AP Calculus I
Notes 2.2
Finding Limits Graphically and Numerically

Consider: $f(x) = \frac{x^2-1}{x-1}$ What is the domain of $f(x)$?

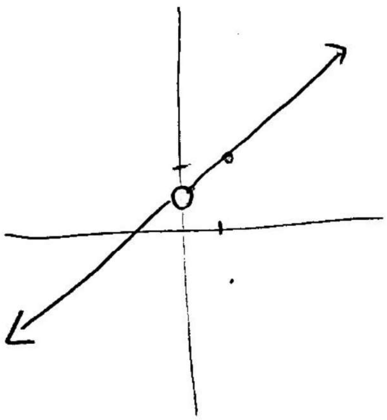
$\frac{(x+1)(x-1)}{x-1}$

$x-1 \neq 0$

$x \neq 1$

$(-\infty, 1), (1, \infty)$

Graph:



What happens as x approaches 1?

x approaches from left → ? ← x approaches from right

x	0.75	0.9	0.999	1	1.001	1.1	1.25	1.5
$f(x)$	1.75	1.9	1.999	und	2.001	2.1	2.25	2.5

f(x) approaches 2 (y-value)

f(x) approaches 2

f(1) = undefined

but

As x approaches 1, $f(x)$ moves arbitrarily close to 2.

$$\boxed{\lim_{x \to 1} f(x) = 2}$$

Definition and Notation of a Limit

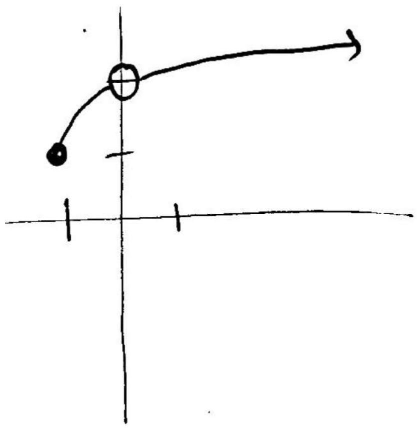
If f(x) becomes arbitrarily close to a single number L as x approaches c from either side, then

$$\lim_{x \to c} f(x) = L$$

Ex. 1: Find $\lim_{x \to 0} \frac{x}{\sqrt{x+1}-1}$ by completing the table, and graph to verify.

x	-.15	-.1	-.01	-.001	0	.001	.01	.1	.15
$f(x)$	1.922	1.949	1.995	1.9995	und	2.0005	2.005	2.05	2.07

f(x) approaches 2 → ← f(x) approaches 2



f(0) = undefined

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1}-1} = 2$$

Three-Pronged Approach to Finding Limits

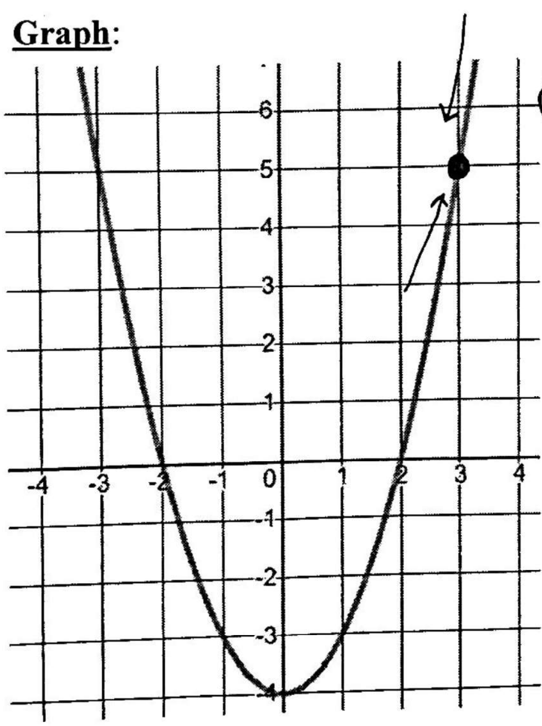
1) Numerical Approach (Table of Values)
2) Graphical Approach
3) Analytic Approach (Algebra or Calculus)

****Note:** 1 and 2 are means of ***approximating*** a limit. **

as x approaches 3, what does y approach?

Ex. 2: Find $\lim_{x \to 3} (x^2 - 4)$

a) **Graph**:



approaching the y-value of 5

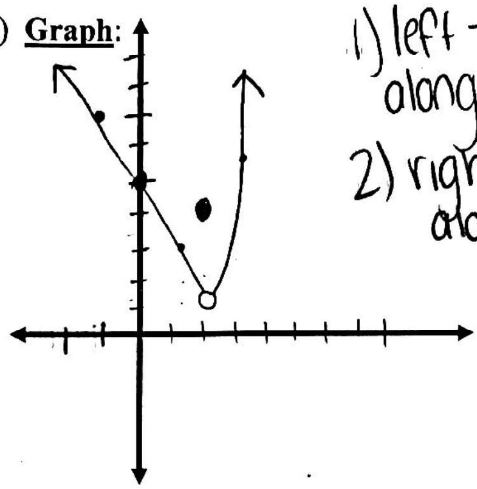
b) $\lim_{x \to 3} (x^2 - 4) = 5$

c) $f(3) = 5$

Ex. 3: Given $f(x) = \begin{cases} -2x+5, & x<2 \\ 4, & x=2 \\ x^2-3, & x>2 \end{cases}$, find the following:

1st piece (linear) on either side

a) **Graph**:



