1 Preparation for Calculus

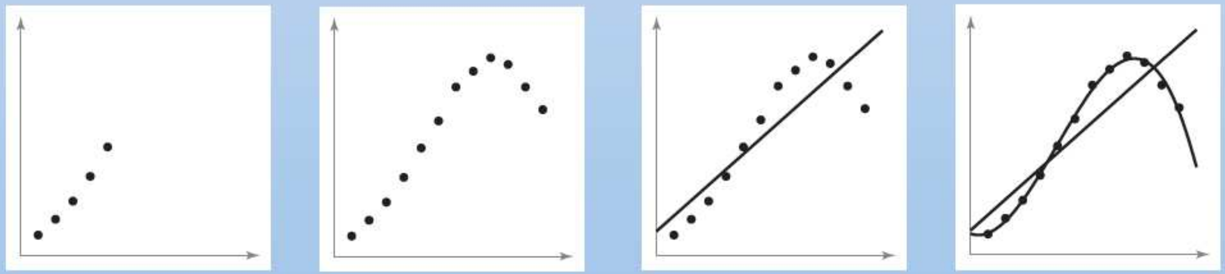
This chapter reviews several concepts that will help you prepare for your study of calculus. These concepts include sketching the graphs of equations and functions, and fitting mathematical models to data. It is important to review these concepts before moving on to calculus.

In this chapter, you should learn the following.

- How to identify the characteristics of equations and sketch their graphs. (1.1)
- How to find and graph equations of lines, including parallel and perpendicular lines, using the concept of slope. (1.2)
- How to evaluate and graph functions and their transformations. (1.3)
- How to fit mathematical models to real-life data sets. (1.4)
- How to determine whether a function has an inverse function. The properties of inverse trigonometric functions. (1.5)
- The properties of the natural exponential and natural logarithmic functions. (1.6)

Jeremy Walker/Getty Images

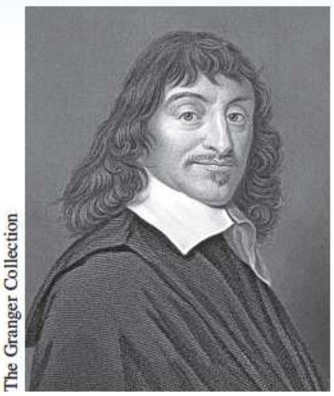
In 2006, China surpassed the United States as the world's biggest emitter of carbon dioxide, the main greenhouse gas. Given the carbon dioxide concentrations in the atmosphere for several years, can older mathematical models still accurately predict future atmospheric concentrations compared with more recent models? (See Section 1.1, Example 6.)



Mathematical models are commonly used to describe data sets. These models can be represented by many different types of functions, such as linear, quadratic, cubic, rational, and trigonometric functions. (See Section 1.4.)

1.1 Graphs and Models

- **Sketch the graph of an equation.**
- **Find the intercepts of a graph.**
- **Test a graph for symmetry with respect to an axis and the origin.**
- **Find the points of intersection of two graphs.**
- **Interpret mathematical models for real-life data.**



RENÉ DESCARTES (1596–1650)

Descartes made many contributions to philosophy, science, and mathematics. The idea of representing points in the plane by pairs of real numbers and representing curves in the plane by equations was described by Descartes in his book *La Géométrie*, published in 1637.

The Graph of an Equation

In 1637, the French mathematician René Descartes revolutionized the study of mathematics by joining its two major fields—algebra and geometry. With Descartes's coordinate plane, geometric concepts could be formulated analytically and algebraic concepts could be viewed graphically. The power of this approach is such that within a century, much of calculus had been developed.

The same approach can be followed in your study of calculus. That is, by viewing calculus from multiple perspectives—*graphically*, *analytically*, and *numerically*—you will increase your understanding of core concepts.

Consider the equation $3x + y = 7$. The point $(2, 1)$ is a **solution point** of the equation because the equation is satisfied (is true) when 2 is substituted for x and 1 is substituted for y. This equation has many other solutions, such as $(1, 4)$ and $(0, 7)$. To find other solutions systematically, solve the original equation for y.

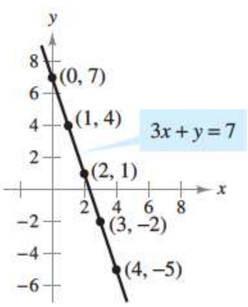
$$y = 7 - 3x \qquad \text{Analytic approach}$$

Then construct a table of values by substituting several values for x.

x	0	1	2	3	4
y	7	4	1	-2	-5

Numerical approach

From the table, you can see that $(0, 7)$, $(1, 4)$, $(2, 1)$, $(3, -2)$, and $(4, -5)$ are solutions of the original equation $3x + y = 7$. Like many equations, this equation has an infinite number of solutions. The set of all solution points is the **graph** of the equation, as shown in Figure 1.1.



Graphical approach: $3x + y = 7$
Figure 1.1

NOTE Even though we refer to the sketch shown in Figure 1.1 as the graph of $3x + y = 7$, it really represents only a *portion* of the graph. The entire graph would extend beyond the page. ■

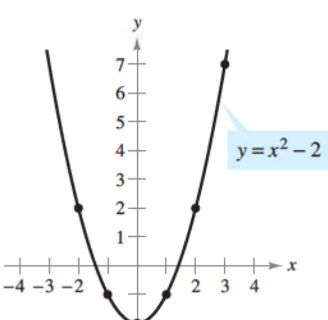
In this course, you will study many sketching techniques. The simplest is point plotting—that is, you plot points until the basic shape of the graph seems apparent.

EXAMPLE 1 Sketching a Graph by Point Plotting

Sketch the graph of $y = x^2 - 2$.

Solution First construct a table of values. Then plot the points shown in the table.

x	-2	-1	0	1	2	3
y	2	-1	-2	-1	2	7



The parabola $y = x^2 - 2$
Figure 1.2

Finally, connect the points with a *smooth curve*, as shown in Figure 1.2. This graph is a **parabola.** It is one of the conics you will study in Chapter 10. ■

One disadvantage of point plotting is that to get a good idea about the shape of a graph, you may need to plot many points. With only a few points, you could misrepresent the graph. For instance, suppose that to sketch the graph of

$$y = \frac{1}{30}x(39 - 10x^2 + x^4)$$

you plotted only five points:

$$(-3, -3), (-1, -1), (0, 0), (1, 1), \text{ and } (3, 3)$$

as shown in Figure 1.3(a). From these five points, you might conclude that the graph is a line. This, however, is not correct. By plotting several more points, you can see that the graph is more complicated, as shown in Figure 1.3(b).

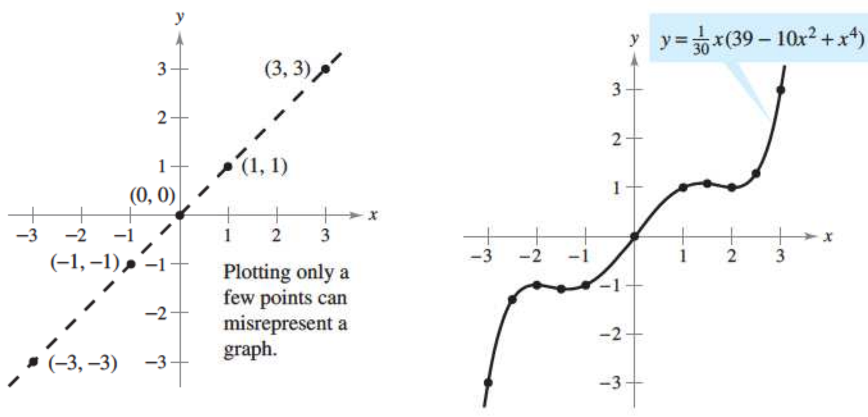


Figure 1.3

EXPLORATION

Comparing Graphical and Analytic Approaches Use a graphing utility to graph each equation. In each case, find a viewing window that shows the important characteristics of the graph.

a. $y = x^3 - 3x^2 + 2x + 5$

b. $y = x^3 - 3x^2 + 2x + 25$

c. $y = -x^3 - 3x^2 + 20x + 5$

d. $y = 3x^3 - 40x^2 + 50x - 45$

e. $y = -(x + 12)^3$

f. $y = (x - 2)(x - 4)(x - 6)$

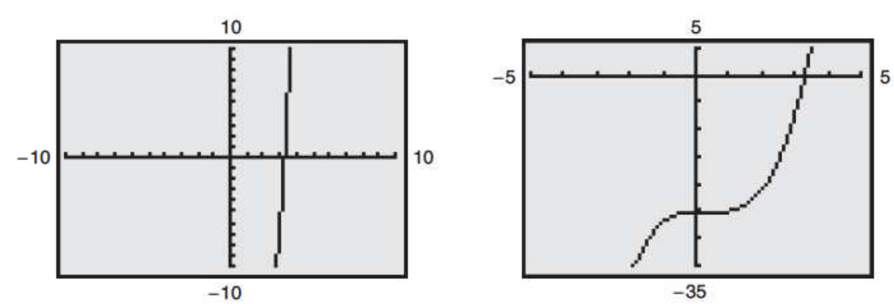
A purely graphical approach to this problem would involve a simple "guess, check, and revise" strategy. What types of things do you think an analytic approach might involve? For instance, does the graph have symmetry? Does the graph have turns? If so, where are they?

As you proceed through Chapters 2, 3, and 4 of this text, you will study many new analytic tools that will help you analyze graphs of equations such as these.

TECHNOLOGY Technology has made sketching of graphs easier. Even with technology, however, it is possible to misrepresent a graph badly. For instance, each of the graphing utility screens in Figure 1.4 shows a portion of the graph of

$$y = x^3 - x^2 - 25.$$

From the screen on the left, you might assume that the graph is a line. From the screen on the right, however, you can see that the graph is not a line. So, whether you are sketching a graph by hand or using a graphing utility, you must realize that different "viewing windows" can produce very different views of a graph. In choosing a viewing window, your goal is to show a view of the graph that fits well in the context of the problem.



Graphing utility screens of $y = x^3 - x^2 - 25$
Figure 1.4

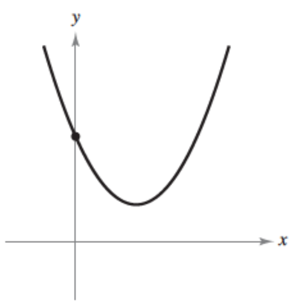
NOTE In this text, the term *graphing utility* means either a graphing calculator or computer graphing software such as *Maple*, *Mathematica*, or the *TI-89*.

Intercepts of a Graph

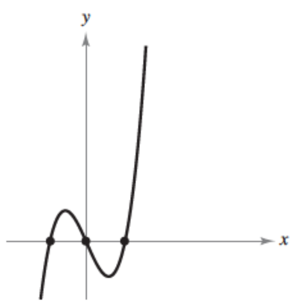
Two types of solution points that are especially useful in graphing an equation are those having zero as their x- or y-coordinate. Such points are called **intercepts** because they are the points at which the graph intersects the x- or y-axis. The point $(a, 0)$ is an **x-intercept** of the graph of an equation if it is a solution point of the equation. To find the x-intercepts of a graph, let y be zero and solve the equation for x. The point $(0, b)$ is a **y-intercept** of the graph of an equation if it is a solution point of the equation. To find the y-intercepts of a graph, let x be zero and solve the equation for y.

NOTE Some texts denote the x-intercept as the x-coordinate of the point $(a, 0)$ rather than the point itself. Unless it is necessary to make a distinction, we will use the term *intercept* to mean either the point or the coordinate.

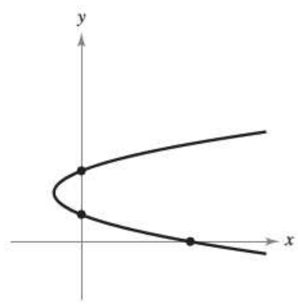
It is possible for a graph to have no intercepts, or it might have several. For instance, consider the four graphs shown in Figure 1.5.



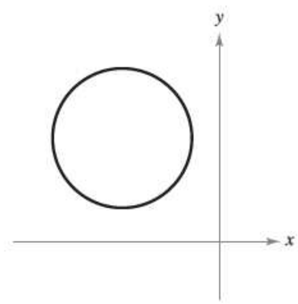
No x-intercepts
One y-intercept



Three x-intercepts
One y-intercept



One x-intercept
Two y-intercepts



No intercepts

Figure 1.5

EXAMPLE 2 Finding x- and y-intercepts

Find the x- and y-intercepts of the graph of $y = x^3 - 4x$.

Solution To find the x-intercepts, let y be zero and solve for x.

$$x^3 - 4x = 0 \qquad \text{Let } y \text{ be zero.}$$
$$x(x - 2)(x + 2) = 0 \qquad \text{Factor.}$$
$$x = 0, 2, \text{ or } -2 \qquad \text{Solve for } x.$$

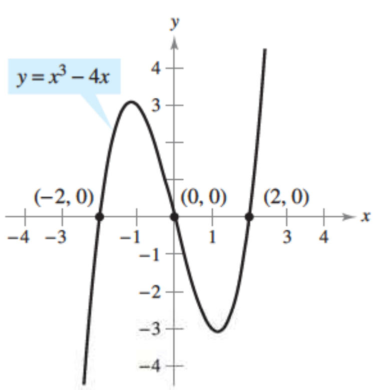
Because this equation has three solutions, you can conclude that the graph has three x-intercepts:

$(0, 0), (2, 0),$ and $(-2, 0)$. $\qquad$ x-intercepts

To find the y-intercepts, let x be zero. Doing so produces $y = 0$. So, the y-intercept is

$(0, 0)$. $\qquad$ y-intercept

(See Figure 1.6.)



Intercepts of a graph
Figure 1.6

TECHNOLOGY Example 2 uses an analytic approach to finding intercepts. When an analytic approach is not possible, you can use a graphical approach by finding the points where the graph intersects the axes. Use a graphing utility to approximate the intercepts.

Symmetry of a Graph

Knowing the symmetry of a graph *before* attempting to sketch it is useful because you need only half as many points to sketch the graph. The following three types of symmetry can be used to help sketch the graphs of equations (see Figure 1.7).

1. A graph is **symmetric with respect to the y-axis** if, whenever (x, y) is a point on the graph, $(-x, y)$ is also a point on the graph. This means that the portion of the graph to the left of the y-axis is a mirror image of the portion to the right of the y-axis.
2. A graph is **symmetric with respect to the x-axis** if, whenever (x, y) is a point on the graph, $(x, -y)$ is also a point on the graph. This means that the portion of the graph above the x-axis is a mirror image of the portion below the x-axis.
3. A graph is **symmetric with respect to the origin** if, whenever (x, y) is a point on the graph, $(-x, -y)$ is also a point on the graph. This means that the graph is unchanged by a rotation of 180° about the origin.

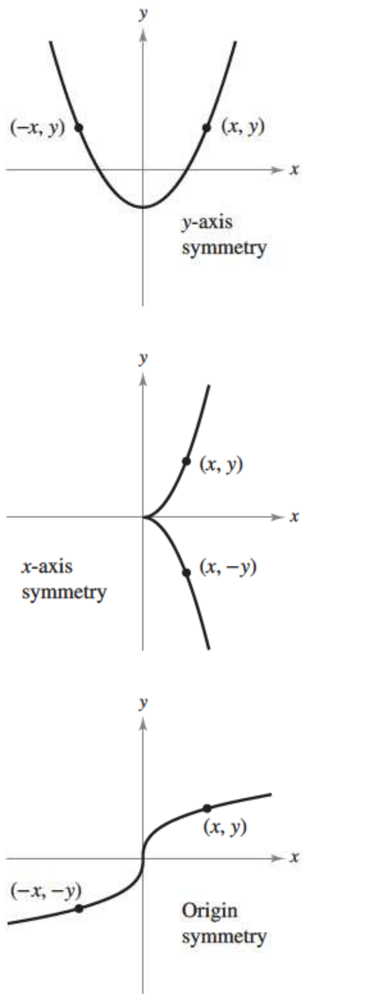


Figure 1.7

TESTS FOR SYMMETRY

1. The graph of an equation in x and y is symmetric with respect to the y-axis if replacing x by $-x$ yields an equivalent equation.
2. The graph of an equation in x and y is symmetric with respect to the x-axis if replacing y by $-y$ yields an equivalent equation.
3. The graph of an equation in x and y is symmetric with respect to the origin if replacing x by $-x$ and y by $-y$ yields an equivalent equation.

The graph of a polynomial has symmetry with respect to the y-axis if each term has an even exponent (or is a constant). For instance, the graph of $y = 2x^4 - x^2 + 2$ has symmetry with respect to the y-axis. Similarly, the graph of a polynomial has symmetry with respect to the origin if each term has an odd exponent, as illustrated in Example 3.

EXAMPLE 3 Testing for Symmetry

Test the graph of $y = 2x^3 - x$ for symmetry with respect to the y-axis and to the origin.

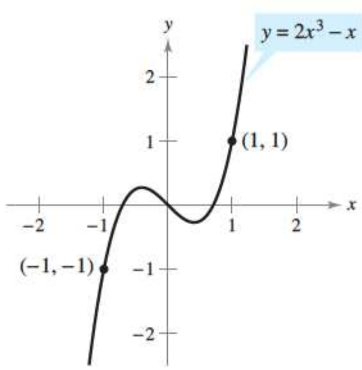
Solution

y-axis Symmetry:

$y = 2x^3 - x$	Write original equation.
$y = 2(-x)^3 - (-x)$	Replace x by $-x$.
$y = -2x^3 + x$	Simplify. It is not an equivalent equation.

Origin Symmetry:

$y = 2x^3 - x$	Write original equation.
$-y = 2(-x)^3 - (-x)$	Replace x by $-x$ and y by $-y$.
$-y = -2x^3 + x$	Simplify.
$y = 2x^3 - x$	Equivalent equation



Origin symmetry
Figure 1.8

Because replacing both x by $-x$ and y by $-y$ yields an equivalent equation, you can conclude that the graph of $y = 2x^3 - x$ is symmetric with respect to the origin, as shown in Figure 1.8. ■

EXAMPLE 4 Using Intercepts and Symmetry to Sketch a Graph

Sketch the graph of $x - y^2 = 1$.

Solution The graph is symmetric with respect to the x-axis because replacing y by $-y$ yields an equivalent equation.

$$x - y^2 = 1 \quad \text{Write original equation.}$$
$$x - (-y)^2 = 1 \quad \text{Replace } y \text{ by } -y.$$
$$x - y^2 = 1 \quad \text{Equivalent equation}$$

This means that the portion of the graph below the x-axis is a mirror image of the portion above the x-axis. To sketch the graph, first sketch the portion above the x-axis. Then reflect in the x-axis to obtain the entire graph, as shown in Figure 1.9. ■

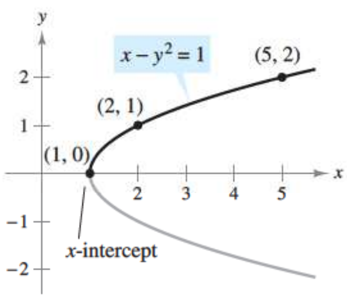


Figure 1.9

TECHNOLOGY Graphing utilities are designed so that they most easily graph equations in which y is a function of x (see Section 1.3 for a definition of **function**). To graph other types of equations, you need to split the graph into two or more parts *or* you need to use a different graphing mode. For instance, to graph the equation in Example 4, you can split it into two parts.

$$y_1 = \sqrt{x - 1} \quad \text{Top portion of graph}$$
$$y_2 = -\sqrt{x - 1} \quad \text{Bottom portion of graph}$$

Points of Intersection

A **point of intersection** of the graphs of two equations is a point that satisfies both equations. You can find the point(s) of intersection of two graphs by solving their equations simultaneously.

EXAMPLE 5 Finding Points of Intersection

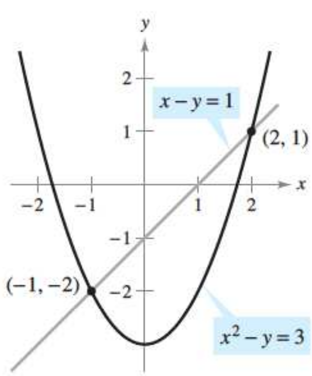
Find all points of intersection of the graphs of $x^2 - y = 3$ and $x - y = 1$.

Solution Begin by sketching the graphs of both equations on the *same* rectangular coordinate system, as shown in Figure 1.10. Having done this, it appears that the graphs have two points of intersection. You can find these two points, as follows.

$$y = x^2 - 3 \quad \text{Solve first equation for } y.$$
$$y = x - 1 \quad \text{Solve second equation for } y.$$
$$x^2 - 3 = x - 1 \quad \text{Equate } y\text{-values.}$$
$$x^2 - x - 2 = 0 \quad \text{Write in general form.}$$
$$(x - 2)(x + 1) = 0 \quad \text{Factor.}$$
$$x = 2 \text{ or } -1 \quad \text{Solve for } x.$$

The corresponding values of y are obtained by substituting $x = 2$ and $x = -1$ into either of the original equations. Doing this produces two points of intersection:

$$(2, 1) \text{ and } (-1, -2). \quad \text{Points of intersection}$$ ■



Two points of intersection
Figure 1.10

STUDY TIP You can check the points of intersection in Example 5 by substituting into *both* of the original equations or by using the *intersect* feature of a graphing utility.

The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

Mathematical Models

Real-life applications of mathematics often use equations as **mathematical models.** In developing a mathematical model to represent actual data, you should strive for two (often conflicting) goals—accuracy and simplicity. That is, you want the model to be simple enough to be workable, yet accurate enough to produce meaningful results. Section 1.4 explores these goals more completely.

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The Mauna Loa Observatory in Hawaii has been measuring the increasing concentration of carbon dioxide in Earth's atmosphere since 1958. Carbon dioxide is the main greenhouse gas responsible for global climate warming.

EXAMPLE 6 Comparing Two Mathematical Models

The Mauna Loa Observatory in Hawaii records the carbon dioxide concentration y (in parts per million) in Earth's atmosphere. The January readings for various years are shown in Figure 1.11. In the July 1990 issue of *Scientific American*, these data were used to predict the carbon dioxide level in Earth's atmosphere in the year 2035. The article used the quadratic model

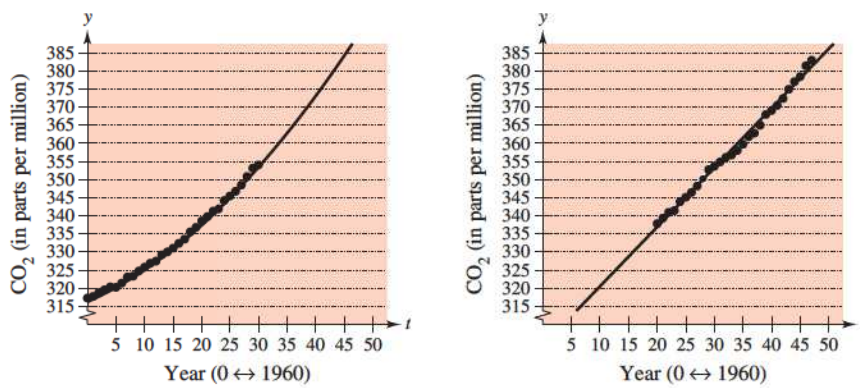
$$y = 316.2 + 0.70t + 0.018t^2 \qquad \text{Quadratic model for 1960–1990 data}$$

where $t = 0$ represents 1960, as shown in Figure 1.11(a).

The data shown in Figure 1.11(b) represent the years 1980 through 2007 and can be modeled by

$$y = 304.1 + 1.64t \qquad \text{Linear model for 1980–2007 data}$$

where $t = 0$ represents 1960. What was the prediction given in the *Scientific American* article in 1990? Given the new data for 1990 through 2007, does this prediction for the year 2035 seem accurate?



(a) (b)

Figure 1.11

Solution To answer the first question, substitute $t = 75$ (for 2035) into the quadratic model.

$$y = 316.2 + 0.70(75) + 0.018(75)^2 = 469.95 \qquad \text{Quadratic model}$$

So, the prediction in the *Scientific American* article was that the carbon dioxide concentration in Earth's atmosphere would reach about 470 parts per million in the year 2035. Using the linear model for the 1980–2007 data, the prediction for the year 2035 is

$$y = 304.1 + 1.64(75) = 427.1. \qquad \text{Linear model}$$

So, based on the linear model for 1980–2007, it appears that the 1990 prediction was too high. ■

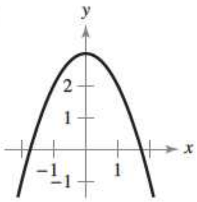
NOTE The models in Example 6 were developed using a procedure called *least squares regression* (see Section 13.9). The quadratic and linear models have correlations given by $r^2 = 0.997$ and $r^2 = 0.994$, respectively. The closer r^2 is to 1, the "better" the model.

1.1 Exercises

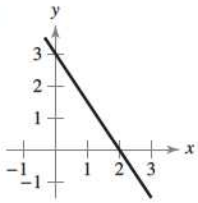
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]

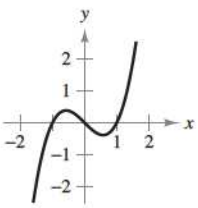
(a)



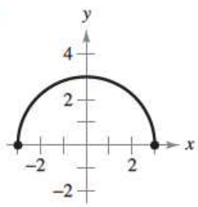
(b)



(c)



(d)



1. $y = -\frac{3}{2}x + 3$ **2.** $y = \sqrt{9 - x^2}$

3. $y = 3 - x^2$ **4.** $y = x^3 - x$

In Exercises 5–14, sketch the graph of the equation by point plotting.

5. $y = \frac{1}{2}x + 2$ **6.** $y = 5 - 2x$

7. $y = 4 - x^2$ **8.** $y = (x - 3)^2$

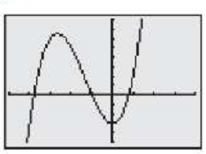
9. $y = |x + 2|$ **10.** $y = |x| - 1$

11. $y = \sqrt{x} - 6$ **12.** $y = \sqrt{x + 2}$

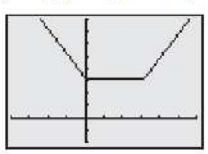
13. $y = \dfrac{3}{x}$ **14.** $y = \dfrac{1}{x + 2}$

In Exercises 15 and 16, describe the viewing window that yields the figure.

15. $y = x^3 + 4x^2 - 3$



16. $y = |x| + |x - 16|$



In Exercises 17 and 18, use a graphing utility to graph the equation. Move the cursor along the curve to approximate the unknown coordinate of each solution point accurate to two decimal places.

17. $y = \sqrt{5 - x}$ (a) $(2, y)$ (b) $(x, 3)$

18. $y = x^5 - 5x$ (a) $(-0.5, y)$ (b) $(x, -4)$

In Exercises 19–28, find any intercepts.

19. $y = 2x - 5$ **20.** $y = 4x^2 + 3$

21. $y = x^2 + x - 2$ **22.** $y^2 = x^3 - 4x$

23. $y = x\sqrt{16 - x^2}$ **24.** $y = (x - 1)\sqrt{x^2 + 1}$

25. $y = \dfrac{2 - \sqrt{x}}{5x}$ **26.** $y = \dfrac{x^2 + 3x}{(3x + 1)^2}$

27. $x^2y - x^2 + 4y = 0$ **28.** $y = 2x - \sqrt{x^2 + 1}$

In Exercises 29–40, test for symmetry with respect to each axis and to the origin.

29. $y = x^2 - 6$ **30.** $y = x^2 - x$

31. $y^2 = x^3 - 8x$ **32.** $y = x^3 + x$

33. $xy = 4$ **34.** $xy^2 = -10$

35. $y = 4 - \sqrt{x + 3}$ **36.** $xy - \sqrt{4 - x^2} = 0$

37. $y = \dfrac{x}{x^2 + 1}$ **38.** $y = \dfrac{x^2}{x^2 + 1}$

39. $y = |x^3 + x|$ **40.** $|y| - x = 3$

In Exercises 41–58, sketch the graph of the equation. Identify any intercepts and test for symmetry.

41. $y = 2 - 3x$ **42.** $y = -\frac{3}{2}x + 6$

43. $y = \frac{1}{2}x - 4$ **44.** $y = \frac{2}{3}x + 1$

45. $y = 9 - x^2$ **46.** $y = x^2 + 3$

47. $y = (x + 3)^2$ **48.** $y = 2x^2 + x$

49. $y = x^3 + 2$ **50.** $y = x^3 - 4x$

51. $y = x\sqrt{x + 5}$ **52.** $y = \sqrt{25 - x^2}$

53. $x = y^3$ **54.** $x = y^2 - 4$

55. $y = \dfrac{8}{x}$ **56.** $y = \dfrac{10}{x^2 + 1}$

57. $y = 6 - |x|$ **58.** $y = |6 - x|$

In Exercises 59–62, use a graphing utility to graph the equation. Identify any intercepts and test for symmetry.

