$x = -2$

**WRITING ABOUT CONCEPTS**

101. Describe the differences between the rectangular coordinate system and the polar coordinate system.
102. Give the equations for the coordinate conversion from rectangular to polar coordinates and vice versa.
103. How are the slopes of tangent lines determined in polar coordinates? What are tangent lines at the pole and how are they determined?

**CAPSTONE**

104. Describe the graphs of the following polar equations.

(a)  $r = 7$

(b)  $r^2 = 7$

(c)  $r = \frac{7}{\cos \theta}$

(d)  $r = \frac{7}{\sin \theta}$

(e)  $r = 7 \cos \theta$

(f)  $r = 7 \sin \theta$

105. Sketch the graph of  $r = 4 \sin \theta$  over each interval.

(a)  $0 \leq \theta \leq \frac{\pi}{2}$       (b)  $\frac{\pi}{2} \leq \theta \leq \pi$       (c)  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

106. **Think About It** Use a graphing utility to graph the polar equation  $r = 6[1 + \cos(\theta - \phi)]$  for (a)  $\phi = 0$ , (b)  $\phi = \pi/4$ , and (c)  $\phi = \pi/2$ . Use the graphs to describe the effect of the angle  $\phi$ . Write the equation as a function of  $\sin \theta$  for part (c).

- 107.** Verify that if the curve whose polar equation is  $r = f(\theta)$  is rotated about the pole through an angle  $\phi$ , then an equation for the rotated curve is  $r = f(\theta - \phi)$ .
- 108.** The polar form of an equation of a curve is  $r = f(\sin \theta)$ . Show that the form becomes
- $r = f(-\cos \theta)$  if the curve is rotated counterclockwise  $\pi/2$  radians about the pole.
  - $r = f(-\sin \theta)$  if the curve is rotated counterclockwise  $\pi$  radians about the pole.
  - $r = f(\cos \theta)$  if the curve is rotated counterclockwise  $3\pi/2$  radians about the pole.

In Exercises 109–112, use the results of Exercises 107 and 108.



- 109.** Write an equation for the limaçon  $r = 2 - \sin \theta$  after it has been rotated by the given amount. Use a graphing utility to graph the rotated limaçon.

(a)  $\frac{\pi}{4}$     (b)  $\frac{\pi}{2}$     (c)  $\pi$     (d)  $\frac{3\pi}{2}$



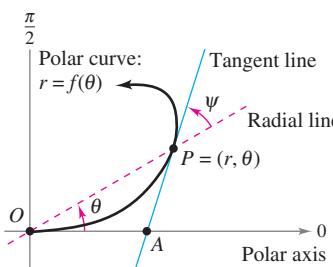
- 110.** Write an equation for the rose curve  $r = 2 \sin 2\theta$  after it has been rotated by the given amount. Verify the results by using a graphing utility to graph the rotated rose curve.

(a)  $\frac{\pi}{6}$     (b)  $\frac{\pi}{2}$     (c)  $\frac{2\pi}{3}$     (d)  $\pi$

- 111.** Sketch the graph of each equation.

(a)  $r = 1 - \sin \theta$     (b)  $r = 1 - \sin\left(\theta - \frac{\pi}{4}\right)$

- 112.** Prove that the tangent of the angle  $\psi$  ( $0 \leq \psi \leq \pi/2$ ) between the radial line and the tangent line at the point  $(r, \theta)$  on the graph of  $r = f(\theta)$  (see figure) is given by  $\tan \psi = |r/(dr/d\theta)|$ .



- 113–118.** In Exercises 113–118, use the result of Exercise 112 to find the angle  $\psi$  between the radial and tangent lines to the graph for the indicated value of  $\theta$ . Use a graphing utility to graph the polar equation, the radial line, and the tangent line for the indicated value of  $\theta$ . Identify the angle  $\psi$ .

Polar Equation	Value of $\theta$
113. $r = 2(1 - \cos \theta)$	$\theta = \pi$
114. $r = 3(1 - \cos \theta)$	$\theta = 3\pi/4$
115. $r = 2 \cos 3\theta$	$\theta = \pi/4$
116. $r = 4 \sin 2\theta$	$\theta = \pi/6$
117. $r = \frac{6}{1 - \cos \theta}$	$\theta = 2\pi/3$
118. $r = 5$	$\theta = \pi/6$

**True or False?** In Exercises 119–122, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

- If  $(r_1, \theta_1)$  and  $(r_2, \theta_2)$  represent the same point on the polar coordinate system, then  $|r_1| = |r_2|$ .
- If  $(r, \theta_1)$  and  $(r, \theta_2)$  represent the same point on the polar coordinate system, then  $\theta_1 = \theta_2 + 2\pi n$  for some integer  $n$ .
- If  $x > 0$ , then the point  $(x, y)$  on the rectangular coordinate system can be represented by  $(r, \theta)$  on the polar coordinate system, where  $r = \sqrt{x^2 + y^2}$  and  $\theta = \arctan(y/x)$ .
- The polar equations  $r = \sin 2\theta$ ,  $r = -\sin 2\theta$ , and  $r = \sin(-2\theta)$  all have the same graph.

## SECTION PROJECT

### Anamorphic Art

Anamorphic art appears distorted, but when the art is viewed from a particular point or is viewed with a device such as a mirror, it appears to be normal. Use the anamorphic transformations

$$r = y + 16 \quad \text{and} \quad \theta = -\frac{\pi}{8}x, \quad -\frac{3\pi}{4} \leq \theta \leq \frac{3\pi}{4}$$

to sketch the transformed polar image of the rectangular graph. When the reflection (in a cylindrical mirror centered at the pole) of each polar image is viewed from the polar axis, the viewer will see the original rectangular image.

- (a)  $y = 3$     (b)  $x = 2$     (c)  $y = x + 5$     (d)  $x^2 + (y - 5)^2 = 5^2$

From the Millington-Barnard Collection of Scientific Apparatus, ca 1855  
The University of Mississippi Museum, Oxford, Mississippi



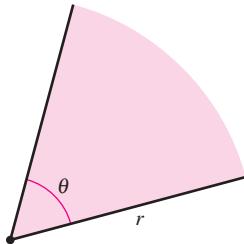
This example of anamorphic art is from the Millington-Barnard Collection at the University of Mississippi. When the reflection of the transformed “polar painting” is viewed in the mirror, the viewer sees the distorted art in its proper proportions.

**FOR FURTHER INFORMATION** For more information on anamorphic art, see the article “Anamorphisms” by Philip Hickin in the *Mathematical Gazette*.

## 10.5 Area and Arc Length in Polar Coordinates

- Find the area of a region bounded by a polar graph.
- Find the points of intersection of two polar graphs.
- Find the arc length of a polar graph.
- Find the area of a surface of revolution (polar form).

### Area of a Polar Region



The area of a sector of a circle is  $A = \frac{1}{2}\theta r^2$ .

Figure 10.49

The development of a formula for the area of a polar region parallels that for the area of a region on the rectangular coordinate system, but uses sectors of a circle instead of rectangles as the basic elements of area. In Figure 10.49, note that the area of a circular sector of radius  $r$  is given by  $\frac{1}{2}\theta r^2$ , provided  $\theta$  is measured in radians.

Consider the function given by  $r = f(\theta)$ , where  $f$  is continuous and nonnegative on the interval given by  $\alpha \leq \theta \leq \beta$ . The region bounded by the graph of  $f$  and the radial lines  $\theta = \alpha$  and  $\theta = \beta$  is shown in Figure 10.50(a). To find the area of this region, partition the interval  $[\alpha, \beta]$  into  $n$  equal subintervals

$$\alpha = \theta_0 < \theta_1 < \theta_2 < \dots < \theta_{n-1} < \theta_n = \beta.$$

Then approximate the area of the region by the sum of the areas of the  $n$  sectors, as shown in Figure 10.50(b).

$$\text{Radius of } i\text{th sector} = f(\theta_i)$$

$$\text{Central angle of } i\text{th sector} = \frac{\beta - \alpha}{n} = \Delta\theta$$

$$A \approx \sum_{i=1}^n \left(\frac{1}{2}\right) \Delta\theta [f(\theta_i)]^2$$

Taking the limit as  $n \rightarrow \infty$  produces

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \frac{1}{2} \sum_{i=1}^n [f(\theta_i)]^2 \Delta\theta \\ &= \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \end{aligned}$$

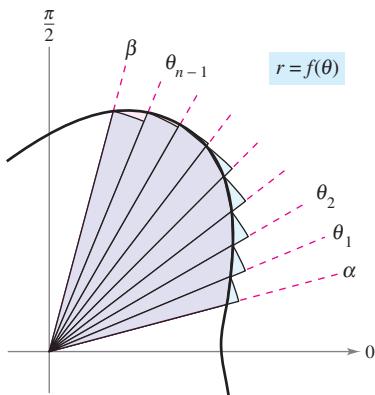
which leads to the following theorem.

### THEOREM 10.13 AREA IN POLAR COORDINATES

If  $f$  is continuous and nonnegative on the interval  $[\alpha, \beta]$ ,  $0 < \beta - \alpha \leq 2\pi$ , then the area of the region bounded by the graph of  $r = f(\theta)$  between the radial lines  $\theta = \alpha$  and  $\theta = \beta$  is given by

$$\begin{aligned} A &= \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \\ &= \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta. \quad 0 < \beta - \alpha \leq 2\pi \end{aligned}$$

(a)



(b)

Figure 10.50

**NOTE** You can use the same formula to find the area of a region bounded by the graph of a continuous *nonpositive* function. However, the formula is not necessarily valid if  $f$  takes on both positive *and* negative values in the interval  $[\alpha, \beta]$ . ■

## EXAMPLE 1 Finding the Area of a Polar Region

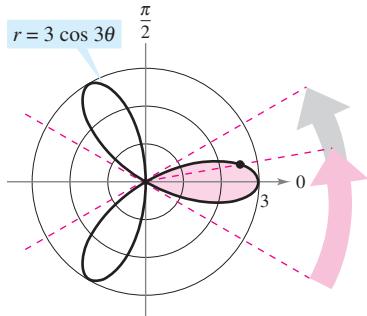
Find the area of one petal of the rose curve given by  $r = 3 \cos 3\theta$ .

**Solution** In Figure 10.51, you can see that the petal on the right is traced as  $\theta$  increases from  $-\pi/6$  to  $\pi/6$ . So, the area is

$$\begin{aligned} A &= \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta = \frac{1}{2} \int_{-\pi/6}^{\pi/6} (3 \cos 3\theta)^2 d\theta \\ &= \frac{9}{2} \int_{-\pi/6}^{\pi/6} \frac{1 + \cos 6\theta}{2} d\theta \\ &= \frac{9}{4} \left[ \theta + \frac{\sin 6\theta}{6} \right]_{-\pi/6}^{\pi/6} \\ &= \frac{9}{4} \left( \frac{\pi}{6} + \frac{\pi}{6} \right) = \frac{3\pi}{4}. \end{aligned}$$

Formula for area in polar coordinates

Trigonometric identity



The area of one petal of the rose curve that lies between the radial lines  $\theta = -\pi/6$  and  $\theta = \pi/6$  is  $3\pi/4$ .

Figure 10.51

**NOTE** To find the area of the region lying inside all three petals of the rose curve in Example 1, you could not simply integrate between 0 and  $2\pi$ . In doing this you would obtain  $9\pi/2$ , which is twice the area of the three petals. The duplication occurs because the rose curve is traced twice as  $\theta$  increases from 0 to  $2\pi$ . ■

## EXAMPLE 2 Finding the Area Bounded by a Single Curve

Find the area of the region lying between the inner and outer loops of the limaçon  $r = 1 - 2 \sin \theta$ .

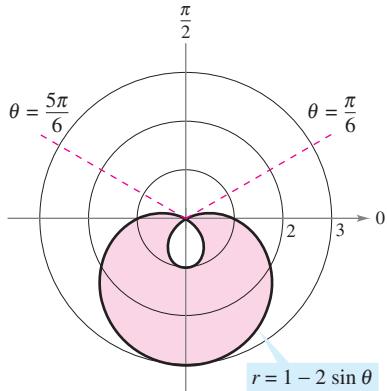
**Solution** In Figure 10.52, note that the inner loop is traced as  $\theta$  increases from  $\pi/6$  to  $5\pi/6$ . So, the area inside the *inner loop* is

$$\begin{aligned} A_1 &= \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta = \frac{1}{2} \int_{\pi/6}^{5\pi/6} (1 - 2 \sin \theta)^2 d\theta \\ &= \frac{1}{2} \int_{\pi/6}^{5\pi/6} (1 - 4 \sin \theta + 4 \sin^2 \theta) d\theta \\ &= \frac{1}{2} \int_{\pi/6}^{5\pi/6} \left[ 1 - 4 \sin \theta + 4 \left( \frac{1 - \cos 2\theta}{2} \right) \right] d\theta \\ &= \frac{1}{2} \int_{\pi/6}^{5\pi/6} (3 - 4 \sin \theta - 2 \cos 2\theta) d\theta \\ &= \frac{1}{2} \left[ 3\theta + 4 \cos \theta - \sin 2\theta \right]_{\pi/6}^{5\pi/6} \\ &= \frac{1}{2} (2\pi - 3\sqrt{3}) \\ &= \pi - \frac{3\sqrt{3}}{2}. \end{aligned}$$

Formula for area in polar coordinates

Trigonometric identity

Simplify.



