

4

Applications of Differentiation

This chapter discusses several applications of the derivative of a function. These applications fall into three basic categories—curve sketching, optimization, and approximation techniques.

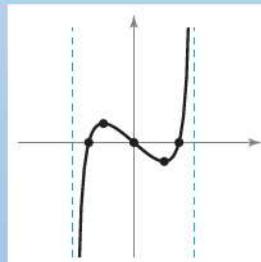
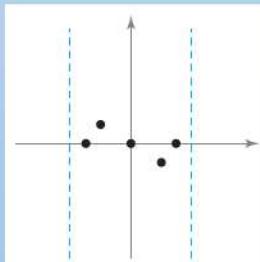
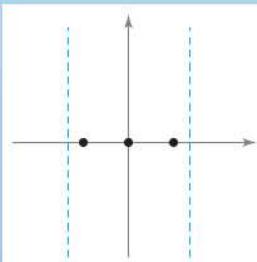
In this chapter, you should learn the following.

- How to use a derivative to locate the minimum and maximum values of a function on a closed interval. (4.1)
- How numerous results in this chapter depend on two important theorems called *Rolle's Theorem* and the *Mean Value Theorem*. (4.2)
- How to use the first derivative to determine whether a function is increasing or decreasing. (4.3)
- How to use the second derivative to determine whether the graph of a function is concave upward or concave downward. (4.4)
- How to find horizontal asymptotes of the graph of a function. (4.5)
- How to graph a function using the techniques from Chapters 1–4. (4.6)
- How to solve optimization problems. (4.7)
- How to use approximation techniques to solve problems. (4.8)



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A small aircraft starts its descent from an altitude of 1 mile, 4 miles west of the runway. Given a function that models the glide path of the plane, when would the plane be descending at the greatest rate? (See Section 4.4, Exercise 91.)



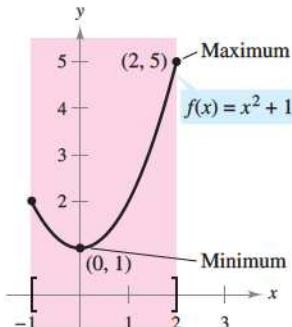
In Chapter 4, you will use calculus to analyze graphs of functions. For example, you can use the derivative of a function to determine the function's maximum and minimum values. You can use limits to identify any asymptotes of the function's graph. In Section 4.6, you will combine these techniques to sketch the graph of a function.

4.1 Extrema on an Interval

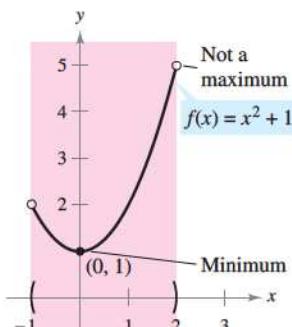
- Understand the definition of extrema of a function on an interval.
- Understand the definition of relative extrema of a function on an open interval.
- Find extrema on a closed interval.

Extrema of a Function

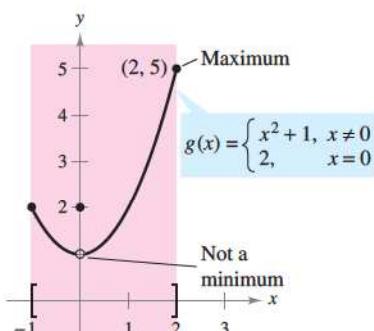
In calculus, much effort is devoted to determining the behavior of a function f on an interval I . Does f have a maximum value on I ? Does it have a minimum value? Where is the function increasing? Where is it decreasing? In this chapter you will learn how derivatives can be used to answer these questions. You will also see why these questions are important in real-life applications.



(a) f is continuous, $[-1, 2]$ is closed.



(b) f is continuous, $(-1, 2)$ is open.



(c) g is not continuous, $[-1, 2]$ is closed.
Extrema can occur at interior points or endpoints of an interval. Extrema that occur at the endpoints are called **endpoint extrema**.

Figure 4.1

DEFINITION OF EXTREMA

Let f be defined on an interval I containing c .

1. $f(c)$ is the **minimum of f on I** if $f(c) \leq f(x)$ for all x in I .
2. $f(c)$ is the **maximum of f on I** if $f(c) \geq f(x)$ for all x in I .

The minimum and maximum of a function on an interval are the **extreme values**, or **extrema** (the singular form of extrema is **extremum**), of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum**, or the **global minimum** and **global maximum**, on the interval.

A function need not have a minimum or a maximum on an interval. For instance, in Figure 4.1(a) and (b), you can see that the function $f(x) = x^2 + 1$ has both a minimum and a maximum on the closed interval $[-1, 2]$, but does not have a maximum on the open interval $(-1, 2)$. Moreover, in Figure 4.1(c), you can see that continuity (or the lack of it) can affect the existence of an extremum on the interval. This suggests the following theorem. (Although the Extreme Value Theorem is intuitively plausible, a proof of this theorem is not within the scope of this text.)

THEOREM 4.1 THE EXTREME VALUE THEOREM

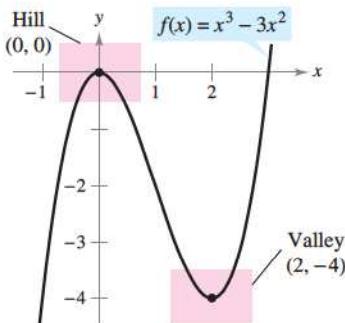
If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

EXPLORATION

Finding Minimum and Maximum Values The Extreme Value Theorem (like the Intermediate Value Theorem) is an *existence theorem* because it tells of the existence of minimum and maximum values but does not show how to find these values. Use the extreme-value capability of a graphing utility to find the minimum and maximum values of each of the following functions. In each case, do you think the x -values are exact or approximate? Explain your reasoning.

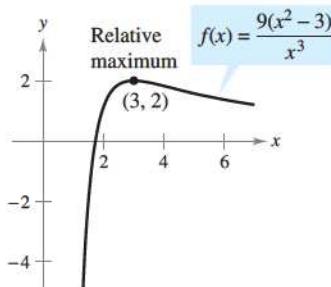
- a. $f(x) = x^2 - 4x + 5$ on the closed interval $[-1, 3]$
- b. $f(x) = x^3 - 2x^2 - 3x - 2$ on the closed interval $[-1, 3]$

Relative Extrema and Critical Numbers

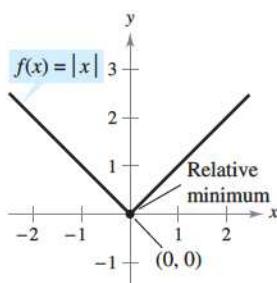


f has a relative maximum at $(0, 0)$ and a relative minimum at $(2, -4)$.

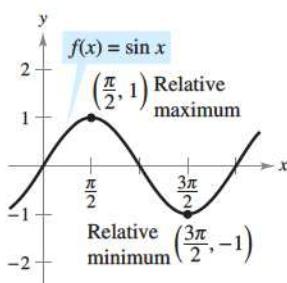
Figure 4.2



(a) $f'(3) = 0$



(b) $f'(0)$ does not exist.



(c) $f'\left(\frac{\pi}{2}\right) = 0; f'\left(\frac{3\pi}{2}\right) = 0$

Figure 4.3

In Figure 4.2, the graph of $f(x) = x^3 - 3x^2$ has a **relative maximum** at the point $(0, 0)$ and a **relative minimum** at the point $(2, -4)$. Informally, for a continuous function, you can think of a relative maximum as occurring on a “hill” on the graph, and a relative minimum as occurring in a “valley” on the graph. Such a hill and valley can occur in two ways. If the hill (or valley) is smooth and rounded, the graph has a horizontal tangent line at the high point (or low point). If the hill (or valley) is sharp and peaked, the graph represents a function that is not differentiable at the high point (or low point).

DEFINITION OF RELATIVE EXTREMA

- If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** of f , or you can say that f has a **relative maximum at $(c, f(c))$** .
- If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** of f , or you can say that f has a **relative minimum at $(c, f(c))$** .

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima. Relative maximum and relative minimum are sometimes called **local maximum** and **local minimum**, respectively.

Example 1 examines the derivatives of functions at *given* relative extrema. (Much more is said about *finding* the relative extrema of a function in Section 4.3.)

EXAMPLE 1 The Value of the Derivative at Relative Extrema

Find the value of the derivative at each relative extremum shown in Figure 4.3.

Solution

- a. The derivative of $f(x) = \frac{9(x^2 - 3)}{x^3}$ is

$$\begin{aligned} f'(x) &= \frac{x^3(18x) - (9)(x^2 - 3)(3x^2)}{(x^3)^2} && \text{Differentiate using Quotient Rule.} \\ &= \frac{9(9 - x^2)}{x^4}. && \text{Simplify.} \end{aligned}$$

At the point $(3, 2)$, the value of the derivative is $f'(3) = 0$ [see Figure 4.3(a)].

- b. At $x = 0$, the derivative of $f(x) = |x|$ does not exist because the following one-sided limits differ [see Figure 4.3(b)].

$$\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1 \quad \text{Limit from the left}$$

$$\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1 \quad \text{Limit from the right}$$

- c. The derivative of $f(x) = \sin x$ is

$$f'(x) = \cos x.$$

At the point $(\pi/2, 1)$, the value of the derivative is $f'(\pi/2) = \cos(\pi/2) = 0$. At the point $(3\pi/2, -1)$, the value of the derivative is $f'(3\pi/2) = \cos(3\pi/2) = 0$ [see Figure 4.3(c)]. ■

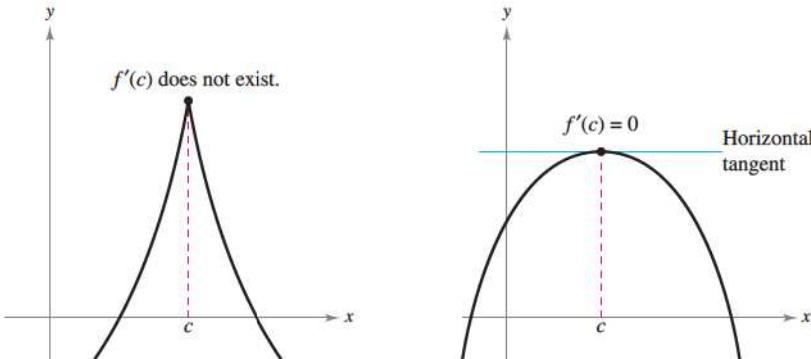
Note in Example 1 that at each relative extremum, the derivative either is zero or does not exist. The x -values at these special points are called **critical numbers**. Figure 4.4 illustrates the two types of critical numbers. Notice in the definition that the critical number c has to be in the domain of f , but c does not have to be in the domain of f' .

TECHNOLOGY Use a graphing utility to examine the graphs of the following four functions. Only one of the functions has critical numbers. Which is it?

$$\begin{aligned}f(x) &= e^x \\f(x) &= \ln x \\f(x) &= \sin x \\f(x) &= \tan x\end{aligned}$$

DEFINITION OF CRITICAL NUMBER

Let f be defined at c . If $f'(c) = 0$ or if f is not differentiable at c , then c is a **critical number** of f .



c is a critical number of f .

Figure 4.4

THEOREM 4.2 RELATIVE EXTREMA OCCUR ONLY AT CRITICAL NUMBERS

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f .



PIERRE DE FERMAT (1601–1665)

For Fermat, who was trained as a lawyer, mathematics was more of a hobby than a profession. Nevertheless, Fermat made many contributions to analytic geometry, number theory, calculus, and probability. In letters to friends, he wrote of many of the fundamental ideas of calculus, long before Newton or Leibniz. For instance, Theorem 4.2 is sometimes attributed to Fermat.

PROOF

Case 1: If f is not differentiable at $x = c$, then, by definition, c is a critical number of f and the theorem is valid.

Case 2: If f is differentiable at $x = c$, then $f'(c)$ must be positive, negative, or 0. Suppose $f'(c)$ is positive. Then

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} > 0$$

which implies that there exists an interval (a, b) containing c such that

$$\frac{f(x) - f(c)}{x - c} > 0, \text{ for all } x \neq c \text{ in } (a, b). \quad [\text{See Exercise 84(b), Section 2.2.}]$$

Because this quotient is positive, the signs of the denominator and numerator must agree. This produces the following inequalities for x -values in the interval (a, b) .

Left of c : $x < c$ and $f(x) < f(c) \Rightarrow f(c)$ is not a relative minimum.

Right of c : $x > c$ and $f(x) > f(c) \Rightarrow f(c)$ is not a relative maximum.

