

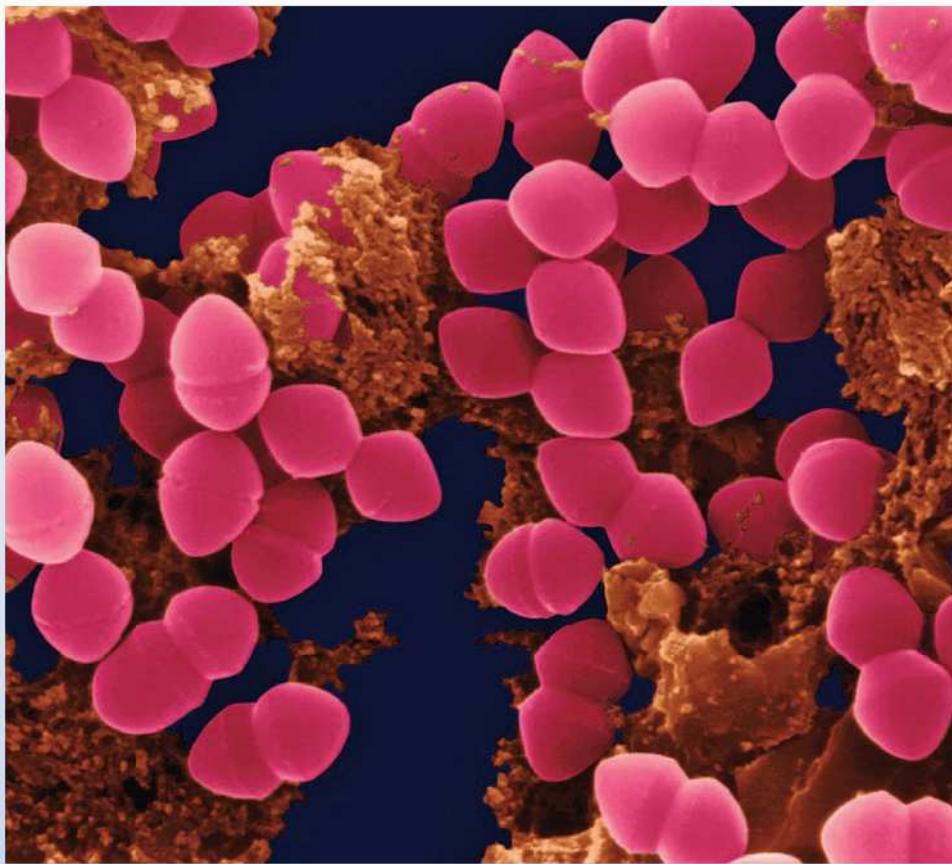
6

Differential Equations

In this chapter, you will study one of the most important applications of calculus—*differential equations*. You will learn several methods for solving different types of differential equations, such as homogeneous, first-order linear, and Bernoulli. Then you will apply these methods to solve differential equations in applied problems.

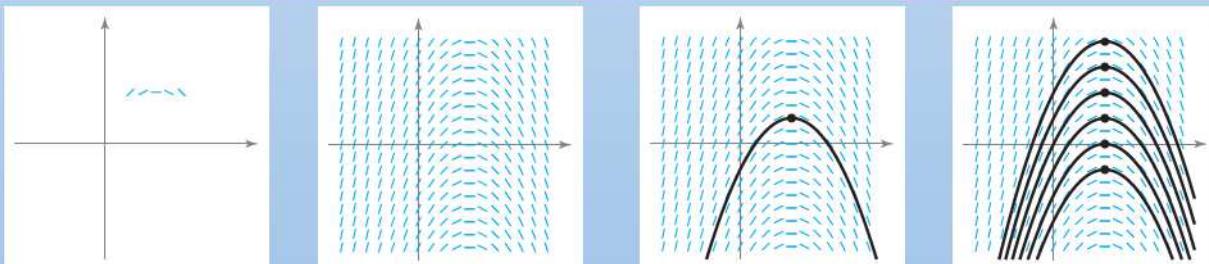
In this chapter, you should learn the following.

- How to sketch a slope field of a differential equation, and find a particular solution. (6.1)
- How to use an exponential function to model growth and decay. (6.2)
- How to use separation of variables to solve a differential equation. (6.3)
- How to solve and analyze logistic differential equations. (6.4)
- How to solve a first-order linear differential equation and a Bernoulli differential equation. (6.5)
- How to analyze predator-prey and competing-species differential equations. (6.6)



Dr. Dennis Kunkel/Getty Images

Depending on the type of bacteria, the time it takes for a culture's weight to double can vary greatly from several minutes to several days. How could you use a differential equation to model the growth rate of a bacteria culture's weight? (See Section 6.4, Exercise 34.)



A function $y = f(x)$ is a solution of a differential equation if the equation is satisfied when y and its derivatives are replaced by $f(x)$ and its derivatives. One way to solve a differential equation is to use slope fields, which show the general shape of all solutions of a differential equation. (See Section 6.1.)

6.1**Slope Fields and Euler's Method**

- Use initial conditions to find particular solutions of differential equations.
- Use slope fields to approximate solutions of differential equations.
- Use Euler's Method to approximate solutions of differential equations.

General and Particular Solutions

In this text, you will learn that physical phenomena can be described by differential equations. Recall that a **differential equation** in x and y is an equation that involves x , y , and derivatives of y . In Section 6.2, you will see that problems involving radioactive decay, population growth, and Newton's Law of Cooling can be formulated in terms of differential equations.

A function $y = f(x)$ is called a **solution** of a differential equation if the equation is satisfied when y and its derivatives are replaced by $f(x)$ and its derivatives. For example, differentiation and substitution would show that $y = e^{-2x}$ is a solution of the differential equation $y' + 2y = 0$. It can be shown that every solution of this differential equation is of the form

$$y = Ce^{-2x}$$

General solution of $y' + 2y = 0$

where C is any real number. This solution is called the **general solution**. Some differential equations have **singular solutions** that cannot be written as special cases of the general solution. However, such solutions are not considered in this text. The **order** of a differential equation is determined by the highest-order derivative in the equation. For instance, $y' = 4y$ is a first-order differential equation. First-order linear differential equations are discussed in Section 6.5.

In Section 5.1, Example 8, you saw that the second-order differential equation $s''(t) = -32$ has the general solution

$$s(t) = -16t^2 + C_1t + C_2$$

General solution of $s''(t) = -32$

which contains two arbitrary constants. It can be shown that a differential equation of order n has a general solution with n arbitrary constants.

EXAMPLE 1 Verifying Solutions

Determine whether the function is a solution of the differential equation $y'' - y = 0$.

- a. $y = \sin x$ b. $y = 4e^{-x}$ c. $y = Ce^x$

Solution

- a. Because $y = \sin x$, $y' = \cos x$, and $y'' = -\sin x$, it follows that

$$y'' - y = -\sin x - \sin x = -2 \sin x \neq 0.$$

So, $y = \sin x$ is *not* a solution.

- b. Because $y = 4e^{-x}$, $y' = -4e^{-x}$, and $y'' = 4e^{-x}$, it follows that

$$y'' - y = 4e^{-x} - 4e^{-x} = 0.$$

So, $y = 4e^{-x}$ is a solution.

- c. Because $y = Ce^x$, $y' = Ce^x$, and $y'' = Ce^x$, it follows that

$$y'' - y = Ce^x - Ce^x = 0.$$

So, $y = Ce^x$ is a solution for any value of C .



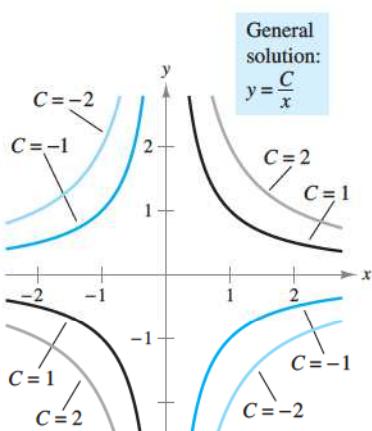
Solution curves for $xy' + y = 0$

Figure 6.1

Geometrically, the general solution of a first-order differential equation represents a family of curves known as **solution curves**, one for each value assigned to the arbitrary constant. For instance, you can verify that every function of the form

$$y = \frac{C}{x}$$

General solution of $xy' + y = 0$

is a solution of the differential equation $xy' + y = 0$. Figure 6.1 shows four of the solution curves corresponding to different values of C .

As discussed in Section 5.1, **particular solutions** of a differential equation are obtained from **initial conditions** that give the values of the dependent variable or one of its derivatives for particular values of the independent variable. The term “initial condition” stems from the fact that, often in problems involving time, the value of the dependent variable or one of its derivatives is known at the *initial* time $t = 0$. For instance, the second-order differential equation $s''(t) = -32$ having the general solution

$$s(t) = -16t^2 + C_1 t + C_2$$

General solution of $s''(t) = -32$

might have the following initial conditions.

$$s(0) = 80, \quad s'(0) = 64$$

Initial conditions

In this case, the initial conditions yield the particular solution

$$s(t) = -16t^2 + 64t + 80.$$

Particular solution

EXAMPLE 2 Finding a Particular Solution

For the differential equation $xy' - 3y = 0$, verify that $y = Cx^3$ is a solution, and find the particular solution determined by the initial condition $y = 2$ when $x = -3$.

Solution You know that $y = Cx^3$ is a solution because $y' = 3Cx^2$ and

$$\begin{aligned} xy' - 3y &= x(3Cx^2) - 3(Cx^3) \\ &= 0. \end{aligned}$$

Furthermore, the initial condition $y = 2$ when $x = -3$ yields

$$y = Cx^3 \quad \text{General solution}$$

$$2 = C(-3)^3 \quad \text{Substitute initial condition.}$$

$$-\frac{2}{27} = C \quad \text{Solve for } C.$$

and you can conclude that the particular solution is

$$y = -\frac{2x^3}{27}. \quad \text{Particular solution}$$

Try checking this solution by substituting for y and y' in the original differential equation. ■

NOTE To determine a particular solution, the number of initial conditions must match the number of constants in the general solution. ■

The icon  indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple and Mathematica.

Slope Fields

Solving a differential equation analytically can be difficult or even impossible. However, there is a graphical approach you can use to learn a lot about the solution of a differential equation. Consider a differential equation of the form

$$y' = F(x, y) \quad \text{Differential equation}$$

where $F(x, y)$ is some expression in x and y . At each point (x, y) in the xy -plane where F is defined, the differential equation determines the slope $y' = F(x, y)$ of the solution at that point. If you draw short line segments with slope $F(x, y)$ at selected points (x, y) in the domain of F , then these line segments form a **slope field**, or a *direction field*, for the differential equation $y' = F(x, y)$. Each line segment has the same slope as the solution curve through that point. A slope field shows the general shape of all the solutions and can be helpful in getting a visual perspective of the directions of the solutions of a differential equation.

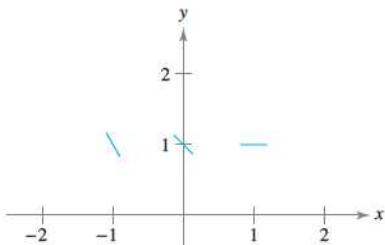


Figure 6.2

EXAMPLE 3 Sketching a Slope Field

Sketch a slope field for the differential equation $y' = x - y$ for the points $(-1, 1)$, $(0, 1)$, and $(1, 1)$.

Solution The slope of the solution curve at any point (x, y) is $F(x, y) = x - y$. So, the slope at $(-1, 1)$ is $y' = -1 - 1 = -2$, the slope at $(0, 1)$ is $y' = 0 - 1 = -1$, and the slope at $(1, 1)$ is $y' = 1 - 1 = 0$. Draw short line segments at the three points with their respective slopes, as shown in Figure 6.2.

EXAMPLE 4 Identifying Slope Fields for Differential Equations

Match each slope field with its differential equation.

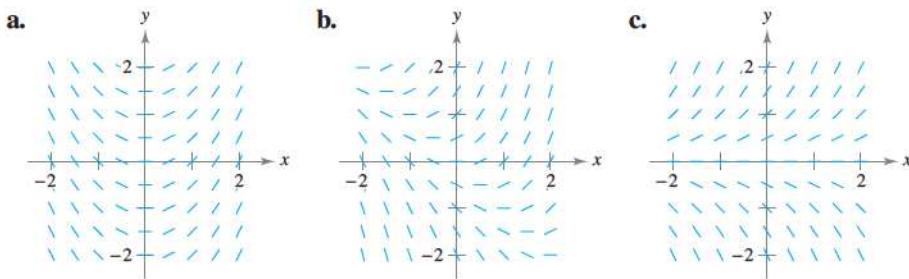


Figure 6.3

i. $y' = x + y$

ii. $y' = x$

iii. $y' = y$

Solution

- a. In Figure 6.3(a), you can see that the slope at any point along the y -axis is 0. The only equation that satisfies this condition is $y' = x$. So, the graph matches equation (ii).
- b. In Figure 6.3(b), you can see that the slope at the point $(1, -1)$ is 0. The only equation that satisfies this condition is $y' = x + y$. So, the graph matches equation (i).
- c. In Figure 6.3(c), you can see that the slope at any point along the x -axis is 0. The only equation that satisfies this condition is $y' = y$. So, the graph matches equation (iii).

A solution curve of a differential equation $y' = F(x, y)$ is simply a curve in the xy -plane whose tangent line at each point (x, y) has slope equal to $F(x, y)$. This is illustrated in Example 5.

EXAMPLE 5 Sketching a Solution Using a Slope Field

Sketch a slope field for the differential equation

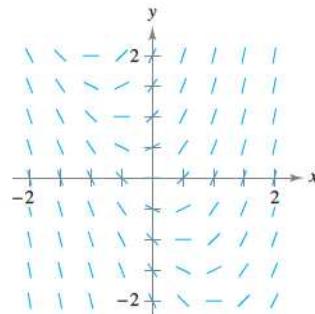
$$y' = 2x + y.$$

Use the slope field to sketch the solution that passes through the point $(1, 1)$.

Solution Make a table showing the slopes at several points. The table shown is a small sample. The slopes at many other points should be calculated to get a representative slope field.

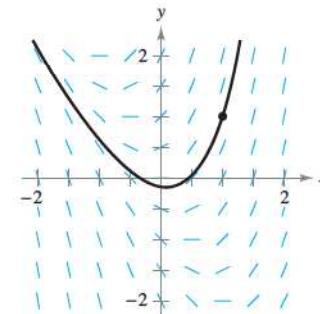
x	-2	-2	-1	-1	0	0	1	1	2	2
y	-1	1	-1	1	-1	1	-1	1	-1	1
$y' = 2x + y$	-5	-3	-3	-1	-1	1	1	3	3	5

Next draw line segments at the points with their respective slopes, as shown in Figure 6.4.



Slope field for $y' = 2x + y$

Figure 6.4



Particular solution for $y' = 2x + y$

passing through $(1, 1)$

Figure 6.5

After the slope field is drawn, start at the initial point $(1, 1)$ and move to the right in the direction of the line segment. Continue to draw the solution curve so that it moves parallel to the nearby line segments. Do the same to the left of $(1, 1)$. The resulting solution is shown in Figure 6.5. ■

In Example 5, note that the slope field shows that y' increases to infinity as x increases.

NOTE Drawing a slope field by hand is tedious. In practice, slope fields are usually drawn using a graphing utility. ■

Euler's Method

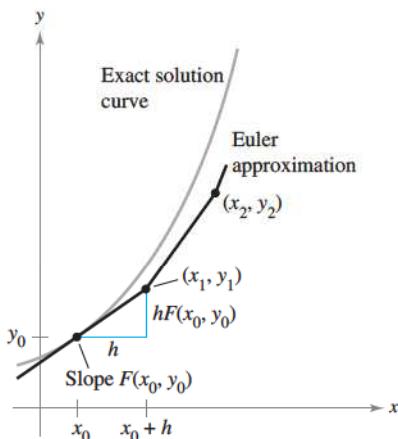


Figure 6.6

Euler's Method is a numerical approach to approximating the particular solution of the differential equation

$$y' = F(x, y)$$

that passes through the point (x_0, y_0) . From the given information, you know that the graph of the solution passes through the point (x_0, y_0) and has a slope of $F(x_0, y_0)$ at that point. This gives you a “starting point” for approximating the solution.

From this starting point, you can proceed in the direction indicated by the slope. Using a small step h , move along the tangent line until you arrive at the point (x_1, y_1) , where

$$x_1 = x_0 + h \quad \text{and} \quad y_1 = y_0 + hF(x_0, y_0)$$

as shown in Figure 6.6. If you think of (x_1, y_1) as a new starting point, you can repeat the process to obtain a second point (x_2, y_2) . The values of x_i and y_i are as follows.

$$\begin{aligned} x_1 &= x_0 + h & y_1 &= y_0 + hF(x_0, y_0) \\ x_2 &= x_1 + h & y_2 &= y_1 + hF(x_1, y_1) \\ &\vdots & &\vdots \\ x_n &= x_{n-1} + h & y_n &= y_{n-1} + hF(x_{n-1}, y_{n-1}) \end{aligned}$$

NOTE You can obtain better approximations of the exact solution by choosing smaller and smaller step sizes. ■

EXAMPLE 6 Approximating a Solution Using Euler's Method

Use Euler's Method to approximate the particular solution of the differential equation

$$y' = x - y$$

passing through the point $(0, 1)$. Use a step of $h = 0.1$.

Solution Using $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = x - y$, you have $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, . . . , and

$$y_1 = y_0 + hF(x_0, y_0) = 1 + (0.1)(0 - 1) = 0.9$$

$$y_2 = y_1 + hF(x_1, y_1) = 0.9 + (0.1)(0.1 - 0.9) = 0.82$$

$$y_3 = y_2 + hF(x_2, y_2) = 0.82 + (0.1)(0.2 - 0.82) = 0.758.$$

The first 10 approximations are shown in the table. You can plot these values to see a graph of the approximate solution, as shown in Figure 6.7.

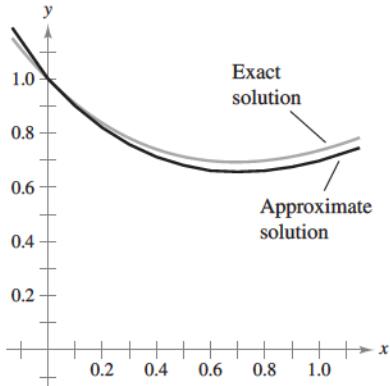


Figure 6.7

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	1	0.900	0.820	0.758	0.712	0.681	0.663	0.657	0.661	0.675	0.697

NOTE For the differential equation in Example 6, you can verify the exact solution to be $y = x - 1 + 2e^{-x}$. Figure 6.7 compares this exact solution with the approximate solution obtained in Example 6. ■

6.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, verify the solution of the differential equation.

Solution

1. $y = Ce^{4x}$ $y' = 4y$
2. $y = e^{-2x}$ $3y' + 5y = -e^{-2x}$
3. $x^2 + y^2 = Cy$ $y' = 2xy/(x^2 - y^2)$
4. $y^2 - 2 \ln y = x^2$ $\frac{dy}{dx} = \frac{xy}{y^2 - 1}$
5. $y = C_1 \sin x - C_2 \cos x$ $y'' + y = 0$
6. $y = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x$ $y'' + 2y' + 2y = 0$
7. $y = -\cos x \ln |\sec x + \tan x|$ $y'' + y = \tan x$
8. $y = \frac{2}{5}(e^{-4x} + e^x)$ $y'' + 4y' = 2e^x$

In Exercises 9–12, verify the particular solution of the differential equation.

Solution

9. $y = \sin x \cos x - \cos^2 x$ $Differential\ Equation\ and\ Initial\ Condition$
 $2y + y' = 2 \sin(2x) - 1$
 $y\left(\frac{\pi}{4}\right) = 0$
10. $y = \frac{1}{2}x^2 - 2 \cos x - 3$ $y' = x + 2 \sin x$
 $y(0) = -5$
11. $y = 4e^{-6x^2}$ $y' = -12xy$
 $y(0) = 4$
12. $y = e^{-\cos x}$ $y' = y \sin x$
 $y\left(\frac{\pi}{2}\right) = 1$

In Exercises 13–20, determine whether the function is a solution of the differential equation $y^{(4)} - 16y = 0$.

13. $y = 3 \cos x$
14. $y = 2 \sin x$
15. $y = 3 \cos 2x$
16. $y = 3 \sin 2x$
17. $y = e^{-2x}$
18. $y = 5 \ln x$
19. $y = C_1 e^{2x} + C_2 e^{-2x} + C_3 \sin 2x + C_4 \cos 2x$
20. $y = 3e^{2x} - 4 \sin 2x$

In Exercises 21–28, determine whether the function is a solution of the differential equation $xy' - 2y = x^3 e^x$.

21. $y = x^2$
22. $y = x^3$
23. $y = x^2 e^x$
24. $y = x^2(2 + e^x)$
25. $y = \sin x$
26. $y = \cos x$
27. $y = \ln x$
28. $y = x^2 e^x - 5x^2$

In Exercises 29–32, some of the curves corresponding to different values of C in the general solution of the differential equation are given. Find the particular solution that passes through the point shown on the graph.

Solution

29. $y = Ce^{-x/2}$
30. $y(x^2 + y) = C$
31. $y^2 = Cx^3$
32. $2x^2 - y^2 = C$

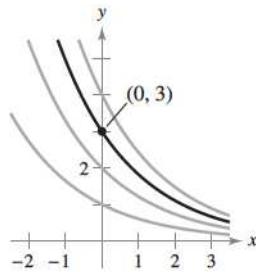


Figure for 29

Differential Equation

29. $2y' + y = 0$
30. $2xy + (x^2 + 2y)y' = 0$
31. $2xy' - 3y = 0$
32. $yy' - 2x = 0$

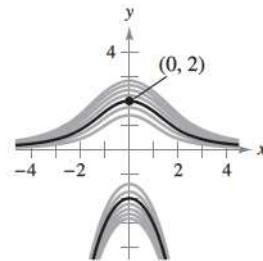


Figure for 30

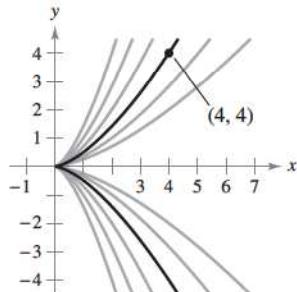


Figure for 31

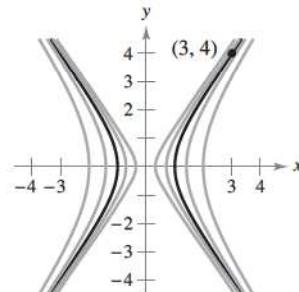


Figure for 32

In Exercises 33 and 34, the general solution of the differential equation is given. Use a graphing utility to graph the particular solutions for the given values of C .

33. $4yy' - x = 0$
 34. $yy' + x = 0$
- $$4y^2 - x^2 = C$$
- $$C = 0, C = \pm 1, C = \pm 4$$
- $$x^2 + y^2 = C$$
- $$C = 0, C = 1, C = 4$$

In Exercises 35–40, verify that the general solution satisfies the differential equation. Then find the particular solution that satisfies the initial condition.

35. $y = Ce^{-2x}$
36. $3x^2 + 2y^2 = C$
37. $y' + 2y = 0$
38. $3x + 2yy' = 0$
39. $y = 3$ when $x = 0$
40. $y = 3$ when $x = 1$
41. $y = C_1 \sin 3x + C_2 \cos 3x$
42. $xy'' + y' = 0$
43. $y = 2$ when $x = \pi/6$
44. $y = 0$ when $x = 2$
45. $y' = 1$ when $x = \pi/6$
46. $y' = \frac{1}{2}$ when $x = 2$

39. $y = C_1x + C_2x^3$
 $x^2y'' - 3xy' + 3y = 0$
 $y = 0$ when $x = 2$
 $y' = 4$ when $x = 2$

40. $y = e^{2x/3}(C_1 + C_2x)$
 $9y'' - 12y' + 4y = 0$
 $y = 4$ when $x = 0$
 $y = 0$ when $x = 3$

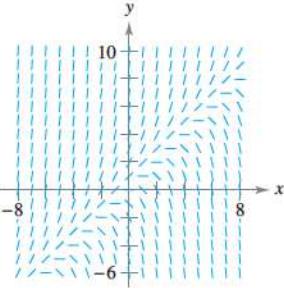
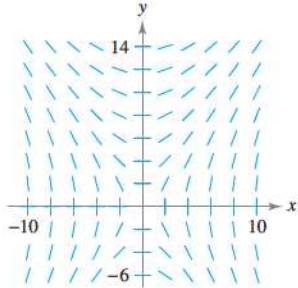
In Exercises 41–52, use integration to find a general solution of the differential equation.