59. $y^2 - x = 9$ **60.** $x^2 + 4y^2 = 4$

61. $x + 3y^2 = 6$ **62.** $3x - 4y^2 = 8$

In Exercises 63–70, find the points of intersection of the graphs of the equations.

63. $x + y = 8$
$4x - y = 7$

64. $3x - 2y = -4$
$4x + 2y = -10$

65. $x^2 + y = 6$
$x + y = 4$

66. $x = 3 - y^2$
$y = x - 1$

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system. The solutions of other exercises may also be facilitated by use of appropriate technology.

67. $x^2 + y^2 = 5$
$x - y = 1$

68. $x^2 + y^2 = 25$
$-3x + y = 15$

69. $y = x^3$
$y = x$

70. $y = x^3 - 4x$
$y = -(x + 2)$

In Exercises 71–74, use a graphing utility to find the points of intersection of the graphs. Check your results analytically.

71. $y = x^3 - 2x^2 + x - 1$
$y = -x^2 + 3x - 1$

72. $y = x^4 - 2x^2 + 1$
$y = 1 - x^2$

73. $y = \sqrt{x + 6}$
$y = \sqrt{-x^2 - 4x}$

74. $y = -|2x - 3| + 6$
$y = 6 - x$

75. ***Modeling Data*** The table shows the Consumer Price Index (CPI) for selected years. *(Source: Bureau of Labor Statistics)*

Year	1975	1980	1985	1990	1995	2000	2005
CPI	53.8	82.4	107.6	130.7	152.4	172.2	195.3

(a) Use the regression capabilities of a graphing utility to find a mathematical model of the form $y = at^2 + bt + c$ for the data. In the model, y represents the CPI and t represents the year, with $t = 5$ corresponding to 1975.

(b) Use a graphing utility to plot the data and graph the model. Compare the data with the model.

(c) Use the model to predict the CPI for the year 2010.

76. ***Modeling Data*** The table shows the numbers of cellular phone subscribers (in millions) in the United States for selected years. *(Source: Cellular Telecommunications and Internet Association)*

Year	1990	1993	1996	1999	2002	2005
Number	5	16	44	86	141	208

(a) Use the regression capabilities of a graphing utility to find a mathematical model of the form $y = at^2 + bt + c$ for the data. In the model, y represents the number of subscribers and t represents the year, with $t = 0$ corresponding to 1990.

(b) Use a graphing utility to plot the data and graph the model. Compare the data with the model.

(c) Use the model to predict the number of cellular phone subscribers in the United States in the year 2015.

77. ***Break-Even Point*** Find the sales necessary to break even $(R = C)$ if the cost C of producing x units is

$C = 5.5\sqrt{x} + 10{,}000$ Cost equation

and the revenue R from selling x units is

$R = 3.29x.$ Revenue equation

78. ***Copper Wire*** The resistance y in ohms of 1000 feet of solid copper wire at 77°F can be approximated by the model

$$y = \frac{10{,}770}{x^2} - 0.37, \quad 5 \le x \le 100$$

where x is the diameter of the wire in mils (0.001 in.). Use a graphing utility to graph the model. If the diameter of the wire is doubled, the resistance is changed by about what factor?

WRITING ABOUT CONCEPTS

In Exercises 79 and 80, write an equation whose graph has the given property. (There may be more than one correct answer.)

79. The graph has intercepts at $x = -4$, $x = 3$, and $x = 8$.

80. The graph has intercepts at $x = -\frac{3}{2}$, $x = 4$, and $x = \frac{5}{2}$.

81. (a) Prove that if a graph is symmetric with respect to the x-axis and to the y-axis, then it is symmetric with respect to the origin. Give an example to show that the converse is not true.

(b) Prove that if a graph is symmetric with respect to one axis and to the origin, then it is symmetric with respect to the other axis.

CAPSTONE

82. Match the equation or equations with the given characteristic.

(i) $y = 3x^3 - 3x$ (ii) $y = (x + 3)^2$ (iii) $y = 3x - 3$
(iv) $y = \sqrt[3]{x}$ (v) $y = 3x^2 + 3$ (vi) $y = \sqrt{x + 3}$

(a) Symmetric with respect to the y-axis
(b) Three x-intercepts
(c) Symmetric with respect to the x-axis
(d) $(-2, 1)$ is a point on the graph
(e) Symmetric with respect to the origin
(f) Graph passes through the origin

True or False? **In Exercises 83–86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

83. If $(-4, -5)$ is a point on a graph that is symmetric with respect to the x-axis, then $(4, -5)$ is also a point on the graph.

84. If $(-4, -5)$ is a point on a graph that is symmetric with respect to the y-axis, then $(4, -5)$ is also a point on the graph.

85. If $b^2 - 4ac > 0$ and $a \neq 0$, then the graph of $y = ax^2 + bx + c$ has two x-intercepts.

86. If $b^2 - 4ac = 0$ and $a \neq 0$, then the graph of $y = ax^2 + bx + c$ has only one x-intercept.

In Exercises 87 and 88, find an equation of the graph that consists of all points (x, y) having the given distance from the origin. (For a review of the Distance Formula, see Appendix C.)

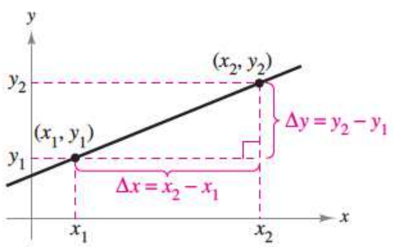
87. The distance from the origin is twice the distance from $(0, 3)$.

88. The distance from the origin is $K (K \neq 1)$ times the distance from $(2, 0)$.

1.2 Linear Models and Rates of Change

- Find the slope of a line passing through two points.
- Write the equation of a line given a point and the slope.
- Interpret slope as a ratio or as a rate in a real-life application.
- Sketch the graph of a linear equation in slope-intercept form.
- Write equations of lines that are parallel or perpendicular to a given line.

The Slope of a Line



$\Delta y = y_2 - y_1 =$ change in y
$\Delta x = x_2 - x_1 =$ change in x
Figure 1.12

The **slope** of a nonvertical line is a measure of the number of units the line rises (or falls) vertically for each unit of horizontal change from left to right. Consider the two points (x_1, y_1) and (x_2, y_2) on the line in Figure 1.12. As you move from left to right along this line, a vertical change of

$$\Delta y = y_2 - y_1 \qquad \text{Change in } y$$

units corresponds to a horizontal change of

$$\Delta x = x_2 - x_1 \qquad \text{Change in } x$$

units. (Δ is the Greek uppercase letter *delta*, and the symbols Δy and Δx are read "delta y" and "delta x.")

DEFINITION OF THE SLOPE OF A LINE

The **slope** m of the nonvertical line passing through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2.$$

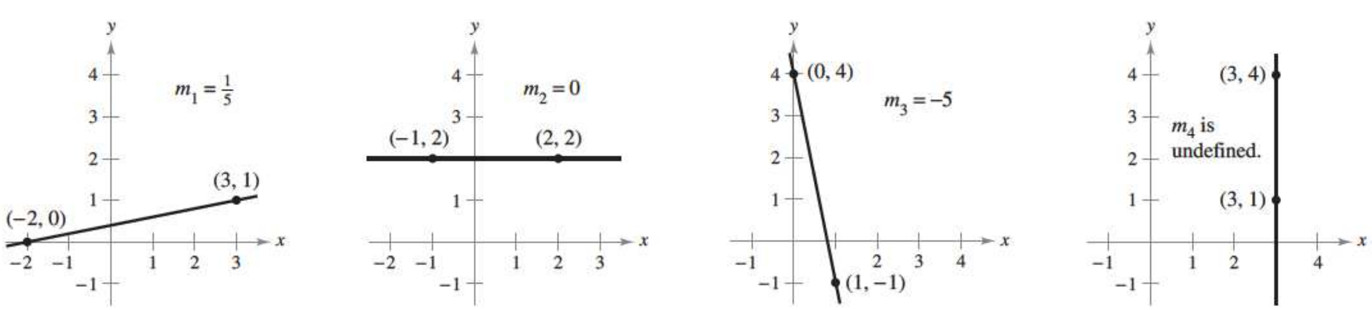
Slope is not defined for vertical lines.

NOTE When using the formula for slope, note that

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_1 - y_2)}{-(x_1 - x_2)} = \frac{y_1 - y_2}{x_1 - x_2}.$$

So, it does not matter in which order you subtract *as long as* you are consistent and both "subtracted coordinates" come from the same point. ■

Figure 1.13 shows four lines: one has a positive slope, one has a slope of zero, one has a negative slope, and one has an "undefined" slope. In general, the greater the absolute value of the slope of a line, the steeper the line is. For instance, in Figure 1.13, the line with a slope of -5 is steeper than the line with a slope of $\frac{1}{5}$.



If m is positive, then the line rises from left to right.

If m is zero, then the line is horizontal.

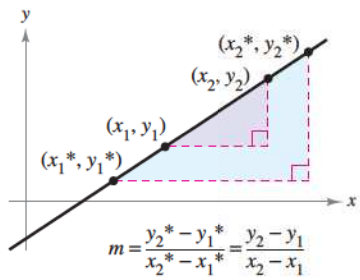
If m is negative, then the line falls from left to right.

If m is undefined, then the line is vertical.

Figure 1.13

Equations of Lines

Any two points on a nonvertical line can be used to calculate its slope. This can be verified from the similar triangles shown in Figure 1.14. (Recall that the ratios of corresponding sides of similar triangles are equal.)



Any two points on a nonvertical line can be used to determine its slope.
Figure 1.14

> **EXPLORATION**
>
> ***Investigating Equations of Lines*** Use a graphing utility to graph each of the linear equations. Which point is common to all seven lines? Which value in the equation determines the slope of each line?
>
> **a.** $y - 4 = -2(x + 1)$
> **b.** $y - 4 = -1(x + 1)$
> **c.** $y - 4 = -\frac{1}{2}(x + 1)$
> **d.** $y - 4 = 0(x + 1)$
> **e.** $y - 4 = \frac{1}{2}(x + 1)$
> **f.** $y - 4 = 1(x + 1)$
> **g.** $y - 4 = 2(x + 1)$
>
> Use your results to write an equation of a line passing through $(-1, 4)$ with a slope of m.

You can write an equation of a nonvertical line if you know the slope of the line and the coordinates of one point on the line. Suppose the slope is m and the point is (x_1, y_1). If (x, y) is any other point on the line, then

$$\frac{y - y_1}{x - x_1} = m.$$

This equation, involving the two variables x and y, can be rewritten in the form $y - y_1 = m(x - x_1)$, which is called the **point-slope form of the equation of a line.**

> **POINT-SLOPE FORM OF THE EQUATION OF A LINE**
>
> An equation of the line with slope m passing through the point (x_1, y_1) is given by
>
> $$y - y_1 = m(x - x_1).$$

EXAMPLE 1 Finding an Equation of a Line

Find an equation of the line that has a slope of 3 and passes through the point $(1, -2)$.

Solution

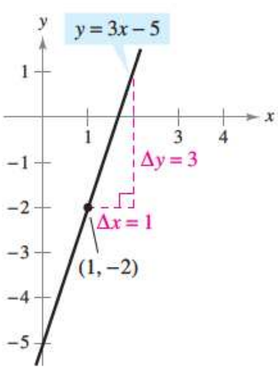
$y - y_1 = m(x - x_1)$ Point-slope form

$y - (-2) = 3(x - 1)$ Substitute -2 for y_1, 1 for x_1, and 3 for m.

$y + 2 = 3x - 3$ Simplify.

$y = 3x - 5$ Solve for y.

(See Figure 1.15.) ■



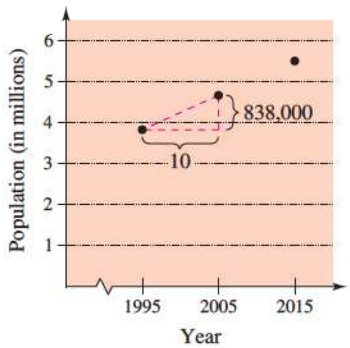
The line with a slope of 3 passing through the point $(1, -2)$
Figure 1.15

NOTE Remember that only nonvertical lines have a slope. Consequently, vertical lines cannot be written in point-slope form. For instance, the equation of the vertical line passing through the point $(1, -2)$ is $x = 1$. ■

Ratios and Rates of Change

The slope of a line can be interpreted as either a *ratio* or a *rate*. If the x- and y-axes have the same unit of measure, the slope has no units and is a **ratio.** If the x- and y-axes have different units of measure, the slope is a rate or **rate of change.** In your study of calculus, you will encounter applications involving both interpretations of slope.

EXAMPLE 2 Population Growth and Engineering Design



Population of Colorado
Figure 1.16

a. The population of Colorado was 3,827,000 in 1995 and 4,665,000 in 2005. Over this 10-year period, the average rate of change of the population was

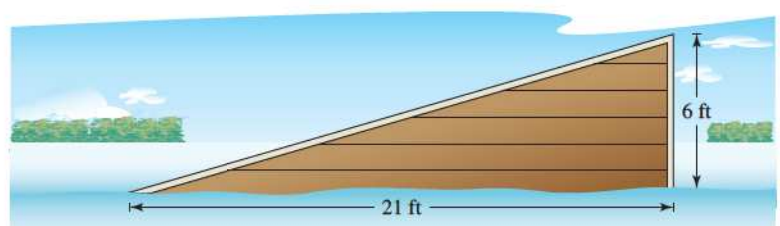
$$\begin{aligned}\text{Rate of change} &= \frac{\text{change in population}}{\text{change in years}}\\ &= \frac{4{,}665{,}000 - 3{,}827{,}000}{2005 - 1995}\\ &= 83{,}800 \text{ people per year.}\end{aligned}$$

If Colorado's population continues to increase at this same rate for the next 10 years, it will have a 2015 population of 5,503,000 (see Figure 1.16). *(Source: U.S. Census Bureau)*

b. In tournament water-ski jumping, the ramp rises to a height of 6 feet on a raft that is 21 feet long, as shown in Figure 1.17. The slope of the ski ramp is the ratio of its height (the rise) to the length of its base (the run).

$$\begin{aligned}\text{Slope of ramp} &= \frac{\text{rise}}{\text{run}} && \text{Rise is vertical change, run is horizontal change.}\\ &= \frac{6 \text{ feet}}{21 \text{ feet}}\\ &= \frac{2}{7}\end{aligned}$$

In this case, note that the slope is a ratio and has no units.



Dimensions of a water-ski ramp
Figure 1.17 ■

The rate of change found in Example 2(a) is an **average rate of change.** An average rate of change is always calculated over an interval. In this case, the interval is [1995, 2005]. In Chapter 3 you will study another type of rate of change called an *instantaneous rate of change*.

Graphing Linear Models

Many problems in analytic geometry can be classified in two basic categories: (1) Given a graph, what is its equation? and (2) Given an equation, what is its graph? The point-slope equation of a line can be used to solve problems in the first category. However, this form is not especially useful for solving problems in the second category. The form that is better suited to sketching the graph of a line is the **slope-intercept form of the equation of a line.**

> **SLOPE-INTERCEPT FORM OF THE EQUATION OF A LINE**
>
> The graph of the linear equation
>
> $$y = mx + b$$
>
> is a line having a *slope* of m and a *y-intercept* at $(0, b)$.

EXAMPLE 3 Sketching Lines in the Plane

Sketch the graph of each equation.

a. $y = 2x + 1$ **b.** $y = 2$ **c.** $3y + x - 6 = 0$

Solution

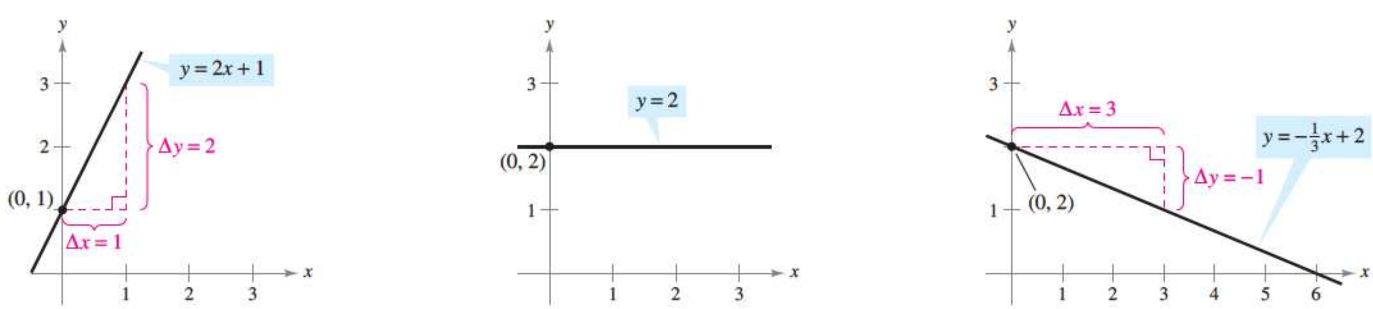
a. Because $b = 1$, the y-intercept is $(0, 1)$. Because the slope is $m = 2$, you know that the line rises two units for each unit it moves to the right, as shown in Figure 1.18(a).

b. Because $b = 2$, the y-intercept is $(0, 2)$. Because the slope is $m = 0$, you know that the line is horizontal, as shown in Figure 1.18(b).

c. Begin by writing the equation in slope-intercept form.

$$3y + x - 6 = 0 \quad \text{Write original equation.}$$
$$3y = -x + 6 \quad \text{Isolate } y\text{-term on the left.}$$
$$y = -\frac{1}{3}x + 2 \quad \text{Slope-intercept form}$$

In this form, you can see that the y-intercept is $(0, 2)$ and the slope is $m = -\frac{1}{3}$. This means that the line falls one unit for every three units it moves to the right, as shown in Figure 1.18(c).



(a) $m = 2$; line rises (b) $m = 0$; line is horizontal (c) $m = -\frac{1}{3}$; line falls

Figure 1.18

Because the slope of a vertical line is not defined, its equation cannot be written in the slope-intercept form. However, the equation of *any* line can be written in the **general form**

$$Ax + By + C = 0$$

General form of the equation of a line

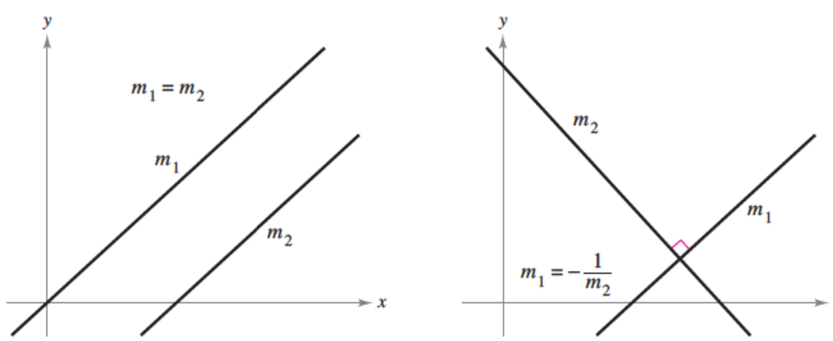
where A and B are not *both* zero. For instance, the vertical line given by $x = a$ can be represented by the general form $x - a = 0$.

SUMMARY OF EQUATIONS OF LINES

1. General form: $Ax + By + C = 0$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Point-slope form: $y - y_1 = m(x - x_1)$
5. Slope-intercept form: $y = mx + b$

Parallel and Perpendicular Lines

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular, as shown in Figure 1.19. Specifically, nonvertical lines with the same slope are parallel and nonvertical lines whose slopes are negative reciprocals are perpendicular.



Parallel lines Perpendicular lines
Figure 1.19

STUDY TIP In mathematics, the phrase "if and only if" is a way of stating two implications in one statement. For instance, the first statement at the right could be rewritten as the following two implications.

a. If two distinct nonvertical lines are parallel, then their slopes are equal.

b. If two distinct nonvertical lines have equal slopes, then they are parallel.

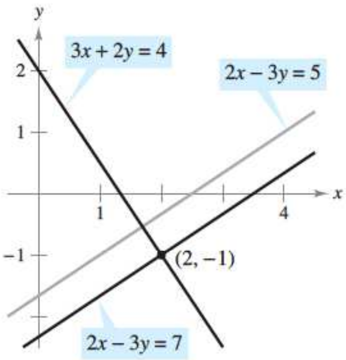
PARALLEL AND PERPENDICULAR LINES

1. Two distinct nonvertical lines are **parallel** if and only if their slopes are equal—that is, if and only if

   $$m_1 = m_2.$$

2. Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other—that is, if and only if

   $$m_1 = -\frac{1}{m_2}.$$



Lines parallel and perpendicular to $2x - 3y = 5$
Figure 1.20

EXAMPLE 4 Finding Parallel and Perpendicular Lines

Find the general forms of the equations of the lines that pass through the point $(2, -1)$ and are

a. parallel to the line $2x - 3y = 5$. **b.** perpendicular to the line $2x - 3y = 5$.

(See Figure 1.20.)

Solution By writing the linear equation $2x - 3y = 5$ in slope-intercept form, $y = \frac{2}{3}x - \frac{5}{3}$, you can see that the given line has a slope of $m = \frac{2}{3}$.

a. The line through $(2, -1)$ that is parallel to the given line also has a slope of $\frac{2}{3}$.

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$
$$y - (-1) = \tfrac{2}{3}(x - 2) \quad \text{Substitute.}$$
$$3(y + 1) = 2(x - 2) \quad \text{Simplify.}$$
$$2x - 3y - 7 = 0 \quad \text{General form}$$

Note the similarity to the original equation.

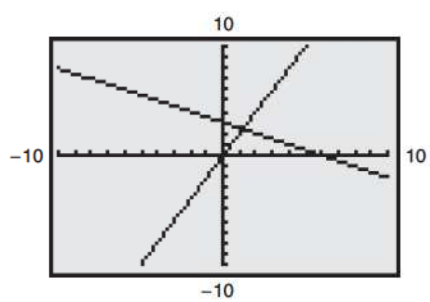
b. Using the negative reciprocal of the slope of the given line, you can determine that the slope of a line perpendicular to the given line is $-\frac{3}{2}$. So, the line through the point $(2, -1)$ that is perpendicular to the given line has the following equation.

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$
$$y - (-1) = -\tfrac{3}{2}(x - 2) \quad \text{Substitute.}$$
$$2(y + 1) = -3(x - 2) \quad \text{Simplify.}$$
$$3x + 2y - 4 = 0 \quad \text{General form}$$

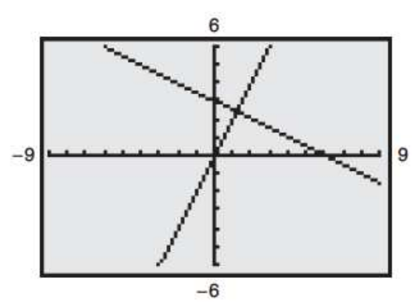
TECHNOLOGY PITFALL The slope of a line will appear distorted if you use different tick-mark spacing on the x- and y-axes. For instance, the graphing calculator screens in Figures 1.21(a) and 1.21(b) both show the lines given by

$$y = 2x \quad \text{and} \quad y = -\tfrac{1}{2}x + 3.$$

Because these lines have slopes that are negative reciprocals, they must be perpendicular. In Figure 1.21(a), however, the lines don't appear to be perpendicular because the tick-mark spacing on the x-axis is not the same as that on the y-axis. In Figure 1.21(b), the lines appear perpendicular because the tick-mark spacing on the x-axis is the same as on the y-axis. This type of viewing window is said to have a *square setting*.



(a) Tick-mark spacing on the x-axis is not the same as tick-mark spacing on the y-axis.



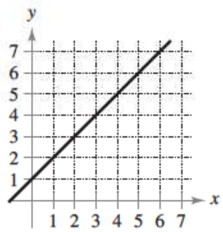
(b) Tick-mark spacing on the x-axis is the same as tick-mark spacing on the y-axis.

Figure 1.21

1.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, estimate the slope of the line from its graph. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

1.


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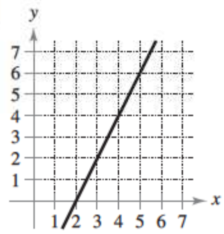

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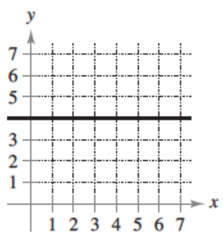

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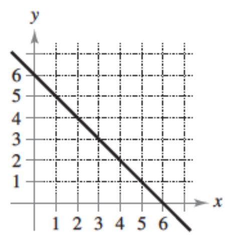

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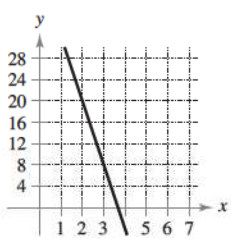

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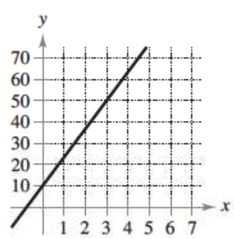


In Exercises 7 and 8, sketch the lines through the point with the given slopes. Make the sketches on the same set of coordinate axes.

	Point	*Slopes*			
7.	$(3, 4)$	(a) 1	(b) -2	(c) $-\frac{3}{2}$	(d) Undefined
8.	$(-2, 5)$	(a) 3	(b) -3	(c) $\frac{1}{3}$	(d) 0

In Exercises 9–14, plot the pair of points and find the slope of the line passing through them.

9. $(3, -4), (5, 2)$ **10.** $(1, 1), (-2, 7)$

11. $(4, 6), (4, 1)$

12. $(3, -5), (5, -5)$

13. $\left(-\frac{1}{2}, \frac{2}{3}\right), \left(-\frac{3}{4}, \frac{1}{6}\right)$

14. $\left(\frac{7}{8}, \frac{3}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)$

In Exercises 15–18, use the point on the line and the slope of the line to find three additional points that the line passes through. (There is more than one correct answer.)

	Point	*Slope*		*Point*	*Slope*
15.	$(6, 2)$	$m = 0$	**16.**	$(-4, 3)$	m is undefined.
17.	$(1, 7)$	$m = -3$	**18.**	$(-2, -2)$	$m = 2$

19. *Conveyor Design* A moving conveyor is built to rise 1 meter for each 3 meters of horizontal change.

(a) Find the slope of the conveyor.

(b) Suppose the conveyor runs between two floors in a factory. Find the length of the conveyor if the vertical distance between floors is 10 feet.

20. *Rate of Change* Each of the following is the slope of a line representing daily revenue y in terms of time x in days. Use the slope to interpret any change in daily revenue for a one-day increase in time.

(a) $m = 800$ (b) $m = 250$ (c) $m = 0$

21. *Modeling Data* The table shows the populations y (in millions) of the United States for 2000 through 2005. The variable t represents the time in years, with $t = 0$ corresponding to 2000. *(Source: U.S. Bureau of the Census)*

t	0	1	2	3	4	5
y	282.4	285.3	288.2	291.1	293.9	296.6

(a) Plot the data by hand and connect adjacent points with a line segment.

(b) Use the slope of each line segment to determine the year when the population increased least rapidly.

22. *Modeling Data* The table shows the rate r (in miles per hour) that a vehicle is traveling after t seconds.

t	5	10	15	20	25	30
r	57	74	85	84	61	43

(a) Plot the data by hand and connect adjacent points with a line segment.

(b) Use the slope of each line segment to determine the interval when the vehicle's rate changed most rapidly. How did the rate change?

In Exercises 23–28, find the slope and the y-intercept (if possible) of the line.

23. $y = 4x - 3$ **24.** $-x + y = 1$

25. $x + 5y = 20$ **26.** $6x - 5y = 15$

27. $x = 4$

28. $y = -1$

In Exercises 29–34, find an equation of the line that passes through the point and has the given slope. Sketch the line.

	Point	*Slope*		*Point*	*Slope*
29.	$(0, 3)$	$m = \frac{3}{4}$	**30.**	$(-5, -2)$	m is undefined.
31.	$(0, 0)$	$m = \frac{2}{3}$	**32.**	$(0, 4)$	$m = 0$
33.	$(3, -2)$	$m = 3$	**34.**	$(-2, 4)$	$m = -\frac{3}{5}$

In Exercises 35–44, find an equation of the line that passes through the points, and sketch the line.