So, the assumption that $f'(c) > 0$ contradicts the hypothesis that $f(c)$ is a relative extremum. Assuming that $f'(c) < 0$ produces a similar contradiction, you are left with only one possibility—namely, $f'(c) = 0$. So, by definition, c is a critical number of f and the theorem is valid. ■

Finding Extrema on a Closed Interval

Theorem 4.2 states that the relative extrema of a function can occur *only* at the critical numbers of the function. Knowing this, you can use the following guidelines to find extrema on a closed interval.

GUIDELINES FOR FINDING EXTREMA ON A CLOSED INTERVAL

To find the extrema of a continuous function f on a closed interval $[a, b]$, use the following steps.

1. Find the critical numbers of f in (a, b) .
2. Evaluate f at each critical number in (a, b) .
3. Evaluate f at each endpoint of $[a, b]$.
4. The least of these values is the minimum. The greatest is the maximum.

The next three examples show how to apply these guidelines. Be sure you see that finding the critical numbers of the function is only part of the procedure. Evaluating the function at the critical numbers *and* the endpoints is the other part.

EXAMPLE 2 Finding Extrema on a Closed Interval

Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$.

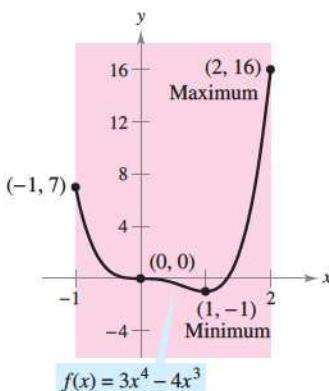
Solution Begin by differentiating the function.

$$\begin{aligned} f(x) &= 3x^4 - 4x^3 && \text{Write original function.} \\ f'(x) &= 12x^3 - 12x^2 && \text{Differentiate.} \end{aligned}$$

To find the critical numbers of f in the interval $(-1, 2)$, you must find all x -values for which $f'(x) = 0$ and all x -values for which $f'(x)$ does not exist.

$$\begin{aligned} f'(x) &= 12x^3 - 12x^2 = 0 && \text{Set } f'(x) \text{ equal to 0.} \\ 12x^2(x - 1) &= 0 && \text{Factor.} \\ x = 0, 1 & && \text{Critical numbers} \end{aligned}$$

Because f' is defined for all x , you can conclude that these are the only critical numbers of f . By evaluating f at these two critical numbers and at the endpoints of $[-1, 2]$, you can determine that the maximum is $f(2) = 16$ and the minimum is $f(1) = -1$, as shown in the table. The graph of f is shown in Figure 4.5.

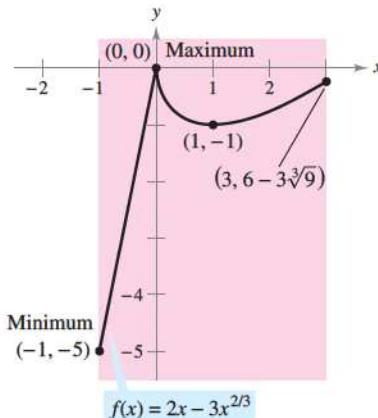


On the closed interval $[-1, 2]$, f has a minimum at $(1, -1)$ and a maximum at $(2, 16)$.

Figure 4.5

Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = 7$	$f(0) = 0$	$f(1) = -1$ Minimum	$f(2) = 16$ Maximum

In Figure 4.5, note that the critical number $x = 0$ does not yield a relative minimum or a relative maximum. This tells you that the converse of Theorem 4.2 is not true. In other words, *the critical numbers of a function need not produce relative extrema*.



On the closed interval $[-1, 3]$, f has a minimum at $(-1, -5)$ and a maximum at $(0, 0)$.

Figure 4.6

EXAMPLE 3 Finding Extrema on a Closed Interval

Find the extrema of $f(x) = 2x - 3x^{2/3}$ on the interval $[-1, 3]$.

Solution Begin by differentiating the function.

$$f(x) = 2x - 3x^{2/3}$$

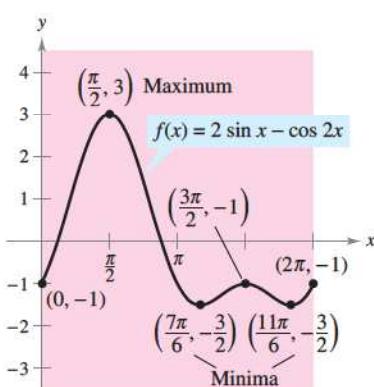
Write original function.

$$f'(x) = 2 - \frac{2}{x^{1/3}} = 2\left(\frac{x^{1/3} - 1}{x^{1/3}}\right)$$

Differentiate.

From this derivative, you can see that the function has two critical numbers in the interval $[-1, 3]$. The number 1 is a critical number because $f'(1) = 0$, and the number 0 is a critical number because $f'(0)$ does not exist. By evaluating f at these two numbers and at the endpoints of the interval, you can conclude that the minimum is $f(-1) = -5$ and the maximum is $f(0) = 0$, as shown in the table. The graph of f is shown in Figure 4.6.

Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = -5$ Minimum	$f(0) = 0$ Maximum	$f(1) = -1$	$f(3) = 6 - 3\sqrt[3]{9} \approx -0.24$



On the closed interval $[0, 2\pi]$, f has minima at $(7\pi/6, -3/2)$ and $(11\pi/6, -3/2)$ and a maximum at $(\pi/2, 3)$.

Figure 4.7

EXAMPLE 4 Finding Extrema on a Closed Interval

Find the extrema of $f(x) = 2 \sin x - \cos 2x$ on the interval $[0, 2\pi]$.

Solution This function is differentiable for all real x , so you can find all critical numbers by differentiating the function and setting $f'(x)$ equal to zero, as shown.

$$f(x) = 2 \sin x - \cos 2x$$

Write original function.

$$f'(x) = 2 \cos x + 2 \sin 2x = 0$$

Set $f'(x)$ equal to 0.

$$2 \cos x + 4 \cos x \sin x = 0$$

$\sin 2x = 2 \cos x \sin x$

$$2(\cos x)(1 + 2 \sin x) = 0$$

Factor.

In the interval $[0, 2\pi]$, the factor $\cos x$ is zero when $x = \pi/2$ and when $x = 3\pi/2$. The factor $(1 + 2 \sin x)$ is zero when $x = 7\pi/6$ and when $x = 11\pi/6$. By evaluating f at these four critical numbers and at the endpoints of the interval, you can conclude that the maximum is $f(\pi/2) = 3$ and the minimum occurs at two points, $f(7\pi/6) = -3/2$ and $f(11\pi/6) = -3/2$, as shown in the table. The graph of f is shown in Figure 4.7.

Left Endpoint	Critical Number	Critical Number	Critical Number	Critical Number	Right Endpoint
$f(0) = -1$	$f\left(\frac{\pi}{2}\right) = 3$ Maximum	$f\left(\frac{7\pi}{6}\right) = -\frac{3}{2}$ Minimum	$f\left(\frac{3\pi}{2}\right) = -1$	$f\left(\frac{11\pi}{6}\right) = -\frac{3}{2}$ Minimum	$f(2\pi) = -1$

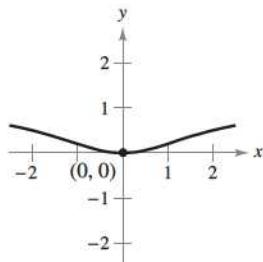
The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple and Mathematica.

4.1 Exercises

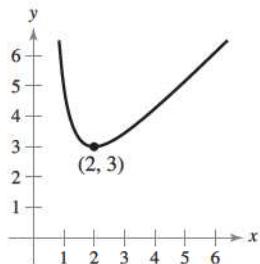
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the value of the derivative (if it exists) at each given extremum.

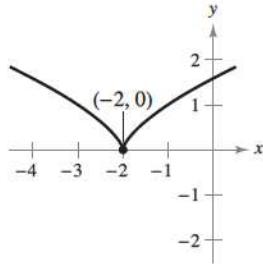
1. $f(x) = \frac{x^2}{x^2 + 4}$



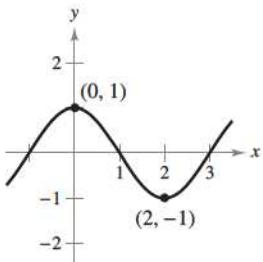
3. $g(x) = x + \frac{4}{x^2}$



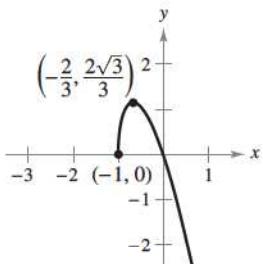
5. $f(x) = (x + 2)^{2/3}$



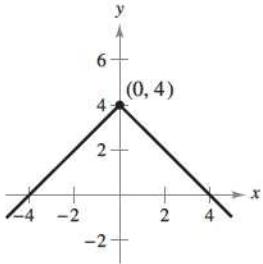
2. $f(x) = \cos \frac{\pi x}{2}$



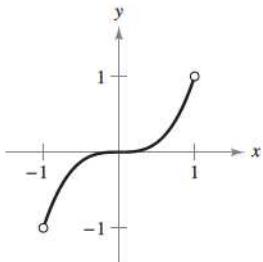
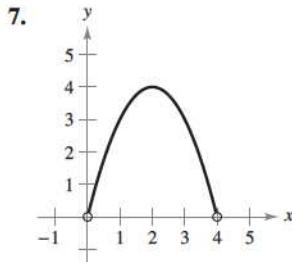
4. $f(x) = -3x\sqrt{x+1}$



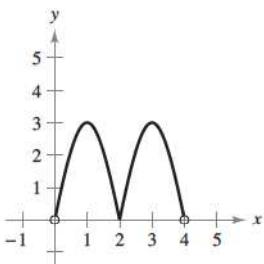
6. $f(x) = 4 - |x|$



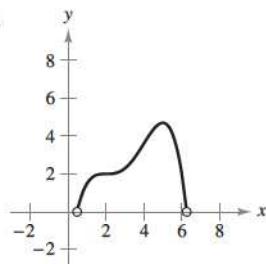
In Exercises 7–10, approximate the critical numbers of the function shown in the graph. Determine whether the function has a relative maximum, a relative minimum, an absolute maximum, an absolute minimum, or none of these at each critical number on the interval shown.



9.



10.



In Exercises 11–20, find any critical numbers of the function.

11. $f(x) = x^3 - 3x^2$

12. $g(x) = x^4 - 4x^2$

13. $g(t) = t\sqrt{4-t}$, $t < 3$

14. $f(x) = \frac{4x}{x^2 + 1}$

15. $h(x) = \sin^2 x + \cos x$

16. $f(\theta) = 2 \sec \theta + \tan \theta$

$0 < x < 2\pi$

$0 < \theta < 2\pi$

17. $f(t) = te^{-2t}$

18. $g(t) = 2t \ln t$

19. $f(x) = x^2 \log_2(x^2 + 1)$

20. $g(x) = 4x^2(3^x)$

In Exercises 21–42, locate the absolute extrema of the function on the closed interval.

21. $f(x) = 3 - x$, $[-1, 2]$

22. $f(x) = \frac{2x + 5}{3}$, $[0, 5]$

23. $g(x) = x^2 - 2x$, $[0, 4]$

24. $h(x) = -x^2 + 3x - 5$, $[-2, 1]$

25. $f(x) = x^3 - \frac{3}{2}x^2$, $[-1, 2]$

26. $f(x) = x^3 - 12x$, $[0, 4]$

27. $y = 3x^{2/3} - 2x$, $[-1, 1]$

28. $g(x) = \sqrt[3]{x}$, $[-1, 1]$

29. $h(s) = \frac{1}{s-2}$, $[0, 1]$

30. $h(t) = \frac{t}{t-2}$, $[3, 5]$

31. $g(t) = \frac{t^2}{t^2 + 3}$, $[-1, 1]$

32. $y = 3 - |t - 3|$, $[-1, 5]$

33. $f(x) = \cos \pi x$, $\left[0, \frac{1}{6}\right]$

34. $g(x) = \sec x$, $\left[-\frac{\pi}{6}, \frac{\pi}{3}\right]$

35. $y = 3 \cos x$, $[0, 2\pi]$

36. $y = \tan\left(\frac{\pi x}{8}\right)$, $[0, 2]$

37. $f(x) = \arctan x^2$, $[-2, 1]$

38. $g(x) = \frac{\ln x}{x}$, $[1, 4]$

39. $h(x) = 5e^x - e^{2x}$, $[-1, 2]$

40. $y = x^2 - 8 \ln x$, $[1, 5]$

41. $y = e^x \sin x$, $[0, \pi]$

42. $y = x \ln(x + 3)$, $[0, 3]$

In Exercises 43 and 44, locate the absolute extrema of the function (if any exist) over each interval.

