AP Calculus I
Notes 2.2
Finding Limits Graphically and Numerically

Consider: $f(x) = \frac{x^2-1}{x-1}$ What is the domain of $f(x)$?

Graph:

What happens as x approaches 1?

x approaches from left → ← *x* approaches from right

x	0.75	0.9	0.999	1	1.001	1.1	1.25	1.5
$\boldsymbol{f(x)}$								

As x approaches 1, $f(x)$ moves arbitrarily close to ____________.

Definition and Notation of a Limit

Ex. 1: Find $\lim_{x \to 0} \frac{x}{\sqrt{x+1}-1}$ by completing the table, and graph to verify.

x									
$f(x)$									

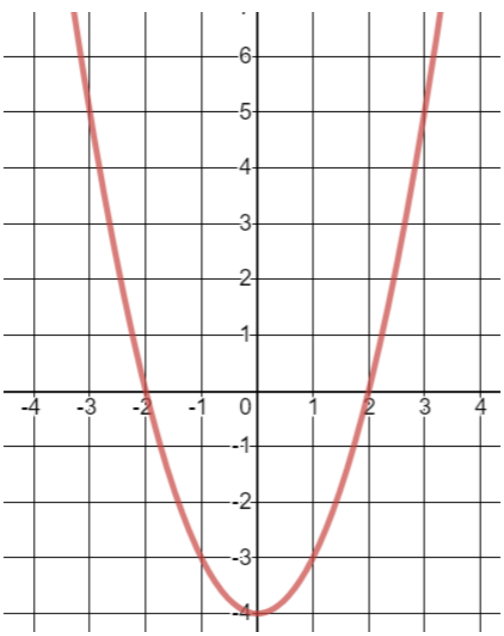
Three-Pronged Approach to Finding Limits

1. **1)** Numerical Approach (Table of Values)
2. **2)** Graphical Approach
3. **3)** Analytic Approach (Algebra or Calculus)

****Note:** 1 and 2 are means of ***approximating*** a limit. **

Ex. 2: Find $\lim_{x \to 3} (x^2 - 4)$

a) **Graph**:

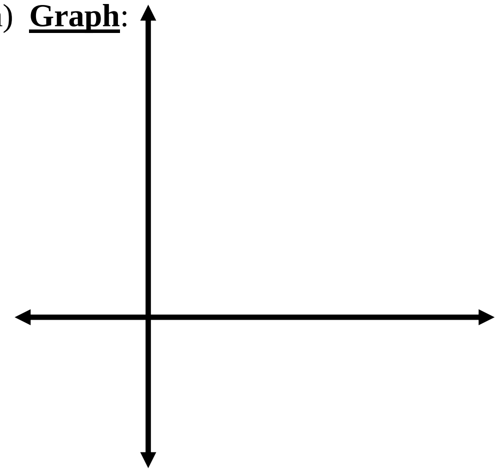


b) $\lim_{x \to 3} (x^2 - 4)$

c) $f(3) =$

Ex. 3: Given $f(x) = \begin{cases} -2x + 5, & x < 2 \\ 4, & x = 2 \\ x^2 - 3, & x > 2 \end{cases}$, find the following:

a) **Graph**:



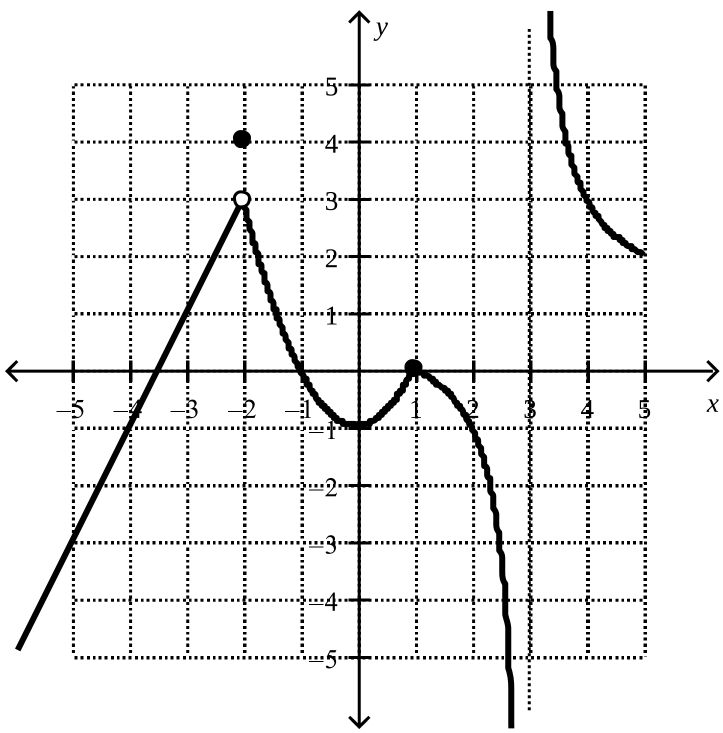
b) $\lim_{x \to 1} f(x) =$

c) $\lim_{x \to 2} f(x) =$

d) $f(2) =$

***** The existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as x approaches c.*****

Ex. 4: Evaluate the following, or state that it does not exist



a) $\lim_{x \to -2} f(x) =$

b) $f(-2) =$

c) $\lim_{x \to 1} f(x) =$

d) $f(1) =$

e) $\lim_{x \to 3} f(x) =$

f) $\lim_{x \to 0} f\big(f(x)\big) =$

g) Find the value(s) of c where $\lim_{x \to c} f(x) = 3$

h) $\lim_{x \to -2} f\big(f(x)\big) =$

Some limits do not exist.

<u>Ex. 5:</u> Find $\lim_{x\to 0} \frac{|x|}{x}$

<u>Graph</u>:

<u>Ex. 6:</u> Find $\lim_{x\to 0} \frac{1}{x^2}$

<u>Graph</u>:

<u>Ex. 7:</u> Find $\lim_{x\to 0} \sin\frac{1}{x}$

<u>Graph</u>:

<u>Common Types of Behavior Associated with the Nonexistence of a Limit</u>

1) $f(x)$ approaches a ________________________ from the right side of *c* than from the left side.
2) $f(x)$ increases or decreases ____________________ as *x* approaches *c*.
3) $f(x)$ _________________ between two fixed values as *x* approaches *c*.

AP Calculus I
Notes 2.3
Evaluating Limits Analytically

In some cases, the limit can be evaluated by direct substitution.

$\lim_{x \to c} f(x) =$_________ iff $f(x)$ is ____________________________.

Ex. 1: Find each limit:

a) $\lim_{x \to 1} \frac{x^2+x+2}{x+1}$

b) $\lim_{x \to 2} (-x^2 + 3)$

Ex. 2: If $\lim_{x \to c} f(x) = 3$ and $\lim_{x \to c} g(x) = 2$, find:

a) $\lim_{x \to c} [4g(x)]$

b) $\lim_{x \to c} [f(x) - g(x)]$

We have studied functions whose limits are evaluated by direct substitution. This is not always possible.

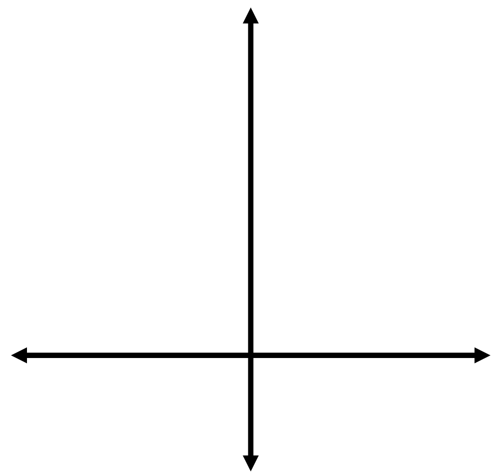
Theorem – Functions that Agree at All But One Point

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as $x \to c$ exists, then the limit of $f(x)$ also exists and $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$. This theorem permits us to evaluate limits for functions that are not continuous.

