AP Calculus I
Notes 4.1
Extrema on an Interval

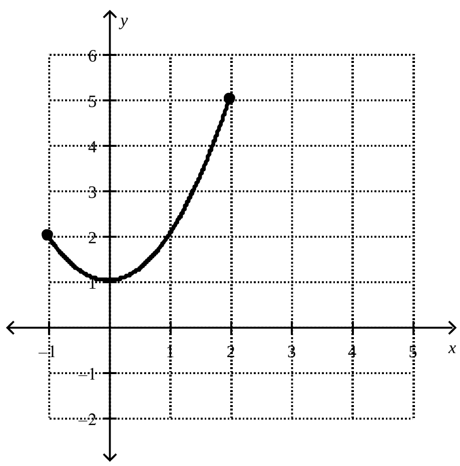
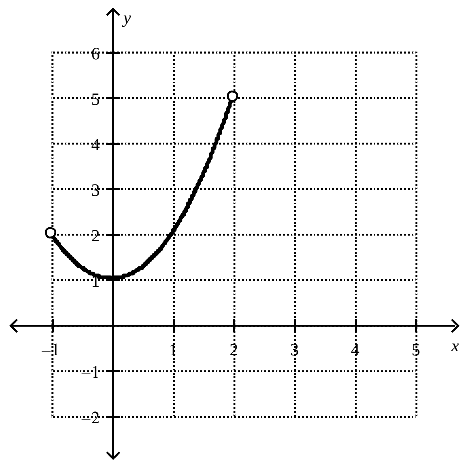
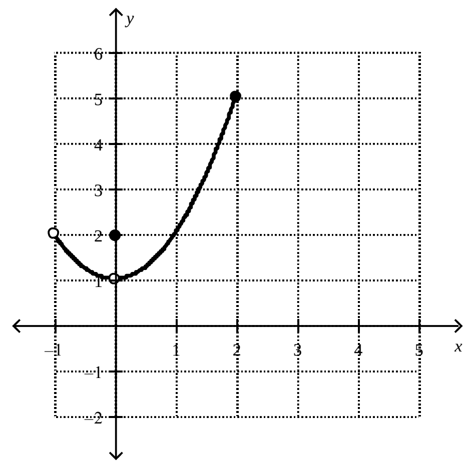
<u>Definition of Extrema</u>

Let f be defined on an interval l containing c.

1) $f(c)$ is the **<u>absolute minimum</u>** of f if $f(c) \leq f(x)$ for all x.
2) $f(c)$ is the **<u>absolute maximum</u>** of f if $f(c) \geq f(x)$ for all x.

The minimum and maximum of a function on an interval are the **extreme values**, or **extrema**, of the function on the interval. The overall minimum and maximum of a function are called the **absolute minimum** and **absolute maximum** on the interval.

<u>Ex. 1:</u> Consider the following variations of the function $f(x) = x^2 + 1$:

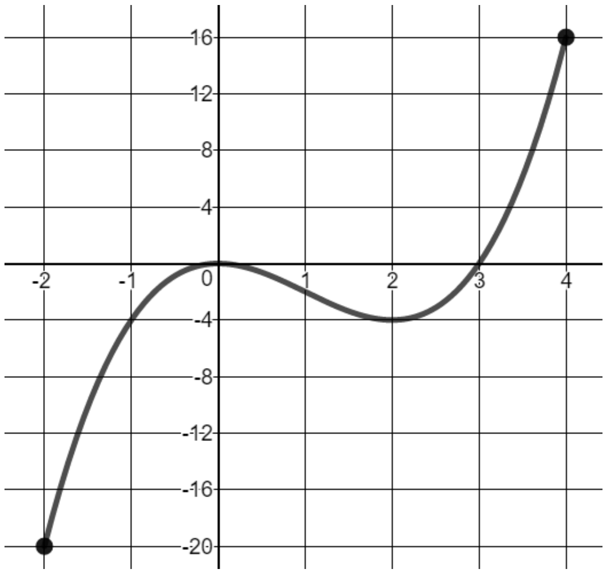




<u>The Extreme Value Theorem:</u>

There is **guaranteed** to be an absolute minimum and an absolute maximum for f on a **closed** interval $[a, b]$ as long as f is ___________________________ on the closed interval $[a, b]$.

Relative Extrema:

Consider the graph below of $f(x) = x^3 - 3x^2$:



Determine if $f(x)$ has absolute extrema from $-2 \le x \le 4$.

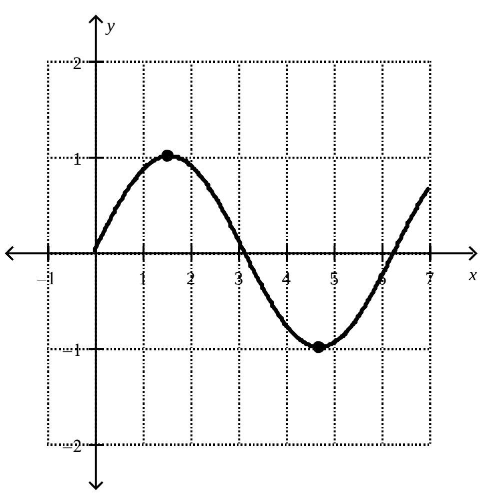
The graph of $f(x) = x^3 - 3x^2$ has a **relative minimum** at the point _____________

The graph of $f(x) = x^3 - 3x^2$ has a **relative maximum** at the point _____________

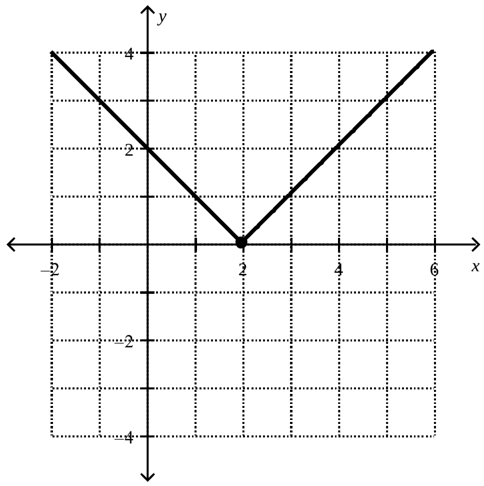
Think of the relative maximum occurring on a _____________ of the graph and the relative minimum occurring in a ____________________ of the graph.