The area between the inner and outer loops is approximately 8.34.

Figure 10.52

In a similar way, you can integrate from  $5\pi/6$  to  $13\pi/6$  to find that the area of the region lying inside the *outer loop* is  $A_2 = 2\pi + (3\sqrt{3}/2)$ . The area of the region lying between the two loops is the difference of  $A_2$  and  $A_1$ .

$$A = A_2 - A_1 = \left( 2\pi + \frac{3\sqrt{3}}{2} \right) - \left( \pi - \frac{3\sqrt{3}}{2} \right) = \pi + 3\sqrt{3} \approx 8.34$$

■

## Points of Intersection of Polar Graphs

Because a point may be represented in different ways in polar coordinates, care must be taken in determining the points of intersection of two polar graphs. For example, consider the points of intersection of the graphs of

$$r = 1 - 2 \cos \theta \quad \text{and} \quad r = 1$$

as shown in Figure 10.53. If, as with rectangular equations, you attempted to find the points of intersection by solving the two equations simultaneously, you would obtain

$$r = 1 - 2 \cos \theta$$

First equation

$$1 = 1 - 2 \cos \theta$$

Substitute  $r = 1$  from 2nd equation into 1st equation.

$$\cos \theta = 0$$

Simplify.

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

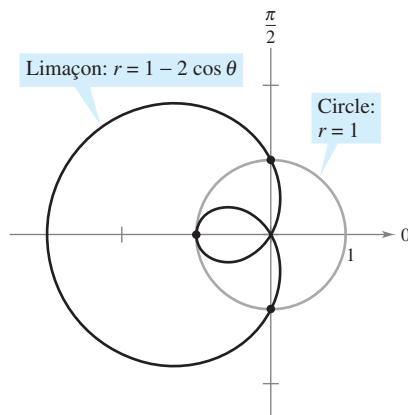
Solve for  $\theta$ .

**FOR FURTHER INFORMATION** For more information on using technology to find points of intersection, see the article “Finding Points of Intersection of Polar-Coordinate Graphs” by Warren W. Esty in *Mathematics Teacher*. To view this article, go to the website [www.matharticles.com](http://www.matharticles.com).

The corresponding points of intersection are  $(1, \pi/2)$  and  $(1, 3\pi/2)$ . However, from Figure 10.53 you can see that there is a *third* point of intersection that did not show up when the two polar equations were solved simultaneously. (This is one reason why you should sketch a graph when finding the area of a polar region.) The reason the third point was not found is that it does not occur with the same coordinates in the two graphs. On the graph of  $r = 1$ , the point occurs with coordinates  $(1, \pi)$ , but on the graph of  $r = 1 - 2 \cos \theta$ , the point occurs with coordinates  $(-1, 0)$ .

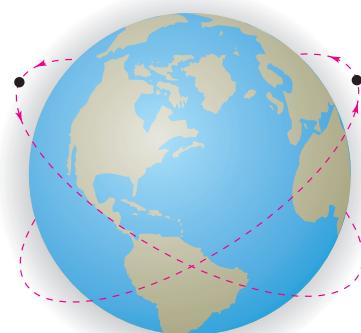
You can compare the problem of finding points of intersection of two polar graphs with that of finding collision points of two satellites in intersecting orbits about Earth, as shown in Figure 10.54. The satellites will not collide as long as they reach the points of intersection at different times ( $\theta$ -values). Collisions will occur only at the points of intersection that are “simultaneous points”—those reached at the same time ( $\theta$ -value).

**NOTE** Because the pole can be represented by  $(0, \theta)$ , where  $\theta$  is *any* angle, you should check separately for the pole when finding points of intersection. ■



Three points of intersection:  $(1, \pi/2)$ ,  $(-1, 0)$ ,  $(1, 3\pi/2)$

Figure 10.53



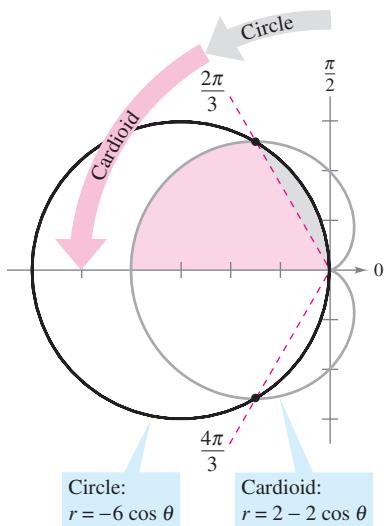
The paths of satellites can cross without causing a collision.

Figure 10.54

**EXAMPLE 3** Finding the Area of a Region Between Two Curves

Find the area of the region common to the two regions bounded by the following curves.

$$\begin{array}{ll} r = -6 \cos \theta & \text{Circle} \\ r = 2 - 2 \cos \theta & \text{Cardioid} \end{array}$$



**Figure 10.55**

**Solution** Because both curves are symmetric with respect to the  $x$ -axis, you can work with the upper half-plane, as shown in Figure 10.55. The gray shaded region lies between the circle and the radial line  $\theta = 2\pi/3$ . Because the circle has coordinates  $(0, \pi/2)$  at the pole, you can integrate between  $\pi/2$  and  $2\pi/3$  to obtain the area of this region. The region that is shaded red is bounded by the radial lines  $\theta = 2\pi/3$  and  $\theta = \pi$  and the cardioid. So, you can find the area of this second region by integrating between  $2\pi/3$  and  $\pi$ . The sum of these two integrals gives the area of the common region lying above the radial line  $\theta = \pi$ .

$$\begin{aligned} A &= \frac{1}{2} \overbrace{\int_{\pi/2}^{2\pi/3} (-6 \cos \theta)^2 d\theta + \frac{1}{2} \int_{2\pi/3}^{\pi} (2 - 2 \cos \theta)^2 d\theta}^{\text{Region between circle}} + \overbrace{\int_{2\pi/3}^{\pi} (4 - 8 \cos \theta + 4 \cos^2 \theta) d\theta}^{\text{Region between cardioid and radial lines } \theta = 2\pi/3 \text{ and } \theta = \pi} \\ &= 18 \int_{\pi/2}^{2\pi/3} \cos^2 \theta d\theta + \frac{1}{2} \int_{2\pi/3}^{\pi} (4 - 8 \cos \theta + 4 \cos^2 \theta) d\theta \\ &= 9 \int_{\pi/2}^{2\pi/3} (1 + \cos 2\theta) d\theta + \int_{2\pi/3}^{\pi} (3 - 4 \cos \theta + \cos 2\theta) d\theta \\ &= 9 \left[ \theta + \frac{\sin 2\theta}{2} \right]_{\pi/2}^{2\pi/3} + \left[ 3\theta - 4 \sin \theta + \frac{\sin 2\theta}{2} \right]_{2\pi/3}^{\pi} \\ &= 9 \left( \frac{2\pi}{3} - \frac{\sqrt{3}}{4} - \frac{\pi}{2} \right) + \left( 3\pi - 2\pi + 2\sqrt{3} + \frac{\sqrt{3}}{4} \right) \\ &= \frac{5\pi}{2} \\ &\approx 7.85 \end{aligned}$$

Finally, multiplying by 2, you can conclude that the total area is  $5\pi$ . ■

**NOTE** To check the reasonableness of the result obtained in Example 3, note that the area of the circular region is  $\pi r^2 = 9\pi$ . So, it seems reasonable that the area of the region lying inside the circle and the cardioid is  $5\pi$ . ■

To see the benefit of polar coordinates for finding the area in Example 3, consider the following integral, which gives the comparable area in rectangular coordinates.

$$\frac{A}{2} = \int_{-4}^{-3/2} \sqrt{2\sqrt{1-2x} - x^2 - 2x + 2} dx + \int_{-3/2}^0 \sqrt{-x^2 - 6x} dx$$

Use the integration capabilities of a graphing utility to show that you obtain the same area as that found in Example 3.

## Arc Length in Polar Form

**NOTE** When applying the arc length formula to a polar curve, be sure that the curve is traced out only once on the interval of integration. For instance, the rose curve given by  $r = \cos 3\theta$  is traced out once on the interval  $0 \leq \theta \leq \pi$ , but is traced out twice on the interval  $0 \leq \theta \leq 2\pi$ .

The formula for the length of a polar arc can be obtained from the arc length formula for a curve described by parametric equations. (See Exercise 89.)

### THEOREM 10.14 ARC LENGTH OF A POLAR CURVE

Let  $f$  be a function whose derivative is continuous on an interval  $\alpha \leq \theta \leq \beta$ .

The length of the graph of  $r = f(\theta)$  from  $\theta = \alpha$  to  $\theta = \beta$  is

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

### EXAMPLE 4 Finding the Length of a Polar Curve

Find the length of the arc from  $\theta = 0$  to  $\theta = 2\pi$  for the cardioid

$$r = f(\theta) = 2 - 2 \cos \theta$$

as shown in Figure 10.56.

**Solution** Because  $f'(\theta) = 2 \sin \theta$ , you can find the arc length as follows.

$$\begin{aligned} s &= \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta && \text{Formula for arc length of a polar curve} \\ &= \int_0^{2\pi} \sqrt{(2 - 2 \cos \theta)^2 + (2 \sin \theta)^2} d\theta \\ &= 2\sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos \theta} d\theta && \text{Simplify.} \\ &= 2\sqrt{2} \int_0^{2\pi} \sqrt{2 \sin^2 \frac{\theta}{2}} d\theta && \text{Trigonometric identity} \\ &= 4 \int_0^{2\pi} \sin \frac{\theta}{2} d\theta && \sin \frac{\theta}{2} \geq 0 \text{ for } 0 \leq \theta \leq 2\pi \\ &= 8 \left[ -\cos \frac{\theta}{2} \right]_0^{2\pi} \\ &= 8(1 + 1) \\ &= 16 \end{aligned}$$

In the fifth step of the solution, it is legitimate to write

$$\sqrt{2 \sin^2(\theta/2)} = \sqrt{2} \sin(\theta/2)$$

rather than

$$\sqrt{2 \sin^2(\theta/2)} = \sqrt{2} |\sin(\theta/2)|$$

because  $\sin(\theta/2) \geq 0$  for  $0 \leq \theta \leq 2\pi$ . ■

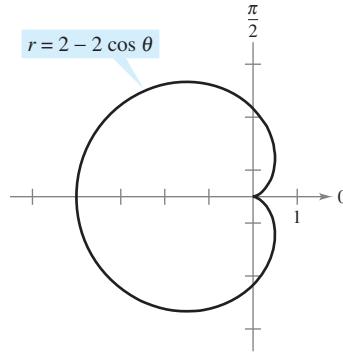


Figure 10.56

**NOTE** Using Figure 10.56, you can determine the reasonableness of this answer by comparing it with the circumference of a circle. For example, a circle of radius  $\frac{5}{2}$  has a circumference of  $5\pi \approx 15.7$ . ■

## Area of a Surface of Revolution

The polar coordinate versions of the formulas for the area of a surface of revolution can be obtained from the parametric versions given in Theorem 10.9, using the equations  $x = r \cos \theta$  and  $y = r \sin \theta$ .

### THEOREM 10.15 AREA OF A SURFACE OF REVOLUTION

**NOTE** When using Theorem 10.15, check to see that the graph of  $r = f(\theta)$  is traced only once on the interval  $\alpha \leq \theta \leq \beta$ . For example, the circle given by  $r = \cos \theta$  is traced only once on the interval  $0 \leq \theta \leq \pi$ .

Let  $f$  be a function whose derivative is continuous on an interval  $\alpha \leq \theta \leq \beta$ . The area of the surface formed by revolving the graph of  $r = f(\theta)$  from  $\theta = \alpha$  to  $\theta = \beta$  about the indicated line is as follows.