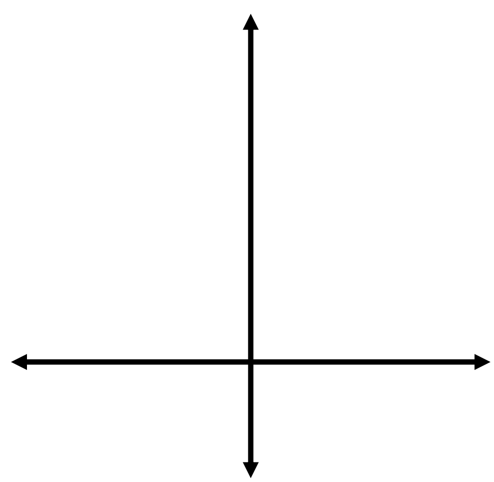
In these examples, direct substitution produces the meaningless form $\frac{0}{0}$. We call these expressions an <u>indeterminate form</u> because you cannot evaluate its limit from this form alone.

Method 1: Cancellation Technique

Ex. 3: Find $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$



Ex. 4: Find $\lim_{x\to 1} \frac{x^3-1}{x-1}$



Method 2: Rationalization Technique

Ex. 5: Find $\lim_{x \to 0} \frac{1-\sqrt{x+1}}{x}$

Ex. 6: Find $\lim_{x \to -2} \frac{5x+10}{3-\sqrt{x+11}}$

Additional Techniques

Ex. 7: Evaluate the following limit: $\lim_{\Delta x \to 0} \frac{3(x+\Delta x)^2-1-(3x^2-1)}{\Delta x}$

Ex. 8: Find each limit:

a) $\lim_{t \to 3} \frac{\frac{1}{t+1} - \frac{1}{4}}{t-3}$

b) $\lim_{k \to 0} \frac{1-e^{-k}}{e^{k}-1}$

c) $\lim_{\theta \to \frac{\pi}{3}} (sin^2 \theta)$

d) $\lim_{\theta \to \frac{\pi}{6}} \frac{\tan\theta \cos 2\theta}{\sin\theta}$

AP Calculus I
Notes 2.4
Continuity and One-Sided Limits

<u>Continuous</u> – A function is continuous at $x = c$ if there is no interruption in the graph of f at c. (No holes, jump discontinuities or infinite discontinuities).

<u>Definitions</u> – A function f is <u>continuous at c</u> iff:

1. $f(c)$ is defined
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

A function is continuous on an open interval (a, b) if it is continuous at each point in the interval. If a function f is defined on I (except possibly at c) and f is not continuous at c, then f has a discontinuity at c. There are two categories of discontinuity:

1) Removable discontinuity

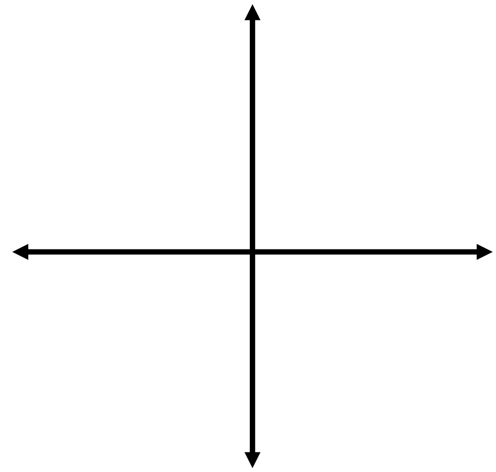
- f can be made continuous by approximately defining (or redefining) $f(c)$.

2) Nonremovable discontinuity

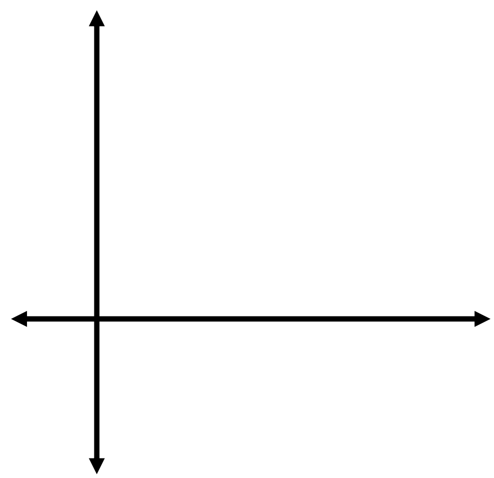
- f cannot be made continuous by defining $f(c)$.