41. $\frac{dy}{dx} = 6x^2$
42. $\frac{dy}{dx} = 2x^3 - 3x$
43. $\frac{dy}{dx} = \frac{x}{1+x^2}$
44. $\frac{dy}{dx} = \frac{e^x}{4+e^x}$
45. $\frac{dy}{dx} = \frac{x-2}{x}$
46. $\frac{dy}{dx} = x \cos x^2$
47. $\frac{dy}{dx} = \sin 2x$
48. $\frac{dy}{dx} = \tan^2 x$
49. $\frac{dy}{dx} = x\sqrt{x-6}$
50. $\frac{dy}{dx} = 2x\sqrt{3-x}$
51. $\frac{dy}{dx} = xe^{x^2}$
52. $\frac{dy}{dx} = 5e^{-x/2}$

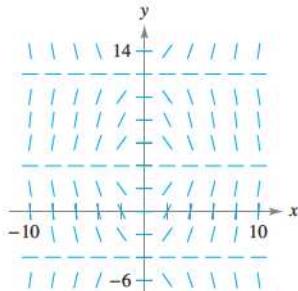
Slope Fields In Exercises 53–56, a differential equation and its slope field are given. Complete the table by determining the slopes (if possible) in the slope field at the given points.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx						

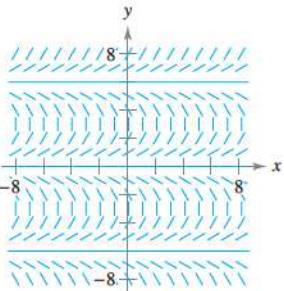
53. $\frac{dy}{dx} = \frac{2x}{y}$ 54. $\frac{dy}{dx} = y - x$



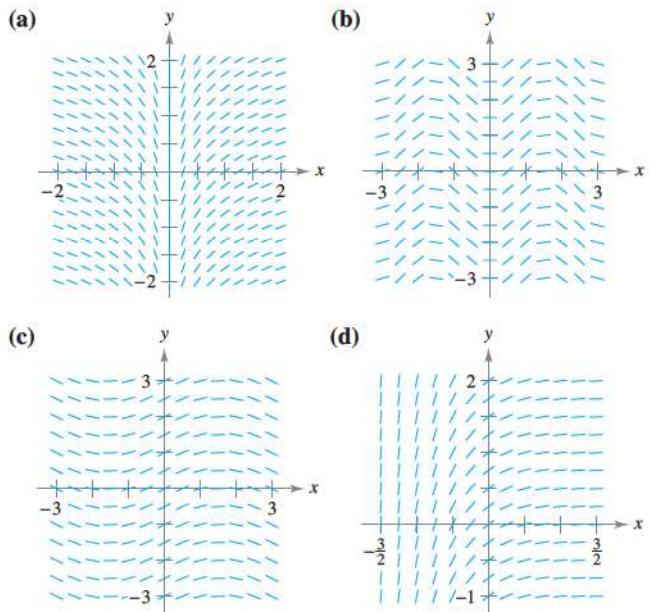
55. $\frac{dy}{dx} = x \cos \frac{\pi y}{8}$



56. $\frac{dy}{dx} = \tan\left(\frac{\pi y}{6}\right)$



In Exercises 57–60, match the differential equation with its slope field. [The slope fields are labeled (a), (b), (c), and (d).]

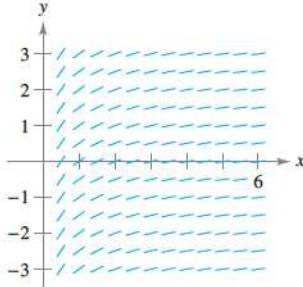


57. $\frac{dy}{dx} = \sin(2x)$ 58. $\frac{dy}{dx} = \frac{1}{2} \cos x$
59. $\frac{dy}{dx} = e^{-2x}$ 60. $\frac{dy}{dx} = \frac{1}{x}$

Slope Fields In Exercises 61–64, (a) sketch the slope field for the differential equation, (b) use the slope field to sketch the solution that passes through the given point, and (c) discuss the graph of the solution as $x \rightarrow \infty$ and $x \rightarrow -\infty$. Use a graphing utility to verify your results.

61. $y' = 3 - x$, (4, 2)
62. $y' = \frac{1}{3}x^2 - \frac{1}{2}x$, (1, 1)
63. $y' = y - 4x$, (2, 2)
64. $y' = y + xy$, (0, -4)

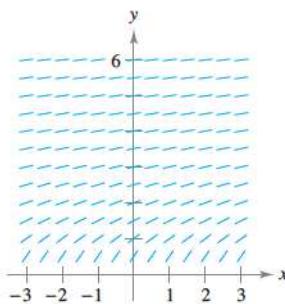
65. **Slope Field** Use the slope field for the differential equation $y' = 1/x$, where $x > 0$, to sketch the graph of the solution that satisfies each given initial condition. Then make a conjecture about the behavior of a particular solution of $y' = 1/x$ as $x \rightarrow \infty$. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.



(a) (1, 0)

(b) (2, -1)

- 66. Slope Field** Use the slope field for the differential equation $y' = 1/y$, where $y > 0$, to sketch the graph of the solution that satisfies each initial condition. Then make a conjecture about the behavior of a particular solution of $y' = 1/y$ as $x \rightarrow \infty$. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

(a) $(0, 1)$ (b) $(1, 1)$

- CAS Slope Fields** In Exercises 67–72, use a computer algebra system to (a) graph the slope field for the differential equation and (b) graph the solution satisfying the specified initial condition.

67. $\frac{dy}{dx} = 0.25y, \quad y(0) = 4$

68. $\frac{dy}{dx} = 4 - y, \quad y(0) = 6$

69. $\frac{dy}{dx} = 0.02y(10 - y), \quad y(0) = 2$

70. $\frac{dy}{dx} = 0.2x(2 - y), \quad y(0) = 9$

71. $\frac{dy}{dx} = 0.4y(3 - x), \quad y(0) = 1$

72. $\frac{dy}{dx} = \frac{1}{2}e^{-x/8} \sin \frac{\pi y}{4}, \quad y(0) = 2$

- Euler's Method** In Exercises 73–78, use Euler's Method to make a table of values for the approximate solution of the differential equation with the specified initial value. Use n steps of size h .

73. $y' = x + y, \quad y(0) = 2, \quad n = 10, \quad h = 0.1$

74. $y' = x + y, \quad y(0) = 2, \quad n = 20, \quad h = 0.05$

75. $y' = 3x - 2y, \quad y(0) = 3, \quad n = 10, \quad h = 0.05$

76. $y' = 0.5x(3 - y), \quad y(0) = 1, \quad n = 5, \quad h = 0.4$

77. $y' = e^{xy}, \quad y(0) = 1, \quad n = 10, \quad h = 0.1$

78. $y' = \cos x + \sin y, \quad y(0) = 5, \quad n = 10, \quad h = 0.1$

- In Exercises 79–81, complete the table using the exact solution of the differential equation and two approximations obtained using Euler's Method to approximate the particular solution of the differential equation. Use $h = 0.2$ and $h = 0.1$ and compute each approximation to four decimal places.

x	0	0.2	0.4	0.6	0.8	1
$y(x)$ (exact)						
$y(x)$ ($h = 0.2$)						
$y(x)$ ($h = 0.1$)						

Table for 79–81

Differential Equation	Initial Condition	Exact Solution
79. $\frac{dy}{dx} = y$	$(0, 3)$	$y = 3e^x$
80. $\frac{dy}{dx} = \frac{2x}{y}$	$(0, 2)$	$y = \sqrt{2x^2 + 4}$
81. $\frac{dy}{dx} = y + \cos(x)$	$(0, 0)$	$y = \frac{1}{2}(\sin x - \cos x + e^x)$

82. Compare the values of the approximations in Exercises 79–81 with the values given by the exact solution. How does the error change as h increases?

- AP** 83. **Temperature** At time $t = 0$ minutes, the temperature of an object is 140°F . The temperature of the object is changing at the rate given by the differential equation

$$\frac{dy}{dt} = -\frac{1}{2}(y - 72).$$

- (a) Use a graphing utility and Euler's Method to approximate the particular solutions of this differential equation at $t = 1, 2$, and 3 . Use a step size of $h = 0.1$. (A graphing utility program for Euler's Method is available on the book's website.)

- (b) Compare your results with the exact solution

$$y = 72 + 68e^{-t/2}.$$

- (c) Repeat parts (a) and (b) using a step size of $h = 0.05$. Compare the results.

CAPSTONE

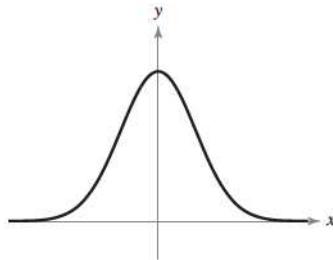
84. The graph shows a solution of one of the following differential equations. Determine the correct equation. Explain your reasoning.

(a) $y' = xy$

(b) $y' = \frac{4x}{y}$

(c) $y' = -4xy$

(d) $y' = 4 - xy$



WRITING ABOUT CONCEPTS

85. In your own words, describe the difference between a general solution of a differential equation and a particular solution.
86. Explain how to interpret a slope field.
87. Describe how to use Euler's Method to approximate a particular solution of a differential equation.
88. It is known that $y = Ce^{kx}$ is a solution of the differential equation $y' = 0.07y$. Is it possible to determine C or k from the information given? If so, find its value.

True or False? In Exercises 89–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

89. If $y = f(x)$ is a solution of a first-order differential equation, then $y = f(x) + C$ is also a solution.
90. The general solution of a differential equation is $y = -4.9x^2 + C_1x + C_2$. To find a particular solution, you must be given two initial conditions.
91. Slope fields represent the general solutions of differential equations.
92. A slope field shows that the slope at the point $(1, 1)$ is 6. This slope field represents the family of solutions for the differential equation $y' = 4x + 2y$.

93. **Errors and Euler's Method** The exact solution of the differential equation

$$\frac{dy}{dx} = -2y$$

where $y(0) = 4$, is $y = 4e^{-2x}$.

- (a) Use a graphing utility to complete the table, where y is the exact value of the solution, y_1 is the approximate solution using Euler's Method with $h = 0.1$, y_2 is the approximate solution using Euler's Method with $h = 0.2$, e_1 is the absolute error $|y - y_1|$, e_2 is the absolute error $|y - y_2|$, and r is the ratio e_1/e_2 .

x	0	0.2	0.4	0.6	0.8	1
y						
y_1						
y_2						
e_1						
e_2						
r						

- (b) What can you conclude about the ratio r as h changes?
(c) Predict the absolute error when $h = 0.05$.

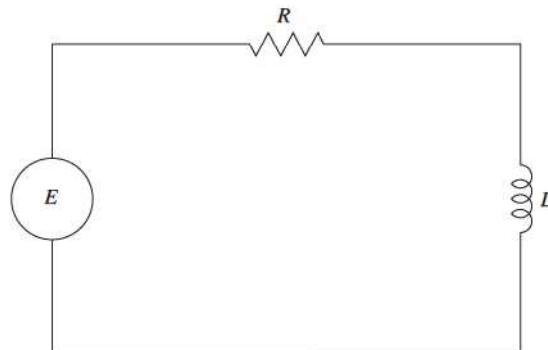


94. **Errors and Euler's Method** Repeat Exercise 93 for which the exact solution of the differential equation

$$\frac{dy}{dx} = x - y$$

where $y(0) = 1$, is $y = x - 1 + 2e^{-x}$.

95. **Electric Circuits** The diagram shows a simple electric circuit consisting of a power source, a resistor, and an inductor.



A model of the current I , in amperes (A), at time t is given by the first-order differential equation

$$L \frac{dI}{dt} + RI = E(t)$$

where $E(t)$ is the voltage (V) produced by the power source, R is the resistance, in ohms (Ω), and L is the inductance, in henrys (H). Suppose the electric circuit consists of a 24-V power source, a $12\text{-}\Omega$ resistor, and a 4-H inductor.

- (a) Sketch a slope field for the differential equation.
(b) What is the limiting value of the current? Explain.

96. **Think About It** It is known that $y = e^{kt}$ is a solution of the differential equation $y'' - 16y = 0$. Find the values of k .

97. **Think About It** It is known that $y = A \sin \omega t$ is a solution of the differential equation $y'' + 16y = 0$. Find the values of ω .

PUTNAM EXAM CHALLENGE

98. Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x)$$

where $g(x) \geq 0$ for all real x . Prove that $|f(x)|$ is bounded.

99. Prove that if the family of integral curves of the differential equation

$$\frac{dy}{dx} + p(x)y = q(x), \quad p(x) \cdot q(x) \neq 0$$

is cut by the line $x = k$, the tangents at the points of intersection are concurrent.

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6.2**Differential Equations: Growth and Decay**

- Use separation of variables to solve a simple differential equation.
- Use exponential functions to model growth and decay in applied problems.

Differential Equations

In the preceding section, you learned to analyze visually the solutions of differential equations using slope fields and to approximate solutions numerically using Euler's Method. Analytically, you have learned to solve only two types of differential equations—those of the forms $y' = f(x)$ and $y'' = f(x)$. In this section, you will learn how to solve a more general type of differential equation. The strategy is to rewrite the equation so that each variable occurs on only one side of the equation. This strategy is called *separation of variables*. (You will study this strategy in detail in Section 6.3.)

EXAMPLE 1 Solving a Differential Equation

$$y' = \frac{2x}{y} \quad \text{Original equation}$$

$yy' = 2x$ Multiply both sides by y .

$$\int yy' dx = \int 2x dx \quad \text{Integrate with respect to } x.$$

$$\int y dy = \int 2x dx \quad dy = y' dx$$

$$\frac{1}{2}y^2 = x^2 + C_1 \quad \text{Apply Power Rule.}$$

$$y^2 - 2x^2 = C \quad \text{Rewrite, letting } C = 2C_1.$$

So, the general solution is given by $y^2 - 2x^2 = C$. ■

STUDY TIP You can use implicit differentiation to check the solution in Example 1.

EXPLORATION

In Example 1, the general solution of the differential equation is

$$y^2 - 2x^2 = C.$$

Use a graphing utility to sketch the particular solutions for $C = \pm 2$, $C = \pm 1$, and $C = 0$. Describe the solutions graphically. Is the following statement true of each solution?

The slope of the graph at the point (x, y) is equal to twice the ratio of x and y .

Explain your reasoning. Are all curves for which this statement is true represented by the general solution?

Notice that when you integrate both sides of the equation in Example 1, you don't need to add a constant of integration to both sides. If you did, you would obtain the same result.

$$\begin{aligned} \int y dy &= \int 2x dx \\ \frac{1}{2}y^2 + C_2 &= x^2 + C_3 \\ \frac{1}{2}y^2 &= x^2 + (C_3 - C_2) \\ \frac{1}{2}y^2 &= x^2 + C_1 \end{aligned}$$

Some people prefer to use Leibniz notation and differentials when applying separation of variables. The solution of Example 1 is shown below using this notation.

$$\begin{aligned} \frac{dy}{dx} &= \frac{2x}{y} \\ y dy &= 2x dx \\ \int y dy &= \int 2x dx \\ \frac{1}{2}y^2 &= x^2 + C_1 \\ y^2 - 2x^2 &= C \end{aligned}$$

Growth and Decay Models

In many applications, the rate of change of a variable y is proportional to the value of y . If y is a function of time t , the proportion can be written as follows.

$$\frac{dy}{dt} = ky$$

Rate of change of y is proportional to y .

The general solution of this differential equation is given in the following theorem.

THEOREM 6.1 EXPONENTIAL GROWTH AND DECAY MODEL

If y is a differentiable function of t such that $y > 0$ and $y' = ky$ for some constant k , then

$$y = Ce^{kt}$$

C is the **initial value** of y , and k is the **proportionality constant**. **Exponential growth** occurs when $k > 0$, and **exponential decay** occurs when $k < 0$.

PROOF

$$y' = ky \quad \text{Write original equation.}$$

$$\frac{y'}{y} = k \quad \text{Separate variables.}$$

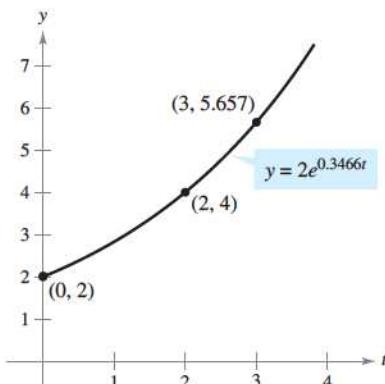
$$\int \frac{y'}{y} dt = \int k dt \quad \text{Integrate with respect to } t.$$

$$\int \frac{1}{y} dy = \int k dt \quad dy = y' dt$$

$$\ln y = kt + C_1 \quad \text{Find antiderivative of each side.}$$

$$y = e^{kt+C_1} \quad \text{Solve for } y.$$

$$y = Ce^{kt} \quad \text{Let } C = e^{C_1}.$$



If the rate of change of y is proportional to y , then y follows an exponential model.

Figure 6.8

STUDY TIP Using logarithmic properties, note that the value of k in Example 2 can also be written as $\ln(\sqrt{2})$. So, the model becomes $y = 2e^{(\ln\sqrt{2})t}$, which can then be rewritten as $y = 2(\sqrt{2})^t$.

EXAMPLE 2 Using an Exponential Growth Model

The rate of change of y is proportional to y . When $t = 0$, $y = 2$, and when $t = 2$, $y = 4$. What is the value of y when $t = 3$?

Solution Because $y' = ky$, you know that y and t are related by the equation $y = Ce^{kt}$. You can find the values of the constants C and k by applying the initial conditions.

$$2 = Ce^0 \Rightarrow C = 2 \quad \text{When } t = 0, y = 2.$$

$$4 = 2e^{2k} \Rightarrow k = \frac{1}{2} \ln 2 \approx 0.3466 \quad \text{When } t = 2, y = 4.$$

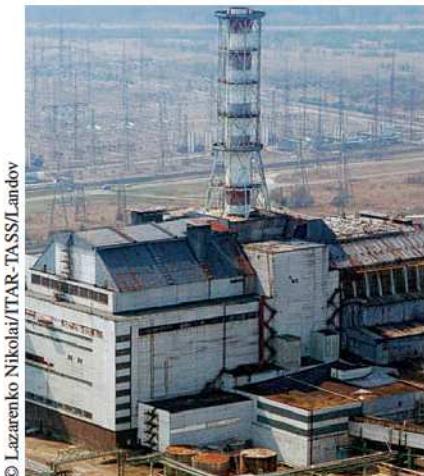
So, the model is $y \approx 2e^{0.3466t}$. When $t = 3$, the value of y is $2e^{0.3466(3)} \approx 5.657$ (see Figure 6.8).

TECHNOLOGY Most graphing utilities have curve-fitting capabilities that can be used to find models that represent data. Use the *exponential regression* feature of a graphing utility and the information in Example 2 to find a model for the data. How does your model compare with the given model?

Radioactive decay is measured in terms of *half-life*—the number of years required for half of the atoms in a sample of radioactive material to decay. The rate of decay is proportional to the amount present. The half-lives of some common radioactive isotopes are shown below.

Uranium (^{238}U)	4,470,000,000 years
Plutonium (^{239}Pu)	24,100 years
Carbon (^{14}C)	5715 years
Radium (^{226}Ra)	1599 years
Einsteinium (^{254}Es)	276 days
Nobelium (^{257}No)	25 seconds

EXAMPLE 3 Radioactive Decay



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Suppose that 10 grams of the plutonium isotope ^{239}Pu was released in the Chernobyl nuclear accident. How long will it take for the 10 grams to decay to 1 gram?

Solution Let y represent the mass (in grams) of the plutonium. Because the rate of decay is proportional to y , you know that

$$y = Ce^{kt}$$

where t is the time in years. To find the values of the constants C and k , apply the initial conditions. Using the fact that $y = 10$ when $t = 0$, you can write

$$10 = Ce^{k(0)} = Ce^0$$

which implies that $C = 10$. Next, using the fact that the half-life of ^{239}Pu is 24,100 years, you have $y = 10/2 = 5$ when $t = 24,100$, so you can write

$$5 = 10e^{k(24,100)}$$

$$\frac{1}{2} = e^{24,100k}$$

$$\frac{1}{24,100} \ln \frac{1}{2} = k$$

$$-0.000028761 \approx k.$$

So, the model is

$$y = 10e^{-0.000028761t}. \quad \text{Half-life model}$$

To find the time it would take for 10 grams to decay to 1 gram, you can solve for t in the equation

$$1 = 10e^{-0.000028761t}.$$

The solution is approximately 80,059 years. ■

NOTE The exponential decay model in Example 3 could also be written as $y = 10\left(\frac{1}{2}\right)^{t/24,100}$. This model is much easier to derive, but for some applications it is not as convenient to use.

From Example 3, notice that in an exponential growth or decay problem, it is easy to solve for C when you are given the value of y at $t = 0$. The next example demonstrates a procedure for solving for C and k when you do not know the value of y at $t = 0$.

EXAMPLE 4 Population Growth

Suppose an experimental population of fruit flies increases according to the law of exponential growth. There were 100 flies after the second day of the experiment and 300 flies after the fourth day. Approximately how many flies were in the original population?

Solution Let $y = Ce^{kt}$ be the number of flies at time t , where t is measured in days. Note that y is continuous whereas the number of flies is discrete. Because $y = 100$ when $t = 2$ and $y = 300$ when $t = 4$, you can write

$$100 = Ce^{2k} \quad \text{and} \quad 300 = Ce^{4k}.$$

From the first equation, you know that $C = 100e^{-2k}$. Substituting this value into the second equation produces the following.

$$300 = 100e^{-2k}e^{4k}$$

$$300 = 100e^{2k}$$

$$\ln 3 = 2k$$

$$\frac{1}{2} \ln 3 = k$$

$$0.5493 \approx k$$

So, the exponential growth model is

$$y = Ce^{0.5493t}.$$

To solve for C , reapply the condition $y = 100$ when $t = 2$ and obtain

$$100 = Ce^{0.5493(2)}$$

$$C = 100e^{-1.0986} \approx 33.$$

So, the original population (when $t = 0$) consisted of approximately $y = C = 33$ flies, as shown in Figure 6.9.

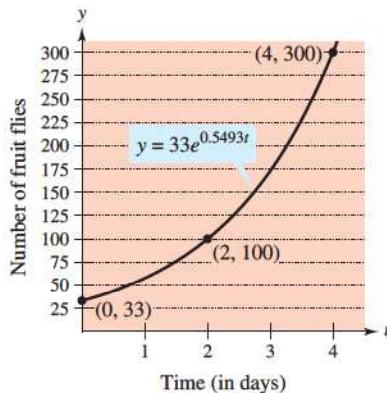


Figure 6.9

EXAMPLE 5 Declining Sales

Four months after it stops advertising, a manufacturing company notices that its sales have dropped from 100,000 units per month to 80,000 units per month. If the sales follow an exponential pattern of decline, what will they be after another 2 months?

Solution Use the exponential decay model $y = Ce^{kt}$, where t is measured in months. From the initial condition ($t = 0$), you know that $C = 100,000$. Moreover, because $y = 80,000$ when $t = 4$, you have

$$80,000 = 100,000e^{4k}$$

$$0.8 = e^{4k}$$

$$\ln(0.8) = 4k$$

$$-0.0558 \approx k.$$

So, after 2 more months ($t = 6$), you can expect the monthly sales rate to be

$$y \approx 100,000e^{-0.0558(6)}$$

$$\approx 71,500 \text{ units.}$$

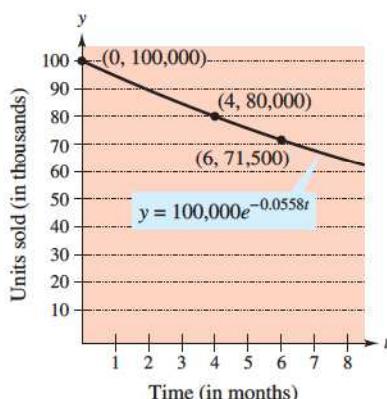


Figure 6.10

See Figure 6.10. ■

In Examples 2 through 5, you did not actually have to solve the differential equation

$$y' = ky.$$

(This was done once in the proof of Theorem 6.1.) The next example demonstrates a problem whose solution involves the separation of variables technique. The example concerns **Newton's Law of Cooling**, which states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium.

EXAMPLE 6 Newton's Law of Cooling

Let y represent the temperature (in °F) of an object in a room whose temperature is kept at a constant 60°. If the object cools from 100° to 90° in 10 minutes, how much longer will it take for its temperature to decrease to 80°?

Solution From Newton's Law of Cooling, you know that the rate of change in y is proportional to the difference between y and 60. This can be written as

$$y' = k(y - 60), \quad 80 \leq y \leq 100.$$

To solve this differential equation, use separation of variables, as follows.

$$\frac{dy}{dt} = k(y - 60) \quad \text{Differential equation}$$

$$\left(\frac{1}{y - 60}\right) dy = k dt \quad \text{Separate variables.}$$

$$\int \frac{1}{y - 60} dy = \int k dt \quad \text{Integrate each side.}$$

$$\ln|y - 60| = kt + C_1 \quad \text{Find antiderivative of each side.}$$

Because $y > 60$, $|y - 60| = y - 60$, and you can omit the absolute value signs. Using exponential notation, you have

$$y - 60 = e^{kt+C_1} \quad \Rightarrow \quad y = 60 + Ce^{kt}. \quad C = e^{C_1}$$

Using $y = 100$ when $t = 0$, you obtain $100 = 60 + Ce^{k(0)} = 60 + C$, which implies that $C = 40$. Because $y = 90$ when $t = 10$,

$$90 = 60 + 40e^{k(10)}$$

$$30 = 40e^{10k}$$

$$k = \frac{1}{10} \ln \frac{3}{4} \approx -0.02877.$$

So, the model is

$$y = 60 + 40e^{-0.02877t} \quad \text{Cooling model}$$

and finally, when $y = 80$, you obtain

$$80 = 60 + 40e^{-0.02877t}$$

$$20 = 40e^{-0.02877t}$$

$$\frac{1}{2} = e^{-0.02877t}$$

$$\ln \frac{1}{2} = -0.02877t$$

$$t \approx 24.09 \text{ minutes.}$$

So, it will require about 14.09 more minutes for the object to cool to a temperature of 80° (see Figure 6.11). ■

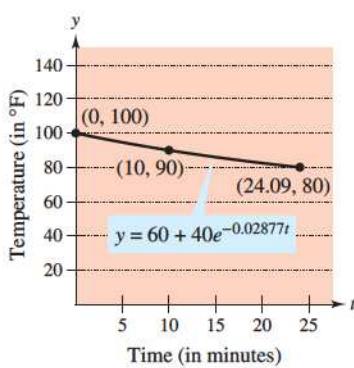


Figure 6.11

6.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, solve the differential equation.