1.  $S = 2\pi \int_{\alpha}^{\beta} f(\theta) \sin \theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta$  About the polar axis
2.  $S = 2\pi \int_{\alpha}^{\beta} f(\theta) \cos \theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta$  About the line  $\theta = \frac{\pi}{2}$

### EXAMPLE 5 Finding the Area of a Surface of Revolution

Find the area of the surface formed by revolving the circle  $r = f(\theta) = \cos \theta$  about the line  $\theta = \pi/2$ , as shown in Figure 10.57.

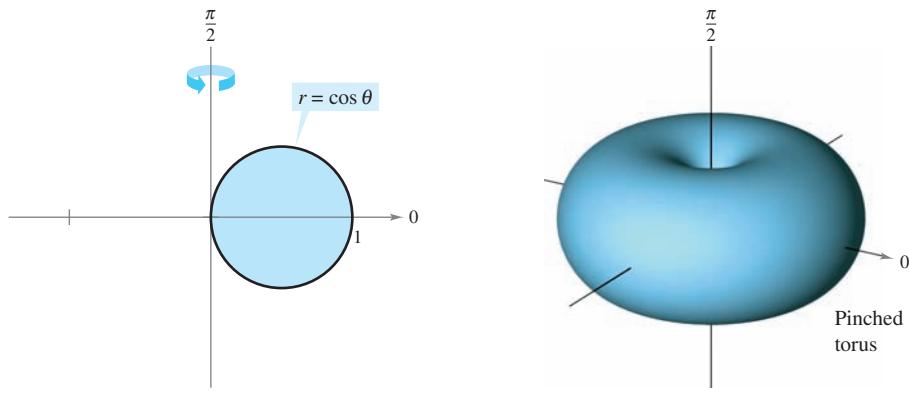


Figure 10.57

**Solution** You can use the second formula given in Theorem 10.15 with  $f'(\theta) = -\sin \theta$ . Because the circle is traced once as  $\theta$  increases from 0 to  $\pi$ , you have

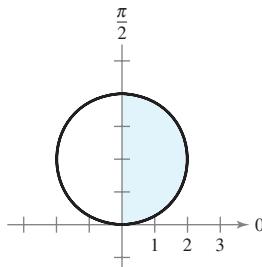
$$\begin{aligned}
 S &= 2\pi \int_{\alpha}^{\beta} f(\theta) \cos \theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta && \text{Formula for area of a surface of revolution} \\
 &= 2\pi \int_0^{\pi} \cos \theta (\cos \theta) \sqrt{\cos^2 \theta + \sin^2 \theta} d\theta \\
 &= 2\pi \int_0^{\pi} \cos^2 \theta d\theta && \text{Trigonometric identity} \\
 &= \pi \int_0^{\pi} (1 + \cos 2\theta) d\theta && \text{Trigonometric identity} \\
 &= \pi \left[ \theta + \frac{\sin 2\theta}{2} \right]_0^{\pi} = \pi^2. && \blacksquare
 \end{aligned}$$

## 10.5 Exercises

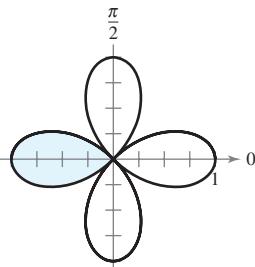
See [www.CalcChat.com](http://www.CalcChat.com) for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, write an integral that represents the area of the shaded region of the figure. Do not evaluate the integral.

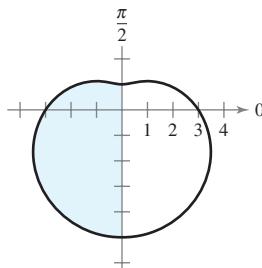
1.  $r = 4 \sin \theta$



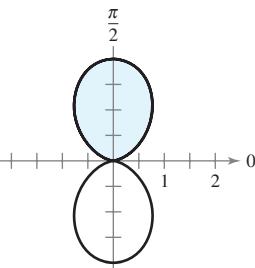
2.  $r = \cos 2\theta$



3.  $r = 3 - 2 \sin \theta$



4.  $r = 1 - \cos 2\theta$



In Exercises 5–16, find the area of the region.

5. Interior of  $r = 6 \sin \theta$

6. Interior of  $r = 3 \cos \theta$

7. One petal of  $r = 2 \cos 3\theta$

8. One petal of  $r = 4 \sin 3\theta$

9. One petal of  $r = \sin 2\theta$

10. One petal of  $r = \cos 5\theta$

11. Interior of  $r = 1 - \sin \theta$

12. Interior of  $r = 1 - \sin \theta$  (above the polar axis)

13. Interior of  $r = 5 + 2 \sin \theta$

14. Interior of  $r = 4 - 4 \cos \theta$

15. Interior of  $r^2 = 4 \cos 2\theta$

16. Interior of  $r^2 = 6 \sin 2\theta$

In Exercises 17–24, use a graphing utility to graph the polar equation and find the area of the given region.

17. Inner loop of  $r = 1 + 2 \cos \theta$

18. Inner loop of  $r = 2 - 4 \cos \theta$

19. Inner loop of  $r = 1 + 2 \sin \theta$

20. Inner loop of  $r = 4 - 6 \sin \theta$

21. Between the loops of  $r = 1 + 2 \cos \theta$

22. Between the loops of  $r = 2(1 + 2 \sin \theta)$

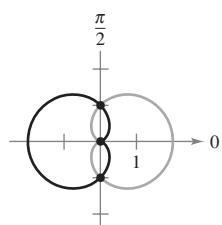
23. Between the loops of  $r = 3 - 6 \sin \theta$

24. Between the loops of  $r = \frac{1}{2} + \cos \theta$

In Exercises 25–34, find the points of intersection of the graphs of the equations.

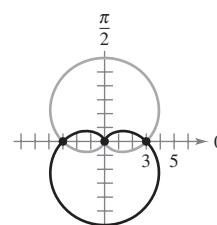
25.  $r = 1 + \cos \theta$

$r = 1 - \cos \theta$



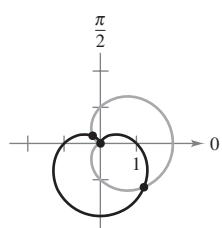
26.  $r = 3(1 + \sin \theta)$

$r = 3(1 - \sin \theta)$



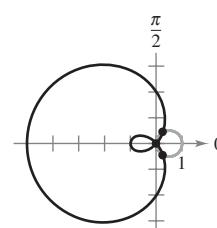
27.  $r = 1 + \cos \theta$

$r = 1 - \sin \theta$



28.  $r = 2 - 3 \cos \theta$

$r = \cos \theta$



29.  $r = 4 - 5 \sin \theta$

$r = 3 \sin \theta$

31.  $r = \frac{\theta}{2}$

$r = 2$

33.  $r = 2 \sin 2\theta$

$r = 1$

30.  $r = 1 + \cos \theta$

$r = 3 \cos \theta$

32.  $\theta = \frac{\pi}{4}$

$r = 2$

34.  $r = 3 + \sin \theta$

$r = 2 \csc \theta$

In Exercises 35 and 36, use a graphing utility to graph the polar equations and approximate the points of intersection of the graphs. Confirm your results analytically.

35.  $r = 2 + 3 \cos \theta$

$$r = \frac{\sec \theta}{2}$$

36.  $r = 3(1 - \cos \theta)$

$$r = \frac{6}{1 - \cos \theta}$$

Writing In Exercises 37 and 38, use a graphing utility to graph the polar equations and find the points of intersection of the graphs. Watch the graphs as they are traced in the viewing window. Explain why the pole is not a point of intersection obtained by solving the equations simultaneously.

37.  $r = \cos \theta$

$r = 2 - 3 \sin \theta$

38.  $r = 4 \sin \theta$

$r = 2(1 + \sin \theta)$



In Exercises 39–46, use a graphing utility to graph the polar equations and find the area of the given region.

39. Common interior of  $r = 4 \sin 2\theta$  and  $r = 2$
40. Common interior of  $r = 2(1 + \cos \theta)$  and  $r = 2(1 - \cos \theta)$
41. Common interior of  $r = 3 - 2 \sin \theta$  and  $r = -3 + 2 \sin \theta$
42. Common interior of  $r = 5 - 3 \sin \theta$  and  $r = 5 - 3 \cos \theta$
43. Common interior of  $r = 4 \sin \theta$  and  $r = 2$
44. Common interior of  $r = 2 \cos \theta$  and  $r = 2 \sin \theta$
45. Inside  $r = 2 \cos \theta$  and outside  $r = 1$
46. Inside  $r = 3 \sin \theta$  and outside  $r = 1 + \sin \theta$

In Exercises 47–50, find the area of the region.

47. Inside  $r = a(1 + \cos \theta)$  and outside  $r = a \cos \theta$
48. Inside  $r = 2a \cos \theta$  and outside  $r = a$
49. Common interior of  $r = a(1 + \cos \theta)$  and  $r = a \sin \theta$
50. Common interior of  $r = a \cos \theta$  and  $r = a \sin \theta$ , where  $a > 0$

**51. Antenna Radiation** The radiation from a transmitting antenna is not uniform in all directions. The intensity from a particular antenna is modeled by  $r = a \cos^2 \theta$ .

- (a) Convert the polar equation to rectangular form.
- (b) Use a graphing utility to graph the model for  $a = 4$  and  $a = 6$ .
- (c) Find the area of the geographical region between the two curves in part (b).

**52. Area** The area inside one or more of the three interlocking circles  $r = 2a \cos \theta$ ,  $r = 2a \sin \theta$ , and  $r = a$  is divided into seven regions. Find the area of each region.

**53. Conjecture** Find the area of the region enclosed by

$$r = a \cos(n\theta)$$

for  $n = 1, 2, 3, \dots$ . Use the results to make a conjecture about the area enclosed by the function if  $n$  is even and if  $n$  is odd.

**54. Area** Sketch the strophoid

$$r = \sec \theta - 2 \cos \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Convert this equation to rectangular coordinates. Find the area enclosed by the loop.

In Exercises 55–60, find the length of the curve over the given interval.

Polar Equation	Interval
55. $r = 8$	$0 \leq \theta \leq 2\pi$
56. $r = a$	$0 \leq \theta \leq 2\pi$
57. $r = 4 \sin \theta$	$0 \leq \theta \leq \pi$
58. $r = 2a \cos \theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
59. $r = 1 + \sin \theta$	$0 \leq \theta \leq 2\pi$
60. $r = 8(1 + \cos \theta)$	$0 \leq \theta \leq 2\pi$



In Exercises 61–66, use a graphing utility to graph the polar equation over the given interval. Use the integration capabilities of the graphing utility to approximate the length of the curve accurate to two decimal places.

61.  $r = 2\theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$
62.  $r = \sec \theta, \quad 0 \leq \theta \leq \frac{\pi}{3}$
63.  $r = \frac{1}{\theta}, \quad \pi \leq \theta \leq 2\pi$
64.  $r = e^\theta, \quad 0 \leq \theta \leq \pi$
65.  $r = \sin(3 \cos \theta), \quad 0 \leq \theta \leq \pi$
66.  $r = 2 \sin(2 \cos \theta), \quad 0 \leq \theta \leq \pi$

In Exercises 67–70, find the area of the surface formed by revolving the curve about the given line.

Polar Equation	Interval	Axis of Revolution
67. $r = 6 \cos \theta$	$0 \leq \theta \leq \frac{\pi}{2}$	Polar axis
68. $r = a \cos \theta$	$0 \leq \theta \leq \frac{\pi}{2}$	$\theta = \frac{\pi}{2}$
69. $r = e^{a\theta}$	$0 \leq \theta \leq \frac{\pi}{2}$	$\theta = \frac{\pi}{2}$
70. $r = a(1 + \cos \theta)$	$0 \leq \theta \leq \pi$	Polar axis



In Exercises 71 and 72, use the integration capabilities of a graphing utility to approximate to two decimal places the area of the surface formed by revolving the curve about the polar axis.

71.  $r = 4 \cos 2\theta, \quad 0 \leq \theta \leq \frac{\pi}{4}$
72.  $r = \theta, \quad 0 \leq \theta \leq \pi$

### WRITING ABOUT CONCEPTS

73. Explain why finding points of intersection of polar graphs may require further analysis beyond solving two equations simultaneously.
74. Which integral yields the arc length of  $r = 3(1 - \cos 2\theta)$ ? State why the other integrals are incorrect.
  - (a)  $3 \int_0^{2\pi} \sqrt{(1 - \cos 2\theta)^2 + 4 \sin^2 2\theta} d\theta$
  - (b)  $12 \int_0^{\pi/4} \sqrt{(1 - \cos 2\theta)^2 + 4 \sin^2 2\theta} d\theta$
  - (c)  $3 \int_0^{\pi} \sqrt{(1 - \cos 2\theta)^2 + 4 \sin^2 2\theta} d\theta$
  - (d)  $6 \int_0^{\pi/2} \sqrt{(1 - \cos 2\theta)^2 + 4 \sin^2 2\theta} d\theta$
75. Give the integral formulas for the area of the surface of revolution formed when the graph of  $r = f(\theta)$  is revolved about (a) the  $x$ -axis and (b) the  $y$ -axis.