43. $f(x) = 2x - 3$

44. $f(x) = \sqrt{9 - x^2}$

(a) $[0, 2]$ (b) $[0, 2]$

(a) $[-3, 3]$ (b) $[-3, 0]$

(c) $(0, 2]$ (d) $(0, 2)$

(c) $(-3, 3)$ (d) $[1, 3]$

In Exercises 45–50, use a graphing utility to graph the function. Locate the absolute extrema of the function on the given interval.

Function	Interval
45. $f(x) = \begin{cases} 2x + 2, & 0 \leq x \leq 1 \\ 4x^2, & 1 < x \leq 3 \end{cases}$	[0, 3]
46. $f(x) = \begin{cases} 2 - x^2, & 1 \leq x < 3 \\ 2 - 3x, & 3 \leq x \leq 5 \end{cases}$	[1, 5]
47. $f(x) = \frac{3}{x - 1}$	(1, 4]
48. $f(x) = \frac{2}{2 - x}$	[0, 2)
49. $f(x) = x^4 - 2x^3 + x + 1$	[-1, 3]
50. $f(x) = \sqrt{x} + \cos \frac{x}{2}$	[0, 2π]

CAS In Exercises 51–56, (a) use a computer algebra system to graph the function and approximate any absolute extrema on the given interval. (b) Use the utility to find any critical numbers, and use them to find any absolute extrema not located at the endpoints. Compare the results with those in part (a).

Function	Interval
51. $f(x) = 3.2x^5 + 5x^3 - 3.5x$	[0, 1]
52. $f(x) = \frac{4}{3}x\sqrt{3-x}$	[0, 3]
53. $f(x) = (x^2 - 2x)\ln(x+3)$	[0, 3]
54. $f(x) = \sqrt{x+4}e^{x^2/10}$	[-2, 2]
55. $f(x) = 2x \arctan(x-1)$	[0, 2]
56. $f(x) = (x-4) \arcsin \frac{x}{4}$	[-2, 4]

CAS In Exercises 57–60, use a computer algebra system to find the maximum value of $|f''(x)|$ on the closed interval. (This value is used in the error estimate for the Trapezoidal Rule, as discussed in Section 5.6.)

57. $f(x) = \sqrt{1+x^3}$, [0, 2]
 58. $f(x) = \frac{1}{x^2+1}$, $\left[\frac{1}{2}, 3\right]$
 59. $f(x) = e^{-x^2/2}$, [0, 1]
 60. $f(x) = x \ln(x+1)$, [0, 2]

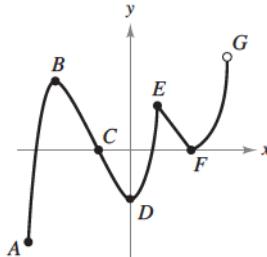
CAS In Exercises 61 and 62, use a computer algebra system to find the maximum value of $|f^{(4)}(x)|$ on the closed interval. (This value is used in the error estimate for Simpson's Rule, as discussed in Section 5.6.)

61. $f(x) = (x+1)^{2/3}$, [0, 2] 62. $f(x) = \frac{1}{x^2+1}$, [-1, 1]

63. **Writing** Write a short paragraph explaining why a continuous function on an open interval may not have a maximum or minimum. Illustrate your explanation with a sketch of the graph of such a function.

CAPSTONE

64. Decide whether each labeled point is an absolute maximum or minimum, a relative maximum or minimum, or neither.

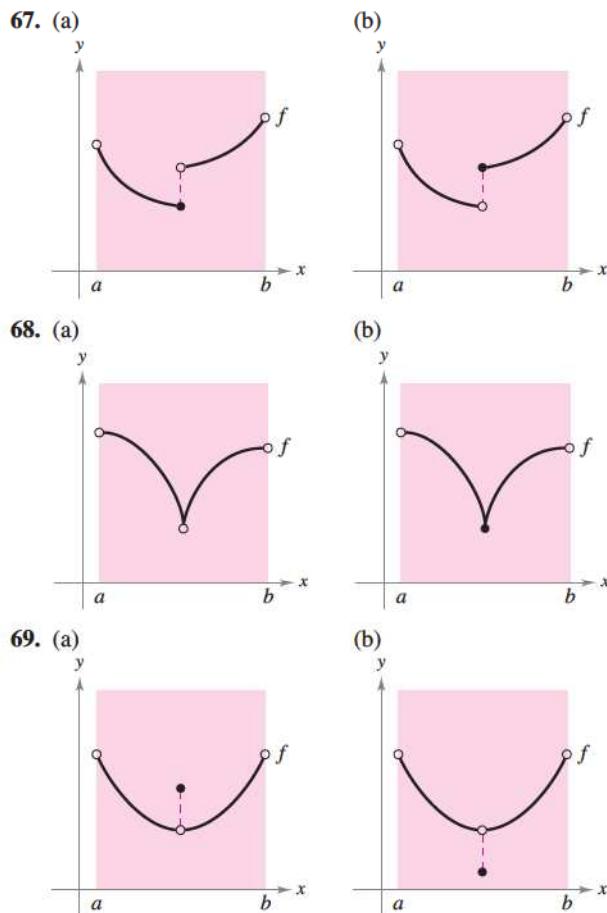


WRITING ABOUT CONCEPTS

In Exercises 65 and 66, graph a function on the interval $[-2, 5]$ having the given characteristics.

65. Absolute maximum at $x = -2$, absolute minimum at $x = 1$, relative maximum at $x = 3$
 66. Relative minimum at $x = -1$, critical number (but no extremum) at $x = 0$, absolute maximum at $x = 2$, absolute minimum at $x = 5$

In Exercises 67–69, determine from the graph whether f has a minimum in the open interval (a, b) .



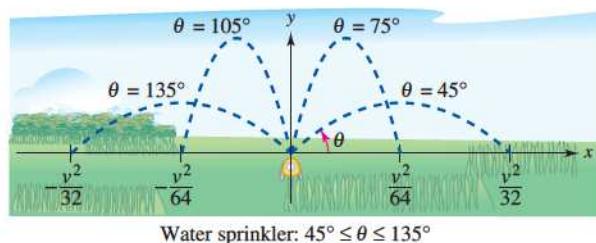
WRITING ABOUT CONCEPTS (continued)

70. Explain why the function $f(x) = \tan x$ has a maximum on $[0, \pi/4]$ but not on $[0, \pi]$.

71. **Lawn Sprinkler** A lawn sprinkler is constructed in such a way that $d\theta/dt$ is constant, where θ ranges between 45° and 135° (see figure). The distance the water travels horizontally is

$$x = \frac{v^2 \sin 2\theta}{32}, \quad 45^\circ \leq \theta \leq 135^\circ$$

where v is the speed of the water. Find dx/dt and explain why this lawn sprinkler does not water evenly. What part of the lawn receives the most water?

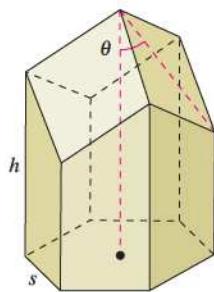


- FOR FURTHER INFORMATION** For more information on the "calculus of lawn sprinklers," see the article "Design of an Oscillating Sprinkler" by Bart Braden in *Mathematics Magazine*. To view this article, go to the website www.matharticles.com.

72. **Honeycomb** The surface area of a cell in a honeycomb is

$$S = 6hs + \frac{3s^2}{2} \left(\frac{\sqrt{3} - \cos \theta}{\sin \theta} \right)$$

where h and s are positive constants and θ is the angle at which the upper faces meet the altitude of the cell (see figure). Find the angle θ ($\pi/6 \leq \theta \leq \pi/2$) that minimizes the surface area S .



- FOR FURTHER INFORMATION** For more information on the geometric structure of a honeycomb cell, see the article "The Design of Honeycombs" by Anthony L. Peressini in UMAP Module 502, published by COMAP, Inc., Suite 210, 57 Bedford Street, Lexington, MA.

True or False? In Exercises 73–76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

73. The maximum of a function that is continuous on a closed interval can occur at two different values in the interval.

74. If a function is continuous on a closed interval, then it must have a minimum on the interval.

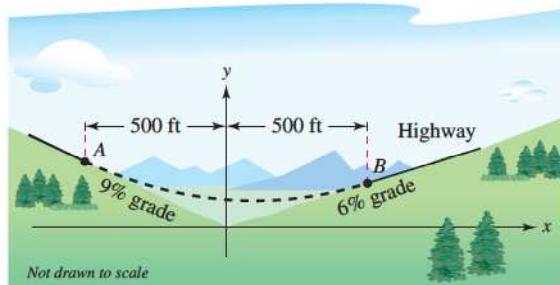
75. If $x = c$ is a critical number of the function f , then it is also a critical number of the function $g(x) = f(x) + k$, where k is a constant.

76. If $x = c$ is a critical number of the function f , then it is also a critical number of the function $g(x) = f(x - k)$, where k is a constant.

77. Let the function f be differentiable on an interval I containing c . If f has a maximum value at $x = c$, show that $-f$ has a minimum value at $x = c$.

78. Consider the cubic function $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. Show that f can have zero, one, or two critical numbers and give an example of each case.

79. **Highway Design** In order to build a highway, it is necessary to fill a section of a valley where the grades (slopes) of the sides are 9% and 6% (see figure). The top of the filled region will have the shape of a parabolic arc that is tangent to the two slopes at the points A and B . The horizontal distances from A to the y -axis and from B to the y -axis are both 500 feet.



- Find the coordinates of A and B .
- Find a quadratic function $y = ax^2 + bx + c$, $-500 \leq x \leq 500$, that describes the top of the filled region.
- Construct a table giving the depths d of the fill for $x = -500, -400, -300, -200, -100, 0, 100, 200, 300, 400$, and 500 .
- What will be the lowest point on the completed highway? Will it be directly over the point where the two hillsides come together?

PUTNAM EXAM CHALLENGE

80. Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region $R = \{(x, y); 0 \leq x \leq a, 0 \leq y \leq f(x)\}$ has perimeter k units and area k square units for some real number k .

This problem was composed by the Committee on the Putnam Prize Competition.
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4.2 Rolle's Theorem and the Mean Value Theorem

- Understand and use Rolle's Theorem.
- Understand and use the Mean Value Theorem.

Rolle's Theorem

ROLLE'S THEOREM

French mathematician Michel Rolle first published the theorem that bears his name in 1691. Before this time, however, Rolle was one of the most vocal critics of calculus, stating that it gave erroneous results and was based on unsound reasoning. Later in life, Rolle came to see the usefulness of calculus.

The Extreme Value Theorem (Section 4.1) states that a continuous function on a closed interval $[a, b]$ must have both a minimum and a maximum on the interval. Both of these values, however, can occur at the endpoints. **Rolle's Theorem**, named after the French mathematician Michel Rolle (1652–1719), gives conditions that guarantee the existence of an extreme value in the *interior* of a closed interval.

EXPLORATION

Extreme Values in a Closed Interval Sketch a rectangular coordinate plane on a piece of paper. Label the points $(1, 3)$ and $(5, 3)$. Using a pencil or pen, draw the graph of a differentiable function f that starts at $(1, 3)$ and ends at $(5, 3)$. Is there at least one point on the graph for which the derivative is zero? Would it be possible to draw the graph so that there *isn't* a point for which the derivative is zero? Explain your reasoning.

THEOREM 4.3 ROLLE'S THEOREM

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) . If

$$f(a) = f(b)$$

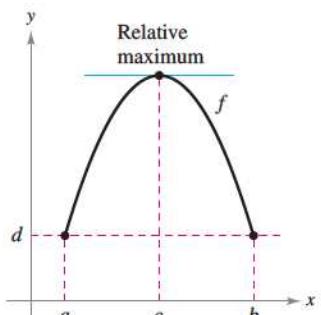
then there is at least one number c in (a, b) such that $f'(c) = 0$.

PROOF Let $f(a) = d = f(b)$.

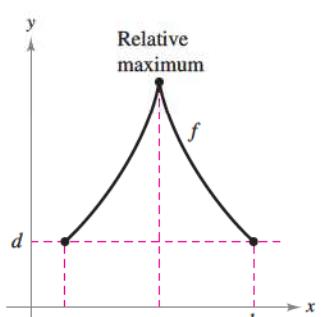
Case 1: If $f(x) = d$ for all x in $[a, b]$, then f is constant on the interval and, by Theorem 3.2, $f'(x) = 0$ for all x in (a, b) .

Case 2: Suppose $f(x) > d$ for some x in (a, b) . By the Extreme Value Theorem, you know that f has a maximum at some c in the interval. Moreover, because $f(c) > d$, this maximum does not occur at either endpoint. So, f has a maximum in the *open* interval (a, b) . This implies that $f(c)$ is a *relative* maximum and, by Theorem 4.2, c is a critical number of f . Finally, because f is differentiable at c , you can conclude that $f'(c) = 0$.