Ex.1: For each, sketch a graph, give the domain, and discuss the continuity. If ever discontinuous, tell whether it is removable or nonremovable using the definition of continuity.

a) $f(x) = \begin{cases} x^2, & x > 1 \\ \frac{1}{x}, & -2 \leq x < 1 \end{cases}$



b) $h(x) = \begin{cases} x + 1, & x \leq 0 \\ x^2 + 1, & 0 < x < 2 \\ 3, & x = 2 \\ 9 - 2x, & 2 < x < 4 \\ h(4 - x), & x > 4 \end{cases}$



Ex. 2: Determine if the following piecewise functions are continuous everywhere. Justify your answer.

a) $f(x) = \begin{cases} \frac{x^2+x-12}{x-3} & x \neq 3 \\ 4 & x = 3 \end{cases}$

b) $h(t) = \begin{cases} \frac{t^2-9}{t-3} & t \neq 3 \\ 6 & t = 3 \end{cases}$

One-Sided Limits

Limit from the right ➔ x approaches c from values greater than c. $\lim_{x\to c^+} f(x) = L$

Limit from the left ➔ x approaches c from values less than c. $\lim_{x\to c^-} f(x) = L$

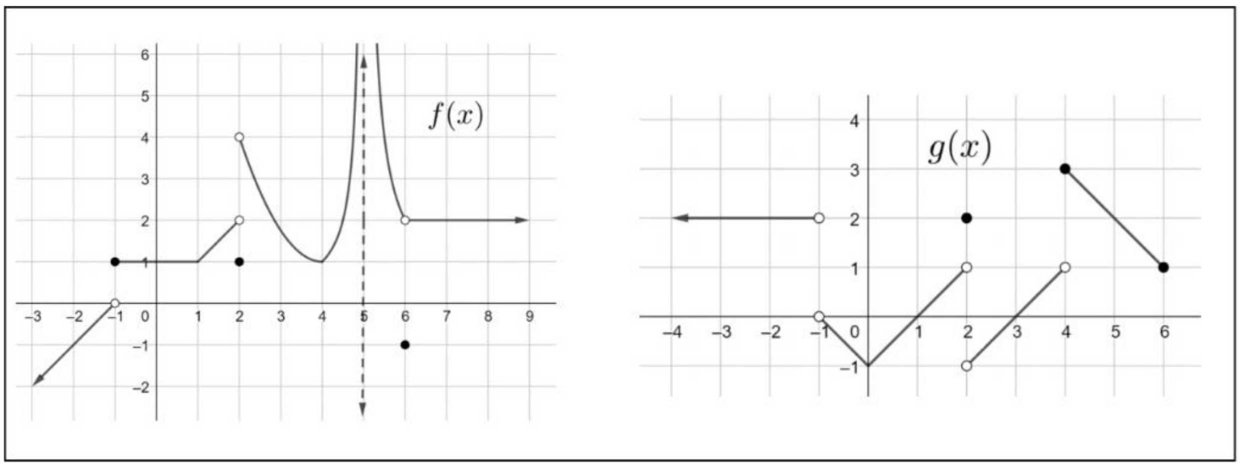
Ex. 3: Evaluate : a) $\lim_{x\to 3^-} (x^2 - 3x + 1)$ b) $\lim_{x\to 1^-} \left.\begin{matrix} x^2 - 5 & x < 1 \\ 2x + 5 & x \geq 1 \end{matrix}\right\}$ c) $\lim_{x\to 1^+} \left.\begin{matrix} x^2 - 5 & x < 1 \\ 2x + 5 & x \geq 1 \end{matrix}\right\}$

Theorem – The Existence of a Limit

Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L iff $\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = L$.

Ex. 4: Determine the value of c that will make g continuous everywhere. Justify your response.

$$g(x) = \begin{cases} 2\cos x + c & x < 0 \\ e^x + \sqrt{x+1} & x \geq 0 \end{cases}$$



Quick Check 4: (Advanced Limits) Use the graphs of $f(x)$ and $g(x)$ above to answer the following.