Ex. 2: Find the value of the derivative, if it exists, at each of the relative extrema below:

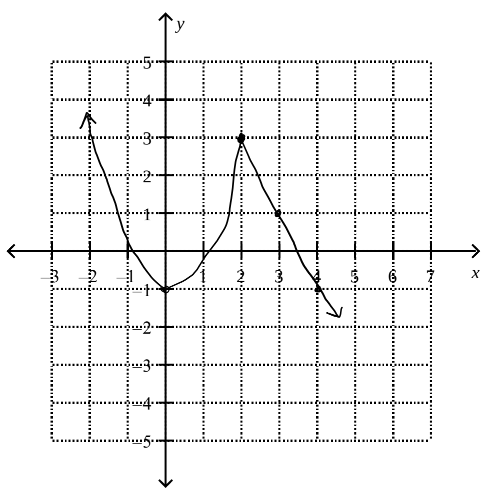
a) $f(x) = \sin x$



b) $f(x) = |x - 2|$



c) $g(x) = \begin{cases} x^2 - 1 & x < 2 \\ 7 - 2x & x \geq 2 \end{cases}$



Note that in each of these examples of relative extrema, the derivative is either _____ or _____________

Definition of Critical Number:

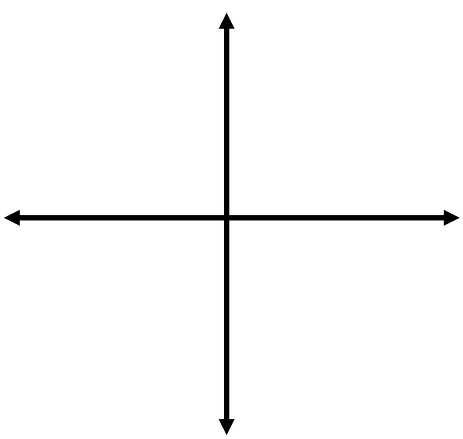
Let f be defined at c. If $f'(c) = 0$ or if f' is undefined at c, then c is a **critical number** of f.

Ex. 3: Find the critical values of $f(x) = \sin^2 x + \sin x$ on the interval $[0, 2\pi]$.

Theorem – Relative Extrema Occur Only at Critical Numbers

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.
However, if $x = c$ is a critical number of f, there is **not necessarily** a relative extrema at $x = c$.

Counterexample: $f(x) = x^3$



$f(x)$ has a critical value at ______________ because

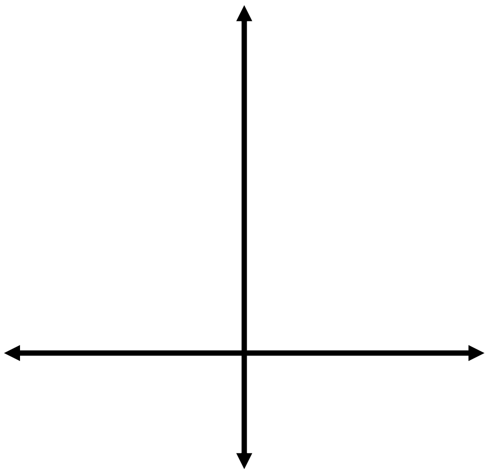
$f(x)$ has a relative extrema at ______________ because

Guidelines for Finding Absolute Extrema on a Closed Interval:

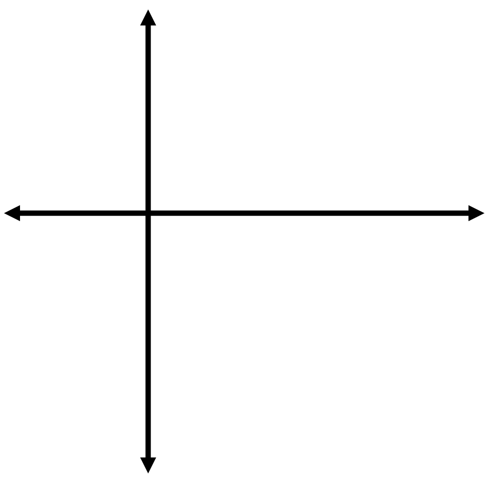
To find the absolute extrema of a continuous function $\boldsymbol{f}$ on a closed interval $[\boldsymbol{a}, \boldsymbol{b}]$, use the following:

1) Find the critical numbers of f in (a, b) and evaluate f at each critical number.
2) Evaluate f at each endpoint of $[a, b]$. The least of these values is the minimum and the greatest is the maximum.

Ex. 4: Find the absolute extrema of $f(x) = 3x^4 - 4x^3$ on $[-1,2]$. Are the extrema guaranteed?



Ex. 5: Find the maximum value of $f(x) = 2x - 3x^{\frac{2}{3}}$ on the interval $[-1,8]$, if any.