Case 3: If $f(x) < d$ for some x in (a, b) , you can use an argument similar to that in Case 2, but involving the minimum instead of the maximum. ■



(a) f is continuous on $[a, b]$ and differentiable on (a, b) .



(b) f is continuous on $[a, b]$.

Figure 4.8

From Rolle's Theorem, you can see that if a function f is continuous on $[a, b]$ and differentiable on (a, b) , and if $f(a) = f(b)$, then there must be at least one x -value between a and b at which the graph of f has a horizontal tangent, as shown in Figure 4.8(a). If the differentiability requirement is dropped from Rolle's Theorem, f will still have a critical number in (a, b) , but it may not yield a horizontal tangent. Such a case is shown in Figure 4.8(b).

EXAMPLE 1 Illustrating Rolle's Theorem

Find the two x -intercepts of

$$f(x) = x^2 - 3x + 2$$

and show that $f'(x) = 0$ at some point between the two x -intercepts.

Solution Note that f is differentiable on the entire real number line. Setting $f(x)$ equal to 0 produces

$$x^2 - 3x + 2 = 0$$

Set $f(x)$ equal to 0.

$$(x - 1)(x - 2) = 0$$

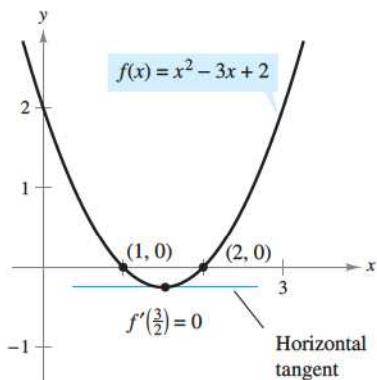
Factor.

So, $f(1) = f(2) = 0$, and from Rolle's Theorem you know that there exists at least one c in the interval $(1, 2)$ such that $f'(c) = 0$. To find such a c , you can solve the equation

$$f'(x) = 2x - 3 = 0$$

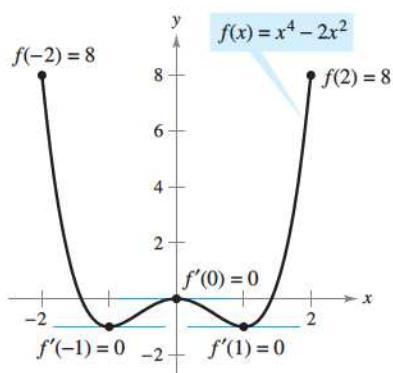
Set $f'(x)$ equal to 0.

and determine that $f'(x) = 0$ when $x = \frac{3}{2}$. Note that this x -value lies in the open interval $(1, 2)$, as shown in Figure 4.9. ■



The x -value for which $f'(x) = 0$ is between the two x -intercepts.

Figure 4.9



$f'(x) = 0$ for more than one x -value in the interval $(-2, 2)$.

Figure 4.10

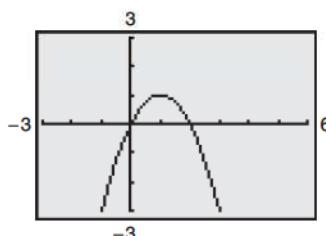


Figure 4.11

EXAMPLE 2 Illustrating Rolle's Theorem

Let $f(x) = x^4 - 2x^2$. Find all values of c in the interval $(-2, 2)$ such that $f'(c) = 0$.

Solution To begin, note that the function satisfies the conditions of Rolle's Theorem. That is, f is continuous on the interval $[-2, 2]$ and differentiable on the interval $(-2, 2)$. Moreover, because $f(-2) = f(2) = 8$, you can conclude that there exists at least one c in $(-2, 2)$ such that $f'(c) = 0$. Setting the derivative equal to 0 produces

$$f'(x) = 4x^3 - 4x = 0$$

Set $f'(x)$ equal to 0.

$$4x(x - 1)(x + 1) = 0$$

Factor.

$$x = 0, 1, -1.$$

x -values for which $f'(x) = 0$

So, in the interval $(-2, 2)$, the derivative is zero at three different values of x , as shown in Figure 4.10. ■

TECHNOLOGY PITFALL A graphing utility can be used to indicate whether the points on the graphs in Examples 1 and 2 are relative minima or relative maxima of the functions. When using a graphing utility, however, you should keep in mind that it can give misleading pictures of graphs. For example, use a graphing utility to graph

$$f(x) = 1 - (x - 1)^2 - \frac{1}{1000(x - 1)^{1/7} + 1}.$$

With most viewing windows, it appears that the function has a maximum of 1 when $x = 1$ (see Figure 4.11). By evaluating the function at $x = 1$, however, you can see that $f(1) = 0$. To determine the behavior of this function near $x = 1$, you need to examine the graph analytically to get the complete picture.

The Mean Value Theorem

Rolle's Theorem can be used to prove another theorem—the **Mean Value Theorem**.

THEOREM 4.4 THE MEAN VALUE THEOREM

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

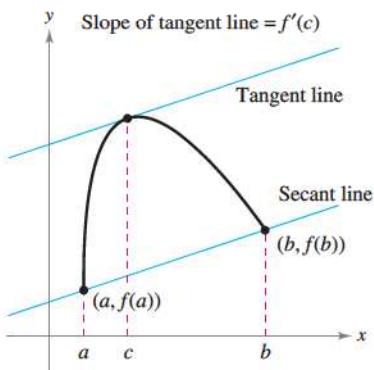


Figure 4.12

PROOF Refer to Figure 4.12. The equation of the secant line that passes through the points $(a, f(a))$ and $(b, f(b))$ is

$$y = \left[\frac{f(b) - f(a)}{b - a} \right] (x - a) + f(a).$$

Let $g(x)$ be the difference between $f(x)$ and y . Then

$$\begin{aligned} g(x) &= f(x) - y \\ &= f(x) - \left[\frac{f(b) - f(a)}{b - a} \right] (x - a) - f(a). \end{aligned}$$

By evaluating g at a and b , you can see that $g(a) = 0 = g(b)$. Because f is continuous on $[a, b]$, it follows that g is also continuous on $[a, b]$. Furthermore, because f is differentiable, g is also differentiable, and you can apply Rolle's Theorem to the function g . So, there exists a number c in (a, b) such that $g'(c) = 0$, which implies that

$$\begin{aligned} 0 &= g'(c) \\ &= f'(c) - \frac{f(b) - f(a)}{b - a}. \end{aligned}$$

So, there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

NOTE The “mean” in the Mean Value Theorem refers to the mean (or average) rate of change of f in the interval $[a, b]$. ■



JOSEPH-LOUIS LAGRANGE (1736–1813)

The Mean Value Theorem was first proved by the famous mathematician Joseph-Louis Lagrange. Born in Italy, Lagrange held a position in the court of Frederick the Great in Berlin for 20 years. Afterward, he moved to France, where he met emperor Napoleon Bonaparte, who is quoted as saying, “Lagrange is the lofty pyramid of the mathematical sciences.”

Although the Mean Value Theorem can be used directly in problem solving, it is used more often to prove other theorems. In fact, some people consider this to be the most important theorem in calculus—it is closely related to the Fundamental Theorem of Calculus discussed in Section 5.4. For now, you can get an idea of the versatility of the Mean Value Theorem by looking at the results stated in Exercises 89–97 in this section.

The Mean Value Theorem has implications for both basic interpretations of the derivative. Geometrically, the theorem guarantees the existence of a tangent line that is parallel to the secant line through the points $(a, f(a))$ and $(b, f(b))$, as shown in Figure 4.12. Example 3 illustrates this geometric interpretation of the Mean Value Theorem. In terms of rates of change, the Mean Value Theorem implies that there must be a point in the open interval (a, b) at which the instantaneous rate of change is equal to the average rate of change over the interval $[a, b]$. This is illustrated in Example 4.

EXAMPLE 3 Finding a Tangent Line

Given $f(x) = 5 - (4/x)$, find all values of c in the open interval $(1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

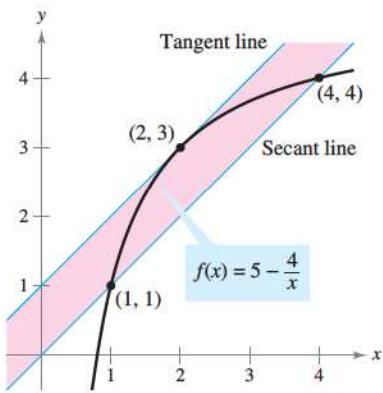
Solution The slope of the secant line through $(1, f(1))$ and $(4, f(4))$ is

$$\frac{f(4) - f(1)}{4 - 1} = \frac{4 - 1}{4 - 1} = 1.$$

Note that the function satisfies the conditions of the Mean Value Theorem. That is, f is continuous on the interval $[1, 4]$ and differentiable on the interval $(1, 4)$. So, there exists at least one number c in $(1, 4)$ such that $f'(c) = 1$. Solving the equation $f'(x) = 1$ yields

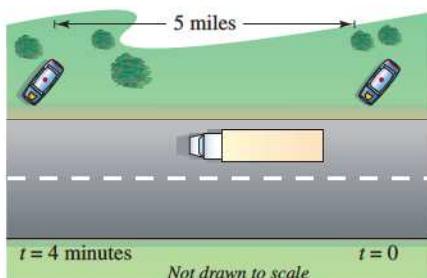
$$f'(x) = \frac{4}{x^2} = 1$$

which implies that $x = \pm 2$. So, in the interval $(1, 4)$, you can conclude that $c = 2$, as shown in Figure 4.13.



The tangent line at $(2, 3)$ is parallel to the secant line through $(1, 1)$ and $(4, 4)$.

Figure 4.13



At some time t , the instantaneous velocity is equal to the average velocity over 4 minutes.

Figure 4.14

EXAMPLE 4 Finding an Instantaneous Rate of Change

Two stationary patrol cars equipped with radar are 5 miles apart on a highway, as shown in Figure 4.14. As a truck passes the first patrol car, its speed is clocked at 55 miles per hour. Four minutes later, when the truck passes the second patrol car, its speed is clocked at 50 miles per hour. Prove that the truck must have exceeded the speed limit (of 55 miles per hour) at some time during the 4 minutes.

Solution Let $t = 0$ be the time (in hours) when the truck passes the first patrol car. The time when the truck passes the second patrol car is

$$t = \frac{4}{60} = \frac{1}{15} \text{ hour.}$$

By letting $s(t)$ represent the distance (in miles) traveled by the truck, you have $s(0) = 0$ and $s\left(\frac{1}{15}\right) = 5$. So, the average velocity of the truck over the five-mile stretch of highway is

$$\text{Average velocity} = \frac{s\left(\frac{1}{15}\right) - s(0)}{\left(\frac{1}{15}\right) - 0} = \frac{5}{1/15} = 75 \text{ miles per hour.}$$

Assuming that the position function is differentiable, you can apply the Mean Value Theorem to conclude that the truck must have been traveling at a rate of 75 miles per hour sometime during the 4 minutes. ■

A useful alternative form of the Mean Value Theorem is as follows: If f is continuous on $[a, b]$ and differentiable on (a, b) , then there exists a number c in (a, b) such that

$$f(b) = f(a) + (b - a)f'(c).$$

Alternative form of Mean Value Theorem

NOTE When doing the exercises for this section, keep in mind that polynomial functions, rational functions, and transcendental functions are differentiable at all points in their domains.

4.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, explain why Rolle's Theorem does not apply to the function even though there exist a and b such that $f(a) = f(b)$.

1. $f(x) = \begin{cases} 1 \\ |x| \end{cases}$, $[-1, 1]$

2. $f(x) = \cot \frac{x}{2}$, $[\pi, 3\pi]$

3. $f(x) = 1 - |x - 1|$, $[0, 2]$

4. $f(x) = \sqrt{(2 - x^{2/3})^3}$, $[-1, 1]$

In Exercises 5–8, find the two x -intercepts of the function f and show that $f'(x) = 0$ at some point between the two x -intercepts.

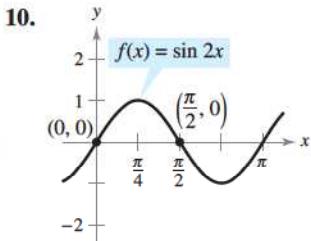
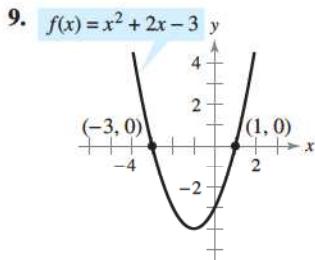
5. $f(x) = x^2 - x - 2$

6. $f(x) = x(x - 3)$

7. $f(x) = x\sqrt{x + 4}$

8. $f(x) = -3x\sqrt{x + 1}$

Rolle's Theorem In Exercises 9 and 10, the graph of f is shown. Apply Rolle's Theorem and find all values of c such that $f'(c) = 0$ at some point between the labeled intercepts.