1) left → approach (1st) along linear
2) right → approach along quadratic

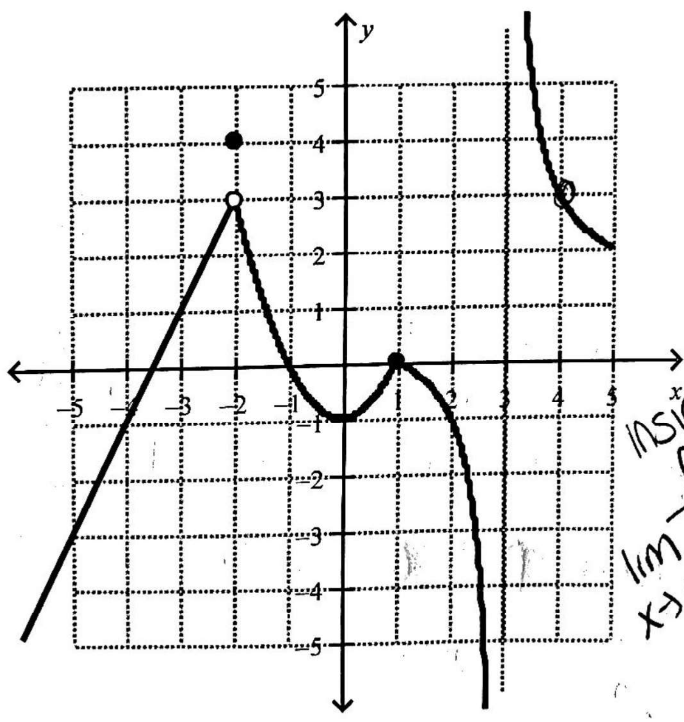
b) $\lim_{x \to 1} f(x) =$ 3

c) $\lim_{x \to 2} f(x) =$ 1

d) $f(2) =$ 4

***** The existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as x approaches c.*****

Ex. 4: Evaluate the following, or state that it does not exist



approaches the hole @ (-2,3) from left & right

a) $\lim_{x \to -2} f(x) =$ 3

point

b) $f(-2) =$ 4

c) $\lim_{x \to 1} f(x) =$ 0

d) $f(1) =$ 0

e) $\lim_{x \to 3} f(x) =$ Does Not Exist ($\pm\infty$)

inside limit first

$\lim_{x \to 0} f(x) = -1$

f) $\lim_{x \to 0} f(f(x)) =$

outside $\lim_{x \to -1} f(x) = 0$

g) Find the value(s) of c where $\lim_{x \to c} f(x) = 3$

$x = -2$ & 4

h) $\lim_{x \to -2} f(f(x)) =$

inside

$\lim_{x \to -2} f(x) = 3$

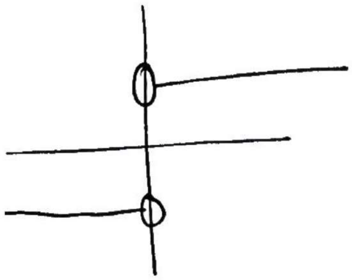
$\lim_{x \to 3} f(x)$: Does not exist

→ $x=0$ makes the denom $=0$ in each example

Some limits do not exist.

Ex. 5: Find $\lim_{x\to 0} \frac{|x|}{x}$ = Does Not Exist (DNE)

Graph:

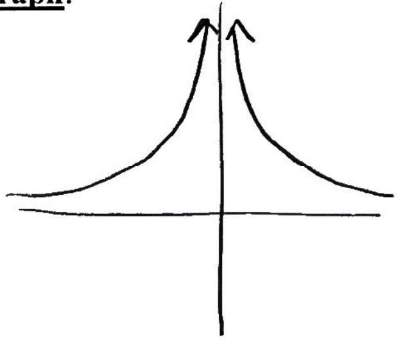


As x approaches zero from the left & right, y approaches 2 different values

(Jump)

Ex. 6: Find $\lim_{x\to 0} \frac{1}{x^2}$ = ∞ (DNE)

Graph:

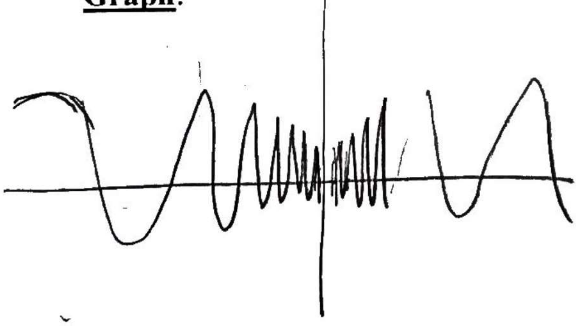


Since the y-values do not approach a single finite value, the limit does not exist.

(Asymptote)

Ex. 7: Find $\lim_{x\to 0} \sin\frac{1}{x}$ = DNE

Graph:



As x approaches zero, f(x) rapidly between -1 and 1. It does not approach a single finite value

(oscilla..)

Common Types of Behavior Associated with the Nonexistence of a Limit

1) $f(x)$ approaches a different value from the right side of c than from the left side.
2) $f(x)$ increases or decreases without bound as x approaches c.
3) $f(x)$ oscillates between two fixed values as x approaches c.

AP Calculus I
Notes 2.3
Evaluating Limits Analytically

In some cases, the limit can be evaluated by direct substitution.

$\lim_{x \to c} f(x) =$ f(c) iff $f(x)$ is continuous.

Ex. 1: Find each limit:

a) $\lim_{x \to 1} \frac{x^2+x+2}{x+1}$

$\frac{(1)^2+1+2}{1+1} = \frac{4}{2}$

$\boxed{=2}$

b) $\lim_{x \to 2} (-x^2 + 3)$

$-1(2)^2+3$

$-4+3 = -1$

Ex. 2: If $\lim_{x \to c} f(x) = 3$ and $\lim_{x \to c} g(x) = 2$, find:

a) $\lim_{x \to c} [4g(x)]$

$4 \lim_{x \to c} g(x)$

$4 \cdot 2 = \boxed{8}$

b) $\lim_{x \to c} [f(x) - g(x)]$

$3 - 2$

$\boxed{1}$

We have studied functions whose limits are evaluated by direct substitution. This is not always possible.

Theorem – Functions that Agree at All But One Point

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as $x \to c$ exists, then the limit of $f(x)$ also exists and $\lim_{x\to c} f(x) = \lim_{x \to c} g(x)$. This theorem permits us to evaluate limits for functions that are not continuous.

In these examples, direct substitution produces the meaningless form $\frac{0}{0}$. We call these expressions an indeterminate form because you cannot evaluate its limit from this form alone.