Summary

a) The Extreme Value Theorem states

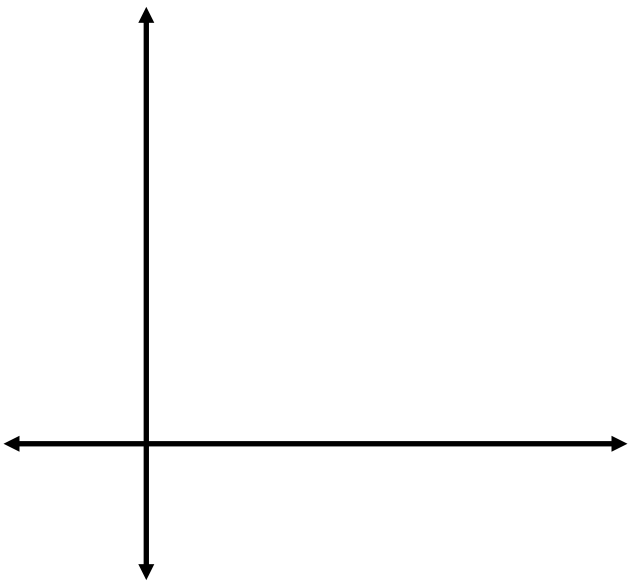
b) A critical number is

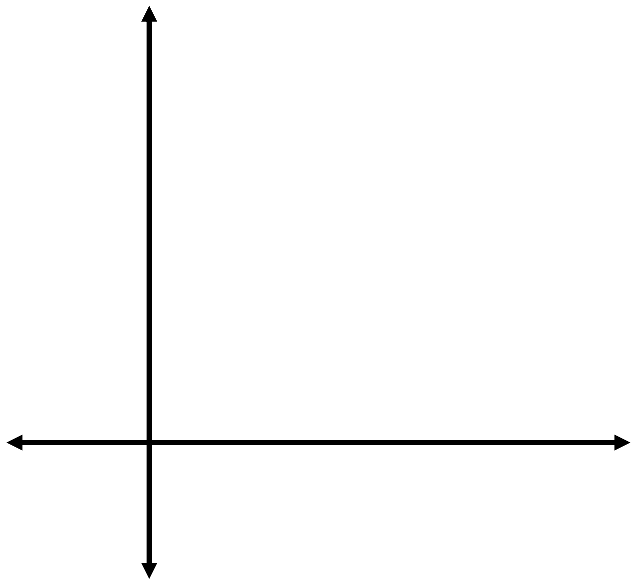
c) Absolute extrema can occur at

d) Relative extrema occur at

AP Calculus I
Notes 4.2
Rolle's Theorem and the Mean Value Theorem

Exploration
Label the points (1,3) and (5,3). Draw any random function that starts at (1,3) and ends at (5,3).

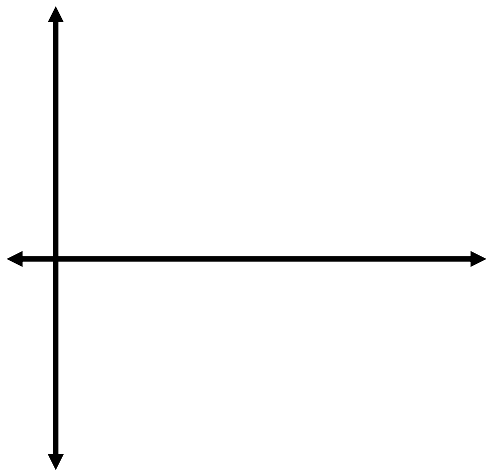




- Is there one point on the graph for which the derivative is zero?
- Could there be more than one? If so, draw an example of this function.
- Could there be none? If so, draw an example of this function.
- Describe the continuity and differentiability of the different graphs sketched.
- Would it be possible to draw a **differentiable** graph so that there *isn't* a point for which the derivative is zero? Explain your reasoning.

Rolle's Theorem:

Ex. 1: Determine whether the function $f(x) = \sin x$ over $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ satisfies the conditions of Rolle's Theorem. If f does, then find the guaranteed point(s).



Ex. 2: Find the two zeroes of the function and determine whether Rolle's Theorem can be applied on the interval of the zeros. If Rolle's Theorem can be applied, find the values of c such that $f'(c) = 0$.

a) $f(x) = (x-3)(x+1)^2$

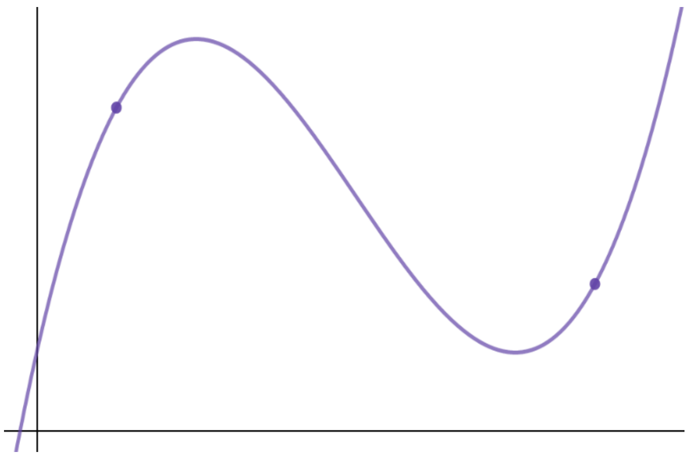
b) $f(x) = x^{\frac{2}{3}} - 4$

The Mean Value Theorem:

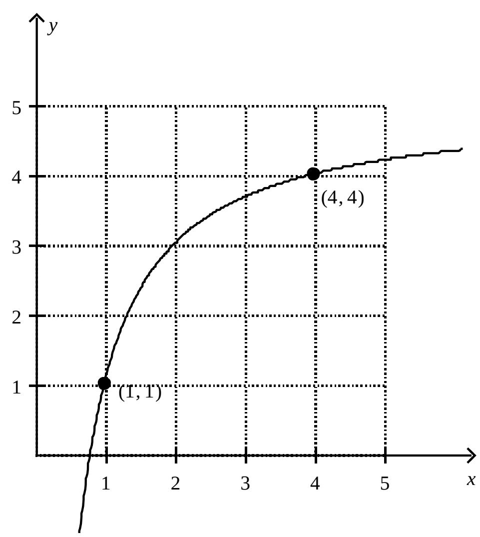
If $\boldsymbol{f}$ is continuous on the closed interval $[\boldsymbol{a}, \boldsymbol{b}]$ and differentiable on the open interval $(\boldsymbol{a}, \boldsymbol{b})$, then there exists a number (or multiple) $\boldsymbol{c}$ in $(\boldsymbol{a}, \boldsymbol{b})$ such that $\boldsymbol{f}'(\boldsymbol{c}) = \frac{\boldsymbol{f}(\boldsymbol{b}) - \boldsymbol{f}(\boldsymbol{a})}{\boldsymbol{b} - \boldsymbol{a}}$.