(a) $\lim_{x \to -1} f(x)g(x)$

(b) $\lim_{x \to 2} g(f(x))$

(c) $\lim_{x \to 0} g(g(x))$

The Intermediate Value Theorem Investigation

1. Consider the function $f(x) = x^4 - 6x + 2$ from [1,2]. Find the values of $f(1)$ and $f(2)$. Explain why you think f must cross the x-axis somewhere between $x = 1$ and $x = 2$. That is, why must there be some number c between $x = 1$ and $x = 2$ such that $f(c) = 0$?

2. Consider the function $g(x) = \begin{cases} x^2 + 2, & -3 \le x \le 0 \\ 2 - x, & 0 < x \le 2 \end{cases}$. Find the values of $g(-3)$ and $g(2)$. Must there be some c value between $x = -3$ and $x = 2$ such that $g(c) = 3$? What about $g(c) = -8$? Graph this function to verify.

3. Now consider the function $h(x) = \begin{cases} x^2 + 2, & -3 \le x \le 0 \\ -2 - x, & 0 < x \le 2 \end{cases}$. Find the values of $h(-3)$ and $h(2)$ and think about whether $h(x)$ will take on the value of 0 somewhere in the interval. Graph the function and try to explain why you came to your conclusion.

<u>The Intermediate Value Theorem</u>

Suppose f is a _____________ function on the closed interval from $[a, b]$. For any number k between ______ and ______, there exists a point c between _______ and ________ such that ______________.

<u>Ex. 6:</u> Certain values of the continuous functions f and g are given below. The function h is given by

$$h(x) = f\big(g(x)\big) - 6.$$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

AP Calculus I
Notes 2.5
Infinite Limits

A limit in which $f(x)$ increases/decreases without bound as x approaches c is called an <u>infinite limit</u>.

<u>Ex. 1:</u> Determine the limit of each function as x approaches 1 from the left and the right:

a) $f(x) = \frac{1}{x-1}$

$\lim_{x\to 1^-} f(x) =$ $\lim_{x\to 1^+} f(x) =$

b) $f(x) = \frac{1}{(x-1)^2}$

$\lim_{x\to 1^-} f(x) =$ $\lim_{x\to 1^+} f(x) =$

If $f(x)$ approaches infinity or negative infinity as x approaches c from the left or the right, then $x = c$ is a <u>vertical asymptote</u> of the graph of $f(x)$.

Vertical Asymptotes are ___________________________ discontinuities. Therefore, the limit at these discontinuities ___________________________.

Ex. 2: Determine all discontinuities of $f(x) = \frac{x^2+2x-8}{x^2-4}$. Then, compute the limit at each of the domain restrictions.

Ex. 3: Find all vertical asymptotes:

a) $f(x) = \frac{x^2+1}{e^x-1}$

b) $f(x) = \cot x$

Determining Infinite Limits Analytically

Ex. 4: Find:

a) $\lim_{x \to 1^-} \frac{x^2 - 3x}{x - 1}$

b) $\lim_{x \to 1^+} \frac{x^2 - 3x}{x - 1}$

Theorem – Properties of Infinite Limits

Let c and L be constants and let $f(x)$ and $g(x)$ be functions so that $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = L$.

1) Sum or Difference: $\lim_{x \to c} [f(x) \pm g(x)] =$

2) Product: $\lim_{x \to c} [f(x)g(x)] =$, if $L > 0$

3) Quotient: $\lim_{x \to c} \left[\frac{f(x)}{g(x)}\right] =$

$\lim_{x \to c} \left[\frac{g(x)}{f(x)}\right] =$

Ex. 5: Determine each limit:

a) $\lim_{x \to 0} \left(1 + \frac{1}{x^2}\right)$

b) $\lim_{x \to \frac{\pi}{2}^+} 3 \tan x$

c) $\lim_{x \to 0^+} 3 \ln x$

AP Calculus I
Notes 4.5
Limits at Infinity

<u>Exploration:</u>

Consider the graph of $f(x) = \frac{3x^2}{x^2+1}$

x	$-\infty \leftarrow$	-100	-10	-1	0	1	10	100	$\rightarrow \infty$
$f(x)$									

$\lim_{x \to -\infty} f(x) =$ $\qquad\qquad$ $\lim_{x \to \infty} f(x) =$

These are **limits at infinity**, and $y = 3$ is a **horizontal asymptote**.

