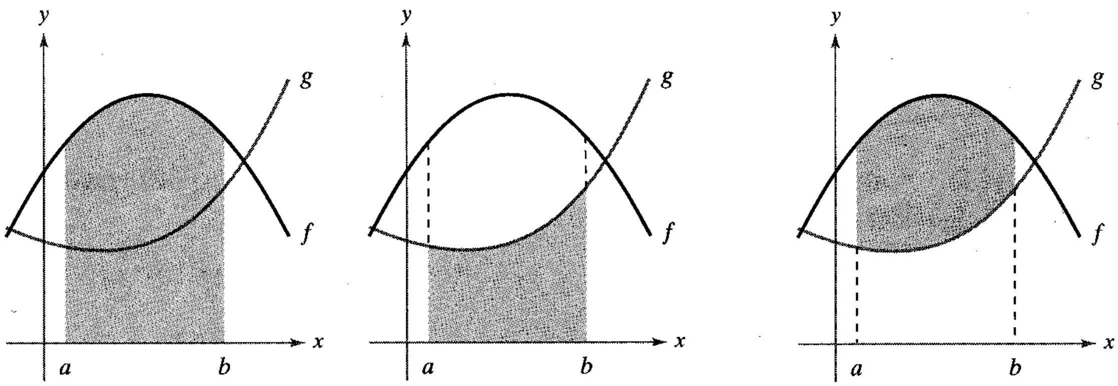
AP Calculus I
Notes 7.1
Area of a Region Between Two Curves

We are able to extend the application of definite integrals from the area *under* a curve to the area of a region *between* two curves. Consider the following graphs:



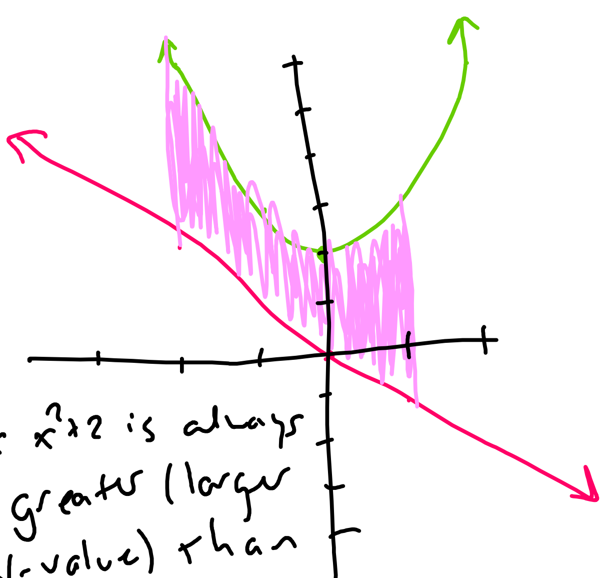
Area = $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \boxed{\int_a^b [f(x) - g(x)]\,dx}$

Area of a Region Between Two Curves

If f and g are continuous on $[a, b]$ and $g(x) \le f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

Area = $\int_a^b [f(x) - g(x)]\,dx$ * provided $f(x) \ge g(x)$ for all x from $[a,b]$

Ex. 1: Find the area of the region bounded by the graphs of $y = x^2 + 2$, $y = -x$, $x = 1$, and $x = -2$.



* x^2+2 is always greater (larger y-value) than $-x$ *

$A = \int_{-2}^{1} [(x^2+2) - (-x)]\,dx$

$= \int_{-2}^{1} (x^2 + x + 2)\,dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x \Big|_{-2}^{1}$

$= \left(\frac{1}{3} + \frac{1}{2} + 2\right) + \left(+\frac{8}{3} - 2 + 4\right)$

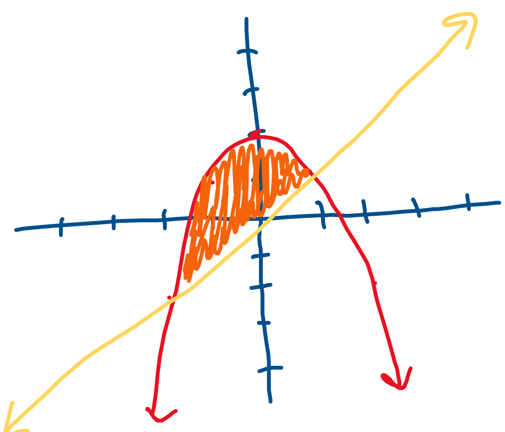
$= \frac{17}{6} + \frac{14}{3} = \frac{45}{6} = \boxed{15/2}$

Area of a Region Between Intersecting Curves

A more common problem involves the area of a region bounded by two intersecting graphs, where the values of a and b must be calculated.

Ex. 2: Find the area of a region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

Quick (inaccurate) sketch



Looks like intersects at $x = -2$ & 1, but can't eyeball it

$f = g$ would give intersection

$2 - x^2 = x$, $0 = x^2 + x - 2$

$0 = (x+2)(x-1)$ $\boxed{x = 1, -2}$ bounds!

$A = \int_{-2}^{1} [\text{larger} - \text{smaller}]\,dx$

how to know which is larger? Use x-value in between! $f(0) = 2 \leftarrow L$, $g(0) = 0 \leftarrow S$

$= \int_{-2}^{1} [2 - x^2 - x]\,dx$

$\boxed{A = 4.5}$

Ex. 3: Find the area of a region bounded by the graphs of $f(x) = \frac{-6}{2x-4}$ and $g(x) = x - 6$.

No need to sketch anymore!

Bounds $\rightarrow$ $f = g$ $\rightarrow$ $\frac{-6}{2x-4} = x - 6$, $-6 = 2x^2 - 16x + 24$

$0 = 2x^2 - 16x + 30$

$0 = 2(x-5)(x-3)$

$\boxed{x = 3, 5}$

Integrand $\rightarrow$ $f(4) = \frac{-6}{4} = -1.5$ ☆ Large

$g(4) = 4 - 6 = -2$ Small

$\text{Area} = \int_3^5 \left[\frac{-6}{2x-4} - (x-6)\right]dx = -\frac{1}{2}\int_3^5 \frac{2}{2x-4}\,dx + \int_3^5 (-x+6)\,dx$

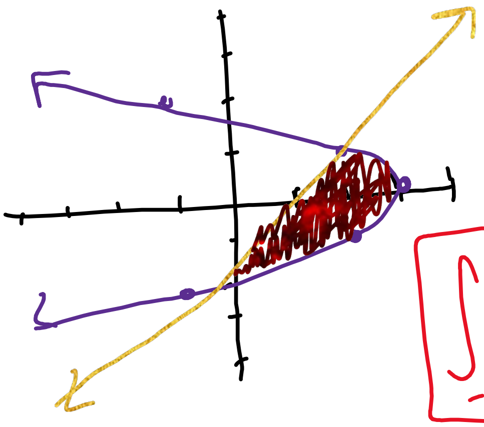
$u = 2x - 4$ $du = 2\,dx$ no u-sub!

$= -3\ln|2x-4| - \frac{1}{2}x^2 + 6x \Big|_3^5 = \left(-3\ln 6 - \frac{25}{2} + 30\right) + \left(+3\ln 2 + \frac{9}{2} - 18\right)$

$= 3\ln\left(\frac{1}{3}\right) - 8 + 12 = \boxed{4 - 3\ln 3}$

x	-1	2	3	2	-1
y	-2	-1	0	1	2

Ex. 4: Setup the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.



no longer want to go larger y − smaller y

Now we want x!!

$3 - y^2 = y + 1$, $y^2 + y - 2 = 0$, $(y+2)(y-1) = 0$

$y = 1, -2$

$$\boxed{\int_{-2}^{1} \left[(3 - y^2) - (y+1)\right] dy}$$

$x_1(0) = 3$ ✩

$x_2(0) = 1$

Ex. 5: Setup the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.

$3x^3 - x^2 - 10x = -x^2 + 2x$

$3x^3 - 12x = 0$, $3x(x^2 - 4) = 0$ $\quad x = 0, \pm 2 \leftarrow$ 3 intersections!