In Exercises 11–26, determine whether Rolle's Theorem can be applied to f on the closed interval $[a, b]$. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$. If Rolle's Theorem cannot be applied, explain why not.

11. $f(x) = -x^2 + 3x$, $[0, 3]$

12. $f(x) = x^2 - 5x + 4$, $[1, 4]$

13. $f(x) = (x - 1)(x - 2)(x - 3)$, $[1, 3]$

14. $f(x) = (x - 3)(x + 1)^2$, $[-1, 3]$

15. $f(x) = x^{2/3} - 1$, $[-8, 8]$

16. $f(x) = 3 - |x - 3|$, $[0, 6]$

17. $f(x) = \frac{x^2 - 2x - 3}{x + 2}$, $[-1, 3]$

18. $f(x) = \frac{x^2 - 1}{x}$, $[-1, 1]$

19. $f(x) = (x^2 - 2x)e^x$, $[0, 2]$

20. $f(x) = x - 2 \ln x$, $[1, 3]$

21. $f(x) = \sin x$, $[0, 2\pi]$

22. $f(x) = \cos x$, $[0, 2\pi]$

23. $f(x) = \frac{6x}{\pi} - 4 \sin^2 x$, $\left[0, \frac{\pi}{6}\right]$

24. $f(x) = \cos 2x$, $[-\pi, \pi]$

25. $f(x) = \tan x$, $[0, \pi]$

26. $f(x) = \sec x$, $[\pi, 2\pi]$



In Exercises 27–32, use a graphing utility to graph the function on the closed interval $[a, b]$. Determine whether Rolle's Theorem can be applied to f on the interval and, if so, find all values of c in the open interval (a, b) such that $f'(c) = 0$.

27. $f(x) = |x| - 1$, $[-1, 1]$

28. $f(x) = x - x^{1/3}$, $[0, 1]$

29. $f(x) = x - \tan \pi x$, $[-\frac{1}{4}, \frac{1}{4}]$

30. $f(x) = \frac{x}{2} - \sin \frac{\pi x}{6}$, $[-1, 0]$

31. $f(x) = 2 + \arcsin(x^2 - 1)$, $[-1, 1]$

32. $f(x) = 2 + (x^2 - 4x)(2^{-x/4})$, $[0, 4]$

33. **Vertical Motion** The height of a ball t seconds after it is thrown upward from a height of 6 feet and with an initial velocity of 48 feet per second is $f(t) = -16t^2 + 48t + 6$.

(a) Verify that $f(1) = f(2)$.

(b) According to Rolle's Theorem, what must the velocity be at some time in the interval $(1, 2)$? Find that time.

34. **Reorder Costs** The ordering and transportation cost C of components used in a manufacturing process is approximated by

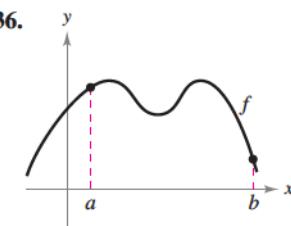
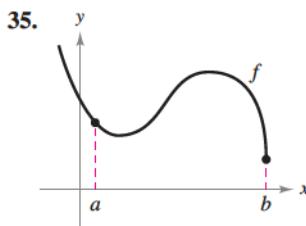
$$C(x) = 10\left(\frac{1}{x} + \frac{x}{x+3}\right)$$

where C is measured in thousands of dollars and x is the order size in hundreds.

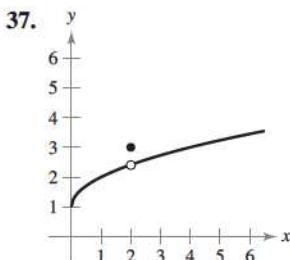
(a) Verify that $C(3) = C(6)$.

(b) According to Rolle's Theorem, the rate of change of cost must be 0 for some order size in the interval $(3, 6)$. Find that order size.

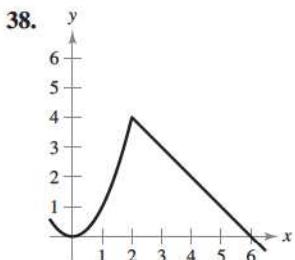
In Exercises 35 and 36, copy the graph and sketch the secant line to the graph through the points $(a, f(a))$ and $(b, f(b))$. Then sketch any tangent lines to the graph for each value of c guaranteed by the Mean Value Theorem. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.



Writing In Exercises 37–40, explain why the Mean Value Theorem does not apply to the function f on the interval $[0, 6]$.



37. $f(x) = \frac{1}{x - 3}$



38. $f(x) = |x - 3|$

A In Exercises 57–62, use a graphing utility to (a) graph the function f on the given interval, (b) find and graph the secant line through points on the graph of f at the endpoints of the given interval, and (c) find and graph any tangent lines to the graph of f that are parallel to the secant line.

57. $f(x) = \frac{x}{x + 1}, [-\frac{1}{2}, 2]$

58. $f(x) = x - 2 \sin x, [-\pi, \pi]$

59. $f(x) = \sqrt{x}, [1, 9]$

60. $f(x) = x^4 - 2x^3 + x^2, [0, 6]$

61. $f(x) = 2e^{x/4} \cos \frac{\pi x}{4}, [0, 2]$

62. $f(x) = \ln|\sec \pi x|, [0, \frac{1}{4}]$

41. **Mean Value Theorem** Consider the graph of the function $f(x) = -x^2 + 5$. (a) Find the equation of the secant line joining the points $(-1, 4)$ and $(2, 1)$. (b) Use the Mean Value Theorem to determine a point c in the interval $(-1, 2)$ such that the tangent line at c is parallel to the secant line. (c) Find the equation of the tangent line through c . (d) Then use a graphing utility to graph f , the secant line, and the tangent line.

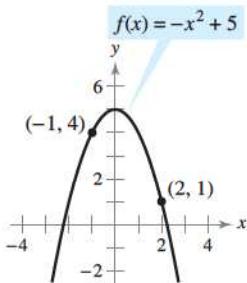


Figure for 41

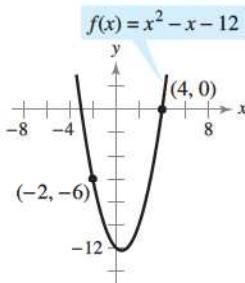


Figure for 42

42. **Mean Value Theorem** Consider the graph of the function $f(x) = x^2 - x - 12$. (a) Find the equation of the secant line joining the points $(-2, -6)$ and $(4, 0)$. (b) Use the Mean Value Theorem to determine a point c in the interval $(-2, 4)$ such that the tangent line at c is parallel to the secant line. (c) Find the equation of the tangent line through c . (d) Then use a graphing utility to graph f , the secant line, and the tangent line.

In Exercises 43–56, determine whether the Mean Value Theorem can be applied to f on the closed interval $[a, b]$. If the Mean Value Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$. If the Mean Value Theorem cannot be applied, explain why not.

43. $f(x) = x^2, [-2, 1]$

44. $f(x) = x^3, [0, 1]$

45. $f(x) = x^3 + 2x, [-1, 1]$

46. $f(x) = x^4 - 8x, [0, 2]$

47. $f(x) = x^{2/3}, [0, 1]$

48. $f(x) = \frac{x + 1}{x}, [-1, 2]$

49. $f(x) = |2x + 1|, [-1, 3]$

50. $f(x) = \sqrt{2 - x}, [-7, 2]$

51. $f(x) = \sin x, [0, \pi]$

52. $f(x) = \cos x + \tan x, [0, \pi]$

53. $f(x) = e^{-3x}, [0, 2]$

54. $f(x) = (x + 3) \ln(x + 3), [-2, -1]$

55. $f(x) = x \log_2 x, [1, 2]$

56. $f(x) = \arctan(1 - x), [0, 1]$

WRITING ABOUT CONCEPTS

63. Let f be continuous on $[a, b]$ and differentiable on (a, b) . If there exists c in (a, b) such that $f'(c) = 0$, does it follow that $f(a) = f(b)$? Explain.

64. Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) . Also, suppose that $f(a) = f(b)$ and that c is a real number in the interval such that $f'(c) = 0$. Find an interval for the function g over which Rolle's Theorem can be applied, and find the corresponding critical number of g (k is a constant).

(a) $g(x) = f(x) + k$

(b) $g(x) = f(x - k)$

(c) $g(x) = f(kx)$

65. The function

$$f(x) = \begin{cases} 0, & x = 0 \\ 1 - x, & 0 < x \leq 1 \end{cases}$$

is differentiable on $(0, 1)$ and satisfies $f(0) = f(1)$. However, its derivative is never zero on $(0, 1)$. Does this contradict Rolle's Theorem? Explain.

66. Can you find a function f such that $f(-2) = -2, f(2) = 6$, and $f'(x) < 1$ for all x ? Why or why not?

67. **Speed** A plane begins its takeoff at 2:00 P.M. on a 2500-mile flight. After 5.5 hours, the plane arrives at its destination. Explain why there are at least two times during the flight when the speed of the plane is 400 miles per hour.

68. **Temperature** When an object is removed from a furnace and placed in an environment with a constant temperature of 90°F , its core temperature is 1500°F . Five hours later the core temperature is 390°F . Explain why there must exist a time in the interval when the temperature is decreasing at a rate of 222°F per hour.

69. **Velocity** Two bicyclists begin a race at 8:00 A.M. They both finish the race 2 hours and 15 minutes later. Prove that at some time during the race, the bicyclists are traveling at the same velocity.

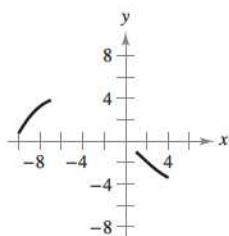
70. **Acceleration** At 9:13 A.M., a sports car is traveling 35 miles per hour. Two minutes later, the car is traveling 85 miles per hour. Prove that at some time during this two-minute interval, the car's acceleration is exactly 1500 miles per hour squared.

71. Consider the function $f(x) = 3 \cos^2\left(\frac{\pi x}{2}\right)$.

- (a) Use a graphing utility to graph f and f' .
 (b) Is f a continuous function? Is f' a continuous function?
 (c) Does Rolle's Theorem apply on the interval $[-1, 1]$? Does it apply on the interval $[1, 2]$? Explain.
 (d) Evaluate, if possible, $\lim_{x \rightarrow 3^-} f'(x)$ and $\lim_{x \rightarrow 3^+} f'(x)$.

CAPSTONE

72. **Graphical Reasoning** The figure shows two parts of the graph of a continuous differentiable function f on $[-10, 4]$. The derivative f' is also continuous. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.



- (a) Explain why f must have at least one zero in $[-10, 4]$.
 (b) Explain why f' must also have at least one zero in the interval $[-10, 4]$. What are these zeros called?
 (c) Make a possible sketch of the function with one zero of f' on the interval $[-10, 4]$.
 (d) Make a possible sketch of the function with two zeros of f' on the interval $[-10, 4]$.
 (e) Were the conditions of continuity of f and f' necessary to do parts (a) through (d)? Explain.

Think About It In Exercises 73 and 74, sketch the graph of an arbitrary function f that satisfies the given condition but does not satisfy the conditions of the Mean Value Theorem on the interval $[-5, 5]$.

73. f is continuous on $[-5, 5]$.

74. f is not continuous on $[-5, 5]$.

In Exercises 75–78, use the Intermediate Value Theorem and Rolle's Theorem to prove that the equation has exactly one real solution.

75. $x^5 + x^3 + x + 1 = 0$

76. $2x^5 + 7x - 1 = 0$

77. $3x + 1 - \sin x = 0$

78. $2x - 2 - \cos x = 0$

79. Determine the values of a , b , and c such that the function f satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 3]$.

$$f(x) = \begin{cases} 1, & x = 0 \\ ax + b, & 0 < x \leq 1 \\ x^2 + 4x + c, & 1 < x \leq 3 \end{cases}$$

80. Determine the values a , b , c , and d such that the function f satisfies the hypotheses of the Mean Value Theorem on the interval $[-1, 2]$.

$$f(x) = \begin{cases} a, & x = -1 \\ 2, & -1 < x \leq 0 \\ bx^2 + c, & 0 < x \leq 1 \\ dx + 4, & 1 < x \leq 2 \end{cases}$$

Differential Equations In Exercises 81–84, find a function f that has the derivative $f'(x)$ and whose graph passes through the given point. Explain your reasoning.