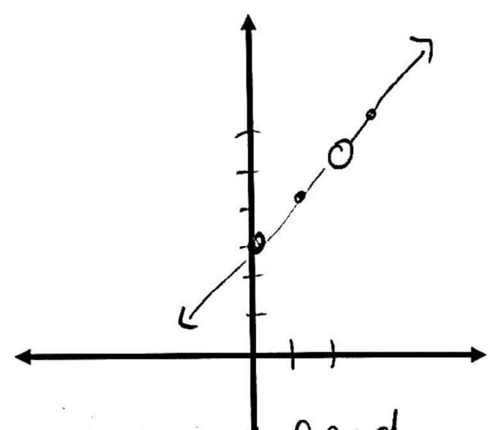
Method 1: Cancellation Technique

Ex. 3: Find $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$ Direct Sub: $\frac{0}{0}$ so

f(2) is undefined

$$\underbrace{\frac{(x+3)(x-2)}{x-2}}_{f(x)} = \underbrace{x+3}_{g(x)}$$ at every x-value other than 2

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} x+3 = 5$$



f(2) = undefined

but

g(2) = 5

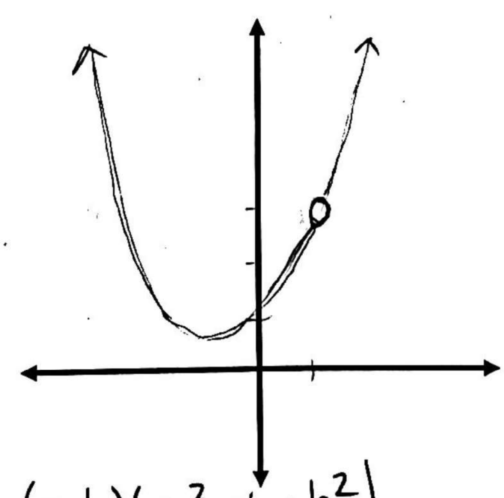
Ex. 4: Find $\lim_{x\to 1} \frac{x^3-1}{x-1}$ Direct Sub: $\frac{0}{0}$

f(1) is undefined

$$\lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}}$$

$$\lim_{x\to 1} x^2+x+1 = \boxed{3}$$

* rewrite the limit *



$a^3+b^3 = (a+b)(a^2-ab+b^2)$

$a^3-b^3 = (a-b)(a^2+ab+b^2)$

Conjugate : Change the sign outside the root

Method 2: Rationalization Technique

Ex. 5: Find $\lim_{x \to 0} \frac{1-\sqrt{x+1}}{x} \to \frac{0}{0}$

Numerator: $(1-\sqrt{x+1})(1+\sqrt{x+1})$

$1 - \cancel{\sqrt{x+1}} + \cancel{\sqrt{x+1}} - \sqrt{x+1}^2$

$1-(x+1) = -x$

$$\lim_{x \to 0} \frac{1-\sqrt{x+1}}{x} \cdot \left(\frac{1+\sqrt{x+1}}{1+\sqrt{x+1}}\right)$$

$$\lim_{x \to 0} \frac{-\cancel{x}}{\cancel{x}(1+\sqrt{x+1})} = \lim_{x \to 0} \frac{-1}{1+\sqrt{x+1}} = \boxed{\frac{-1}{2}}$$

Dont expand denominator! (goal is to cancel)

Ex. 6: Find $\lim_{x \to -2} \frac{5x+10}{3-\sqrt{x+11}}$ $\frac{(3+\sqrt{x+11})}{(3+\sqrt{x+11})}$

Direct Sub: $\frac{0}{0}$

$$\lim_{x \to -2} \frac{(5x+10)(3+\sqrt{x+11})}{9-(x+11)}$$

$$\lim_{x \to -2} \frac{5(x+2)(3+\sqrt{x+11})}{-x-2}$$

$$\lim_{x \to -2} \frac{5\cancel{(x+2)}(3+\sqrt{x+11})}{-1\cancel{(x+2)}}$$

$$\frac{5(6)}{-1} = \boxed{-30}$$

Additional Techniques

Δx is a single variable

Ex. 7: Evaluate the following limit: $\lim_{\Delta x \to 0} \frac{3(x+\Delta x)^2-1-(3x^2-1)}{\Delta x} \to \frac{0}{0}$

$$\lim_{\Delta x \to 0} \frac{3(x^2+2x\Delta x+\Delta x^2 \cancel{-1} -3x^2 \cancel{+1}}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\cancel{3x^2}+6x\Delta x+3\Delta x^2-\cancel{3x^2}}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(6x+3\Delta x)}{\cancel{\Delta x}} = \lim_{\Delta x \to 0} 6x+3\Delta x = \boxed{6x}$$

Complex fractions? Common Denominator

Ex. 8: Find each limit:

a) $\lim_{t\to 3} \frac{\frac{1}{t+1}-\frac{1}{4}}{t-3}$ $\quad \frac{4(t+1)}{4(t+1)}$

Direct Sub: $\frac{\frac{1}{4}-\frac{1}{4}}{0}$

$$\lim_{t\to 3} \frac{4-(t+1)}{4(t-3)(t+1)}$$

Do Not Expand Denominator (goal is to cancel)

$$\lim_{t\to 3} \frac{-t+3}{4(t-3)(t+1)}$$

$$\lim_{t\to 3} \frac{-1\cancel{(t-3)}}{4\cancel{(t-3)}(t+1)}$$

$$\lim_{t\to 3} \frac{-1}{4(t+1)} = \boxed{\frac{-1}{16}}$$

b) $\lim_{k\to 0} \frac{1-e^{-k}}{e^k-1}$ $\quad$ Direct Sub: $\frac{1-1}{1-1}$

$$\lim_{k\to 0} \frac{1-\frac{1}{e^k}}{e^k-1}\cdot\frac{e^k}{e^k}$$

$$\lim_{k\to 0} \frac{\cancel{e^k-1}}{e^k\cancel{(e^k-1)}}$$

$$\lim_{k\to 0} \frac{1}{e^k} = \frac{1}{1} = \boxed{1}$$

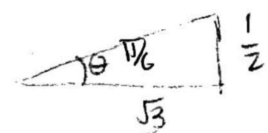
c) $\lim_{\theta\to\frac{\pi}{3}} (sin^2\,\theta)$

$$\sin^2\frac{\pi}{3}$$

$$\left(\frac{\sqrt{3}}{2}\right)^2$$

$$\boxed{=\frac{3}{4}}$$

d) $\lim_{\theta\to\frac{\pi}{6}} \frac{\tan\theta\cos 2\theta}{\sin\theta}$



Direct Sub: $\frac{\frac{1}{\sqrt{3}}\cdot\cos\frac{\pi}{3}}{\frac{1}{2}} = \frac{\frac{1}{\sqrt{3}}\cdot\frac{1}{2}}{\frac{1}{2}}$

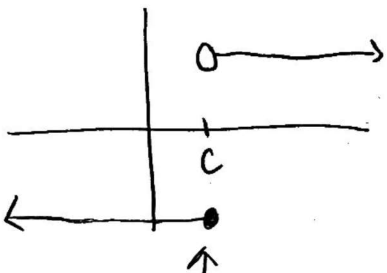
$$=\frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3}$$

AP Calculus I
Notes 2.4
Continuity and One-Sided Limits

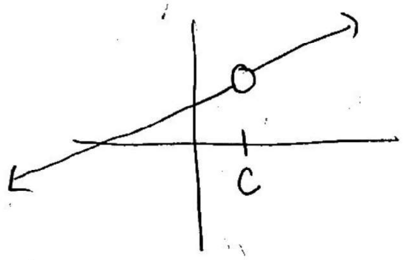
Continuous – A function is continuous at $x = c$ if there is no interruption in the graph of f at c. (No holes, jump discontinuities or infinite discontinuities).