35. $(0, 0), (4, 8)$
36. $(0, 0), (-1, 5)$
37. $(2, 1), (0, -3)$
38. $(-2, -2), (1, 7)$
39. $(2, 8), (5, 0)$
40. $(-3, 6), (1, 2)$
41. $(6, 3), (6, 8)$
42. $(1, -2), (3, -2)$
43. $\left(\frac{1}{2}, \frac{7}{2}\right), \left(0, \frac{3}{4}\right)$
44. $\left(\frac{7}{8}, \frac{3}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)$

45. Find an equation of the vertical line with x-intercept at 3.

46. Show that the line with intercepts $(a, 0)$ and $(0, b)$ has the following equation.

$$\frac{x}{a} + \frac{y}{b} = 1, \quad a \neq 0, b \neq 0$$

In Exercises 47–50, use the result of Exercise 46 to write an equation of the line in general form.

47. x-intercept: $(2, 0)$
y-intercept: $(0, 3)$

48. x-intercept: $\left(-\frac{2}{3}, 0\right)$
y-intercept: $(0, -2)$

49. Point on line: $(1, 2)$
x-intercept: $(a, 0)$
y-intercept: $(0, a)$
$(a \neq 0)$

50. Point on line: $(-3, 4)$
x-intercept: $(a, 0)$
y-intercept: $(0, a)$
$(a \neq 0)$

In Exercises 51–58, sketch a graph of the equation.

51. $y = -3$
52. $x = 4$
53. $y = -2x + 1$
54. $y = \frac{1}{3}x - 1$
55. $y - 2 = \frac{3}{2}(x - 1)$
56. $y - 1 = 3(x + 4)$
57. $2x - y - 3 = 0$
58. $x + 2y + 6 = 0$

59. ***Square Setting*** Use a graphing utility to graph the lines $y = 2x - 3$ and $y = -\frac{1}{2}x + 1$ in each viewing window. Compare the graphs. Do the lines appear perpendicular? Are the lines perpendicular? Explain.

(a)
Xmin = -5
Xmax = 5
Xscl = 1
Ymin = -5
Ymax = 5
Yscl = 1

(b)
Xmin = -6
Xmax = 6
Xscl = 1
Ymin = -4
Ymax = 4
Yscl = 1

CAPSTONE

60. A line is represented by the equation $ax + by = 4$.

(a) When is the line parallel to the x-axis?

(b) When is the line parallel to the y-axis?

(c) Give values for a and b such that the line has a slope of $\frac{5}{8}$.

(d) Give values for a and b such that the line is perpendicular to $y = \frac{2}{5}x + 3$.

(e) Give values for a and b such that the line coincides with the graph of $5x + 6y = 8$.

In Exercises 61–66, write the general forms of the equations of the lines through the point (a) parallel to the given line and (b) perpendicular to the given line.

	Point	*Line*		*Point*	*Line*
61.	$(-7, -2)$	$x = 1$	**62.**	$(-1, 0)$	$y = -3$
63.	$(2, 1)$	$4x - 2y = 3$	**64.**	$(-3, 2)$	$x + y = 7$
65.	$\left(\frac{3}{4}, \frac{7}{8}\right)$	$5x - 3y = 0$	**66.**	$(4, -5)$	$3x + 4y = 7$

Rate of Change **In Exercises 67–70, you are given the dollar value of a product in 2008 *and* the rate at which the value of the product is expected to change during the next 5 years. Write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 0$ represent 2000.)**

	2008 Value	*Rate*
67.	\$1850	\$250 increase per year
68.	\$156	\$4.50 increase per year
69.	\$17,200	\$1600 decrease per year
70.	\$245,000	\$5600 decrease per year

In Exercises 71 and 72, use a graphing utility to graph the parabolas and find their points of intersection. Find an equation of the line through the points of intersection and graph the line in the same viewing window.

71. $y = x^2$
$y = 4x - x^2$

72. $y = x^2 - 4x + 3$
$y = -x^2 + 2x + 3$

In Exercises 73 and 74, determine whether the points are collinear. (Three points are *collinear* if they lie on the same line.)

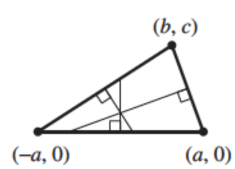
73. $(-2, 1), (-1, 0), (2, -2)$

74. $(0, 4), (7, -6), (-5, 11)$

WRITING ABOUT CONCEPTS

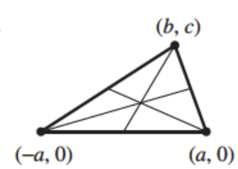
In Exercises 75–77, find the coordinates of the point of intersection of the given segments. Explain your reasoning.

75.



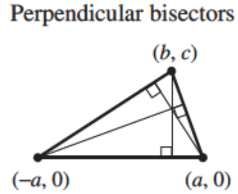
Perpendicular bisectors

76.



Medians

77.



Altitudes

78. Show that the points of intersection in Exercises 75, 76, and 77 are collinear.

79. *Temperature Conversion* Find a linear equation that expresses the relationship between the temperature in degrees Celsius C and degrees Fahrenheit F. Use the fact that water freezes at 0°C (32°F) and boils at 100°C (212°F). Use the equation to convert 72°F to degrees Celsius.

80. *Reimbursed Expenses* A company reimburses its sales representatives \$175 per day for lodging and meals plus 48¢ per mile driven. Write a linear equation giving the daily cost C to the company in terms of x, the number of miles driven. How much does it cost the company if a sales representative drives 137 miles on a given day?

81. *Career Choice* An employee has two options for positions in a large corporation. One position pays \$14.50 per hour *plus* an additional unit rate of \$0.75 per unit produced. The other pays \$11.20 per hour *plus* a unit rate of \$1.30.

(a) Find linear equations for the hourly wages W in terms of x, the number of units produced per hour, for each option.

(b) Use a graphing utility to graph the linear equations and find the point of intersection.

(c) Interpret the meaning of the point of intersection of the graphs in part (b). How would you use this information to select the correct option if the goal were to obtain the highest hourly wage?

82. *Straight-Line Depreciation* A small business purchases a piece of equipment for \$875. After 5 years the equipment will be outdated, having no value.

(a) Write a linear equation giving the value y of the equipment in terms of the time x, $0 \leq x \leq 5$.

(b) Find the value of the equipment when $x = 2$.

(c) Estimate (to two-decimal-place accuracy) the time when the value of the equipment is \$200.

83. *Apartment Rental* A real estate office manages an apartment complex with 50 units. When the rent is \$780 per month, all 50 units are occupied. However, when the rent is \$825, the average number of occupied units drops to 47. Assume that the relationship between the monthly rent p and the demand x is linear. (*Note:* The term *demand* refers to the number of occupied units.)

(a) Write a linear equation giving the demand x in terms of the rent p.

(b) *Linear extrapolation* Use a graphing utility to graph the demand equation and use the *trace* feature to predict the number of units occupied if the rent is raised to \$855.

(c) *Linear interpolation* Predict the number of units occupied if the rent is lowered to \$795. Verify graphically.

84. *Modeling Data* An instructor gives regular 20-point quizzes and 100-point exams in a mathematics course. Average scores for six students, given as ordered pairs (x, y), where x is the average quiz score and y is the average test score, are (18, 87), (10, 55), (19, 96), (16, 79), (13, 76), and (15, 82).

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) Use a graphing utility to plot the points and graph the regression line in the same viewing window.

(c) Use the regression line to predict the average exam score for a student with an average quiz score of 17.

(d) Interpret the meaning of the slope of the regression line.

(e) The instructor adds 4 points to the average test score of everyone in the class. Describe the changes in the positions of the plotted points and the change in the equation of the line.

85. *Tangent Line* Find an equation of the line tangent to the circle $x^2 + y^2 = 169$ at the point $(5, 12)$.

86. *Tangent Line* Find an equation of the line tangent to the circle $(x - 1)^2 + (y - 1)^2 = 25$ at the point $(4, -3)$.

Distance **In Exercises 87–92, find the distance between the point and line, or between the lines, using the formula for the distance between the point (x_1, y_1) and the line $Ax + By + C = 0$.**

$$\textbf{Distance} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

87. Point: $(0, 0)$
Line: $4x + 3y = 10$

88. Point: $(2, 3)$
Line: $4x + 3y = 10$

89. Point: $(-2, 1)$
Line: $x - y - 2 = 0$

90. Point: $(6, 2)$
Line: $x = -1$

91. Line: $x + y = 1$
Line: $x + y = 5$

92. Line: $3x - 4y = 1$
Line: $3x - 4y = 10$

93. Show that the distance between the point (x_1, y_1) and the line $Ax + By + C = 0$ is

$$\text{Distance} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

94. Write the distance d between the point $(3, 1)$ and the line $y = mx + 4$ in terms of m. Use a graphing utility to graph the equation. When is the distance 0? Explain the result geometrically.

95. Prove that the diagonals of a rhombus intersect at right angles. (A rhombus is a quadrilateral with sides of equal lengths.)

96. Prove that the figure formed by connecting consecutive midpoints of the sides of any quadrilateral is a parallelogram.

97. Prove that if the points (x_1, y_1) and (x_2, y_2) lie on the same line as (x_1^*, y_1^*) and (x_2^*, y_2^*), then

$$\frac{y_2^* - y_1^*}{x_2^* - x_1^*} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Assume $x_1 \neq x_2$ and $x_1^* \neq x_2^*$.

98. Prove that if the slopes of two nonvertical lines are negative reciprocals of each other, then the lines are perpendicular.

True or False? **In Exercises 99 and 100, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

99. The lines represented by $ax + by = c_1$ and $bx - ay = c_2$ are perpendicular. Assume $a \neq 0$ and $b \neq 0$.

100. It is possible for two lines with positive slopes to be perpendicular to each other.

1.3 Functions and Their Graphs

- Use function notation to represent and evaluate a function.
- Find the domain and range of a function.
- Sketch the graph of a function.
- Identify different types of transformations of functions.
- Classify functions and recognize combinations of functions.

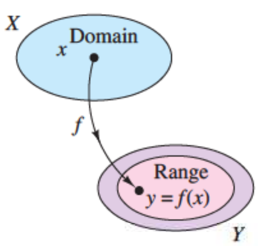
Functions and Function Notation

A **relation** between two sets X and Y is a set of ordered pairs, each of the form (x, y), where x is a member of X and y is a member of Y. A **function** from X to Y is a relation between X and Y having the property that any two ordered pairs with the same x-value also have the same y-value. The variable x is the **independent variable,** and the variable y is the **dependent variable.**

Many real-life situations can be modeled by functions. For instance, the area A of a circle is a function of the circle's radius r.

$$A = \pi r^2 \qquad \text{A is a function of r.}$$

In this case r is the independent variable and A is the dependent variable.



A real-valued function f of a real variable x
Figure 1.22

DEFINITION OF A REAL-VALUED FUNCTION OF A REAL VARIABLE

Let X and Y be sets of real numbers. A **real-valued function f of a real variable x** from X to Y is a correspondence that assigns to each number x in X exactly one number y in Y.

The **domain** of f is the set X. The number y is the **image** of x under f and is denoted by $f(x)$, which is called the **value of f at x.** The **range** of f is a subset of Y and consists of all images of numbers in X (see Figure 1.22).

Functions can be specified in a variety of ways. In this text, however, we will concentrate primarily on functions that are given by equations involving the dependent and independent variables. For instance, the equation

$$x^2 + 2y = 1 \qquad \text{Equation in implicit form}$$

defines y, the dependent variable, as a function of x, the independent variable. To **evaluate** this function (that is, to find the y-value that corresponds to a given x-value), it is convenient to isolate y on the left side of the equation.

$$y = \frac{1}{2}(1 - x^2) \qquad \text{Equation in explicit form}$$

Using f as the name of the function, you can write this equation as

$$f(x) = \frac{1}{2}(1 - x^2). \qquad \text{Function notation}$$

The original equation, $x^2 + 2y = 1$, **implicitly** defines y as a function of x. When you solve the equation for y, you are writing the equation in **explicit** form.

Function notation has the advantage of clearly identifying the dependent variable as $f(x)$ while at the same time telling you that x is the independent variable and that the function itself is "f." The symbol $f(x)$ is read "f of x." Function notation allows you to be less wordy. Instead of asking "What is the value of y that corresponds to $x = 3$?" you can ask, "What is $f(3)$?"

FUNCTION NOTATION

The word *function* was first used by Gottfried Wilhelm Leibniz in 1694 as a term to denote any quantity connected with a curve, such as the coordinates of a point on a curve or the slope of a curve. Forty years later, Leonhard Euler used the word *function* to describe any expression made up of a variable and some constants. He introduced the notation $y = f(x)$.

In an equation that defines a function, the role of the variable x is simply that of a placeholder. For instance, the function given by

$$f(x) = 2x^2 - 4x + 1$$

can be described by the form

$$f(\quad) = 2(\quad)^2 - 4(\quad) + 1$$

where parentheses are used instead of x. To evaluate $f(-2)$, simply place -2 in each set of parentheses.

$$\begin{aligned} f(-2) &= 2(-2)^2 - 4(-2) + 1 && \text{Substitute } -2 \text{ for } x. \\ &= 2(4) + 8 + 1 && \text{Simplify.} \\ &= 17 && \text{Simplify.} \end{aligned}$$

NOTE Although f is often used as a convenient function name and x as the independent variable, you can use other symbols. For instance, the following equations all define the same function.

$$\begin{aligned} f(x) &= x^2 - 4x + 7 && \text{Function name is } f, \text{ independent variable is } x. \\ f(t) &= t^2 - 4t + 7 && \text{Function name is } f, \text{ independent variable is } t. \\ g(s) &= s^2 - 4s + 7 && \text{Function name is } g, \text{ independent variable is } s. \end{aligned}$$

EXAMPLE 1 Evaluating a Function

For the function f defined by $f(x) = x^2 + 7$, evaluate each of the following.

a. $f(3a)$ **b.** $f(b - 1)$ **c.** $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}, \quad \Delta x \neq 0$

Solution

a.
$$\begin{aligned} f(3a) &= (3a)^2 + 7 && \text{Substitute } 3a \text{ for } x. \\ &= 9a^2 + 7 && \text{Simplify.} \end{aligned}$$

b.
$$\begin{aligned} f(b - 1) &= (b - 1)^2 + 7 && \text{Substitute } b - 1 \text{ for } x. \\ &= b^2 - 2b + 1 + 7 && \text{Expand binomial.} \\ &= b^2 - 2b + 8 && \text{Simplify.} \end{aligned}$$

c.
$$\begin{aligned} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \frac{[(x + \Delta x)^2 + 7] - (x^2 + 7)}{\Delta x} \\ &= \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 7 - x^2 - 7}{\Delta x} \\ &= \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \frac{\cancel{\Delta x}(2x + \Delta x)}{\cancel{\Delta x}} \\ &= 2x + \Delta x, \quad \Delta x \neq 0 \end{aligned}$$

STUDY TIP In calculus, it is important to specify clearly the domain of a function or expression. For instance, in Example 1(c), the two expressions

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{and} \quad 2x + \Delta x, \quad \Delta x \neq 0$$

are equivalent because $\Delta x = 0$ is excluded from the domain of each expression. Without a stated domain restriction, the two expressions would not be equivalent.

NOTE The expression in Example 1(c) is called a *difference quotient* and has a special significance in calculus. You will learn more about this in Chapter 3.

The Domain and Range of a Function

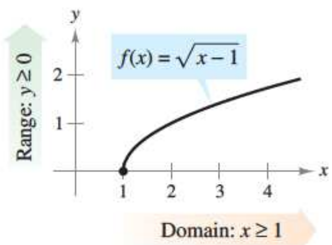
The domain of a function may be described explicitly, or it may be described *implicitly* by an equation used to define the function. The implied domain is the set of all real numbers for which the equation is defined, whereas an explicitly defined domain is one that is given along with the function. For example, the function given by

$$f(x) = \frac{1}{x^2 - 4}, \quad 4 \le x \le 5$$

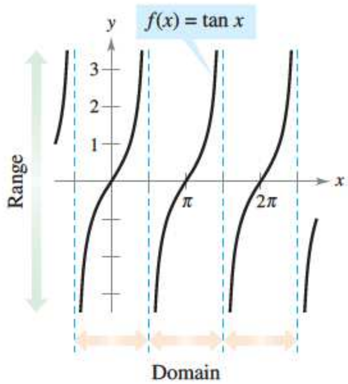
has an explicitly defined domain given by $\{x\colon 4 \le x \le 5\}$. On the other hand, the function given by

$$g(x) = \frac{1}{x^2 - 4}$$

has an implied domain that is the set $\{x\colon x \ne \pm 2\}$.



(a) The domain of f is $[1, \infty)$ and the range is $[0, \infty)$.



(b) The domain of f is all x-values such that $x \ne \frac{\pi}{2} + n\pi$ and the range is $(-\infty, \infty)$.

Figure 1.23

EXAMPLE 2 Finding the Domain and Range of a Function

a. The domain of the function

$$f(x) = \sqrt{x - 1}$$

is the set of all x-values for which $x - 1 \ge 0$, which is the interval $[1, \infty)$. To find the range, observe that $f(x) = \sqrt{x - 1}$ is never negative. So, the range is the interval $[0, \infty)$, as indicated in Figure 1.23(a).

b. The domain of the tangent function, shown in Figure 1.23(b),

$$f(x) = \tan x$$

is the set of all x-values such that

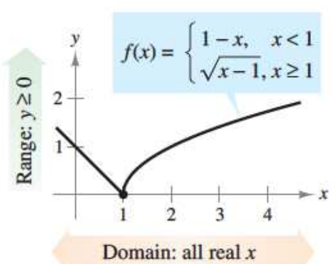
$$x \ne \frac{\pi}{2} + n\pi, \quad n \text{ is an integer.} \qquad \text{Domain of tangent function}$$

The range of this function is the set of all real numbers. For a review of the characteristics of this and other trigonometric functions, see Appendix C.

EXAMPLE 3 A Function Defined by More than One Equation

Determine the domain and range of the function.

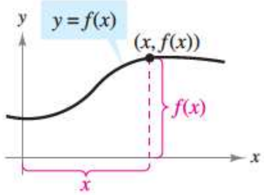
$$f(x) = \begin{cases} 1 - x, & \text{if } x < 1 \\ \sqrt{x - 1}, & \text{if } x \ge 1 \end{cases}$$



The domain of f is $(-\infty, \infty)$ and the range is $[0, \infty)$.
Figure 1.24

Solution Because f is defined for $x < 1$ and $x \ge 1$, the domain is the entire set of real numbers. On the portion of the domain for which $x \ge 1$, the function behaves as in Example 2(a). For $x < 1$, the values of $1 - x$ are positive. So, the range of the function is the interval $[0, \infty)$. (See Figure 1.24.) ■

A function from X to Y is **one-to-one** if to each y-value in the range there corresponds exactly one x-value in the domain. For instance, the function given in Example 2(a) is one-to-one, whereas the functions given in Examples 2(b) and 3 are not one-to-one. A function from X to Y is **onto** if its range consists of all of Y.



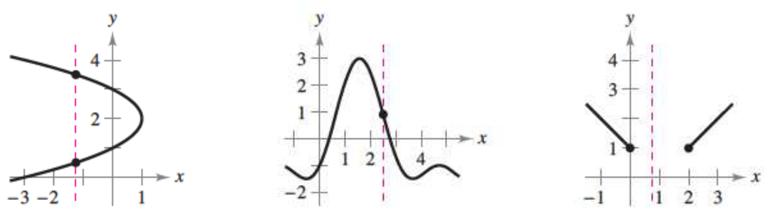
The graph of a function
Figure 1.25

The Graph of a Function

The graph of the function $y = f(x)$ consists of all points $(x, f(x))$, where x is in the domain of f. In Figure 1.25, note that

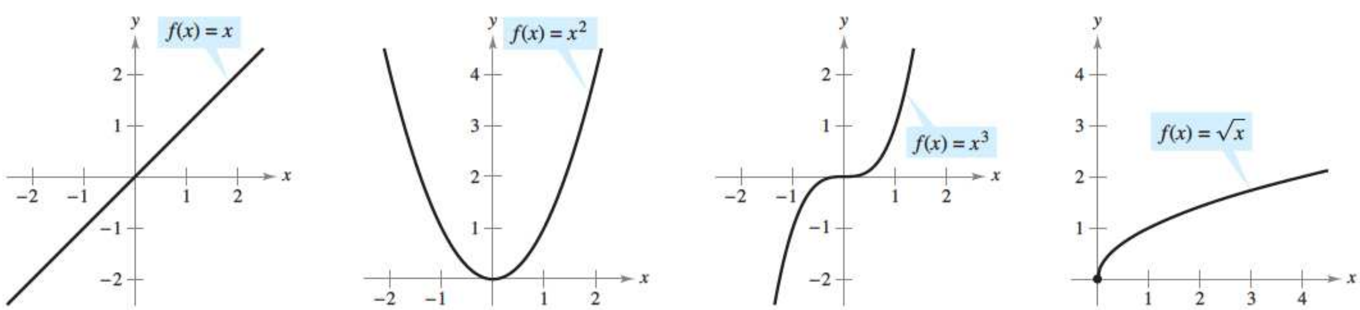
$$x = \text{the directed distance from the } y\text{-axis}$$
$$f(x) = \text{the directed distance from the } x\text{-axis.}$$

A vertical line can intersect the graph of a function of x at most *once*. This observation provides a convenient visual test, called the **Vertical Line Test,** for functions of x. That is, a graph in the coordinate plane is the graph of a function of x if and only if no vertical line intersects the graph at more than one point. For example, in Figure 1.26(a), you can see that the graph does not define y as a function of x because a vertical line intersects the graph twice. In Figures 1.26(b) and (c), the graphs do define y as a function of x.

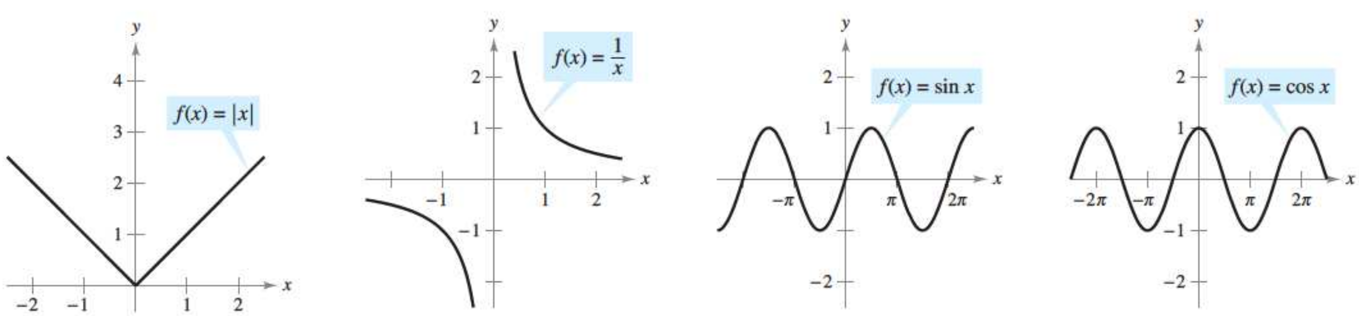


(a) Not a function of x **(b)** A function of x **(c)** A function of x
Figure 1.26

Figure 1.27 shows the graphs of eight basic functions. You should be able to recognize these graphs. (Graphs of the other four basic trigonometric functions are shown in Appendix C.)



Identity function Squaring function Cubing function Square root function

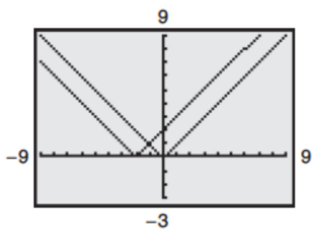


Absolute value function Rational function Sine function Cosine function

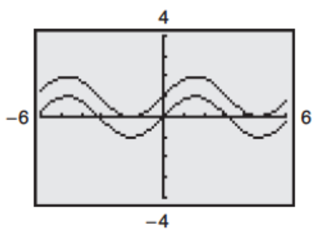
The graphs of eight basic functions
Figure 1.27

EXPLORATION

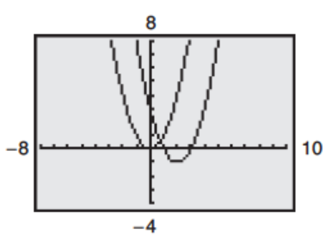
Writing Equations for Functions
Each of the graphing utility screens below shows the graph of one of the eight basic functions shown on page 22. Each screen also shows a transformation of the graph. Describe the transformation. Then use your description to write an equation for the transformation.



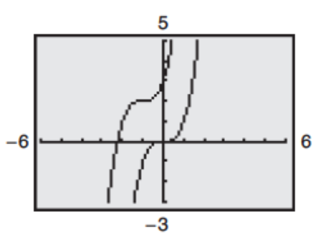
a.



b.



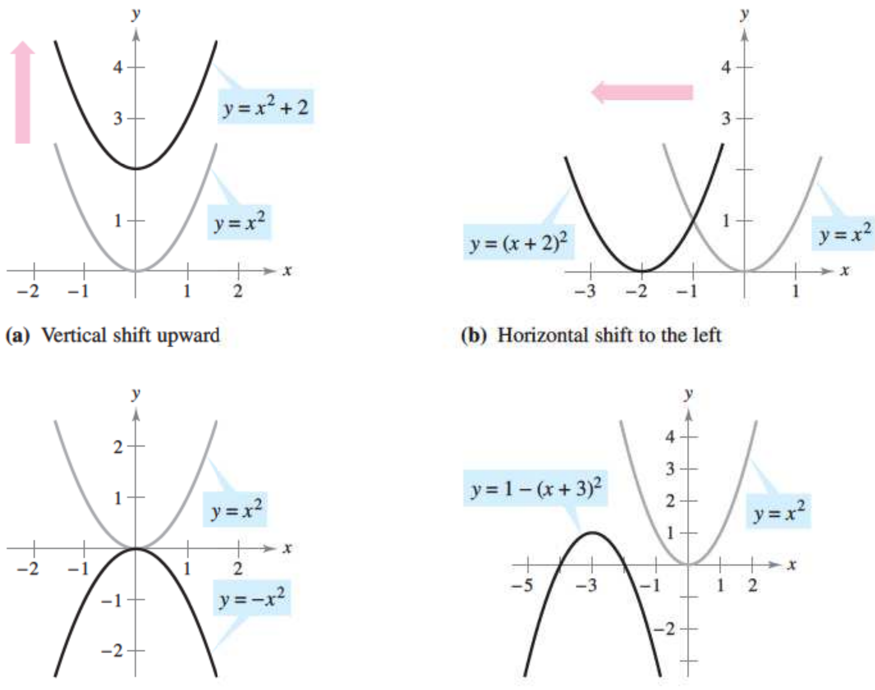
c.



d.

Transformations of Functions

Some families of graphs have the same basic shape. For example, compare the graph of $y = x^2$ with the graphs of the four other quadratic functions shown in Figure 1.28.



(a) Vertical shift upward
(b) Horizontal shift to the left
(c) Reflection
(d) Shift left, reflect, and shift upward

Figure 1.28

Each of the graphs in Figure 1.28 is a **transformation** of the graph of $y = x^2$. The three basic types of transformations illustrated by these graphs are vertical shifts, horizontal shifts, and reflections. Function notation lends itself well to describing transformations of graphs in the plane. For instance, if $f(x) = x^2$ is considered to be the original function in Figure 1.28, the transformations shown can be represented by the following equations.

$y = f(x) + 2$	Vertical shift up two units
$y = f(x + 2)$	Horizontal shift to the left two units
$y = -f(x)$	Reflection about the x-axis
$y = -f(x + 3) + 1$	Shift left three units, reflect about x-axis, and shift up one unit

BASIC TYPES OF TRANSFORMATIONS $(c > 0)$

Original graph:	$y = f(x)$
Horizontal shift c units to the **right:**	$y = f(x - c)$
Horizontal shift c units to the **left:**	$y = f(x + c)$
Vertical shift c units **downward:**	$y = f(x) - c$
Vertical shift c units **upward:**	$y = f(x) + c$
Reflection (about the x-axis):	$y = -f(x)$
Reflection (about the y-axis):	$y = f(-x)$
Reflection (about the origin):	$y = -f(-x)$

Michael Nicholson/Corbis

LEONHARD EULER (1707–1783)

In addition to making major contributions to almost every branch of mathematics, Euler was one of the first to apply calculus to real-life problems in physics. His extensive published writings include such topics as shipbuilding, acoustics, optics, astronomy, mechanics, and magnetism.

Classifications and Combinations of Functions

The modern notion of a function is derived from the efforts of many seventeenth- and eighteenth-century mathematicians. Of particular note was Leonhard Euler, to whom we are indebted for the function notation $y = f(x)$. By the end of the eighteenth century, mathematicians and scientists had concluded that many real-world phenomena could be represented by mathematical models taken from a collection of functions called **elementary functions.** Elementary functions fall into three categories.