<u>Definition of Horizontal Asymptote</u>

The line $y = L$ is a **horizontal asymptote** of the graph of $f(x)$ if $\lim_{x \to -\infty} f(x) = L$ or $\lim_{x \to \infty} f(x) = L$.

<u>Theorem – Limits at Infinity</u>

If r is a positive rational number and c is any real number, then $\lim_{x \to \infty} \frac{c}{x^r} = 0$ and $\lim_{x \to -\infty} \frac{c}{x^r} = 0$.

<u>Ex. 1</u>: Find the limit:

a) $\lim_{x \to \infty} \left(5 - \frac{2}{x^2}\right)$ $\qquad\qquad$ b) $\lim_{x \to \infty} 3e^{-x}$

<u>Ex. 2</u>: Find each of the limits. Then draw a conclusion about an asymptote of the function:

a) $\lim_{x\to\infty} \frac{2x+5}{3x^2+1}$

b) $\lim_{x\to-\infty} \frac{2x+5}{3x^2+1}$

c) $\lim_{x\to\infty} \frac{2x^2+5}{3x^2+1}$

d) $\lim_{x\to\infty} \frac{2x^3+5}{3x^2+1}$

<u>Guidelines for Finding Limits at Infinity of Rational Functions</u>

1. If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is 0.

2. If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.

3. If the degree of the numerator is *greater than* the degree of the denominator, then the limit of the rational function does not exist.

Functions may approach different horizontal asymptotes to the left and to the right.

<u>Ex. 3:</u> Determine each of the limits:

a) $\lim\limits_{x \to \pm\infty} \dfrac{10x-2}{\sqrt{4x^2+1}}$

b) $\lim\limits_{x \to \pm\infty} \dfrac{6}{1+e^{-x}}$

<u>Ex. 4:</u> Determine each of the limits:

a) $\lim\limits_{x \to \infty} \sin x$

b) $\lim\limits_{x \to \infty} \dfrac{\sin x}{x}$

Ex. 5: Evaluate the following:

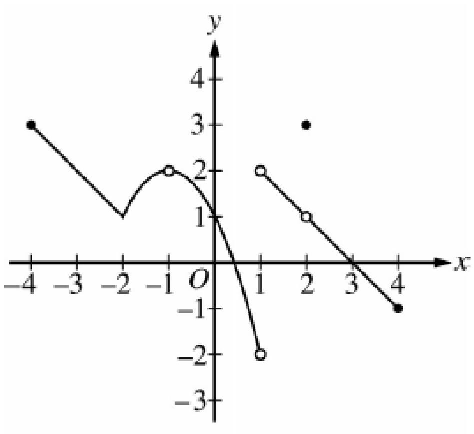
$$\lim_{x\to\infty}\frac{\sqrt{9x^4+1}}{x^2-3x+5} \text{ is}$$

(A) 1

(B) 3

(C) 9

(D) nonexistent



Graph of f

The graph of the function f is shown in the figure above. For how many values of x in the open interval $(-4, 4)$ is f discontinuous?