1. $\frac{dy}{dx} = x + 3$

2. $\frac{dy}{dx} = 6 - x$

3. $\frac{dy}{dx} = y + 3$

4. $\frac{dy}{dx} = 6 - y$

5. $y' = \frac{5x}{y}$

6. $y' = \frac{\sqrt{x}}{7y}$

7. $y' = \sqrt{x}y$

8. $y' = x(1+y)$

9. $(1+x^2)y' - 2xy = 0$

10. $xy + y' = 100x$

In Exercises 11–14, write and solve the differential equation that models the verbal statement.

11. The rate of change of Q with respect to t is inversely proportional to the square of t .

12. The rate of change of P with respect to t is proportional to $25 - t$.

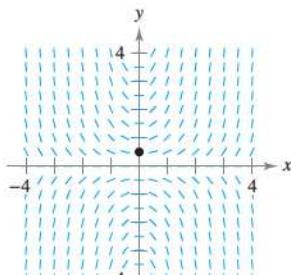
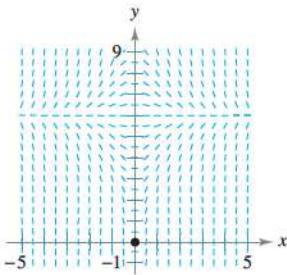
13. The rate of change of N with respect to s is proportional to $500 - s$.

14. The rate of change of y with respect to x varies jointly as x and $L - y$.

Slope Fields In Exercises 15 and 16, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketch in part (a). To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

15. $\frac{dy}{dx} = x(6-y), \quad (0,0)$

16. $\frac{dy}{dx} = xy, \quad (0, \frac{1}{2})$



In Exercises 17–20, find the function $y = f(t)$ passing through the point $(0, 10)$ with the given first derivative. Use a graphing utility to graph the solution.

17. $\frac{dy}{dt} = \frac{1}{2}t$

18. $\frac{dy}{dt} = -\frac{3}{4}\sqrt{t}$

19. $\frac{dy}{dt} = -\frac{1}{2}y$

20. $\frac{dy}{dt} = \frac{3}{4}y$

In Exercises 21–24, write and solve the differential equation that models the verbal statement. Evaluate the solution at the specified value of the independent variable.

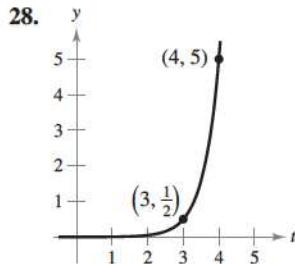
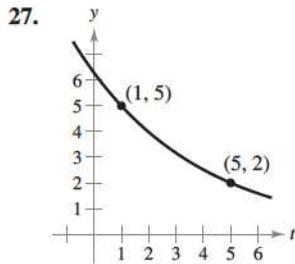
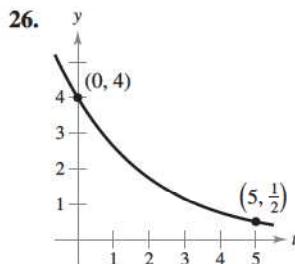
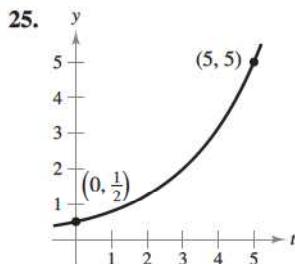
21. The rate of change of y is proportional to y . When $x = 0$, $y = 6$, and when $x = 4$, $y = 15$. What is the value of y when $x = 8$?

22. The rate of change of N is proportional to N . When $t = 0$, $N = 250$, and when $t = 1$, $N = 400$. What is the value of N when $t = 4$?

23. The rate of change of V is proportional to V . When $t = 0$, $V = 20,000$, and when $t = 4$, $V = 12,500$. What is the value of V when $t = 6$?

24. The rate of change of P is proportional to P . When $t = 0$, $P = 5000$, and when $t = 1$, $P = 4750$. What is the value of P when $t = 5$?

In Exercises 25–28, find the exponential function $y = Ce^{kt}$ that passes through the two given points.



WRITING ABOUT CONCEPTS

29. Describe what the values of C and k represent in the exponential growth and decay model, $y = Ce^{kt}$.

30. Give the differential equation that models exponential growth and decay.

In Exercises 31 and 32, determine the quadrants in which the solution of the differential equation is an increasing function. Explain. (Do not solve the differential equation.)

31. $\frac{dy}{dx} = \frac{1}{2}xy$

32. $\frac{dy}{dx} = \frac{1}{2}x^2y$

Radioactive Decay In Exercises 33–40, complete the table for the radioactive isotope.

Isotope	Half-Life (in years)	Initial Quantity	Amount After 1000 Years	Amount After 10,000 Years
33. ^{226}Ra	1599	20 g		
34. ^{226}Ra	1599		1.5 g	
35. ^{226}Ra	1599			0.1 g
36. ^{14}C	5715			3 g
37. ^{14}C	5715	5 g		
38. ^{14}C	5715		1.6 g	
39. ^{239}Pu	24,100		2.1 g	
40. ^{239}Pu	24,100			0.4 g

41. **Radioactive Decay** Radioactive radium has a half-life of approximately 1599 years. What percent of a given amount remains after 100 years?

42. **Carbon Dating** Carbon-14 dating assumes that the carbon dioxide on Earth today has the same radioactive content as it did centuries ago. If this is true, the amount of ^{14}C absorbed by a tree that grew several centuries ago should be the same as the amount of ^{14}C absorbed by a tree growing today. A piece of ancient charcoal contains only 15% as much of the radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal? (The half-life of ^{14}C is 5715 years.)

Compound Interest In Exercises 43–48, complete the table for a savings account in which interest is compounded continuously.

Initial Investment	Annual Rate	Time to Double	Amount After 10 Years
43. \$4000	6%		
44. \$18,000	$5\frac{1}{2}\%$		
45. \$750		$7\frac{3}{4}$ yr	
46. \$12,500		5 yr	
47. \$500			\$1292.85
48. \$2000			\$5436.56

Compound Interest In Exercises 49–52, find the principal P that must be invested at rate r , compounded monthly, so that \$1,000,000 will be available for retirement in t years.

49. $r = 7\frac{1}{2}\%$, $t = 20$ 50. $r = 6\%$, $t = 40$
 51. $r = 8\%$, $t = 35$ 52. $r = 9\%$, $t = 25$

Compound Interest In Exercises 53–56, find the time necessary for \$1000 to double if it is invested at a rate of r compounded (a) annually, (b) monthly, (c) daily, and (d) continuously.

53. $r = 7\%$ 54. $r = 6\%$
 55. $r = 8.5\%$ 56. $r = 5.5\%$

Population In Exercises 57–61, the population (in millions) of a country in 2007 and the expected continuous annual rate of change k of the population are given. (Source: U.S. Census Bureau, International Data Base)

- (a) Find the exponential growth model $P = Ce^{kt}$ for the population by letting $t = 0$ correspond to 2000.
 (b) Use the model to predict the population of the country in 2015.
 (c) Discuss the relationship between the sign of k and the change in the population of the country.

Country	2007 Population	k
57. Latvia	2.3	-0.006
58. Egypt	80.3	0.017
59. Paraguay	6.7	0.024
60. Hungary	10.0	-0.003
61. Uganda	30.3	0.036

CAPSTONE

62. (a) Suppose an insect population increases by a constant number each month. Explain why the number of insects can be represented by a linear function.
 (b) Suppose an insect population increases by a constant percentage each month. Explain why the number of insects can be represented by an exponential function.

63. **Modeling Data** One hundred bacteria are started in a culture and the number N of bacteria is counted each hour for 5 hours. The results are shown in the table, where t is the time in hours.

t	0	1	2	3	4	5
N	100	126	151	198	243	297

- (a) Use the regression capabilities of a graphing utility to find an exponential model for the data.
 (b) Use the model to estimate the time required for the population to quadruple in size.
 64. **Bacteria Growth** The number of bacteria in a culture is increasing according to the law of exponential growth. There are 125 bacteria in the culture after 2 hours and 350 bacteria after 4 hours.
 (a) Find the initial population.
 (b) Write an exponential growth model for the bacteria population. Let t represent time in hours.
 (c) Use the model to determine the number of bacteria after 8 hours.
 (d) After how many hours will the bacteria count be 25,000?

65. **Learning Curve** The management at a certain factory has found that a worker can produce at most 30 units in a day. The learning curve for the number of units N produced per day after a new employee has worked t days is $N = 30(1 - e^{-kt})$. After 20 days on the job, a particular worker produces 19 units.

- (a) Find the learning curve for this worker.
 (b) How many days should pass before this worker is producing 25 units per day?
- 66. Learning Curve** If the management in Exercise 65 requires a new employee to produce at least 20 units per day after 30 days on the job, find (a) the learning curve that describes this minimum requirement and (b) the number of days before a minimal achiever is producing 25 units per day.

- 67. Modeling Data** The table shows the populations P (in millions) of the United States from 1960 to 2000. (Source: U.S. Census Bureau)

Year	1960	1970	1980	1990	2000
Population, P	181	205	228	250	282

- (a) Use the 1960 and 1970 data to find an exponential model P_1 for the data. Let $t = 0$ represent 1960.
 (b) Use a graphing utility to find an exponential model P_2 for all the data. Let $t = 0$ represent 1960.
 (c) Use a graphing utility to plot the data and graph models P_1 and P_2 in the same viewing window. Compare the actual data with the predictions. Which model better fits the data?
 (d) Estimate when the population will be 320 million.

- 68. Modeling Data** The table shows the net receipts and the amounts required to service the national debt (interest on Treasury debt securities) of the United States from 2001 through 2010. The years 2007 through 2010 are estimated, and the monetary amounts are given in billions of dollars. (Source: U.S. Office of Management and Budget)

Year	2001	2002	2003	2004	2005
Receipts	1991.4	1853.4	1782.5	1880.3	2153.9
Interest	359.5	332.5	318.1	321.7	352.3

- (a) Use the regression capabilities of a graphing utility to find an exponential model R for the receipts and a quartic model I for the amount required to service the debt. Let t represent the time in years, with $t = 1$ corresponding to 2001.
 (b) Use a graphing utility to plot the points corresponding to the receipts, and graph the exponential model. Based on the model, what is the continuous rate of growth of the receipts?
 (c) Use a graphing utility to plot the points corresponding to the amounts required to service the debt, and graph the quartic model.
 (d) Find a function $P(t)$ that approximates the percent of the receipts that is required to service the national debt. Use a graphing utility to graph this function.

- 69. Sound Intensity** The level of sound β (in decibels) with an intensity of I is $\beta(I) = 10 \log_{10}(I/I_0)$, where I_0 is an intensity of 10^{-16} watt per square centimeter, corresponding roughly to the faintest sound that can be heard. Determine $\beta(I)$ for the following.

- (a) $I = 10^{-14}$ watt per square centimeter (whisper)
 (b) $I = 10^{-9}$ watt per square centimeter (busy street corner)
 (c) $I = 10^{-6.5}$ watt per square centimeter (air hammer)
 (d) $I = 10^{-4}$ watt per square centimeter (threshold of pain)

- 70. Noise Level** With the installation of noise suppression materials, the noise level in an auditorium was reduced from 93 to 80 decibels. Use the function in Exercise 69 to find the percent decrease in the intensity level of the noise as a result of the installation of these materials.

- 71. Forestry** The value of a tract of timber is $V(t) = 100,000e^{0.8\sqrt{t}}$, where t is the time in years, with $t = 0$ corresponding to 2008. If money earns interest continuously at 10%, the present value of the timber at any time t is $A(t) = V(t)e^{-0.10t}$. Find the year in which the timber should be harvested to maximize the present value function.

- 72. Earthquake Intensity** On the Richter scale, the magnitude R of an earthquake of intensity I is

$$R = \frac{\ln I - \ln I_0}{\ln 10}$$

where I_0 is the minimum intensity used for comparison. Assume that $I_0 = 1$.

- (a) Find the intensity of the 1906 San Francisco earthquake ($R = 8.3$).
 (b) Find the factor by which the intensity is increased if the Richter scale measurement is doubled.
 (c) Find dR/dI .

- 73. Newton's Law of Cooling** When an object is removed from a furnace and placed in an environment with a constant temperature of 80°F , its core temperature is 1500°F . One hour after it is removed, the core temperature is 1120°F . Find the core temperature 5 hours after the object is removed from the furnace.

- 74. Newton's Law of Cooling** A container of hot liquid is placed in a freezer that is kept at a constant temperature of 20°F . The initial temperature of the liquid is 160°F . After 5 minutes, the liquid's temperature is 60°F . How much longer will it take for its temperature to decrease to 30°F ?

True or False? In Exercises 75–78, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

75. In exponential growth, the rate of growth is constant.
 76. In linear growth, the rate of growth is constant.
 77. If prices are rising at a rate of 0.5% per month, then they are rising at a rate of 6% per year.
 78. The differential equation modeling exponential growth is $dy/dx = ky$, where k is a constant.

6.3**Differential Equations: Separation of Variables**

- Recognize and solve differential equations that can be solved by separation of variables.
- Recognize and solve homogeneous differential equations.
- Use differential equations to model and solve applied problems.

Separation of Variables

Consider a differential equation that can be written in the form

$$M(x) + N(y) \frac{dy}{dx} = 0$$

where M is a continuous function of x alone and N is a continuous function of y alone. As you saw in the preceding section, for this type of equation, all x terms can be collected with dx and all y terms with dy , and a solution can be obtained by integration. Such equations are said to be **separable**, and the solution procedure is called **separation of variables**. Below are some examples of differential equations that are separable.

<i>Original Differential Equation</i>	<i>Rewritten with Variables Separated</i>
$x^2 + 3y \frac{dy}{dx} = 0$	$3y dy = -x^2 dx$
$(\sin x)y' = \cos x$	$dy = \cot x dx$
$\frac{xy'}{e^y + 1} = 2$	$\frac{1}{e^y + 1} dy = \frac{2}{x} dx$

**EXAMPLE 1** Separation of Variables

Find the general solution of $(x^2 + 4) \frac{dy}{dx} = xy$.

Solution To begin, note that $y = 0$ is a solution. To find other solutions, assume that $y \neq 0$ and separate variables as shown.

$(x^2 + 4) dy = xy dx$	Differential form
$\frac{dy}{y} = \frac{x}{x^2 + 4} dx$	Separate variables.

Now, integrate to obtain

$$\int \frac{dy}{y} = \int \frac{x}{x^2 + 4} dx \quad \text{Integrate.}$$

$$\ln|y| = \frac{1}{2} \ln(x^2 + 4) + C_1$$

$$\begin{aligned} \ln|y| &= \ln\sqrt{x^2 + 4} + C_1 \\ |y| &= e^{C_1}\sqrt{x^2 + 4} \\ y &= \pm e^{C_1}\sqrt{x^2 + 4}. \end{aligned}$$

Because $y = 0$ is also a solution, you can write the general solution as

$$y = C\sqrt{x^2 + 4}. \quad \text{General solution } (C = \pm e^{C_1})$$

NOTE Be sure to check your solutions throughout this chapter. In Example 1, you can check the solution $y = C\sqrt{x^2 + 4}$ by differentiating and substituting into the original equation.

$$\begin{aligned} (x^2 + 4) \frac{dy}{dx} &= xy \\ (x^2 + 4) \frac{Cx}{\sqrt{x^2 + 4}} &\stackrel{?}{=} x(C\sqrt{x^2 + 4}) \\ Cx\sqrt{x^2 + 4} &= Cx\sqrt{x^2 + 4} \end{aligned}$$

So, the solution checks.

In some cases it is not feasible to write the general solution in the explicit form $y = f(x)$. The next example illustrates such a solution. Implicit differentiation can be used to verify this solution.

FOR FURTHER INFORMATION

For an example (from engineering) of a differential equation that is separable, see the article “Designing a Rose Cutter” by J. S. Hartzler in *The College Mathematics Journal*. To view this article, go to the website www.matharticles.com.

EXAMPLE 2 Finding a Particular Solution

Given the initial condition $y(0) = 1$, find the particular solution of the equation

$$xy \, dx + e^{-x^2}(y^2 - 1) \, dy = 0.$$

Solution Note that $y = 0$ is a solution of the differential equation—but this solution does not satisfy the initial condition. So, you can assume that $y \neq 0$. To separate variables, you must rid the first term of y and the second term of e^{-x^2} . So, you should multiply by e^{x^2}/y and obtain the following.

$$\begin{aligned} xy \, dx + e^{-x^2}(y^2 - 1) \, dy &= 0 \\ e^{-x^2}(y^2 - 1) \, dy &= -xy \, dx \\ \int \left(y - \frac{1}{y}\right) \, dy &= \int -xe^{x^2} \, dx \\ \frac{y^2}{2} - \ln|y| &= -\frac{1}{2}e^{x^2} + C \end{aligned}$$

From the initial condition $y(0) = 1$, you have $\frac{1}{2} - 0 = -\frac{1}{2} + C$, which implies that $C = 1$. So, the particular solution has the implicit form

$$\frac{y^2}{2} - \ln|y| = -\frac{1}{2}e^{x^2} + 1$$

$$y^2 - \ln y^2 + e^{x^2} = 2.$$

You can check this by differentiating and rewriting to get the original equation.

EXAMPLE 3 Finding a Particular Solution Curve

Find the equation of the curve that passes through the point $(1, 3)$ and has a slope of y/x^2 at any point (x, y) .

Solution Because the slope of the curve is given by y/x^2 , you have

$$\frac{dy}{dx} = \frac{y}{x^2}$$

with the initial condition $y(1) = 3$. Separating variables and integrating produces

$$\begin{aligned} \int \frac{dy}{y} &= \int \frac{dx}{x^2}, \quad y \neq 0 \\ \ln|y| &= -\frac{1}{x} + C_1 \\ y &= e^{-(1/x)+C_1} = Ce^{-1/x}. \end{aligned}$$

Because $y = 3$ when $x = 1$, it follows that $3 = Ce^{-1}$ and $C = 3e$. So, the equation of the specified curve is

$$y = (3e)e^{-1/x} = 3e^{(x-1)/x}, \quad x > 0.$$

Because the solution is not defined at $x = 0$ and the initial condition is given at $x = 1$, x is restricted to positive values. See Figure 6.12. ■

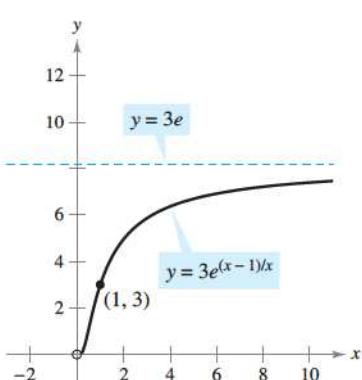


Figure 6.12

Homogeneous Differential Equations

Some differential equations that are not separable in x and y can be made separable by a change of variables. This is true for differential equations of the form $y' = f(x, y)$, where f is a **homogeneous function**. The function given by $f(x, y)$ is **homogeneous of degree n** if

NOTE The notation $f(x, y)$ is used to denote a function of two variables in much the same way as $f(x)$ denotes a function of one variable. You will study functions of two variables in detail in Chapter 13.

$$f(tx, ty) = t^n f(x, y)$$

Homogeneous function of degree n

where n is an integer.

EXAMPLE 4 Verifying Homogeneous Functions

- a. $f(x, y) = x^2y - 4x^3 + 3xy^2$ is a homogeneous function of degree 3 because

$$\begin{aligned} f(tx, ty) &= (tx)^2(ty) - 4(tx)^3 + 3(tx)(ty)^2 \\ &= t^3(x^2y) - t^3(4x^3) + t^3(3xy^2) \\ &= t^3(x^2y - 4x^3 + 3xy^2) \\ &= t^3f(x, y). \end{aligned}$$

- b. $f(x, y) = xe^{x/y} + y \sin(y/x)$ is a homogeneous function of degree 1 because

$$\begin{aligned} f(tx, ty) &= txe^{tx/ty} + ty \sin \frac{ty}{tx} \\ &= t\left(xe^{x/y} + y \sin \frac{y}{x}\right) \\ &= tf(x, y). \end{aligned}$$

- c. $f(x, y) = x + y^2$ is *not* a homogeneous function because

$$f(tx, ty) = tx + t^2y^2 = t(x + ty^2) \neq t^n(x + y^2).$$

- d. $f(x, y) = x/y$ is a homogeneous function of degree 0 because

$$f(tx, ty) = \frac{tx}{ty} = t^0 \frac{x}{y}.$$

DEFINITION OF HOMOGENEOUS DIFFERENTIAL EQUATION

A **homogeneous differential equation** is an equation of the form

$$M(x, y) dx + N(x, y) dy = 0$$

where M and N are homogeneous functions of the same degree.

EXAMPLE 5 Testing for Homogeneous Differential Equations

- a. $(x^2 + xy) dx + y^2 dy = 0$ is homogeneous of degree 2.

- b. $x^3 dx = y^3 dy$ is homogeneous of degree 3.

- c. $(x^2 + 1) dx + y^2 dy = 0$ is *not* a homogeneous differential equation.

To solve a homogeneous differential equation by the method of separation of variables, use the following change of variables theorem.

THEOREM 6.2 CHANGE OF VARIABLES FOR HOMOGENEOUS EQUATIONS

If $M(x, y) dx + N(x, y) dy = 0$ is homogeneous, then it can be transformed into a differential equation whose variables are separable by the substitution

$$y = vx$$

where v is a differentiable function of x .

EXAMPLE 6 Solving a Homogeneous Differential Equation

Find the general solution of

$$(x^2 - y^2) dx + 3xy dy = 0.$$

STUDY TIP The substitution $y = vx$ will yield a differential equation that is separable with respect to the variables x and v . You must write your final solution, however, in terms of x and y .

Solution Because $(x^2 - y^2)$ and $3xy$ are both homogeneous of degree 2, let $y = vx$ to obtain $dy = x dv + v dx$. Then, by substitution, you have

$$\begin{aligned} (x^2 - v^2x^2) dx + 3x(vx)(\underbrace{x dv + v dx}_{} \color{red}{dy}) &= 0 \\ (x^2 + 2v^2x^2) dx + 3x^3v dv &= 0 \\ x^2(1 + 2v^2) dx + x^2(3vx) dv &= 0. \end{aligned}$$

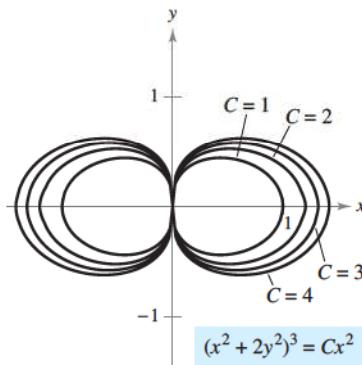
Dividing by x^2 and separating variables produces

$$\begin{aligned} (1 + 2v^2) dx &= -3vx dv \\ \int \frac{dx}{x} &= \int \frac{-3v}{1 + 2v^2} dv \\ \ln|x| &= -\frac{3}{4} \ln(1 + 2v^2) + C_1 \\ 4 \ln|x| &= -3 \ln(1 + 2v^2) + \ln|C| \\ \ln x^4 &= \ln|C(1 + 2v^2)^{-3}| \\ x^4 &= C(1 + 2v^2)^{-3}. \end{aligned}$$

Substituting for v produces the following general solution.

$$\begin{aligned} x^4 &= C \left[1 + 2 \left(\frac{y}{x} \right)^2 \right]^{-3} \\ \left(1 + \frac{2y^2}{x^2} \right)^3 x^4 &= C \\ (x^2 + 2y^2)^3 &= Cx^2 \quad \text{General solution} \end{aligned}$$

You can check this by differentiating and rewriting to get the original equation. ■



General solution of
 $(x^2 - y^2) dx + 3xy dy = 0$

Figure 6.13

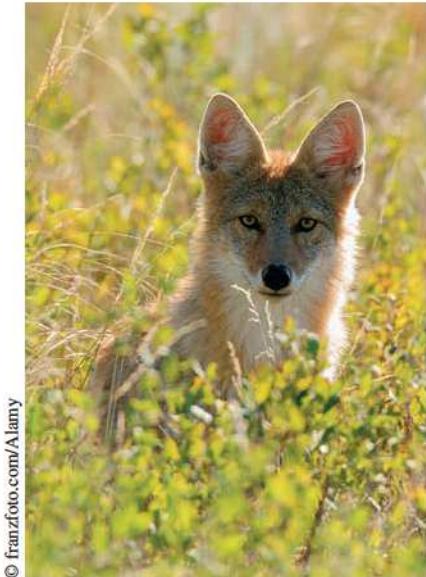
TECHNOLOGY If you have access to a graphing utility, try using it to graph several solutions of the equation in Example 6. For instance, Figure 6.13 shows the graphs of

$$(x^2 + 2y^2)^3 = Cx^2$$

for $C = 1, 2, 3$, and 4 .

Applications

EXAMPLE 7 Wildlife Population



The rate of change of the number of coyotes $N(t)$ in a population is directly proportional to $650 - N(t)$, where t is the time in years. When $t = 0$, the population is 300, and when $t = 2$, the population has increased to 500. Find the population when $t = 3$.