$x = -2$ to $x = 0$

$f(-1) = -3 - 1 + 10 = 6$ ✩

$g(-1) = -1 - 2 = -3$

$+$

$x = 0$ to $x = 2$

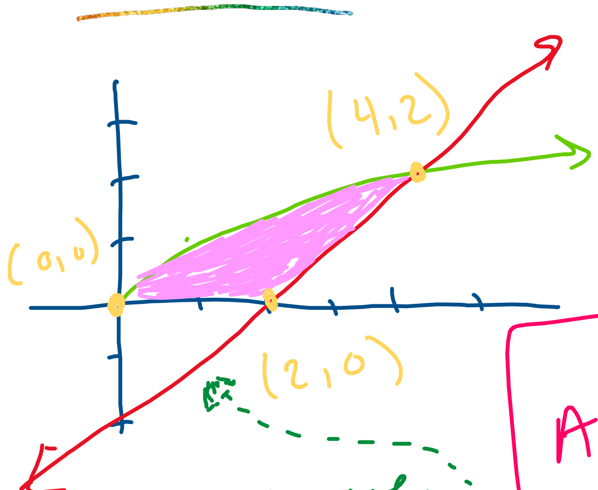
$f(1) = 3 - 1 - 10 = -8$

$g(1) = -1 + 2 = 1$ ✩

$$\boxed{\text{Area} = \int_{-2}^{0} \left[f(x) - g(x)\right] dx + \int_{0}^{2} \left[g(x) - f(x)\right] dx}$$

Ex. 6: Write an expression involving one or more integrals that represents the area of a region in the first quadrant bounded above by $f(x) = \sqrt{x}$ and below by the line $g(x) = x - 2$ and the x −axis.

Sketch!



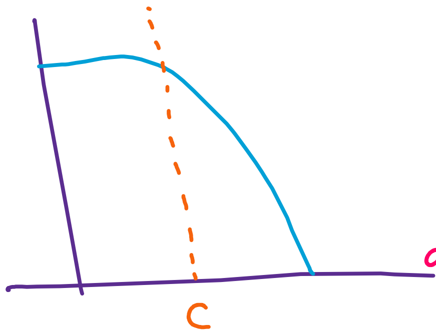
Region isn't always made up of f being greater and g being lower! There are 2 Lowers!!

$$\boxed{A = \int_0^2 \left[f(x) - 0\right] dx + \int_2^4 \left[f(x) - g(x)\right] dx}$$

x-axis is lower from [0, 2]

* could also have $\int_0^4 [f - g] dx$ − Area of small white triangle

Ex. 7: The region R, bounded by the curve $g(x) = \cos x$, the x –axis and the vertical lines $x = 0$ and $x = c$, changes as c increases. If c increases at a rate of π units per second, how fast is the area of region R increasing when $c = \frac{\pi}{6}$?



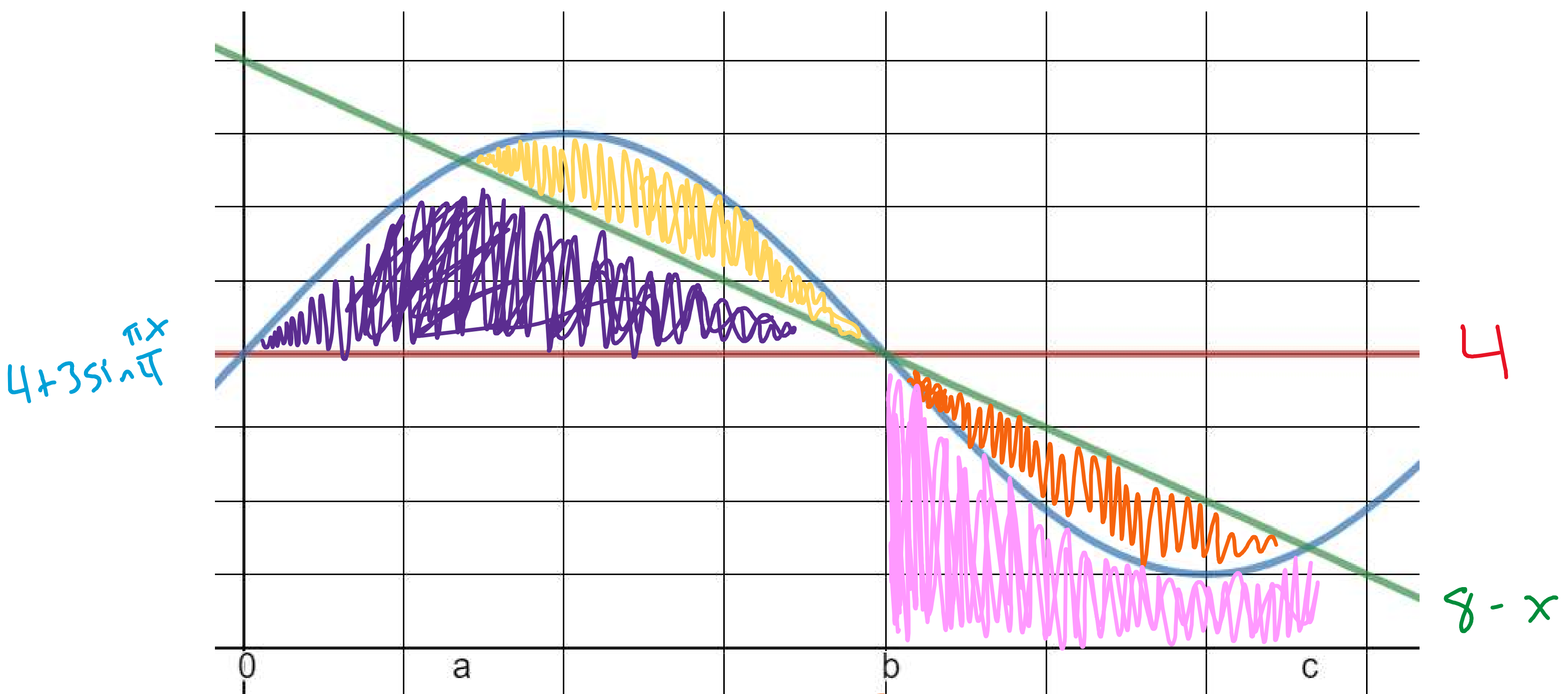
$\frac{dc}{dt} = \pi$ $\frac{dA}{dt} = ??$ when $c = \pi/6$

we need an area equation.... it's not a geometric shape, but now we know area!

$A = \int_0^c \cos x \, dx \rightarrow \frac{dA}{dt} = \cos c \cdot \frac{dc}{dt} - \cancel{\cos 0 \cdot 0}$ 2nd FTC!!

$\frac{dA}{dt} = \cos \frac{\pi}{6} \cdot \pi = \boxed{\frac{\pi\sqrt{3}}{2}}$

Ex. 8: Label the different regions found in the graph for the following integral expressions:



I. $\int_a^b \left(\left(4 + 3\sin\left(\frac{\pi x}{4}\right)\right) - (8 - x)\right) dx$

II. $\int_b^c \left((8 - x) - \left(4 + 3\sin\left(\frac{\pi x}{4}\right)\right)\right) dx$

III. $\int_0^a \left(\left(4 + 3\sin\left(\frac{\pi x}{4}\right)\right) - 4\right) dx + \int_a^b ((8 - x) - 4) dx$

IV. $\int_b^c \left(4 + 3\sin\left(\frac{\pi x}{4}\right)\right) dx$

AP Calculus I
Notes 7.2
Volume: The Disc Method

Area is only one of many applications of the definite integral. Another important application is its use of finding the volume of three-dimensional solids. We begin with solids of revolution. Such solids are commonly used in engineering and manufacturing. (axles, funnels, bottles, etc.)

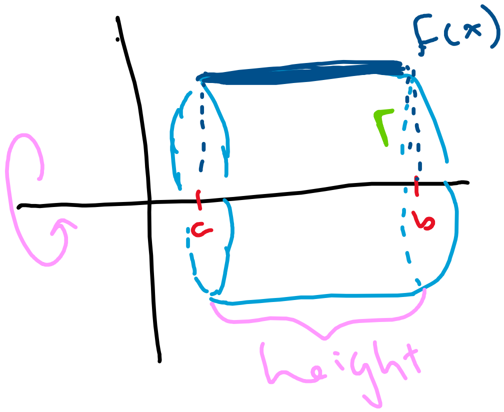
If a region in a plane is revolved about a line, the resulting solid is a **solid of revolution** and the line is called the **axis of revolution**. The simplest such solid is formed by revolving a rectangle about an axis adjacent to one side of the rectangle, creating a cylinder / disk.