81. $f'(x) = 0$, $(2, 5)$ 82. $f'(x) = 4$, $(0, 1)$
 83. $f'(x) = 2x$, $(1, 0)$ 84. $f'(x) = 2x + 3$, $(1, 0)$

True or False? In Exercises 85–88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

85. The Mean Value Theorem can be applied to $f(x) = 1/x$ on the interval $[-1, 1]$.
 86. If the graph of a function has three x -intercepts, then it must have at least two points at which its tangent line is horizontal.
 87. If the graph of a polynomial function has three x -intercepts, then it must have at least two points at which its tangent line is horizontal.
 88. If $f'(x) = 0$ for all x in the domain of f , then f is a constant function.
 89. Prove that if $a > 0$ and n is any positive integer, then the polynomial function $p(x) = x^{2n+1} + ax + b$ cannot have two real roots.
 90. Prove that if $f'(x) = 0$ for all x in an interval (a, b) , then f is constant on (a, b) .
 91. Let $p(x) = Ax^2 + Bx + C$. Prove that for any interval $[a, b]$, the value c guaranteed by the Mean Value Theorem is the midpoint of the interval.
 92. (a) Let $f(x) = x^2$ and $g(x) = -x^3 + x^2 + 3x + 2$. Then $f(-1) = g(-1)$ and $f(2) = g(2)$. Show that there is at least one value c in the interval $(-1, 2)$ where the tangent line to f at $(c, f(c))$ is parallel to the tangent line to g at $(c, g(c))$. Identify c .
 (b) Let f and g be differentiable functions on $[a, b]$ where $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one value c in the interval (a, b) where the tangent line to f at $(c, f(c))$ is parallel to the tangent line to g at $(c, g(c))$.
 93. Prove that if f is differentiable on $(-\infty, \infty)$ and $f'(x) < 1$ for all real numbers, then f has at most one fixed point. A fixed point of a function f is a real number c such that $f(c) = c$.
 94. Use the result of Exercise 93 to show that $f(x) = \frac{1}{2} \cos x$ has at most one fixed point.
 95. Prove that $|\cos a - \cos b| \leq |a - b|$ for all a and b .
 96. Prove that $|\sin a - \sin b| \leq |a - b|$ for all a and b .
 97. Let $0 < a < b$. Use the Mean Value Theorem to show that

$$\sqrt{b} - \sqrt{a} < \frac{b - a}{2\sqrt{a}}.$$

4.3 Increasing and Decreasing Functions and the First Derivative Test

- Determine intervals on which a function is increasing or decreasing.
- Apply the First Derivative Test to find relative extrema of a function.

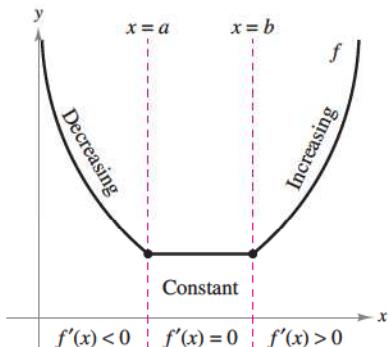
Increasing and Decreasing Functions

In this section you will learn how derivatives can be used to *classify* relative extrema as either relative minima or relative maxima. First, it is important to define increasing and decreasing functions.

DEFINITIONS OF INCREASING AND DECREASING FUNCTIONS

A function f is **increasing** on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function f is **decreasing** on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.



The derivative is related to the slope of a function.

Figure 4.15

A function is increasing if, *as x moves to the right*, its graph moves up, and is decreasing if its graph moves down. For example, the function in Figure 4.15 is decreasing on the interval $(-\infty, a)$, is constant on the interval (a, b) , and is increasing on the interval (b, ∞) . As shown in Theorem 4.5 below, a positive derivative implies that the function is increasing; a negative derivative implies that the function is decreasing; and a zero derivative on an entire interval implies that the function is constant on that interval.

THEOREM 4.5 TEST FOR INCREASING AND DECREASING FUNCTIONS

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) .

1. If $f'(x) > 0$ for all x in (a, b) , then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for all x in (a, b) , then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ for all x in (a, b) , then f is constant on $[a, b]$.

PROOF To prove the first case, assume that $f'(x) > 0$ for all x in the interval (a, b) and let $x_1 < x_2$ be any two points in the interval. By the Mean Value Theorem, you know that there exists a number c such that $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Because $f'(c) > 0$ and $x_2 - x_1 > 0$, you know that

$$f(x_2) - f(x_1) > 0$$

which implies that $f(x_1) < f(x_2)$. So, f is increasing on the interval. The second case has a similar proof (see Exercise 119), and the third case was given as Exercise 90 in Section 4.2. ■

NOTE The conclusions in the first two cases of Theorem 4.5 are valid even if $f'(x) = 0$ at a finite number of x -values in (a, b) . ■

EXAMPLE 1 Intervals on Which f Is Increasing or Decreasing

Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

Solution Note that f is differentiable on the entire real number line. To determine the critical numbers of f , set $f'(x)$ equal to zero.

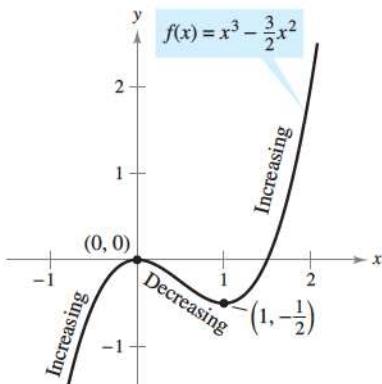


Figure 4.16

$$f(x) = x^3 - \frac{3}{2}x^2 \quad \text{Write original function.}$$

$$f'(x) = 3x^2 - 3x = 0 \quad \text{Differentiate and set } f'(x) \text{ equal to 0.}$$

$$3x(x - 1) = 0 \quad \text{Factor.}$$

$$x = 0, 1 \quad \text{Critical numbers}$$

Because there are no points for which f' does not exist, you can conclude that $x = 0$ and $x = 1$ are the only critical numbers. The table summarizes the testing of the three intervals determined by these two critical numbers.

Interval	$-\infty < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test Value	$x = -1$	$x = \frac{1}{2}$	$x = 2$
Sign of $f'(x)$	$f'(-1) = 6 > 0$	$f'\left(\frac{1}{2}\right) = -\frac{3}{4} < 0$	$f'(2) = 6 > 0$
Conclusion	Increasing	Decreasing	Increasing

So, f is increasing on the intervals $(-\infty, 0)$ and $(1, \infty)$ and decreasing on the interval $(0, 1)$, as shown in Figure 4.16. ■

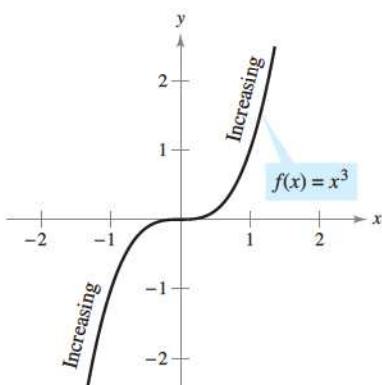
Example 1 gives you one example of how to find intervals on which a function is increasing or decreasing. The guidelines below summarize the steps followed in that example.

GUIDELINES FOR FINDING INTERVALS ON WHICH A FUNCTION IS INCREASING OR DECREASING

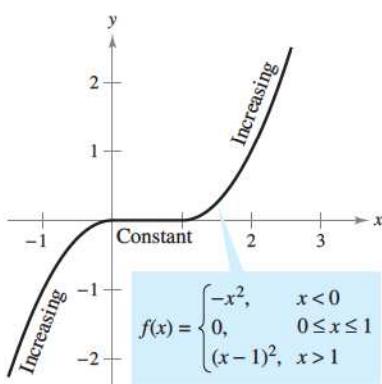
Let f be continuous on the interval (a, b) . To find the open intervals on which f is increasing or decreasing, use the following steps.

1. Locate the critical numbers of f in (a, b) , and use these numbers to determine test intervals.
2. Determine the sign of $f'(x)$ at one test value in each of the intervals.
3. Use Theorem 4.5 to determine whether f is increasing or decreasing on each interval.

These guidelines are also valid if the interval (a, b) is replaced by an interval of the form $(-\infty, b)$, (a, ∞) , or $(-\infty, \infty)$.



(a) Strictly monotonic function

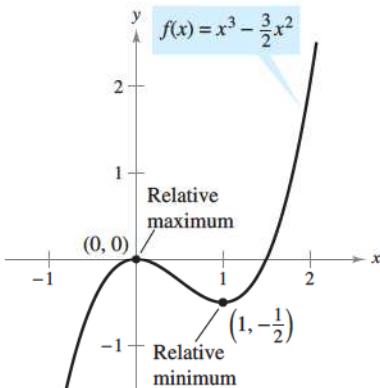


(b) Not strictly monotonic

Figure 4.17

A function is **strictly monotonic** on an interval if it is either increasing on the entire interval or decreasing on the entire interval. For instance, the function $f(x) = x^3$ is strictly monotonic on the entire real number line because it is increasing on the entire real number line, as shown in Figure 4.17(a). The function shown in Figure 4.17(b) is not strictly monotonic on the entire real number line because it is constant on the interval $[0, 1]$.

The First Derivative Test



Relative extrema of f
Figure 4.18

After you have determined the intervals on which a function is increasing or decreasing, it is not difficult to locate the relative extrema of the function. For instance, in Figure 4.18 (from Example 1), the function

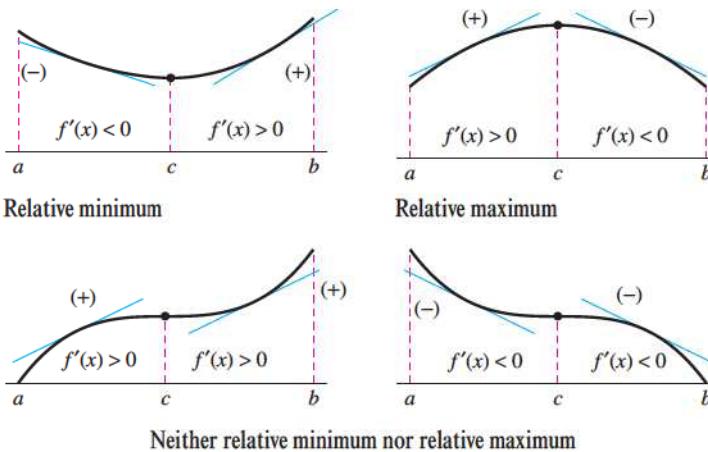
$$f(x) = x^3 - \frac{3}{2}x^2$$

has a relative maximum at the point $(0, 0)$ because f is increasing immediately to the left of $x = 0$ and decreasing immediately to the right of $x = 0$. Similarly, f has a relative minimum at the point $(1, -\frac{1}{2})$ because f is decreasing immediately to the left of $x = 1$ and increasing immediately to the right of $x = 1$. The following theorem, called the First Derivative Test, makes this more explicit.

THEOREM 4.6 THE FIRST DERIVATIVE TEST

Let c be a critical number of a function f that is continuous on an open interval I containing c . If f is differentiable on the interval, except possibly at c , then $f(c)$ can be classified as follows.

- If $f'(x)$ changes from negative to positive at c , then f has a *relative minimum* at $(c, f(c))$.
- If $f'(x)$ changes from positive to negative at c , then f has a *relative maximum* at $(c, f(c))$.
- If $f'(x)$ is positive on both sides of c or negative on both sides of c , then $f(c)$ is neither a relative minimum nor a relative maximum.



PROOF Assume that $f'(x)$ changes from negative to positive at c . Then there exist a and b in I such that

$$f'(x) < 0 \text{ for all } x \text{ in } (a, c)$$

and

$$f'(x) > 0 \text{ for all } x \text{ in } (c, b).$$

By Theorem 4.5, f is decreasing on $[a, c]$ and increasing on $[c, b]$. So, $f(c)$ is a minimum of f on the open interval (a, b) and, consequently, a relative minimum of f . This proves the first case of the theorem. The second case can be proved in a similar way (see Exercise 120). ■

EXAMPLE 2 Applying the First Derivative Test

Find the relative extrema of the function $f(x) = \frac{1}{2}x - \sin x$ in the interval $(0, 2\pi)$.

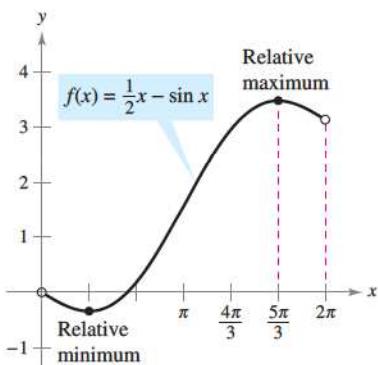
Solution Note that f is continuous on the interval $(0, 2\pi)$. To determine the critical numbers of f in this interval, set $f'(x)$ equal to 0.

$$f'(x) = \frac{1}{2} - \cos x = 0 \quad \text{Set } f'(x) \text{ equal to 0.}$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3} \quad \text{Critical numbers}$$

Because there are no points for which f' does not exist, you can conclude that $x = \pi/3$ and $x = 5\pi/3$ are the only critical numbers. The table summarizes the testing of the three intervals determined by these two critical numbers.