Solution Because the rate of change of the population is proportional to $650 - N(t)$, you can write the following differential equation.

$$\frac{dN}{dt} = k(650 - N)$$

You can solve this equation using separation of variables.

$$dN = k(650 - N) dt \quad \text{Differential form}$$

$$\frac{dN}{650 - N} = k dt \quad \text{Separate variables.}$$

$$-\ln|650 - N| = kt + C_1 \quad \text{Integrate.}$$

$$\ln|650 - N| = -kt - C_1 \quad \text{Multiply each side by } -1.$$

$$650 - N = e^{-kt - C_1} \quad \text{Assume } N < 650.$$

$$N = 650 - Ce^{-kt} \quad \text{General solution}$$

Using $N = 300$ when $t = 0$, you can conclude that $C = 350$, which produces

$$N = 650 - 350e^{-kt}.$$

Then, using $N = 500$ when $t = 2$, it follows that

$$500 = 650 - 350e^{-2k} \Rightarrow e^{-2k} = \frac{3}{7} \Rightarrow k \approx 0.4236.$$

So, the model for the coyote population is

$$N = 650 - 350e^{-0.4236t}. \quad \text{Model for population}$$

When $t = 3$, you can approximate the population to be

$$N = 650 - 350e^{-0.4236(3)} \approx 552 \text{ coyotes.}$$

The model for the population is shown in Figure 6.14. Note that $N = 650$ is the horizontal asymptote of the graph and is the *carrying capacity* of the model. You will learn more about carrying capacity in Section 6.4.

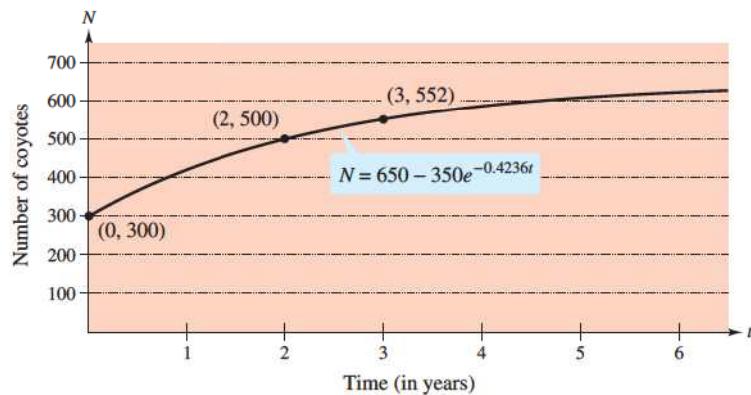
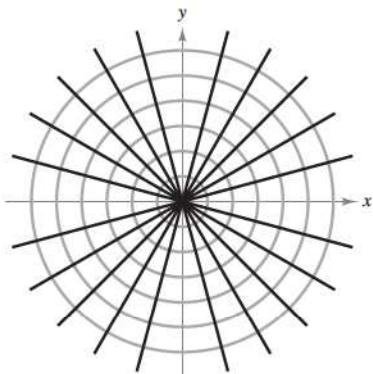


Figure 6.14



Each line $y = Kx$ is an orthogonal trajectory of the family of circles.

Figure 6.15

A common problem in electrostatics, thermodynamics, and hydrodynamics involves finding a family of curves, each of which is orthogonal to all members of a given family of curves. For example, Figure 6.15 shows a family of circles

$$x^2 + y^2 = C \quad \text{Family of circles}$$

each of which intersects the lines in the family

$$y = Kx \quad \text{Family of lines}$$

at right angles. Two such families of curves are said to be **mutually orthogonal**, and each curve in one of the families is called an **orthogonal trajectory** of the other family. In electrostatics, lines of force are orthogonal to the *equipotential curves*. In thermodynamics, the flow of heat across a plane surface is orthogonal to the *isothermal curves*. In hydrodynamics, the flow (stream) lines are orthogonal trajectories of the *velocity potential curves*.

EXAMPLE 8 Finding Orthogonal Trajectories

Describe the orthogonal trajectories for the family of curves given by

$$y = \frac{C}{x}$$

for $C \neq 0$. Sketch several members of each family.

Solution First, solve the given equation for C and write $xy = C$. Then, by differentiating implicitly with respect to x , you obtain the differential equation

$$xy' + y = 0 \quad \text{Differential equation}$$

$$x \frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x}. \quad \text{Slope of given family}$$

Because y' represents the slope of the given family of curves at (x, y) , it follows that the orthogonal family has the negative reciprocal slope x/y . So,

$$\frac{dy}{dx} = \frac{x}{y}. \quad \text{Slope of orthogonal family}$$

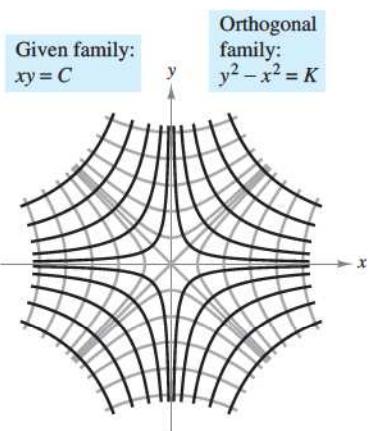
Now you can find the orthogonal family by separating variables and integrating.

$$\int y \, dy = \int x \, dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1$$

$$y^2 - x^2 = K$$

The centers are at the origin, and the transverse axes are vertical for $K > 0$ and horizontal for $K < 0$. If $K = 0$, the orthogonal trajectories are the lines $y = \pm x$. If $K \neq 0$, the orthogonal trajectories are hyperbolas. Several trajectories are shown in Figure 6.16. ■



Orthogonal trajectories
Figure 6.16

EXAMPLE 9 Modeling Advertising Awareness

A new cereal product is introduced through an advertising campaign to a population of 1 million potential customers. The rate at which the population hears about the product is assumed to be proportional to the number of people who are not yet aware of the product. By the end of 1 year, half of the population has heard of the product. How many will have heard of it by the end of 2 years?

Solution Let y be the number (in millions) of people at time t who have heard of the product. This means that $(1 - y)$ is the number of people who have not heard of it, and dy/dt is the rate at which the population hears about the product. From the given assumption, you can write the differential equation as shown.

$$\frac{dy}{dt} = k(1 - y)$$

↑ ↑ ↑
 Rate of change of y is proportional to the difference between 1 and y .

You can solve this equation using separation of variables.

$dy = k(1 - y) dt$	<small>Differential form</small>
$\frac{dy}{1 - y} = k dt$	<small>Separate variables.</small>
$-\ln 1 - y = kt + C_1$	<small>Integrate.</small>
$\ln 1 - y = -kt - C_1$	<small>Multiply each side by -1.</small>
$1 - y = e^{-kt - C_1}$	<small>Assume $y < 1$.</small>
$y = 1 - Ce^{-kt}$	<small>General solution</small>

To solve for the constants C and k , use the initial conditions. That is, because $y = 0$ when $t = 0$, you can determine that $C = 1$. Similarly, because $y = 0.5$ when $t = 1$, it follows that $0.5 = 1 - e^{-k}$, which implies that

$$k = \ln 2 \approx 0.693.$$

So, the particular solution is

$$y = 1 - e^{-0.693t}. \quad \text{Particular solution}$$

This model is shown in Figure 6.17. Using the model, you can determine that the number of people who have heard of the product after 2 years is

$$\begin{aligned} y &= 1 - e^{-0.693(2)} \\ &\approx 0.75 \text{ or } 750,000 \text{ people.} \end{aligned}$$

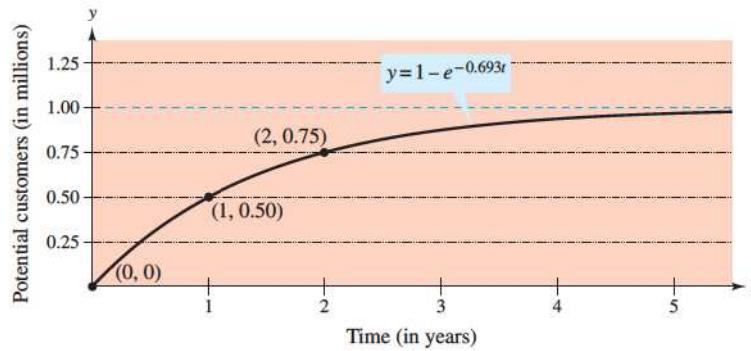


Figure 6.17

EXAMPLE 10 Modeling a Chemical Reaction

During a chemical reaction, substance A is converted into substance B at a rate that is proportional to the square of the amount of A. When $t = 0$, 60 grams of A is present, and after 1 hour ($t = 1$), only 10 grams of A remains unconverted. How much of A is present after 2 hours?

Solution Let y be the unconverted amount of substance A at any time t . From the given assumption about the conversion rate, you can write the differential equation as shown.

$$\frac{dy}{dt} = ky^2$$

Rate of change of y is proportional to the square of y .

You can solve this equation using separation of variables.

$$dy = ky^2 dt \quad \text{Differential form}$$

$$\frac{dy}{y^2} = k dt \quad \text{Separate variables.}$$

$$-\frac{1}{y} = kt + C \quad \text{Integrate.}$$

$$y = \frac{-1}{kt + C} \quad \text{General solution}$$

To solve for the constants C and k , use the initial conditions. That is, because $y = 60$ when $t = 0$, you can determine that $C = -\frac{1}{60}$. Similarly, because $y = 10$ when $t = 1$, it follows that

$$10 = \frac{-1}{k - (1/60)}$$

which implies that $k = -\frac{1}{12}$. So, the particular solution is

$$y = \frac{-1}{(-1/12)t - (1/60)} \quad \text{Substitute for } k \text{ and } C.$$

$$= \frac{60}{5t + 1}. \quad \text{Particular solution}$$

Using the model, you can determine that the unconverted amount of substance A after 2 hours is

$$y = \frac{60}{5(2) + 1}$$

$$\approx 5.45 \text{ grams.}$$

In Figure 6.18, note that the chemical conversion is occurring rapidly during the first hour. Then, as more and more of substance A is converted, the conversion rate slows down. ■

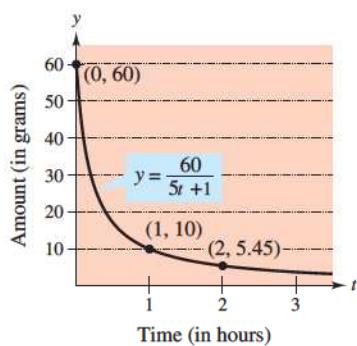


Figure 6.18

EXPLORATION

In Example 10, the rate of conversion was assumed to be proportional to the square of the unconverted amount. How would the result change if the rate of conversion were assumed to be proportional to the unconverted amount?

The next example describes a growth model called a **Gompertz growth model**. This model assumes that the rate of change of y is proportional to the product of y and the natural log of L/y , where L is the population limit.

EXAMPLE 11 Modeling Population Growth

A population of 20 wolves has been introduced into a national park. The forest service estimates that the maximum population the park can sustain is 200 wolves. After 3 years, the population is estimated to be 40 wolves. If the population follows a Gompertz growth model, how many wolves will there be 10 years after their introduction?

Solution Let y be the number of wolves at any time t . From the given assumption about the rate of growth of the population, you can write the differential equation as shown.

$$\frac{dy}{dt} = ky \ln \frac{200}{y}$$

Rate of change of y is proportional to the product of y and the natural log of the ratio of 200 and y .

Using separation of variables or a computer algebra system, you can find the general solution to be

$$y = 200e^{-Ce^{-kt}}. \quad \text{General solution}$$

To solve for the constants C and k , use the initial conditions. That is, because $y = 20$ when $t = 0$, you can determine that

$$C = \ln 10 \approx 2.3026.$$

Similarly, because $y = 40$ when $t = 3$, it follows that

$$40 = 200e^{-2.3026e^{-3k}}$$

which implies that $k \approx 0.1194$. So, the particular solution is

$$y = 200e^{-2.3026e^{-0.1194t}}. \quad \text{Particular solution}$$

Using the model, you can estimate the wolf population after 10 years to be

$$y = 200e^{-2.3026e^{-0.1194(10)}} \approx 100 \text{ wolves.}$$

In Figure 6.19, note that after 10 years the population has reached about half of the estimated maximum population. Try checking the growth model to see that it yields $y = 20$ when $t = 0$ and $y = 40$ when $t = 3$.

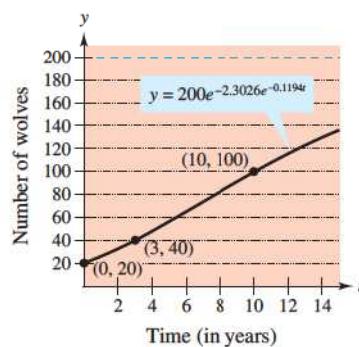


Figure 6.19

In genetics, a commonly used hybrid selection model is based on the differential equation

$$\frac{dy}{dt} = ky(1 - y)(a - by).$$

In this model, y represents the portion of the population that has a certain characteristic and t represents the time (measured in generations). The numbers a , b , and k are constants that depend on the genetic characteristic that is being studied.

EXAMPLE 12 Modeling Hybrid Selection

You are studying a population of beetles to determine how quickly characteristic D will pass from one generation to the next. At the beginning of your study ($t = 0$), you find that half the population has characteristic D. After four generations ($t = 4$), you find that 80% of the population has characteristic D. Use the hybrid selection model above with $a = 2$ and $b = 1$ to find the percent of the population that will have characteristic D after 10 generations.

Solution Using $a = 2$ and $b = 1$, the differential equation for the hybrid selection model is

$$\frac{dy}{dt} = ky(1 - y)(2 - y).$$

Using separation of variables or a computer algebra system, you can find the general solution to be

$$\frac{y(2 - y)}{(1 - y)^2} = Ce^{2kt}. \quad \text{General solution}$$

To solve for the constants C and k , use the initial conditions. That is, because $y = 0.5$ when $t = 0$, you can determine that $C = 3$. Similarly, because $y = 0.8$ when $t = 4$, it follows that

$$\frac{0.8(1.2)}{(0.2)^2} = 3e^{8k}$$

which implies that

$$k = \frac{1}{8} \ln 8 \approx 0.2599.$$

So, the particular solution is

$$\frac{y(2 - y)}{(1 - y)^2} = 3e^{0.5199t}. \quad \text{Particular solution}$$

Using the model, you can estimate the percent of the population that will have characteristic D after 10 generations to be given by

$$\frac{y(2 - y)}{(1 - y)^2} = 3e^{0.5199(10)}.$$

Using a computer algebra system, you can solve this equation for y to obtain $y \approx 0.96$ or 96% of the population. The graph of the model is shown in Figure 6.20. ■

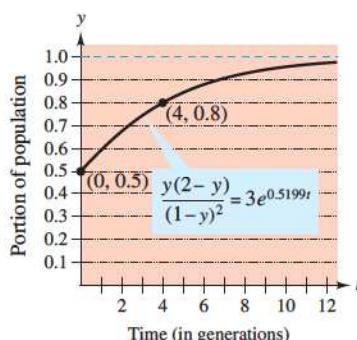


Figure 6.20

6.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–14, find the general solution of the differential equation.

1. $\frac{dy}{dx} = \frac{x}{y}$

2. $\frac{dy}{dx} = \frac{3x^2}{y^2}$

3. $x^2 + 5y \frac{dy}{dx} = 0$

4. $\frac{dy}{dx} = \frac{x^2 - 3}{6y^2}$

5. $\frac{dr}{ds} = 0.75r$

6. $\frac{dr}{ds} = 0.75s$

7. $(2 + x)y' = 3y$

8. $xy' = y$

9. $yy' = 4 \sin x$

10. $yy' = -8 \cos(\pi x)$

11. $\sqrt{1 - 4x^2}y' = x$

12. $\sqrt{x^2 - 16}y' = 11x$

13. $y \ln x - xy' = 0$

14. $12yy' - 7e^x = 0$

In Exercises 15–24, find the particular solution that satisfies the initial condition.

Differential Equation

15. $yy' - 2e^x = 0$

Initial Condition

$y(0) = 3$

16. $\sqrt{x} + \sqrt{y}y' = 0$

$y(1) = 9$

17. $y(x + 1) + y' = 0$

$y(-2) = 1$

18. $2xy' - \ln x^2 = 0$

$y(1) = 2$

19. $y(1 + x^2)y' - x(1 + y^2) = 0$

$y(0) = \sqrt{3}$

20. $y\sqrt{1 - x^2}y' - x\sqrt{1 - y^2} = 0$

$y(0) = 1$

21. $\frac{du}{dv} = uv \sin v^2$

$u(0) = 1$

22. $\frac{dr}{ds} = e^{r-2s}$

$r(0) = 0$

23. $dP - kP dt = 0$

$P(0) = P_0$

24. $dT + k(T - 70) dt = 0$

$T(0) = 140$

In Exercises 25–28, find an equation of the graph that passes through the point and has the given slope.

25. $(0, 2), \quad y' = \frac{x}{4y}$

26. $(1, 1), \quad y' = -\frac{9x}{16y}$

27. $(9, 1), \quad y' = \frac{y}{2x}$

28. $(8, 2), \quad y' = \frac{2y}{3x}$

In Exercises 29 and 30, find all functions f having the indicated property.

29. The tangent to the graph of f at the point (x, y) intersects the x -axis at $(x + 2, 0)$.

30. All tangents to the graph of f pass through the origin.

In Exercises 31–38, determine whether the function is homogeneous, and if it is, determine its degree.

31. $f(x, y) = x^3 - 4xy^2 + y^3$

32. $f(x, y) = x^3 + 3x^2y^2 - 2y^2$

33. $f(x, y) = \frac{x^2y^2}{\sqrt{x^2 + y^2}}$

34. $f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$

35. $f(x, y) = 2 \ln xy$

36. $f(x, y) = \tan(x + y)$

37. $f(x, y) = 2 \ln \frac{x}{y}$

38. $f(x, y) = \tan \frac{y}{x}$

In Exercises 39–44, solve the homogeneous differential equation.

39. $y' = \frac{x + y}{2x}$

40. $y' = \frac{x^3 + y^3}{xy^2}$

41. $y' = \frac{x - y}{x + y}$

42. $y' = \frac{x^2 + y^2}{2xy}$

43. $y' = \frac{xy}{x^2 - y^2}$

44. $y' = \frac{2x + 3y}{x}$

In Exercises 45–48, find the particular solution that satisfies the initial condition.

Differential Equation

45. $x dy - (2xe^{-y/x} + y) dx = 0$

$y(1) = 0$

46. $-y^2 dx + x(x + y) dy = 0$

$y(1) = 1$

47. $\left(x \sec \frac{y}{x} + y\right) dx - x dy = 0$

$y(1) = 0$

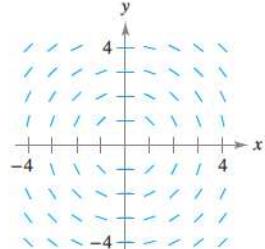
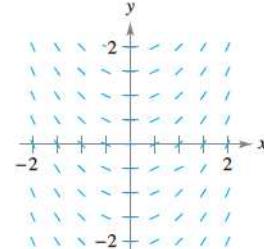
48. $(2x^2 + y^2) dx + xy dy = 0$

$y(1) = 0$

Slope Fields In Exercises 49–52, sketch a few solutions of the differential equation on the slope field and then find the general solution analytically. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

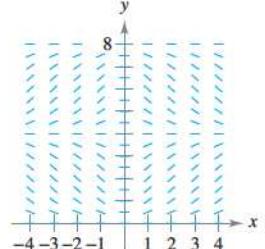
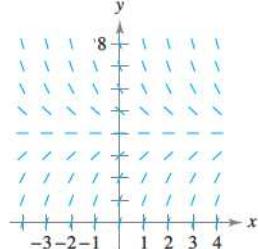
49. $\frac{dy}{dx} = x$

50. $\frac{dy}{dx} = -\frac{x}{y}$



51. $\frac{dy}{dx} = 4 - y$

52. $\frac{dy}{dx} = 0.25x(4 - y)$



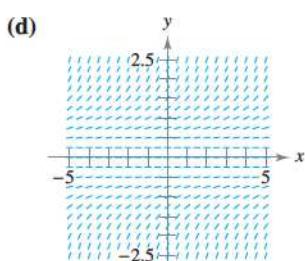
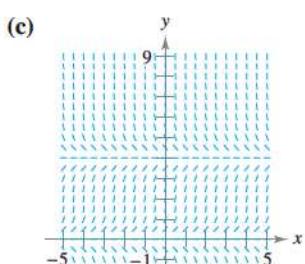
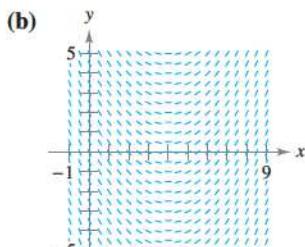
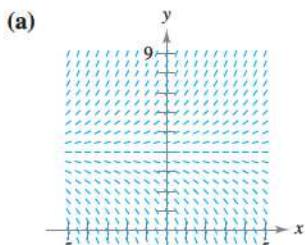
Euler's Method In Exercises 53–56, (a) use Euler's Method with a step size of $h = 0.1$ to approximate the particular solution of the initial value problem at the given x -value, (b) find the exact solution of the differential equation analytically, and (c) compare the solutions at the given x -value.

Differential Equation	Initial Condition	x -value
53. $\frac{dy}{dx} = -6xy$	(0, 5)	$x = 1$
54. $\frac{dy}{dx} + 6xy^2 = 0$	(0, 3)	$x = 1$
55. $\frac{dy}{dx} = \frac{2x + 12}{3y^2 - 4}$	(1, 2)	$x = 2$
56. $\frac{dy}{dx} = 2x(1 + y^2)$	(1, 0)	$x = 1.5$

57. **Radioactive Decay** The rate of decomposition of radioactive radium is proportional to the amount present at any time. The half-life of radioactive radium is 1599 years. What percent of a present amount will remain after 50 years?

58. **Chemical Reaction** In a chemical reaction, a certain compound changes into another compound at a rate proportional to the unchanged amount. If initially there is 40 grams of the original compound, and there is 35 grams after 1 hour, when will 75 percent of the compound be changed?

A **Slope Fields** In Exercises 59–62, (a) write a differential equation for the statement, (b) match the differential equation with a possible slope field, and (c) verify your result by using a graphing utility to graph a slope field for the differential equation. [The slope fields are labeled (a), (b), (c), and (d).] To print an enlarged copy of the graph, go to the website www.mathgraphs.com.



59. The rate of change of y with respect to x is proportional to the difference between y and 4.

60. The rate of change of y with respect to x is proportional to the difference between x and 4.

61. The rate of change of y with respect to x is proportional to the product of y and the difference between y and 4.

62. The rate of change of y with respect to x is proportional to y^2 .

63. **Weight Gain** A calf that weighs 60 pounds at birth gains weight at the rate

$$\frac{dw}{dt} = k(1200 - w)$$

where w is weight in pounds and t is time in years.

(a) Solve the differential equation.

A (b) Use a graphing utility to graph the solution when $k = 0.8$, 0.9, and 1.

(c) If the animal is sold when its weight reaches 800 pounds, find the time of sale for each of the models in part (b).

(d) What is the maximum weight of the animal for each of the models?

64. **Weight Gain** A calf that weighs w_0 pounds at birth gains weight at the rate

$$\frac{dw}{dt} = 1200 - w$$

where w is weight in pounds and t is time in years. Solve the differential equation.

A In Exercises 65–70, find the orthogonal trajectories of the family. Use a graphing utility to graph several members of each family.

65. $x^2 + y^2 = C$

66. $x^2 - 2y^2 = C$

67. $x^2 = Cy$

68. $y^2 = 2Cx$

69. $y^2 = Cx^3$

70. $y = Ce^x$

71. **Biology** At any time t , the rate of growth of the population N of deer in a state park is proportional to the product of N and $L - N$, where $L = 500$ is the maximum number of deer the park can sustain. When $t = 0$, $N = 100$, and when $t = 4$, $N = 200$. Write N as a function of t .

- 72. Sales Growth** The rate of change in sales S (in thousands of units) of a new product is proportional to the product of S and $L - S$. L (in thousands of units) is the estimated maximum level of sales, and $S = 10$ when $t = 0$. Write and solve the differential equation for this sales model.

 **Chemical Reaction** In Exercises 73 and 74, use the chemical reaction model given in Example 10 to find the amount y as a function of t , and use a graphing utility to graph the function.

73. $y = 45$ grams when $t = 0$; $y = 4$ grams when $t = 2$
 74. $y = 75$ grams when $t = 0$; $y = 12$ grams when $t = 1$

In Exercises 75 and 76, use the Gompertz growth model described in Example 11 to find the growth function, and sketch its graph.

75. $L = 500$; $y = 100$ when $t = 0$; $y = 150$ when $t = 2$
 76. $L = 5000$; $y = 500$ when $t = 0$; $y = 625$ when $t = 1$

77. **Biology** A population of eight beavers has been introduced into a new wetlands area. Biologists estimate that the maximum population the wetlands can sustain is 60 beavers. After 3 years, the population is 15 beavers. If the population follows a Gompertz growth model, how many beavers will be present in the wetlands after 10 years?

78. **Biology** A population of 30 rabbits has been introduced into a new region. It is estimated that the maximum population the region can sustain is 400 rabbits. After 1 year, the population is estimated to be 90 rabbits. If the population follows a Gompertz growth model, how many rabbits will be present after 3 years?

Biology In Exercises 79 and 80, use the hybrid selection model described in Example 12 to find the percent of the population that has the given characteristic.

79. You are studying a population of mayflies to determine how quickly characteristic A will pass from one generation to the next. At the start of the study, half the population has characteristic A. After four generations, 75% of the population has characteristic A. Find the percent of the population that will have characteristic A after 10 generations. (Assume $a = 2$ and $b = 1$.)