**CAPSTONE**

- 76.** For each polar equation, sketch its graph, determine the interval that traces the graph only once, and find the area of the region bounded by the graph using a geometric formula and integration.

(a)  $r = 10 \cos \theta$       (b)  $r = 5 \sin \theta$

- 77. Surface Area of a Torus** Find the surface area of the torus generated by revolving the circle given by  $r = 2$  about the line  $r = 5 \sec \theta$ .

- 78. Surface Area of a Torus** Find the surface area of the torus generated by revolving the circle given by  $r = a$  about the line  $r = b \sec \theta$ , where  $0 < a < b$ .

- 79. Approximating Area** Consider the circle  $r = 8 \cos \theta$ .

- (a) Find the area of the circle.  
 (b) Complete the table giving the areas  $A$  of the sectors of the circle between  $\theta = 0$  and the values of  $\theta$  in the table.

$\theta$	0.2	0.4	0.6	0.8	1.0	1.2	1.4
$A$							

- (c) Use the table in part (b) to approximate the values of  $\theta$  for which the sector of the circle composes  $\frac{1}{4}$ ,  $\frac{1}{2}$ , and  $\frac{3}{4}$  of the total area of the circle.  
 (d) Use a graphing utility to approximate, to two decimal places, the angles  $\theta$  for which the sector of the circle composes  $\frac{1}{4}$ ,  $\frac{1}{2}$ , and  $\frac{3}{4}$  of the total area of the circle.  
 (e) Do the results of part (d) depend on the radius of the circle? Explain.

- 80. Approximate Area** Consider the circle  $r = 3 \sin \theta$ .

- (a) Find the area of the circle.  
 (b) Complete the table giving the areas  $A$  of the sectors of the circle between  $\theta = 0$  and the values of  $\theta$  in the table.

$\theta$	0.2	0.4	0.6	0.8	1.0	1.2	1.4
$A$							

- (c) Use the table in part (b) to approximate the values of  $\theta$  for which the sector of the circle composes  $\frac{1}{8}$ ,  $\frac{1}{4}$ , and  $\frac{1}{2}$  of the total area of the circle.  
 (d) Use a graphing utility to approximate, to two decimal places, the angles  $\theta$  for which the sector of the circle composes  $\frac{1}{8}$ ,  $\frac{1}{4}$ , and  $\frac{1}{2}$  of the total area of the circle.

- 81.** What conic section does the following polar equation represent?

$$r = a \sin \theta + b \cos \theta$$

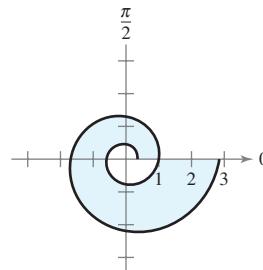
- 82. Area** Find the area of the circle given by  $r = \sin \theta + \cos \theta$ . Check your result by converting the polar equation to rectangular form, then using the formula for the area of a circle.

- 83. Spiral of Archimedes** The curve represented by the equation  $r = a\theta$ , where  $a$  is a constant, is called the spiral of Archimedes.

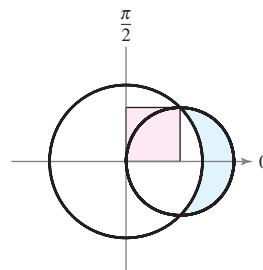


- (a) Use a graphing utility to graph  $r = \theta$ , where  $\theta \geq 0$ . What happens to the graph of  $r = a\theta$  as  $a$  increases? What happens if  $\theta \leq 0$ ?  
 (b) Determine the points on the spiral  $r = a\theta$  ( $a > 0$ ,  $\theta \geq 0$ ), where the curve crosses the polar axis.  
 (c) Find the length of  $r = \theta$  over the interval  $0 \leq \theta \leq 2\pi$ .  
 (d) Find the area under the curve  $r = \theta$  for  $0 \leq \theta \leq 2\pi$ .

- 84. Logarithmic Spiral** The curve represented by the equation  $r = ae^{b\theta}$ , where  $a$  and  $b$  are constants, is called a **logarithmic spiral**. The figure shows the graph of  $r = e^{\theta/6}$ ,  $-2\pi \leq \theta \leq 2\pi$ . Find the area of the shaded region.



- 85.** The larger circle in the figure is the graph of  $r = 1$ . Find the polar equation of the smaller circle such that the shaded regions are equal.



- 86. Folium of Descartes** A curve called the **folium of Descartes** can be represented by the parametric equations

$$x = \frac{3t}{1+t^3} \quad \text{and} \quad y = \frac{3t^2}{1+t^3}.$$

- (a) Convert the parametric equations to polar form.  
 (b) Sketch the graph of the polar equation from part (a).  
 (c) Use a graphing utility to approximate the area enclosed by the loop of the curve.

**True or False?** In Exercises 87 and 88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

- 87.** If  $f(\theta) > 0$  for all  $\theta$  and  $g(\theta) < 0$  for all  $\theta$ , then the graphs of  $r = f(\theta)$  and  $r = g(\theta)$  do not intersect.

- 88.** If  $f(\theta) = g(\theta)$  for  $\theta = 0$ ,  $\pi/2$ , and  $3\pi/2$ , then the graphs of  $r = f(\theta)$  and  $r = g(\theta)$  have at least four points of intersection.

- 89.** Use the formula for the arc length of a curve in parametric form to derive the formula for the arc length of a polar curve.

## 10.6 Polar Equations of Conics and Kepler's Laws

- Analyze and write polar equations of conics.
- Understand and use Kepler's Laws of planetary motion.

### Polar Equations of Conics

#### EXPLORATION

**Graphing Conics** Set a graphing utility to *polar* mode and enter polar equations of the form

$$r = \frac{a}{1 \pm b \cos \theta}$$

or

$$r = \frac{a}{1 \pm b \sin \theta}.$$

As long as  $a \neq 0$ , the graph should be a conic. What values of  $a$  and  $b$  produce parabolas? What values produce ellipses? What values produce hyperbolas?

In this chapter you have seen that the rectangular equations of ellipses and hyperbolas take simple forms when the origin lies at their *centers*. As it happens, there are many important applications of conics in which it is more convenient to use one of the foci as the reference point (the origin) for the coordinate system. For example, the sun lies at a focus of Earth's orbit. Similarly, the light source of a parabolic reflector lies at its focus. In this section you will see that polar equations of conics take simple forms if one of the foci lies at the pole.

The following theorem uses the concept of *eccentricity*, as defined in Section 10.1, to classify the three basic types of conics. A proof of this theorem is given in Appendix A.

#### THEOREM 10.16 CLASSIFICATION OF CONICS BY ECCENTRICITY

Let  $F$  be a fixed point (*focus*) and let  $D$  be a fixed line (*directrix*) in the plane. Let  $P$  be another point in the plane and let  $e$  (*eccentricity*) be the ratio of the distance between  $P$  and  $F$  to the distance between  $P$  and  $D$ . The collection of all points  $P$  with a given eccentricity is a conic.

1. The conic is an ellipse if  $0 < e < 1$ .
2. The conic is a parabola if  $e = 1$ .
3. The conic is a hyperbola if  $e > 1$ .

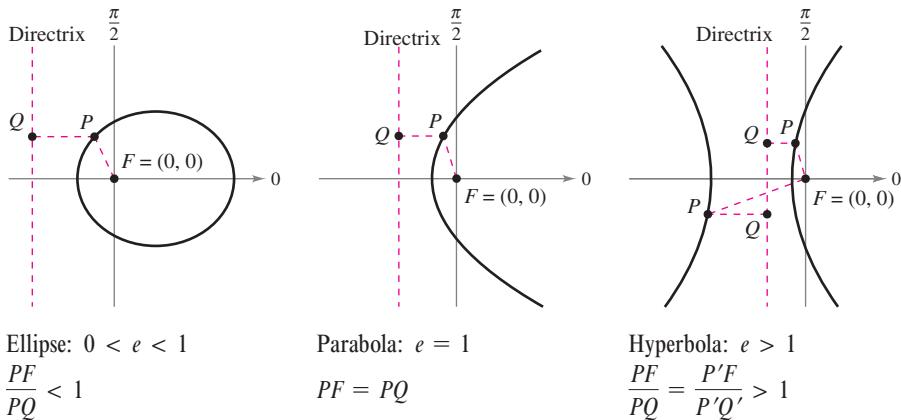


Figure 10.58

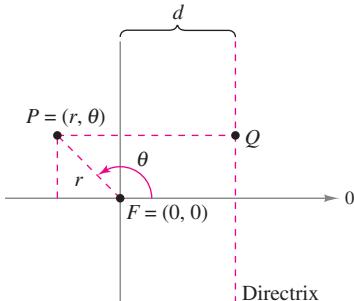
In Figure 10.58, note that for each type of conic the pole corresponds to the fixed point (focus) given in the definition. The benefit of this location can be seen in the proof of the following theorem.

**THEOREM 10.17 POLAR EQUATIONS OF CONICS**

The graph of a polar equation of the form

$$r = \frac{ed}{1 \pm e \cos \theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e \sin \theta}$$

is a conic, where  $e > 0$  is the eccentricity and  $|d|$  is the distance between the focus at the pole and its corresponding directrix.



**Figure 10.59**

**PROOF** The following is a proof for  $r = ed/(1 + e \cos \theta)$  with  $d > 0$ . In Figure 10.59, consider a vertical directrix  $d$  units to the right of the focus  $F = (0, 0)$ . If  $P = (r, \theta)$  is a point on the graph of  $r = ed/(1 + e \cos \theta)$ , the distance between  $P$  and the directrix can be shown to be

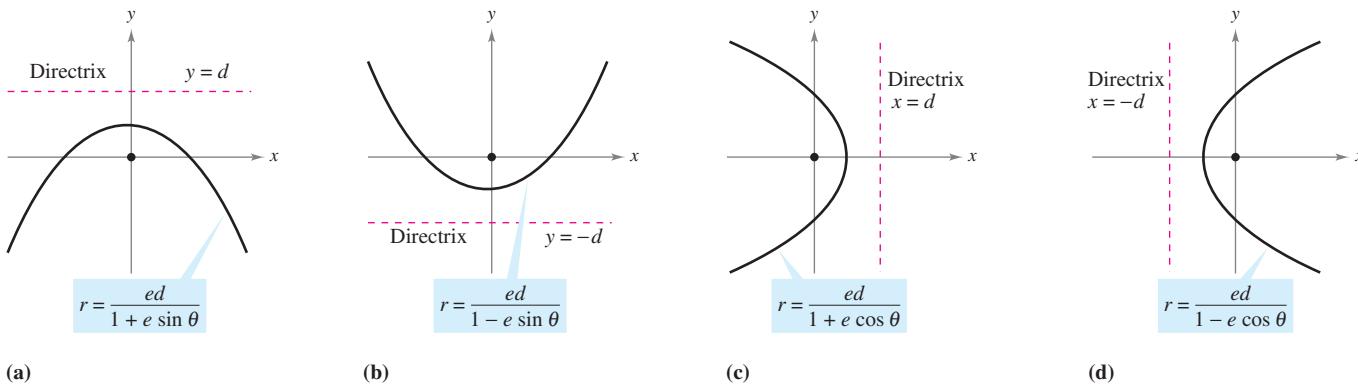
$$PQ = |d - x| = |d - r \cos \theta| = \left| \frac{r(1 + e \cos \theta)}{e} - r \cos \theta \right| = \left| \frac{r}{e} \right|.$$

Because the distance between  $P$  and the pole is simply  $PF = |r|$ , the ratio of  $PF$  to  $PQ$  is  $PF/PQ = |r|/|r/e| = |e| = e$  and, by Theorem 10.16, the graph of the equation must be a conic. The proofs of the other cases are similar. ■

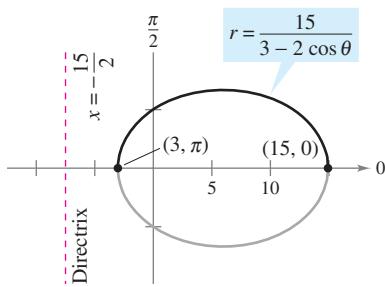
The four types of equations indicated in Theorem 10.17 can be classified as follows, where  $d > 0$ .