Volume = $\pi r^2 h$

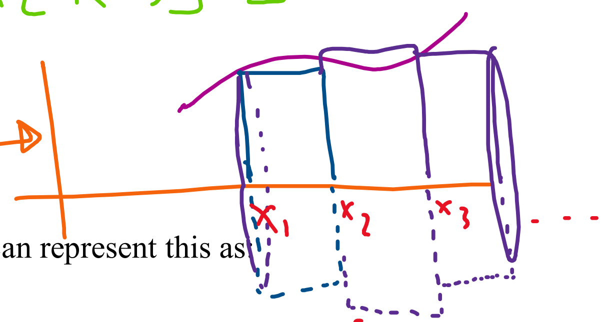
radius = $R(x)$ ← distance from curve to axis of revolution

height = $b - a = \Delta x$

$V = \pi [R(x)]^2 \Delta x$



For a curve →



Since we add up the numerous rectangles being revolved, we can represent this as

$$V \approx \sum_{k=1}^{n} \pi [R(x_k)]^2 \Delta x$$

we want infinitely many infinitesimally thin rectangles

As $\Delta x \to 0$, we know that Δx approaches dx. Therefore, we can represent the volume of a solid as:

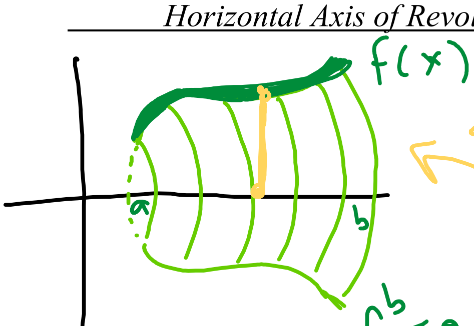
$$V = \int_a^b \pi [R(x)]^2 \, dx$$

$R(x)$ is length of each radius (curve − axis of revolution)

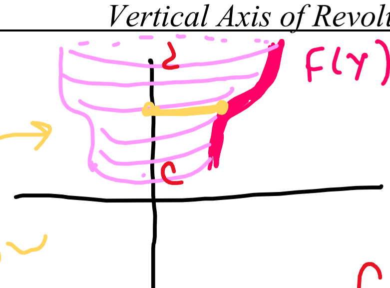
The Disc Method

To find the volume of a solid of revolution with the **disc method**, use one of the following:

Horizontal Axis of Revolution | *Vertical Axis of Revolution*



sample radii in yellow



$$V = \pi \int_a^b [R(x)]^2 \, dx$$

$$V = \pi \int_c^d [R(y)]^2 \, dy$$

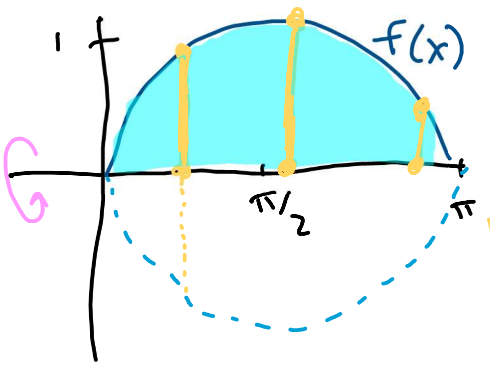
Ex. 1: Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = \sqrt{\sin x}$ and the x −axis $(0 \le x \le \pi)$ about the x −axis.

States in terms of x!!

bounds are x's!

Radius is in terms of x!

Use dx!



$$V = \pi \int_a^b [R(x)]^2\,dx$$

$$V = \pi \int_0^\pi (\sqrt{\sin x} - 0)^2\,dx$$

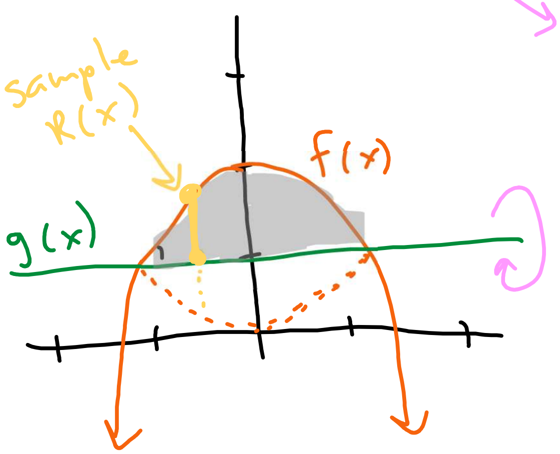
$$= \pi \int_0^\pi \sin x\,dx = \pi\left[-\cos x \Big|_0^\pi\right]$$

$$= \pi(1 - -1) = \boxed{2\pi}$$

3 sample radii are shown. They have different lengths, but each are calculated by the distance from the curve $f(x)$ and x-axis $R(x) = \sqrt{\sin x} - 0$

Ex. 2: Find the volume of the solid formed by revolving the region bounded by $f(x) = 2 - x^2$ and $g(x) = 1$ about the line $y = 1$.

since $y=1$ is a horizontal line (like the x-axis) this is in terms of x!!



$$V = \pi \int_a^b [R(x)]^2\,dx$$

$2 - x^2 = 1$

$1 = x^2$

$x = \pm 1$

$$= \pi \int_{-1}^{1} (1 - x^2)^2\,dx$$

$u = 1 - x^2$

$du = -2x\,dx$

NOPE!

$$= \pi \int_{-1}^{1} (1 - 2x^2 + x^4)\,dx$$

$$= \pi\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \Big|_{-1}^{1}\right]$$

$$= \pi\left[\left(1 - \frac{2}{3} + \frac{1}{5}\right) - \left(-1 + \frac{2}{3} - \frac{1}{5}\right)\right]$$

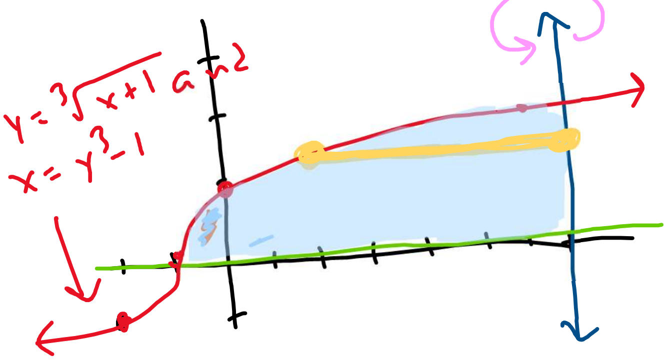
$$= \pi\left[\frac{8}{15} - -\frac{8}{15}\right] = \boxed{\frac{16\pi}{15}}$$

$R(x)$ is length of each radius. So it is the distance from $f(x)$ to the axis of revolution $g(x)$

$R(x) = 2 - x^2 - 1$

$= 1 - x^2$

Ex. 3: Find the volume of the solid formed by revolving the region bounded from above by $y = \sqrt[3]{x+1}$, from the right by $x = 7$ and below by $y = 0$ about the line $x = 7$.



since vertical like y-axis → terms of y!!

$$V = \pi \int_c^d [R(y)]^2 \, dy$$

bounds are y's
function is y's

* bounds aren't $x=-1$ to $x=7$.....
they are $y=0$ to $y=$ ___

intersection of $x=7$ and $y=\sqrt[3]{x+1}$

$y = \sqrt[3]{7+1} = 2$

$R(y) =$ curve − axis

$R(y) = \sqrt[3]{x+1} - 7$ wait.... in terms of x

$y = \sqrt[3]{x+1}$

$y^3 = x+1$

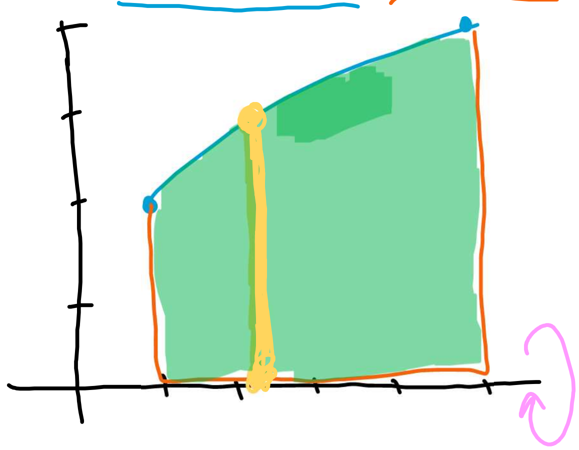
$x = y^3 - 1$

$R(y) = y^3 - 1 - 7$

$\boxed{R(y) = y^3 - 8}$

$$\boxed{V = \pi \int_0^2 (y^3-8)^2 \, dy = \frac{576\pi}{7}}$$

Ex. 4: Setup the volume of the solid formed by revolving the region bounded by the graph of $y = \sqrt{2x-1} + 1$, $x = 1$, $x = 5$ and $y = 0$ about the x –axis. in terms of x!



$$V = \pi \int_a^b [R(x)]^2 \, dx$$

$$= \boxed{\pi \int_1^5 \left(\sqrt{2x-1}+1\right)^2 dx}$$

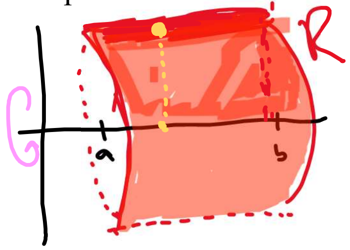
$R(x) = \sqrt{2x-1} + 1 - 0$

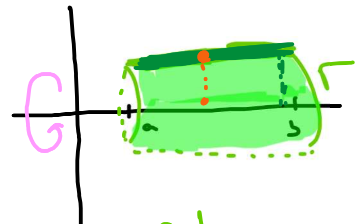
$= \sqrt{2x-1} + 1$

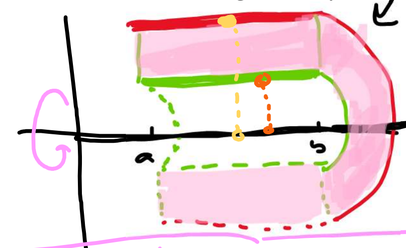
The Washer Method → difference of 2 disk methods