80. A research team is studying a population of snails to determine how quickly characteristic B will pass from one generation to the next. At the start of the study, 40% of the snails have characteristic B. After five generations, 80% of the population has characteristic B. Find the percent of the population that will have characteristic B after eight generations. (Assume $a = 2$ and $b = 1$.)

81. **Chemical Mixture** A 100-gallon tank is full of a solution containing 25 pounds of a concentrate. Starting at time $t = 0$, distilled water is admitted to the tank at the rate of 5 gallons per minute, and the well-stirred solution is withdrawn at the same rate, as shown in the figure.



- (a) Find the amount Q of the concentrate in the solution as a function of t . (Hint: $Q' + Q/20 = 0$)

- (b) Find the time when the amount of concentrate in the tank reaches 15 pounds.

82. **Chemical Mixture** A 200-gallon tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 pound of concentrate per gallon enters the tank at the rate of 5 gallons per minute, and the well-stirred mixture is withdrawn at the same rate. Find the amount Q of concentrate in the tank after 30 minutes. (Hint: $Q' + Q/20 = \frac{5}{2}$)

83. **Chemical Reaction** In a chemical reaction, a compound changes into another compound at a rate proportional to the unchanged amount, according to the model

$$\frac{dy}{dt} = ky.$$

- (a) Solve the differential equation.

- (b) If the initial amount of the original compound is 20 grams, and the amount remaining after 1 hour is 16 grams, when will 75% of the compound have been changed?

84. **Safety** Assume that the rate of change in the number of miles s of road cleared per hour by a snowplow is inversely proportional to the depth h of snow. That is,

$$\frac{ds}{dh} = \frac{k}{h}.$$

Find s as a function of h if $s = 25$ miles when $h = 2$ inches and $s = 12$ miles when $h = 6$ inches ($2 \leq h \leq 15$).

85. **Chemistry** A wet towel hung from a clothesline to dry loses moisture through evaporation at a rate proportional to its moisture content. If after 1 hour the towel has lost 40% of its original moisture content, after how long will it have lost 80%?

86. **Biology** Let x and y be the sizes of two internal organs of a particular mammal at time t . Empirical data indicate that the relative growth rates of these two organs are equal, and can be modeled by

$$\frac{1}{x} \frac{dx}{dt} = \frac{1}{y} \frac{dy}{dt}.$$

Use this differential equation to write y as a function of x .

- 87. Population Growth** When predicting population growth, demographers must consider birth and death rates as well as the net change caused by the difference between the rates of immigration and emigration. Let P be the population at time t and let N be the net increase per unit time due to the difference between immigration and emigration. So, the rate of growth of the population is given by

$$\frac{dP}{dt} = kP + N, \quad N \text{ is constant.}$$

Solve this differential equation to find P as a function of time.

- 88. Meteorology** The barometric pressure y (in inches of mercury) at an altitude of x miles above sea level decreases at a rate proportional to the current pressure according to the model

$$\frac{dy}{dx} = -0.2y$$

where $y = 29.92$ inches when $x = 0$. Find the barometric pressure (a) at the top of Mt. St. Helens (8364 feet) and (b) at the top of Mt. McKinley (20,320 feet).

- 89. Investment** A large corporation starts at time $t = 0$ to invest part of its receipts at a rate of P dollars per year in a fund for future corporate expansion. Assume that the fund earns r percent interest per year compounded continuously. So, the rate of growth of the amount A in the fund is given by

$$\frac{dA}{dt} = rA + P$$

where $A = 0$ when $t = 0$. Solve this differential equation for A as a function of t .

Investment In Exercises 90–92, use the result of Exercise 89.

90. Find A for each situation.

- (a) $P = \$275,000$, $r = 8\%$, and $t = 10$ years
 (b) $P = \$550,000$, $r = 5.9\%$, and $t = 25$ years

91. Find P if the corporation needs $\$260,000,000$ in 8 years and the fund earns $7\frac{1}{4}\%$ interest compounded continuously.

92. Find t if the corporation needs $\$1,000,000$ and it can invest $\$125,000$ per year in a fund earning 8% interest compounded continuously.

In Exercises 93 and 94, use the Gompertz growth model described in Example 11.



93. (a) Use a graphing utility to graph the slope field for the growth model when $k = 0.02$ and $L = 5000$.

- (b) Describe the behavior of the graph as $t \rightarrow \infty$.
 (c) Solve the growth model for $L = 5000$, $y_0 = 500$, and $k = 0.02$.
 (d) Graph the equation you found in part (c). Determine the concavity of the graph.



94. (a) Use a graphing utility to graph the slope field for the growth model when $k = 0.05$ and $L = 1000$.

- (b) Describe the behavior of the graph as $t \rightarrow \infty$.

- (c) Solve the growth model for $L = 1000$, $y_0 = 100$, and $k = 0.05$.
 (d) Graph the equation you found in part (c). Determine the concavity of the graph.

WRITING ABOUT CONCEPTS

95. In your own words, describe how to recognize and solve differential equations that can be solved by separation of variables.
 96. State the test for determining if a differential equation is homogeneous. Give an example.
 97. In your own words, describe the relationship between two families of curves that are mutually orthogonal.

CAPSTONE

98. Determine whether the differential equation is separable. If the equation is separable, rewrite it in the form $N(y)dy = M(x)dx$. (Do not solve the differential equation.)
 (a) $y(1+x)dx + x dy = 0$
 (b) $y' = y^{1/2}$
 (c) $y' + xy = 5$
 (d) $y' = x - xy - y + 1$

True or False? In Exercises 99–102, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

99. The function $y = 0$ is always a solution of a differential equation that can be solved by separation of variables.

100. The differential equation

$$y' = xy - 2y + x - 2$$

can be written in separated variables form.

101. The function

$$f(x, y) = x^2 - 4xy + 6y^2 + 1$$

is homogeneous.

102. The families

$$x^2 + y^2 = 2Cy \text{ and } x^2 + y^2 = 2Kx$$

are mutually orthogonal.

PUTNAM EXAM CHALLENGE

103. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b) .

This problem was composed by the Committee on the Putnam Prize Competition.
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6.4 The Logistic Equation

- Solve and analyze logistic differential equations.
- Use logistic differential equations to model and solve applied problems.

Logistic Differential Equation

In Section 6.2, the exponential growth model was derived from the fact that the rate of change of a variable y is proportional to the value of y . You observed that the differential equation $dy/dt = ky$ has the general solution $y = Ce^{kt}$. Exponential growth is unlimited, but when describing a population, there often exists some upper limit L past which growth cannot occur. This upper limit L is called the **carrying capacity**, which is the maximum population $y(t)$ that can be sustained or supported as time t increases. A model that is often used to describe this type of growth is the **logistic differential equation**

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \quad \text{Logistic differential equation}$$

where k and L are positive constants. A population that satisfies this equation does not grow without bound, but approaches the carrying capacity L as t increases.

From the equation, you can see that if y is between 0 and the carrying capacity L , then $dy/dt > 0$, and the population increases. If y is greater than L , then $dy/dt < 0$, and the population decreases. The general solution of the logistic differential equation is derived in the next example.

EXAMPLE 1 Deriving the General Solution

Solve the logistic differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$.

Solution Begin by separating variables.

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \quad \text{Write differential equation.}$$

$$\frac{1}{y(1 - y/L)} dy = k dt \quad \text{Separate variables.}$$

$$\int \frac{1}{y(1 - y/L)} dy = \int k dt \quad \text{Integrate each side.}$$

$$\int \left(\frac{1}{y} + \frac{1}{L-y}\right) dy = \int k dt \quad \text{Rewrite left side using partial fractions.}$$

$$\ln|y| - \ln|L-y| = kt + C \quad \text{Find antiderivative of each side.}$$

$$\ln\left|\frac{L-y}{y}\right| = -kt - C \quad \text{Multiply each side by } -1 \text{ and simplify.}$$

$$\left|\frac{L-y}{y}\right| = e^{-kt-C} = e^{-C}e^{-kt} \quad \text{Exponentiate each side.}$$

$$\frac{L-y}{y} = be^{-kt} \quad \text{Let } \pm e^{-C} = b.$$

Solving this equation for y produces $y = \frac{L}{1+be^{-kt}}$. ■

EXPLORATION

Use a graphing utility to investigate the effects of the values of L , b , and k on the graph of

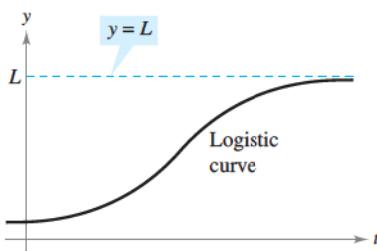
$$y = \frac{L}{1 + be^{-kt}}.$$

Include some examples to support your results.

From Example 1, you can conclude that all solutions of the logistic differential equation are of the general form

$$y = \frac{L}{1 + be^{-kt}}.$$

The graph of the function y is called the *logistic curve*, as shown in Figure 6.21. In the next example, you will verify a particular solution of a logistic differential equation and find the initial condition.



Note that as $t \rightarrow \infty$, $y \rightarrow L$.

Figure 6.21

EXAMPLE 2 Verifying a Particular Solution

Verify that the equation

$$y = \frac{4}{1 + 2e^{-3t}}$$

satisfies the logistic differential equation, and find the initial condition.

Solution Comparing the given equation with the general form derived in Example 1, you know that $L = 4$, $b = 2$, and $k = 3$. You can verify that y satisfies the logistic differential equation as follows.

$$\begin{aligned} y &= 4(1 + 2e^{-3t})^{-1} && \text{Rewrite using negative exponent.} \\ y' &= 4(-1)(1 + 2e^{-3t})^{-2}(-6e^{-3t}) && \text{Apply Power Rule.} \\ &= 3\left(\frac{4}{1 + 2e^{-3t}}\right)\left(\frac{2e^{-3t}}{1 + 2e^{-3t}}\right) && \text{Rewrite.} \\ &= 3y\left(\frac{2e^{-3t}}{1 + 2e^{-3t}}\right) && \text{Rewrite using } y = \frac{4}{1 + 2e^{-3t}}. \\ &= 3y\left(1 - \frac{1}{1 + 2e^{-3t}}\right) && \text{Rewrite fraction using long division.} \\ &= 3y\left(1 - \frac{4}{4(1 + 2e^{-3t})}\right) && \text{Multiply fraction by } \frac{4}{4}. \\ &= 3y\left(1 - \frac{y}{4}\right) && \text{Rewrite using } y = \frac{4}{1 + 2e^{-3t}}. \end{aligned}$$

So, y satisfies the logistic differential equation $y' = 3y\left(1 - \frac{y}{4}\right)$. The initial condition can be found by letting $t = 0$ in the given equation.

$$y = \frac{4}{1 + 2e^{-3(0)}} = \frac{4}{3}$$

Let $t = 0$ and simplify.

$$\text{So, the initial condition is } y(0) = \frac{4}{3}.$$

EXAMPLE 3 Verifying the Upper Limit

Verify that the upper limit of $y = \frac{4}{1 + 2e^{-3t}}$ is 4.

Solution In Figure 6.22, you can see that the values of y appear to approach 4 as t increases without bound. You can come to this conclusion numerically, as shown in the table.

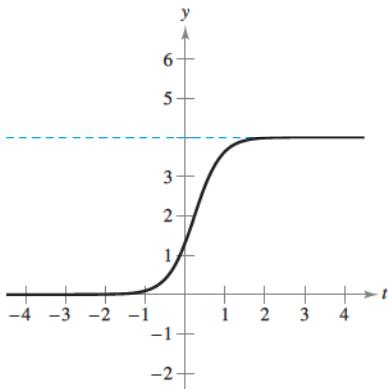


Figure 6.22

<i>t</i>	0	1	2	5	10	100
<i>y</i>	1.3333	3.6378	3.9803	4.0000	4.0000	4.0000

Finally, you can obtain the same results analytically, as follows.

$$\lim_{t \rightarrow \infty} y = \lim_{t \rightarrow \infty} \frac{4}{1 + 2e^{-3t}} = \frac{\lim_{t \rightarrow \infty} 4}{\lim_{t \rightarrow \infty} (1 + 2e^{-3t})} = \frac{4}{1 + 0} = 4$$

The upper limit of y is 4, which is also the carrying capacity $L = 4$.

EXAMPLE 4 Determining the Point of Inflection

Sketch a graph of $y = \frac{4}{1 + 2e^{-3t}}$. Calculate y'' in terms of y and y' . Then determine the point of inflection.

Solution From Example 2, you know that

$$y' = 3y\left(1 - \frac{y}{4}\right).$$

Now calculate y'' in terms of y and y' .

$$y'' = 3y\left(-\frac{y'}{4}\right) + \left(1 - \frac{y}{4}\right)3y' \quad \text{Differentiate using Product Rule.}$$

$$y'' = 3y'\left(1 - \frac{y}{2}\right) \quad \text{Factor and simplify.}$$

When $2 < y < 4$, $y'' < 0$ and the graph of y is concave downward. When $0 < y < 2$, $y'' > 0$ and the graph of y is concave upward. So, a point of inflection must occur at $y = 2$. The corresponding t -value is

$$2 = \frac{4}{1 + 2e^{-3t}} \Rightarrow 1 + 2e^{-3t} = 2 \Rightarrow e^{-3t} = \frac{1}{2} \Rightarrow t = \frac{1}{3} \ln 2.$$

The point of inflection is $(\frac{1}{3} \ln 2, 2)$, as shown in Figure 6.23. ■

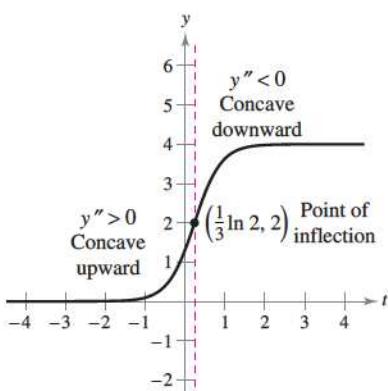


Figure 6.23

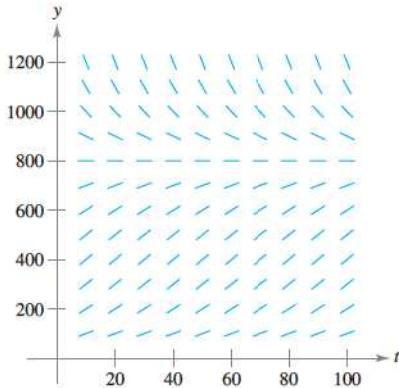
NOTE In Example 4, the point of inflection occurs at $y = \frac{L}{2}$. This is true for any logistic growth curve for which the solution starts below the carrying capacity L (see Exercise 37). ■

EXAMPLE 5 Graphing a Slope Field and Solution Curves

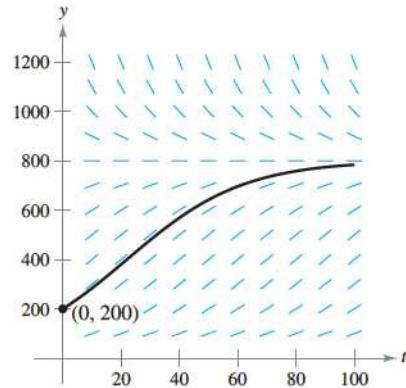
Graph a slope field for the logistic differential equation $y' = 0.05y\left(1 - \frac{y}{800}\right)$.

Then graph solution curves for the initial conditions $y(0) = 200$, $y(0) = 1200$, and $y(0) = 800$.

Solution You can use a graphing utility to graph the slope field shown in Figure 6.24. The solution curves for the initial conditions $y(0) = 200$, $y(0) = 1200$, and $y(0) = 800$ are shown in Figures 6.25–6.27.



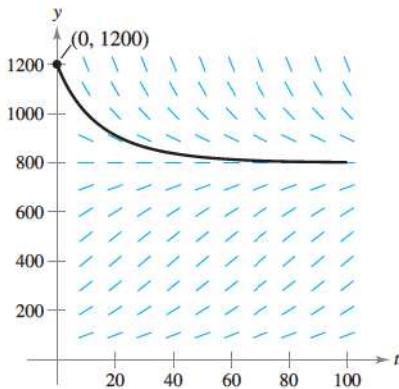
Slope field for
 $y' = 0.05y\left(1 - \frac{y}{800}\right)$



Particular solution for
 $y' = 0.05y\left(1 - \frac{y}{800}\right)$
and initial condition $y(0) = 200$

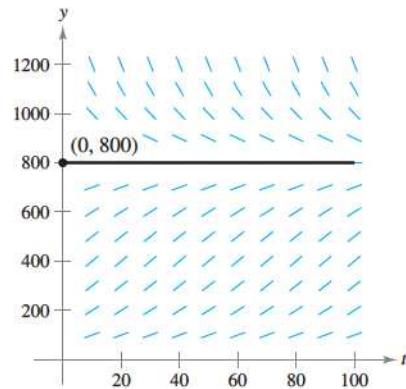
Figure 6.24

Figure 6.25



Particular solution for
 $y' = 0.05y\left(1 - \frac{y}{800}\right)$
and initial condition $y(0) = 1200$

Figure 6.26



Particular solution for
 $y' = 0.05y\left(1 - \frac{y}{800}\right)$
and initial condition $y(0) = 800$

Figure 6.27

Note that as t increases without bound, the solution curves in Figures 6.25–6.27 all tend to the same limit, which is the carrying capacity of 800. ■

EXAMPLE 6 Solving a Logistic Differential Equation

A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population p is

$$\frac{dp}{dt} = kp \left(1 - \frac{p}{4000}\right), \quad 40 \leq p \leq 4000$$

where t is the number of years.

- Write a model for the elk population in terms of t .
- Graph the slope field for the differential equation and the solution that passes through the point $(0, 40)$.
- Use the model to estimate the elk population after 15 years.
- Find the limit of the model as $t \rightarrow \infty$.

Solution

- You know that $L = 4000$. So, the solution of the equation is of the form

$$p = \frac{4000}{1 + be^{-kt}}.$$

Because $p(0) = 40$, you can solve for b as follows.

$$\begin{aligned} 40 &= \frac{4000}{1 + be^{-k(0)}} \\ 40 &= \frac{4000}{1 + b} \quad \Rightarrow \quad b = 99 \end{aligned}$$

Then, because $p = 104$ when $t = 5$, you can solve for k .

$$104 = \frac{4000}{1 + 99e^{-k(5)}} \quad \Rightarrow \quad k \approx 0.194$$

So, a model for the elk population is given by $p = \frac{4000}{1 + 99e^{-0.194t}}$.

- Using a graphing utility, you can graph the slope field of

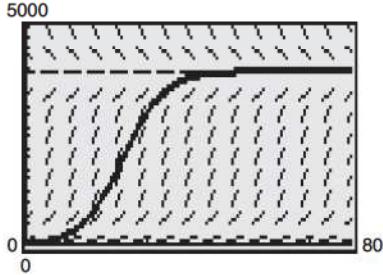
$$\frac{dp}{dt} = 0.194p \left(1 - \frac{p}{4000}\right)$$

and the solution that passes through $(0, 40)$, as shown in Figure 6.28.

- To estimate the elk population after 15 years, substitute 15 for t in the model.

$$\begin{aligned} p &= \frac{4000}{1 + 99e^{-0.194(15)}} && \text{Substitute 15 for } t. \\ &= \frac{4000}{1 + 99e^{-2.91}} \approx 626 && \text{Simplify.} \end{aligned}$$

- As t increases without bound, the denominator of $\frac{4000}{1 + 99e^{-0.194t}}$ gets closer and closer to 1. So, $\lim_{t \rightarrow \infty} \frac{4000}{1 + 99e^{-0.194t}} = 4000$. ■



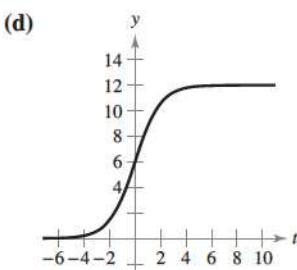
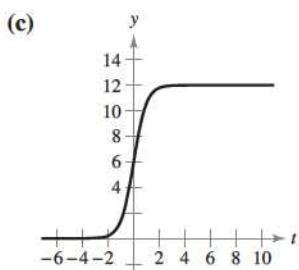
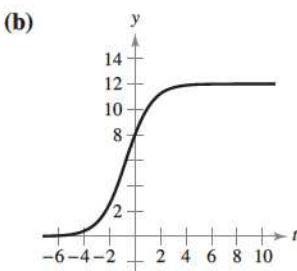
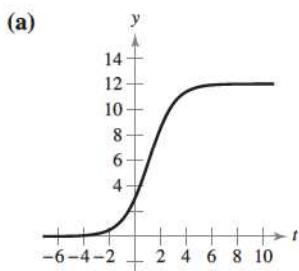
Slope field for
 $\frac{dp}{dt} = 0.194p \left(1 - \frac{p}{4000}\right)$
and the solution passing through $(0, 40)$

Figure 6.28

6.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, match the logistic equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]



1. $y = \frac{12}{1 + e^{-t}}$

2. $y = \frac{12}{1 + 3e^{-t}}$

3. $y = \frac{12}{1 + \frac{1}{2}e^{-t}}$

4. $y = \frac{12}{1 + e^{-2t}}$

In Exercises 5–8, verify that the equation satisfies the logistic differential equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right).$$

Then find the initial condition.

5. $y = \frac{8}{1 + e^{-2t}}$

6. $y = \frac{10}{1 + 3e^{-4t}}$

7. $y = \frac{12}{1 + 6e^{-t}}$

8. $y = \frac{14}{1 + 5e^{-3t}}$

In Exercises 9–12, the logistic equation models the growth of a population. Use the equation to (a) find the value of k , (b) find the carrying capacity, (c) find the initial population, (d) determine when the population will reach 50% of its carrying capacity, and (e) write a logistic differential equation that has the solution $P(t)$.

9. $P(t) = \frac{2100}{1 + 29e^{-0.75t}}$

10. $P(t) = \frac{5000}{1 + 39e^{-0.2t}}$

11. $P(t) = \frac{6000}{1 + 4999e^{-0.8t}}$

12. $P(t) = \frac{1000}{1 + 8e^{-0.2t}}$

In Exercises 13–16, the logistic differential equation models the growth rate of a population. Use the equation to (a) find the value of k , (b) find the carrying capacity, (c) use a computer algebra system to graph a slope field, and (d) determine the value of P at which the population growth rate is the greatest.

13. $\frac{dP}{dt} = 3P\left(1 - \frac{P}{100}\right)$

14. $\frac{dP}{dt} = 0.5P\left(1 - \frac{P}{250}\right)$

15. $\frac{dP}{dt} = 0.1P - 0.0004P^2$

16. $\frac{dP}{dt} = 0.4P - 0.00025P^2$

In Exercises 17–20, find the logistic equation that satisfies the initial condition. Then use the logistic equation to find y when $t = 5$ and $t = 100$.

Logistic Differential Equation

Initial Condition

17. $\frac{dy}{dt} = y\left(1 - \frac{y}{36}\right)$

(0, 4)

18. $\frac{dy}{dt} = 2.8y\left(1 - \frac{y}{10}\right)$

(0, 7)

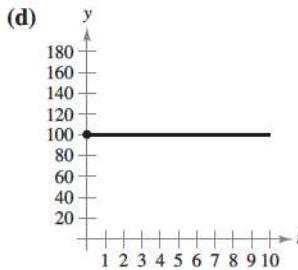
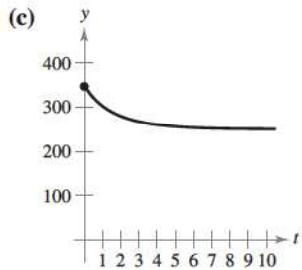
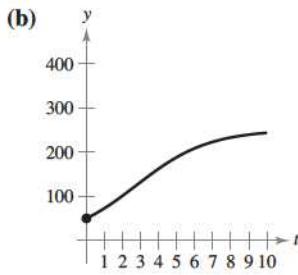
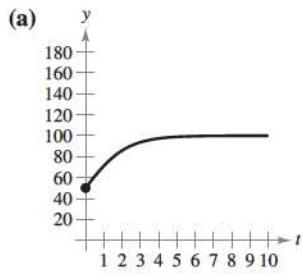
19. $\frac{dy}{dt} = \frac{4y}{5} - \frac{y^2}{150}$

(0, 8)

20. $\frac{dy}{dt} = \frac{3y}{20} - \frac{y^2}{1600}$

(0, 15)

In Exercises 21–24, match the logistic differential equation and initial condition with the graph of its solution. [The graphs are labeled (a), (b), (c), and (d).]



21. $\frac{dy}{dt} = 0.5y\left(1 - \frac{y}{250}\right), (0, 350)$

22. $\frac{dy}{dt} = 0.9y\left(1 - \frac{y}{100}\right), (0, 100)$

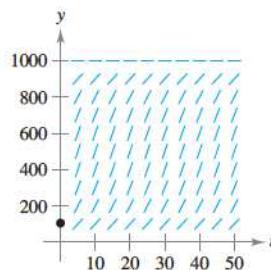
23. $\frac{dy}{dt} = 0.5y\left(1 - \frac{y}{250}\right)$, $(0, 50)$

24. $\frac{dy}{dt} = 0.9y\left(1 - \frac{y}{100}\right)$, $(0, 50)$

 **Slope Fields** In Exercises 25–28, a logistic differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketch in part (a). To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

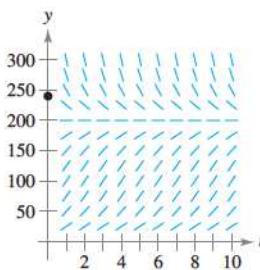
25. $\frac{dy}{dt} = 0.2y\left(1 - \frac{y}{1000}\right)$,

$(0, 105)$



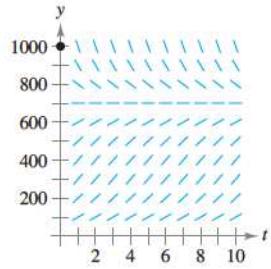
26. $\frac{dy}{dt} = 0.9y\left(1 - \frac{y}{200}\right)$,

$(0, 240)$



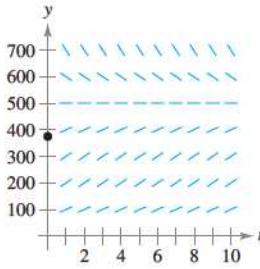
27. $\frac{dy}{dt} = 0.6y\left(1 - \frac{y}{700}\right)$,

$(0, 1000)$



28. $\frac{dy}{dt} = 0.4y\left(1 - \frac{y}{500}\right)$,

$(0, 375)$



WRITING ABOUT CONCEPTS

29. Describe what the value of L represents in the logistic differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$.