- a. Horizontal directrix above the pole:  $r = \frac{ed}{1 + e \sin \theta}$
- b. Horizontal directrix below the pole:  $r = \frac{ed}{1 - e \sin \theta}$
- c. Vertical directrix to the right of the pole:  $r = \frac{ed}{1 + e \cos \theta}$
- d. Vertical directrix to the left of the pole:  $r = \frac{ed}{1 - e \cos \theta}$

Figure 10.60 illustrates these four possibilities for a parabola.



**Figure 10.60**  
The four types of polar equations for a parabola

**EXAMPLE 1** Determining a Conic from Its Equation


The graph of the conic is an ellipse with  $e = \frac{2}{3}$ .

**Figure 10.61**

Sketch the graph of the conic given by  $r = \frac{15}{3 - 2 \cos \theta}$ .

**Solution** To determine the type of conic, rewrite the equation as

$$\begin{aligned} r &= \frac{15}{3 - 2 \cos \theta} \\ &= \frac{5}{1 - (2/3) \cos \theta}. \end{aligned}$$

Write original equation.

Divide numerator and denominator by 3.

So, the graph is an ellipse with  $e = \frac{2}{3}$ . You can sketch the upper half of the ellipse by plotting points from  $\theta = 0$  to  $\theta = \pi$ , as shown in Figure 10.61. Then, using symmetry with respect to the polar axis, you can sketch the lower half. ■

For the ellipse in Figure 10.61, the major axis is horizontal and the vertices lie at  $(15, 0)$  and  $(3, \pi)$ . So, the length of the *major* axis is  $2a = 18$ . To find the length of the minor axis, you can use the equations  $e = c/a$  and  $b^2 = a^2 - c^2$  to conclude that

$$b^2 = a^2 - c^2 = a^2 - (ea)^2 = a^2(1 - e^2).$$

Ellipse

Because  $e = \frac{2}{3}$ , you have

$$b^2 = 9^2 \left[ 1 - \left( \frac{2}{3} \right)^2 \right] = 45$$

which implies that  $b = \sqrt{45} = 3\sqrt{5}$ . So, the length of the minor axis is  $2b = 6\sqrt{5}$ . A similar analysis for hyperbolas yields

$$b^2 = c^2 - a^2 = (ea)^2 - a^2 = a^2(e^2 - 1).$$

Hyperbola


**EXAMPLE 2** Sketching a Conic from Its Polar Equation

Sketch the graph of the polar equation  $r = \frac{32}{3 + 5 \sin \theta}$ .

**Solution** Dividing the numerator and denominator by 3 produces

$$r = \frac{32/3}{1 + (5/3) \sin \theta}.$$

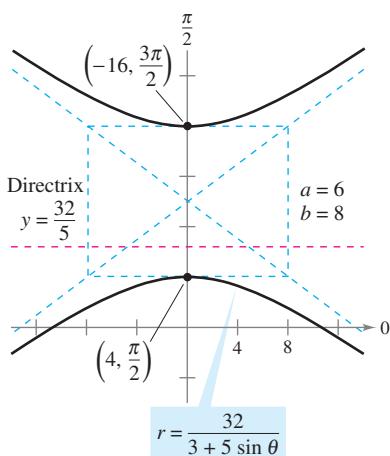
Because  $e = \frac{5}{3} > 1$ , the graph is a hyperbola. Because  $d = \frac{32}{5}$ , the directrix is the line  $y = \frac{32}{5}$ . The transverse axis of the hyperbola lies on the line  $\theta = \pi/2$ , and the vertices occur at

$$(r, \theta) = \left( 4, \frac{\pi}{2} \right) \quad \text{and} \quad (r, \theta) = \left( -16, \frac{3\pi}{2} \right).$$

Because the length of the transverse axis is 12, you can see that  $a = 6$ . To find  $b$ , write

$$b^2 = a^2(e^2 - 1) = 6^2 \left[ \left( \frac{5}{3} \right)^2 - 1 \right] = 64.$$

Therefore,  $b = 8$ . Finally, you can use  $a$  and  $b$  to determine the asymptotes of the hyperbola and obtain the sketch shown in Figure 10.62. ■



The graph of the conic is a hyperbola with  $e = \frac{5}{3}$ .

**Figure 10.62**

Mary Evans Picture Library/The Image Works



JOHANNES KEPLER (1571–1630)

Kepler formulated his three laws from the extensive data recorded by Danish astronomer Tycho Brahe, and from direct observation of the orbit of Mars.

## Kepler's Laws

Kepler's Laws, named after the German astronomer Johannes Kepler, can be used to describe the orbits of the planets about the sun.

1. Each planet moves in an elliptical orbit with the sun as a focus.
2. A ray from the sun to the planet sweeps out equal areas of the ellipse in equal times.
3. The square of the period is proportional to the cube of the mean distance between the planet and the sun.\*

Although Kepler derived these laws empirically, they were later validated by Newton. In fact, Newton was able to show that each law can be deduced from a set of universal laws of motion and gravitation that govern the movement of all heavenly bodies, including comets and satellites. This is shown in the next example, involving the comet named after the English mathematician and physicist Edmund Halley (1656–1742).

### EXAMPLE 3 Halley's Comet

Halley's comet has an elliptical orbit with the sun at one focus and has an eccentricity of  $e \approx 0.967$ . The length of the major axis of the orbit is approximately 35.88 astronomical units (AU). (An astronomical unit is defined as the mean distance between Earth and the sun, 93 million miles.) Find a polar equation for the orbit. How close does Halley's comet come to the sun?

**Solution** Using a vertical axis, you can choose an equation of the form

$$r = \frac{ed}{1 + e \sin \theta}.$$

Because the vertices of the ellipse occur when  $\theta = \pi/2$  and  $\theta = 3\pi/2$ , you can determine the length of the major axis to be the sum of the  $r$ -values of the vertices, as shown in Figure 10.63. That is,

$$2a = \frac{0.967d}{1 + 0.967} + \frac{0.967d}{1 - 0.967}$$

$$35.88 \approx 29.79d.$$

$$2a \approx 35.88$$

So,  $d \approx 1.204$  and  $ed \approx (0.967)(1.204) \approx 1.164$ . Using this value in the equation produces

$$r = \frac{1.164}{1 + 0.967 \sin \theta}$$

where  $r$  is measured in astronomical units. To find the closest point to the sun (the focus), you can write  $c = ea \approx (0.967)(17.94) \approx 17.35$ . Because  $c$  is the distance between the focus and the center, the closest point is

$$a - c \approx 17.94 - 17.35$$

$$\approx 0.59 \text{ AU}$$

$$\approx 55,000,000 \text{ miles.}$$

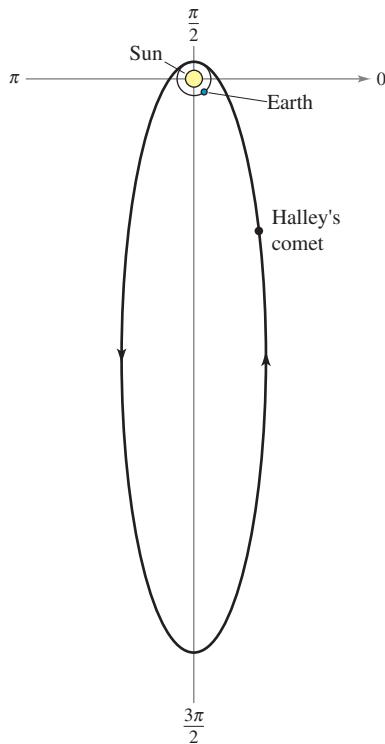
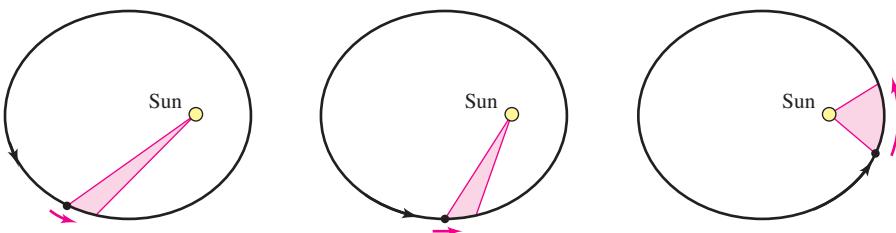


Figure 10.63

\* If Earth is used as a reference with a period of 1 year and a distance of 1 astronomical unit, the proportionality constant is 1. For example, because Mars has a mean distance to the sun of  $D = 1.524 \text{ AU}$ , its period  $P$  is given by  $D^3 = P^2$ . So, the period for Mars is  $P = 1.88$ .

Kepler's Second Law states that as a planet moves about the sun, a ray from the sun to the planet sweeps out equal areas in equal times. This law can also be applied to comets or asteroids with elliptical orbits. For example, Figure 10.64 shows the orbit of the asteroid Apollo about the sun. Applying Kepler's Second Law to this asteroid, you know that the closer it is to the sun, the greater its velocity, because a short ray must be moving quickly to sweep out as much area as a long ray.



A ray from the sun to the asteroid Apollo sweeps out equal areas in equal times.

Figure 10.64

#### EXAMPLE 4 The Asteroid Apollo

The asteroid Apollo has a period of 661 Earth days, and its orbit is approximated by the ellipse

$$r = \frac{1}{1 + (5/9) \cos \theta} = \frac{9}{9 + 5 \cos \theta}$$

where  $r$  is measured in astronomical units. How long does it take Apollo to move from the position given by  $\theta = -\pi/2$  to  $\theta = \pi/2$ , as shown in Figure 10.65?

**Solution** Begin by finding the area swept out as  $\theta$  increases from  $-\pi/2$  to  $\pi/2$ .

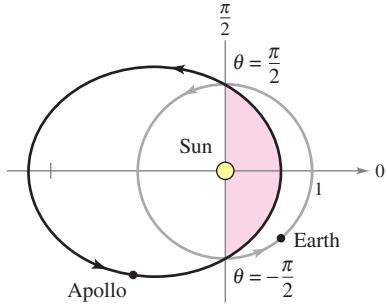


Figure 10.65

$$\begin{aligned} A &= \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta && \text{Formula for area of a polar graph} \\ &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left( \frac{9}{9 + 5 \cos \theta} \right)^2 d\theta \end{aligned}$$

Using the substitution  $u = \tan(\theta/2)$ , as discussed in Section 8.6, you obtain

$$A = \frac{81}{112} \left[ \frac{-5 \sin \theta}{9 + 5 \cos \theta} + \frac{18}{\sqrt{56}} \arctan \frac{\sqrt{56} \tan(\theta/2)}{14} \right]_{-\pi/2}^{\pi/2} \approx 0.90429.$$

Because the major axis of the ellipse has length  $2a = 81/28$  and the eccentricity is  $e = 5/9$ , you can determine that  $b = a\sqrt{1 - e^2} = 9/\sqrt{56}$ . So, the area of the ellipse is

$$\text{Area of ellipse} = \pi ab = \pi \left( \frac{81}{56} \right) \left( \frac{9}{\sqrt{56}} \right) \approx 5.46507.$$

Because the time required to complete the orbit is 661 days, you can apply Kepler's Second Law to conclude that the time  $t$  required to move from the position  $\theta = -\pi/2$  to  $\theta = \pi/2$  is given by

$$\frac{t}{661} = \frac{\text{area of elliptical segment}}{\text{area of ellipse}} \approx \frac{0.90429}{5.46507}$$

which implies that  $t \approx 109$  days. ■

## 10.6 Exercises

See [www.CalcChat.com](http://www.CalcChat.com) for worked-out solutions to odd-numbered exercises.

**Graphical Reasoning** In Exercises 1–4, use a graphing utility to graph the polar equation when (a)  $e = 1$ , (b)  $e = 0.5$ , and (c)  $e = 1.5$ . Identify the conic.

$$1. r = \frac{2e}{1 + e \cos \theta}$$

$$2. r = \frac{2e}{1 - e \cos \theta}$$

$$3. r = \frac{2e}{1 - e \sin \theta}$$

$$4. r = \frac{2e}{1 + e \sin \theta}$$

**Writing** Consider the polar equation

$$r = \frac{4}{1 + e \sin \theta}.$$

(a) Use a graphing utility to graph the equation for  $e = 0.1$ ,  $e = 0.25$ ,  $e = 0.5$ ,  $e = 0.75$ , and  $e = 0.9$ . Identify the conic and discuss the change in its shape as  $e \rightarrow 1^-$  and  $e \rightarrow 0^+$ .

(b) Use a graphing utility to graph the equation for  $e = 1$ . Identify the conic.