In other words, if $\boldsymbol{f}$ is continuous and differentiable, then there is some (or many) number(s) $\boldsymbol{c}$ in $(\boldsymbol{a}, \boldsymbol{b})$ such that the ______________________ rate of change is equal to the _________________ rate of change.

Additionally, Extreme Value, Rolle's and the Mean Value Theorem are **existence** theorems. This means there is **guaranteed** to exist at least one point where the conclusion is met, but you may **not** know where.



Ex. 3: Given $f(x) = 5 - \frac{4}{x}$, find all values of c in the open interval $(1,4)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$, if they exist.



Ex. 4: 2019 New FRQ

Consider the function $f'(x) = x^k - 3x^2 + 1$ where k is a nonzero constant.

a) Let $k = 2$. Explain why there must be a value c, for $-1 < c < 2$, such that $f''(c) = -2$.

b) Let $k = 3$. Explain why there must be a value r, for $2 < r < 4$, such that $f'(r) = 0$.

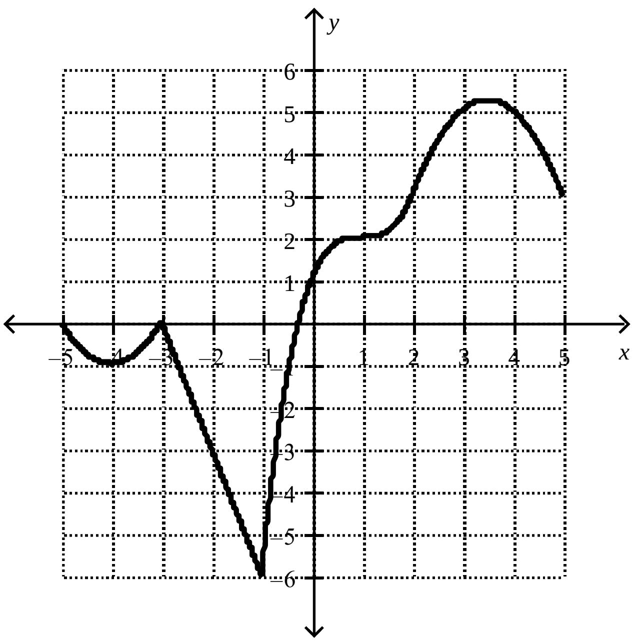
c) Let $k = 4$. Write an equation for the line tangent to the graph of f' at the point whose x −coordinate is 2.

d) Let $k = 6$. Find the maximum and minimum value of f' on the interval $[0,2]$.

AP Calculus I
Notes 4.3
Increasing/ Decreasing Functions and the First Derivative Test

Exploration

Examine the graph of $f(x)$ below:



1. On which intervals is $f(x)$ increasing?

2. On which intervals is $f(x)$ decreasing?

3. What separates these intervals?

4. What is true about the slopes of the tangent lines at each value of x in the intervals in which the function is increasing?

5. What is true about the slopes of the tangent lines at each value of x in the intervals in which the function is decreasing?

Theorem – Test for Increasing and Decreasing Functions

Let f be a function that is continuous on the interval $[a, b]$ and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.

Guidelines for Finding Intervals on Which a Function is Increasing or Decreasing

Let f be continuous on the interval (a, b). To find the intervals on which f is increasing or decreasing:

Ex. 1: Find the intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

Ex. 2: Find the intervals on which $f(x) = \cos^3 x$ is increasing or decreasing from $[0, 2\pi)$.

Ex. 3: The derivative of a function f is given by $f'(x) = e^{\sin x} - \cos x - 1$ for $0 < x < 9$. On what intervals is f decreasing?

(A) $0 < x < 0.633$ and $4.115 < x < 6.916$

(B) $0 < x < 1.947$ and $5.744 < x < 8.230$

(C) $0.633 < x < 4.115$ and $6.916 < x < 9$

(D) $1.947 < x < 5.744$ and $8.230 < x < 9$

Theorem – The First Derivative Test

If c is a critical number of a function f, then $f(c)$ can be classified as follows:

1. If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a ______________________________ of f.

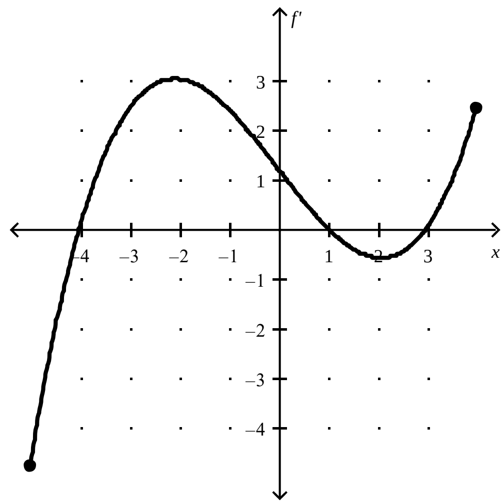
2. If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a ______________________________ of f.

Ex. 4: Find the relative extrema of $f(x) = xe^{2x}$.

Ex. 5: Find the relative extrema of $h(\theta) = (\theta - 2)^3(\theta + 3)^2$.

Ex. 6: Let $g(x)$ be a continuous function, where $g < 0$ for all values of x. Find the relative extrema of the function f given $f'(x) = \frac{4g(x)}{\sqrt[3]{x^2-4}}$.

Ex. 7: Given a graph of f':



a) Is f continuous? Is f differentiable? Justify your answer.

b) Over what interval is f increasing/decreasing? Justify your answer.

c) At what value(s) of x does f have relative mins/maxs? Justify your answer.

Ex. 8: The following table shows certain values of f', the derivative of the differentiable function f.

x	-2	0	1	3	5	7
f'	4	2	0	-4	3	-5

Which of the following statements are true about the graph of f? Justify your answers.

a) There is at least one value of c, such that $-2 < c < 7$, where $f'(c)$ does not exist.

b) f is increasing from $-2 < x < 1$.

c) There is a relative maximum of f at $x = 1$.

d) There is a relative minimum of f at some value $x = c$ such that $3 < c < 5$.