Definitions – A function f is continuous at c iff:

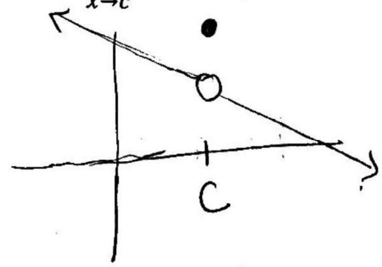
1. $f(c)$ is defined
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$



f(c) is defined
(but lets look @ other rules)



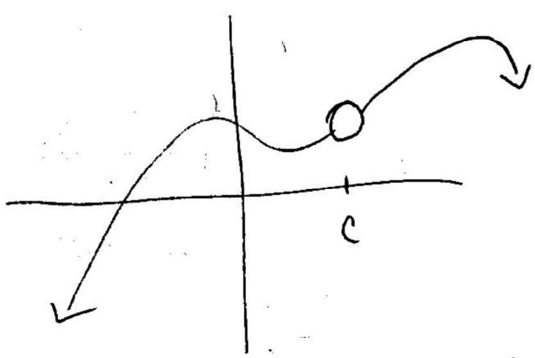
limit exists
(but is f(c) defined?)



limit exists & f(c) is defined
but not

A function is continuous on an open interval (a, b) if it is continuous at each point in the interval. If a function f is defined on I (except possibly at c) and f is not continuous at c, then f has a discontinuity at c. There are two categories of discontinuity:

1) Removable discontinuity (Hole)

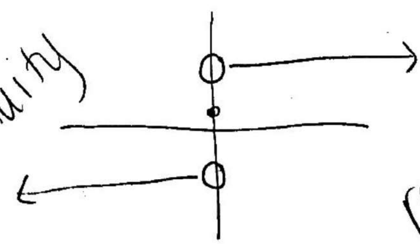


If we "fill in the hole" w/ a point, we can make it continuous

- f can be made continuous by approximately defining (or redefining) $f(c)$.

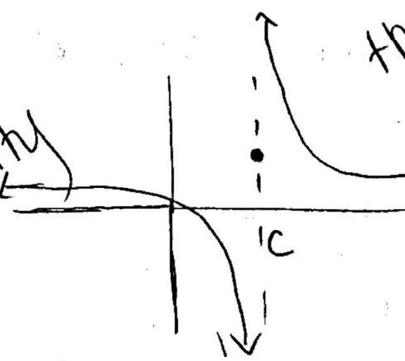
2) Nonremovable discontinuity

Jump Discontinuity



redefining the point won't fix it

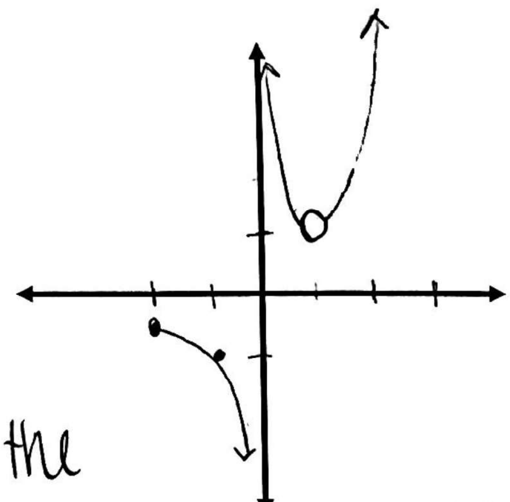
Infinite Discontinuity



- f cannot be made continuous by defining $f(c)$.

Ex.1: For each, sketch a graph, give the domain, and discuss the continuity. If ever discontinuous, tell whether it is removable or nonremovable using the definition of continuity.

a) $f(x) = \begin{cases} x^2, & x > 1 \\ \frac{1}{x}, & -2 \le x < 1 \end{cases}$



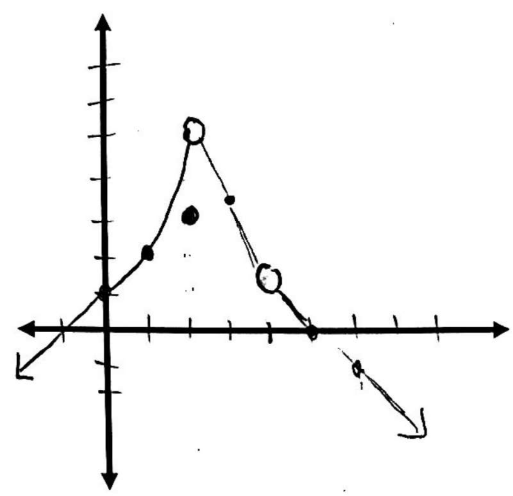
There is a nonremovable discontinuity at $x=0$
b/c $f(0)$ is undef & $\lim_{x\to 0} f(x)$ DNE

The function is not continuous at $x=1$
b/c $f(1)$ is undefined. Since $\lim_{x\to 1} f(x)=1$ the discontinuity is removable

b) $h(x) = \begin{cases} x+1, & x \le 0 \\ x^2+1, & 0 < x < 2 \\ 3, & x = 2 \\ 9-2x, & 2 < x < 4 \\ h(4-x), & x > 4 \end{cases}$

last piece

4	h(0)	1
5	h(-1)	0
6	h(-2)	-1



$\underline{x=0}$ Continuous because
$h(0) = \lim_{x\to 0} h(x) = 1$

$\underline{x=2}$ Not continuous
$\lim_{x\to 2} h(x) = 5$, $h(2)=3$

$5 \ne 3$ limit exists
$\therefore$ removable discontinuity

$\underline{x=4}$ Not continuous
$\lim_{x\to 4} h(x) = 1$ but $h(4)$ is undefined
$\longrightarrow$ removable discontinuity