(c) Use a graphing utility to graph the equation for  $e = 1.1$ ,  $e = 1.5$ , and  $e = 2$ . Identify the conic and discuss the change in its shape as  $e \rightarrow 1^+$  and  $e \rightarrow \infty$ .

6. Consider the polar equation

$$r = \frac{4}{1 - 0.4 \cos \theta}.$$

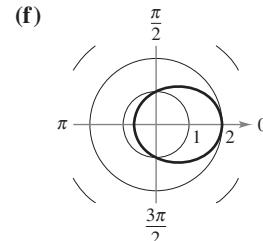
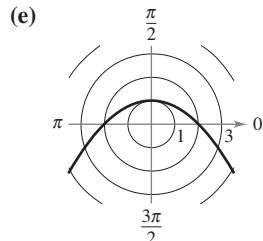
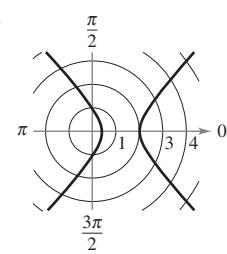
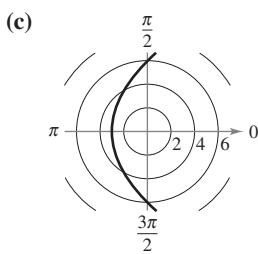
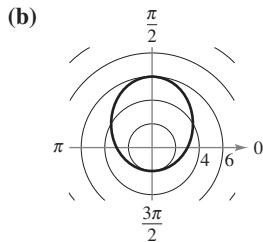
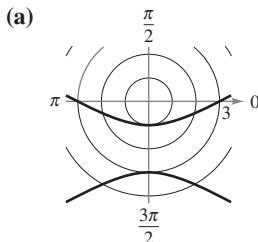
(a) Identify the conic without graphing the equation.

(b) Without graphing the following polar equations, describe how each differs from the polar equation above.

$$r = \frac{4}{1 + 0.4 \cos \theta}, \quad r = \frac{4}{1 - 0.4 \sin \theta}$$

(c) Verify the results of part (b) graphically.

In Exercises 7–12, match the polar equation with the correct graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]



$$7. r = \frac{6}{1 - \cos \theta}$$

$$9. r = \frac{3}{1 - 2 \sin \theta}$$

$$11. r = \frac{6}{2 - \sin \theta}$$

$$8. r = \frac{2}{2 - \cos \theta}$$

$$10. r = \frac{2}{1 + \sin \theta}$$

$$12. r = \frac{2}{2 + 3 \cos \theta}$$

In Exercises 13–26, find the eccentricity and the distance from the pole to the directrix of the conic. Then sketch and identify the graph. Use a graphing utility to confirm your results.

$$13. r = \frac{1}{1 - \cos \theta}$$

$$15. r = \frac{-4}{1 - \sin \theta}$$

$$17. r = \frac{6}{2 + \cos \theta}$$

$$18. r = \frac{10}{5 + 4 \sin \theta}$$

$$19. r(2 + \sin \theta) = 4$$

$$20. r(3 - 2 \cos \theta) = 6$$

$$21. r = \frac{5}{-1 + 2 \cos \theta}$$

$$22. r = \frac{-6}{3 + 7 \sin \theta}$$

$$23. r = \frac{3}{2 + 6 \sin \theta}$$

$$24. r = \frac{8}{1 + 4 \cos \theta}$$

$$25. r = \frac{300}{-12 + 6 \sin \theta}$$

$$26. r = \frac{180}{15 - 3.75 \cos \theta}$$

$$14. r = \frac{1}{1 + \sin \theta}$$

$$16. r = \frac{4}{1 + \cos \theta}$$

In Exercises 27–30, use a graphing utility to graph the polar equation. Identify the graph and find its eccentricity.

$$27. r = \frac{3}{-4 + 2 \sin \theta}$$

$$29. r = \frac{-10}{1 - \cos \theta}$$

$$28. r = \frac{-15}{2 + 8 \sin \theta}$$

$$30. r = \frac{6}{6 + 7 \cos \theta}$$



**In Exercises 31–34, use a graphing utility to graph the conic. Describe how the graph differs from the graph in the indicated exercise.**

31.  $r = \frac{-4}{1 - \sin(\theta - \pi/4)}$  (See Exercise 15.)

32.  $r = \frac{4}{1 + \cos(\theta - \pi/3)}$  (See Exercise 16.)

33.  $r = \frac{6}{2 + \cos(\theta + \pi/6)}$  (See Exercise 17.)

34.  $r = \frac{-6}{3 + 7 \sin(\theta + 2\pi/3)}$  (See Exercise 22.)

35. Write the equation for the ellipse rotated  $\pi/6$  radian clockwise from the ellipse

$$r = \frac{8}{8 + 5 \cos \theta}$$

36. Write the equation for the parabola rotated  $\pi/4$  radian counterclockwise from the parabola

$$r = \frac{9}{1 + \sin \theta}$$

**In Exercises 37–48, find a polar equation for the conic with its focus at the pole. (For convenience, the equation for the directrix is given in rectangular form.)**

<u>Conic</u>	<u>Eccentricity</u>	<u>Directrix</u>
37. Parabola	$e = 1$	$x = -3$
38. Parabola	$e = 1$	$y = 4$
39. Ellipse	$e = \frac{1}{2}$	$y = 1$
40. Ellipse	$e = \frac{3}{4}$	$y = -2$
41. Hyperbola	$e = 2$	$x = 1$
42. Hyperbola	$e = \frac{3}{2}$	$x = -1$

<u>Conic</u>	<u>Vertex or Vertices</u>
43. Parabola	$\left(1, -\frac{\pi}{2}\right)$
44. Parabola	$(5, \pi)$
45. Ellipse	$(2, 0), (8, \pi)$
46. Ellipse	$\left(2, \frac{\pi}{2}\right), \left(4, \frac{3\pi}{2}\right)$
47. Hyperbola	$\left(1, \frac{3\pi}{2}\right), \left(9, \frac{3\pi}{2}\right)$
48. Hyperbola	$(2, 0), (10, 0)$

49. Find a polar equation for the ellipse with focus  $(0, 0)$ , eccentricity  $\frac{1}{2}$ , and a directrix at  $r = 4 \sec \theta$ .
50. Find a polar equation for the hyperbola with focus  $(0, 0)$ , eccentricity 2, and a directrix at  $r = -8 \csc \theta$ .

### WRITING ABOUT CONCEPTS

51. Classify the conics by their eccentricities.

52. Identify each conic.

(a)  $r = \frac{5}{1 - 2 \cos \theta}$

(b)  $r = \frac{5}{10 - \sin \theta}$

(c)  $r = \frac{5}{3 - 3 \cos \theta}$

(d)  $r = \frac{5}{1 - 3 \sin(\theta - \pi/4)}$

53. Describe what happens to the distance between the directrix and the center of an ellipse if the foci remain fixed and  $e$  approaches 0.

### CAPSTONE

54. Explain how the graph of each conic differs from the graph of  $r = \frac{4}{1 + \sin \theta}$ .

(a)  $r = \frac{4}{1 - \cos \theta}$

(b)  $r = \frac{4}{1 - \sin \theta}$

(c)  $r = \frac{4}{1 + \cos \theta}$

(d)  $r = \frac{4}{1 - \sin(\theta - \pi/4)}$

55. Show that the polar equation for  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is

$$r^2 = \frac{b^2}{1 - e^2 \cos^2 \theta}. \quad \text{Ellipse}$$

56. Show that the polar equation for  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is

$$r^2 = \frac{-b^2}{1 - e^2 \cos^2 \theta}. \quad \text{Hyperbola}$$

**In Exercises 57–60, use the results of Exercises 55 and 56 to write the polar form of the equation of the conic.**

57. Ellipse: focus at  $(4, 0)$ ; vertices at  $(5, 0), (5, \pi)$

58. Hyperbola: focus at  $(5, 0)$ ; vertices at  $(4, 0), (4, \pi)$

59.  $\frac{x^2}{9} - \frac{y^2}{16} = 1$

60.  $\frac{x^2}{4} + y^2 = 1$



**In Exercises 61–64, use the integration capabilities of a graphing utility to approximate to two decimal places the area of the region bounded by the graph of the polar equation.**

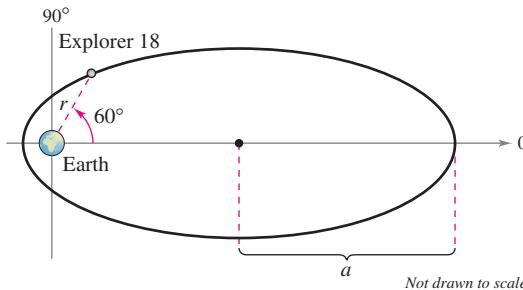
61.  $r = \frac{3}{2 - \cos \theta}$

62.  $r = \frac{9}{4 + \cos \theta}$

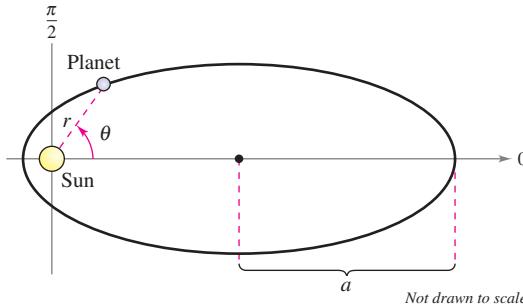
63.  $r = \frac{2}{3 - 2 \sin \theta}$

64.  $r = \frac{3}{6 + 5 \sin \theta}$

- 65. Explorer 18** On November 27, 1963, the United States launched Explorer 18. Its low and high points above the surface of Earth were approximately 119 miles and 123,000 miles (see figure). The center of Earth is the focus of the orbit. Find the polar equation for the orbit and find the distance between the surface of Earth and the satellite when  $\theta = 60^\circ$ . (Assume that the radius of Earth is 4000 miles.)



- 66. Planetary Motion** The planets travel in elliptical orbits with the sun as a focus, as shown in the figure.



- (a) Show that the polar equation of the orbit is given by

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta}$$

where  $e$  is the eccentricity.

- (b) Show that the minimum distance (*perihelion*) from the sun to the planet is  $r = a(1 - e)$  and the maximum distance (*aphelion*) is  $r = a(1 + e)$ .

In Exercises 67–70, use Exercise 66 to find the polar equation of the elliptical orbit of the planet, and the perihelion and aphelion distances.

67. Earth  $a = 1.496 \times 10^8$  kilometers

$$e = 0.0167$$

68. Saturn  $a = 1.427 \times 10^9$  kilometers

$$e = 0.0542$$

69. Neptune  $a = 4.498 \times 10^9$  kilometers

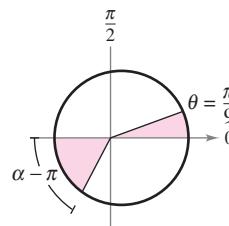
$$e = 0.0086$$

70. Mercury  $a = 5.791 \times 10^7$  kilometers

$$e = 0.2056$$

- CAS 71. Planetary Motion** In Exercise 69, the polar equation for the elliptical orbit of Neptune was found. Use the equation and a computer algebra system to perform each of the following.

- (a) Approximate the area swept out by a ray from the sun to the planet as  $\theta$  increases from 0 to  $\pi/9$ . Use this result to determine the number of years required for the planet to move through this arc if the period of one revolution around the sun is 165 years.
- (b) By trial and error, approximate the angle  $\alpha$  such that the area swept out by a ray from the sun to the planet as  $\theta$  increases from  $\pi$  to  $\alpha$  equals the area found in part (a) (see figure). Does the ray sweep through a larger or smaller angle than in part (a) to generate the same area? Why is this the case?



- (c) Approximate the distances the planet traveled in parts (a) and (b). Use these distances to approximate the average number of kilometers per year the planet traveled in the two cases.

- 72. Comet Hale-Bopp** The comet Hale-Bopp has an elliptical orbit with the sun at one focus and has an eccentricity of  $e \approx 0.995$ . The length of the major axis of the orbit is approximately 500 astronomical units.

- (a) Find the length of its minor axis.  
 (b) Find a polar equation for the orbit.  
 (c) Find the perihelion and aphelion distances.

In Exercises 73 and 74, let  $r_0$  represent the distance from the focus to the nearest vertex, and let  $r_1$  represent the distance from the focus to the farthest vertex.

73. Show that the eccentricity of an ellipse can be written as

$$e = \frac{r_1 - r_0}{r_1 + r_0}. \text{ Then show that } \frac{r_1}{r_0} = \frac{1 + e}{1 - e}.$$

74. Show that the eccentricity of a hyperbola can be written as

$$e = \frac{r_1 + r_0}{r_1 - r_0}. \text{ Then show that } \frac{r_1}{r_0} = \frac{e + 1}{e - 1}.$$

In Exercises 75 and 76, show that the graphs of the given equations intersect at right angles.