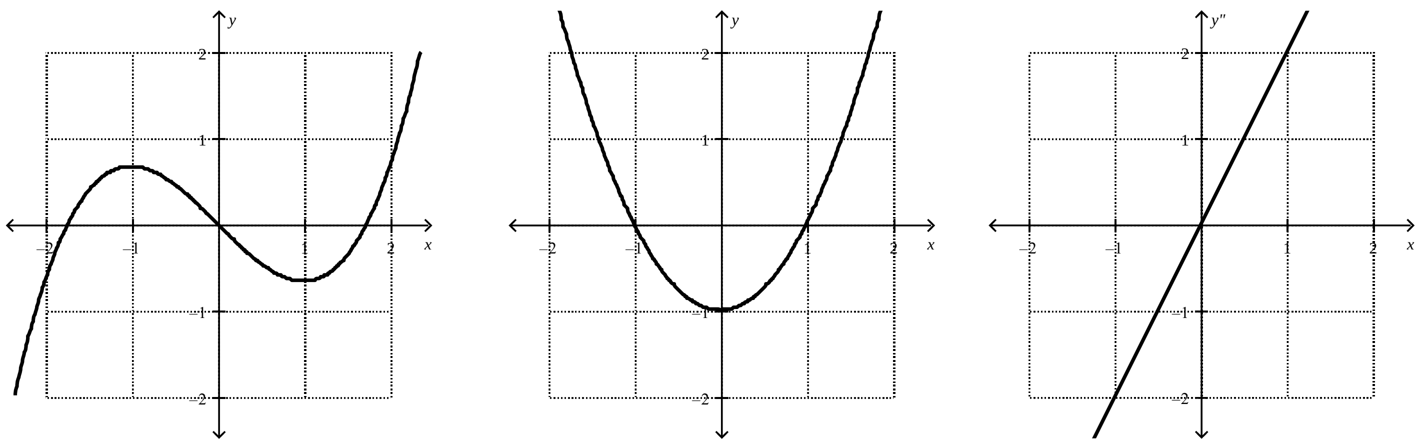
AP Calculus I
Notes 4.4
Concavity and the Second Derivative Test

Definitions of Concavity

Let f be differentiable, the graph of f is

- **Concave upward** if f' is increasing on the interval (the slopes are increasing).
- **Concave downward** if f' is decreasing on the interval (the slopes are decreasing).

Exploration



1. On what interval is the graph of f concave up?
2. On what interval is the graph of f concave down?

Think of concave up as a ________________ and concave down as a ________________________.

1. Let f be differentiable at c. If the graph of f is **concave upward** at $(c, f(c))$, the graph of f lies *above* the tangent line at $(c, f(c))$ on some open interval containing c.

2. Let f be differentiable at c. If the graph of f is **concave downward** at $(c, f(c))$, the graph of f lies *below* the tangent line at $(c, f(c))$ on some open interval containing c.

Theorem – Test for Concavity

1. Since concave up means f' is increasing and increasing means ________________________________ ,

2. Since concave down means f' is decreasing and decreasing means ____________________________ ,

Three definitions of Concave up (and down):

1) 2) 3)

Ex. 1: Determine the intervals on which the graph of $f(x) = e^{-x^2/2}$ is concave upward or downward.

Ex. 2: Determine the open intervals in which the graph of $f(x) = 2\ln|x^2 + 1| + 3\arctan x$ is concave upward or concave downward.

Theorem – Points of Inflection

A point of inflection occurs when the concavity changes from upward or downward or vice versa. So, the point of inflection occurs when f'' changes from ____________ to _______________ or vice versa. Also, the point of inflection occurs when f' changes from ____________ to _______________ or vice versa. If $\left(c, f(c)\right)$ is a point of inflection of a graph of f, then either $f''(c) = 0$ or $f''(c)$ is undefined.

Ex. 3: Determine the points of inflection and discuss the concavity of the graph of $f(x) = x^5 - 5x^4 + x$.

Ex. 4: Find the points of inflection of $f(x)$, where $x > -1$, given $f'(x) = x\sqrt{x+1}$.

Concavity also allows us to determine relative extrema of a function.

Theorem – The Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an interval containing c.

1. If $f''(c) > 0$, then $f(c)$ is a ________________________ of f.

2. If $f''(c) < 0$, then $f(c)$ is a ________________________ of f.

3. If $f''(c) = 0$, the test fails. In such cases, use the First Derivative Test.

Ex. 5: Find the relative extrema of the function $B(t) = -3t^4 + 6t^2$ using the second derivative test.

Ex. 6: For all x in the closed interval $[2,5]$, the function f has a positive first derivative and a negative second derivative. Which of the following could be the table of values for f?

x	$f(x)$
2	7
3	9
4	12
5	16

x	$f(x)$
2	7
3	9
4	11
5	13

x	$f(x)$
2	7
3	11
4	14
5	16

x	$f(x)$
2	16
3	14
4	11
5	7

x	$f(x)$
2	16
3	13
4	10
5	7

AP Calculus I
Graph of f'

Quick Definition Review

A function $f(x)$ is increasing when $f'(x)$ ____.

- If the graph shown is $f(x)$, ____.
- If the graph shown is $f'(x)$, ____.

A function $f(x)$ is decreasing when $f'(x)$____.

- If the graph shown is $f(x)$, ____.
- If the graph shown is $f'(x)$, ____.

A function $f(x)$ has a local maximum when $f'(x)$______________________________________.

- If the graph shown is $f(x)$, ____.
- If the graph shown is $f'(x)$, ____.

A function $f(x)$ has a local minimum when $f'(x)$______________________________________.

- If the graph shown is $f(x)$, ____.
- If the graph shown is $f'(x)$, ____.

A function $f(x)$ has a point of inflection when $f''(x)$___________________________________.

- If the graph shown is $f(x)$, ____.
- If the graph shown is $f'(x)$, ____.
- If the graph shown is $f''(x)$, ____.