Ex. 2: Determine if the following piecewise functions are continuous everywhere. Justify your answer.

a) $f(x) = \begin{cases} \frac{x^2+x-12}{x-3} & x \neq 3 \\ 4 & x = 3 \end{cases}$

$f(3) = 4$

$\lim_{x \to 3} f(x)$ ← f approaches 3 along the top piece

$\lim_{x \to 3} \frac{x^2+x-12}{x-3}$

$\lim_{x \to 3} \frac{\cancel{(x-3)}(x+4)}{\cancel{x-3}}$

$\lim_{x \to 3} x+4 = 7$

$4 \neq 7$

Not continuous @ $x=3$

(Removable)

b) $h(t) = \begin{cases} \frac{t^2-9}{t-3} & t \neq 3 \\ 6 & t = 3 \end{cases}$

$h(3) = 6$

$\lim_{t \to 3} \frac{(t+3)(t-3)}{t-3}$

$\lim_{t \to 3} t+3 = 6$

$6 = 6$ so continuous everywhere

Top piece: linear function w/ a hole @ (3,6)

Bottom Piece: a point "filling in" the hole

One-Sided Limits

Limit from the right ➔ x approaches c from values greater than c. $\lim_{x \to c^+} f(x) = L$

Limit from the left ➔ x approaches c from values less than c. $\lim_{x \to c^-} f(x) = L$

left c right

Ex. 3: Evaluate : a) $\lim_{x \to 3^-} (x^2 - 3x + 1)$

$3^2 - 3(3) + 1$

$\boxed{= 1}$

b) $\lim_{x \to 1^-} \begin{cases} x^2 - 5 & x < 1 \\ 2x + 5 & x \geq 1 \end{cases}$

$1^2 - 5$

$= 4$

c) $\lim_{x \to 1^+} \begin{cases} x^2 - 5 & x < 1 \\ 2x + 5 & x \geq 1 \end{cases}$

$2(1) + 5$

$= 7$

Both 1-sided limits exist, but they dont match so overall limit DNE

$\lim_{x \to 1} f(x) = \text{DNE}$ (jum

Theorem – The Existence of a Limit

Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L iff $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = L$.

$f(c) = L$?? not necessarily

Ex. 4: Determine the value of c that will make g continuous everywhere. Justify your response.

$$g(x) = \begin{cases} 2\cos x + c & x < 0 \\ e^x + \sqrt{x+1} & x \geq 0 \end{cases}$$

l & r limits must be =

$\lim_{x\to 0^-} 2\cos x + c$

$2\cos 0 + c$

$2 + c$

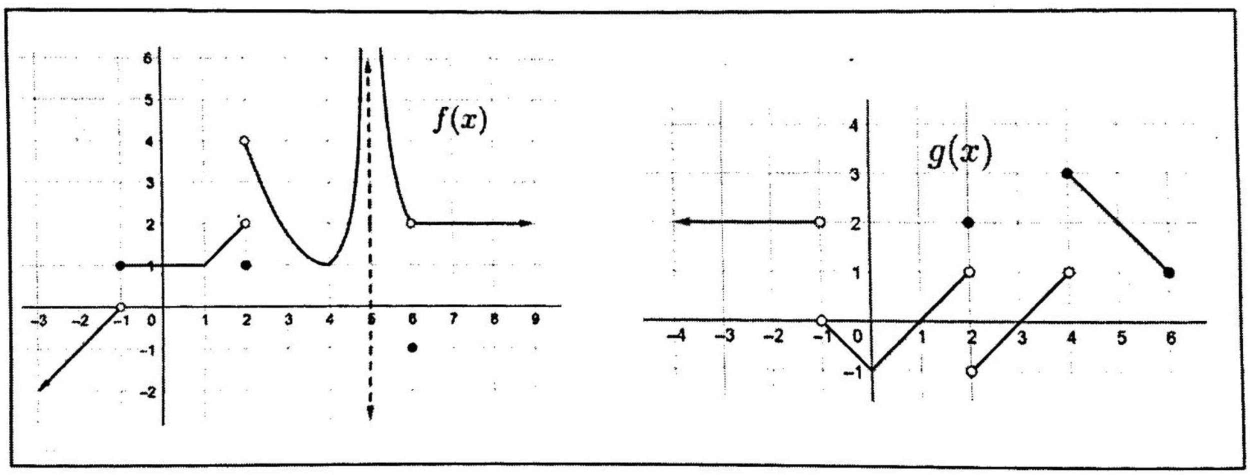
$\lim_{x\to 0^+} e^x + \sqrt{x+1}$

$1 + \sqrt{1}$

2

$2 + c = 2$

$\boxed{c = 0}$



Quick Check 4: (Advanced Limits) Use the graphs of $f(x)$ and $g(x)$ above to answer the following.

jumps!

(a) $\lim_{x\to -1} f(x)g(x)$

$\lim_{x\to -1^-} f(x)\cdot g(x)$

$0 \cdot 2 = 0$

$\lim_{x\to -1^+} f(x)\cdot g(x)$

$1 \cdot 0 = 0$

$\therefore \lim_{x\to -1} f(x)\cdot g(x) = 0$

(b) $\lim_{x\to 2} g(f(x))$ $= 1$

left: ins $\lim_{x\to 2^-} f(x) = 2$ from below

out $\lim_{x\to 2^-} g(x) = 1$

right: $\lim_{x\to 2^+} f(x) = 4$ from below

ins

out $\lim_{x\to 4^-} = 1$

(c) $\lim_{x\to 0} g(g(x))$ $\to g(0) = -1$

ins $\lim_{x\to 0} g(x) = -1$ from above

out $\lim_{x\to -1^+} g(x) = 0$

The Intermediate Value Theorem Investigation

1. Consider the function $f(x) = x^4 - 6x + 2$ from $[1,2]$. Find the values of $f(1)$ and $f(2)$. Explain why you think f must cross the x-axis somewhere between $x = 1$ and $x = 2$. That is, why must there be some number c between $x = 1$ and $x = 2$ such that $f(c) = 0$?

$f(1) = -3$ $\quad$ $1 \le c \le 2$

$f(2) = 6$

Yes, since $f(1) < 0$ & $f(2) > 0$ f must pass through 0 somewhere in the middle.