Should revolve all the way around

The disc method can be extended to cover solids of revolution with holes by replacing the representative disc with a representative **washer**. If $\boldsymbol{r}$ is the inner radius and $\boldsymbol{R}$ is the outer radius, then the volume is given by:



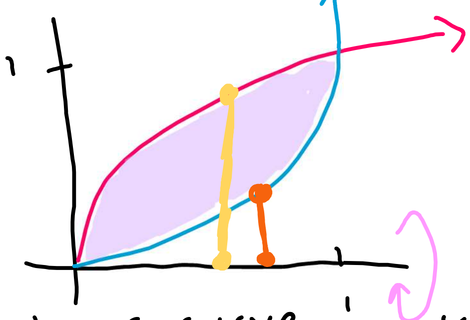




$$\pi\int_a^b R(x)^2\,dx \;-\; \pi\int_a^b r(x)^2\,dx = \boxed{\pi\int_a^b \left[R(x)^2 - r(x)^2\right]dx}$$

Ex. 5: Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x −axis. → Means in terms of x!!

Since the region isn't flush up against the axis of revolution, there will be space removed from the solid. This is washer!



Radius = curve − axis still

Outer = Further from axis = $R(x) = \sqrt{x} - 0$

Inner = Closer to axis = $r(x) = x^2 - 0$

(notice this doesn't span the region)

$$V = \pi\int_0^1 \left[(\sqrt{x})^2 - (x^2)^2\right]dx$$

$$= \pi\int_0^1 (x - x^4)\,dx$$

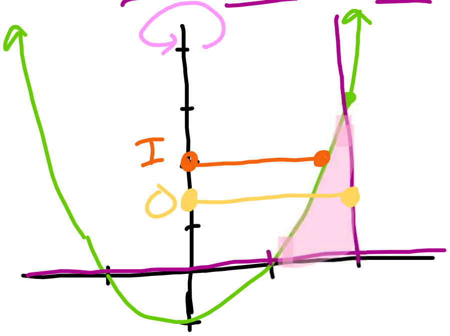
$$= \pi\left[\tfrac{1}{2}x^2 - \tfrac{1}{5}x^5 \Big|_0^1\right] = \pi\left(\tfrac{1}{2} - \tfrac{1}{5}\right) = \boxed{\tfrac{3}{10}\pi}$$

Integrating with Respect to y

Ex. 6: Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 - 1$, $y = 0$, $x = 1$, and $x = 2$ about the y −axis. in terms of y!!

$$y = x^2 - 1 \rightarrow y + 1 = x^2 \rightarrow x = \cancel{\pm}\sqrt{y+1}$$

positive x side



$$V = \pi\int_0^3 \left[(2)^2 - (\sqrt{y+1})^2\right]dy$$

* bounds are y's!! *

$$= \pi\int_0^3 (4 - (y+1))\,dy = \pi\int_0^3 (3 - y)\,dy$$

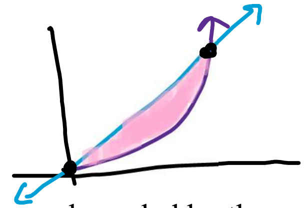
$$= \pi\left[3y - \tfrac{1}{2}y^2 \Big|_0^3\right]$$

$$= \pi\left[(9 - 9/2) - (0 - 0)\right] = \boxed{\frac{9\pi}{2}}$$

To help know which is Inner, draw out from axis First part of the region you touch is Inner!

O: curve − axis → $2 - 0 = 2$

I: curve − axis → $\sqrt{y+1} - 0 = \sqrt{y+1}$



$x^2 = 4x$ $\quad (0,0)$

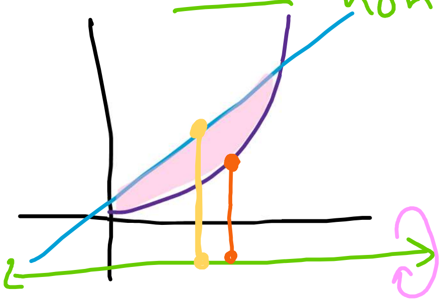
$x^2 - 4x = 0$ $\quad (4,16)$

$x(x-4) = 0$

Volumes when Not Revolved About the Axes

Ex. 7: Find the volume of the solid that results when the area bounded by the curve $y = x^2$ and the curve $y = 4x$ is revolved about the following lines:

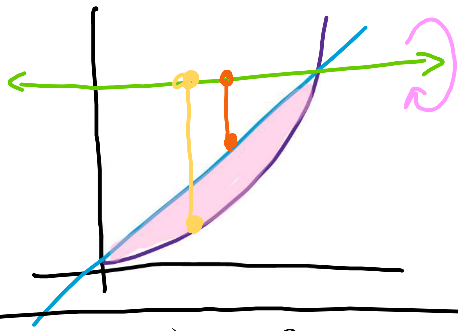
a) $y = -2$ horizontal like x-axis → in terms of x!!



Outer: $R(x) = 4x - {-2}$ Inner: $r(x) = x^2 - {-2}$

$$V = \pi \int_0^4 \left[(4x+2)^2 - (x^2+2)^2 \right] dx = \boxed{562.973}$$

b) $y = 16$ in terms of x!!

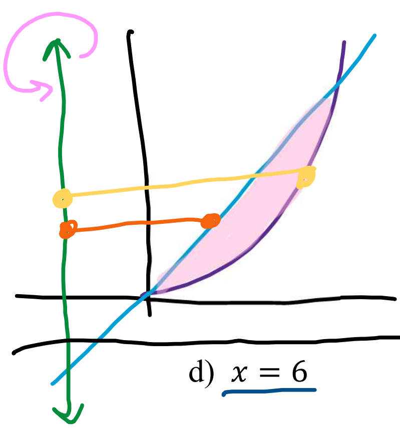


Outer: $R(x) = x^2 - 16$ Inner: $r(x) = 4x - 16$

(Notice function used for R switched in a) & b))

$$V = \pi \int_0^4 \left[(x^2-16)^2 - (4x-16)^2 \right] dx = \boxed{643.398}$$

c) $x = -3$ vertical like y-axis, so in terms of y!!

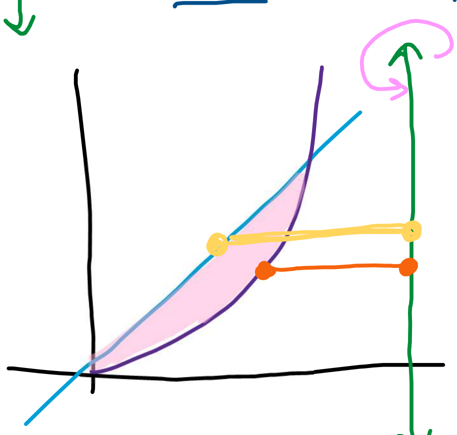


$y = x^2 \to x = \sqrt{y}$ $\quad$ $y = 4x \to x = \frac{1}{4}y$

Outer: $R(y) = \sqrt{y} - {-3}$ Inner: $r(y) = \frac{1}{4}y - {-3}$

$$V = \pi \int_0^{16} \left[(\sqrt{y}+3)^2 - (\tfrac{1}{4}y+3)^2 \right] dy = \boxed{335.103}$$

d) $x = 6$ in terms of y!! (use y-values for bounds!)