30. It is known that $y = \frac{L}{1 + be^{-kt}}$ is a solution of the logistic differential equation $\frac{dy}{dt} = 0.75y\left(1 - \frac{y}{2500}\right)$. Is it possible to determine L , k , and b from the information given? If so, find their values. If not, which value(s) cannot be determined and what information do you need to determine the value(s)?

31. Is the logistic differential equation separable? Explain.

CAPSTONE

32. Suppose the growth of a population is modeled by a logistic equation. As the population increases, its rate of growth decreases. What do you think causes this to occur in real-life situations such as animal or human populations?

33. **Endangered Species** A conservation organization releases 25 Florida panthers into a game preserve. After 2 years, there are 39 panthers in the preserve. The Florida preserve has a carrying capacity of 200 panthers.

- (a) Write a logistic equation that models the population of panthers in the preserve.
 (b) Find the population after 5 years.
 (c) When will the population reach 100?
 (d) Write a logistic differential equation that models the growth rate of the panther population. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answer.
 (e) After how many years is the panther population growing most rapidly? Explain.

34. **Bacteria Growth** At time $t = 0$, a bacterial culture weighs 1 gram. Two hours later, the culture weighs 4 grams. The maximum weight of the culture is 20 grams.

- (a) Write a logistic equation that models the weight of the bacterial culture.
 (b) Find the culture's weight after 5 hours.
 (c) When will the culture's weight reach 18 grams?
 (d) Write a logistic differential equation that models the growth rate of the culture's weight. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answer.
 (e) After how many hours is the culture's weight increasing most rapidly? Explain.

True or False? In Exercises 35 and 36, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

35. For the logistic differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$, if $y > L$, then $dy/dt > 0$ and the population increases.

36. For the logistic differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$, if $0 < y < L$, then $dy/dt > 0$ and the population increases.

37. For any logistic growth curve, show that the point of inflection occurs at $y = \frac{L}{2}$ when the solution starts below the carrying capacity L .

38. Show that if $y = \frac{1}{1 + be^{-kt}}$, then $\frac{dy}{dt} = ky(1 - y)$.

6.5**First-Order Linear Differential Equations**

- Solve a first-order linear differential equation.
- Solve a Bernoulli differential equation.
- Use linear differential equations to solve applied problems.

First-Order Linear Differential Equations

In this section, you will see how to solve a very important class of first-order differential equations—first-order linear differential equations.

DEFINITION OF FIRST-ORDER LINEAR DIFFERENTIAL EQUATION

A **first-order linear differential equation** is an equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions of x . This first-order linear differential equation is said to be in **standard form**.

NOTE It is instructive to see why the integrating factor helps solve a linear differential equation of the form $y' + P(x)y = Q(x)$. When both sides of the equation are multiplied by the integrating factor $u(x) = e^{\int P(x) dx}$, the left-hand side becomes the derivative of a product.

$$y'e^{\int P(x) dx} + P(x)y e^{\int P(x) dx} = Q(x)e^{\int P(x) dx}$$

$$[ye^{\int P(x) dx}]' = Q(x)e^{\int P(x) dx}$$

Integrating both sides of this second equation and dividing by $u(x)$ produces the general solution.

To solve a linear differential equation, write it in standard form to identify the functions $P(x)$ and $Q(x)$. Then integrate $P(x)$ and form the expression

$$u(x) = e^{\int P(x) dx} \quad \text{Integrating factor}$$

which is called an **integrating factor**. The general solution of the equation is

$$y = \frac{1}{u(x)} \int Q(x)u(x) dx. \quad \text{General solution}$$

EXAMPLE 1 Solving a Linear Differential Equation

Find the general solution of

$$y' + y = e^x.$$

Solution For this equation, $P(x) = 1$ and $Q(x) = e^x$. So, the integrating factor is

$$\begin{aligned} u(x) &= e^{\int P(x) dx} && \text{Integrating factor} \\ &= e^{\int dx} \\ &= e^x. \end{aligned}$$

This implies that the general solution is

$$\begin{aligned} y &= \frac{1}{u(x)} \int Q(x)u(x) dx \\ &= \frac{1}{e^x} \int e^x(e^x) dx \\ &= e^{-x} \left(\frac{1}{2}e^{2x} + C \right) \\ &= \frac{1}{2}e^x + Ce^{-x}. \end{aligned} \quad \text{General solution}$$

THEOREM 6.3 SOLUTION OF A FIRST-ORDER LINEAR DIFFERENTIAL EQUATION

An integrating factor for the first-order linear differential equation

$$y' + P(x)y = Q(x)$$

is $u(x) = e^{\int P(x) dx}$. The solution of the differential equation is

$$ye^{\int P(x) dx} = \int Q(x)e^{\int P(x) dx} dx + C.$$

**ANNA JOHNSON PELL WHEELER
(1883–1966)**

Anna Johnson Pell Wheeler was awarded a master's degree from the University of Iowa for her thesis *The Extension of Galois Theory to Linear Differential Equations* in 1904. Influenced by David Hilbert, she worked on integral equations while studying infinite linear spaces.

STUDY TIP Rather than memorizing the formula in Theorem 6.3, just remember that multiplication by the integrating factor $e^{\int P(x) dx}$ converts the left side of the differential equation into the derivative of the product $ye^{\int P(x) dx}$.

EXAMPLE 2 Solving a First-Order Linear Differential Equation

Find the general solution of

$$xy' - 2y = x^2.$$

Solution The standard form of the given equation is

$$y' + P(x)y = Q(x)$$

$$y' - \left(\frac{2}{x}\right)y = x. \quad \text{Standard form}$$

So, $P(x) = -2/x$, and you have

$$\begin{aligned} \int P(x) dx &= -\int \frac{2}{x} dx \\ &= -\ln x^2 \\ e^{\int P(x) dx} &= e^{-\ln x^2} \\ &= \frac{1}{e^{\ln x^2}} \\ &= \frac{1}{x^2}. \quad \text{Integrating factor} \end{aligned}$$

So, multiplying each side of the standard form by $1/x^2$ yields

$$\begin{aligned} \frac{y'}{x^2} - \frac{2y}{x^3} &= \frac{1}{x} \\ \frac{d}{dx} \left[\frac{y}{x^2} \right] &= \frac{1}{x} \\ \frac{y}{x^2} &= \int \frac{1}{x} dx \\ \frac{y}{x^2} &= \ln |x| + C \\ y &= x^2(\ln |x| + C). \quad \text{General solution} \end{aligned}$$

Several solution curves (for $C = -2, -1, 0, 1, 2, 3$, and 4) are shown in Figure 6.29.

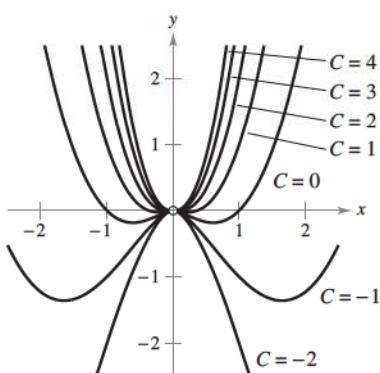


Figure 6.29

EXAMPLE 3 Solving a First-Order Linear Differential Equation

Find the general solution of

$$y' - y \tan t = 1, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

Solution The equation is already in the standard form $y' + P(t)y = Q(t)$. So, $P(t) = -\tan t$, and

$$\int P(t) dt = -\int \tan t dt = \ln |\cos t|.$$

Because $-\pi/2 < t < \pi/2$, you can drop the absolute value signs and conclude that the integrating factor is

$$\begin{aligned} e^{\int P(t) dt} &= e^{\ln(\cos t)} \\ &= \cos t. \end{aligned} \quad \text{Integrating factor}$$

So, multiplying $y' - y \tan t = 1$ by $\cos t$ produces

$$\begin{aligned} \frac{d}{dt}[y \cos t] &= \cos t \\ y \cos t &= \int \cos t dt \\ y \cos t &= \sin t + C \\ y &= \tan t + C \sec t. \end{aligned} \quad \text{General solution}$$

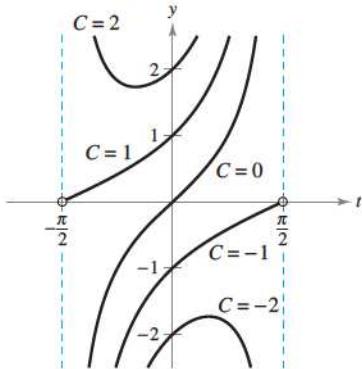


Figure 6.30

Several solution curves are shown in Figure 6.30. ■

Bernoulli Equation

A well-known nonlinear equation that reduces to a linear one with an appropriate substitution is the **Bernoulli equation**, named after James Bernoulli (1654–1705).

$$y' + P(x)y = Q(x)y^n \quad \text{Bernoulli equation}$$

This equation is linear if $n = 0$, and has separable variables if $n = 1$. So, in the following development, assume that $n \neq 0$ and $n \neq 1$. Begin by multiplying by y^{-n} and $(1 - n)$ to obtain

$$\begin{aligned} y^{-n}y' + P(x)y^{1-n} &= Q(x) \\ (1 - n)y^{-n}y' + (1 - n)P(x)y^{1-n} &= (1 - n)Q(x) \\ \frac{d}{dx}[y^{1-n}] + (1 - n)P(x)y^{1-n} &= (1 - n)Q(x) \end{aligned}$$

which is a linear equation in the variable y^{1-n} . Letting $z = y^{1-n}$ produces the linear equation

$$\frac{dz}{dx} + (1 - n)P(x)z = (1 - n)Q(x).$$

Finally, by Theorem 6.3, the *general solution of the Bernoulli equation* is

$$y^{1-n}e^{\int(1-n)P(x)dx} = \int (1 - n)Q(x)e^{\int(1-n)P(x)dx} dx + C.$$

EXAMPLE 4 Solving a Bernoulli Equation

Find the general solution of

$$y' + xy = xe^{-x^2}y^{-3}.$$

Solution For this Bernoulli equation, let $n = -3$, and use the substitution

$$z = y^4$$

Let $z = y^{1-n} = y^{1-(-3)}$.

$$z' = 4y^3y'.$$

Differentiate.

Multiplying the original equation by $4y^3$ produces

$$y' + xy = xe^{-x^2}y^{-3}$$

Write original equation.

$$4y^3y' + 4xy^4 = 4xe^{-x^2}$$

Multiply each side by $4y^3$.

$$z' + 4xz = 4xe^{-x^2}.$$

Linear equation: $z' + P(x)z = Q(x)$

This equation is linear in z . Using $P(x) = 4x$ produces

$$\int P(x) dx = \int 4x dx$$

$$= 2x^2$$

which implies that e^{2x^2} is an integrating factor. Multiplying the linear equation by this factor produces

$$z' + 4xz = 4xe^{-x^2}$$

Linear equation

$$z'e^{2x^2} + 4xze^{2x^2} = 4xe^{-x^2}$$

Multiply by integrating factor.

$$\frac{d}{dx}[ze^{2x^2}] = 4xe^{-x^2}$$

Write left side as derivative.

$$ze^{2x^2} = \int 4xe^{-x^2} dx$$

Integrate each side.

$$ze^{2x^2} = 2e^{-x^2} + C$$

Divide each side by e^{2x^2} .

$$z = 2e^{-x^2} + Ce^{-2x^2}.$$

Finally, substituting $z = y^4$, the general solution is

$$y^4 = 2e^{-x^2} + Ce^{-2x^2}.$$

General solution ■

So far you have studied several types of first-order differential equations. Of these, the separable variables case is usually the simplest, and solution by an integrating factor is ordinarily used only as a last resort.

SUMMARY OF FIRST-ORDER DIFFERENTIAL EQUATIONS
Method

1. Separable variables:

Form of Equation

$$M(x)dx + N(y)dy = 0$$

2. Homogeneous:

$M(x, y)dx + N(x, y)dy = 0$, where M and N are n th-degree homogeneous functions

3. Linear:

$$y' + P(x)y = Q(x)$$

4. Bernoulli equation:

$$y' + P(x)y = Q(x)y^n$$

Applications

One type of problem that can be described in terms of a differential equation involves chemical mixtures, as illustrated in the next example.

EXAMPLE 5 A Mixture Problem



Figure 6.31

A tank contains 50 gallons of a solution composed of 90% water and 10% alcohol. A second solution containing 50% water and 50% alcohol is added to the tank at the rate of 4 gallons per minute. As the second solution is being added, the tank is being drained at a rate of 5 gallons per minute, as shown in Figure 6.31. Assuming the solution in the tank is stirred constantly, how much alcohol is in the tank after 10 minutes?

Solution Let y be the number of gallons of alcohol in the tank at any time t . You know that $y = 5$ when $t = 0$. Because the number of gallons of solution in the tank at any time is $50 - t$, and the tank loses 5 gallons of solution per minute, it must lose

$$\left(\frac{5}{50-t}\right)y$$

gallons of alcohol per minute. Furthermore, because the tank is gaining 2 gallons of alcohol per minute, the rate of change of alcohol in the tank is given by

$$\frac{dy}{dt} = 2 - \left(\frac{5}{50-t}\right)y \quad \Rightarrow \quad \frac{dy}{dt} + \left(\frac{5}{50-t}\right)y = 2.$$

To solve this linear differential equation, let $P(t) = 5/(50 - t)$ and obtain

$$\int P(t) dt = \int \frac{5}{50-t} dt = -5 \ln |50-t|.$$

Because $t < 50$, you can drop the absolute value signs and conclude that

$$e^{\int P(t) dt} = e^{-5 \ln(50-t)} = \frac{1}{(50-t)^5}.$$

So, the general solution is

$$\begin{aligned} \frac{y}{(50-t)^5} &= \int \frac{2}{(50-t)^5} dt = \frac{1}{2(50-t)^4} + C \\ y &= \frac{50-t}{2} + C(50-t)^5. \end{aligned}$$

Because $y = 5$ when $t = 0$, you have

$$5 = \frac{50}{2} + C(50)^5 \quad \Rightarrow \quad -\frac{20}{50^5} = C$$

which means that the particular solution is

$$y = \frac{50-t}{2} - 20\left(\frac{50-t}{50}\right)^5.$$

Finally, when $t = 10$, the amount of alcohol in the tank is

$$y = \frac{50-10}{2} - 20\left(\frac{50-10}{50}\right)^5 \approx 13.45 \text{ gal}$$

which represents a solution containing 33.6% alcohol. ■

In most falling-body problems discussed so far in the text, air resistance has been neglected. The next example includes this factor. In the example, the air resistance on the falling object is assumed to be proportional to its velocity v . If g is the gravitational constant, the downward force F on a falling object of mass m is given by the difference $mg - kv$. If a is the acceleration of the object, then by Newton's Second Law of Motion

$$\begin{aligned} F &= ma \\ &= m \frac{dv}{dt} \end{aligned}$$

which yields the following differential equation.

$$m \frac{dv}{dt} = mg - kv \quad \Rightarrow \quad \frac{dv}{dt} + \frac{kv}{m} = g$$

EXAMPLE 6 A Falling Object with Air Resistance

An object of mass m is dropped from a hovering helicopter. Find its velocity as a function of time t . Assume that the air resistance is proportional to the velocity of the object.

Solution The velocity v satisfies the equation

$$\frac{dv}{dt} + \frac{kv}{m} = g$$

where g is the gravitational constant and k is the constant of proportionality. Letting $b = k/m$, you can *separate variables* to obtain

$$\begin{aligned} dv &= (g - bv) dt \\ \int \frac{dv}{g - bv} &= \int dt \\ -\frac{1}{b} \ln |g - bv| &= t + C_1 \\ \ln |g - bv| &= -bt - bC_1 \\ g - bv &= Ce^{-bt}. \quad C = e^{-bC_1} \end{aligned}$$

Because the object was dropped, $v = 0$ when $t = 0$; so $g = C$, and it follows that

$$-bv = -g + ge^{-bt} \quad \Rightarrow \quad v = \frac{g - ge^{-bt}}{b} = \frac{mg}{k} (1 - e^{-kt/m}). \quad \blacksquare$$

NOTE Notice in Example 6 that the velocity approaches a limit of mg/k as a result of the air resistance. For falling-body problems in which air resistance is neglected, the velocity increases without bound. ■

A simple electric circuit consists of an electric current I (in amperes), a resistance R (in ohms), an inductance L (in henrys), and a constant electromotive force E (in volts), as shown in Figure 6.32. According to Kirchhoff's Second Law, if the switch S is closed when $t = 0$, the applied electromotive force (voltage) is equal to the sum of the voltage drops in the rest of the circuit. This in turn means that the current I satisfies the differential equation

$$L \frac{dI}{dt} + RI = E.$$

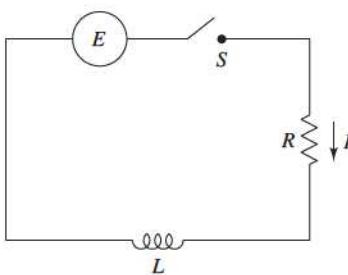


Figure 6.32

EXAMPLE 7 An Electric Circuit Problem

Find the current I as a function of time t (in seconds), given that I satisfies the differential equation $L(dI/dt) + RI = \sin 2t$, where R and L are nonzero constants.

TECHNOLOGY The integral in Example 7 was found using a computer algebra system. If you have access to *Maple*, *Mathematica*, or the *TI-89*, try using it to integrate

$$\frac{1}{L} \int e^{(R/L)t} \sin 2t \, dt.$$

In Chapter 8 you will learn how to integrate functions of this type using integration by parts.

Solution In standard form, the given linear equation is

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{1}{L} \sin 2t.$$

Let $P(t) = R/L$, so that $e^{\int P(t) dt} = e^{(R/L)t}$, and, by Theorem 6.3,

$$\begin{aligned} Ie^{(R/L)t} &= \frac{1}{L} \int e^{(R/L)t} \sin 2t \, dt \\ &= \frac{1}{4L^2 + R^2} e^{(R/L)t} (R \sin 2t - 2L \cos 2t) + C. \end{aligned}$$

So the general solution is

$$\begin{aligned} I &= e^{-(R/L)t} \left[\frac{1}{4L^2 + R^2} e^{(R/L)t} (R \sin 2t - 2L \cos 2t) + C \right] \\ I &= \frac{1}{4L^2 + R^2} (R \sin 2t - 2L \cos 2t) + Ce^{-(R/L)t}. \end{aligned}$$

6.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, determine whether the differential equation is linear. Explain your reasoning.

- | | |
|---------------------------|----------------------------|
| 1. $xy' + y = xe^x$ | 2. $2xy - y' \ln x = y$ |
| 3. $y' - y \sin x = xy^2$ | 4. $\frac{2 - y'}{y} = 5x$ |

In Exercises 5–14, solve the first-order linear differential equation.

- | | |
|---------------------------------------------------------|---------------------------------------------------------|
| 5. $\frac{dy}{dx} + \left(\frac{1}{x}\right)y = 6x + 2$ | 6. $\frac{dy}{dx} + \left(\frac{2}{x}\right)y = 3x - 5$ |
| 7. $y' - y = 16$ | 8. $y' + 2xy = 10x$ |
| 9. $(y + 1) \cos x \, dx - dy = 0$ | 10. $(y - 1) \sin x \, dx - dy = 0$ |
| 11. $(x - 1)y' + y = x^2 - 1$ | 12. $y' + 3y = e^{3x}$ |
| 13. $y' - 3x^2y = e^{x^3}$ | 14. $y' + y \tan x = \sec x$ |

A Slope Fields In Exercises 15 and 16, (a) sketch an approximate solution of the differential equation satisfying the initial condition by hand on the slope field, (b) find the particular solution that satisfies the given initial condition, and (c) use a graphing utility to graph the particular solution. Compare the graph with the hand-drawn graph in part (a). To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

Differential Equation	Initial Condition
15. $\frac{dy}{dx} = e^x - y$	$(0, 1)$
16. $y' + \left(\frac{1}{x}\right)y = \sin x^2$	$(\sqrt{\pi}, 0)$

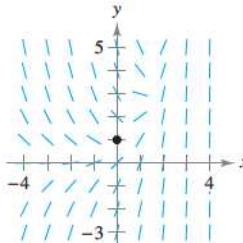


Figure for 15

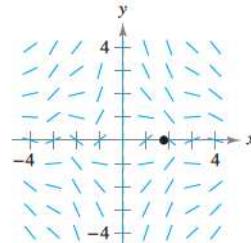


Figure for 16

In Exercises 17–24, find the particular solution of the differential equation that satisfies the boundary condition.

Differential Equation	Boundary Condition
17. $y' \cos^2 x + y - 1 = 0$	$y(0) = 5$
18. $x^3y' + 2y = e^{1/x^2}$	$y(1) = e$
19. $y' + y \tan x = \sec x + \cos x$	$y(0) = 1$
20. $y' + y \sec x = \sec x$	$y(0) = 4$
21. $y' + \left(\frac{1}{x}\right)y = 0$	$y(2) = 2$
22. $y' + (2x - 1)y = 0$	$y(1) = 2$
23. $x \, dy = (x + y + 2) \, dx$	$y(1) = 10$
24. $2x \, y' - y = x^3 - x$	$y(4) = 2$

In Exercises 25–32, solve the Bernoulli differential equation.

25. $y' + 3x^2y = x^2y^3$

26. $y' + xy = xy^{-1}$

27. $y' + \left(\frac{1}{x}\right)y = xy^2$

29. $xy' + y = xy^3$

31. $y' - y = e^{x/3}y$

28. $y' + \left(\frac{1}{x}\right)y = x\sqrt{y}$

30. $y' - y = y^3$

32. $yy' - 2y^2 = e^x$

 **Slope Fields** In Exercises 33–36, (a) use a graphing utility to graph the slope field for the differential equation, (b) find the particular solutions of the differential equation passing through the given points, and (c) use a graphing utility to graph the particular solutions on the slope field.

Differential Equation

33. $\frac{dy}{dx} - \frac{1}{x}y = x^2$

34. $\frac{dy}{dx} + 4x^3y = x^3$

35. $\frac{dy}{dx} + (\cot x)y = 2$

36. $\frac{dy}{dx} + 2xy = xy^2$

Points

(−2, 4), (2, 8)

(0, $\frac{7}{2}$), (0, $-\frac{1}{2}$)

(1, 1), (3, −1)

(0, 3), (0, 1)

37. **Intravenous Feeding** Glucose is added intravenously to the bloodstream at the rate of q units per minute, and the body removes glucose from the bloodstream at a rate proportional to the amount present. Assume that $Q(t)$ is the amount of glucose in the bloodstream at time t .

- Determine the differential equation describing the rate of change of glucose in the bloodstream with respect to time.
- Solve the differential equation from part (a), letting $Q = Q_0$ when $t = 0$.
- Find the limit of $Q(t)$ as $t \rightarrow \infty$.

38. **Learning Curve** The management at a certain factory has found that the maximum number of units a worker can produce in a day is 75. The rate of increase in the number of units N produced with respect to time t in days by a new employee is proportional to $75 - N$.

- Determine the differential equation describing the rate of change of performance with respect to time.
- Solve the differential equation from part (a).
- Find the particular solution for a new employee who produced 20 units on the first day at the factory and 35 units on the twentieth day.

Mixture In Exercises 39–44, consider a tank that at time $t = 0$ contains v_0 gallons of a solution that, by weight, contains q_0 pounds of soluble concentrate. Another solution containing q_1 pounds of the concentrate per gallon is running into the tank at the rate of r_1 gallons per minute. The solution in the tank is kept well stirred and is withdrawn at the rate of r_2 gallons per minute.

39. If Q is the amount of concentrate in the solution at any time t , show that

$$\frac{dQ}{dt} + \frac{r_2 Q}{v_0 + (r_1 - r_2)t} = q_1 r_1.$$

40. If Q is the amount of concentrate in the solution at any time t , write the differential equation for the rate of change of Q with respect to t if $r_1 = r_2 = r$.

41. A 200-gallon tank is full of a solution containing 25 pounds of concentrate. Starting at time $t = 0$, distilled water is admitted to the tank at a rate of 10 gallons per minute, and the well-stirred solution is withdrawn at the same rate, as shown in the figure.

10 gal/min



- Find the amount of concentrate Q in the solution as a function of t .
- Find the time at which the amount of concentrate in the tank reaches 15 pounds.
- Find the quantity of the concentrate in the solution as $t \rightarrow \infty$.

42. Repeat Exercise 41, assuming that the solution entering the tank contains 0.04 pound of concentrate per gallon.

43. A 200-gallon tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 pound of concentrate per gallon enters the tank at the rate of 5 gallons per minute, and the well-stirred mixture is withdrawn at the rate of 3 gallons per minute, as shown in the figure.

5 gal/min



- After how many minutes will the tank be full?
- When the tank is full, how many pounds of concentrate will it contain?