True b/c polynomials are continuous

2. Consider the function $g(x) = \begin{cases} x^2 + 2, & -3 \le x \le 0 \\ 2 - x, & 0 < x \le 2 \end{cases}$. Find the values of $g(-3)$ and $g(2)$. Must there be some c value between $x = -3$ and $x = 2$ such that $g(c) = 3$? What about $g(c) = -8$? Graph this function to verify.

$g(-3) = 11$

$g(2) = 0$

Yes, since $11 < 3 < 0$ it must reach the y-value at some point

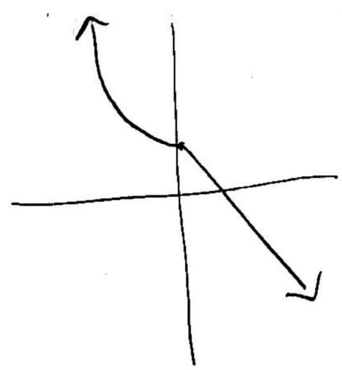
not guaranteed since -8 isnt between 11 & 0

Continuous?

$\lim_{x \to 0^-} g(x) = 2$

$g(0) = 2$

$\lim_{x \to 0^+} g(x) = 2$

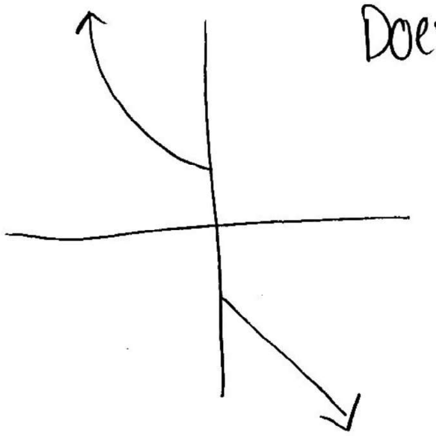


3. Now consider the function $h(x) = \begin{cases} x^2 + 2, & -3 \le x \le 0 \\ -2 - x, & 0 < x \le 2 \end{cases}$. Find the values of $h(-3)$ and $h(2)$ and think about whether $h(x)$ will take on the value of 0 somewhere in the interval. Graph the function and try to explain why you came to your conclusion.

$h(-3) = 11$

$h(2) = -4$

$-4 < 0 < 11$



Does not pass through 0

$\lim_{x \to 0} h(x)$ does not exist

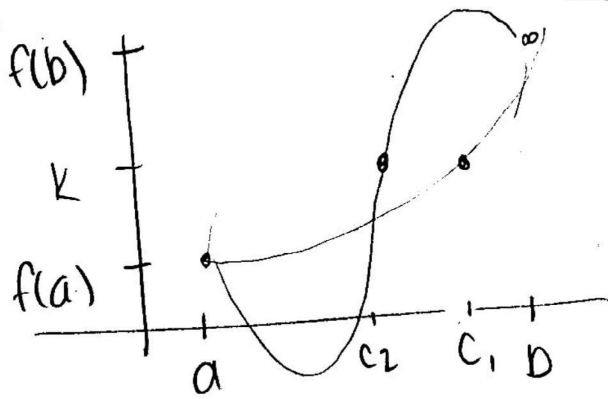
So we cant guarantee

The Intermediate Value Theorem

Suppose f is a continuous function on the closed interval from $[a, b]$. For any number k between $f(a)$ and $f(b)$, there exists a point c between a and b such that $f(c) = k$.

y-values!

x-values



* guarantees y-values *

Ex. 6: Certain values of the continuous functions f and g are given below. The function h is given by

$$h(x) = f(g(x)) - 6.$$ *Existence theorem*

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	−4	4	2
4	−1	3	6	7

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

$h(1) = f(g(1)) - 6$

$f(2) - 6 = 9 - 6 = 3$

$h(3) = f(g(3)) - 6$

$f(4) - 6 = -1 - 6 = -7$

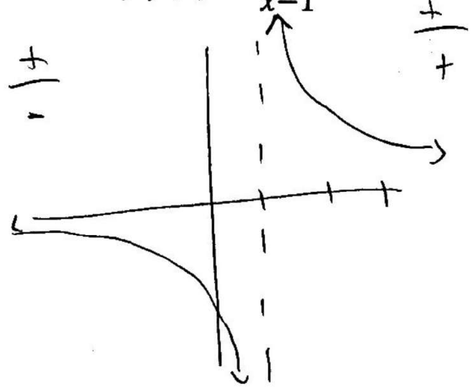
Since h is continuous on $1 < r < 3$ and $-7 < -5 < 3$, the Intermediate Value Theorem guarantees there must be an r on $1 < r < 3$ such that $h(r) = -5$

AP Calculus I
Notes 2.5
Infinite Limits

A limit in which $f(x)$ increases/decreases without bound as x approaches c is called an infinite limit.

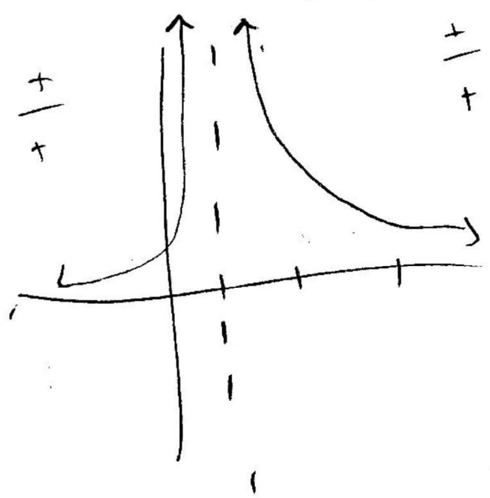
Ex. 1: Determine the limit of each function as x approaches 1 from the left and the right:

a) $f(x) = \dfrac{1}{x-1}$



$\lim_{x \to 1^-} f(x) = -\infty$ $\lim_{x \to 1^+} f(x) = \infty$

b) $f(x) = \dfrac{1}{(x-1)^2}$



$\lim_{x \to 1^-} f(x) = \infty$ $\lim_{x \to 1^+} f(x) = \infty$

If $f(x)$ approaches infinity or negative infinity as x approaches c from the left or the right, then $x = c$ is a vertical asymptote of the graph of $f(x)$.

Vertical Asymptotes are infinite/non-removable discontinuities. Therefore, the limit at these discontinuities Do Not Exist.