Outer: $R(y) = \frac{1}{4}y - 6$ Inner: $r(y) = \sqrt{y} - 6$

$$V = \pi \int_0^{16} \left[(\tfrac{1}{4}y-6)^2 - (\sqrt{y}-6)^2 \right] dy$$

$$= \boxed{268.083}$$

Ex. 8: Let R be the region in the first and second quadrants bounded above by the graph of $f(x) = \frac{12}{1+x^2}$ and below by the horizontal line $g(x) = 3$.

a) Find the area of R. (NC)

$f = g$ at $\frac{12}{1+x^2} = 3$, $4 = 1 + x^2$, $3 = x^2$

$x = \pm\sqrt{3}$

$f(0) = 12 \leftarrow$ Larger

$g(0) = 3$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \left[\frac{12}{1+x^2} - 3\right] dx$$

$$= 12\int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{1+x^2}\,dx - \int_{-\sqrt{3}}^{\sqrt{3}} 3\,dx = 12 \cdot \frac{1}{1}\arctan\frac{x}{1} - 3x \Big|_{-\sqrt{3}}^{\sqrt{3}}$$

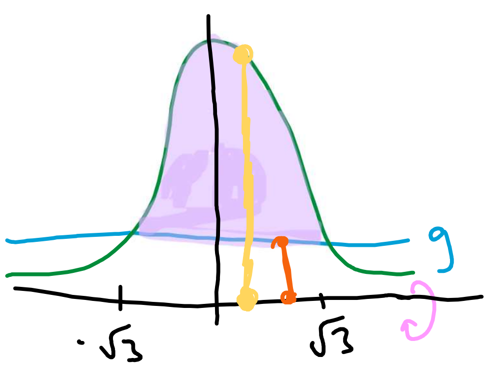
$a^2 = 1$ $u^2 = x^2$

$a = 1$ $u = x$ $du = 1dx$ ✓

$$= \boxed{(12\arctan\sqrt{3} - 3\sqrt{3}) - (12\arctan -\sqrt{3} + 3\sqrt{3})}$$

b) Find the volume of the solid generated when R is rotated about the x −axis. (CA)

in terms of x!!



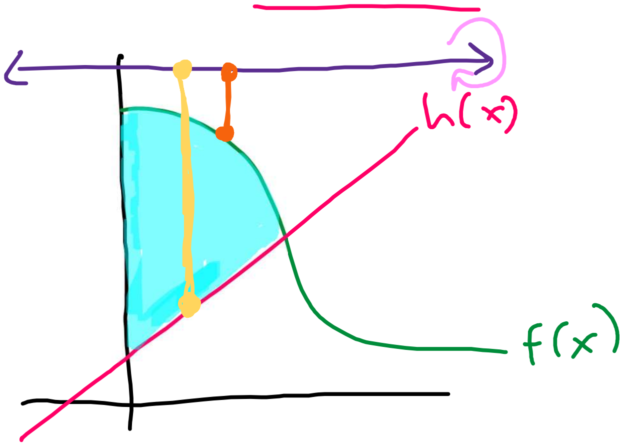
Outer: $R(x) = f(x) - 0$

Inner: $r(x) = g(x) - 0$

$$V = \pi\int_{-\sqrt{3}}^{\sqrt{3}} \left[(f(x))^2 - (g(x))^2\right] dx = \boxed{571.686}$$

c) Let S be the region in the first quadrant bounded above by $f(x) = \frac{12}{1+x^2}$ and below by $h(x) = 3 + 3x$. Find the volume of the solid generated when S is rotated about $y = 15$. (CA)

like x-axis!



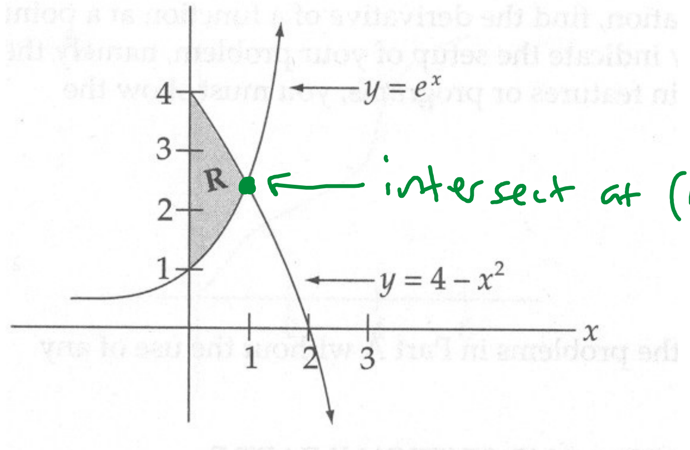
$f = g$ at $x = 1$

Outer: $R(x) = h(x) - 15$

Inner: $r(x) = f(x) - 15$

$$V = \pi\int_0^1 \left[(h(x) - 15)^2 - (f(x) - 15)^2\right] dx = \boxed{239.373}$$

Ex. 9: Let R be the region in the first quadrant shown in the figure below. Calc Active

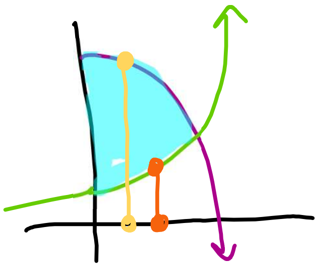


intersect at (1.0580064, 2.8806225)

a) Find the area of R.

$$A = \int_0^{1.0580064} (4 - x^2 - e^x)\,dx = \boxed{1.957}$$

b) Find the volume of the solid generated when R is revolved about the x −axis.

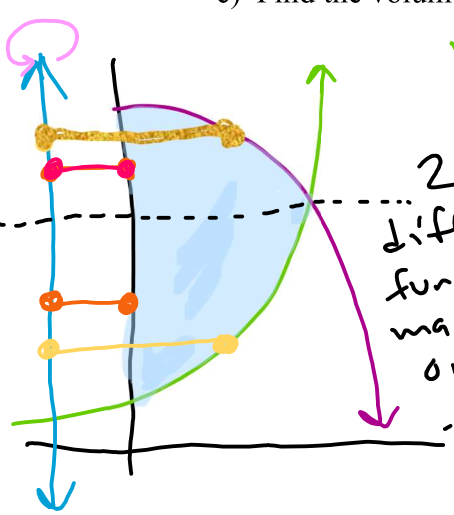


Outer: $R(x) = 4 - x^2 - 0$

Inner: $r(x) = e^x - 0$

$$V = \pi \int_0^{1.0580064} \left[(4 - x^2)^2 - (e^x)^2\right] dx = \boxed{32.629}$$

c) Find the volume of the solid generated when R is revolved about the line $x = -3$. in terms of y



2 different functions make up outside! ∴ 2 integrals!

$y = e^x \rightarrow x = \ln y$ | $y = 4 - x^2 \rightarrow x = \sqrt{4 - y}$

$y = 1 \rightarrow y = 2.8806225$ | $y = 2.8806225 \rightarrow y = 4$

$R(x) = \ln y - -3$ | $R(x) = \sqrt{4 - y} - -3$

$r(x) = 0 - -3$ | $r(x) = 0 - -3$

$$V = \pi \int_1^{2.8806225} \left[(\ln y + 3)^2 - (3)^2\right] dy + \pi \int_{2.8806225}^{4} \left[(\sqrt{4 - y} + 3)^2 - (3)^2\right] dy = \boxed{41.647}$$

Volumes of Solids with Known Cross Sections

1. For cross sections of area $A(x)$ taken perpendicular to the x –axis, $V = \int_a^b A(x)dx$.
2. For cross sections of area $A(y)$ taken perpendicular to the y –axis, $V = \int_c^d A(y)dy$.

On the AP Exam, cross-sections will be squares, equilateral triangles, rectangles, semicircles, or a new function altogether.

Formulas to Know: * side is distance from curve to curve*

Area of an equilateral triangle: $A = (side)^2 \frac{\sqrt{3}}{4}$

Area of a square: $A = (side)^2$

Area of a semicircle: $A = \frac{1}{2}\pi\left(\frac{side}{2}\right)^2 = \frac{\pi}{8}(side)^2$

Area of a rectangle: $A = (side)(height)$

Area of an isosceles right triangle with leg on region: $A = \frac{1}{2}(side)^2$

Cross Sections Connection to Disc Method:

Disk Method: $V = \int_a^b \pi[R(x)]^2 dx$

Cross Section: $V = \int_a^b A(x)\,dx$

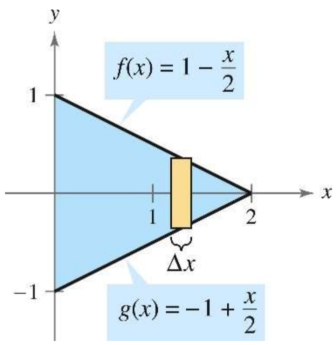
In Disk method, each cross section is a circle (slice through the cylinder and look from side... it's a circle).