75.  $r = \frac{ed}{1 + \sin \theta}$  and  $r = \frac{ed}{1 - \sin \theta}$

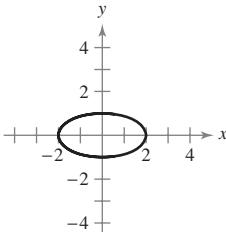
76.  $r = \frac{c}{1 + \cos \theta}$  and  $r = \frac{d}{1 - \cos \theta}$

# 10 REVIEW EXERCISES

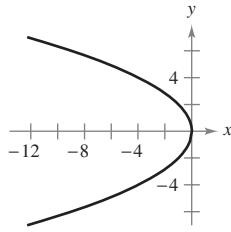
See [www.CalcChat.com](http://www.CalcChat.com) for worked-out solutions to odd-numbered exercises.

**In Exercises 1–6, match the equation with the correct graph.**  
[The graphs are labeled (a), (b), (c), (d), (e), and (f).]

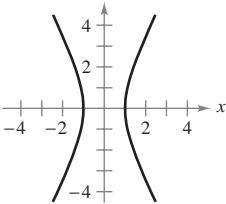
(a)



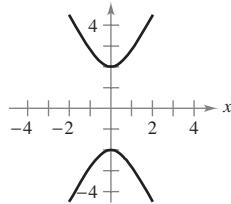
(b)



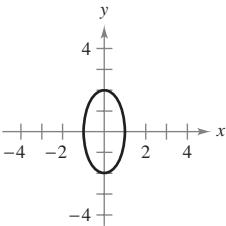
(c)



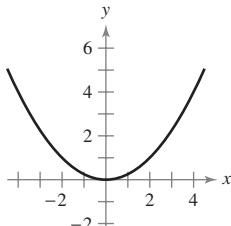
(d)



(e)



(f)



1.  $4x^2 + y^2 = 4$

2.  $4x^2 - y^2 = 4$

3.  $y^2 = -4x$

4.  $y^2 - 4x^2 = 4$

5.  $x^2 + 4y^2 = 4$

6.  $x^2 = 4y$

**In Exercises 7–12, analyze the equation and sketch its graph.**  
Use a graphing utility to confirm your results.

7.  $16x^2 + 16y^2 - 16x + 24y - 3 = 0$

8.  $y^2 - 12y - 8x + 20 = 0$

9.  $3x^2 - 2y^2 + 24x + 12y + 24 = 0$

10.  $5x^2 + y^2 - 20x + 19 = 0$

11.  $3x^2 + 2y^2 - 12x + 12y + 29 = 0$

12.  $12x^2 - 12y^2 - 12x + 24y - 45 = 0$

**In Exercises 13 and 14, find an equation of the parabola.**

13. Vertex:  $(0, 2)$ ; directrix:  $x = -3$

14. Vertex:  $(2, 6)$ ; focus:  $(2, 4)$

**In Exercises 15 and 16, find an equation of the ellipse.**

15. Vertices:  $(-5, 0), (7, 0)$ ; foci:  $(-3, 0), (5, 0)$

16. Center:  $(0, 0)$ ; solution points:  $(1, 2), (2, 0)$

**In Exercises 17 and 18, find an equation of the hyperbola.**

17. Vertices:  $(\pm 7, 0)$ ; foci:  $(\pm 9, 0)$

18. Foci:  $(0, \pm 8)$ ; asymptotes:  $y = \pm 4x$



**In Exercises 19 and 20, use a graphing utility to approximate the perimeter of the ellipse.**

19.  $\frac{x^2}{9} + \frac{y^2}{4} = 1$

20.  $\frac{x^2}{4} + \frac{y^2}{25} = 1$

21. A line is tangent to the parabola  $y = x^2 - 2x + 2$  and perpendicular to the line  $y = x - 2$ . Find the equation of the line.

22. A line is tangent to the parabola  $3x^2 + y = x - 6$  and perpendicular to the line  $2x + y = 5$ . Find the equation of the line.

23. **Satellite Antenna** A cross section of a large parabolic antenna is modeled by the graph of

$$y = \frac{x^2}{200}, \quad -100 \leq x \leq 100.$$

The receiving and transmitting equipment is positioned at the focus.

(a) Find the coordinates of the focus.

(b) Find the surface area of the antenna.

24. **Fire Truck** Consider a fire truck with a water tank 16 feet long whose vertical cross sections are ellipses modeled by the equation

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

(a) Find the volume of the tank.

(b) Find the force on the end of the tank when it is full of water.  
(The density of water is 62.4 pounds per cubic foot.)

(c) Find the depth of the water in the tank if it is  $\frac{3}{4}$  full (by volume) and the truck is on level ground.

(d) Approximate the tank's surface area.

**In Exercises 25–32, sketch the curve represented by the parametric equations (indicate the orientation of the curve), and write the corresponding rectangular equation by eliminating the parameter.**

25.  $x = 1 + 8t, y = 3 - 4t$

26.  $x = t - 6, y = t^2$

27.  $x = e^t - 1, y = e^{3t}$

28.  $x = e^{4t}, y = t + 4$

29.  $x = 6 \cos \theta, y = 6 \sin \theta$

30.  $x = 2 + 5 \cos t, y = 3 + 2 \sin t$

31.  $x = 2 + \sec \theta, y = 3 + \tan \theta$

32.  $x = 5 \sin^3 \theta, y = 5 \cos^3 \theta$

In Exercises 33–36, find a parametric representation of the line or conic.

33. Line: passes through  $(-2, 6)$  and  $(3, 2)$

34. Circle: center at  $(-4, -5)$ ; radius 3

35. Ellipse: center at  $(-3, 4)$ ; horizontal major axis of length 8 and minor axis of length 6

36. Hyperbola: vertices at  $(0, \pm 4)$ ; foci at  $(0, \pm 5)$

37. **Rotary Engine** The rotary engine was developed by Felix Wankel in the 1950s. It features a rotor, which is a modified equilateral triangle. The rotor moves in a chamber that, in two dimensions, is an epitrochoid. Use a graphing utility to graph the chamber modeled by the parametric equations

$$x = \cos 3\theta + 5 \cos \theta$$

and

$$y = \sin 3\theta + 5 \sin \theta.$$

38. **Serpentine Curve** Consider the parametric equations  $x = 2 \cot \theta$  and  $y = 4 \sin \theta \cos \theta$ ,  $0 < \theta < \pi$ .

(a) Use a graphing utility to graph the curve.

(b) Eliminate the parameter to show that the rectangular equation of the serpentine curve is  $(4 + x^2)y = 8x$ .

In Exercises 39–48, (a) find  $dy/dx$  and all points of horizontal tangency, (b) eliminate the parameter where possible, and (c) sketch the curve represented by the parametric equations.

39.  $x = 2 + 5t$ ,  $y = 1 - 4t$     40.  $x = t - 6$ ,  $y = t^2$

41.  $x = \frac{1}{t}$ ,  $y = 2t + 3$     42.  $x = \frac{1}{t}$ ,  $y = t^2$

43.  $x = \frac{1}{2t+1}$     44.  $x = 2t - 1$

$$y = \frac{1}{t^2 - 2t}$$

45.  $x = 5 + \cos \theta$

$$y = 3 + 4 \sin \theta$$

46.  $x = 10 \cos \theta$

$$y = 10 \sin \theta$$

47.  $x = \cos^3 \theta$

$$y = 4 \sin^3 \theta$$

48.  $x = e^t$

$$y = e^{-t}$$

In Exercises 49–52, find all points (if any) of horizontal and vertical tangency to the curve. Use a graphing utility to confirm your results.

49.  $x = 5 - t$ ,  $y = 2t^2$

50.  $x = t + 2$ ,  $y = t^3 - 2t$

51.  $x = 2 + 2 \sin \theta$ ,  $y = 1 + \cos \theta$

52.  $x = 2 - 2 \cos \theta$ ,  $y = 2 \sin 2\theta$



In Exercises 53 and 54, (a) use a graphing utility to graph the curve represented by the parametric equations, (b) use a graphing utility to find  $dx/d\theta$ ,  $dy/d\theta$ , and  $dy/dx$  for  $\theta = \pi/6$ , and (c) use a graphing utility to graph the tangent line to the curve when  $\theta = \pi/6$ .

53.  $x = \cot \theta$

$$y = \sin 2\theta$$

54.  $x = 2\theta - \sin \theta$

$$y = 2 - \cos \theta$$

**Arc Length** In Exercises 55 and 56, find the arc length of the curve on the given interval.

55.  $x = r(\cos \theta + \theta \sin \theta)$

$$y = r(\sin \theta - \theta \cos \theta)$$

$$0 \leq \theta \leq \pi$$

56.  $x = 6 \cos \theta$

$$y = 6 \sin \theta$$

$$0 \leq \theta \leq \pi$$

**Surface Area** In Exercises 57 and 58, find the area of the surface generated by revolving the curve about (a) the  $x$ -axis and (b) the  $y$ -axis.

57.  $x = t$ ,  $y = 3t$ ,  $0 \leq t \leq 2$

58.  $x = 2 \cos \theta$ ,  $y = 2 \sin \theta$ ,  $0 \leq \theta \leq \frac{\pi}{2}$

**Area** In Exercises 59 and 60, find the area of the region.

59.  $x = 3 \sin \theta$

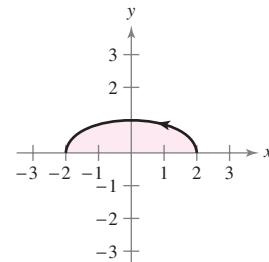
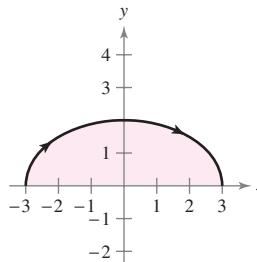
$$y = 2 \cos \theta$$

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

60.  $x = 2 \cos \theta$

$$y = \sin \theta$$

$$0 \leq \theta \leq \pi$$



In Exercises 61–64, plot the point in polar coordinates and find the corresponding rectangular coordinates of the point.

61.  $\left(5, \frac{3\pi}{2}\right)$

62.  $\left(-6, \frac{7\pi}{6}\right)$

63.  $(\sqrt{3}, 1.56)$

64.  $(-2, -2.45)$

In Exercises 65–68, the rectangular coordinates of a point are given. Plot the point and find two sets of polar coordinates of the point for  $0 \leq \theta < 2\pi$ .

65.  $(4, -4)$

66.  $(0, -7)$

67.  $(-1, 3)$

68.  $(-\sqrt{3}, -\sqrt{3})$

**In Exercises 69–76, convert the polar equation to rectangular form.**

69.  $r = 3 \cos \theta$

70.  $r = 10$

71.  $r = -2(1 + \cos \theta)$

72.  $r = \frac{1}{2 - \cos \theta}$

73.  $r^2 = \cos 2\theta$

74.  $r = 4 \sec\left(\theta - \frac{\pi}{3}\right)$

75.  $r = 4 \cos 2\theta \sec \theta$

76.  $\theta = \frac{3\pi}{4}$

**In Exercises 77–80, convert the rectangular equation to polar form.**

77.  $(x^2 + y^2)^2 = ax^2y$

78.  $x^2 + y^2 - 4x = 0$

79.  $x^2 + y^2 = a^2 \left(\arctan \frac{y}{x}\right)^2$

80.  $(x^2 + y^2) \left(\arctan \frac{y}{x}\right)^2 = a^2$

**In Exercises 81–92, sketch a graph of the polar equation.**

81.  $r = 6$

82.  $\theta = \frac{\pi}{10}$

83.  $r = -\sec \theta$

84.  $r = 5 \csc \theta$

85.  $r = -2(1 + \cos \theta)$

86.  $r = 3 - 4 \cos \theta$

87.  $r = 4 - 3 \cos \theta$

88.  $r = 4\theta$

89.  $r = -3 \cos 2\theta$

90.  $r = \cos 5\theta$

91.  $r^2 = 4 \sin^2 2\theta$

92.  $r^2 = \cos 2\theta$

 **In Exercises 93–96, use a graphing utility to graph the polar equation.**

93.  $r = \frac{3}{\cos(\theta - \pi/4)}$

94.  $r = 2 \sin \theta \cos^2 \theta$

95.  $r = 4 \cos 2\theta \sec \theta$

96.  $r = 4(\sec \theta - \cos \theta)$

 **In Exercises 97 and 98, (a) find the tangents at the pole, (b) find all points of vertical and horizontal tangency, and (c) use a graphing utility to graph the polar equation and draw a tangent line to the graph for  $\theta = \pi/6$ .**