Ex. 2: Determine all discontinuities of $f(x) = \frac{x^2+2x-8}{x^2-4}$. Then, compute the limit at each of the domain restrictions.

$x \neq -2, 2$

$\boxed{x=-2}$

$$\lim_{x\to -2} \frac{x^2+2x-8}{x^2-4}$$

$$\frac{4-4-8}{4-4} = \frac{-8}{0}$$

undefined, limit DNE

$\frac{\#}{0} \to$ Vertical Asymptote @ $x=-2$

$\boxed{x=2}$

$$\lim_{x\to 2} \frac{x^2+2x-8}{x^2-4} \to \frac{0}{0}$$

$$\lim_{x\to 2} \frac{(x+4)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$$

$$\lim_{x\to 2} \frac{x+4}{x+2} = \frac{3}{2}$$

f(2) is undefined but limit exists so theres a hole at $(2, 3/2)$

Ex. 3: Find all vertical asymptotes:

a) $f(x) = \frac{x^2+1}{e^x-1}$

$e^x - 1 = 0$

ln $\to e^x = 1$

$x = \ln 1 = 0$

$x=0$

$$\lim_{x\to 0} \frac{x^2+1}{e^x-1} \to \frac{1}{0}$$

Limit DNE, vert asymptote @ $x=0$

b) $f(x) = \cot x$

$f(x) = \frac{\cos x}{\sin x}$

$\sin x = 0$ @ $x = 0, \pi, 2\pi, \ldots$

Vertical asymp. @ $x = 0 + \pi n$

Determining Infinite Limits Analytically

Ex. 4: Find:

a) $\lim\limits_{x \to 1^-} \frac{x^2-3x}{x-1} \to \frac{-2}{0}$ Overall limit DNE

The graph has a V.A so $+\infty$ or $-\infty$

use x-values close to 1 (.9, .99, etc)

$\frac{(<1)^2-3(<1)}{<1-1} \to \frac{<1-<3}{\text{negative}}$

$\frac{\sim -2}{\text{small negative}} \to \boxed{+\infty}$

b) $\lim\limits_{x \to 1^+} \frac{x^2-3x}{x-1} \to \frac{-2}{0}$

$\frac{(>1)^2-3(>1)}{>1-1}$

$\frac{\sim -2}{\text{small positive}}$

$\to \boxed{-\infty}$

Theorem – Properties of Infinite Limits

Let c and L be constants and let $f(x)$ and $g(x)$ be functions so that $\lim\limits_{x\to c} f(x) = \infty$ and $\lim\limits_{x\to c} g(x) = L$.

1) **Sum or Difference:** $\lim\limits_{x\to c} [f(x) \pm g(x)] = \infty$
$\infty \pm L$

2) **Product:** $\lim\limits_{x\to c} [f(x)g(x)] = \infty$, if $L > 0$
$\infty \cdot L$

3) **Quotient:** $\lim\limits_{x\to c} \left[\frac{f(x)}{g(x)}\right] = \frac{\infty}{L} = \infty$

$\lim\limits_{x\to c} \left[\frac{g(x)}{f(x)}\right] = \frac{L}{\infty} = 0$

$\infty \cdot 0 =$ indeterminat

$\infty - \infty =$ indet.

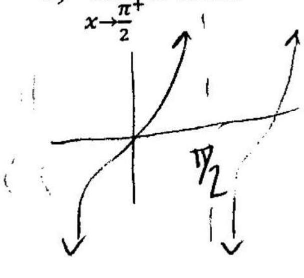
Ex. 5: Determine each limit:

a) $\lim\limits_{x\to 0} \left(1 + \frac{1}{x^2}\right)$

$1 + \infty$

$= \infty$

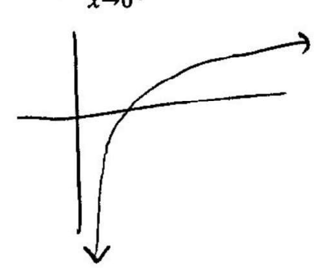
b) $\lim\limits_{x\to \frac{\pi}{2}^+} 3\tan x$



$3 \cdot -\infty$

$= -\infty$

c) $\lim\limits_{x\to 0^+} 3\ln x$



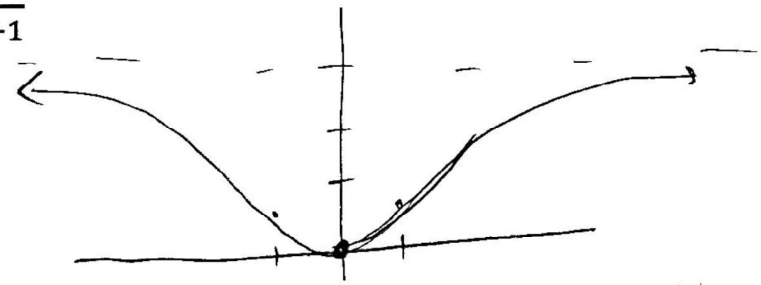
$= -\infty$

AP Calculus I
Notes 4.5
Limits at Infinity

$x \to \pm\infty$

Exploration:

Consider the graph of $f(x) = \frac{3x^2}{x^2+1}$



x	$-\infty \leftarrow$	-100	-10	-1	0	1	10	100	$\to \infty$
$f(x)$		-2.9997	-2.9703	$-3/2$	0	$3/2$	2.9703	2.997	3

$\lim_{x\to-\infty} f(x) = 3$ $\lim_{x\to\infty} f(x) = 3$

These are **limits at infinity**, and $y = 3$ is a **horizontal asymptote.**

Definition of Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of $f(x)$ if $\lim_{x\to-\infty} f(x) = L$ or $\lim_{x\to\infty} f(x) = L$.

Theorem – Limits at Infinity

If r is a positive rational number and c is any real number, then $\lim_{x\to\infty} \frac{c}{x^r} = 0$ and $\lim_{x\to-\infty} \frac{c}{x^r} = 0$.