1. Algebraic functions (polynomial, radical, rational)
2. Trigonometric functions (sine, cosine, tangent, and so on)
3. Exponential and logarithmic functions

You can review the trigonometric functions in Appendix C. The other nonalgebraic functions, such as the inverse trigonometric functions and the exponential and logarithmic functions, are introduced in Sections 1.5 and 1.6.

The most common type of algebraic function is a **polynomial function**

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer. The numbers a_i are **coefficients,** with a_n the **leading coefficient** and a_0 the **constant term** of the polynomial function. If $a_n \neq 0$, then n is the degree of the polynomial function. The zero polynomial $f(x) = 0$ is not assigned a degree. It is common practice to use subscript notation for coefficients of general polynomial functions, but for polynomial functions of low degree, the following simpler forms are often used. (Note that $a \neq 0$.)

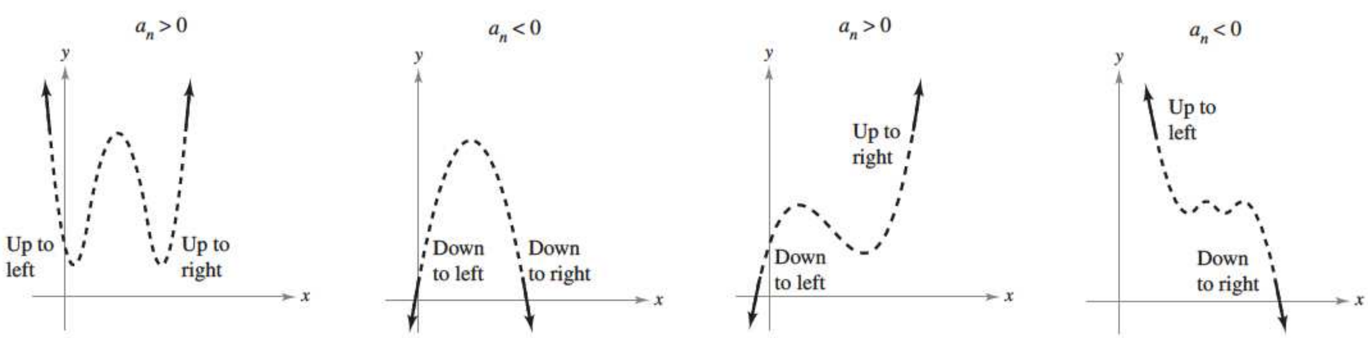
Zeroth degree:	$f(x) = a$	Constant function
First degree:	$f(x) = ax + b$	Linear function
Second degree:	$f(x) = ax^2 + bx + c$	Quadratic function
Third degree:	$f(x) = ax^3 + bx^2 + cx + d$	Cubic function

FOR FURTHER INFORMATION For more on the history of the concept of a function, see the article "Evolution of the Function Concept: A Brief Survey" by Israel Kleiner in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

Although the graph of a polynomial function can have several turns, eventually the graph will rise or fall without bound as x moves to the right or left. Whether the graph of

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$

eventually rises or falls can be determined by the function's degree (odd or even) and by the leading coefficient a_n, as indicated in Figure 1.29. Note that the dashed portions of the graphs indicate that the **Leading Coefficient Test** determines *only* the right and left behavior of the graph.



Graphs of polynomial functions of even degree ($n \geq 2$)

Graphs of polynomial functions of odd degree

The Leading Coefficient Test for polynomial functions
Figure 1.29

Just as a rational number can be written as the quotient of two integers, a **rational function** can be written as the quotient of two polynomials. Specifically, a function f is rational if it has the form

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0$$

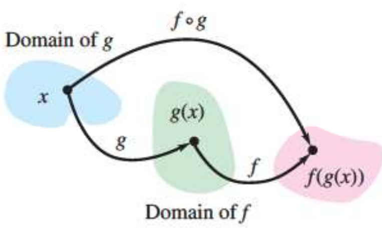
where $p(x)$ and $q(x)$ are polynomials.

Polynomial functions and rational functions are examples of **algebraic functions.** An algebraic function of x is one that can be expressed as a finite number of sums, differences, multiples, quotients, and radicals involving x^n. For example, $f(x) = \sqrt{x+1}$ is algebraic. Functions that are not algebraic are **transcendental.** For instance, the trigonometric functions are transcendental.

Two functions can be combined in various ways to create new functions. For example, given $f(x) = 2x - 3$ and $g(x) = x^2 + 1$, you can form the functions shown.

$$(f+g)(x) = f(x) + g(x) = (2x-3) + (x^2+1) \qquad \text{Sum}$$
$$(f-g)(x) = f(x) - g(x) = (2x-3) - (x^2+1) \qquad \text{Difference}$$
$$(fg)(x) = f(x)g(x) = (2x-3)(x^2+1) \qquad \text{Product}$$
$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{2x-3}{x^2+1} \qquad \text{Quotient}$$

You can combine two functions in yet another way, called **composition.** The resulting function is called a **composite function.**



The domain of the composite function $f \circ g$
Figure 1.30

DEFINITION OF COMPOSITE FUNCTION

Let f and g be functions. The function given by $(f \circ g)(x) = f(g(x))$ is called the **composite** of f with g. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f (see Figure 1.30).

The composite of f with g may not be equal to the composite of g with f.

EXAMPLE 4 Finding Composites of Functions

Given $f(x) = 2x - 3$ and $g(x) = \cos x$, find the following.

a. $f \circ g$ **b.** $g \circ f$

Solution

a.
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g\\ &= f(\cos x) && \text{Substitute } \cos x \text{ for } g(x).\\ &= 2(\cos x) - 3 && \text{Definition of } f(x)\\ &= 2\cos x - 3 && \text{Simplify.}\end{aligned}$$

b.
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f\\ &= g(2x-3) && \text{Substitute } 2x-3 \text{ for } f(x).\\ &= \cos(2x-3) && \text{Definition of } g(x)\end{aligned}$$

Note that $(f \circ g)(x) \neq (g \circ f)(x)$. ■

EXPLORATION

Graph each of the following functions with a graphing utility. Determine whether the function is *even*, *odd*, or *neither*.

$f(x) = x^2 - x^4$

$g(x) = 2x^3 + 1$

$h(x) = x^5 - 2x^3 + x$

$j(x) = 2 - x^6 - x^8$

$k(x) = x^5 - 2x^4 + x - 2$

$p(x) = x^9 + 3x^5 - x^3 + x$

Describe a way to identify a function as odd or even by inspecting its equation.

In Section 1.1, an x-intercept of a graph was defined to be a point $(a, 0)$ at which the graph crosses the x-axis. If the graph represents a function f, the number a is a **zero** of f. In other words, *the zeros of a function f are the solutions of the equation $f(x) = 0$.* For example, the function $f(x) = x - 4$ has a zero at $x = 4$ because $f(4) = 0$.

In Section 1.1 you also studied different types of symmetry. In the terminology of functions, a function is **even** if its graph is symmetric with respect to the y-axis, and is **odd** if its graph is symmetric with respect to the origin. The symmetry tests in Section 1.1 yield the following test for even and odd functions.

TEST FOR EVEN AND ODD FUNCTIONS

The function $y = f(x)$ is **even** if $f(-x) = f(x)$.

The function $y = f(x)$ is **odd** if $f(-x) = -f(x)$.

NOTE Except for the constant function $f(x) = 0$, the graph of a function of x cannot have symmetry with respect to the x-axis because it then would fail the Vertical Line Test for the graph of the function.

EXAMPLE 5 Even and Odd Functions and Zeros of Functions

Determine whether each function is even, odd, or neither. Then find the zeros of the function.

a. $f(x) = x^3 - x$ **b.** $g(x) = 1 + \cos x$

Solution

a. This function is odd because

$$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x).$$

The zeros of f are found as shown.

$$x^3 - x = 0 \qquad \text{Let } f(x) = 0.$$
$$x(x^2 - 1) = 0 \qquad \text{Factor.}$$
$$x(x - 1)(x + 1) = 0 \qquad \text{Factor.}$$
$$x = 0, 1, -1 \qquad \text{Zeros of } f$$

See Figure 1.31(a).

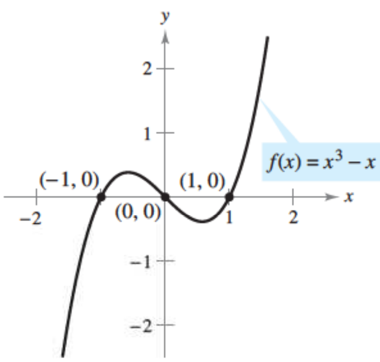
b. This function is even because

$$g(-x) = 1 + \cos(-x) = 1 + \cos x = g(x). \qquad \cos(-x) = \cos(x)$$

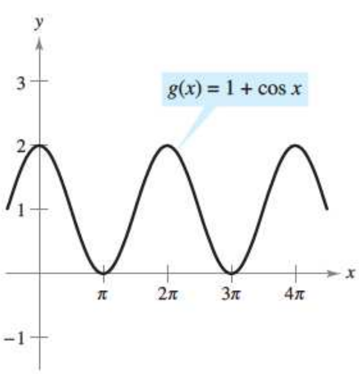
The zeros of g are found as shown.

$$1 + \cos x = 0 \qquad \text{Let } g(x) = 0.$$
$$\cos x = -1 \qquad \text{Subtract 1 from each side.}$$
$$x = (2n + 1)\pi,\ n \text{ is an integer.} \qquad \text{Zeros of } g$$

See Figure 1.31(b).



(a) Odd function



(b) Even function

Figure 1.31

NOTE Each of the functions in Example 5 is either even or odd. However, some functions, such as $f(x) = x^2 + x + 1$, are neither even nor odd.

1.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use the graphs of f and g to answer the following.

(a) Identify the domains and ranges of f and g.

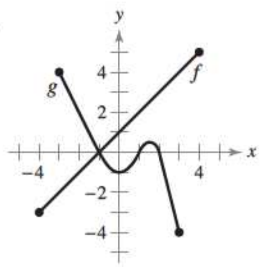
(b) Identify $f(-2)$ and $g(3)$.

(c) For what value(s) of x is $f(x) = g(x)$?

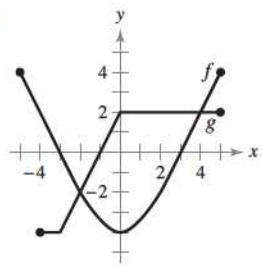
(d) Estimate the solution(s) of $f(x) = 2$.

(e) Estimate the solutions of $g(x) = 0$.

1.



2.



In Exercises 3–12, evaluate (if possible) the function at the given value(s) of the independent variable. Simplify the results.

3. $f(x) = 7x - 4$
(a) $f(0)$
(b) $f(-3)$
(c) $f(b)$
(d) $f(x - 1)$

4. $f(x) = \sqrt{x + 5}$
(a) $f(-4)$
(b) $f(11)$
(c) $f(-8)$
(d) $f(x + \Delta x)$

5. $g(x) = 5 - x^2$
(a) $g(0)$
(b) $g(\sqrt{5})$
(c) $g(-2)$
(d) $g(t - 1)$

6. $g(x) = x^2(x - 4)$
(a) $g(4)$
(b) $g(\frac{3}{2})$
(c) $g(c)$
(d) $g(t + 4)$

7. $f(x) = \cos 2x$
(a) $f(0)$
(b) $f(-\pi/4)$
(c) $f(\pi/3)$

8. $f(x) = \sin x$
(a) $f(\pi)$
(b) $f(5\pi/4)$
(c) $f(2\pi/3)$

9. $f(x) = x^3$
$$\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

10. $f(x) = 3x - 1$
$$\frac{f(x) - f(1)}{x - 1}$$

11. $f(x) = \dfrac{1}{\sqrt{x - 1}}$
$$\frac{f(x) - f(2)}{x - 2}$$

12. $f(x) = x^3 - x$
$$\frac{f(x) - f(1)}{x - 1}$$

In Exercises 13–20, find the domain and range of the function.

13. $f(x) = 4x^2$

14. $g(x) = x^2 - 5$

15. $g(x) = \sqrt{6x}$

16. $h(x) = -\sqrt{x + 3}$

17. $f(t) = \sec \dfrac{\pi t}{4}$

18. $h(t) = \cot t$

19. $f(x) = \dfrac{3}{x}$

20. $g(x) = \dfrac{2}{x - 1}$

In Exercises 21–26, find the domain of the function.

21. $f(x) = \sqrt{x} + \sqrt{1 - x}$

22. $f(x) = \sqrt{x^2 - 3x + 2}$

23. $g(x) = \dfrac{2}{1 - \cos x}$

24. $h(x) = \dfrac{1}{\sin x - \frac{1}{2}}$

25. $f(x) = \dfrac{1}{|x + 3|}$

26. $g(x) = \dfrac{1}{|x^2 - 4|}$

In Exercises 27–30, evaluate the function as indicated. Determine its domain and range.

27. $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \ge 0 \end{cases}$

(a) $f(-1)$ (b) $f(0)$ (c) $f(2)$ (d) $f(t^2 + 1)$

28. $f(x) = \begin{cases} x^2 + 2, & x \le 1 \\ 2x^2 + 2, & x > 1 \end{cases}$

(a) $f(-2)$ (b) $f(0)$ (c) $f(1)$ (d) $f(s^2 + 2)$

29. $f(x) = \begin{cases} |x| + 1, & x < 1 \\ -x + 1, & x \ge 1 \end{cases}$

(a) $f(-3)$ (b) $f(1)$ (c) $f(3)$ (d) $f(b^2 + 1)$

30. $f(x) = \begin{cases} \sqrt{x + 4}, & x \le 5 \\ (x - 5)^2, & x > 5 \end{cases}$

(a) $f(-3)$ (b) $f(0)$ (c) $f(5)$ (d) $f(10)$

In Exercises 31–38, sketch a graph of the function and find its domain and range. Use a graphing utility to verify your graph.

31. $f(x) = 4 - x$

32. $g(x) = \dfrac{4}{x}$

33. $h(x) = \sqrt{x - 6}$

34. $f(x) = \frac{1}{4}x^3 + 3$

35. $f(x) = \sqrt{9 - x^2}$

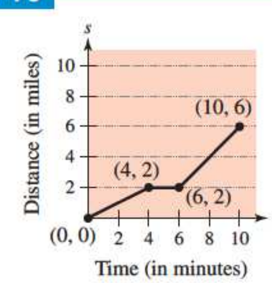
36. $f(x) = x + \sqrt{4 - x^2}$

37. $g(t) = 3 \sin \pi t$

38. $h(\theta) = -5 \cos \dfrac{\theta}{2}$

WRITING ABOUT CONCEPTS

39. The graph of the distance that a student drives in a 10-minute trip to school is shown in the figure. Give a verbal description of characteristics of the student's drive to school.

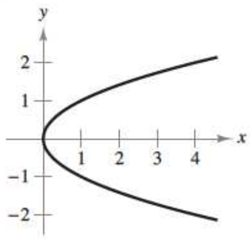


WRITING ABOUT CONCEPTS (continued)

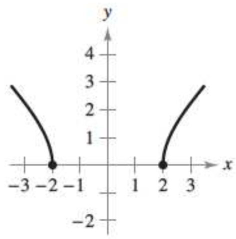
40. A student who commutes 27 miles to attend college remembers, after driving a few minutes, that a term paper that is due has been forgotten. Driving faster than usual, the student returns home, picks up the paper, and once again starts toward school. Sketch a possible graph of the student's distance from home as a function of time.

In Exercises 41–44, use the Vertical Line Test to determine whether y is a function of x. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

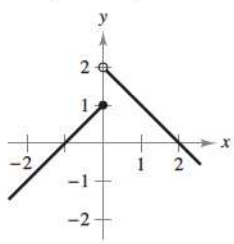
41. $x - y^2 = 0$



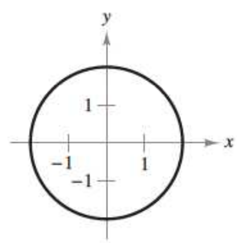
42. $\sqrt{x^2 - 4} - y = 0$



43. $y = \begin{cases} x + 1, & x \le 0 \\ -x + 2, & x > 0 \end{cases}$



44. $x^2 + y^2 = 4$



In Exercises 45–48, determine whether y is a function of x.

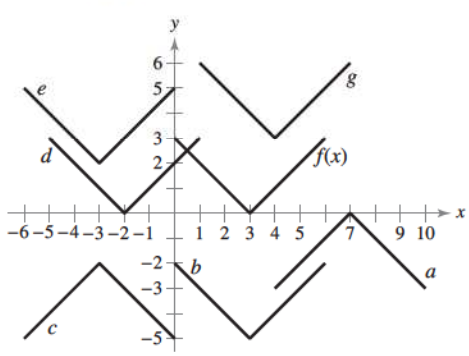
45. $x^2 + y^2 = 16$

46. $x^2 + y = 16$

47. $y^2 = x^2 - 1$

48. $x^2y - x^2 + 4y = 0$

In Exercises 49–54, use the graph of $y = f(x)$ to match the function with its graph.



49. $y = f(x + 5)$

50. $y = f(x) - 5$

51. $y = -f(-x) - 2$

52. $y = -f(x - 4)$

53. $y = f(x + 6) + 2$

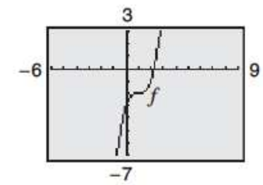
54. $y = f(x - 1) + 3$

55. Use the graph of f shown in the figure to sketch the graph of each function. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(a) $f(x + 3)$ (b) $f(x - 1)$

(c) $f(x) + 2$ (d) $f(x) - 4$

(e) $3f(x)$ (f) $\frac{1}{4}f(x)$

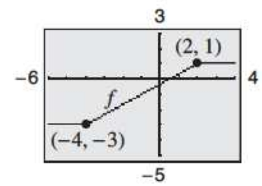


56. Use the graph of f shown in the figure to sketch the graph of each function. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(a) $f(x - 4)$ (b) $f(x + 2)$

(c) $f(x) + 4$ (d) $f(x) - 1$

(e) $2f(x)$ (f) $\frac{1}{2}f(x)$



57. Use the graph of $f(x) = \sqrt{x}$ to sketch the graph of each function. In each case, describe the transformation.

(a) $y = \sqrt{x} + 2$ (b) $y = -\sqrt{x}$ (c) $y = \sqrt{x - 2}$

58. Specify a sequence of transformations that will yield each graph of h from the graph of the function $f(x) = \sin x$.

(a) $h(x) = \sin\left(x + \frac{\pi}{2}\right) + 1$ (b) $h(x) = -\sin(x - 1)$

59. Given $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, evaluate each expression.

(a) $f(g(1))$ (b) $g(f(1))$ (c) $g(f(0))$

(d) $f(g(-4))$ (e) $f(g(x))$ (f) $g(f(x))$

60. Given $f(x) = \sin x$ and $g(x) = \pi x$, evaluate each expression.

(a) $f(g(2))$ (b) $f\left(g\left(\frac{1}{2}\right)\right)$ (c) $g(f(0))$

(d) $g\left(f\left(\frac{\pi}{4}\right)\right)$ (e) $f(g(x))$ (f) $g(f(x))$

In Exercises 61–64, find the composite functions $(f \circ g)$ and $(g \circ f)$. What is the domain of each composite function? Are the two composite functions equal?

61. $f(x) = x^2$
$g(x) = \sqrt{x}$

62. $f(x) = x^2 - 1$
$g(x) = \cos x$

63. $f(x) = \dfrac{3}{x}$
$g(x) = x^2 - 1$

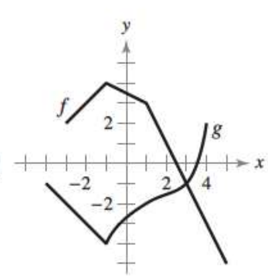
64. $f(x) = \dfrac{1}{x}$
$g(x) = \sqrt{x + 2}$

65. Use the graphs of f and g to evaluate each expression. If the result is undefined, explain why.

(a) $(f \circ g)(3)$ (b) $g(f(2))$

(c) $g(f(5))$ (d) $(f \circ g)(-3)$

(e) $(g \circ f)(-1)$ (f) $f(g(-1))$



66. *Ripples* A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius (in feet) of the outer ripple is given by $r(t) = 0.6t$, where t is the time in seconds after the pebble strikes the water. The area of the circle is given by the function $A(r) = \pi r^2$. Find and interpret $(A \circ r)(t)$.

Think About It **In Exercises 67 and 68, $F(x) = f \circ g \circ h$. Identify functions for f, g, and h. (There are many correct answers.)**

67. $F(x) = \sqrt{2x - 2}$

68. $F(x) = -4\sin(1 - x)$

In Exercises 69–72, determine whether the function is even, odd, or neither. Use a graphing utility to verify your result.

69. $f(x) = x^2(4 - x^2)$

70. $f(x) = \sqrt[3]{x}$

71. $f(x) = x \cos x$

72. $f(x) = \sin^2 x$

Think About It **In Exercises 73 and 74, find the coordinates of a second point on the graph of a function f if the given point is on the graph and the function is (a) even and (b) odd.**

73. $\left(-\frac{3}{2}, 4\right)$

74. $(4, 9)$

75. The graphs of f, g, and h are shown in the figure. Decide whether each function is even, odd, or neither.

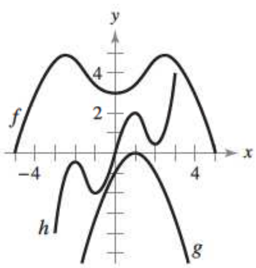


Figure for 75

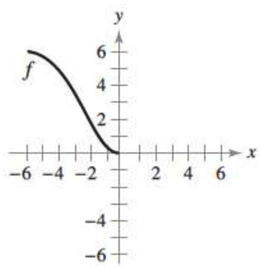


Figure for 76

76. The domain of the function f shown in the figure is $-6 \le x \le 6$.

(a) Complete the graph of f given that f is even.

(b) Complete the graph of f given that f is odd.

Writing Functions **In Exercises 77–80, write an equation for a function that has the given graph.**

77. Line segment connecting $(-2, 4)$ and $(0, -6)$

78. Line segment connecting $(3, 1)$ and $(5, 8)$

79. The bottom half of the parabola $x + y^2 = 0$

80. The bottom half of the circle $x^2 + y^2 = 36$

In Exercises 81–84, sketch a possible graph of the situation.

81. The speed of an airplane as a function of time during a 5-hour flight

82. The height of a baseball as a function of horizontal distance during a home run

83. The amount of a certain brand of sneaker sold by a sporting goods store as a function of the price of the sneaker

84. The value of a new car as a function of time over a period of 8 years

85. Find the value of c such that the domain of

$$f(x) = \sqrt{c - x^2}$$

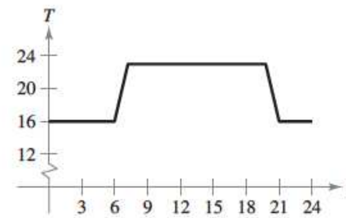
is $[-5, 5]$.

86. Find all values of c such that the domain of

$$f(x) = \frac{x + 3}{x^2 + 3cx + 6}$$

is the set of all real numbers.

87. *Graphical Reasoning* An electronically controlled thermostat is programmed to lower the temperature during the night automatically (see figure). The temperature T in degrees Celsius is given in terms of t, the time in hours on a 24-hour clock.



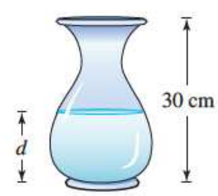
(a) Approximate $T(4)$ and $T(15)$.

(b) The thermostat is reprogrammed to produce a temperature $H(t) = T(t - 1)$. How does this change the temperature? Explain.

(c) The thermostat is reprogrammed to produce a temperature $H(t) = T(t) - 1$. How does this change the temperature? Explain.

CAPSTONE

88. Water runs into a vase of height 30 centimeters at a constant rate. The vase is full after 5 seconds. Use this information and the shape of the vase shown to answer the questions if d is the depth of the water in centimeters and t is the time in seconds (see figure).



(a) Explain why d is a function of t.

(b) Determine the domain and range of the function.

(c) Sketch a possible graph of the function.

(d) Use the graph in part (c) to approximate $d(4)$. What does this represent?

89. ***Modeling Data*** The table shows the average numbers of acres per farm in the United States for selected years. *(Source: U.S. Department of Agriculture)*

Year	1955	1965	1975	1985	1995	2005
Acreage	258	340	420	441	438	444

(a) Plot the data, where A is the acreage and t is the time in years, with $t = 5$ corresponding to 1955. Sketch a freehand curve that approximates the data.

(b) Use the curve in part (a) to approximate $A(20)$.

90. ***Automobile Aerodynamics*** The horsepower H required to overcome wind drag on a certain automobile is approximated by

$$H(x) = 0.002x^2 + 0.005x - 0.029, \quad 10 \le x \le 100$$

where x is the speed of the car in miles per hour.

(a) Use a graphing utility to graph H.

(b) Rewrite the power function so that x represents the speed in kilometers per hour. [Find $H(x/1.6)$.]

91. ***Think About It*** Write the function

$$f(x) = |x| + |x - 2|$$

without using absolute value signs. (For a review of absolute value, see Appendix C.)

92. ***Writing*** Use a graphing utility to graph the polynomial functions $p_1(x) = x^3 - x + 1$ and $p_2(x) = x^3 - x$. How many zeros does each function have? Is there a cubic polynomial that has no zeros? Explain.

93. Prove that the function is odd.

$$f(x) = a_{2n+1}x^{2n+1} + \cdots + a_3x^3 + a_1x$$

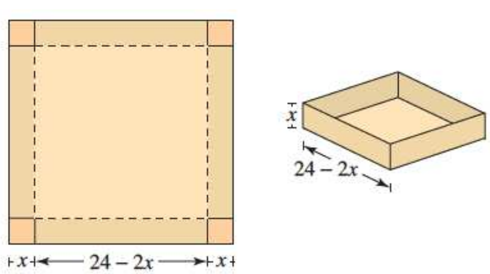
94. Prove that the function is even.

$$f(x) = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdots + a_2x^2 + a_0$$

95. Prove that the product of two even (or two odd) functions is even.

96. Prove that the product of an odd function and an even function is odd.

97. ***Volume*** An open box of maximum volume is to be made from a square piece of material 24 centimeters on a side by cutting equal squares from the corners and turning up the sides (see figure).



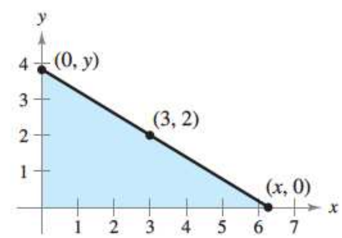
(a) Write the volume V as a function of x, the length of the corner squares. What is the domain of the function?

(b) Use a graphing utility to graph the volume function and approximate the dimensions of the box that yield a maximum volume.

(c) Use the *table* feature of a graphing utility to verify your answer in part (b). (The first two rows of the table are shown.)

Height, x	Length and Width	Volume, V
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$

98. ***Length*** A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(3, 2)$ (see figure). Write the length L of the hypotenuse as a function of x.



True or False? **In Exercises 99–102, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

99. If $f(a) = f(b)$, then $a = b$.

100. A vertical line can intersect the graph of a function at most once.

101. If $f(x) = f(-x)$ for all x in the domain of f, then the graph of f is symmetric with respect to the y-axis.

102. If f is a function, then $f(ax) = af(x)$.

PUTNAM EXAM CHALLENGE

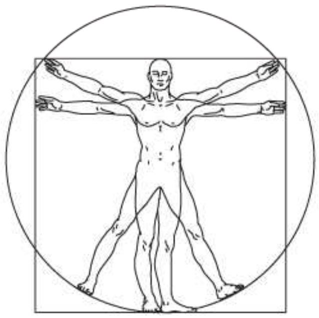
103. Let R be the region consisting of the points (x, y) of the Cartesian plane satisfying both $|x| - |y| \le 1$ and $|y| \le 1$. Sketch the region R and find its area.

104. Consider a polynomial $f(x)$ with real coefficients having the property $f(g(x)) = g(f(x))$ for every polynomial $g(x)$ with real coefficients. Determine and prove the nature of $f(x)$.

1.4 Fitting Models to Data

- Fit a linear model to a real-life data set.
- Fit a quadratic model to a real-life data set.
- Fit a trigonometric model to a real-life data set.