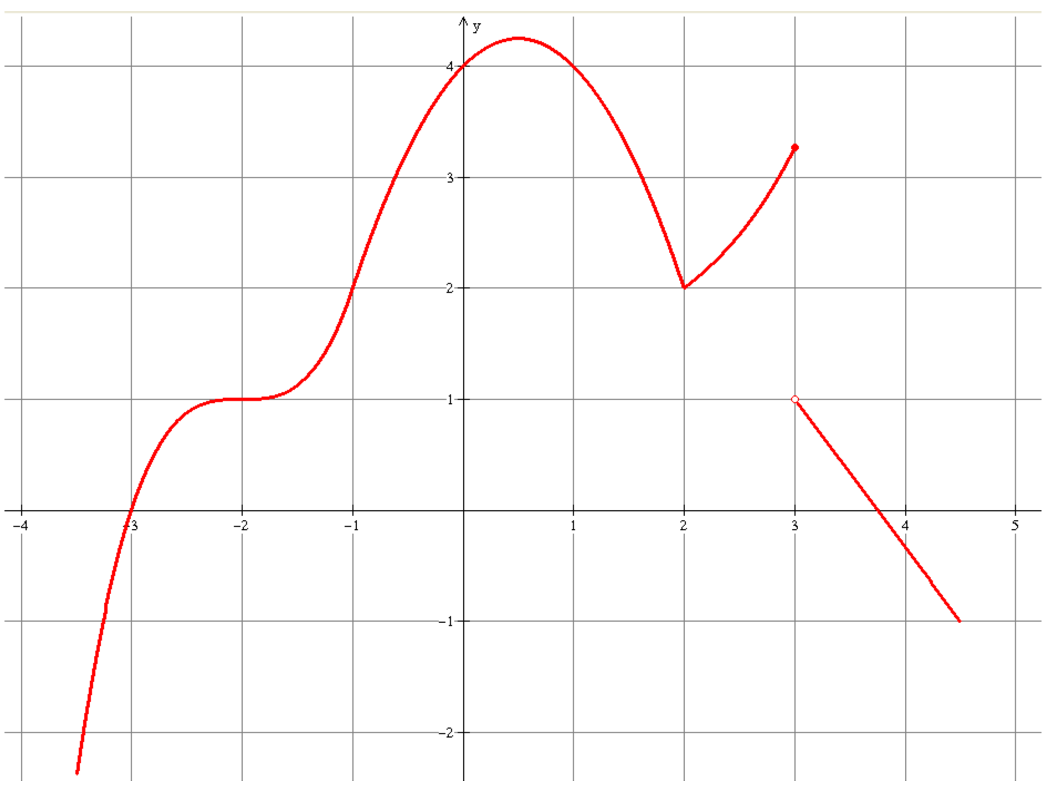
A function $f(x)$ is concave up when $f''(x)$____.

- If the graph shown is $f(x)$, ____.
- If the graph shown is $f'(x)$, ____.
- If the graph shown is $f''(x)$, ____.

A function $f(x)$ is concave down when $f''(x)$____.

- If the graph shown is $f(x)$, ____.
- If the graph shown is $f'(x)$, ____.
- If the graph shown is $f''(x)$, ____.

<u>Ex. 1:</u> Label the graph of $f(x)$:



Ex. 2: Now, let's make the same graph a graph of $f'(x)$.

a) Find the value(s) of x for which $f(x)$ is not differentiable. Is $f(x)$ continuous there?

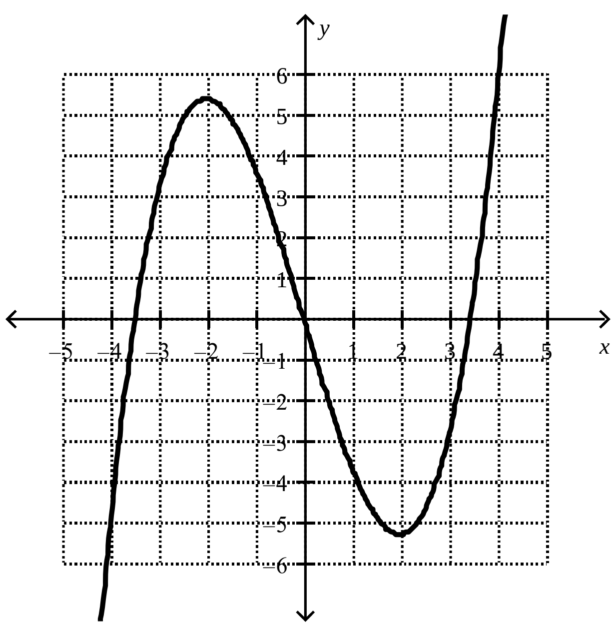
b) Find $f(-2)$, $f'(-2)$, and $f''(-2)$, if possible.

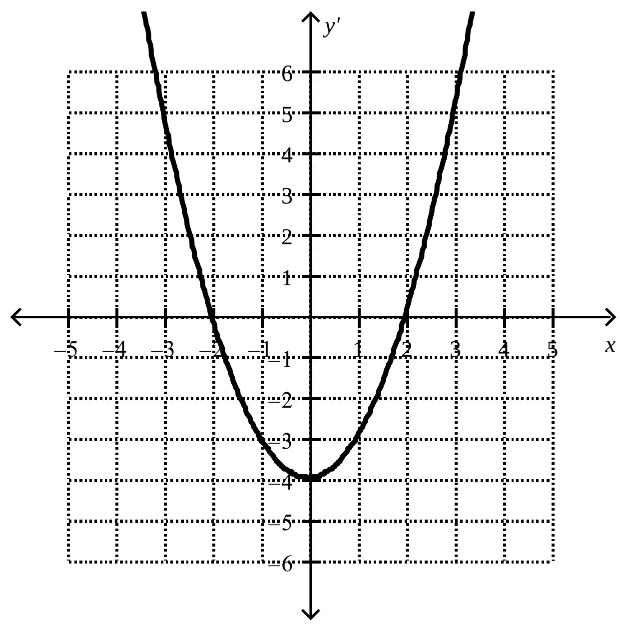
c) Find the interval(s) in which $f(x)$ is increasing/decreasing. Justify your answer.

d) Find the interval(s) in which $f(x)$ is concave up/concave down. Justify your answer.