44. Repeat Exercise 43, assuming that the solution entering the tank contains 1 pound of concentrate per gallon.

Falling Object In Exercises 45 and 46, consider an eight-pound object dropped from a height of 5000 feet, where the air resistance is proportional to the velocity.

45. Write the velocity of the object as a function of time if the velocity after 5 seconds is approximately -101 feet per second. What is the limiting value of the velocity function?
46. Use the result of Exercise 45 to write the position of the object as a function of time. Approximate the velocity of the object when it reaches ground level.

Electric Circuits In Exercises 47 and 48, use the differential equation for electric circuits given by $L \frac{dI}{dt} + RI = E$. In this equation, I is the current, R is the resistance, L is the inductance, and E is the electromotive force (voltage).

47. Solve the differential equation for the current given a constant voltage E_0 .
48. Use the result of Exercise 47 to find the equation for the current if $I(0) = 0$, $E_0 = 120$ volts, $R = 600$ ohms, and $L = 4$ henrys. When does the current reach 90% of its limiting value?

WRITING ABOUT CONCEPTS

49. What does the term “first-order” refer to in a first-order linear differential equation?
50. Give the standard form of a first-order linear differential equation. What is its integrating factor?
51. Give the standard form of the Bernoulli equation. Describe how one reduces it to a linear equation.

CAPSTONE

52. Suppose the expression $u(x)$ is an integrating factor for $y' + P(x)y = Q(x)$. Which of the following is equal to $u'(x)$? Verify your answer.
- (a) $P(x)u(x)$ (b) $P'(x)u(x)$
 (c) $Q(x)u(x)$ (d) $Q'(x)u(x)$

SECTION PROJECT

Weight Loss

A person's weight depends on both the number of calories consumed and the energy used. Moreover, the amount of energy used depends on a person's weight—the average amount of energy used by a person is 17.5 calories per pound per day. So, the more weight a person loses, the less energy a person uses (assuming that the person maintains a constant level of activity). An equation that can be used to model weight loss is

$$\left(\frac{dw}{dt}\right) = \frac{C}{3500} - \frac{17.5}{3500}w$$

where w is the person's weight (in pounds), t is the time in days, and C is the constant daily calorie consumption.

In Exercises 53–56, match the differential equation with its solution.

<u>Differential Equation</u>	<u>Solution</u>
53. $y' - 2x = 0$	(a) $y = Ce^{x^2}$
54. $y' - 2y = 0$	(b) $y = -\frac{1}{2} + Ce^{x^2}$
55. $y' - 2xy = 0$	(c) $y = x^2 + C$
56. $y' - 2xy = x$	(d) $y = Ce^{2x}$

In Exercises 57–68, solve the first-order differential equation by any appropriate method.

57. $\frac{dy}{dx} = \frac{e^{2x+y}}{e^{x-y}}$
58. $\frac{dy}{dx} = \frac{x-3}{y(y+4)}$
59. $y \cos x - \cos x + \frac{dy}{dx} = 0$
60. $y' = 2x\sqrt{1-y^2}$
61. $(3y^2 + 4xy)dx + (2xy + x^2)dy = 0$
62. $(x+y)dx - x dy = 0$
63. $(2y - e^x)dx + x dy = 0$
64. $(y^2 + xy)dx - x^2 dy = 0$
65. $(x^2y^4 - 1)dx + x^3y^3 dy = 0$
66. $y dx + (3x + 4y) dy = 0$
67. $3(y - 4x^2)dx + x dy = 0$
68. $x dx + (y + e^y)(x^2 + 1)dy = 0$

True or False? In Exercises 69 and 70, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

69. $y' + x\sqrt{y} = x^2$ is a first-order linear differential equation.
70. $y' + xy = e^x y$ is a first-order linear differential equation.

- (a) Find the general solution of the differential equation.
 (b) Consider a person who weighs 180 pounds and begins a diet of 2500 calories per day. How long will it take the person to lose 10 pounds? How long will it take the person to lose 35 pounds?
 (c) Use a graphing utility to graph the solution. What is the “limiting” weight of the person?
 (d) Repeat parts (b) and (c) for a person who weighs 200 pounds when the diet is started.

■ **FOR FURTHER INFORMATION** For more information on modeling weight loss, see the article “A Linear Diet Model” by Arthur C. Segal in *The College Mathematics Journal*.

6.6**Predator-Prey Differential Equations**

- Analyze predator-prey differential equations.
- Analyze competing-species differential equations.

Predator-Prey Differential Equations

ALFRED LOTKA (1880–1949)
VITO VOLTERRA (1860–1940)

Although Alfred Lotka and Vito Volterra both worked on other problems, they are most known for their work on predator-prey equations. Lotka was also a statistician, and Volterra did work in the development of integral equations and functional analysis.

In the 1920s, mathematicians Alfred Lotka (1880–1949) and Vito Volterra (1860–1940) independently developed mathematical models to represent many of the different ways in which two species can interact with each other. Two common ways in which species interact with each other are as predator and prey, and as competing species.

Consider a predator-prey relationship involving foxes (predators) and rabbits (prey). Assume that the rabbits are the primary food source for the foxes, the rabbits have an unlimited food supply, and there is no threat to the rabbits other than from the foxes. Let x represent the number of rabbits, let y represent the number of foxes, and let t represent time. If there are no foxes, then the rabbit population will grow according to the exponential growth model $dx/dt = ax$, $a > 0$.

If there are foxes but no rabbits, the foxes have no food and their population will decay according to the exponential decay model $dy/dt = -my$, $m > 0$.

If both foxes and rabbits are present, there is an interaction rate of *decline* for the rabbit population given by $-bxy$, and an interaction rate of *increase* in the fox population given by nxy , where $b, n > 0$. So, the rates of change of each population can be modeled by the following predator-prey system of differential equations.

$$\begin{aligned}\frac{dx}{dt} &= ax - bxy && \text{Rate of change of prey} \\ \frac{dy}{dt} &= -my + nxy && \text{Rate of change of predators}\end{aligned}$$

These equations are called **predator-prey equations** or **Lotka-Volterra equations**. The equations are **autonomous** because the rates of change do not depend explicitly on time t .

In general, it is not possible to solve the predator-prey equations explicitly for x and y . However, you can use techniques such as Euler's Method to approximate solutions. Also, you can discover properties of the solutions by analyzing the differential equations.

EXAMPLE 1 Analyzing Predator-Prey Equations

Write the predator-prey equations for $a = 0.04$, $b = 0.002$, $m = 0.08$, and $n = 0.0004$. Then find the values of x and y for which $dx/dt = dy/dt = 0$.

Solution For $a = 0.04$, $b = 0.002$, $m = 0.08$, and $n = 0.0004$, the predator-prey equations are:

$$\begin{aligned}\frac{dx}{dt} &= 0.04x - 0.002xy && \text{Rate of change of prey} \\ \frac{dy}{dt} &= -0.08y + 0.0004xy && \text{Rate of change of predators}\end{aligned}$$

Solving $dx/dt = x(0.04 - 0.002y) = 0$ and $dy/dt = y(-0.08 + 0.0004x) = 0$, you can see that $dx/dt = dy/dt = 0$ when $(x, y) = (0, 0)$ and when $(x, y) = (200, 20)$.

In general, for the predator-prey equations

$$\frac{dx}{dt} = ax - bxy \quad \text{and} \quad \frac{dy}{dt} = -my + nxy$$

$\frac{dx}{dt} = 0$ when $x = 0$ or $y = \frac{a}{b}$, and $\frac{dy}{dt} = 0$ when $y = 0$ or $x = \frac{m}{n}$. So, at the points $(0, 0)$ and $\left(\frac{m}{n}, \frac{a}{b}\right)$, the prey and predator populations are constant. These points are called **critical points** or **equilibrium points** of the predator-prey equations.

EXAMPLE 2 Analyzing Predator-Prey Equations Graphically

Assume that the predator-prey equations from Example 1

$$\frac{dx}{dt} = 0.04x - 0.002xy \quad \text{and} \quad \frac{dy}{dt} = -0.08y + 0.0004xy$$

model a predator-prey relationship involving foxes and rabbits, where x is the number of rabbits and y is the number of foxes after t months. Use a graphing utility to graph the functions x and y when $0 \leq t \leq 240$ and the initial conditions are 200 rabbits and 10 foxes. What do you observe?

Solution The graphs of x and y are shown in Figure 6.33. Some observations are:

- The rabbit and fox populations oscillate periodically between their respective minimum and maximum values.
- The rabbit population oscillates from about 125 rabbits to about 300 rabbits.
- The fox population oscillates from about 10 foxes to about 35 foxes.
- About 20 months after the rabbit population peaks, the fox population peaks.
- The period of each population appears to be about 115 months.

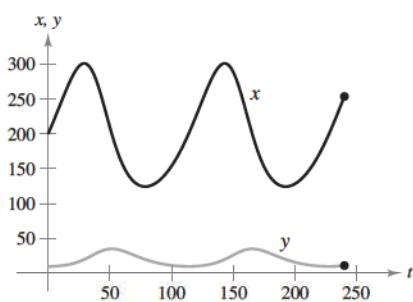


Figure 6.33

You have used slope fields (or direction fields) to analyze solutions of differential equations. In Example 2, the graph shows the curves plotted together with time t along the horizontal axis. You can also use the predator-prey equations dy/dt and dx/dt to graph a slope field. The slope field is graphed using the x -axis to represent the prey and the y -axis to represent the predators.

NOTE If you are using a graphing utility, you may need to rewrite the equations as a function of x :

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-my + nxy}{ax - bxy}.$$

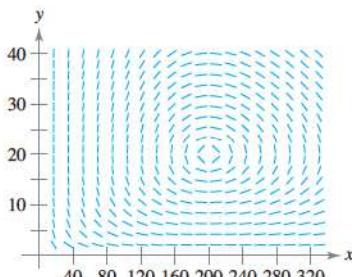


Figure 6.34

EXAMPLE 3 Predator-Prey Equations and Slope Fields

Use a graphing utility to graph the slope field of the predator-prey equations given in Example 2.

Solution The slope field is shown in Figure 6.34. The x -axis represents the rabbit population, and the y -axis represents the fox population.

EXAMPLE 4 Graphing a Solution Curve

Use the predator-prey equations

$$\frac{dx}{dt} = 0.04x - 0.002xy \quad \text{and} \quad \frac{dy}{dt} = -0.08y + 0.0004xy$$

and the slope field from Example 3 to graph the solution curve using the initial conditions of 200 rabbits and 10 foxes. Describe the changes in the populations as you trace the solution curve.

Solution The graph of the solution is a closed curve, as shown in Figures 6.35 and 6.36.

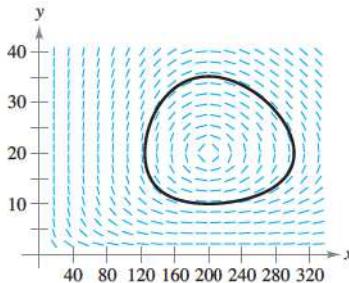


Figure 6.35

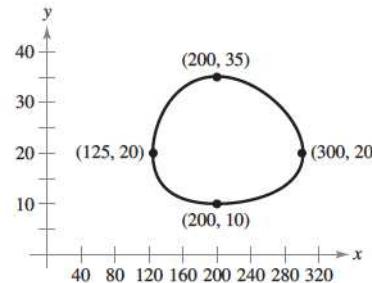


Figure 6.36

At $(200, 10)$, $dy/dt = 0$ and $dx/dt = 4$. So, the rabbit population is increasing at $(200, 10)$. This means that you should trace the curve counterclockwise as t increases. As you trace the curve, note the changes listed in Figure 6.37.

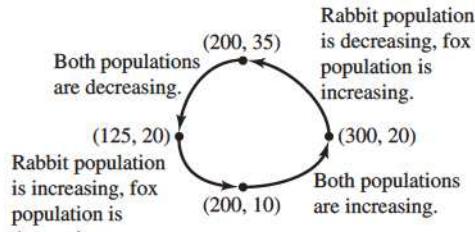


Figure 6.37

Although it is generally not possible to solve predator-prey equations explicitly for x and y , you can separate variables to derive an implicit solution. Begin by writing the equations dy/dt and dx/dt as a function of x .

$$\frac{dy}{dx} = \frac{y(-m + nx)}{x(a - by)}$$

Factor numerator and denominator.

$$x(a - by) dy = y(-m + nx) dx$$

Differential form

$$\frac{a - by}{y} dy = \frac{-m + nx}{x} dx$$

Separate variables.

$$\int \frac{a - by}{y} dy = \int \frac{-m + nx}{x} dx$$

Integrate.

$$a \ln y - by = -m \ln x + nx + C$$

Assume x and y are positive.

$$a \ln y + m \ln x - by - nx = C$$

General solution

The constant C is determined by the initial conditions.

NOTE The general solution $a \ln y + m \ln x - by - nx = C$ can be rewritten as $\ln(y^a x^m) = C + by + nx$ or as $y^a x^m = C_1 e^{by + nx}$.

Competing Species

Consider two species that compete with each other for the food available in their common environment. Assume that their populations are given by x and y at time t . If there were no interaction or competition between the species, then the populations x and y would each have logistic growth and would satisfy the following differential equations.

$$\frac{dx}{dt} = ax - bx^2 \quad \text{Rate of change of first species without interaction}$$

$$\frac{dy}{dt} = my - ny^2 \quad \text{Rate of change of second species without interaction}$$

If the species interact, then their competition for resources causes a rate of decline in each population proportional to the product xy . Using a negative interaction factor leads to the following **competing-species equations** (where a , b , c , m , n , and p are positive constants).

$$\frac{dx}{dt} = ax - bx^2 - cxy \quad \text{Rate of change of first species with interaction}$$

$$\frac{dy}{dt} = my - ny^2 - pxy \quad \text{Rate of change of second species with interaction}$$

In this text it is assumed that competing-species equations have four critical points, as shown in Example 5.

EXAMPLE 5 Deriving the Critical Points

Show that the critical points of the competing-species equations

$$\frac{dx}{dt} = ax - bx^2 - cxy \quad \text{and} \quad \frac{dy}{dt} = my - ny^2 - pxy$$

are $(0, 0)$, $\left(0, \frac{m}{n}\right)$, $\left(\frac{a}{b}, 0\right)$, and $\left(\frac{an - mc}{bn - cp}, \frac{bm - ap}{bn - cp}\right)$.

Solution Set dx/dt and dy/dt equal to 0 and then factor to obtain the following system.

$$x(a - bx - cy) = 0 \quad \text{Set } dx/dt \text{ equal to 0 and factor out } x.$$

$$y(m - ny - px) = 0 \quad \text{Set } dy/dt \text{ equal to 0 and factor out } y.$$

If $x = 0$, then $y = 0$ or $y = m/n$. If $y = 0$, then $x = 0$ or $x = a/b$. So three of the critical points are $(0, 0)$, $(0, m/n)$, and $(a/b, 0)$.

At each of these critical points, one of the populations is 0. These points represent the possibility that both species cannot coexist. The fourth critical point is obtained by solving the system

$$a - bx - cy = 0$$

$$m - ny - px = 0.$$

The solution of this system is

$$(x, y) = \left(\frac{an - mc}{bn - cp}, \frac{bm - ap}{bn - cp}\right).$$

Assuming this point exists and lies in Quadrant I of the xy -plane, the point represents the possibility that both species can coexist. ■

EXAMPLE 6 Competing Species: One Species Survives

Consider the competing-species equations given by

$$\frac{dx}{dt} = 10x - x^2 - 2xy \quad \text{and} \quad \frac{dy}{dt} = 10y - y^2 - 2xy.$$

- a. Find the critical points.

- b. Use a graphing utility to graph the solution of the equations when $0 \leq t \leq 3$ and the initial conditions are $x(0) = 10$ and $y(0) = 15$. What do you observe?

Solution

- a. Note that $a = 10$, $b = 1$, $c = 2$, $m = 10$, $n = 1$, and $p = 2$. So, the critical points are $(0,0)$, $(0, 10)$, $(10, 0)$, and $\left(\frac{10-20}{1-4}, \frac{10-20}{1-4}\right) = \left(\frac{10}{3}, \frac{10}{3}\right)$.
- b. The solution of the competing-species equations is shown in Figure 6.38. From the graph, it appears that one species survives. The population of the surviving species, represented by the graph of y , appears to remain constant at 10.

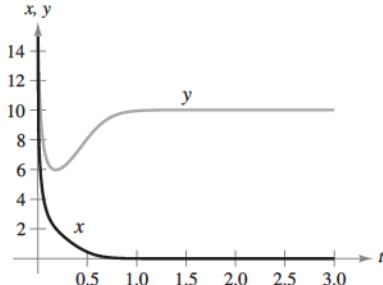


Figure 6.38

EXAMPLE 7 Competing Species: Both Species Survive

Consider the competing-species equations given by

$$\frac{dx}{dt} = 10x - 3x^2 - xy \quad \text{and} \quad \frac{dy}{dt} = 14y - 3y^2 - xy.$$

- a. Find the critical points.

- b. Use a graphing utility to graph the solution of the equations when $0 \leq t \leq 15$ and the initial conditions are $x(0) = 10$ and $y(0) = 15$. What do you observe?

Solution

- a. Note that $a = 10$, $b = 3$, $c = 1$, $m = 14$, $n = 3$, and $p = 1$. So, the critical points are $(0,0)$, $\left(0, \frac{14}{3}\right)$, $\left(\frac{10}{3}, 0\right)$, and $\left(\frac{30-14}{9-1}, \frac{42-10}{9-1}\right) = (2, 4)$.
- b. The solution of the competing-species equations is shown in Figure 6.39. From the graph, it appears that both species survive. The population represented by y appears to remain constant at 4. The population represented by x appears to remain constant at 2.

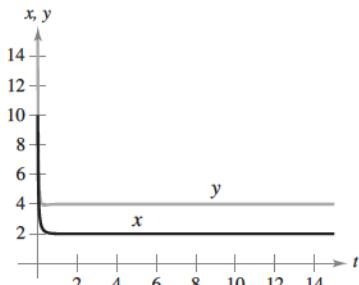


Figure 6.39

Examples 6 and 7 imply general conclusions about competing-species equations that have precisely four critical points. In general, it can be shown that if $bn > cp$, both species survive. If $bn < cp$, then one species will survive and the other will not.

You can also use slope fields to analyze solutions of competing-species equations, as shown in Figures 6.40 (Example 6) and 6.41 (Example 7).

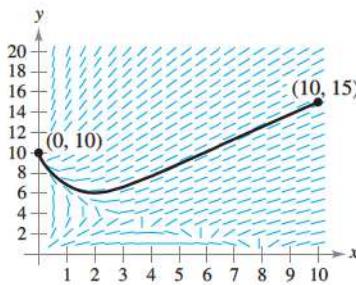


Figure 6.40

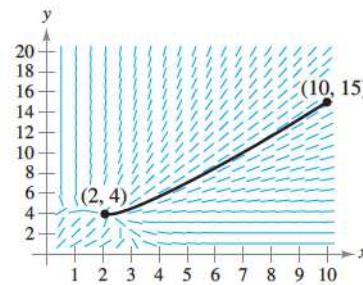


Figure 6.41

6.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, use the given values to write the predator-prey equations $dx/dt = ax - bxy$ and $dy/dt = -my + nxy$. Then find the values of x and y for which $dx/dt = dy/dt = 0$.

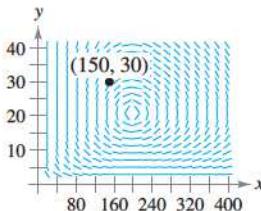
1. $a = 0.9, b = 0.05, m = 0.6, n = 0.008$
2. $a = 0.75, b = 0.006, m = 0.9, n = 0.003$
3. $a = 0.5, b = 0.01, m = 0.49, n = 0.007$
4. $a = 1.2, b = 0.04, m = 1.2, n = 0.02$

A **Slope Fields** In Exercises 5 and 6, predator-prey equations, a point, and a slope field are given. (a) Sketch a solution of the predator-prey equations on the slope field that passes through the given point. (b) Use a graphing utility to graph the solution. Compare the result with the sketch in part (a). To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

5. $\frac{dx}{dt} = 0.04x - 0.002xy$

$$\frac{dy}{dt} = -0.08y + 0.0004xy$$

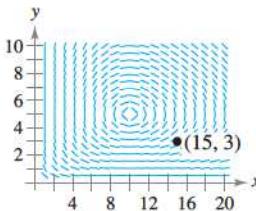
(150, 30)



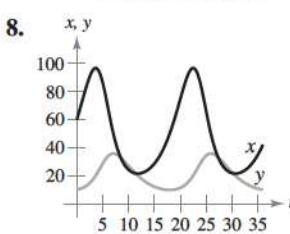
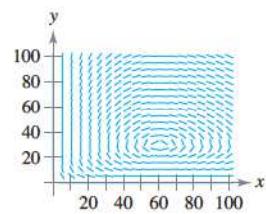
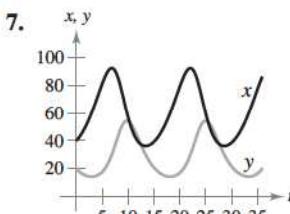
6. $\frac{dx}{dt} = 0.03x - 0.006xy$

$$\frac{dy}{dt} = -0.04y + 0.004xy$$

(15, 3)



In Exercises 7 and 8, two graphs are given. The first is a graph of the functions x and y of a set of predator-prey equations, where x is the number of prey and y is the number of predators at time t . The second graph is the corresponding slope field of the predator-prey equations. (a) Identify the initial conditions. (b) Sketch a solution of the predator-prey equations on the slope field that passes through the initial conditions. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.



Rabbits and Foxes In Exercises 9–12, consider a predator-prey relationship involving foxes (predators) and rabbits (prey). Let x represent the number of rabbits, let y represent the number of foxes, and let t represent the time in months. Assume that the following predator-prey equations model the rates of change of each population.

$$\frac{dx}{dt} = 0.8x - 0.04xy \quad \text{Rate of change of prey population}$$

$$\frac{dy}{dt} = -0.3y + 0.006xy \quad \text{Rate of change of predator population}$$

When $t = 0, x = 55$ and $y = 10$.

9. Find the critical points of the predator-prey equations.

- A** 10. Use a graphing utility to graph the functions x and y when $0 \leq t \leq 36$. Describe the behavior of each solution as t increases.

- A** 11. Use a graphing utility to graph a slope field of the predator-prey equations when $0 \leq x \leq 150$ and $0 \leq y \leq 50$.

- A** 12. Use the predator-prey equations and the slope field in Exercise 11 to graph the solution curve using the initial conditions. Describe the changes in the rabbit and fox populations as you trace the solution curve.

Prairie Dogs and Black-Footed Ferrets In Exercises 13–16, consider a predator-prey relationship involving black-footed ferrets (predators) and prairie dogs (prey). Let x represent the number of prairie dogs, let y represent the number of black-footed ferrets, and let t represent the time in months. Assume that the following predator-prey equations model the rates of change of each population.

$$\frac{dx}{dt} = 0.1x - 0.00008xy \quad \text{Rate of change of prey population}$$

$$\frac{dy}{dt} = -0.4y + 0.00004xy \quad \text{Rate of change of predator population}$$

When $t = 0, x = 4000$ and $y = 1000$.

13. Find the critical points of the predator-prey equations.

- A** 14. Use a graphing utility to graph the functions x and y when $0 \leq t \leq 240$. Describe the behavior of each solution as t increases.

- A** 15. Use a graphing utility to graph a slope field of the predator-prey equations when $0 \leq x \leq 25,000$ and $0 \leq y \leq 5000$.

- A** 16. Use the predator-prey equations and the slope field in Exercise 15 to graph the solution curve using the initial conditions. Describe the changes in the prairie dog and black-footed ferret populations as you trace the solution curve.

- A** 17. **Critical Point as the Initial Condition** In Exercise 9, you found the critical points of the predator-prey system. Assume that the critical point given by $(m/n, a/b)$ is the initial condition and repeat Exercises 10–12. Compare the results.

- F** 18. *Critical Point as the Initial Condition* In Exercise 13, you found the critical points of the predator-prey system. Assume that the critical point given by $(m/n, a/b)$ is the initial condition and repeat Exercises 14–16. Compare the results.

In Exercises 19–22, use the given values to write the competing-species equations $dx/dt = ax - bx^2 - cxy$ and $dy/dt = my - ny^2 - pxy$. Then find the values of x and y for which $dx/dt = dy/dt = 0$.

19. $a = 2, b = 3, c = 2, m = 2, n = 3, p = 2$
20. $a = 1, b = 0.5, c = 0.5, m = 2.5, n = 2, p = 0.5$
21. $a = 0.15, b = 0.6, c = 0.75, m = 0.15, n = 1.2, p = 0.45$
22. $a = 0.025, b = 0.1, c = 0.2, m = 0.3, n = 0.45, p = 0.1$

Bass and Trout In Exercises 23 and 24, consider a competing-species relationship involving bass and trout. Assume the bass and trout compete for the same resources. Let x represent the number of bass (in thousands), let y represent the number of trout (in thousands), and let t represent the time in months. Assume that the following competing-species equations model the rates of change of the two populations.

$$\frac{dx}{dt} = 0.8x - 0.4x^2 - 0.1xy \quad \text{Rate of change of bass population}$$

$$\frac{dy}{dt} = 0.3y - 0.6y^2 - 0.1xy \quad \text{Rate of change of trout population}$$

When $t = 0, x = 9$ and $y = 5$.

23. Find the critical points of the competing-species equations.

- F** 24. Use a graphing utility to graph the functions x and y when $0 \leq t \leq 36$. Describe the behavior of each solution as t increases.