Fitting a Linear Model to Data



A computer graphics drawing based on the pen and ink drawing of Leonardo da Vinci's famous study of human proportions, called *Vitruvian Man*

A basic premise of science is that much of the physical world can be described mathematically and that many physical phenomena are predictable. This scientific outlook was part of the scientific revolution that took place in Europe during the late 1500s. Two early publications connected with this revolution were *On the Revolutions of the Heavenly Spheres* by the Polish astronomer Nicolaus Copernicus and *On the Structure of the Human Body* by the Belgian anatomist Andreas Vesalius. Each of these books was published in 1543, and each broke with prior tradition by suggesting the use of a scientific method rather than unquestioned reliance on authority.

One method of modern science is gathering data and then describing the data with a mathematical model. For instance, the data given in Example 1 are inspired by Leonardo da Vinci's famous drawing that indicates that a person's height and arm span are equal.

EXAMPLE 1 Fitting a Linear Model to Data

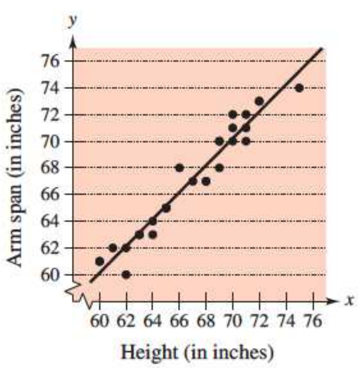
A class of 28 people collected the following data, which represent their heights x and arm spans y (rounded to the nearest inch).

$$(60, 61), (65, 65), (68, 67), (72, 73), (61, 62), (63, 63), (70, 71),$$
$$(75, 74), (71, 72), (62, 60), (65, 65), (66, 68), (62, 62), (72, 73),$$
$$(70, 70), (69, 68), (69, 70), (60, 61), (63, 63), (64, 64), (71, 71),$$
$$(68, 67), (69, 70), (70, 72), (65, 65), (64, 63), (71, 70), (67, 67)$$

Find a linear model to represent these data.

Solution There are different ways to model these data with an equation. The simplest would be to observe that x and y are about the same and list the model as simply $y = x$. A more careful analysis would be to use a procedure from statistics called linear regression. (You will study this procedure in Section 13.9.) The least squares regression line for these data is

$$y = 1.006x - 0.23. \qquad \text{Least squares regression line}$$



Linear model and data
Figure 1.32

The graph of the model and the data are shown in Figure 1.32. From this model, you can see that a person's arm span tends to be about the same as his or her height. ■

TECHNOLOGY Many scientific and graphing calculators have built-in least squares regression programs. Typically, you enter the data into the calculator and then run the linear regression program. The program usually displays the slope and y-intercept of the best-fitting line and the *correlation coefficient* r. The correlation coefficient gives a measure of how well the model fits the data. The closer $|r|$ is to 1, the better the model fits the data. For instance, the correlation coefficient for the model in Example 1 is $r \approx 0.97$, which indicates that the model is a good fit for the data. If the r-value is positive, the variables have a positive correlation, as in Example 1. If the r-value is negative, the variables have a negative correlation.

Fitting a Quadratic Model to Data

A function that gives the height s of a falling object in terms of the time t is called a position function. If air resistance is not considered, the position of a falling object can be modeled by $s(t) = \frac{1}{2}gt^2 + v_0t + s_0$, where g is the acceleration due to gravity, v_0 is the initial velocity, and s_0 is the initial height. The value of g depends on where the object is dropped. On Earth, g is approximately -32 feet per second per second, or -9.8 meters per second per second.

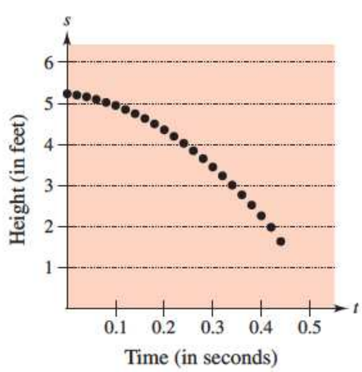
To discover the value of g experimentally, you could record the heights of a falling object at several increments, as shown in Example 2.

EXAMPLE 2 Fitting a Quadratic Model to Data

A basketball is dropped from a height of about $5\frac{1}{4}$ feet. The height of the basketball is recorded 23 times at intervals of about 0.02 second.* The results are shown in the table.

Time	0.0	0.02	0.04	0.06	0.08	0.099996
Height	5.23594	5.20353	5.16031	5.0991	5.02707	4.95146
Time	0.119996	0.139992	0.159988	0.179988	0.199984	0.219984
Height	4.85062	4.74979	4.63096	4.50132	4.35728	4.19523
Time	0.23998	0.25993	0.27998	0.299976	0.319972	0.339961
Height	4.02958	3.84593	3.65507	3.44981	3.23375	3.01048
Time	0.359961	0.379951	0.399941	0.419941	0.439941	
Height	2.76921	2.52074	2.25786	1.98058	1.63488	

Find a model to fit these data. Then use the model to predict the time when the basketball hits the ground.



Scatter plot of data
Figure 1.33

Solution Draw a scatter plot of the data, as shown in Figure 1.33. From the scatter plot, you can see that the data do not appear to be linear. It does appear, however, that they might be quadratic. To find a quadratic model, enter the data into a calculator or computer that has a quadratic regression program. You should obtain the model

$$s = -15.45t^2 - 1.302t + 5.2340. \qquad \text{Least squares regression quadratic}$$

Using this model, you can predict the time when the basketball hits the ground by substituting 0 for s and solving the resulting equation for t.

$$0 = -15.45t^2 - 1.302t + 5.2340 \qquad \text{Let } s = 0.$$

$$t = \frac{1.302 \pm \sqrt{(-1.302)^2 - 4(-15.45)(5.2340)}}{2(-15.45)} \qquad \text{Quadratic Formula}$$

$$t \approx 0.54 \qquad \text{Choose positive solution.}$$

So, the basketball hits the ground about 0.54 second after it is dropped. In other words, the basketball continues to fall for about 0.1 second more before hitting the ground. ■

** Data were collected with a Texas Instruments CBL (Calculator-Based Laboratory) System.*

The plane of Earth's orbit about the sun and its axis of rotation are not perpendicular. Instead, Earth's axis is tilted with respect to its orbit. The result is that the amount of daylight received by locations on Earth varies with the time of year. That is, it varies with the position of Earth in its orbit.

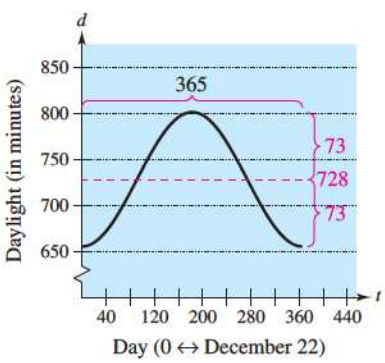
Fitting a Trigonometric Model to Data

What is mathematical modeling? This is one of the questions that is asked in the book *Guide to Mathematical Modelling*. Here is part of the answer.*

1. Mathematical modeling consists of applying your mathematical skills to obtain useful answers to real problems.
2. Learning to apply mathematical skills is very different from learning mathematics itself.
3. Models are used in a very wide range of applications, some of which do not appear initially to be mathematical in nature.
4. Models often allow quick and cheap evaluation of alternatives, leading to optimal solutions that are not otherwise obvious.
5. There are no precise rules in mathematical modeling and no "correct" answers.
6. Modeling can be learned only by *doing*.

EXAMPLE 3 Fitting a Trigonometric Model to Data

The number of hours of daylight on Earth depends on the latitude and the time of year. Here are the numbers of minutes of daylight at a location of 20°N latitude on the longest and shortest days of the year: June 21, 801 minutes; December 22, 655 minutes. Use these data to write a model for the amount of daylight d (in minutes) on each day of the year at a location of 20°N latitude. How could you check the accuracy of your model?



Graph of model
Figure 1.34

Solution Here is one way to create a model. You can hypothesize that the model is a sine function whose period is 365 days. Using the given data, you can conclude that the amplitude of the graph is $(801 - 655)/2$, or 73. So, one possible model is

$$d = 728 - 73 \sin\left(\frac{2\pi t}{365} + \frac{\pi}{2}\right).$$

In this model, t represents the number of the day of the year, with December 22 represented by $t = 0$. A graph of this model is shown in Figure 1.34. To check the accuracy of this model, a weather almanac was used to find the numbers of minutes of daylight on different days of the year at the location of 20°N latitude.

Date	*Value of t*	*Actual Daylight*	*Daylight Given by Model*
Dec 22	0	655 min	655 min
Jan 1	10	657 min	656 min
Feb 1	41	676 min	672 min
Mar 1	69	705 min	701 min
Apr 1	100	740 min	739 min
May 1	130	772 min	773 min
Jun 1	161	796 min	796 min
Jun 21	181	801 min	801 min
Jul 1	191	799 min	800 min
Aug 1	222	782 min	785 min
Sep 1	253	752 min	754 min
Oct 1	283	718 min	716 min
Nov 1	314	685 min	681 min
Dec 1	344	661 min	660 min

You can see that the model is fairly accurate. ■

NOTE For a review of trigonometric functions, see Appendix C.

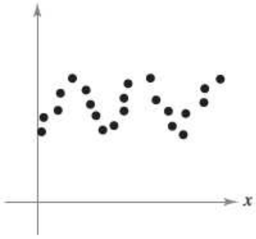
* *Text from Dilwyn Edwards and Mike Hamson,* Guide to Mathematical Modelling *(Boca Raton: CRC Press, 1990), p. 4. Used by permission of the authors.*

1.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, a scatter plot of data is given. Determine whether the data can be modeled by a linear function, a quadratic function, or a trigonometric function, or that there appears to be no relationship between x and y. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

1.



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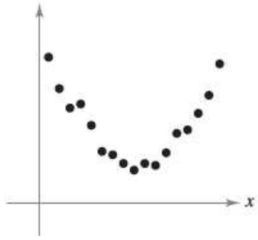

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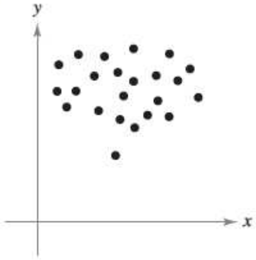
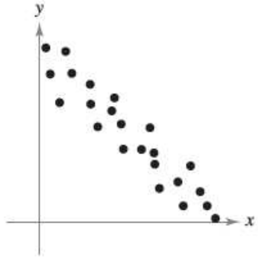


4.



5. ***Carcinogens*** Each ordered pair gives the exposure index x of a carcinogenic substance and the cancer mortality y per 100,000 people in the population.

(3.50, 150.1), (3.58, 133.1), (4.42, 132.9),
(2.26, 116.7), (2.63, 140.7), (4.85, 165.5),
(12.65, 210.7), (7.42, 181.0), (9.35, 213.4)

(a) Plot the data. From the graph, do the data appear to be approximately linear?

(b) Visually find a linear model for the data. Graph the model.

(c) Use the model to approximate y if $x = 3$.

6. ***Quiz Scores*** The ordered pairs represent the scores on two consecutive 15-point quizzes for a class of 18 students.

(7, 13), (9, 7), (14, 14), (15, 15), (10, 15), (9, 7),
(14, 11), (14, 15), (8, 10), (15, 9), (10, 11), (9, 10),
(11, 14), (7, 14), (11, 10), (14, 11), (10, 15), (9, 6)

(a) Plot the data. From the graph, does the relationship between consecutive scores appear to be approximately linear?

(b) If the data appear to be approximately linear, find a linear model for the data. If not, give some possible explanations.

7. ***Hooke's Law*** Hooke's Law states that the force F required to compress or stretch a spring (within its elastic limits) is proportional to the distance d that the spring is compressed or stretched from its original length. That is, $F = kd$, where k is a measure of the stiffness of the spring and is called the *spring constant*. The table shows the elongation d in centimeters of a spring when a force of F newtons is applied.

F	20	40	60	80	100
d	1.4	2.5	4.0	5.3	6.6

(a) Use the regression capabilities of a graphing utility to find a linear model for the data.

(b) Use a graphing utility to plot the data and graph the model. How well does the model fit the data? Explain your reasoning.

(c) Use the model to estimate the elongation of the spring when a force of 55 newtons is applied.

8. ***Falling Object*** In an experiment, students measured the speed s (in meters per second) of a falling object t seconds after it was released. The results are shown in the table.

t	0	1	2	3	4
s	0	11.0	19.4	29.2	39.4

(a) Use the regression capabilities of a graphing utility to find a linear model for the data.

(b) Use a graphing utility to plot the data and graph the model. How well does the model fit the data? Explain your reasoning.

(c) Use the model to estimate the speed of the object after 2.5 seconds.

9. ***Energy Consumption and Gross National Product*** The data show the per capita energy consumptions (in millions of Btu) and the per capita gross national products (in thousands of U.S. dollars) for several countries in 2004. *(Source: U.S. Census Bureau)*

Argentina	(71, 12.53)	Bangladesh	(5, 1.97)
Chile	(75, 10.61)	Ecuador	(29, 3.77)
Greece	(136, 22.23)	Hong Kong	(159, 31.56)
Hungary	(106, 15.8)	India	(15, 3.12)
Mexico	(63, 9.64)	Poland	(95, 12.73)
Portugal	(106, 19.24)	South Korea	(186, 20.53)
Spain	(159, 24.75)	Turkey	(51, 7.72)
United Kingdom	(167, 31.43)	Venezuela	(115, 5.83)

(a) Use the regression capabilities of a graphing utility to find a linear model for the data. What is the correlation coefficient?

(b) Use a graphing utility to plot the data and graph the model.

(c) Interpret the graph in part (b). Use the graph to identify the four countries that differ most from the linear model.

(d) Delete the data for the four countries identified in part (c). Fit a linear model to the remaining data and give the correlation coefficient.

10. *Brinell Hardness* The data in the table show the Brinell hardness H of 0.35 carbon steel when hardened and tempered at temperature t (degrees Fahrenheit). *(Source: Standard Handbook for Mechanical Engineers)*

t	200	400	600	800	1000	1200
H	534	495	415	352	269	217

(a) Use the regression capabilities of a graphing utility to find a linear model for the data.

(b) Use a graphing utility to plot the data and graph the model. How well does the model fit the data? Explain your reasoning.

(c) Use the model to estimate the hardness when t is 500°F.

11. *Automobile Costs* The data in the table show the variable costs of operating an automobile in the United States for several recent years. The functions y_1, y_2, and y_3 represent the costs in cents per mile for gas, maintenance, and tires, respectively. *(Source: Bureau of Transportation Statistics)*

Year	y_1	y_2	y_3
0	5.60	3.30	1.70
1	6.90	3.60	1.70
2	7.90	3.90	1.80
3	5.90	4.10	1.80
4	7.20	4.10	1.80
5	6.50	5.40	0.70
6	9.50	4.90	0.70
7	8.90	4.90	0.70

(a) Use the regression capabilities of a graphing utility to find cubic models for y_1 and y_3 and a linear model for y_2.

(b) Use a graphing utility to graph y_1, y_2, y_3, and $y_1 + y_2 + y_3$ in the same viewing window. Use the model to estimate the total variable cost per mile in year 12.

12. *Beam Strength* Students in a lab measured the breaking strength S (in pounds) of wood 2 inches thick, x inches high, and 12 inches long. The results are shown in the table.

x	4	6	8	10	12
S	2370	5460	10,310	16,250	23,860

(a) Use the regression capabilities of a graphing utility to fit a quadratic model to the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the model to approximate the breaking strength when $x = 2$.

13. *Car Performance* The times t (in seconds) required to attain speeds of s miles per hour from a standing start for a Honda Accord Hybrid are shown in the table. *(Source: Car & Driver)*

s	30	40	50	60	70	80	90
t	2.5	3.5	5.0	6.7	8.7	11.5	14.4

(a) Use the regression capabilities of a graphing utility to find a quadratic model for the data.

(b) Use a graphing utility to plot the data and graph the model.

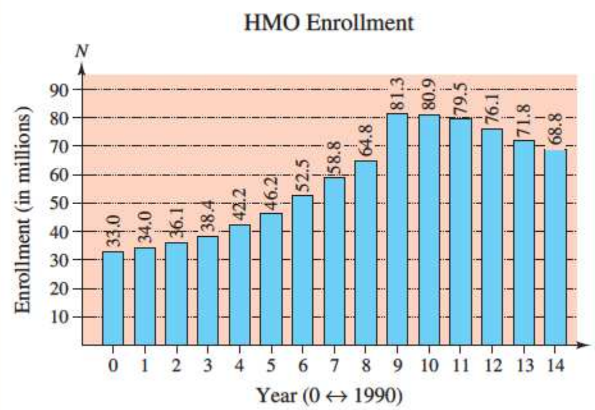
(c) Use the graph in part (b) to state why the model is not appropriate for determining the times required to attain speeds of less than 20 miles per hour.

(d) Because the test began from a standing start, add the point (0, 0) to the data. Fit a quadratic model to the revised data and graph the new model.

(e) Does the quadratic model in part (d) more accurately model the behavior of the car? Explain.

CAPSTONE

14. *Health Maintenance Organizations* The bar graph shows the numbers of people N (in millions) receiving care in HMOs for the years 1990 through 2004. *(Source: HealthLeaders-InterStudy)*



(a) Let t be the time in years, with $t = 0$ corresponding to 1990. Use the regression capabilities of a graphing utility to find linear and cubic models for the data.

(b) Use a graphing utility to graph the data and the linear and cubic models.

(c) Use the graphs in part (b) to determine which is the better model.

(d) Use a graphing utility to find and graph a quadratic model for the data. How well does the model fit the data? Explain your reasoning.

(e) Use the linear and cubic models to estimate the number of people receiving care in HMOs in the year 2007. What do you notice?

(f) Use a graphing utility to find other models for the data. Which models do you think best represent the data? Explain.

15. ***Car Performance*** A V8 car engine is coupled to a dynamometer, and the horsepower y is measured at different engine speeds x (in thousands of revolutions per minute). The results are shown in the table.

x	1	2	3	4	5	6
y	40	85	140	200	225	245

(a) Use the regression capabilities of a graphing utility to find a cubic model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the model to approximate the horsepower when the engine is running at 4500 revolutions per minute.

16. ***Boiling Temperature*** The table shows the temperatures T (°F) at which water boils at selected pressures p (pounds per square inch). *(Source: Standard Handbook for Mechanical Engineers)*

p	5	10	14.696 (1 atmosphere)	20
T	162.24°	193.21°	212.00°	227.96°

p	30	40	60	80	100
T	250.33°	267.25°	292.71°	312.03°	327.81°

(a) Use the regression capabilities of a graphing utility to find a cubic model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the graph to estimate the pressure required for the boiling point of water to exceed 300°F.

(d) Explain why the model would not be accurate for pressures exceeding 100 pounds per square inch.

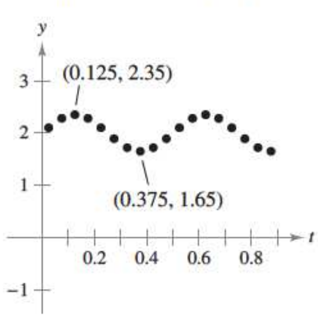
17. ***Harmonic Motion*** The motion of an oscillating weight suspended by a spring was measured by a motion detector. The data collected and the approximate maximum (positive and negative) displacements from equilibrium are shown in the figure. The displacement y is measured in centimeters and the time t is measured in seconds.

(a) Is y a function of t? Explain.

(b) Approximate the amplitude and period of the oscillations.

(c) Find a model for the data.

(d) Use a graphing utility to graph the model in part (c). Compare the result with the data in the figure.



18. ***Temperature*** The table shows the normal daily high temperatures for Miami M and Syracuse S (in degrees Fahrenheit) for month t, with $t = 1$ corresponding to January. *(Source: NOAA)*

t	1	2	3	4	5	6
M	76.5	77.7	80.7	83.8	87.2	89.5
S	31.4	33.5	43.1	55.7	68.5	77.0

t	7	8	9	10	11	12
M	90.9	90.6	89.0	85.4	81.2	77.5
S	81.7	79.6	71.4	59.8	47.4	36.3

(a) A model for Miami is

$$M(t) = 83.70 + 7.46\sin(0.4912t - 1.95).$$

Find a model for Syracuse.

(b) Use a graphing utility to graph the data and the model for the temperatures in Miami. How well does the model fit?

(c) Use a graphing utility to graph the data and the model for the temperatures in Syracuse. How well does the model fit?

(d) Use the models to estimate the average annual temperature in each city. Which term of the model did you use? Explain.

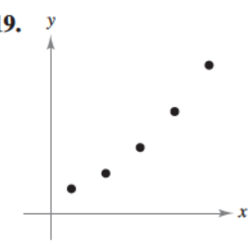
(e) What is the period of each model? Is it what you expected? Explain.

(f) Which city has a greater variability in temperature throughout the year? Which factor of the models determines this variability? Explain.

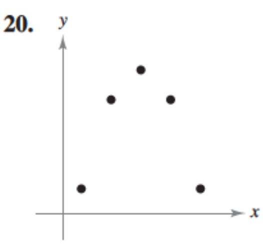
WRITING ABOUT CONCEPTS

In Exercises 19 and 20, describe a possible real-life situation for each data set. Then describe how a model could be used in the real-life setting.

19.



20.



PUTNAM EXAM CHALLENGE

21. For $i = 1, 2$, let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \le a_2$, $b_1 \le b_2$, $c_1 \le c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \le A_2$?

1.5 Inverse Functions

- Verify that one function is the inverse function of another function.
- Determine whether a function has an inverse function.
- Develop properties of the six inverse trigonometric functions.

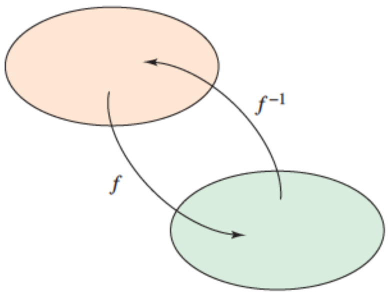
Inverse Functions

Recall from Section 1.3 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 3$ from $A = \{1, 2, 3, 4\}$ to $B = \{4, 5, 6, 7\}$ can be written as

$$f: \{(1, 4), (2, 5), (3, 6), (4, 7)\}.$$

By interchanging the first and second coordinates of each ordered pair, you can form the **inverse function** of f. This function is denoted by f^{-1}. It is a function from B to A, and can be written as

$$f^{-1}: \{(4, 1), (5, 2), (6, 3), (7, 4)\}.$$



Domain of f = range of f^{-1}
Domain of f^{-1} = range of f
Figure 1.35

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 1.35. The functions f and f^{-1} have the effect of "undoing" each other. That is, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x$$

EXPLORATION

Finding Inverse Functions
Explain how to "undo" each of the following functions. Then use your explanation to write the inverse function of f.

a. $f(x) = x - 5$

b. $f(x) = 6x$

c. $f(x) = \dfrac{x}{2}$

d. $f(x) = 3x + 2$

e. $f(x) = x^3$

f. $f(x) = 4(x - 2)$

Use a graphing utility to graph each function and its inverse function in the same "square" viewing window. What observation can you make about each pair of graphs?

DEFINITION OF INVERSE FUNCTION

A function g is the **inverse function** of the function f if

$$f(g(x)) = x \quad \text{for each } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \quad \text{for each } x \text{ in the domain of } f.$$

The function g is denoted by f^{-1} (read "f inverse").

NOTE Although the notation used to denote an inverse function resembles *exponential notation*, it is a different use of -1 as a superscript. That is, in general, $f^{-1}(x) \neq 1/f(x)$. ■

Here are some important observations about inverse functions.

1. If g is the inverse function of f, then f is the inverse function of g.
2. The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f.
3. A function need not have an inverse function, but if it does, the inverse function is unique (see Exercise 159).

You can think of f^{-1} as undoing what has been done by f. For example, subtraction can be used to undo addition, and division can be used to undo multiplication. Use the definition of an inverse function to check the following.

$$f(x) = x + c \quad \text{and} \quad f^{-1}(x) = x - c \quad \text{are inverse functions of each other.}$$

$$f(x) = cx \quad \text{and} \quad f^{-1}(x) = \frac{x}{c},\ c \neq 0, \quad \text{are inverse functions of each other.}$$

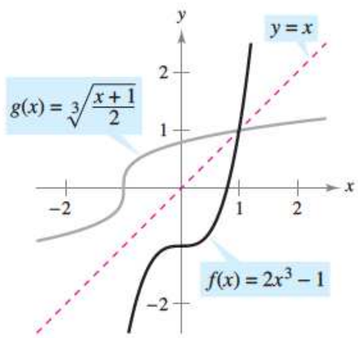
EXAMPLE 1 Verifying Inverse Functions

Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

Solution Because the domains and ranges of both f and g consist of all real numbers, you can conclude that both composite functions exist for all x. The composition of f with g is given by

$$\begin{aligned} f(g(x)) &= 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 \\ &= 2\left(\frac{x+1}{2}\right) - 1 \\ &= x + 1 - 1 \\ &= x. \end{aligned}$$



f and *g* are inverse functions of each other.
Figure 1.36

The composition of g with f is given by

$$\begin{aligned} g(f(x)) &= \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}} \\ &= \sqrt[3]{\frac{2x^3}{2}} \\ &= \sqrt[3]{x^3} \\ &= x. \end{aligned}$$

Because $f(g(x)) = x$ and $g(f(x)) = x$, you can conclude that f and g are inverse functions of each other (see Figure 1.36). ■

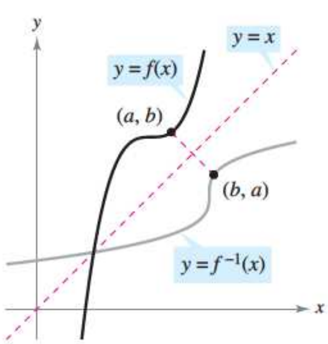
STUDY TIP In Example 1, try comparing the functions f and g verbally.

For f: First cube x, then multiply by 2, then subtract 1.

For g: First add 1, then divide by 2, then take the cube root.

Do you see the "undoing pattern"? ■

In Figure 1.36, the graphs of f and $g = f^{-1}$ appear to be mirror images of each other with respect to the line $y = x$. The graph of f^{-1} is a **reflection** of the graph of f in the line $y = x$. This idea is generalized as follows.



The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.
Figure 1.37

REFLECTIVE PROPERTY OF INVERSE FUNCTIONS

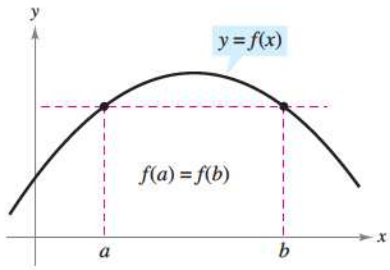
The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

To see this, suppose (a, b) is on the graph of f. Then $f(a) = b$ and you can write

$$f^{-1}(b) = f^{-1}(f(a)) = a.$$

So, (b, a) is on the graph of f^{-1}, as shown in Figure 1.37. A similar argument will verify this result in the other direction.

Existence of an Inverse Function



If a horizontal line intersects the graph of f twice, then f is not one-to-one.
Figure 1.38

Not every function has an inverse, and the Reflective Property of Inverse Functions suggests a graphical test for those that do—the **Horizontal Line Test** for an inverse function. This test states that a function f has an inverse function if and only if every horizontal line intersects the graph of f at most once (see Figure 1.38). The following formally states why the Horizontal Line Test is valid.

THE EXISTENCE OF AN INVERSE FUNCTION

A function has an inverse function if and only if it is one-to-one.

EXAMPLE 2 The Existence of an Inverse Function

Which of the functions has an inverse function?

a. $f(x) = x^3 - 1$ **b.** $f(x) = x^3 - x + 1$

Solution

a. From the graph of f shown in Figure 1.39(a), it appears that f is one-to-one over its entire domain. To verify this, suppose that there exist x_1 and x_2 such that $f(x_1) = f(x_2)$. By showing that $x_1 = x_2$, it follows that f is one-to-one.

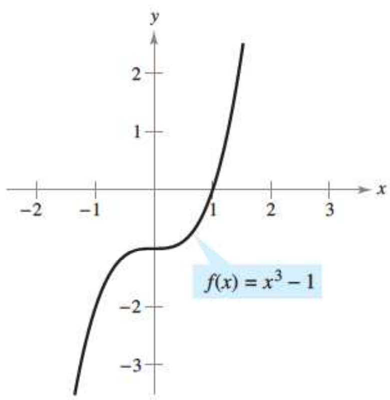
$$\begin{aligned} f(x_1) &= f(x_2) \\ x_1^3 - 1 &= x_2^3 - 1 \\ x_1^3 &= x_2^3 \\ \sqrt[3]{x_1^3} &= \sqrt[3]{x_2^3} \\ x_1 &= x_2 \end{aligned}$$

Because f is one-to-one, you can conclude that f must have an inverse function.