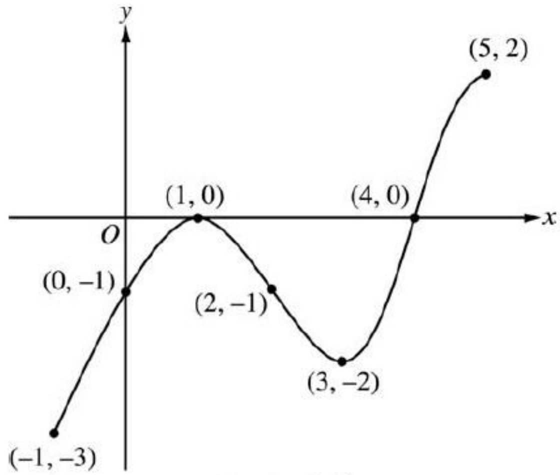
e) Find all relative extrema and points of inflection of $f(x)$. Justify your answer.

Ex. 3: Given the function $f(x) = \frac{1}{3}x^3 - 4x$, find the intervals of increasing, decreasing, concave up, concave down, relative maxima, relative minima, and points of inflection. Then, use the graph to verify all your answers.





Ex. 4:



Graph of f'

The figure above shows the graph of f', the derivative of the function f, on the closed interval $-1 \leq x \leq 5$. The graph of f' has horizontal tangent lines at $x = 1$ and $x = 3$. The function f is twice differentiable with $f(2) = 6$.

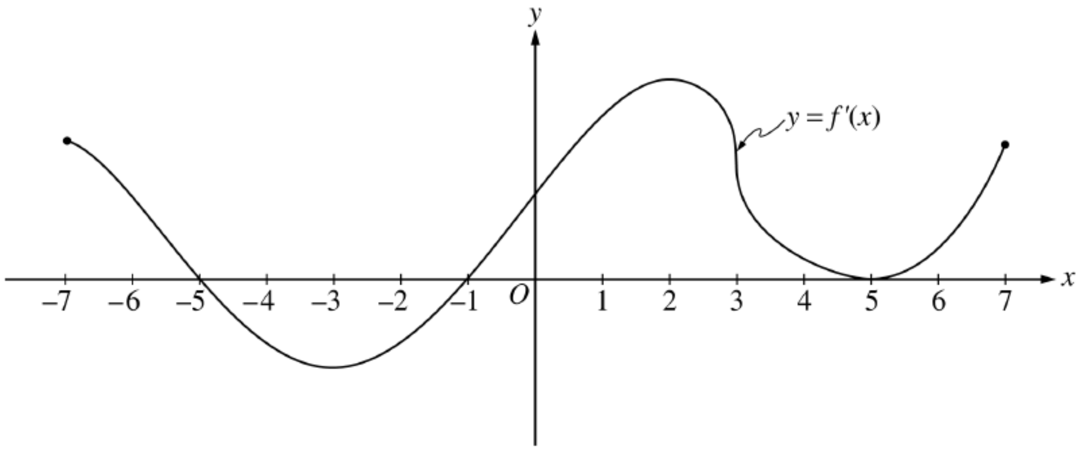
a) Find the x –coordinate of each of the points of inflection of the graph of $f(x)$. Give a reason for your answer.

b) Is $f(4) < 6$? Justify your response.

c) At what value of x does f attain its minimum value on the closed interval $-1 \leq x \leq 5$? Show the analysis that leads to your answer.

d) Let g be the function defined by $g(x) = xf(-2x) + \ln\left(f'(-3x)\right)$. Find an equation for the line tangent to the graph of g at $x = -1$.

Ex. 5:



The figure above shows the graph of f', the derivative of the function f, for $-7 \le x \le 7$. The graph of f' has horizontal tangent lines at $x = -3$, $x = 2$ and $x = 5$, and a vertical tangent line at $x = 3$.

a) Find all values of x, for $-7 < x < 7$, at which f attains a relative min. Justify your answer.

b) Find all values of x, for $-7 < x < 7$, at which f attains a relative max. Justify your answer.

c) Find all values of x, for $-7 < x < 7$, at which $f''(x) < 0$.

d) At which value of x, for $-7 \le x \le 7$, does f attains its absolute maximum. Justify your answer.

Ex. 6: The function f is continuous on the closed interval $[-3,3]$ such that $f(-3) = 4$ and $f(3) = 1$. The functions f' and f'' have the properties given in the table below.

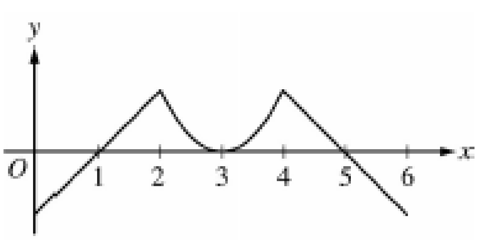
x	$-3 < x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x < 3$
$f'(x)$	Positive	Fails to Exist	Negative	0	Negative
$f''(x)$	Positive	Fails to Exist	Positive	0	Negative

a) At what value of x does f attain its relative minimum and maximum value, if any?

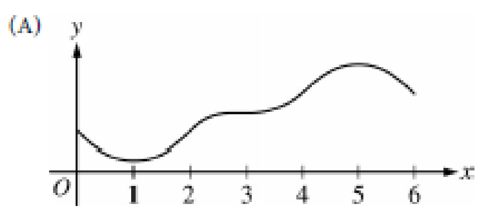
b) Is f guaranteed a value of c, where $-3 \leq c \leq 3$, such that $f(c) \geq f(x)$? If so, find the value of $x = c$, if possible. Show the analysis that leads to your answer.

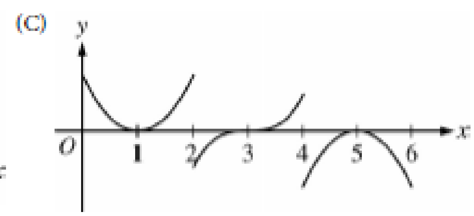
c) Find the x —coordinate of each point of inflection of the graph of f. Justify your answer.