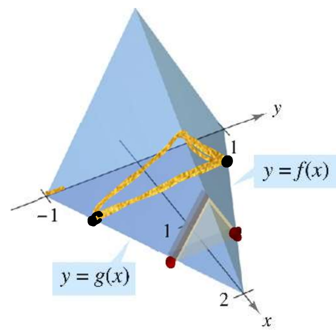
So Area of circle is πr^2. In Disk method, radius is distance from curve to axis, so

$V = \int_a^b A(x)dx = \int_a^b \pi[R(x)]^2 dx$ ← Disk ✓

Ex. 10: Find the volume of the solid whose base is the region bounded by the lines $f(x) = 1 - \frac{x}{2}$, $g(x) = -1 + \frac{x}{2}$, and $x = 0$. The cross sections perpendicular to the x −axis are equilateral triangles.

in terms of x!





$$V = \int_a^b A(x)\,dx = \int_a^b \frac{\sqrt{3}}{4}(\text{side})^2\,dx$$

$$V = \int_0^2 \frac{\sqrt{3}}{4}(2-x)^2\,dx \qquad u = 2-x \quad du = -1\,dx$$

side goes from f to g

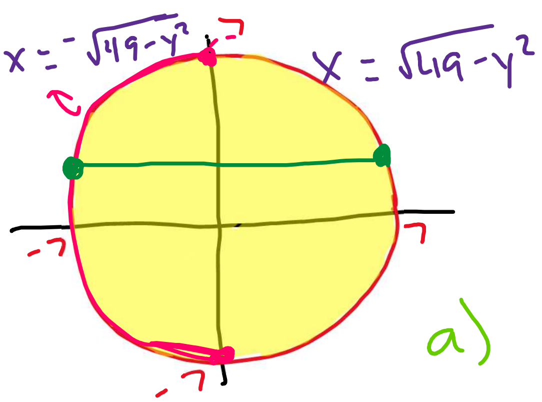
$$\text{side} = \left(1 - \frac{x}{2}\right) - \left(-1 + \frac{x}{2}\right) = 2 - x$$

$$V = -\frac{\sqrt{3}}{4}\int_2^0 u^2\,du = \frac{\sqrt{3}}{4}\int_0^2 u^2\,du$$

$$= \frac{\sqrt{3}}{4}\cdot\frac{1}{3}u^3\Big|_0^2 = \frac{\sqrt{3}}{12}(8) = \boxed{\frac{2\sqrt{3}}{3}}$$

Ex. 11: Find the volume of a solid whose base is the circle $x^2 + y^2 = 49$, and where cross-sections area all a) squares whose sides, b) semi-circles whose diameters, and c) rectangles whose height is always three times the side length and whose sides lie perpendicular to the y −axis on the base of the circle.

in terms of y!!



$$x^2 + y^2 = 49 \rightarrow x^2 = 49 - y^2$$

$$x = \pm\sqrt{49-y^2}$$

a) $V = \int_a^b A(y)\,dy$ area of square = $(\text{side})^2$

side = curve − curve

$$= \sqrt{49-y^2} - -\sqrt{49-y^2}$$

$$= 2\sqrt{49-y^2}$$

$$V = \int_{-7}^{7}\left(2\sqrt{49-y^2}\right)^2 dy = \boxed{1829.333}$$

b) $V = \int_a^b A(y)\,dy$ area of semicircle = $\frac{\pi}{8}(\text{side})^2$

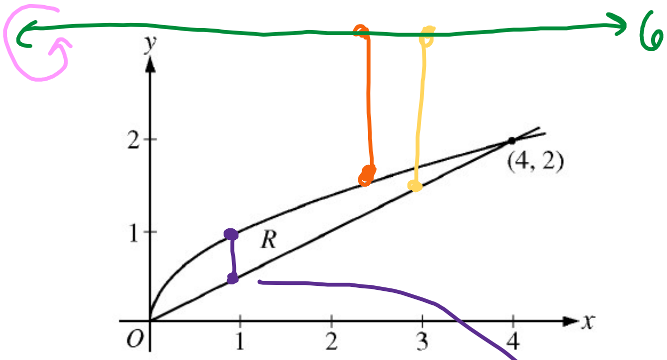
$$= \frac{\pi}{8}\int_{-7}^{7}\left(2\sqrt{49-y^2}\right)^2 dy = \boxed{718.378}$$

c) $V = \int_a^b A(y)\,dy$ area of rectangle = length · height

$$V = \int_{-7}^{7}\left(2\sqrt{49-y^2}\right)3\left(2\sqrt{49-y^2}\right)dy = \boxed{5488}$$

Ex. 12: NC

c)



$f(x) = \sqrt{x}$

$g(x) = \frac{1}{2}x$

b) side $= \sqrt{x} - \frac{1}{2}x$

a) Find the area of R.

b) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x −axis are isosceles right triangles with a leg on the region R. Find the volume of this solid.

c) Write, but do not evaluate, an integral expression for the volume of the solid when R is rotated about the horizontal line $y = 6$.

a) Area $= \int_0^4 [\sqrt{x} - \frac{1}{2}x]dx = \frac{2}{3}x^{3/2} - \frac{1}{4}x^2 \Big|_0^4 = \left(\frac{16}{3} - 4\right) - (0 - 0) = \boxed{4/3}$

b) $V = \int_0^4 A(x)\,dx = \int_0^4 \frac{1}{2}(\text{side})^2 dx = \frac{1}{2}\int_0^4 (\sqrt{x} - \frac{1}{2}x)^2 dx = \frac{1}{2}\int_0^4 [x - x^{3/2} + \frac{1}{4}x^2]dx$

$= \frac{1}{2}\left[\frac{1}{2}x^2 - \frac{2}{5}x^{5/2} + \frac{1}{12}x^3 \Big|_0^4\right] = \boxed{\frac{1}{2}\left[\left(\frac{1}{2}(4)^2 - \frac{2}{5}(4)^{5/2} + \frac{1}{12}(4)^3\right) - (0 - 0 + 0)\right]}$

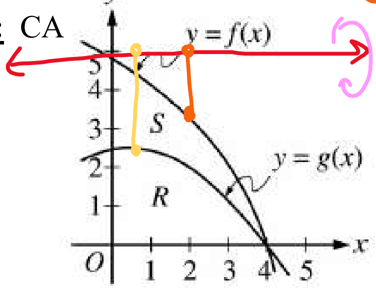
c) Outer: $R(x) = g(x) - 6$

Inner: $r(x) = f(x) - 6$

$\boxed{V = \pi \int_0^6 [(g(x) - 6)^2 - (f(x) - 6)^2]dx}$

Ex. 13: CA

B)



A) $\int_0^4 [f(x) - g(x)]dx =$

$\int_0^4 f(x)dx - \int_0^4 g(x)dx = \boxed{12.142 - 6.93865 = 5.204}$

$\int_0^4 f(x)dx = 12.142$

Let R be the region in the first quadrant bounded by the graph of g, and let S be the region in the first quadrant between the graphs of f and g, as shown in the figure above. The region in the first quadrant bounded by the graph of f and the coordinate axes has area 12.142. The function g is given by $g(x) = (\sqrt{x+6})\cos\left(\frac{\pi x}{8}\right)$, and the function f is not explicitly given. The graphs of f and g intersect at the point $(4, 0)$.

(A) Find the area of S.

(B) A solid is generated when S is revolved about the horizontal line $y = 5$. Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid.

(C) Region R is the base of an art sculpture. At all points in R at a distance x from the y-axis, the height of the sculpture is given by $h(x) = 4 - x$. Find the volume of the art sculpture.

B) $R(x) = g(x) - 5$

$r(x) = f(x) - 5$

$\boxed{V = \pi \int_0^4 [(g(x) - 5)^2 - (f(x) - 5)^2]dx}$

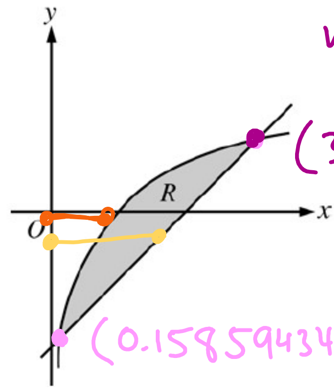
c) $\int_a^b A(x)\,dx \xrightarrow[\text{height} \cdot \text{length}]{\text{rectangle}} \int_0^4 [h(x) \cdot g(x)]dx = \boxed{17.243}$

height   length

Ex. 14: CA

a) Area $= \int_a^c [\ln x - (x-2)]\,dx$

$= \boxed{1.949}$



not π!! $\downarrow$ $(3.1461932, 1.1461932) = (c, d)$

$(0.15859434, -1.841406) = (a, b)$

Let R be the shaded region bounded by the graph of $y = \ln x$ and the line $y = x - 2$, as shown above.

a) Find the area of R.

b) Find the volume of the solid generated when R is rotated about the y −axis. in terms of y!!

c) The region R is the base of a solid. Cross sections perpendicular to the x −axis have an area modeled by $A(h) = 8\cos^2(1.4h + 2)$. Based on this model, find the volume of the solid. in terms of x!!

give us area formula $\rightarrow$

b) $y = \ln x \rightarrow x = e^y$ $\quad R(y) = y + 2 - 0$

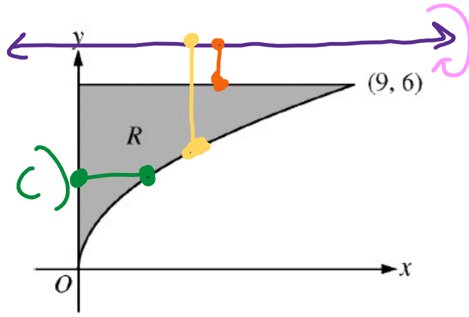
$y = x - 2 \rightarrow x = y + 2$ $\quad r(y) = e^y - 0$