A relative minimum occurs where f changes from decreasing to increasing, and a relative maximum occurs where f changes from increasing to decreasing.

Figure 4.19

Interval	$0 < x < \frac{\pi}{3}$	$\frac{\pi}{3} < x < \frac{5\pi}{3}$	$\frac{5\pi}{3} < x < 2\pi$
Test Value	$x = \frac{\pi}{4}$	$x = \pi$	$x = \frac{7\pi}{4}$
Sign of $f'(x)$	$f'(\frac{\pi}{4}) < 0$	$f'(\pi) > 0$	$f'(\frac{7\pi}{4}) < 0$
Conclusion	Decreasing	Increasing	Decreasing

By applying the First Derivative Test, you can conclude that f has a relative minimum at the point where

$$x = \frac{\pi}{3} \quad \text{x-value where relative minimum occurs}$$

and a relative maximum at the point where

$$x = \frac{5\pi}{3} \quad \text{x-value where relative maximum occurs}$$

as shown in Figure 4.19. ■

EXPLORATION

Comparing Graphical and Analytic Approaches From Section 4.2, you know that, by itself, a graphing utility can give misleading information about the relative extrema of a graph. Used in conjunction with an analytic approach, however, a graphing utility can provide a good way to reinforce your conclusions. Try using a graphing utility to graph the function in Example 2. Then use the *zoom* and *trace* features to estimate the relative extrema. How close are your graphical approximations?

Note that in Examples 1 and 2 the given functions are differentiable on the entire real number line. For such functions, the only critical numbers are those for which $f'(x) = 0$. Example 3 concerns a function that has two types of critical numbers—those for which $f'(x) = 0$ and those for which f is not differentiable.

EXAMPLE 3 Applying the First Derivative Test

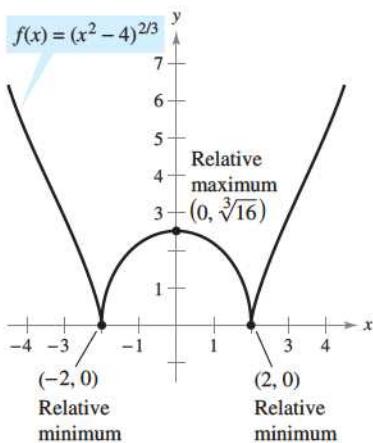
Find the relative extrema of

$$f(x) = (x^2 - 4)^{2/3}.$$

Solution Begin by noting that f is continuous on the entire real number line. The derivative of f

$$\begin{aligned} f'(x) &= \frac{2}{3}(x^2 - 4)^{-1/3}(2x) && \text{General Power Rule} \\ &= \frac{4x}{3(x^2 - 4)^{1/3}} && \text{Simplify.} \end{aligned}$$

is 0 when $x = 0$ and does not exist when $x = \pm 2$. So, the critical numbers are $x = -2$, $x = 0$, and $x = 2$. The table summarizes the testing of the four intervals determined by these three critical numbers.



You can apply the First Derivative Test to find relative extrema.

Figure 4.20

Interval	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test Value	$x = -3$	$x = -1$	$x = 1$	$x = 3$
Sign of $f'(x)$	$f'(-3) < 0$	$f'(-1) > 0$	$f'(1) < 0$	$f'(3) > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

By applying the First Derivative Test, you can conclude that f has a relative minimum at the point $(-2, 0)$, a relative maximum at the point $(0, \sqrt[3]{16})$, and another relative minimum at the point $(2, 0)$, as shown in Figure 4.20. ■

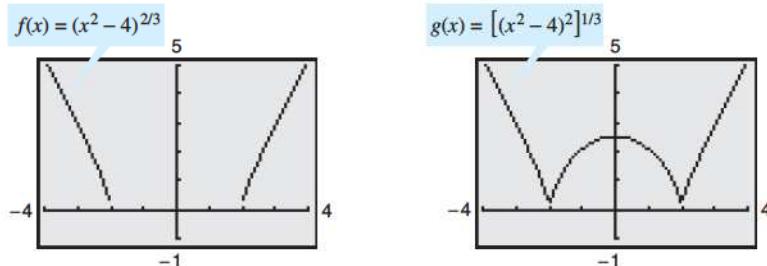
TECHNOLOGY PITFALL When using a graphing utility to graph a function involving radicals or rational exponents, be sure you understand the way the utility evaluates radical expressions. For instance, even though

$$f(x) = (x^2 - 4)^{2/3}$$

and

$$g(x) = [(x^2 - 4)^2]^{1/3}$$

are the same algebraically, some graphing utilities distinguish between these two functions. Which of the graphs shown in Figure 4.21 is incorrect? Why did the graphing utility produce an incorrect graph?



Which graph is incorrect?

Figure 4.21

When using the First Derivative Test, be sure to consider the domain of the function. For instance, in the next example, the function

$$f(x) = \frac{x^4 + 1}{x^2}$$

is not defined when $x = 0$. This x -value must be used with the critical numbers to determine the test intervals.

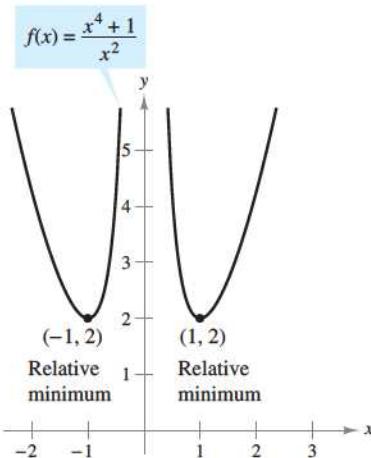
EXAMPLE 4 Applying the First Derivative Test

Find the relative extrema of $f(x) = \frac{x^4 + 1}{x^2}$.

Solution

$$\begin{aligned} f(x) &= x^2 + x^{-2} && \text{Rewrite original function.} \\ f'(x) &= 2x - 2x^{-3} && \text{Differentiate.} \\ &= 2x - \frac{2}{x^3} && \text{Rewrite with positive exponent.} \\ &= \frac{2(x^4 - 1)}{x^3} && \text{Simplify.} \\ &= \frac{2(x^2 + 1)(x - 1)(x + 1)}{x^3} && \text{Factor.} \end{aligned}$$

So, $f'(x)$ is zero at $x = \pm 1$. Moreover, because $x = 0$ is not in the domain of f , you should use this x -value along with the critical numbers to determine the test intervals.



x -values that are not in the domain of f , as well as critical numbers, determine test intervals for f' .

Figure 4.22

$$x = \pm 1$$

$$x = 0$$

$$\text{Critical numbers, } f'(\pm 1) = 0$$

$$0 \text{ is not in the domain of } f.$$

The table summarizes the testing of the four intervals determined by these three x -values.

Interval	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test Value	$x = -2$	$x = -\frac{1}{2}$	$x = \frac{1}{2}$	$x = 2$
Sign of $f'(x)$	$f'(-2) < 0$	$f'\left(-\frac{1}{2}\right) > 0$	$f'\left(\frac{1}{2}\right) < 0$	$f'(2) > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

By applying the First Derivative Test, you can conclude that f has one relative minimum at the point $(-1, 2)$ and another at the point $(1, 2)$, as shown in Figure 4.22. ■

TECHNOLOGY The most difficult step in applying the First Derivative Test is finding the values for which the derivative is equal to 0. For instance, the values of x for which the derivative of

$$f(x) = \frac{x^4 + 1}{x^2 + 1}$$

is equal to zero are $x = 0$ and $x = \pm\sqrt{\sqrt{2} - 1}$. If you have access to technology that can perform symbolic differentiation and solve equations, use it to apply the First Derivative Test to this function.

EXAMPLE 5 The Path of a Projectile

Thomas Kienzle/AP Photos

If a projectile is propelled from ground level and air resistance is neglected, the object will travel farthest with an initial angle of 45° . If, however, the projectile is propelled from a point above ground level, the angle that yields a maximum horizontal distance is not 45° (see Example 5).

Neglecting air resistance, the path of a projectile that is propelled at an angle θ is

$$y = \frac{g \sec^2 \theta}{2v_0^2} x^2 + (\tan \theta)x + h, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

where y is the height, x is the horizontal distance, g is the acceleration due to gravity, v_0 is the initial velocity, and h is the initial height. (This equation is derived in Section 12.3.) Let $g = -32$ feet per second per second, $v_0 = 24$ feet per second, and $h = 9$ feet. What value of θ will produce a maximum horizontal distance?

Solution To find the distance the projectile travels, let $y = 0$ and use the Quadratic Formula to solve for x .

$$\begin{aligned} \frac{g \sec^2 \theta}{2v_0^2} x^2 + (\tan \theta)x + h &= 0 \\ \frac{-32 \sec^2 \theta}{2(24^2)} x^2 + (\tan \theta)x + 9 &= 0 \\ -\frac{\sec^2 \theta}{36} x^2 + (\tan \theta)x + 9 &= 0 \\ x &= \frac{-\tan \theta \pm \sqrt{\tan^2 \theta + \sec^2 \theta}}{-\sec^2 \theta / 18} \\ x &= 18 \cos \theta (\sin \theta + \sqrt{\sin^2 \theta + 1}), \quad x \geq 0 \end{aligned}$$

At this point, you need to find the value of θ that produces a maximum value of x . Applying the First Derivative Test by hand would be very tedious. Using technology to solve the equation $dx/d\theta = 0$, however, eliminates most of the messy computations. The result is that the maximum value of x occurs when

$$\theta \approx 0.61548 \text{ radian, or } 35.3^\circ.$$

This conclusion is reinforced by sketching the path of the projectile for different values of θ , as shown in Figure 4.23. Of the three paths shown, note that the distance traveled is greatest for $\theta = 35^\circ$.

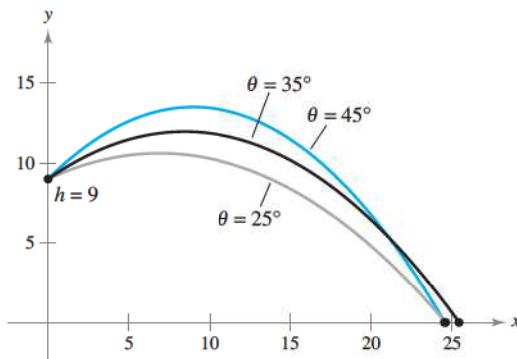
The path of a projectile with initial angle θ

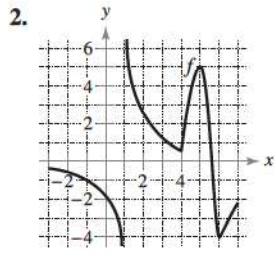
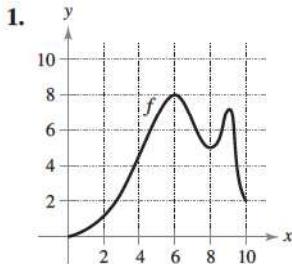
Figure 4.23

NOTE A computer simulation of this example is given in the premium eBook for this text. Using that simulation, you can experimentally discover that the maximum value of x occurs when $\theta \approx 35.3^\circ$.

4.3 Exercises

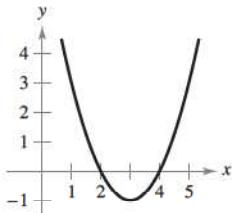
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use the graph of f to find (a) the largest open interval on which f is increasing, and (b) the largest open interval on which f is decreasing.

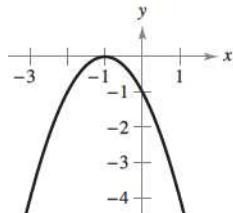


In Exercises 3–8, use the graph to estimate the open intervals on which the function is increasing or decreasing. Then find the open intervals analytically.

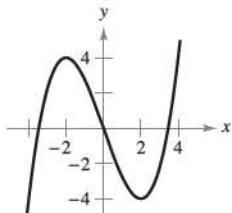
3. $f(x) = x^2 - 6x + 8$



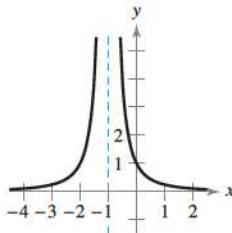
4. $y = -(x+1)^2$



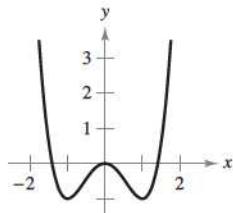
5. $y = \frac{x^3}{4} - 3x$



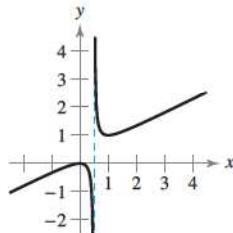
7. $f(x) = \frac{1}{(x+1)^2}$



6. $f(x) = x^4 - 2x^2$



8. $y = \frac{x^2}{2x-1}$



In Exercises 9–20, identify the open intervals on which the function is increasing or decreasing.