(A) one

(B) two

(C) three

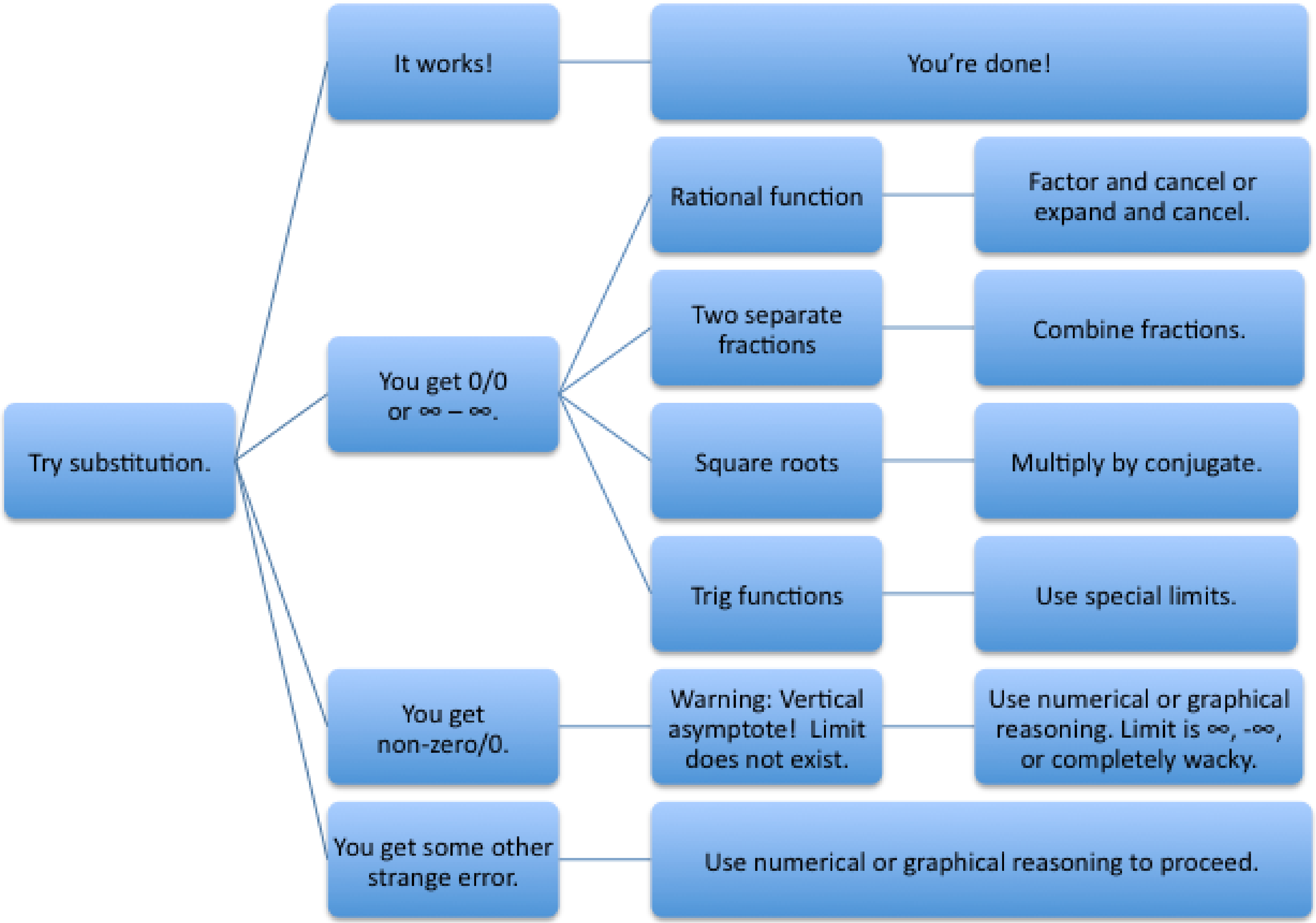
(D) four

Ex. 6: Describe the graphical representation of the following limit statements:

a) $\lim_{x\to 2^+} f(x) = \infty$

b) $\lim_{x\to\infty} f(x) = 1$

c) $\lim_{x\to 1} f(x) = 0$



Try substitution.
It works!
You're done!
You get 0/0 or ∞ – ∞.
Rational function
Factor and cancel or expand and cancel.
Two separate fractions
Combine fractions.
Square roots
Multiply by conjugate.
Trig functions
Use special limits.
You get non-zero/0.
Warning: Vertical asymptote! Limit does not exist.
Use numerical or graphical reasoning. Limit is ∞, -∞, or completely wacky.
You get some other strange error.
Use numerical or graphical reasoning to proceed.