$V = \pi \int_b^d [(y+2)^2 - (e^y)^2]\,dy = \boxed{17.099}$

c) $\int_a^b A(x)\,dx = \int_a^c A(h)\,dh = \boxed{13.672}$

Ex. 15: NC

b)



c)

$y = 2\sqrt{x}$, $\sqrt{x} = \frac{y}{2}$, $x = \frac{y^2}{4}$

Let R be the region in the first quadrant bounded by the graph of $y = 2\sqrt{x}$, the horizontal line $y = 6$, and the y −axis, as shown above.

a) Find the area of R.

b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the line $y = 7$. in terms of x!!

c) Region R is the base of a solid. For each y, where $0 \le y \le 6$, the cross section of the solid taken perpendicular to the y −axis is a rectangle whose height is 3 times the length of its base in region R. Write, but do not evaluate, an integral expression that gives the volume of the solid.

in terms of y!! $\leftarrow$

a) $\int_0^9 (6 - 2\sqrt{x})\,dx = 6x - \frac{4}{3}x^{3/2} \Big|_0^9 = \boxed{\left(54 - \frac{4}{3}(9)^{3/2}\right) - 0}$

b) $R(x) = 2\sqrt{x} - 7$ $\quad r(x) = 6 - 7$ $\quad \boxed{V = \pi \int_0^9 [(2\sqrt{x} - 7)^2 - (6-7)^2]\,dx}$

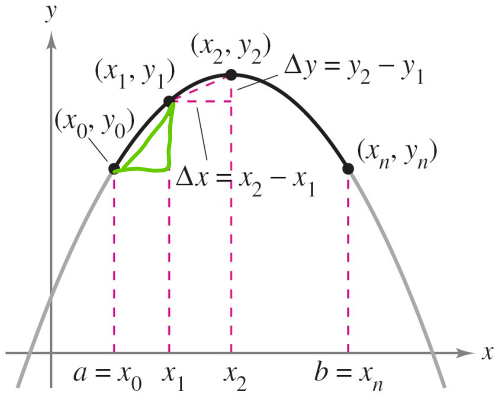
c) $V = \int_a^b A(y)\,dy \xrightarrow{\text{rectangle}} \dfrac{}{\text{height} \cdot \text{length}}$ $\quad$ side length $= \frac{y^2}{4} - 0$

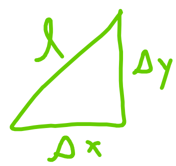
$(3 \cdot \text{length}) \cdot \text{length}$

$\boxed{V = \int_0^6 \left(\frac{3y^2}{4}\right)\left(\frac{y^2}{4}\right) dy}$

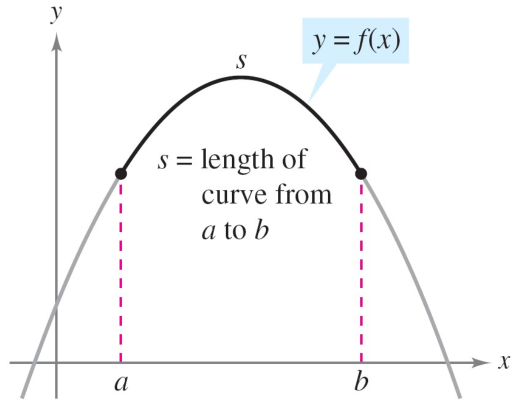
AP Calculus II
Notes 7.4
Arc Length

Definite integrals are used to find the arc lengths of curves and the areas of surfaces of revolution. In either case, an arc (a segment of a curve) is approximated by straight line segments.





$\Delta x^2 + \Delta y^2 = \ell^2$



Consider the graph $y = f(x)$ that is continuously differentiable on $[a, b]$. You can approximate the graph of f by n line segments whose endpoints are determined by the partition $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.

Now, by letting $\Delta x = x_i - x_{i-1}$ and $\Delta y = y_i - y_{i-1}$, you can approximate the length of the graph by:

Length of each hypotenuse => $\ell = \sqrt{\Delta x^2 + \Delta y^2}$

Length of curve ≈ sum of lengths of all hypotenuses

Length ≈ $\sum_{k=1}^{n} \sqrt{\Delta x^2 + \Delta y^2}$ → we know we want ∞ many hypotenuses that are thin...

Length = $\underbrace{\lim_{n\to\infty} \sum_{k=1}^{n}}_{\text{will be } \int} \sqrt{\underbrace{\Delta x^2 + \Delta y^2}_{\text{will be } dx,\ dy}}$ but no dx on outside... so let's use some algebra

Length → $\lim_{n\to\infty} \sum^{n} \sqrt{\Delta x^2 + \Delta y^2} \cdot \frac{\Delta x}{\Delta x} = \lim_{n\to\infty} \sum^{n} \frac{\sqrt{\Delta x^2 + \Delta y^2}}{\Delta x} \cdot \Delta x$

$= \lim_{n\to\infty} \sum^{n} \sqrt{\frac{\Delta x^2 + \Delta y^2}{\Delta x^2}} \cdot \Delta x = \lim_{n\to\infty} \sum^{n} \sqrt{\left(\frac{\Delta x}{\Delta x}\right)^2 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$

Theorem – Definition of Arc Length

Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The **arc length** of f between a and b is

Length $= \int_a^b \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$ ← turn $\Delta x \to dx$

Ex. 1: Find the arc length of the graph of $y = 4x^{\frac{3}{2}} + 1$ from $\underline{x=1}$ to $\underline{x=3}$.

$a \qquad b$

$$\text{Length} = \int_a^b \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx \qquad \frac{dy}{dx} = 6x^{1/2}$$

$$= \int_1^3 \sqrt{1 + (6\sqrt{x})^2}\,dx = \frac{1}{36}\int_1^3 36\sqrt{1+36x}\,dx$$

$u = 1+36x$
$du = 36\,dx$
$x=1 \to u=37$
$x=3 \to u=109$

$$= \frac{1}{36}\int_{37}^{109} \sqrt{u}\,du = \frac{1}{36}\cdot\frac{2}{3}u^{3/2}\Big|_{37}^{109}$$

$$= \boxed{\frac{1}{54}\left(109^{3/2} - 37^{3/2}\right)}$$

Ex. 2: Find the length of the curve given by $f(t) = \ln|\cos t| + 3$ from $t = 0$ to $t = \frac{\pi}{4}$.

$$\text{Length} = \int_a^b \sqrt{\left(\frac{dt}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \qquad \frac{dy}{dt} = f' = \frac{1}{\cos t}\cdot -\sin t$$

$$= \int_0^{\pi/4} \sqrt{1 + \left(\frac{-\sin t}{\cos t}\right)^2}\,dt = \int_0^{\pi/4} \sqrt{1+\tan^2 t}\,dt$$

u-sub won't work... can't "distribute" $\sqrt{\ }$ ..... trig identity!!

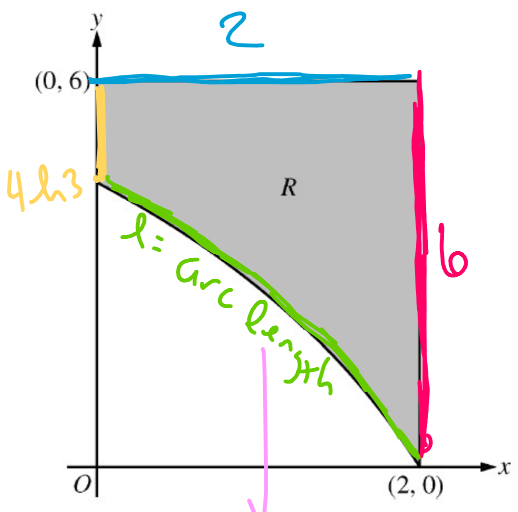
$$= \int_0^{\pi/4} \sqrt{\sec^2 t}\,dt = \int_0^{\pi/4} |\sec t|\,dt$$

$\leftarrow$ since $\sec t > 0$ for $0 \le t \le \pi/4$, drop $|\ |$

$$= \int_0^{\pi/4} \sec t\,dt = \ln|\sec t + \tan t|\Big|_0^{\pi/4}$$

$$\boxed{= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0| = \ln|\sqrt{2}+1|}$$

Ex. 3: Write integral expressions representing the area and the perimeter of the region R, bounded by the lines $y = 4\ln(3-x)$, the y –axis, $y = 6$ and $x = 2$.