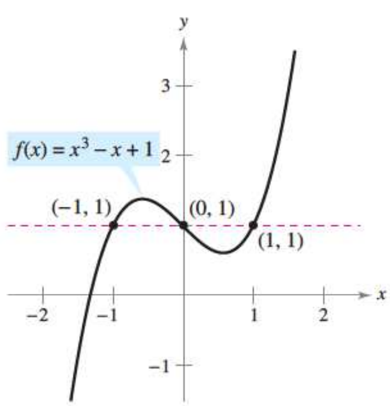
b. From the graph in Figure 1.39(b), you can see that the function does not pass the Horizontal Line Test. In other words, it is not one-to-one. For instance, f has the same value when $x = -1, 0,$ and 1.

$$f(-1) = f(1) = f(0) = 1 \qquad \text{Not one-to-one}$$

Therefore, f does not have an inverse function. ■



(a) Because f is one-to-one over its entire domain, it has an inverse function.

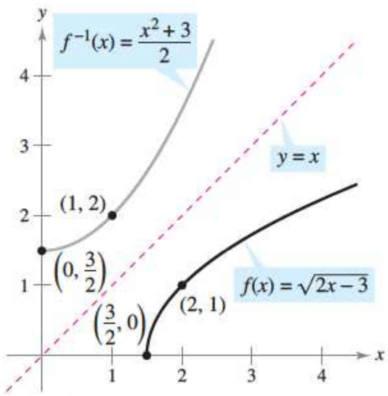


(b) Because f is not one-to-one, it does not have an inverse function.
Figure 1.39

NOTE Often it is easier to prove that a function has an inverse function than to find the inverse function. For instance, by sketching the graph of $f(x) = x^3 + x - 1$, you can see that it is one-to-one. Yet it would be difficult to determine the inverse of this function algebraically. ■

GUIDELINES FOR FINDING AN INVERSE OF A FUNCTION

1. Determine whether the function given by $y = f(x)$ has an inverse function.
2. Solve for x as a function of y: $x = g(y) = f^{-1}(y)$.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of f^{-1} as the range of f.
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.



The domain of f^{-1}, $[0, \infty)$, is the range of f.
Figure 1.40

EXAMPLE 3 Finding an Inverse Function

Find the inverse function of

$$f(x) = \sqrt{2x - 3}.$$

Solution The function has an inverse function because it is one-to-one on its entire domain, $\left[\frac{3}{2}, \infty\right)$, as shown in Figure 1.40. To find an equation for the inverse function, let $y = f(x)$ and solve for x in terms of y.

$$\sqrt{2x - 3} = y \qquad \text{Let } y = f(x).$$

$$2x - 3 = y^2 \qquad \text{Square each side.}$$

$$x = \frac{y^2 + 3}{2} \qquad \text{Solve for } x.$$

$$y = \frac{x^2 + 3}{2} \qquad \text{Interchange } x \text{ and } y.$$

$$f^{-1}(x) = \frac{x^2 + 3}{2} \qquad \text{Replace } y \text{ by } f^{-1}(x).$$

The domain of f^{-1} is the range of f, which is $[0, \infty)$. You can verify this result as shown.

$$f(f^{-1}(x)) = \sqrt{2\left(\frac{x^2 + 3}{2}\right) - 3} = \sqrt{x^2} = x, \quad x \geq 0$$

$$f^{-1}(f(x)) = \frac{\left(\sqrt{2x - 3}\right)^2 + 3}{2} = \frac{2x - 3 + 3}{2} = x, \quad x \geq \frac{3}{2}$$

NOTE Remember that any letter can be used to represent the independent variable. So,

$$f^{-1}(y) = \frac{y^2 + 3}{2}, \quad f^{-1}(x) = \frac{x^2 + 3}{2}, \quad \text{and} \quad f^{-1}(s) = \frac{s^2 + 3}{2}$$

all represent the same function.

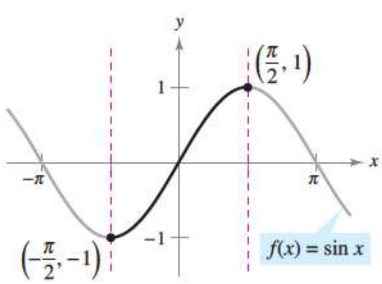
Suppose you are given a function that is *not* one-to-one on its entire domain. By restricting the domain to an interval on which the function *is* one-to-one, you can conclude that the new function has an inverse function on the restricted domain.

EXAMPLE 4 Testing Whether a Function Is One-to-One

Show that the sine function

$$f(x) = \sin x$$

is not one-to-one on the entire real line. Then show that f is one-to-one on the closed interval $[-\pi/2, \pi/2]$.



f is one-to-one on the interval $[-\pi/2, \pi/2]$.
Figure 1.41

Solution It is clear that f is not one-to-one, because many different x-values yield the same y-value. For instance,

$$\sin(0) = 0 = \sin(\pi).$$

Moreover, from the graph of $f(x) = \sin x$ in Figure 1.41, you can see that when f is restricted to the interval $[-\pi/2, \pi/2]$, then the restricted function *is* one-to-one.

Inverse Trigonometric Functions

From the graphs of the six basic trigonometric functions, you can see that they do not have inverse functions. (Graphs of the six basic trigonometric functions are shown in Appendix C.) The functions that are called "inverse trigonometric functions" are actually inverses of trigonometric functions whose domains have been restricted.

For instance, in Example 4, you saw that the sine function is one-to-one on the interval $[-\pi/2, \pi/2]$ (see Figure 1.42). On this interval, you can define the inverse of the *restricted* sine function to be

$$y = \arcsin x \qquad \text{if and only if} \qquad \sin y = x$$

where $-1 \le x \le 1$ and $-\pi/2 \le \arcsin x \le \pi/2$. From Figures 1.42 (a) and (b), you can see that you can obtain the graph of $y = \arcsin x$ by reflecting the graph of $y = \sin x$ in the line $y = x$ on the interval $[-\pi/2, \pi/2]$.

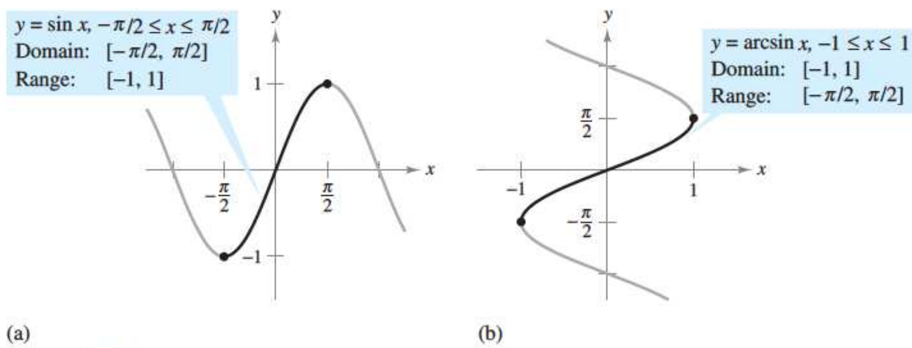


Figure 1.42

Under suitable restrictions, each of the six trigonometric functions is one-to-one and so has an inverse function, as indicated in the following definition. (The term "iff" is used to represent the phrase "if and only if.")

EXPLORATION

Inverse Secant Function In the definition at the right, the inverse secant function is defined by restricting the domain of the secant function to the intervals

$$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right].$$

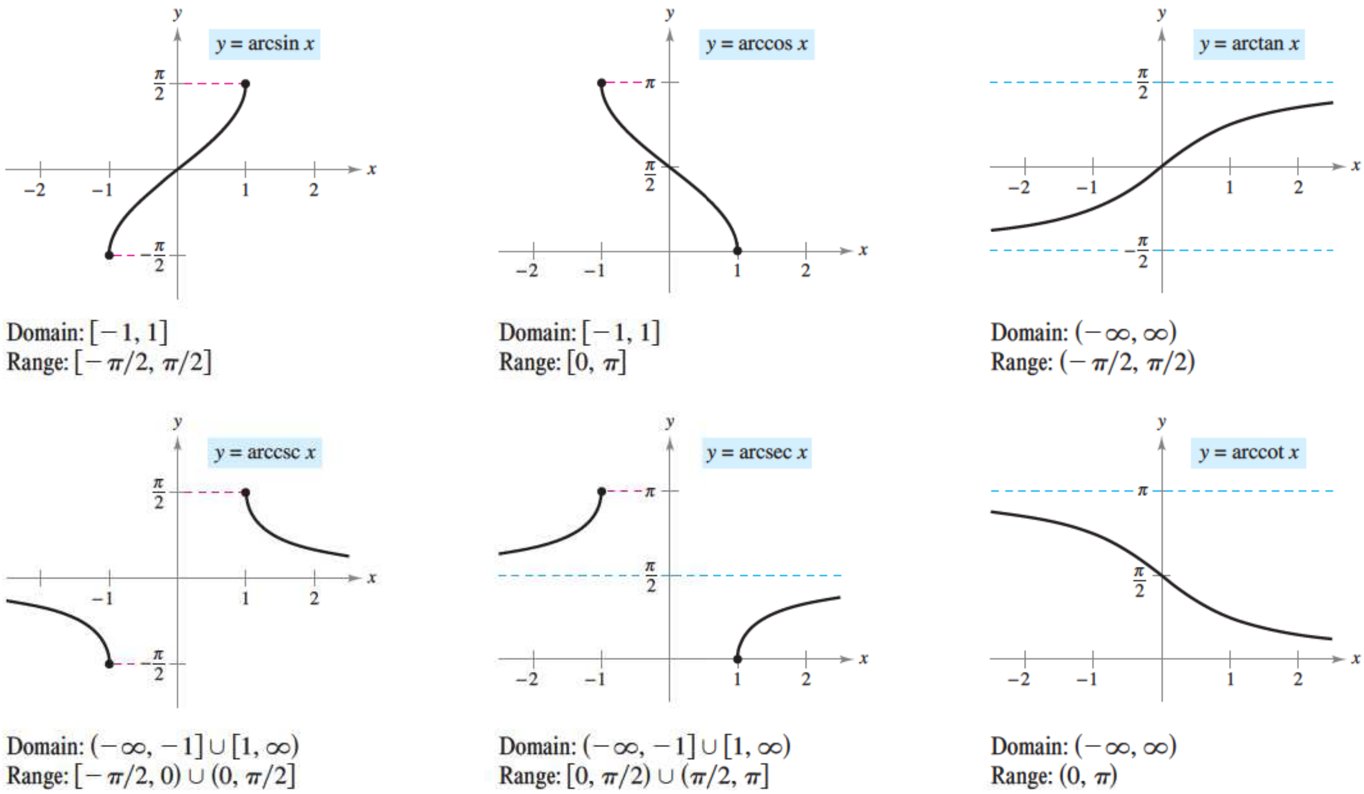
Most other texts and reference books agree with this, but some disagree. What other domains might make sense? Explain your reasoning graphically. Most calculators do not have a key for the inverse secant function. How can you use a calculator to evaluate the inverse secant function?

DEFINITION OF INVERSE TRIGONOMETRIC FUNCTION

Function	*Domain*	*Range*
$y = \arcsin x$ iff $\sin y = x$	$-1 \le x \le 1$	$-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$
$y = \arccos x$ iff $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$
$y = \operatorname{arccot} x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \operatorname{arcsec} x$ iff $\sec y = x$	$\lvert x\rvert \ge 1$	$0 \le y \le \pi, \quad y \ne \dfrac{\pi}{2}$
$y = \operatorname{arccsc} x$ iff $\csc y = x$	$\lvert x\rvert \ge 1$	$-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}, \quad y \ne 0$

NOTE The term arcsin x is read as "the arcsine of x" or sometimes "the angle whose sine is x." An alternative notation for the inverse sine function is $\sin^{-1} x$.

The graphs of the six inverse trigonometric functions are shown in Figure 1.43.



Domain: $[-1, 1]$
Range: $[-\pi/2, \pi/2]$

Domain: $[-1, 1]$
Range: $[0, \pi]$

Domain: $(-\infty, \infty)$
Range: $(-\pi/2, \pi/2)$

Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[-\pi/2, 0) \cup (0, \pi/2]$

Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[0, \pi/2) \cup (\pi/2, \pi]$

Domain: $(-\infty, \infty)$
Range: $(0, \pi)$

Figure 1.43

EXAMPLE 5 Evaluating Inverse Trigonometric Functions

Evaluate each of the following.

a. $\arcsin\left(-\frac{1}{2}\right)$ **b.** $\arccos 0$ **c.** $\arctan \sqrt{3}$ **d.** $\arcsin(0.3)$

NOTE When evaluating inverse trigonometric functions, remember that they denote *angles in radian measure*.

Solution

a. By definition, $y = \arcsin\left(-\frac{1}{2}\right)$ implies that $\sin y = -\frac{1}{2}$. In the interval $[-\pi/2, \pi/2]$, the correct value of y is $-\pi/6$.

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

b. By definition, $y = \arccos 0$ implies that $\cos y = 0$. In the interval $[0, \pi]$, you have $y = \pi/2$.

$$\arccos 0 = \frac{\pi}{2}$$

c. By definition, $y = \arctan \sqrt{3}$ implies that $\tan y = \sqrt{3}$. In the interval $(-\pi/2, \pi/2)$, you have $y = \pi/3$.

$$\arctan \sqrt{3} = \frac{\pi}{3}$$

d. Using a calculator set in *radian* mode produces

$$\arcsin(0.3) \approx 0.3047.$$

EXPLORATION

Graph $y = \arccos(\cos x)$ for $-4\pi \le x \le 4\pi$. Why isn't the graph the same as the graph of $y = x$?

Inverse functions have the properties

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

When applying these properties to inverse trigonometric functions, remember that the trigonometric functions have inverse functions only in restricted domains. For x-values outside these domains, these two properties do not hold. For example, $\arcsin(\sin \pi)$ is equal to 0, not π.

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

1. If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

2. If $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

3. If $|x| \ge 1$ and $0 \le y < \pi/2$ or $\pi/2 < y \le \pi$, then

$$\sec(\operatorname{arcsec} x) = x \quad \text{and} \quad \operatorname{arcsec}(\sec y) = y.$$

Similar properties hold for the other inverse trigonometric functions.

EXAMPLE 6 Solving an Equation

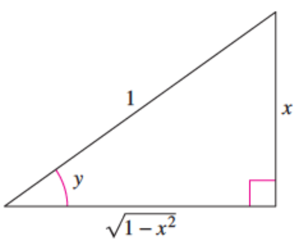
$$\arctan(2x - 3) = \frac{\pi}{4} \qquad \text{Write original equation.}$$

$$\tan[\arctan(2x - 3)] = \tan\frac{\pi}{4} \qquad \text{Take tangent of both sides.}$$

$$2x - 3 = 1 \qquad \tan(\arctan x) = x$$

$$x = 2 \qquad \text{Solve for } x.$$

Some problems in calculus require that you evaluate expressions such as $\cos(\arcsin x)$, as shown in Example 7.



$y = \arcsin x$
Figure 1.44

EXAMPLE 7 Using Right Triangles

a. Given $y = \arcsin x$, where $0 < y < \pi/2$, find $\cos y$.

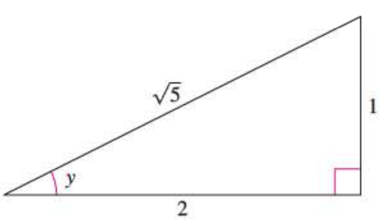
b. Given $y = \operatorname{arcsec}(\sqrt{5}/2)$, find $\tan y$.

Solution

a. Because $y = \arcsin x$, you know that $\sin y = x$. This relationship between x and y can be represented by a right triangle, as shown in Figure 1.44.

$$\cos y = \cos(\arcsin x) = \frac{\text{adj.}}{\text{hyp.}} = \sqrt{1 - x^2}$$

(This result is also valid for $-\pi/2 < y < 0$.)



$y = \operatorname{arcsec}\dfrac{\sqrt{5}}{2}$
Figure 1.45

b. Use the right triangle shown in Figure 1.45.

$$\tan y = \tan\left[\operatorname{arcsec}\left(\frac{\sqrt{5}}{2}\right)\right] = \frac{\text{opp.}}{\text{adj.}} = \frac{1}{2}$$

1.5 Exercises

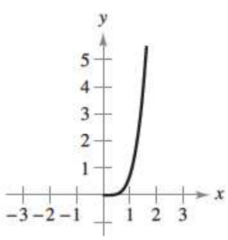
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, show that f and g are inverse functions (a) analytically and (b) graphically.

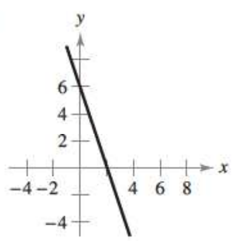
1. $f(x) = 5x + 1, \quad g(x) = \dfrac{x-1}{5}$
2. $f(x) = 3 - 4x, \quad g(x) = \dfrac{3-x}{4}$
3. $f(x) = x^3, \quad g(x) = \sqrt[3]{x}$
4. $f(x) = 1 - x^3, \quad g(x) = \sqrt[3]{1-x}$
5. $f(x) = \sqrt{x-4}, \quad g(x) = x^2 + 4, \quad x \geq 0$
6. $f(x) = 16 - x^2, \quad x \geq 0, \quad g(x) = \sqrt{16-x}$
7. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$
8. $f(x) = \dfrac{1}{1+x}, \quad x \geq 0, \quad g(x) = \dfrac{1-x}{x}, \quad 0 < x \leq 1$

In Exercises 9–12, match the graph of the function with the graph of its inverse function. [The graphs of the inverse functions are labeled (a), (b), (c), and (d).]

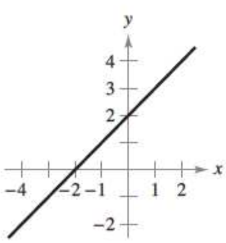
(a)



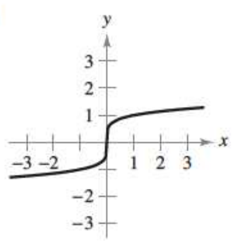
(b)



(c)



(d)



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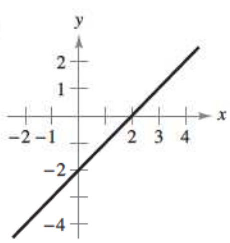

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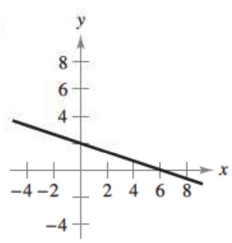
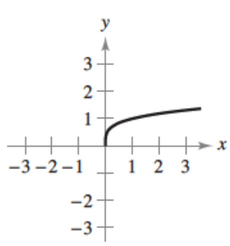


11.



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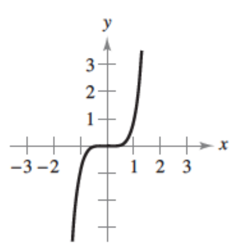
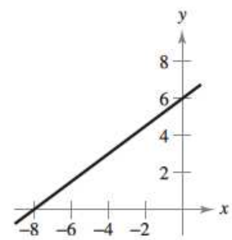
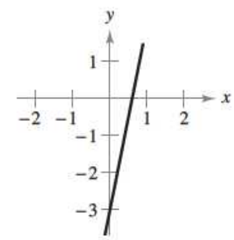


In Exercises 13–16, use the Horizontal Line Test to determine whether the function is one-to-one on its entire domain and therefore has an inverse function. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

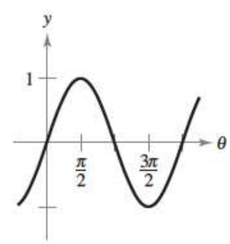
13. $f(x) = \frac{3}{4}x + 6$



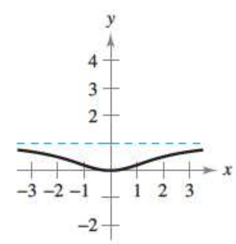
14. $f(x) = 5x - 3$



15. $f(\theta) = \sin \theta$



16. $f(x) = \dfrac{x^2}{x^2+4}$



In Exercises 17–22, use a graphing utility to graph the function. Determine whether the function is one-to-one on its entire domain and therefore has an inverse function.

17. $h(s) = \dfrac{1}{s-2} - 3$
18. $f(x) = \dfrac{1}{x^2+1}$
19. $g(t) = \dfrac{1}{\sqrt{t^2+1}}$
20. $f(x) = 5x\sqrt{x-1}$
21. $g(x) = (x+5)^3$
22. $h(x) = |x+4| - |x-4|$

In Exercises 23–28, determine whether the function is one-to-one on its entire domain and therefore has an inverse function.

23. $f(x) = (x+a)^3 + b$
24. $f(x) = \sin \dfrac{3x}{2}$
25. $f(x) = \dfrac{x^4}{4} - 2x^2$
26. $f(x) = x^3 - 6x^2 + 12x$
27. $f(x) = 2 - x - x^3$
28. $f(x) = \sqrt[3]{x+1}$

In Exercises 29–36, (a) find the inverse function of f, (b) graph f and f^{-1} on the same set of coordinate axes, (c) describe the relationship between the graphs, and (d) state the domains and ranges of f and f^{-1}.

29. $f(x) = 2x - 3$
30. $f(x) = 3x$
31. $f(x) = x^5$
32. $f(x) = x^3 - 1$
33. $f(x) = \sqrt{x}$
34. $f(x) = x^2, \quad x \geq 0$
35. $f(x) = \sqrt{4-x^2}, \quad 0 \leq x \leq 2$
36. $f(x) = \sqrt{x^2-4}, \quad x \geq 2$

In Exercises 37–42, (a) find the inverse function of f, (b) use a graphing utility to graph f and f^{-1} in the same viewing window, (c) describe the relationship between the graphs, and (d) state the domains and ranges of f and f^{-1}.

37. $f(x) = \sqrt[3]{x-1}$

38. $f(x) = 3\sqrt[5]{2x-1}$

39. $f(x) = x^{2/3}, \quad x \geq 0$

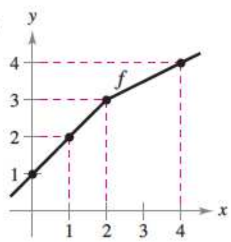
40. $f(x) = x^{3/5}$

41. $f(x) = \dfrac{x}{\sqrt{x^2+7}}$

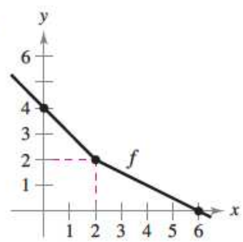
42. $f(x) = \dfrac{x+2}{x}$

In Exercises 43 and 44, use the graph of the function f to make a table of values for the given points. Then make a second table that can be used to find f^{-1}, and sketch the graph of f^{-1}. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

43.



44.



45. *Cost* You need 50 pounds of two commodities costing \$1.25 and \$1.60 per pound.

(a) Verify that the total cost is $y = 1.25x + 1.60(50 - x)$, where x is the number of pounds of the less expensive commodity.

(b) Find the inverse function of the cost function. What does each variable represent in the inverse function?

(c) What is the domain of the inverse function? Validate or explain your answer using the context of the problem.

(d) Determine the number of pounds of the less expensive commodity purchased if the total cost is \$73.

46. *Temperature* The formula $C = \frac{5}{9}(F - 32)$, where $F \geq -459.6$, represents the Celsius temperature C as a function of the Fahrenheit temperature F.

(a) Find the inverse function of C.

(b) What does the inverse function represent?

(c) What is the domain of the inverse function? Validate or explain your answer using the context of the problem.

(d) The temperature is 22°C. What is the corresponding temperature in degrees Fahrenheit?

In Exercises 47 and 48, find f^{-1} over the given interval. Use a graphing utility to graph f and f^{-1} in the same viewing window. Describe the relationship between the graphs.

47. $f(x) = \dfrac{x}{x^2-4}; \quad (-2, 2)$

48. $f(x) = 2 - \dfrac{3}{x^2}; \quad (0, 10)$

Graphical Reasoning **In Exercises 49–52, (a) use a graphing utility to graph the function, (b) use the *drawing* feature of the graphing utility to draw the inverse of the function, and (c) determine whether the graph of the inverse relation is an inverse function. Explain your reasoning.**

49. $f(x) = x^3 + x + 4$

50. $h(x) = x\sqrt{4 - x^2}$

51. $g(x) = \dfrac{3x^2}{x^2+1}$

52. $f(x) = \dfrac{4x}{\sqrt{x^2+15}}$

In Exercises 53–58, show that f is one-to-one on the given interval and therefore has an inverse function on that interval.

	Function	*Interval*
53.	$f(x) = (x-4)^2$	$[4, \infty)$
54.	$f(x) = \lvert x+2 \rvert$	$[-2, \infty)$
55.	$f(x) = \dfrac{4}{x^2}$	$(0, \infty)$
56.	$f(x) = \cot x$	$(0, \pi)$
57.	$f(x) = \cos x$	$[0, \pi]$
58.	$f(x) = \sec x$	$\left[0, \dfrac{\pi}{2}\right)$

In Exercises 59–62, determine whether the function is one-to-one. If it is, find its inverse function.

59. $f(x) = \sqrt{x-2}$

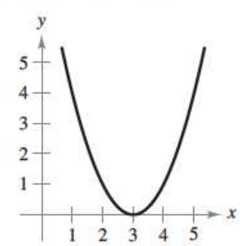
60. $f(x) = -3$

61. $f(x) = \lvert x-2 \rvert, \quad x \leq 2$

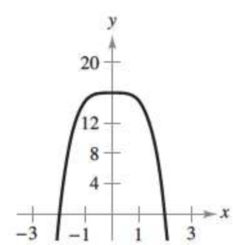
62. $f(x) = ax + b, \quad a \neq 0$

In Exercises 63–66, delete part of the domain so that the function that remains is one-to-one. Find the inverse function of the remaining function and give the domain of the inverse function. (*Note:* There is more than one correct answer.)

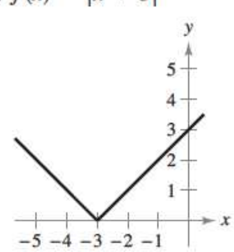
63. $f(x) = (x-3)^2$



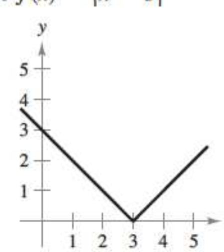
64. $f(x) = 16 - x^4$



65. $f(x) = \lvert x+3 \rvert$



66. $f(x) = \lvert x-3 \rvert$



In Exercises 67–72, (a) sketch a graph of the function f, (b) determine an interval on which f is one-to-one, (c) find the inverse function of f on the interval found in part (b), and (d) give the domain of the inverse function. (*Note:* There is more than one correct answer.)

67. $f(x) = (x + 5)^2$

68. $f(x) = (7 - x)^2$

69. $f(x) = \sqrt{x^2 - 4x}$

70. $f(x) = -\sqrt{25 - x^2}$

71. $f(x) = 3 \cos x$

72. $f(x) = 2 \sin x$

In Exercises 73–78, find $f^{-1}(a)$ for the function f and real number a.

Function	*Real Number*
73. $f(x) = x^3 + 2x - 1$	$a = 2$
74. $f(x) = 2x^5 + x^3 + 1$	$a = -2$
75. $f(x) = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$	$a = \frac{1}{2}$
76. $f(x) = \cos 2x, \quad 0 \le x \le \frac{\pi}{2}$	$a = 1$
77. $f(x) = x^3 - \frac{4}{x}, \quad x > 0$	$a = 6$
78. $f(x) = \sqrt{x - 4}$	$a = 2$

In Exercises 79–82, use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the indicated value.

79. $(f^{-1} \circ g^{-1})(1)$

80. $(g^{-1} \circ f^{-1})(-3)$

81. $(f^{-1} \circ f^{-1})(6)$

82. $(g^{-1} \circ g^{-1})(-4)$

In Exercises 83–86, use the functions $f(x) = x + 4$ and $g(x) = 2x - 5$ to find the indicated function.

83. $g^{-1} \circ f^{-1}$

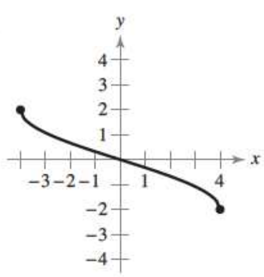
84. $f^{-1} \circ g^{-1}$

85. $(f \circ g)^{-1}$

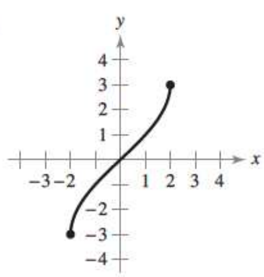
86. $(g \circ f)^{-1}$

In Exercises 87 and 88, (a) use the graph of the function f to determine whether f is one-to-one, (b) state the domain of f^{-1}, and (c) estimate the value of $f^{-1}(2)$.