Bass and Trout In Exercises 25 and 26, consider a competing-species relationship involving bass and trout. Assume the bass and trout compete for the same resources. Let x represent the number of bass (in thousands), let y represent the number of trout (in thousands), and let t represent the time in months. Assume that the following competing-species equations model the rates of change of the two populations.

$$\frac{dx}{dt} = 0.8x - 0.4x^2 - xy \quad \text{Rate of change of bass population}$$

$$\frac{dy}{dt} = 0.3y - 0.6y^2 - xy \quad \text{Rate of change of trout population}$$

When $t = 0, x = 7$ and $y = 6$.

25. Find the critical points of the competing-species equations.

- F** 26. Use a graphing utility to graph the functions x and y when $0 \leq t \leq 36$. Describe the behavior of each solution as t increases.

- F** 27. *Critical Point as the Initial Condition* In Exercise 23, you found the critical points of the competing-species system. Assume that the critical point given by $\left(\frac{an - mc}{bn - cp}, \frac{bm - ap}{bn - cp}\right)$ is the initial condition and repeat Exercise 24. Compare the results.

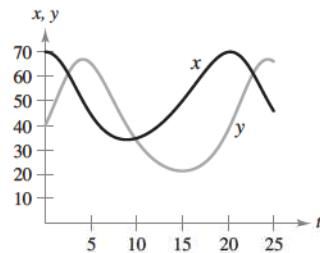
- F** 28. *Critical Point as the Initial Condition* In Exercise 23, you found the critical points of the competing-species system. Assume that the critical point given by $(0, m/n)$ is the initial condition and repeat Exercise 24. Compare the results.

WRITING ABOUT CONCEPTS

29. Are predator-prey equations separable? Explain.
30. Given a set of predator-prey equations, describe how to determine initial values so that both populations remain constant for all $t \geq 0$.
31. Given a set of competing-species equations, describe how to determine initial values so that both populations remain constant for all $t > 0$.

CAPSTONE

32. The populations of two species x and y are shown in the figure. Sketch the graph of the solution curve by hand for $0 \leq t \leq 20$.



- F** 33. *Revising the Predator-Prey Equations* Consider a predator-prey relationship with x prey and y predators at time t . Assume both predator and prey are present. Then the rates of change of the two populations can be modeled by the following revised predator-prey system of differential equations.

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{L}\right) - bxy \quad \text{Rate of change of prey population}$$

$$\frac{dy}{dt} = -my + nxy \quad \text{Rate of change of predator population}$$

- (a) If there are no predators, the prey population will grow according to what model?
- (b) Write the revised predator-prey equations for $a = 0.4$, $L = 100$, $b = 0.01$, $m = 0.3$, and $n = 0.005$. Find the critical numbers.
- (c) Use a graphing utility to graph the functions x and y of the revised predator-prey equations when $0 \leq t \leq 72$ and the initial conditions are $x(0) = 40$ and $y(0) = 80$. Describe the behavior of each solution as t increases.
- (d) Use a graphing utility to graph a slope field of the revised predator-prey equations when $0 \leq x \leq 100$ and $0 \leq y \leq 80$.
- (e) Use the predator-prey equations and the slope field in part (d) to graph the solution curve using the initial conditions in part (c). Describe the changes in the prey and predator populations as you trace the solution curve.

6 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. Determine whether the function $y = x^3$ is a solution of the differential equation $2xy' + 4y = 10x^3$.
2. Determine whether the function $y = 2 \sin 2x$ is a solution of the differential equation $y''' - 8y = 0$.

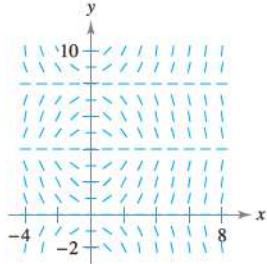
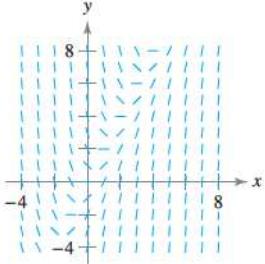
In Exercises 3–10, use integration to find a general solution of the differential equation.

3. $\frac{dy}{dx} = 4x^2 + 7$
4. $\frac{dy}{dx} = 3x^3 - 8x$
5. $\frac{dy}{dx} = \cos 2x$
6. $\frac{dy}{dx} = 2 \sin x$
7. $\frac{dy}{dx} = x\sqrt{x-5}$
8. $\frac{dy}{dx} = 2x\sqrt{x-7}$
9. $\frac{dy}{dx} = e^{2-x}$
10. $\frac{dy}{dx} = 3e^{-x/3}$

Slope Fields In Exercises 11 and 12, a differential equation and its slope field are given. Determine the slopes (if possible) in the slope field at the points given in the table.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx						

11. $\frac{dy}{dx} = 2x - y$
12. $\frac{dy}{dx} = x \sin\left(\frac{\pi y}{4}\right)$



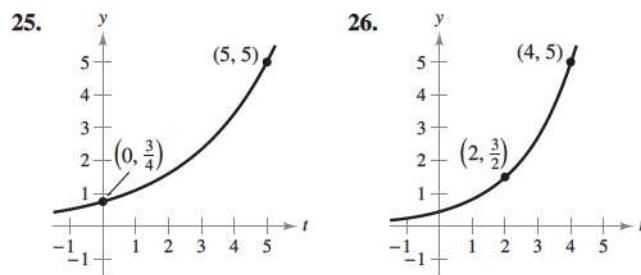
Slope Fields In Exercises 13–18, (a) sketch the slope field for the differential equation, and (b) use the slope field to sketch the solution that passes through the given point. Use a graphing utility to verify your results.

Differential Equation	Point
13. $y' = 3 - x$	(2, 1)
14. $y' = 2x^2 - x$	(0, 2)
15. $y' = \frac{1}{4}x^2 - \frac{1}{3}x$	(0, 3)
16. $y' = y + 4x$	(-1, 1)
17. $y' = \frac{xy}{x^2 + 4}$	(0, 1)
18. $y' = \frac{y}{x^2 + 1}$	(0, -2)

In Exercises 19–24, solve the differential equation.

19. $\frac{dy}{dx} = 8 - x$
20. $\frac{dy}{dx} = y + 8$
21. $\frac{dy}{dx} = (3 + y)^2$
22. $\frac{dy}{dx} = 10\sqrt{y}$
23. $(2 + x)y' - xy = 0$
24. $xy' - (x + 1)y = 0$

In Exercises 25–28, find the exponential function $y = Ce^{kt}$ that passes through the two points.



25. $(0, 1), (5, 5)$
26. $(2, \frac{3}{2}), (4, 5)$

27. $(0, 5), (5, \frac{1}{6})$

28. $(1, 9), (6, 2)$

29. **Air Pressure** Under ideal conditions, air pressure decreases continuously with the height above sea level at a rate proportional to the pressure at that height. The barometer reads 30 inches at sea level and 15 inches at 18,000 feet. Find the barometric pressure at 35,000 feet.

30. **Radioactive Decay** Radioactive radium has a half-life of approximately 1599 years. The initial quantity is 15 grams. How much remains after 750 years?

31. **Sales** The sales S (in thousands of units) of a new product after it has been on the market for t years is given by

$$S = Ce^{kt}.$$

(a) Find S as a function of t if 5000 units have been sold after 1 year and the saturation point for the market is 30,000 units (that is, $\lim_{t \rightarrow \infty} S = 30$).

(b) How many units will have been sold after 5 years?

(c) Use a graphing utility to graph this sales function.

32. **Sales** The sales S (in thousands of units) of a new product after it has been on the market for t years is given by

$$S = 25(1 - e^{kt}).$$

(a) Find S as a function of t if 4000 units have been sold after 1 year.

(b) How many units will saturate this market?

(c) How many units will have been sold after 5 years?

(d) Use a graphing utility to graph this sales function.

33. **Population Growth** A population grows continuously at the rate of 1.85%. How long will it take the population to double?

- 34. Fuel Economy** An automobile gets 28 miles per gallon of gasoline for speeds up to 50 miles per hour. Over 50 miles per hour, the number of miles per gallon drops at the rate of 12 percent for each 10 miles per hour.

(a) s is the speed and y is the number of miles per gallon. Find y as a function of s by solving the differential equation

$$\frac{dy}{ds} = -0.012y, \quad s > 50.$$

(b) Use the function in part (a) to complete the table.

Speed	50	55	60	65	70
Miles per Gallon					

In Exercises 35–40, solve the differential equation.

35. $\frac{dy}{dx} = \frac{x^5 + 7}{x}$

36. $\frac{dy}{dx} = \frac{e^{-2x}}{1 + e^{-2x}}$

37. $y' - 16xy = 0$

38. $y' - e^y \sin x = 0$

39. $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$

40. $\frac{dy}{dx} = \frac{3(x + y)}{x}$

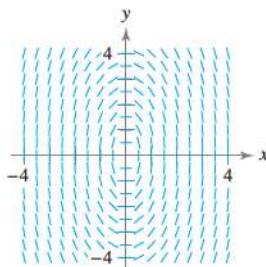
41. Verify that the general solution $y = C_1x + C_2x^3$ satisfies the differential equation $x^2y'' - 3xy' + 3y = 0$. Then find the particular solution that satisfies the initial condition $y = 0$ and $y' = 4$ when $x = 2$.

42. **Vertical Motion** A falling object encounters air resistance that is proportional to its velocity. The acceleration due to gravity is -9.8 meters per second per second. The net change in velocity is $dv/dt = kv - 9.8$.

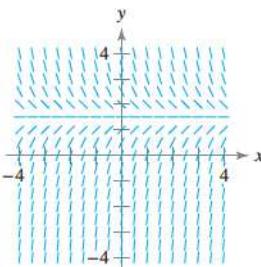
- (a) Find the velocity of the object as a function of time if the initial velocity is v_0 .
- (b) Use the result of part (a) to find the limit of the velocity as t approaches infinity.
- (c) Integrate the velocity function found in part (a) to find the position function s .

Slope Fields In Exercises 43 and 44, sketch a few solutions of the differential equation on the slope field and then find the general solution analytically. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

43. $\frac{dy}{dx} = -\frac{4x}{y}$



44. $\frac{dy}{dx} = 3 - 2y$



In Exercises 45 and 46, the logistic equation models the growth of a population. Use the equation to (a) find the value of k , (b) find the carrying capacity, (c) find the initial population, (d) determine when the population will reach 50% of its carrying capacity, and (e) write a logistic differential equation that has the solution $P(t)$.

45. $P(t) = \frac{5250}{1 + 34e^{-0.55t}}$

46. $P(t) = \frac{4800}{1 + 14e^{-0.15t}}$

In Exercises 47 and 48, find the logistic equation that satisfies the initial condition.

Logistic Differential Equation *Initial Condition*

47. $\frac{dy}{dt} = y\left(1 - \frac{y}{80}\right)$ $(0, 8)$

48. $\frac{dy}{dt} = 1.76y\left(1 - \frac{y}{8}\right)$ $(0, 3)$

49. **Environment** A conservation department releases 1200 brook trout into a lake. It is estimated that the carrying capacity of the lake for the species is 20,400. After the first year, there are 2000 brook trout in the lake.

- (a) Write a logistic equation that models the number of brook trout in the lake.
- (b) Find the number of brook trout in the lake after 8 years.
- (c) When will the number of brook trout reach 10,000?

50. **Environment** Write a logistic differential equation that models the growth rate of the brook trout population in Exercise 49. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answers.

51. **Sales Growth** The rate of change in sales S (in thousands of units) of a new product is proportional to $L - S$. L (in thousands of units) is the estimated maximum level of sales, and $S = 0$ when $t = 0$. Write and solve the differential equation for this sales model.

52. **Sales Growth** Use the result of Exercise 51 to write S as a function of t if (a) $L = 100$, $S = 25$ when $t = 2$, and (b) $L = 500$, $S = 50$ when $t = 1$.

Learning Theory In Exercises 53 and 54, assume that the rate of change in the proportion P of correct responses after n trials is proportional to the product of P and $L - P$, where L is the limiting proportion of correct responses.

53. Write and solve the differential equation for this learning theory model.

54. Use the solution of Exercise 53 to write P as a function of n , and then use a graphing utility to graph the solution.

(a) $L = 1.00$

$P = 0.50$ when $n = 0$

$P = 0.85$ when $n = 4$

(b) $L = 0.80$

$P = 0.25$ when $n = 0$

$P = 0.60$ when $n = 10$



In Exercises 55–58, (a) sketch an approximate solution of the differential equation satisfying the initial condition by hand on the slope field, (b) find the particular solution that satisfies the initial condition, and (c) use a graphing utility to graph the particular solution. Compare the graph with the hand-drawn graph in part (a). To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

<i>Differential Equation</i>	<i>Initial Condition</i>
55. $\frac{dy}{dx} = e^{x/2} - y$	(0, -1)
56. $y' + 2y = \sin x$	(0, 4)
57. $y' = \csc x + y \cot x$	(1, 1)
58. $y' = \csc x - y \cot x$	(1, 2)

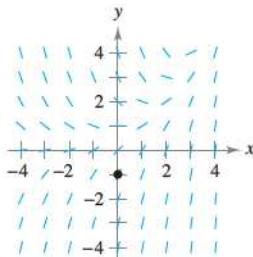


Figure for 55

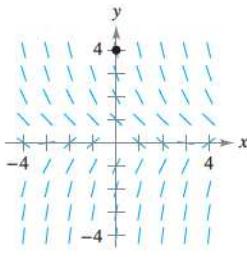


Figure for 56

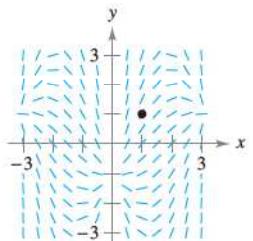


Figure for 57

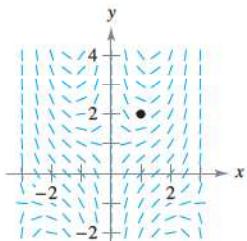


Figure for 58

In Exercises 59–68, solve the first-order linear differential equation.

59. $y' - y = 10$

60. $e^x y' + 4e^x y = 1$

61. $4y' = e^{x/4} + y$

62. $\frac{dy}{dx} - \frac{5y}{x^2} = \frac{1}{x^2}$

63. $(x - 2)y' + y = 1$

64. $(x + 3)y' + 2y = 2(x + 3)^2$

65. $(3y + \sin 2x) dx - dy = 0$

66. $dy = (y \tan x + 2e^x) dx$

67. $y' + 5y = e^{5x}$

68. $xy' - ay = bx^4$

In Exercises 69–72, solve the Bernoulli differential equation.

69. $y' + y = xy^2$ [Hint: $\int xe^{-x} dx = (-x - 1)e^{-x}$]

70. $y' + 2xy = xy^2$

71. $y' + \left(\frac{1}{x}\right)y = \frac{y^3}{x^2}$

72. $xy' + y = xy^2$

In Exercises 73–76, write an example of the given differential equation. Then solve your equation.

73. Homogeneous differential equation

74. Logistic differential equation

75. First-order linear differential equation

76. Bernoulli differential equation

77. **Investment** Let $A(t)$ be the amount in a fund earning interest at an annual rate r compounded continuously. If a continuous cash flow of P dollars per year is withdrawn from the fund, then the rate of change of A is given by the differential equation

$$\frac{dA}{dt} = rA - P$$

where $A = A_0$ when $t = 0$. Solve this differential equation for A as a function of t .

78. **Investment** A retired couple plans to withdraw P dollars per year from a retirement account of \$500,000 earning 10% interest compounded continuously. Use the result of Exercise 77 and a graphing utility to graph the function A for each of the following continuous annual cash flows. Use the graphs to describe what happens to the balance in the fund for each case.

(a) $P = \$40,000$

(b) $P = \$50,000$

(c) $P = \$60,000$

79. **Investment** Use the result of Exercise 77 to find the time necessary to deplete a fund earning 14% interest compounded continuously if $A_0 = \$1,000,000$ and $P = \$200,000$.

In Exercises 80 and 81, (a) use the given values to write a set of predator-prey equations, (b) find the values of x and y for which $x' = y' = 0$, and (c) use a graphing utility to graph the solutions x and y of the predator-prey equations for the given time frame. Describe the behavior of each solution as t increases.

80. Constants: $a = 0.3, b = 0.02, m = 0.4, n = 0.01$

Initial condition: $(20, 20)$

Time frame: $0 \leq t \leq 36$

81. Constants: $a = 0.4, b = 0.04, m = 0.6, n = 0.02$

Initial condition: $(30, 15)$

Time frame: $0 \leq t \leq 24$

In Exercises 82 and 83, (a) use the given values to write a set of competing-species equations, (b) find the values of x and y for which $x' = y' = 0$, and (c) use a graphing utility to graph the solutions x and y of the competing-species equations for the given time frame. Describe the behavior of each solution as t increases.

82. Constants: $a = 3, b = 1, c = 1, m = 2, n = 1, p = 0.5$

Initial condition: $(3, 2)$

Time frame: $0 \leq t \leq 6$

83. Constants: $a = 15, b = 2, c = 4, m = 17, n = 2, p = 4$

Initial condition: $(9, 10)$

Time frame: $0 \leq t \leq 4$

P.S. PROBLEM SOLVING

1. The differential equation

$$\frac{dy}{dt} = ky^{1+\varepsilon}$$

where k and ε are positive constants, is called the **doomsday equation**.

- (a) Solve the doomsday equation

$$\frac{dy}{dt} = y^{1.01}$$

given that $y(0) = 1$. Find the time T at which

$$\lim_{t \rightarrow T^-} y(t) = \infty.$$

- (b) Solve the doomsday equation

$$\frac{dy}{dt} = ky^{1+\varepsilon}$$

given that $y(0) = y_0$. Explain why this equation is called the doomsday equation.

2. A thermometer is taken from a room at 72°F to the outdoors, where the temperature is 20°F . The reading drops to 48°F after 1 minute. Determine the reading on the thermometer after 5 minutes.

3. Let S represent sales of a new product (in thousands of units), let L represent the maximum level of sales (in thousands of units), and let t represent time (in months). The rate of change of S with respect to t varies jointly as the product of S and $L - S$.

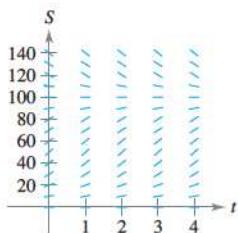
- (a) Write the differential equation for the sales model if $L = 100$, $S = 10$ when $t = 0$, and $S = 20$ when $t = 1$. Verify that

$$S = \frac{L}{1 + Ce^{-kt}}.$$

- (b) After how many months is the growth in sales increasing most rapidly?

- (c) Use a graphing utility to graph the sales function.

- (d) Sketch the solution from part (a) on the slope field shown in the figure below. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.



- (e) If the estimated maximum level of sales is correct, use the slope field to describe the shape of the solution curves for sales if, after some period of time, sales exceed L .

4. Another numerical approach to approximating the particular solution of the differential equation $y' = F(x, y)$ is shown below.

$$x_n = x_{n-1} + h$$

$$y_n = y_{n-1} + hf\left(x_{n-1} + \frac{h}{2}, y_{n-1} + \frac{h}{2}f(x_{n-1}, y_{n-1})\right)$$

This approach is called **modified Euler's Method**.

- (a) Use this method to approximate the solution of the differential equation $y' = x - y$ passing through the point $(0, 1)$. Use a step size of $h = 0.1$.

- (b) Use a graphing utility to graph the exact solution and the approximations found using Euler's Method and modified Euler's Method (see Example 6, page 392). Compare the first 10 approximations found using modified Euler's Method to those found using Euler's Method and to the exact solution $y = x - 1 + 2e^{-x}$. Which approximation appears to be more accurate?

5. Show that the logistic equation $y = L/(1 + be^{-kt})$ can be written as

$$y = \frac{1}{2}L\left[1 + \tanh\left(\frac{1}{2}k\left(t - \frac{\ln b}{k}\right)\right)\right].$$

What can you conclude about the graph of the logistic equation?

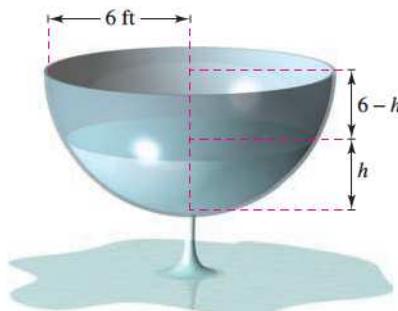
6. Although it is true for some functions f and g , a common mistake in calculus is to believe that the Product Rule for derivatives is $(fg)' = f'g'$.

- (a) Given $g(x) = x$, find f such that $(fg)' = f'g'$.

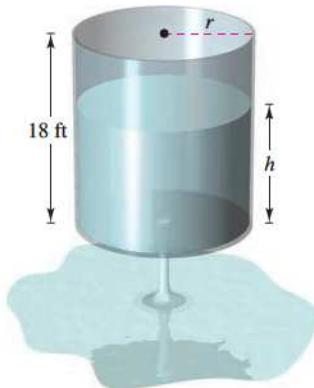
- (b) Given an arbitrary function g , find a function f such that $(fg)' = f'g'$.

- (c) Describe what happens if $g(x) = e^x$.

7. **Torricelli's Law** states that water will flow from an opening at the bottom of a tank with the same speed that it would attain falling from the surface of the water to the opening. One of the forms of Torricelli's Law is $A(h) \frac{dh}{dt} = -k\sqrt{2gh}$, where h is the height of the water in the tank, k is the area of the opening at the bottom of the tank, $A(h)$ is the horizontal cross-sectional area at height h , and g is the acceleration due to gravity ($g \approx 32$ feet per second per second). A hemispherical water tank has a radius of 6 feet. When the tank is full, a circular valve with a radius of 1 inch is opened at the bottom, as shown in the figure. How long will it take for the tank to drain completely?



8. The cylindrical water tank shown in the figure has a height of 18 feet. When the tank is full, a circular valve is opened at the bottom of the tank. After 30 minutes, the depth of the water is 12 feet.



- (a) How long will it take for the tank to drain completely?
 (b) What is the depth of the water in the tank after 1 hour?
 9. Suppose the tank in Exercise 8 has a height of 20 feet and a radius of 8 feet, and the valve is circular with a radius of 2 inches. The tank is full when the valve is opened. How long will it take for the tank to drain completely?
 10. In hilly areas, radio reception may be poor. Consider a situation in which an FM transmitter is located at the point $(-1, 1)$ behind a hill modeled by the graph of

$$y = x - x^2$$

and a radio receiver is on the opposite side of the hill. (Assume that the x -axis represents ground level at the base of the hill.)

- (a) What is the closest position $(x, 0)$ the radio can be to the hill so that reception is unobstructed?
 (b) Write the closest position $(x, 0)$ of the radio with x represented as a function of h if the transmitter is located at $(-1, h)$.
 (c) Use a graphing utility to graph the function for x in part (b). Determine the vertical asymptote of the function and interpret its meaning.

11. Biomass is a measure of the amount of living matter in an ecosystem. Suppose the biomass $s(t)$ in a given ecosystem increases at a rate of about 3.5 tons per year, and decreases by about 1.9% per year. This situation can be modeled by the differential equation

$$\frac{ds}{dt} = 3.5 - 0.019s.$$

- (a) Solve the differential equation.
 (b) Use a graphing utility to graph the slope field for the differential equation. What do you notice?
 (c) Explain what happens as $t \rightarrow \infty$.

In Exercises 12–14, a medical researcher wants to determine the concentration C (in moles per liter) of a tracer drug injected into a moving fluid. Solve this problem by considering a single-compartment dilution model (see figure). Assume that the fluid is continuously mixed and that the volume of the fluid in the compartment is constant.

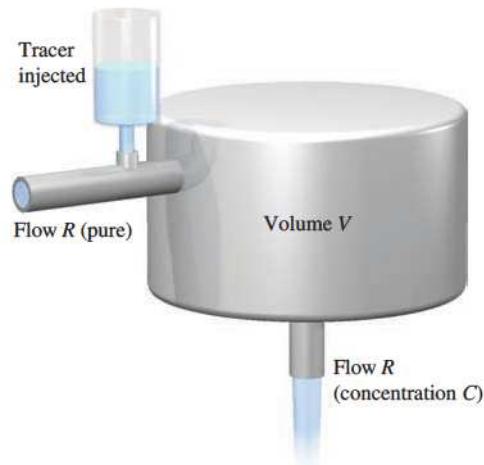


Figure for 12–14

12. If the tracer is injected instantaneously at time $t = 0$, then the concentration of the fluid in the compartment begins diluting according to the differential equation

$$\frac{dC}{dt} = \left(-\frac{R}{V} \right) C, \quad C = C_0 \text{ when } t = 0.$$

- (a) Solve this differential equation to find the concentration C as a function of time t .
 (b) Find the limit of C as $t \rightarrow \infty$.

13. Use the solution of the differential equation in Exercise 12 to find the concentration C as a function of time t , and use a graphing utility to graph the function.

- (a) $V = 2$ liters, $R = 0.5$ liter per minute, and $C_0 = 0.6$ mole per liter
 (b) $V = 2$ liters, $R = 1.5$ liters per minute, and $C_0 = 0.6$ mole per liter

14. In Exercises 12 and 13, it was assumed that there was a single initial injection of the tracer drug into the compartment. Now consider the case in which the tracer is continuously injected (beginning at $t = 0$) at the rate of Q moles per minute. Considering Q to be negligible compared with R , use the differential equation

$$\frac{dC}{dt} = \frac{Q}{V} - \left(\frac{R}{V} \right) C, \quad C = 0 \text{ when } t = 0.$$

- (a) Solve this differential equation to find the concentration C as a function of time t .
 (b) Find the limit of C as $t \rightarrow \infty$.