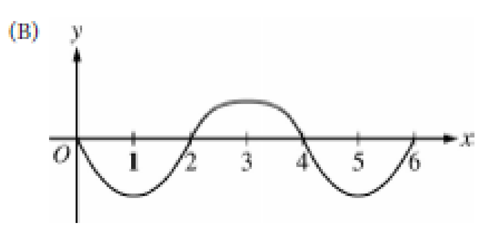
Ex. 7: The graph of f' is shown below. Which of the following could be the graph of f?

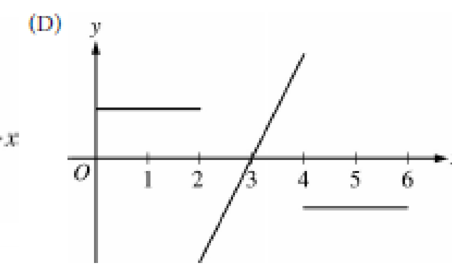


Graph of f'

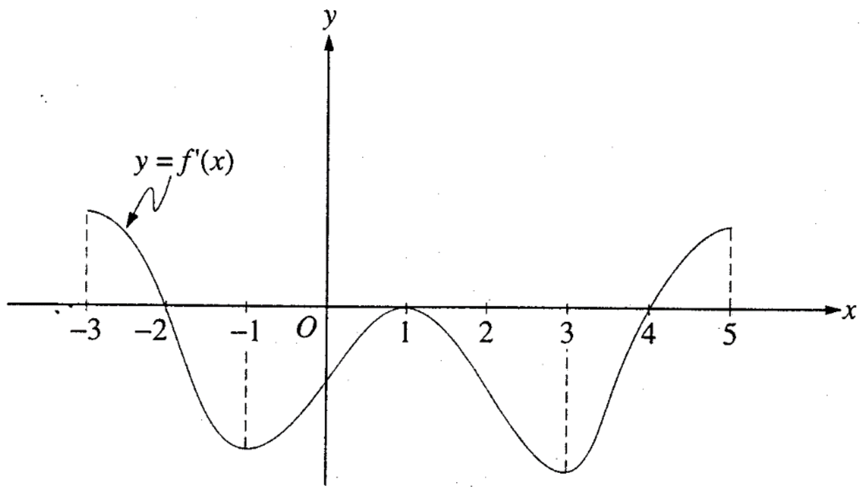








Ex. 8:



The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $-3 < x < 5$.

a) For what values of x does f have a critical value on $-3 < x < 5$? Determine whether each is a relative maximum, relative minimum or neither. Justify your answer.

b) On what intervals is the graph of f concave upward? Justify your answer.

c) Given $f(4) = \pi$ and $g(x) = -2e^{f(x)}$, write the equation of the tangent line to g at $x = 4$.

e) Given g as defined above, for what value(s) of x does g have a relative maximum? Justify your answer.

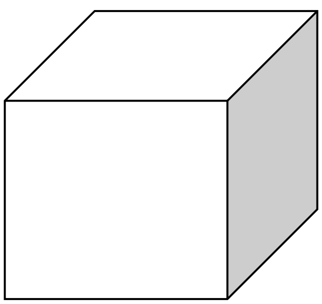
AP Calculus I
Notes 4.7
Optimization

One of the most common applications of calculus involves determining maximum and minimum values.

Problem-Solving Strategy for Applied Minimum and Maximum Problems

1) Assign symbols to all *given* quantities and quantities *to be determined*. When feasible, make a sketch.
2) Write a **primary equation** for the quantity that is to be maximized (or minimized).
3) Reduce the primary equation to one having a *single independent variable*. This may involve the use of **secondary equations** relating the independent variables of the primary equation.
4) Differentiate the equation and then solve for the *critical values*
5) Determine the desired maximum or minimum value depending on the situation.

Ex. 1: A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?



$SA = x^2 + 4xh$

Ex. 2: A rectangular field, bounded on one side by a building, is to be fenced in on the other three sides. If 3000 feet of fence is to be used, find the dimensions of the largest field that can be fenced in.

Ex. 3: A rectangular page is to contain 54 square inches of print. The margins at the top and bottom of the page are to be 1.5 inches, and the margins on the left and right are to be 1 inch. What should the dimensions of the page be so that the least amount of paper is used?

Ex. 4: The product of two positive numbers is 200. Minimize the sum of the first and twice the second.

Ex. 5: What is the radius of a cylindrical soda can with volume of 512 cubic inches that will use the minimum material? (Formulas for a cylinder: $V = \pi r^2 h$ and $SA = 2\pi r^2 + 2\pi rh$)

Ex. 6: Max wants to make a box with no lid from a rectangular sheet of cardboard that is 18 inches by 24 inches. The box is to be made by cutting a square of side x from each corner of the sheet and folding up the sides. Find the value of x that maximizes the volume of the box.

Ex. 7: A Norman window has the shape of a rectangle with a semicircle on top. If the perimeter of the window is 30 feet, find the dimensions of the window that allows the greatest amount of light in.

1. The product of two positive numbers is 200. Minimize the sum of the first and three times the second.

2. A printed page must contain 240 square centimeters of printed material. There are to be margins of 5 centimeters on either side, and margins of 3 centimeters each on the top and the bottom. How long should the printed lines be (horizontally) in order to minimize the amount of paper used?

3. A closed box with a square base is to hold 252 cubic feet. The bottom costs $5 per square foot, the top costs $2 per square foot, and the sides cost $3 per square foot. Find the dimensions that will minimize the cost.

4. A cylindrical can, closed at the top, is to hold 500 cm^3 of liquid. Find the height and radius that minimize the amount of material needed to manufacture the can.

5. A farmer with 750 feet of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible area of the four pens?

6. An open box is formed is formed by cutting squares of equal size from the corners of a 24 by 15-inch piece of metal and folding up the sides. Determine the size of the cutout that maximizes the volume of the box.

7. A computer company determines that its profit equation (in millions of dollars) is given by $P = x^3 - 48x^2 + 720x - 1000$, where x is the number of thousands of units of software sold and $0 \leq x \leq 20$. Maximize the manufacturer's profit.