87.

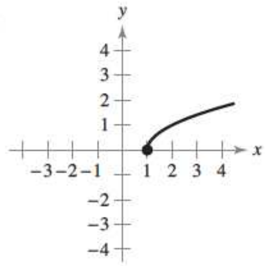


88.



In Exercises 89 and 90, use the graph of the function f to sketch the graph of f^{-1}. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

89.



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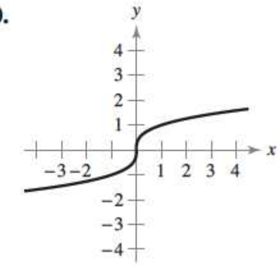


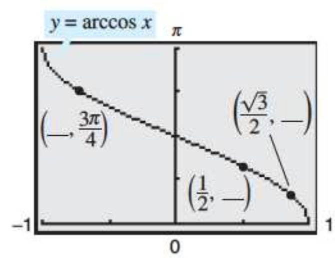
Numerical and Graphical Analysis **In Exercises 91 and 92, (a) use a graphing utility to complete the table, (b) plot the points in the table and graph the function by hand, (c) use a graphing utility to graph the function and compare the result with your hand-drawn graph in part (b), and (d) determine any intercepts and symmetry of the graph.**

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y											

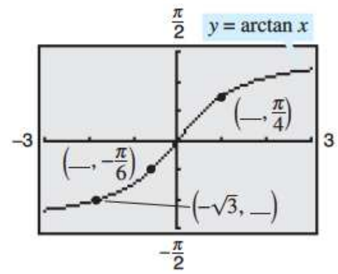
91. $y = \arcsin x$

92. $y = \arccos x$

93. Determine the missing coordinates of the points on the graph of the function.



94. Determine the missing coordinates of the points on the graph of the function.



In Exercises 95–102, evaluate the expression without using a calculator.

95. $\arcsin \frac{1}{2}$

96. $\arcsin 0$

97. $\arccos \frac{1}{2}$

98. $\arccos 0$

99. $\arctan \dfrac{\sqrt{3}}{3}$

100. $\operatorname{arccot}(-\sqrt{3})$

101. $\operatorname{arccsc}(-\sqrt{2})$

102. $\arccos\left(-\dfrac{\sqrt{3}}{2}\right)$

In Exercises 103–106, use a calculator to approximate the value. Round your answer to two decimal places.

103. $\arccos(-0.8)$

104. $\arcsin(-0.39)$

105. $\operatorname{arcsec} 1.269$

106. $\arctan(-3)$

WRITING ABOUT CONCEPTS

107. Describe how to find the inverse function of a one-to-one function given by an equation in x and y. Give an example.

108. Describe the relationship between the graph of a function and the graph of its inverse function.

109. Give an example of a function that does *not* have an inverse function.

110. Explain why $\tan \pi = 0$ does not imply that $\arctan 0 = \pi$.

111. Explain why the domains of the trigonometric functions are restricted when finding the inverse trigonometric functions.

112. Explain how to graph $y = \operatorname{arccot} x$ on a graphing utility that does not have the arccotangent function.

In Exercises 113 and 114, use a graphing utility to confirm that f and g are inverse functions. (Remember to restrict the domain of f properly.)

113. $f(x) = \tan x$
$g(x) = \arctan x$

114. $f(x) = \sin x$
$g(x) = \arcsin x$

In Exercises 115 and 116, use the properties of inverse trigonometric functions to evaluate the expression.

115. $\cos[\arccos(-0.1)]$

116. $\arcsin(\sin 3\pi)$

In Exercises 117–122, evaluate the expression without using a calculator. [*Hint:* Sketch a right triangle, as demonstrated in Example 7(b).]

117. (a) $\sin\left(\arcsin \dfrac{1}{2}\right)$
(b) $\cos\left(\arcsin \dfrac{1}{2}\right)$

118. (a) $\tan\left(\arccos \dfrac{\sqrt{2}}{2}\right)$
(b) $\cos\left(\arcsin \dfrac{5}{13}\right)$

119. (a) $\sin\left(\arctan \dfrac{3}{4}\right)$
(b) $\sec\left(\arcsin \dfrac{4}{5}\right)$

120. (a) $\tan(\operatorname{arccot} 2)$
(b) $\cos(\operatorname{arcsec} \sqrt{5})$

121. (a) $\cot\left[\arcsin\left(-\dfrac{1}{2}\right)\right]$
(b) $\csc\left[\arctan\left(-\dfrac{5}{12}\right)\right]$

122. (a) $\sec\left[\arctan\left(-\dfrac{3}{5}\right)\right]$
(b) $\tan\left[\arcsin\left(-\dfrac{5}{6}\right)\right]$

In Exercises 123–128, use the figure to write the expression in algebraic form given $y = \arccos x$, where $0 < y < \pi/2$.

123. $\cos y$

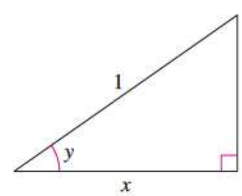
124. $\sin y$

125. $\tan y$

126. $\cot y$

127. $\sec y$

128. $\csc y$



In Exercises 129–138, write the expression in algebraic form. [*Hint:* Sketch a right triangle, as demonstrated in Example 7(a).]

129. $\tan(\arctan x)$

130. $\sin(\arccos x)$

131. $\cos(\arcsin 2x)$

132. $\sec(\arctan 4x)$

133. $\sin(\operatorname{arcsec} x)$

134. $\cos(\operatorname{arccot} x)$

135. $\tan\left(\operatorname{arcsec} \dfrac{x}{3}\right)$

136. $\sec[\arcsin(x - 1)]$

137. $\csc\left(\arctan \dfrac{x}{\sqrt{2}}\right)$

138. $\cos\left(\arcsin \dfrac{x - h}{r}\right)$

In Exercises 139–142, solve the equation for x.

139. $\arcsin(3x - \pi) = \frac{1}{2}$

140. $\arctan(2x - 5) = -1$

141. $\arcsin\sqrt{2x} = \arccos\sqrt{x}$

142. $\arccos x = \operatorname{arcsec} x$

In Exercises 143 and 144, find the point of intersection of the graphs of the functions.

143. $y = \arccos x$

$y = \arctan x$

144. $y = \arcsin x$

$y = \arccos x$

In Exercises 145 and 146, fill in the blank.

145. $\arctan\dfrac{9}{x} = \arcsin\left(\quad\right), \quad x > 0$

146. $\arcsin\dfrac{\sqrt{36 - x^2}}{6} = \arccos\left(\quad\right)$

In Exercises 147 and 148, verify each identity.

147. (a) $\operatorname{arccsc} x = \arcsin\dfrac{1}{x}, \quad |x| \geq 1$

(b) $\arctan x + \arctan\dfrac{1}{x} = \dfrac{\pi}{2}, \quad x > 0$

148. (a) $\arcsin(-x) = -\arcsin x, \quad |x| \leq 1$

(b) $\arccos(-x) = \pi - \arccos x, \quad |x| \leq 1$

In Exercises 149–152, sketch the graph of the function. Use a graphing utility to verify your graph.

149. $f(x) = \arcsin(x - 1)$

150. $f(x) = \arctan x + \dfrac{\pi}{2}$

151. $f(x) = \operatorname{arcsec} 2x$

152. $f(x) = \arccos\dfrac{x}{4}$

153. ***Think About It*** Given that f is a one-to-one function and $f(-3) = 8$, find $f^{-1}(8)$.

154. ***Think About It*** Given $f(x) = 5 + \arccos x$, find $f^{-1}\left(5 + \dfrac{\pi}{2}\right)$.

155. Prove that if f and g are one-to-one functions, then $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

156. Prove that if f has an inverse function, then $(f^{-1})^{-1} = f$.

157. Prove that $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$.

158. Prove that if f is a one-to-one function and $f(x) \neq 0$, then

$$g(x) = \frac{1}{f(x)}$$

is a one-to-one function.

159. Prove that if a function has an inverse function, then the inverse function is unique.

160. Prove that a function has an inverse function if and only if it is one-to-one.

True or False? **In Exercises 161–166, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

161. If f is an even function, then f^{-1} exists.

162. If the inverse function of f exists, then the y-intercept of f is an x-intercept of f^{-1}.

163. $\arcsin^2 x + \arccos^2 x = 1$

164. The range of $y = \arcsin x$ is $[0, \pi]$.

165. If $f(x) = x^n$ where n is odd, then f^{-1} exists.

166. There exists no function f such that $f = f^{-1}$.

167. Verify each identity.

(a) $\operatorname{arccot} x = \begin{cases} \pi + \arctan(1/x), & x < 0 \\ \pi/2, & x = 0 \\ \arctan(1/x), & x > 0 \end{cases}$

(b) $\operatorname{arcsec} x = \arccos(1/x), \quad |x| \geq 1$

(c) $\operatorname{arccsc} x = \arcsin(1/x), \quad |x| \geq 1$

168. Use the results of Exercise 167 and a graphing utility to evaluate the following.

(a) $\operatorname{arccot} 0.5$

(b) $\operatorname{arcsec} 2.7$

(c) $\operatorname{arccsc}(-3.9)$

(d) $\operatorname{arccot}(-1.4)$

169. Prove that

$$\arctan x + \arctan y = \arctan\frac{x + y}{1 - xy}, \quad xy \neq 1.$$

Use this formula to show that

$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}.$$

170. ***Think About It*** Use a graphing utility to graph

$f(x) = \sin x$ and $g(x) = \arcsin(\sin x)$.

Why isn't the graph of g the line $y = x$?

171. Let $f(x) = ax^2 + bx + c$, where $a > 0$ and the domain is all real numbers such that $x \leq -\dfrac{b}{2a}$. Find f^{-1}.

172. Determine conditions on the constants a, b, and c such that the graph of $f(x) = \dfrac{ax + b}{cx - a}$ is symmetric about the line $y = x$.

173. Determine conditions on the constants a, b, c, and d such that $f(x) = \dfrac{ax + b}{cx + d}$ has an inverse function. Then find f^{-1}.

CAPSTONE

174. The point $\left(\dfrac{3\pi}{2}, 0\right)$ is on the graph of $y = \cos x$. Does $\left(0, \dfrac{3\pi}{2}\right)$ lie on the graph of $y = \arccos x$? If not, does this contradict the definition of inverse function?

1.6 Exponential and Logarithmic Functions

- **Develop and use properties of exponential functions.**
- **Understand the definition of the number *e*.**
- **Understand the definition of the natural logarithmic function.**
- **Develop and use properties of the natural logarithmic function.**

Exponential Functions

An **exponential function** involves a constant raised to a power, such as $f(x) = 2^x$. You already know how to evaluate 2^x for *rational* values of x. For instance,

$$2^0 = 1, \quad 2^2 = 4, \quad 2^{-1} = \frac{1}{2}, \quad \text{and} \quad 2^{1/2} = \sqrt{2} \approx 1.4142136.$$

For *irrational* values of x, you can define 2^x by considering a sequence of rational numbers that approach x. A full discussion of this process would not be appropriate here, but the general idea is as follows. Suppose you want to define the number $2^{\sqrt{2}}$. Because $\sqrt{2} = 1.414213\ldots$, you consider the following numbers (which are of the form 2^r, where r is rational).

$$\begin{aligned}
2^1 = 2 &< 2^{\sqrt{2}} < 4 = 2^2 \\
2^{1.4} = 2.639015\ldots &< 2^{\sqrt{2}} < 2.828427\ldots = 2^{1.5} \\
2^{1.41} = 2.657371\ldots &< 2^{\sqrt{2}} < 2.675855\ldots = 2^{1.42} \\
2^{1.414} = 2.664749\ldots &< 2^{\sqrt{2}} < 2.666597\ldots = 2^{1.415} \\
2^{1.4142} = 2.665119\ldots &< 2^{\sqrt{2}} < 2.665303\ldots = 2^{1.4143} \\
2^{1.41421} = 2.665137\ldots &< 2^{\sqrt{2}} < 2.665156\ldots = 2^{1.41422} \\
2^{1.414213} = 2.665143\ldots &< 2^{\sqrt{2}} < 2.665144\ldots = 2^{1.414214}
\end{aligned}$$

From these calculations, it seems reasonable to conclude that

$$2^{\sqrt{2}} \approx 2.66514.$$

In practice, you can use a calculator to approximate numbers such as $2^{\sqrt{2}}$.

In general, you can use any positive base a, $a \neq 1$, to define an exponential function. So, the exponential function with base a is written as $f(x) = a^x$. Exponential functions, even those with irrational values of x, obey the familiar properties of exponents.

PROPERTIES OF EXPONENTS

Let a and b be positive real numbers, and let x and y be any real numbers.

1. $a^0 = 1$ **2.** $a^x a^y = a^{x+y}$ **3.** $(a^x)^y = a^{xy}$ **4.** $(ab)^x = a^x b^x$

5. $\dfrac{a^x}{a^y} = a^{x-y}$ **6.** $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$ **7.** $a^{-x} = \dfrac{1}{a^x}$

EXAMPLE 1 Using Properties of Exponents

a. $(2^2)(2^3) = 2^{2+3} = 2^5$ **b.** $\dfrac{2^2}{2^3} = 2^{2-3} = 2^{-1} = \dfrac{1}{2}$

c. $(3^x)^3 = 3^{3x}$ **d.** $\left(\dfrac{1}{3}\right)^{-x} = (3^{-1})^{-x} = 3^x$

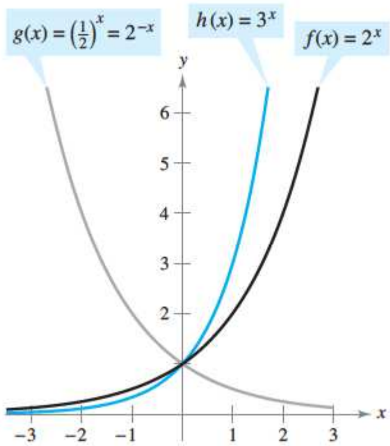


Figure 1.46

EXAMPLE 2 Sketching Graphs of Exponential Functions

Sketch the graphs of the functions

$$f(x) = 2^x, \quad g(x) = \left(\tfrac{1}{2}\right)^x = 2^{-x}, \quad \text{and} \quad h(x) = 3^x.$$

Solution To sketch the graphs of these functions by hand, you can complete a table of values, plot the corresponding points, and connect the points with smooth curves.

x	-3	-2	-1	0	1	2	3	4
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
3^x	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81

Another way to graph these functions is to use a graphing utility. In either case, you should obtain graphs similar to those shown in Figure 1.46. ■

The shapes of the graphs in Figure 1.46 are typical of the exponential functions $y = a^x$ and $y = a^{-x}$ where $a > 1$, as shown in Figure 1.47.

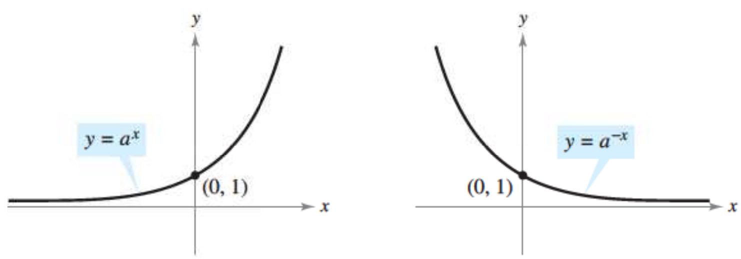


Figure 1.47

PROPERTIES OF EXPONENTIAL FUNCTIONS

Let a be a real number that is greater than 1.

1. The domain of $f(x) = a^x$ and $g(x) = a^{-x}$ is $(-\infty, \infty)$.
2. The range of $f(x) = a^x$ and $g(x) = a^{-x}$ is $(0, \infty)$.
3. The y-intercept of $f(x) = a^x$ and $g(x) = a^{-x}$ is $(0, 1)$.
4. The functions $f(x) = a^x$ and $g(x) = a^{-x}$ are one-to-one.

TECHNOLOGY Functions of the form $h(x) = b^{cx}$ have the same types of properties and graphs as functions of the form $f(x) = a^x$ and $g(x) = a^{-x}$. To see why this is true, notice that

$$b^{cx} = (b^c)^x.$$

For instance, $f(x) = 2^{3x}$ can be written as $f(x) = (2^3)^x$ or $f(x) = 8^x$. Try confirming this by graphing $f(x) = 2^{3x}$ and $g(x) = 8^x$ in the same viewing window.

The Number *e*

In calculus, the natural (or convenient) choice for a base of an exponential number is the irrational number e, whose decimal approximation is

$$e \approx 2.71828182846.$$

This choice may seem anything but natural. However, the convenience of this particular base will become apparent as you continue in this course.

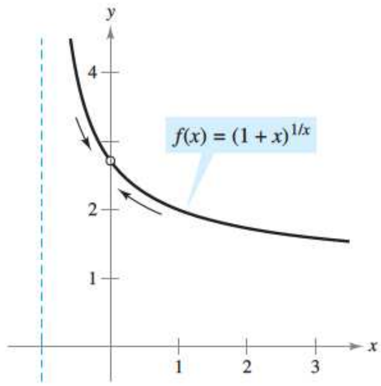


Figure 1.48

EXAMPLE 3 Investigating the Number *e*

Use a graphing utility to graph the function

$$f(x) = (1 + x)^{1/x}.$$

Describe the behavior of the function at values of x that are close to 0.

Solution One way to examine the values of $f(x)$ near 0 is to construct a table.

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$(1 + x)^{1/x}$	2.7320	2.7196	2.7184	2.7181	2.7169	2.7048

From the table, it appears that the closer x gets to 0, the closer $(1 + x)^{1/x}$ gets to e. You can confirm this by graphing the function f, as shown in Figure 1.48. Try using a graphing calculator to obtain this graph. Then zoom in closer and closer to $x = 0$. Although f is not defined when $x = 0$, it is defined for x-values that are arbitrarily close to zero. By zooming in, you can see that the value of $f(x)$ gets closer and closer to $e \approx 2.71828182846$ as x gets closer and closer to 0. Later, when you study limits, you will learn that this result can be written as

$$\lim_{x \to 0} (1 + x)^{1/x} = e$$

which is read as "the limit of $(1 + x)^{1/x}$ as x approaches 0 is e."

EXAMPLE 4 The Graph of the Natural Exponential Function

Sketch the graph of $f(x) = e^x$.

Solution To sketch the graph by hand, you can complete a table of values.

x	-2	-1	0	1	2
e^x	0.135	0.368	1	2.718	7.389

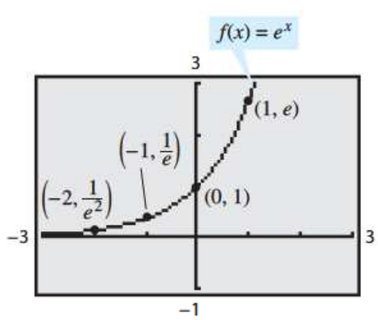


Figure 1.49

You can also use a graphing utility to graph the function. From the values in the table, you can see that a good viewing window for the graph is $-3 \le x \le 3$ and $-1 \le y \le 3$, as shown in Figure 1.49. ■

The Natural Logarithmic Function

Because the natural exponential function $f(x) = e^x$ is one-to-one, it must have an inverse function. Its inverse is called the **natural logarithmic function.** The domain of the natural logarithmic function is the set of positive real numbers.

DEFINITION OF THE NATURAL LOGARITHMIC FUNCTION

Let x be a positive real number. The **natural logarithmic function,** denoted by $\ln x$, is defined as follows. ($\ln x$ is read as "el en of x" or "the natural log of x.")

$$\ln x = b \quad \text{if and only if} \quad e^b = x.$$

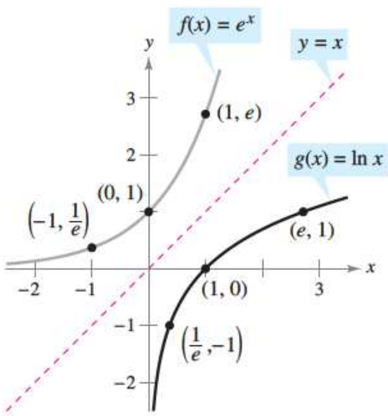


Figure 1.50

This definition tells you that a logarithmic equation can be written in an equivalent exponential form, and vice versa. Here are some examples.

Logarithmic Form	*Exponential Form*
$\ln 1 = 0$	$e^0 = 1$
$\ln e = 1$	$e^1 = e$
$\ln e^{-1} = -1$	$e^{-1} = \dfrac{1}{e}$

Because the function $g(x) = \ln x$ is defined to be the inverse of $f(x) = e^x$, it follows that the graph of the natural logarithmic function is a reflection of the graph of the natural exponential function in the line $y = x$, as shown in Figure 1.50. Several other properties of the natural logarithmic function also follow directly from its definition as the inverse of the natural exponential function.

PROPERTIES OF THE NATURAL LOGARITHMIC FUNCTION

1. The domain of $g(x) = \ln x$ is $(0, \infty)$.
2. The range of $g(x) = \ln x$ is $(-\infty, \infty)$.
3. The x-intercept of $g(x) = \ln x$ is $(1, 0)$.
4. The function $g(x) = \ln x$ is one-to-one.

Because $f(x) = e^x$ and $g(x) = \ln x$ are inverses of each other, you can conclude that

$$\ln e^x = x \quad \text{and} \quad e^{\ln x} = x.$$

EXPLORATION

The graphing utility screen in Figure 1.51 shows the graph of $y_1 = \ln e^x$ or $y_2 = e^{\ln x}$. Which graph is it? What are the domains of y_1 and y_2? Does $\ln e^x = e^{\ln x}$ for all real values of x? Explain.

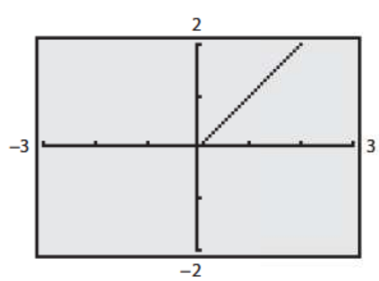


Figure 1.51

Properties of Logarithms

One of the properties of exponents states that when you multiply two exponential functions (having the same base), you add their exponents. For instance,

$$e^x e^y = e^{x+y}.$$

The logarithmic version of this property states that the natural logarithm of the product of two numbers is equal to the sum of the natural logs of the numbers. That is,

$$\ln xy = \ln x + \ln y.$$

This property and the properties dealing with the natural log of a quotient and the natural log of a power are listed here.

PROPERTIES OF LOGARITHMS

Let x, y, and z be real numbers such that $x > 0$ and $y > 0$.

1. $\ln xy = \ln x + \ln y$

2. $\ln \dfrac{x}{y} = \ln x - \ln y$

3. $\ln x^z = z \ln x$

EXAMPLE 5 Expanding Logarithmic Expressions

a. $\ln \dfrac{10}{9} = \ln 10 - \ln 9$ — Property 2

b. $\ln \sqrt{3x+2} = \ln(3x+2)^{1/2}$ — Rewrite with rational exponent.

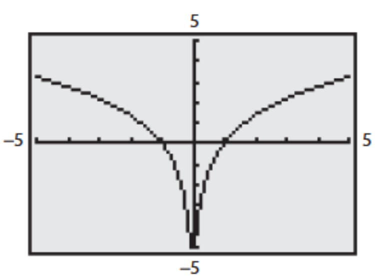
$= \dfrac{1}{2}\ln(3x+2)$ — Property 3

c. $\ln \dfrac{6x}{5} = \ln(6x) - \ln 5$ — Property 2

$= \ln 6 + \ln x - \ln 5$ — Property 1

d.
$$\begin{aligned}\ln \frac{(x^2+3)^2}{x\sqrt[3]{x^2+1}} &= \ln(x^2+3)^2 - \ln\left(x\sqrt[3]{x^2+1}\right)\\ &= 2\ln(x^2+3) - [\ln x + \ln(x^2+1)^{1/3}]\\ &= 2\ln(x^2+3) - \ln x - \ln(x^2+1)^{1/3}\\ &= 2\ln(x^2+3) - \ln x - \frac{1}{3}\ln(x^2+1)\end{aligned}$$

When using the properties of logarithms to rewrite logarithmic functions, you must check to see whether the domain of the rewritten function is the same as the domain of the original function. For instance, the domain of $f(x) = \ln x^2$ is all real numbers except $x = 0$, and the domain of $g(x) = 2\ln x$ is all positive real numbers.



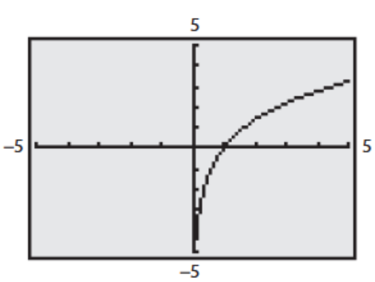


Figure 1.52

TECHNOLOGY Try using a graphing utility to compare the graphs of

$$f(x) = \ln x^2 \quad \text{and} \quad g(x) = 2\ln x.$$

Which of the graphs in Figure 1.52 is the graph of f? Which is the graph of g?

EXAMPLE 6 Solving Exponential and Logarithmic Equations

Solve (a) $7 = e^{x+1}$ and (b) $\ln(2x - 3) = 5$.

Solution

a.
$$\begin{aligned}
7 &= e^{x+1} && \text{Write original equation.}\\
\ln 7 &= \ln(e^{x+1}) && \text{Take natural log of each side.}\\
\ln 7 &= x + 1 && \text{Apply inverse property.}\\
-1 + \ln 7 &= x && \text{Solve for } x.\\
0.946 &\approx x && \text{Use a calculator.}
\end{aligned}$$

b.
$$\begin{aligned}
\ln(2x - 3) &= 5 && \text{Write original equation.}\\
e^{\ln(2x-3)} &= e^5 && \text{Exponentiate each side.}\\
2x - 3 &= e^5 && \text{Apply inverse property.}\\
x &= \frac{1}{2}(e^5 + 3) && \text{Solve for } x.\\
x &\approx 75.707 && \text{Use a calculator.}
\end{aligned}$$

1.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, evaluate the expressions.

1. (a) $25^{3/2}$ (b) $81^{1/2}$ (c) 3^{-2} (d) $27^{-1/3}$

2. (a) $64^{1/3}$ (b) 5^{-4} (c) $\left(\frac{1}{8}\right)^{1/3}$ (d) $\left(\frac{1}{4}\right)^3$

In Exercises 3–6, use the properties of exponents to simplify the expressions.

3. (a) $(5^2)(5^3)$ (b) $(5^2)(5^{-3})$ (c) $\dfrac{5^3}{25^2}$ (d) $\left(\dfrac{1}{4}\right)^2 2^6$

4. (a) $(2^2)^3$ (b) $(5^4)^{1/2}$ (c) $[(27^{-1})(27^{2/3})]^3$ (d) $(25^{3/2})(3^2)$

5. (a) $e^2(e^4)$ (b) $(e^3)^4$ (c) $(e^3)^{-2}$ (d) $\dfrac{e^5}{e^3}$

6. (a) $\left(\dfrac{1}{e}\right)^{-2}$ (b) $\left(\dfrac{e^5}{e^2}\right)^{-1}$ (c) e^0 (d) $\dfrac{1}{e^{-3}}$

In Exercises 7–22, solve for x.

7. $3^x = 81$
8. $4^x = 64$
9. $6^{x-2} = 36$
10. $5^{x+1} = 125$
11. $\left(\frac{1}{2}\right)^x = 32$
12. $\left(\frac{1}{4}\right)^x = 16$
13. $\left(\frac{1}{3}\right)^{x-1} = 27$
14. $\left(\frac{1}{5}\right)^{2x} = 625$
15. $4^3 = (x + 2)^3$
16. $18^2 = (5x - 7)^2$
17. $x^{3/4} = 8$
18. $(x + 3)^{4/3} = 16$
19. $e^x = 5$
20. $e^x = 1$
21. $e^{-2x} = e^5$
22. $e^{3x} = e^{-4}$

In Exercises 23 and 24, compare the given number with the number e. Is the number less than or greater than e?

23. $\left(1 + \dfrac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$

24. $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$

In Exercises 25–34, sketch the graph of the function.

25. $y = 3^x$
26. $y = 3^{x-1}$
27. $y = \left(\frac{1}{3}\right)^x$
28. $y = 2^{-x^2}$
29. $f(x) = 3^{-x^2}$
30. $f(x) = 3^{|x|}$
31. $h(x) = e^{x-2}$
32. $g(x) = -e^{x/2}$
33. $y = e^{-x^2}$
34. $y = e^{-x/4}$

In Exercises 35–40, find the domain of the function.