97.  $r = 1 - 2 \cos \theta$

98.  $r^2 = 4 \sin 2\theta$

**In Exercises 99 and 100, show that the graphs of the polar equations are orthogonal at the points of intersection. Use a graphing utility to confirm your results graphically.**

99.  $r = 1 + \cos \theta$

100.  $r = a \sin \theta$

$r = 1 - \cos \theta$

$r = a \cos \theta$

**In Exercises 101–106, find the area of the region.**

101. One petal of  $r = 3 \cos 5\theta$

102. One petal of  $r = 2 \sin 6\theta$

103. Interior of  $r = 2 + \cos \theta$

104. Interior of  $r = 5(1 - \sin \theta)$

105. Interior of  $r^2 = 4 \sin 2\theta$

106. Common interior of  $r = 4 \cos \theta$  and  $r = 2$

107. Find the points of intersection of the graphs of  $r = 1 - \cos \theta$  and  $r = 1 + \sin \theta$ .

108. Find the points of intersection of the graphs of  $r = 1 + \sin \theta$  and  $r = 3 \sin \theta$ .



**In Exercises 109–112, use a graphing utility to graph the polar equation. Set up an integral for finding the area of the given region and use the integration capabilities of a graphing utility to approximate the integral accurate to two decimal places.**

109. Interior of  $r = \sin \theta \cos^2 \theta$

110. Interior of  $r = 4 \sin 3\theta$

111. Common interior of  $r = 3$  and  $r^2 = 18 \sin 2\theta$

112. Region bounded by the polar axis and  $r = e^\theta$  for  $0 \leq \theta \leq \pi$

**In Exercises 113 and 114, find the length of the curve over the given interval.**

Polar Equation	Interval
113. $r = a(1 - \cos \theta)$	$0 \leq \theta \leq \pi$

114. $r = a \cos 2\theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
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**In Exercises 115 and 116, write an integral that represents the area of the surface formed by revolving the curve about the given line. Use a graphing utility to approximate the integral.**

Polar Equation	Interval	Axis of Revolution
115. $r = 1 + 4 \cos \theta$	$0 \leq \theta \leq \frac{\pi}{2}$	Polar axis
116. $r = 2 \sin \theta$	$0 \leq \theta \leq \frac{\pi}{2}$	$\theta = \frac{\pi}{2}$

**In Exercises 117–122, sketch and identify the graph. Use a graphing utility to confirm your results.**

117.  $r = \frac{6}{1 - \sin \theta}$

118.  $r = \frac{2}{1 + \cos \theta}$

119.  $r = \frac{6}{3 + 2 \cos \theta}$

120.  $r = \frac{4}{5 - 3 \sin \theta}$

121.  $r = \frac{4}{2 - 3 \sin \theta}$

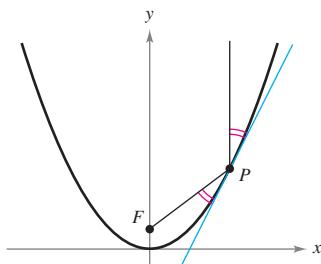
122.  $r = \frac{8}{2 - 5 \cos \theta}$

**In Exercises 123–128, find a polar equation for the line or conic with its focus at the pole.**

123. Circle Center: $(5, \pi/2)$ Solution point: $(0, 0)$ Slope: $\sqrt{3}$	124. Line Solution point: $(0, 0)$ Slope: $\sqrt{3}$
125. Parabola Vertex: $(2, \pi)$	126. Parabola Vertex: $(2, \pi/2)$
127. Ellipse Vertices: $(5, 0), (1, \pi)$	128. Hyperbola Vertices: $(1, 0), (7, 0)$

## P.S. PROBLEM SOLVING

- Consider the parabola  $x^2 = 4y$  and the focal chord  $y = \frac{3}{4}x + 1$ .
  - Sketch the graph of the parabola and the focal chord.
  - Show that the tangent lines to the parabola at the endpoints of the focal chord intersect at right angles.
  - Show that the tangent lines to the parabola at the endpoints of the focal chord intersect on the directrix of the parabola.
- Consider the parabola  $x^2 = 4py$  and one of its focal chords.
  - Show that the tangent lines to the parabola at the endpoints of the focal chord intersect at right angles.
  - Show that the tangent lines to the parabola at the endpoints of the focal chord intersect on the directrix of the parabola.
- Prove Theorem 10.2, Reflective Property of a Parabola, as shown in the figure.



- Consider the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

with foci  $F_1$  and  $F_2$ , as shown in the figure. Let  $T$  be the tangent line at a point  $M$  on the hyperbola. Show that incoming rays of light aimed at one focus are reflected by a hyperbolic mirror toward the other focus.

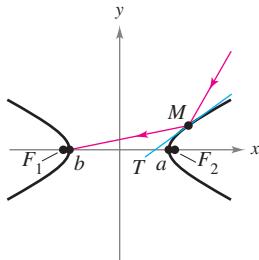


Figure for 4

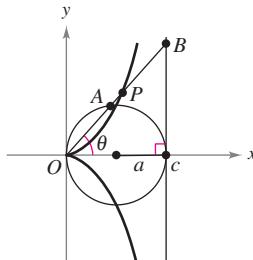


Figure for 5

- Consider a circle of radius  $a$  tangent to the  $y$ -axis and the line  $x = 2a$ , as shown in the figure. Let  $A$  be the point where the segment  $OB$  intersects the circle. The **cissoid of Diocles** consists of all points  $P$  such that  $OP = AB$ .
  - Find a polar equation of the cissoid.
  - Find a set of parametric equations for the cissoid that does not contain trigonometric functions.
  - Find a rectangular equation of the cissoid.

- Consider the region bounded by the ellipse  $x^2/a^2 + y^2/b^2 = 1$ , with eccentricity  $e = c/a$ .

- Show that the area of the region is  $\pi ab$ .
- Show that the solid (oblate spheroid) generated by revolving the region about the minor axis of the ellipse has a volume of  $V = 4\pi^2 b/3$  and a surface area of

$$S = 2\pi a^2 + \pi \left( \frac{b^2}{e} \right) \ln \left( \frac{1+e}{1-e} \right).$$

- Show that the solid (prolate spheroid) generated by revolving the region about the major axis of the ellipse has a volume of  $V = 4\pi ab^2/3$  and a surface area of

$$S = 2\pi b^2 + 2\pi \left( \frac{ab}{e} \right) \arcsin e.$$

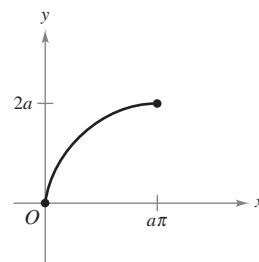
- The curve given by the parametric equations

$$x(t) = \frac{1-t^2}{1+t^2} \quad \text{and} \quad y(t) = \frac{t(1-t^2)}{1+t^2}$$

is called a **strophoid**.

- Find a rectangular equation of the strophoid.
- Find a polar equation of the strophoid.
- Sketch a graph of the strophoid.
- Find the equations of the two tangent lines at the origin.
- Find the points on the graph at which the tangent lines are horizontal.

- Find a rectangular equation of the portion of the cycloid given by the parametric equations  $x = a(\theta - \sin \theta)$  and  $y = a(1 - \cos \theta)$ ,  $0 \leq \theta \leq \pi$ , as shown in the figure.

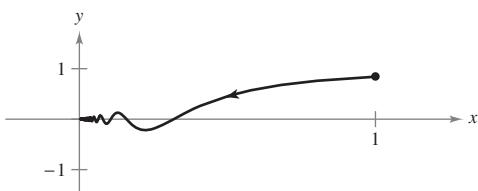


- Consider the **cornu spiral** given by

$$x(t) = \int_0^t \cos \left( \frac{\pi u^2}{2} \right) du \quad \text{and} \quad y(t) = \int_0^t \sin \left( \frac{\pi u^2}{2} \right) du.$$

- (a) Use a graphing utility to graph the spiral over the interval  $-\pi \leq t \leq \pi$ .
- (b) Show that the cornu spiral is symmetric with respect to the origin.
- (c) Find the length of the cornu spiral from  $t = 0$  to  $t = a$ . What is the length of the spiral from  $t = -\pi$  to  $t = \pi$ ?

10. A particle is moving along the path described by the parametric equations  $x = 1/t$  and  $y = \sin t/t$ , for  $1 \leq t < \infty$ , as shown in the figure. Find the length of this path.



11. Let  $a$  and  $b$  be positive constants. Find the area of the region in the first quadrant bounded by the graph of the polar equation

$$r = \frac{ab}{(a \sin \theta + b \cos \theta)}, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

12. Consider the right triangle shown in the figure.

(a) Show that the area of the triangle is  $A(\alpha) = \frac{1}{2} \int_0^\alpha \sec^2 \theta d\theta$ .

(b) Show that  $\tan \alpha = \int_0^\alpha \sec^2 \theta d\theta$ .

- (c) Use part (b) to derive the formula for the derivative of the tangent function.

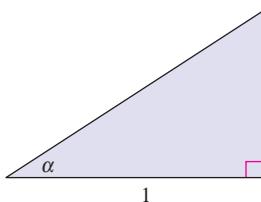


Figure for 12

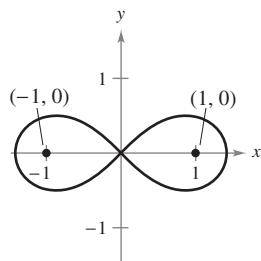
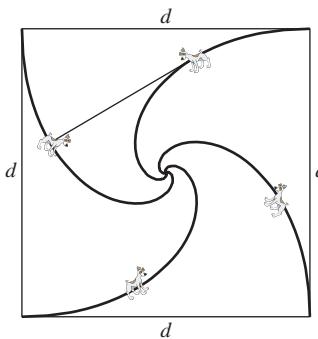


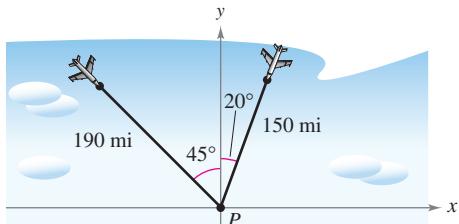
Figure for 13

13. Determine the polar equation of the set of all points  $(r, \theta)$ , the product of whose distances from the points  $(1, 0)$  and  $(-1, 0)$  is equal to 1, as shown in the figure.

14. Four dogs are located at the corners of a square with sides of length  $d$ . The dogs all move counterclockwise at the same speed directly toward the next dog, as shown in the figure. Find the polar equation of a dog's path as it spirals toward the center of the square.



15. An air traffic controller spots two planes at the same altitude flying toward each other (see figure). Their flight paths are  $20^\circ$  and  $315^\circ$ . One plane is 150 miles from point  $P$  with a speed of 375 miles per hour. The other is 190 miles from point  $P$  with a speed of 450 miles per hour.



- (a) Find parametric equations for the path of each plane where  $t$  is the time in hours, with  $t = 0$  corresponding to the time at which the air traffic controller spots the planes.

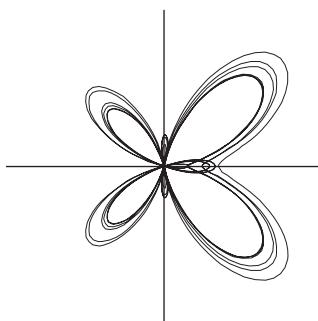
- (b) Use the result of part (a) to write the distance between the planes as a function of  $t$ .

- (c) Use a graphing utility to graph the function in part (b). When will the distance between the planes be minimum? If the planes must keep a separation of at least 3 miles, is the requirement met?

16. Use a graphing utility to graph the curve shown below. The curve is given by

$$r = e^{\cos \theta} - 2 \cos 4\theta + \sin^5 \frac{\theta}{12}.$$

Over what interval must  $\theta$  vary to produce the curve?



**FOR FURTHER INFORMATION** For more information on this curve, see the article “A Study in Step Size” by Temple H. Fay in *Mathematics Magazine*. To view this article, go to the website [www.matharticles.com](http://www.matharticles.com).

17. Use a graphing utility to graph the polar equation  $r = \cos 5\theta + n \cos \theta$  for  $0 \leq \theta < \pi$  and for the integers  $n = -5$  to  $n = 5$ . What values of  $n$  produce the “heart” portion of the curve? What values of  $n$  produce the “bell” portion? (This curve, created by Michael W. Chamberlin, appeared in *The College Mathematics Journal*)