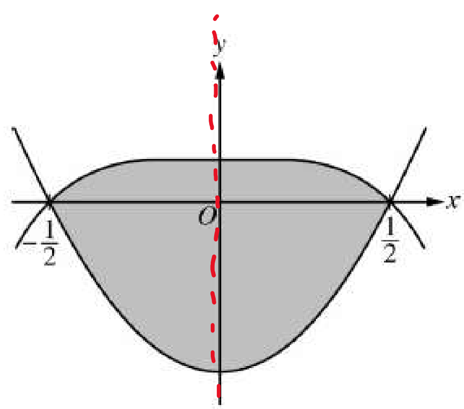
Area = $\boxed{\int_0^2 [6 - 4\ln(3-x)]\,dx}$

Perimeter = $6 + 2 + 6 - 4\ln 3 + l$

$= \boxed{14 - 4\ln 3 + \int_0^2 \sqrt{1 + \left(\frac{-4}{3-x}\right)^2}\,dx}$

$\frac{dy}{dx} = 4 \cdot \frac{1}{3-x} \cdot -1$

Ex. 4:



symmetry!

use $2 \cdot [x = 0 \text{ to } x = \tfrac{1}{2}]$
instead of $[x = -\tfrac{1}{2} \text{ to } x = \tfrac{1}{2}]$

length $= \int_a^b \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx$

The shaded region in the figure above is bounded by the graphs of $y = x^2 - \frac{1}{4}$ and $y = \frac{1}{16} - x^4$ for $-\frac{1}{2} \le x \le \frac{1}{2}$. Which of the following expressions gives the perimeter of the region?

$\frac{dy}{dx} = 2x$

$\frac{dy}{dx} = -4x^3$

(A) $2\int_0^{1/2} \sqrt{4x^2 + 16x^6}\,dx$

(B) $2\int_0^{1/2} \sqrt{1 + 4x^2 + 16x^6}\,dx$

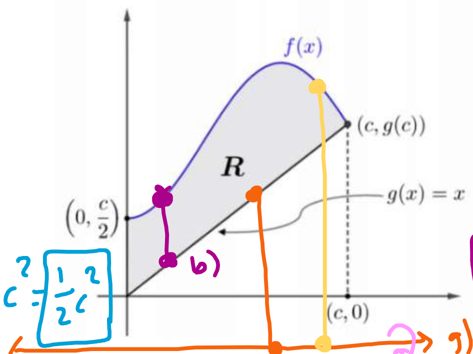
(C) $2\int_0^{1/2} \sqrt{1 + 4x^2}\,dx + 2\int_0^{1/2} \sqrt{1 + 16x^6}\,dx$

(D) $2\int_0^{1/2} \sqrt{1 + \left(x^2 - \frac{1}{4}\right)^2}\,dx + 2\int_0^{1/2} \sqrt{1 + \left(\frac{1}{16} - x^4\right)^2}\,dx$

Ex. 5:

a) Area: $\int_0^c [f(x) - x]\,dx$

$= \int_0^c f(x)\,dx - \frac{1}{2}x^2\Big|_0^c = c^2 - \frac{1}{2}c^2 = \boxed{\frac{1}{2}c^2}$



b) $V = \int_a^b A(x)\,dx$

$V = \int_0^c [\text{height} \cdot \text{length}]\,dx$

$\boxed{V = \int_0^c f'(x)\cdot(f(x) - x)\,dx}$

BC1 The functions f and g are twice differentiable for all real values of x where $g(x) = x$. A portion of the graphs of f and g are given in the figure above. It is known that the average value of f over the interval $[0, c]$ is equal to c.

$\frac{1}{c-0}\int_0^c f(x)\,dx = c$, $\int_0^c f(x)\,dx = c^2$

(a) Find the area of the region R in terms of c.

(b) The region R is the base of a solid whose cross sections perpendicular to the x-axis form rectangles with height $f'(x)$. Find the volume of this solid in terms of c. (Just setup)

(c) The function H is defined by $H(x) = 2x^2 - \int_0^{2x} f(t)\,dt$. Find $H'\left(\frac{c}{2}\right)$ and $H''\left(\frac{c}{2}\right)$ in terms of c. Does the graph of H have a relative minimum, relative maximum, or neither at $x = \frac{c}{2}$. Give a reason

(e) Write but do not evaluate an expression involving one or more integrals that gives the perimeter of R.

(g) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

c) $H'(x) = 4x - [f(2x)\cdot 2 - f(0)\cdot 0] = 4x - 2f(2x)$ * 2nd FTC!!

$H''(x) = 4 - 4f'(2x)$ → $H'(\frac{c}{2}) = 2c - 2f(c) = 2c - 2c = 0$

$H''(\frac{c}{2}) = 4 - 4f'(c)$, $f'(c) < 0$ so $4 - 4f'(c) > 0$

Since $H'(c/2) = 0$ and $H''(c/2) > 0$, H has a $\boxed{\text{rel min}}$ at $x = c/2$, because H has a critical value and is concave up

e) Perimeter $= \boxed{\frac{c}{2} + \int_0^c \sqrt{1 + (f'(x))^2}\,dx + \int_0^c \sqrt{1 + (1)^2}\,dx}$

$g'(x) = 1$

g) Outer: $R(x) = f(x) - {}^-2$ Inner: $r(x) = x - {}^-2$

$$\boxed{V = \pi \int_0^c \left[(f(x)+2)^2 - (x+2)^2\right] dx}$$

Ex. 6:

x	1	3	5	7
$f(x)$	4	6	7	5
$f'(x)$	2	1	0	-1

$$\text{Length} = \int_a^b \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx$$

The table above gives selected values for a differentiable function f and its first derivative. Using a left Riemann sum with 3 subintervals of equal length, which of the following is an approximation of the length of the graph of f on the interval $[1, 7]$?

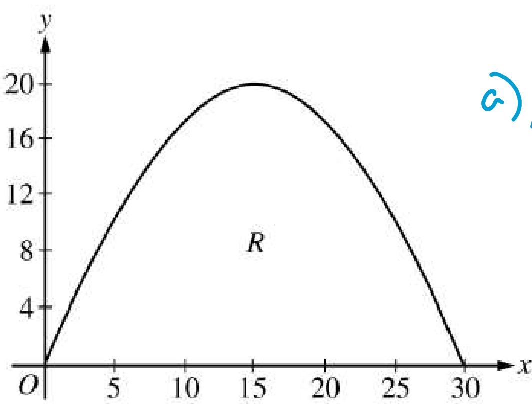
(A) 6 (B) 34 (C) $2\sqrt{3} + 2\sqrt{2} + 2$ (D) $2\sqrt{5} + 2\sqrt{2} + 2$ (E) $2\sqrt{5} + 4\sqrt{2} + 2$

$$\int_1^7 \sqrt{1 + (f'(x))^2}\,dx \approx (3-1)\sqrt{1+(2)^2} + (5-3)\sqrt{1+1^2} + (7-5)\sqrt{1+0^2}$$

$$\approx 2\sqrt{5} + 2\sqrt{2} + 2$$

Ex. 7:

CA



a) $A = 600 - \int_0^{30} (f(x) - 0)\,dx$

$\boxed{A = 218.028}$

A baker is creating a birthday cake. The base of the cake is the region R in the first quadrant under the graph of $y = f(x)$ for $0 \le x \le 30$, where $f(x) = 20\sin\left(\frac{\pi x}{30}\right)$. Both x and y are measured in centimeters. The region R is shown in the figure above. The derivative of f is $f'(x) = \frac{2\pi}{3}\cos\left(\frac{\pi x}{30}\right)$.

(a) The region R is cut out of a 30-centimeter-by-20-centimeter rectangular sheet of cardboard, and the remaining cardboard is discarded. Find the area of the discarded cardboard.

(b) The cake is a solid with base R. Cross sections of the cake perpendicular to the x-axis are semicircles. If the baker uses 0.05 gram of unsweetened chocolate for each cubic centimeter of cake, how many grams of unsweetened chocolate will be in the cake?

(c) Find the perimeter of the base of the cake.

b) Volume $= \int_a^b A(x)\,dx$ semicircles $\to \frac{\pi}{8}(\text{side})^2$, side $= f(x) - 0$ $\to \frac{\pi}{8}\int_0^{30} (f(x) - 0)^2\,dx$

Volume $= 2356.19449$, grams $= 0.05 \cdot$ Volume $= \boxed{117.810\text{g}}$

* Do NOT say $\int_0^{30} \frac{\pi}{8}(f(x))^2\,dx = 2356.19449 \cdot 0.05 = 117.810$ since the *
* integral is not equal to 117.810 as this work states. *
* *

c) Perimeter $= 30 + \int_0^{30} \sqrt{1 + (f'(x))^2}\,dx = \boxed{81.804\text{cm}}$

given!