35. $f(x) = \dfrac{1}{3 + e^x}$
36. $f(x) = \dfrac{1}{2 - e^x}$
37. $f(x) = \sqrt{1 - 4^x}$
38. $f(x) = \sqrt{1 + 3^{-x}}$
39. $f(x) = \sin e^{-x}$
40. $f(x) = \cos e^{-x}$

41. Use a graphing utility to graph $f(x) = e^x$ and the given function in the same viewing window. How are the two graphs related?

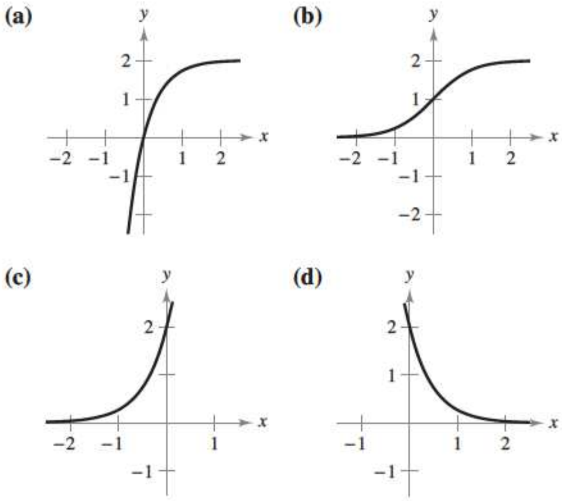
(a) $g(x) = e^{x-2}$ (b) $h(x) = -\frac{1}{2}e^x$ (c) $q(x) = e^{-x} + 3$

42. Use a graphing utility to graph the function. Describe the shape of the graph for very large and very small values of x.

(a) $f(x) = \dfrac{8}{1 + e^{-0.5x}}$

(b) $g(x) = \dfrac{8}{1 + e^{-0.5/x}}$

In Exercises 43–46, match the equation with the correct graph. Assume that a and C are positive real numbers. [The graphs are labeled (a), (b), (c), and (d).]



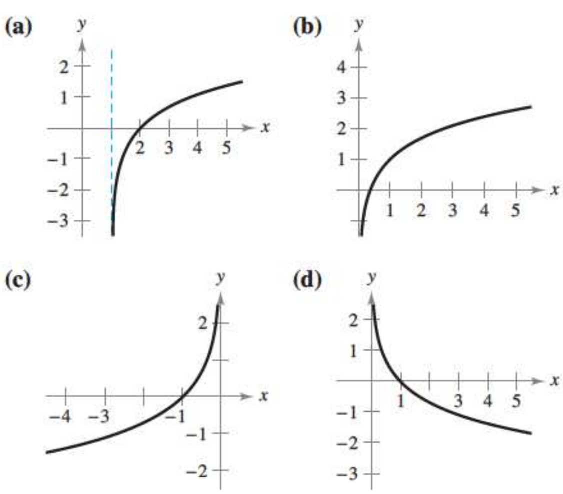
43. $y = Ce^{ax}$

44. $y = Ce^{-ax}$

45. $y = C(1 - e^{-ax})$

46. $y = \dfrac{C}{1 + e^{-ax}}$

In Exercises 47–50, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]



47. $f(x) = \ln x + 1$

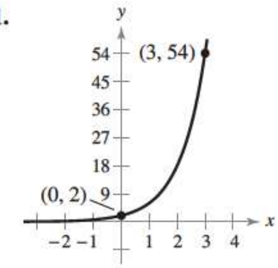
48. $f(x) = -\ln x$

49. $f(x) = \ln(x - 1)$

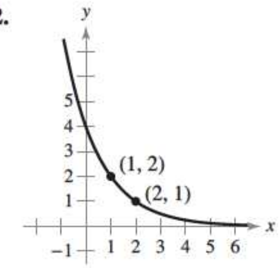
50. $f(x) = -\ln(-x)$

In Exercises 51 and 52, find the exponential function $y = Ca^x$ that fits the graph.

51.



52.



In Exercises 53–56, write the exponential equation as a logarithmic equation, or vice versa.

53. $e^0 = 1$

54. $e^{-2} = 0.1353\ .\ .\ .$

55. $\ln 2 = 0.6931\ .\ .\ .$

56. $\ln 0.5 = -0.6931\ .\ .\ .$

In Exercises 57–62, sketch the graph of the function and state its domain.

57. $f(x) = 3 \ln x$

58. $f(x) = -2 \ln x$

59. $f(x) = \ln 2x$

60. $f(x) = \ln|x|$

61. $f(x) = \ln(x - 1)$

62. $f(x) = 2 + \ln x$

In Exercises 63–66, write an equation for the function having the given characteristics.

63. The shape of $f(x) = e^x$, but shifted eight units upward and reflected in the x-axis

64. The shape of $f(x) = e^x$, but shifted two units to the left and six units downward

65. The shape of $f(x) = \ln x$, but shifted five units to the right and one unit downward

66. The shape of $f(x) = \ln x$, but shifted three units upward and reflected in the y-axis

In Exercises 67–70, show that the functions f and g are inverses of each other by graphing them in the same viewing window.

67. $f(x) = e^{2x}$, $g(x) = \ln \sqrt{x}$

68. $f(x) = e^{x/3}$, $g(x) = \ln x^3$

69. $f(x) = e^x - 1$, $g(x) = \ln(x + 1)$

70. $f(x) = e^{x-1}$, $g(x) = 1 + \ln x$

In Exercises 71–74, (a) find the inverse of the function, (b) use a graphing utility to graph f and f^{-1} in the same viewing window, and (c) verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

71. $f(x) = e^{4x-1}$

72. $f(x) = 3e^{-x}$

73. $f(x) = 2 \ln(x - 1)$

74. $f(x) = 3 + \ln(2x)$

In Exercises 75–80, apply the inverse properties of $\ln x$ and e^x to simplify the given expression.

75. $\ln e^{x^2}$

76. $\ln e^{2x-1}$

77. $e^{\ln(5x+2)}$

78. $e^{\ln \sqrt{x}}$

79. $-1 + \ln e^{2x}$

80. $-8 + e^{\ln x^3}$

In Exercises 81 and 82, use the properties of logarithms to approximate the indicated logarithms, given that $\ln 2 \approx 0.6931$ and $\ln 3 \approx 1.0986$.

81. (a) $\ln 6$ (b) $\ln \frac{2}{3}$ (c) $\ln 81$ (d) $\ln \sqrt{3}$

82. (a) $\ln 0.25$ (b) $\ln 24$ (c) $\ln \sqrt[3]{12}$ (d) $\ln \frac{1}{72}$

WRITING ABOUT CONCEPTS

83. In your own words, state the properties of the natural logarithmic function.

84. Explain why $\ln e^x = x$.

85. In your own words, state the properties of the natural exponential function.

86. The table of values below was obtained by evaluating a function. Determine which of the statements may be true and which must be false, and explain why.

(a) y is an exponential function of x.

(b) y is a logarithmic function of x.

(c) x is an exponential function of y.

(d) y is a linear function of x.

x	1	2	8
y	0	1	3

In Exercises 87–96, use the properties of logarithms to expand the logarithmic expression.

87. $\ln \dfrac{x}{4}$

88. $\ln \sqrt{x^5}$

89. $\ln \dfrac{xy}{z}$

90. $\ln(xyz)$

91. $\ln\left(x\sqrt{x^2+5}\right)$

92. $\ln \sqrt[3]{z+1}$

93. $\ln \sqrt{\dfrac{x-1}{x}}$

94. $\ln z(z-1)^2$

95. $\ln(3e^2)$

96. $\ln \dfrac{1}{e}$

In Exercises 97–104, write the expression as the logarithm of a single quantity.

97. $\ln x + \ln 7$

98. $\ln y + \ln x^2$

99. $\ln(x-2) - \ln(x+2)$

100. $3 \ln x + 2 \ln y - 4 \ln z$

101. $\frac{1}{3}[2 \ln(x+3) + \ln x - \ln(x^2-1)]$

102. $2[\ln x - \ln(x+1) - \ln(x-1)]$

103. $2 \ln 3 - \frac{1}{2} \ln(x^2+1)$

104. $\frac{3}{2}[\ln(x^2+1) - \ln(x+1) - \ln(x-1)]$

In Exercises 105–108, solve for x accurate to three decimal places.

105. (a) $e^{\ln x} = 4$

(b) $\ln e^{2x} = 3$

106. (a) $e^{\ln 2x} = 12$

(b) $\ln e^{-x} = 0$

107. (a) $\ln x = 2$

(b) $e^x = 4$

108. (a) $\ln x^2 = 8$

(b) $e^{-2x} = 5$

In Exercises 109–112, solve the inequality for x.

109. $e^x > 5$

110. $e^{1-x} < 6$

111. $-2 < \ln x < 0$

112. $1 < \ln x < 100$

In Exercises 113 and 114, show that $f = g$ by using a graphing utility to graph f and g in the same viewing window. (Assume $x > 0$.)

113. $f(x) = \ln(x^2/4)$

$g(x) = 2 \ln x - \ln 4$

114. $f(x) = \ln \sqrt{x(x^2+1)}$

$g(x) = \frac{1}{2}[\ln x + \ln(x^2+1)]$

115. Prove that $\ln(x/y) = \ln x - \ln y$, $x > 0, y > 0$.

116. Prove that $\ln x^y = y \ln x$.

117. Graph the functions

$$f(x) = 6^x \quad \text{and} \quad g(x) = x^6$$

in the same viewing window. Where do these graphs intersect? As x increases, which function grows more rapidly?

118. Graph the functions

$$f(x) = \ln x \quad \text{and} \quad g(x) = x^{1/4}$$

in the same viewing window. Where do these graphs intersect? As x increases, which function grows more rapidly?

119. Let $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$.

(a) Use a graphing utility to graph f and determine its domain.

(b) Show that f is an odd function.

(c) Find the inverse function of f.

CAPSTONE

120. Describe the relationship between the graphs of $f(x) = \ln x$ and $g(x) = e^x$.

1 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, find the intercepts (if any).

1. $y = 5x - 8$

2. $y = (x - 2)(x - 6)$

3. $y = \dfrac{x - 3}{x - 4}$

4. $xy = 4$

In Exercises 5 and 6, check for symmetry with respect to both axes and to the origin.

5. $x^2y - x^2 + 4y = 0$

6. $y = x^4 - x^2 + 3$

In Exercises 7–14, sketch the graph of the equation.

7. $y = \frac{1}{2}(-x + 3)$

8. $6x - 3y = 12$

9. $-\frac{1}{3}x + \frac{5}{6}y = 1$

10. $0.02x + 0.15y = 0.25$

11. $y = 9 - 8x - x^2$

12. $y = 6x - x^2$

13. $y = 2\sqrt{4 - x}$

14. $y = |x - 4| - 4$

In Exercises 15 and 16, use a graphing utility to find the point(s) of intersection of the graphs of the equations.

15. $5x + 3y = -1$
$x - y = -5$

16. $x - y + 1 = 0$
$y - x^2 = 7$

In Exercises 17 and 18, plot the points and find the slope of the line passing through the points.

17. $\left(\frac{3}{2}, 1\right), \left(5, \frac{5}{2}\right)$

18. $(-7, 8), (-1, 8)$

In Exercises 19 and 20, use the concept of slope to find t such that the three points are collinear.

19. $(-8, 5), (0, t), (2, -1)$

20. $(-3, 3), (t, -1), (8, 6)$

In Exercises 21–24, find an equation of the line that passes through the point with the given slope. Sketch the line.

21. $(3, -5),\quad m = \frac{7}{4}$

22. $(-8, 1),\quad m$ is undefined.

23. $(-3, 0),\quad m = -\frac{2}{3}$

24. $(5, 4),\quad m = 0$

25. Find equations of the lines passing through $(-3, 5)$ and having the following characteristics.

(a) Slope of $\frac{7}{16}$

(b) Parallel to the line $5x - 3y = 3$

(c) Passing through the origin

(d) Parallel to the y-axis

26. Find equations of the lines passing through $(2, 4)$ and having the following characteristics.

(a) Slope of $-\frac{2}{3}$

(b) Perpendicular to the line $x + y = 0$

(c) Passing through the point $(6, 1)$

(d) Parallel to the x-axis

27. ***Rate of Change*** The purchase price of a new machine is \$12,500, and its value will decrease by \$850 per year. Use this information to write a linear equation that gives the value V of the machine t years after it is purchased. Find its value at the end of 3 years.

28. ***Break-Even Analysis*** A contractor purchases a piece of equipment for \$36,500 that costs an average of \$9.25 per hour for fuel and maintenance. The equipment operator is paid \$13.50 per hour, and customers are charged \$30 per hour.

(a) Write an equation for the cost C of operating this equipment for t hours.

(b) Write an equation for the revenue R derived from t hours of use.

(c) Find the break-even point for this equipment by finding the time at which $R = C$.

In Exercises 29–32, sketch the graph of the equation and use the Vertical Line Test to determine whether the equation expresses y as a function of x.

29. $x - y^2 = 6$

30. $x^2 - y = 0$

31. $y = |x - 2|/(x - 2)$

32. $x = 9 - y^2$

33. Evaluate (if possible) the function $f(x) = 1/x$ at the specified values of the independent variable, and simplify the results.

(a) $f(0)$ (b) $\dfrac{f(1 + \Delta x) - f(1)}{\Delta x}$

34. Evaluate (if possible) the function at each value of the independent variable.

$$f(x) = \begin{cases} x^2 + 2, & x < 0 \\ |x - 2|, & x \ge 0 \end{cases}$$

(a) $f(-4)$ (b) $f(0)$ (c) $f(1)$

35. Find the domain and range of each function.

(a) $y = \sqrt{36 - x^2}$ (b) $y = \dfrac{7}{2x - 10}$ (c) $y = \begin{cases} x^2, & x < 0 \\ 2 - x, & x \ge 0 \end{cases}$

36. Given $f(x) = 1 - x^2$ and $g(x) = 2x + 1$, find the following.

(a) $f(x) - g(x)$ (b) $f(x)g(x)$ (c) $g(f(x))$

37. Sketch (on the same set of coordinate axes) graphs of f for $c = -2, 0,$ and 2.

(a) $f(x) = x^3 + c$

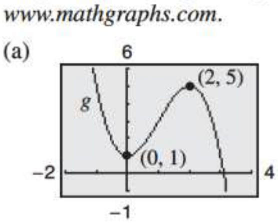
(b) $f(x) = (x - c)^3$

(c) $f(x) = (x - 2)^3 + c$

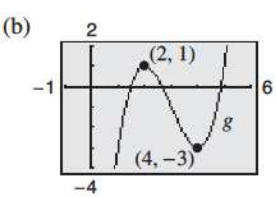
(d) $f(x) = cx^3$

38. Use a graphing utility to graph $f(x) = x^3 - 3x^2$. Use the graph to write an equation for the function g shown in the figure. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*

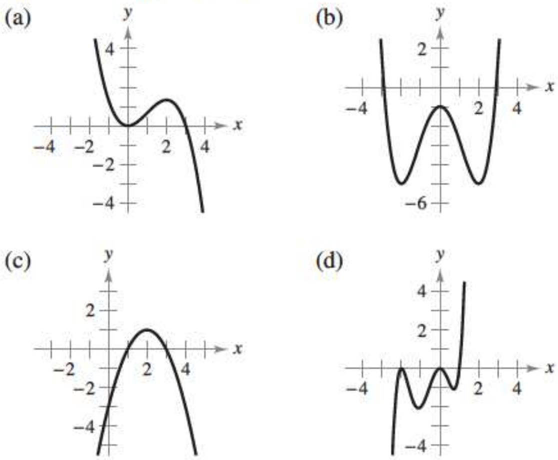
(a)



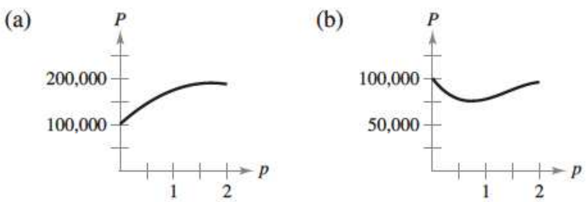
(b)



39. ***Think About It*** What is the minimum degree of the polynomial function whose graph approximates the given graph? What sign must the leading coefficient have?



40. ***Writing*** The following graphs give the profits P for two small companies over a period p of 2 years. Create a story to describe the behavior of each profit function for some hypothetical product the company produces.



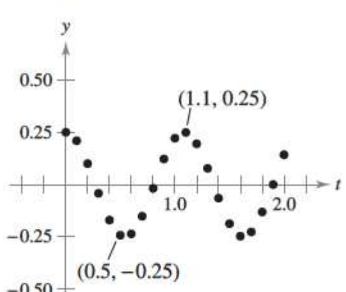
41. ***Harmonic Motion*** The motion of an oscillating weight suspended by a spring was measured by a motion detector. The data collected and the approximate maximum (positive and negative) displacements from equilibrium are shown in the figure. The displacement y is measured in feet and the time t is measured in seconds.

(a) Is y a function of t? Explain.

(b) Approximate the amplitude and period of the oscillations.

(c) Find a model for the data.

(d) Use a graphing utility to graph the model in part (c). Compare the result with the data in the figure.



42. ***Stress Test*** A machine part was tested by bending it x centimeters 10 times per minute until the time y (in hours) of failure. The results are recorded in the table.

x	3	6	9	12	15	18	21	24	27	30
y	61	56	53	55	48	35	36	33	44	23

Table for 42

(a) Use the regression capabilities of a graphing utility to find a linear model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the graph to determine whether there may have been an error made in conducting one of the tests or in recording the results. If so, eliminate the erroneous point and find the model for the remaining data.

In Exercises 43–48, (a) find the inverse of the function, (b) use a graphing utility to graph f and f^{-1} in the same viewing window, and (c) verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

43. $f(x) = \frac{1}{2}x - 3$ **44.** $f(x) = 5x - 7$

45. $f(x) = \sqrt{x + 1}$ **46.** $f(x) = x^3 + 2$

47. $f(x) = \sqrt[3]{x + 1}$ **48.** $f(x) = x^2 - 5, \quad x \geq 0$

In Exercises 49 and 50, sketch the graph of the function by hand.

49. $f(x) = 2 \arctan(x + 3)$ **50.** $h(x) = -3 \arcsin 2x$

In Exercises 51 and 52, evaluate the expression without using a calculator. (*Hint:* Make a sketch of a right triangle.)

51. $\sin\left(\arcsin \frac{1}{2}\right)$ **52.** $\tan(\operatorname{arccot} 2)$

In Exercises 53 and 54, sketch the graph of the function by hand.

53. $f(x) = \ln x + 3$ **54.** $f(x) = \ln(x - 3)$

In Exercises 55 and 56, use the properties of logarithms to expand the logarithmic function.

55. $\ln \sqrt[5]{\dfrac{4x^2 - 1}{4x^2 + 1}}$ **56.** $\ln[(x^2 + 1)(x - 1)]$

In Exercises 57 and 58, write the expression as the logarithm of a single quantity.

57. $\ln 3 + \frac{1}{3} \ln(4 - x^2) - \ln x$

58. $3[\ln x - 2 \ln(x^2 + 1)] + 2 \ln 5$

In Exercises 59 and 60, solve the equation for x.

59. $\ln \sqrt{x + 1} = 2$ **60.** $\ln x + \ln(x - 3) = 0$

In Exercises 61 and 62, (a) find the inverse function of f, (b) use a graphing utility to graph f and f^{-1} in the same viewing window, and (c) verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

61. $f(x) = \ln \sqrt{x}$ **62.** $f(x) = e^{1-x}$

In Exercises 63 and 64, sketch the graph of the function by hand.

63. $y = e^{-x/2}$ **64.** $y = 4e^{-x^2}$

P.S. PROBLEM SOLVING

1. Consider the circle $x^2 + y^2 - 6x - 8y = 0$, as shown in the figure.

(a) Find the center and radius of the circle.

(b) Find an equation of the tangent line to the circle at the point $(0, 0)$.

(c) Find an equation of the tangent line to the circle at the point $(6, 0)$.

(d) Where do the two tangent lines intersect?

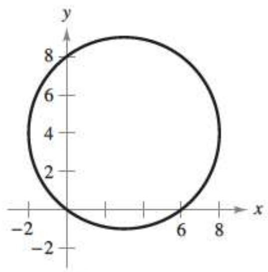


Figure for 1

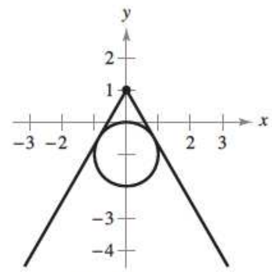


Figure for 2

2. There are two tangent lines from the point $(0, 1)$ to the circle $x^2 + (y + 1)^2 = 1$ (see figure). Find equations of these two lines by using the fact that each tangent line intersects the circle at *exactly* one point.

3. The Heaviside function $H(x)$ is widely used in engineering applications.

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Sketch the graph of the Heaviside function and the graphs of the following functions by hand.

(a) $H(x) - 2$ (b) $H(x - 2)$ (c) $-H(x)$

(d) $H(-x)$ (e) $\frac{1}{2}H(x)$ (f) $-H(x - 2) + 2$

Institute of Electrical Engineers, London

OLIVER HEAVISIDE (1850–1925)

Heaviside was a British mathematician and physicist who contributed to the field of applied mathematics, especially applications of mathematics to electrical engineering. The *Heaviside function* is a classic type of "on-off" function that has applications to electricity and computer science.

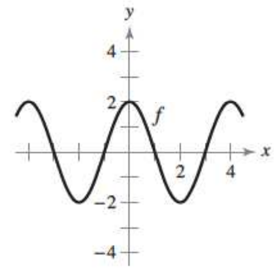
4. Consider the graph of the function f shown below. Use this graph to sketch the graphs of the following functions. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(a) $f(x + 1)$ (b) $f(x) + 1$

(c) $2f(x)$ (d) $f(-x)$

(e) $-f(x)$ (f) $|f(x)|$

(g) $f(|x|)$



5. A rancher plans to fence a rectangular pasture adjacent to a river. The rancher has 100 meters of fencing, and no fencing is needed along the river (see figure).

(a) Write the area A of the pasture as a function of x, the length of the side parallel to the river. What is the domain of A?

(b) Graph the area function $A(x)$ and estimate the dimensions that yield the maximum amount of area for the pasture.

(c) Find the dimensions that yield the maximum amount of area for the pasture by completing the square.

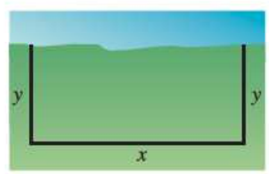


Figure for 5

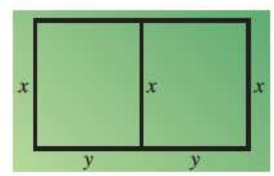


Figure for 6

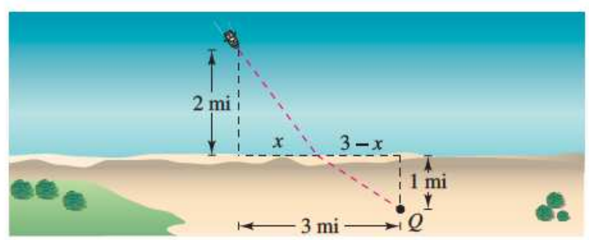
6. A rancher has 300 feet of fencing to enclose two adjacent pastures.

(a) Write the total area A of the two pastures as a function of x (see figure). What is the domain of A?

(b) Graph the area function and estimate the dimensions that yield the maximum amount of area for the pastures.

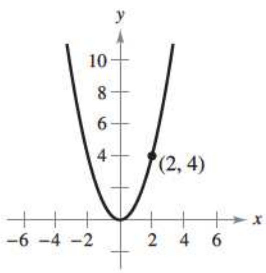
(c) Find the dimensions that yield the maximum amount of area for the pastures by completing the square.

7. You are in a boat 2 miles from the nearest point on the coast. You are to go to a point Q located 3 miles down the coast and 1 mile inland (see figure). You can row at 2 miles per hour and walk at 4 miles per hour. Write the total time T of the trip as a function of x.



8. Graph the function $f(x) = e^x - e^{-x}$. From the graph, the function appears to be one-to-one. Assuming that the function has an inverse, find $f^{-1}(x)$.

9. One of the fundamental themes of calculus is to find the slope of the tangent line to a curve at a point. To see how this can be done, consider the point $(2, 4)$ on the graph of $f(x) = x^2$.



(a) Find the slope of the line joining $(2, 4)$ and $(3, 9)$. Is the slope of the tangent line at $(2, 4)$ greater than or less than this number?

(b) Find the slope of the line joining $(2, 4)$ and $(1, 1)$. Is the slope of the tangent line at $(2, 4)$ greater than or less than this number?

(c) Find the slope of the line joining $(2, 4)$ and $(2.1, 4.41)$. Is the slope of the tangent line at $(2, 4)$ greater than or less than this number?

(d) Find the slope of the line joining $(2, 4)$ and $(2 + h, f(2 + h))$ in terms of the nonzero number h. Verify that $h = 1, -1$, and 0.1 yield the solutions to parts (a)–(c) above.

(e) What is the slope of the tangent line at $(2, 4)$? Explain how you arrived at your answer.

10. Sketch the graph of the function $f(x) = \sqrt{x}$ and label the point $(4, 2)$ on the graph.

(a) Find the slope of the line joining $(4, 2)$ and $(9, 3)$. Is the slope of the tangent line at $(4, 2)$ greater than or less than this number?

(b) Find the slope of the line joining $(4, 2)$ and $(1, 1)$. Is the slope of the tangent line at $(4, 2)$ greater than or less than this number?

(c) Find the slope of the line joining $(4, 2)$ and $(4.41, 2.1)$. Is the slope of the tangent line at $(4, 2)$ greater than or less than this number?

(d) Find the slope of the line joining $(4, 2)$ and $(4 + h, f(4 + h))$ in terms of the nonzero number h.

(e) What is the slope of the tangent line at the point $(4, 2)$? Explain how you arrived at your answer.

11. Explain how you would graph the equation

$$y + |y| = x + |x|.$$

Then sketch the graph.

12. A large room contains two speakers that are 3 meters apart. The sound intensity I of one speaker is twice that of the other, as shown in the figure. (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.) Suppose the listener is free to move about the room to find those positions that receive equal amounts of sound from both speakers. Such a location satisfies two conditions: (1) the sound intensity at the listener's position is directly proportional to the sound level of the source, and (2) the sound intensity is inversely proportional to the square of the distance from the source.

(a) Find the points on the x-axis that receive equal amounts of sound from both speakers.

(b) Find and graph the equation of all locations (x, y) where one could stand and receive equal amounts of sound from both speakers.

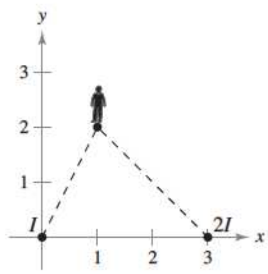


Figure for 12

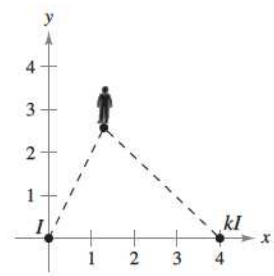


Figure for 13

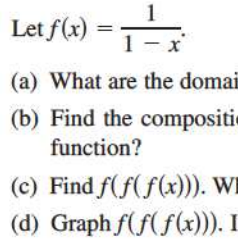
13. Suppose the speakers in Exercise 12 are 4 meters apart and the sound intensity of one speaker is k times that of the other, as shown in the figure. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(a) Find the equation of all locations (x, y) where one could stand and receive equal amounts of sound from both speakers.

(b) Graph the equation for the case $k = 3$.

(c) Describe the set of locations of equal sound as k becomes very large.

14. Let d_1 and d_2 be the distances from the point (x, y) to the points $(-1, 0)$ and $(1, 0)$, respectively, as shown in the figure. Show that the equation of the graph of all points (x, y) satisfying $d_1 d_2 = 1$ is $(x^2 + y^2)^2 = 2(x^2 - y^2)$. This curve is called a **lemniscate.** Graph the lemniscate and identify three points on the graph.



15. Let $f(x) = \dfrac{1}{1 - x}$.

(a) What are the domain and range of f?

(b) Find the composition $f(f(x))$. What is the domain of this function?

(c) Find $f(f(f(x)))$. What is the domain of this function?

(d) Graph $f(f(f(x)))$. Is the graph a line? Why or why not?