Ex. 1: Find the limit:

a) $\lim_{x\to\infty} \left(5 - \frac{2}{x^2}\right)$

$5 - \frac{2}{\infty^2}$

$5 - 0 = \boxed{5}$

b) $\lim_{x\to\infty} 3e^{-x}$

$\lim_{x\to\infty} \frac{3}{e^x}$

$\frac{3}{e^\infty} = \boxed{0}$

Ex. 2: Find each of the limits. Then draw a conclusion about an asymptote of the function:

a) $\lim_{x\to\infty} \frac{2x+5}{3x^2+1} \to \frac{\infty}{\infty}$ indeterminate

Divide everything by highest exponent

$$\lim_{x\to\infty} \frac{2x+5}{3x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$\lim_{x\to\infty} \frac{\frac{2}{x}+\frac{5}{x^2}}{3+\frac{1}{x^2}} \quad \frac{0+0}{3+0} = \boxed{0}$$

b) $\lim_{x\to-\infty} \frac{2x+5}{3x^2+1}$ ← Bottom Heavy

$$\lim_{x\to-\infty} \frac{\frac{2}{x}+\frac{5}{x^2}}{3+\frac{1}{x^2}} = \frac{0+0}{3+0} = \boxed{0}$$

c) $\lim_{x\to\infty} \frac{2x^2+5}{3x^2+1} \to \frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{2x^2+5}{3x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$\lim_{x\to\infty} \frac{2+\frac{5}{x^2}}{3+\frac{1}{x^2}} = \frac{2+0}{3+0} = \frac{2}{3}$$

Ratio of Coefficients

d) $\lim_{x\to\infty} \frac{2x^3+5}{3x^2+1} \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$ ← Top Heavy

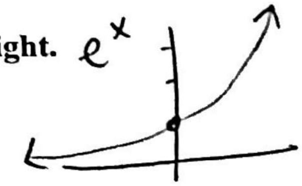
$$\lim_{x\to\infty} \frac{2+\frac{5}{x^3}}{\frac{3}{x}+\frac{1}{x^3}} \to \frac{2}{0}$$

DNE, $+\infty$

Guidelines for Finding Limits at Infinity of Rational Functions

1. If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is 0.
2. If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is *greater than* the degree of the denominator, then the limit of the rational function does not exist.

Functions may approach different horizontal asymptotes to the left and to the right. e^x



Ex. 3: Determine each of the limits:

a) $\lim\limits_{x\to\pm\infty} \frac{10x-2}{\sqrt{4x^2+1}}$

$\lim\limits_{x\to\infty} \frac{10x-2}{\sqrt{4x^2+1}}$ constants are insignificant

$\frac{10x}{\sqrt{4x^2}} = \frac{10x}{|2x|}$ $\frac{+}{+}$

denom stays positive!

$\boxed{= 5}$

$\lim\limits_{x\to-\infty} \frac{10x-2}{\sqrt{4x^2+1}} \to \lim\limits_{x\to-\infty} \frac{10x}{|2x|}$

$\frac{-10}{2}$

$\boxed{= -5}$

b) $\lim\limits_{x\to\pm\infty} \frac{6}{1+e^{-x}}$

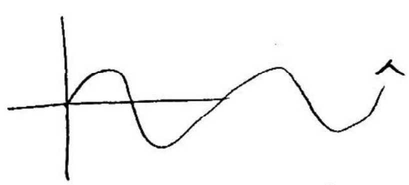
$\lim\limits_{x\to+\infty} \frac{6}{1+\frac{1}{e^x}} = \frac{6}{1+0} = \boxed{6}$

$\lim\limits_{x\to-\infty} \frac{6}{1+\frac{1}{e^x}} = \frac{6}{1+\infty} = \boxed{0}$

$\frac{1}{0} \to \infty$

Ex. 4: Determine each of the limits:

a) $\lim\limits_{x\to\infty} \sin x$ = DNE



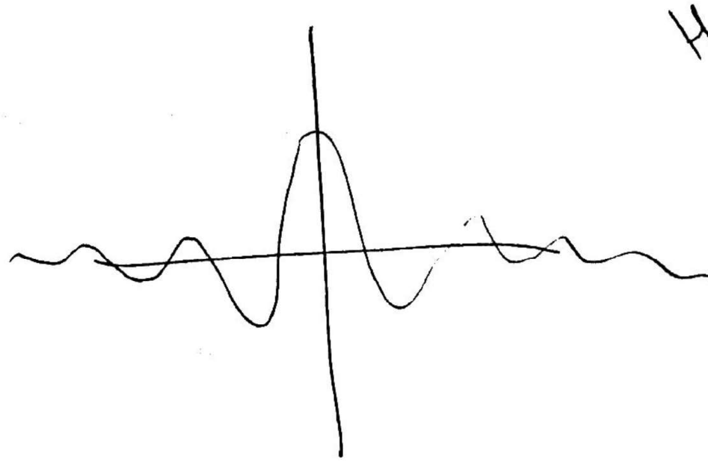
oscillates

b) $\lim\limits_{x\to\infty} \frac{\sin x}{x}$ → stays between -1 $ 1

$\to \infty$

$= 0$

* Bottom Heavy *



Ex. 5: Evaluate the following:

$\lim_{x\to\infty} \frac{\sqrt{9x^4+1}}{x^2-3x+5}$ is

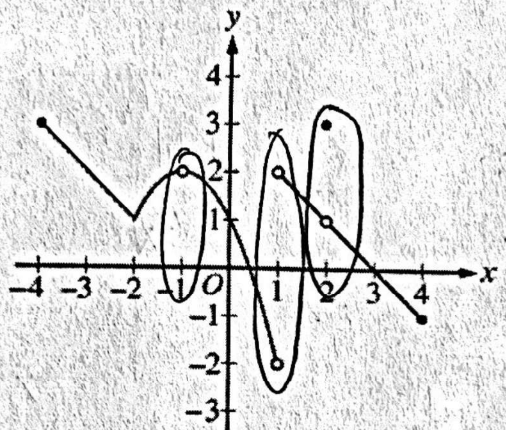
$$\frac{|3x^2|}{x^2}$$

(A) 1

(B) 3

(C) 9

(D) nonexistent



Graph of f

The graph of the function f is shown in the figure above. For how many values of x in the open interval $(-4, 4)$ is f discontinuous?

(A) one

(B) two

(C) three

(D) four

Ex. 6: Describe the graphical representation of the following limit statements:

a) $\lim_{x\to 2^+} f(x) = \infty$ vert asymp @ x=2

b) $\lim_{x\to\infty} f(x) = 1$ Horiz asympt @ y=1

c) $\lim_{x\to 1} f(x) = 0$

as x approaches 1, y approaches 0

a point or a hole