9. $g(x) = x^2 - 2x - 8$

10. $h(x) = 27x - x^3$

11. $y = x\sqrt{16-x^2}$

12. $y = x + \frac{4}{x}$

13. $f(x) = \sin x - 1, \quad 0 < x < 2\pi$

14. $h(x) = \cos \frac{x}{2}, \quad 0 < x < 2\pi$

15. $y = x - 2 \cos x, \quad 0 < x < 2\pi$

16. $f(x) = \cos^2 x - \cos x, \quad 0 < x < 2\pi$

17. $g(x) = e^{-x} + e^{3x}$

18. $h(x) = \sqrt{x} e^{-x}$

19. $f(x) = x^2 \ln\left(\frac{x}{2}\right)$

20. $f(x) = \frac{\ln x}{\sqrt{x}}$

In Exercises 21–58, find the critical numbers of f (if any). Find the open intervals on which the function is increasing or decreasing and locate all relative extrema. Use a graphing utility to confirm your results.

21. $f(x) = x^2 - 4x$

22. $f(x) = x^2 + 6x + 10$

23. $f(x) = -2x^2 + 4x + 3$

24. $f(x) = -(x^2 + 8x + 12)$

25. $f(x) = 2x^3 + 3x^2 - 12x$

26. $f(x) = x^3 - 6x^2 + 15$

27. $f(x) = (x-1)^2(x+3)$

28. $f(x) = (x+2)^2(x-1)$

29. $f(x) = \frac{x^5 - 5x}{5}$

30. $f(x) = x^4 - 32x + 4$

31. $f(x) = x^{1/3} + 1$

32. $f(x) = x^{2/3} - 4$

33. $f(x) = (x+2)^{2/3}$

34. $f(x) = (x-3)^{1/3}$

35. $f(x) = 5 - |x-5|$

36. $f(x) = |x+3| - 1$

37. $f(x) = 2x + \frac{1}{x}$

38. $f(x) = \frac{x}{x+3}$

39. $f(x) = \frac{x^2}{x^2 - 9}$

40. $f(x) = \frac{x+4}{x^2}$

41. $f(x) = \frac{x^2 - 2x + 1}{x+1}$

42. $f(x) = \frac{x^2 - 3x - 4}{x-2}$

43. $f(x) = \begin{cases} 4 - x^2, & x \leq 0 \\ -2x, & x > 0 \end{cases}$

44. $f(x) = \begin{cases} 2x+1, & x \leq -1 \\ x^2 - 2, & x > -1 \end{cases}$

45. $f(x) = \begin{cases} 3x+1, & x \leq 1 \\ 5 - x^2, & x > 1 \end{cases}$

46. $f(x) = \begin{cases} -x^3 + 1, & x \leq 0 \\ -x^2 + 2x, & x > 0 \end{cases}$

47. $f(x) = (3-x)e^{x-3}$

48. $f(x) = (x-1)e^x$

49. $f(x) = 4(x - \arcsin x)$

50. $f(x) = x \arctan x$

51. $f(x) = (x)3^{-x}$

52. $f(x) = 2^{x^2-3}$

53. $f(x) = x - \log_4 x$

54. $f(x) = \frac{x^3}{3} - \ln x$

55. $f(x) = \frac{e^{2x}}{e^{2x} + 1}$

56. $f(x) = \ln(2 - \ln x)$

57. $f(x) = e^{-1/(x-2)}$

58. $f(x) = e^{\arctan x}$

In Exercises 59–66, consider the function on the interval $(0, 2\pi)$. For each function, (a) find the open interval(s) on which the function is increasing or decreasing, (b) apply the First Derivative Test to identify all relative extrema, and (c) use a graphing utility to confirm your results.

59. $f(x) = \frac{x}{2} + \cos x$

60. $f(x) = \sin x \cos x + 5$

61. $f(x) = \sin x + \cos x$

63. $f(x) = \cos^2(2x)$

65. $f(x) = \sin^2 x + \sin x$

62. $f(x) = x + 2 \sin x$

64. $f(x) = \sqrt{3} \sin x + \cos x$

66. $f(x) = \frac{\sin x}{1 + \cos^2 x}$

CAS In Exercises 67–72, (a) use a computer algebra system to differentiate the function, (b) sketch the graphs of f and f' on the same set of coordinate axes over the given interval, (c) find the critical numbers of f in the open interval, (d) find the interval(s) on which f' is positive and the interval(s) on which it is negative, and (e) compare the behavior of f and the sign of f' .

67. $f(x) = 2x\sqrt{9 - x^2}, [-3, 3]$

68. $f(x) = 10(5 - \sqrt{x^2 - 3x + 16}), [0, 5]$

69. $f(t) = t^2 \sin t, [0, 2\pi]$

70. $f(x) = \frac{x}{2} + \cos \frac{x}{2}, [0, 4\pi]$

71. $f(x) = \frac{1}{2}(x^2 - \ln x), (0, 3]$

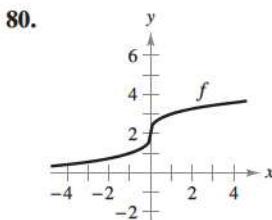
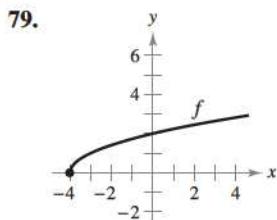
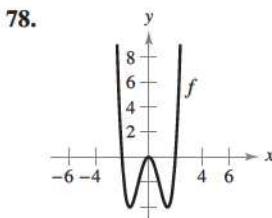
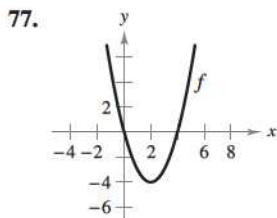
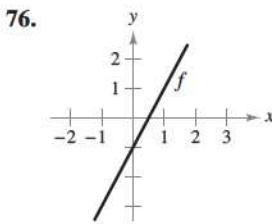
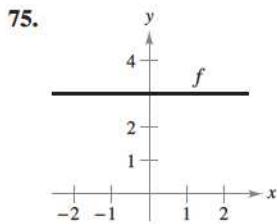
72. $f(x) = (4 - x^2)e^x, [0, 2]$

In Exercises 73 and 74, use symmetry, extrema, and zeros to sketch the graph of f . How do the functions f and g differ? Explain.

73. $f(x) = \frac{x^5 - 4x^3 + 3x}{x^2 - 1}, g(x) = x(x^2 - 3)$

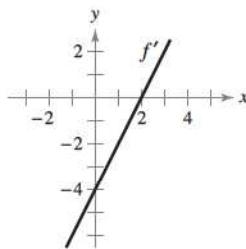
74. $f(t) = \cos^2 t - \sin^2 t, g(t) = 1 - 2 \sin^2 t \quad (-2, 2)$

Think About It In Exercises 75–80, the graph of f is shown in the figure. Sketch a graph of the derivative of f . To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

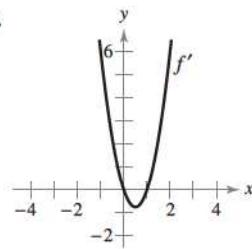


In Exercises 81–84, use the graph of f' to (a) identify the interval(s) on which f is increasing or decreasing, and (b) estimate the value(s) of x at which f has a relative maximum or minimum.

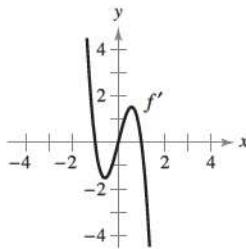
81.



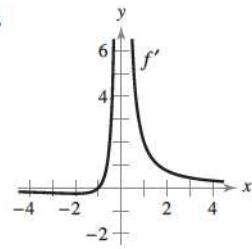
82.



83.

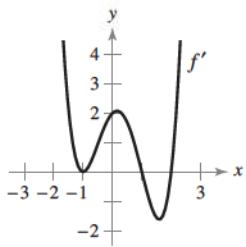


84.

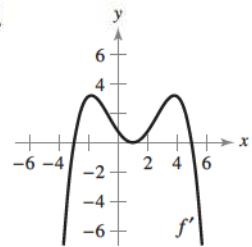


In Exercises 85 and 86, use the graph of f' to (a) identify the critical numbers of f , and (b) determine whether f has a relative maximum, a relative minimum, or neither at each critical number.

85.



86.



WRITING ABOUT CONCEPTS

In Exercises 87–92, assume that f is differentiable for all x , where $f'(x) > 0$ on $(-\infty, -4)$, $f'(x) < 0$ on $(-4, 6)$, and $f'(x) > 0$ on $(6, \infty)$. Supply the appropriate inequality symbol for the given value of c .

Function	Sign of $g'(c)$
87. $g(x) = f(x) + 5$	$g'(0) \quad ? \quad 0$
88. $g(x) = 3f(x) - 3$	$g'(-5) \quad ? \quad 0$
89. $g(x) = -f(x)$	$g'(-6) \quad ? \quad 0$
90. $g(x) = -f(x)$	$g'(0) \quad ? \quad 0$
91. $g(x) = f(x - 10)$	$g'(0) \quad ? \quad 0$
92. $g(x) = f(x - 10)$	$g'(8) \quad ? \quad 0$

93. Sketch the graph of the arbitrary function f such that

$$f'(x) \begin{cases} > 0, & x < 4 \\ \text{undefined,} & x = 4. \\ < 0, & x > 4 \end{cases}$$

CAPSTONE

- 94.** A differentiable function f has one critical number at $x = 5$. Identify the relative extrema of f at the critical number if $f'(4) = -2.5$ and $f'(6) = 3$.

Think About It In Exercises 95 and 96, the function f is differentiable on the given interval. The table shows $f'(x)$ for selected values of x . (a) Sketch the graph of f , (b) approximate the critical numbers, and (c) identify the relative extrema.

- 95.** f is differentiable on $[-1, 1]$.

x	-1	-0.75	-0.50	-0.25
$f'(x)$	-10	-3.2	-0.5	0.8

x	0	0.25	0.50	0.75	1
$f'(x)$	5.6	3.6	-0.2	-6.7	-20.1

- 96.** f is differentiable on $[0, \pi]$.

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$f'(x)$	3.14	-0.23	-2.45	-3.11	0.69

x	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$f'(x)$	3.00	1.37	-1.14	-2.84

- 97. Rolling a Ball Bearing** A ball bearing is placed on an inclined plane and begins to roll. The angle of elevation of the plane is θ . The distance (in meters) the ball bearing rolls in t seconds is $s(t) = 4.9(\sin \theta)t^2$.

- (a) Determine the speed of the ball bearing after t seconds.
(b) Complete the table and use it to determine the value of θ that produces the maximum speed at a particular time.

θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π
$s'(t)$							

- 98. Numerical, Graphical, and Analytic Analysis** The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is

$$C(t) = \frac{3t}{27 + t^3}, \quad t \geq 0.$$

- (a) Complete the table and use it to approximate the time when the concentration is greatest.

t	0	0.5	1	1.5	2	2.5	3
$C(t)$							

- AP** (b) Use a graphing utility to graph the concentration function and use the graph to approximate the time when the concentration is greatest.

- (c) Use calculus to determine analytically the time when the concentration is greatest.

- 99. Numerical, Graphical, and Analytic Analysis** Consider the functions $f(x) = x$ and $g(x) = \sin x$ on the interval $(0, \pi)$.

- (a) Complete the table and make a conjecture about which is the greater function on the interval $(0, \pi)$.

x	0.5	1	1.5	2	2.5	3
$f(x)$						
$g(x)$						

- AP** (b) Use a graphing utility to graph the functions and use the graphs to make a conjecture about which is the greater function on the interval $(0, \pi)$.

- (c) Prove that $f(x) > g(x)$ on the interval $(0, \pi)$. [Hint: Show that $h'(x) > 0$, where $h = f - g$.]

- 100. Numerical, Graphical, and Analytic Analysis** Consider the functions $f(x) = x$ and $g(x) = \tan x$ on the interval $(0, \pi/2)$.

- (a) Complete the table and make a conjecture about which is the greater function on the interval $(0, \pi/2)$.

x	0.25	0.5	0.75	1	1.25	1.5
$f(x)$						
$g(x)$						

- AP** (b) Use a graphing utility to graph the functions and use the graphs to make a conjecture about which is the greater function on the interval $(0, \pi/2)$.

- (c) Prove that $f(x) < g(x)$ on the interval $(0, \pi/2)$. [Hint: Show that $h'(x) > 0$, where $h = g - f$.]

- 101. Trachea Contraction** Coughing forces the trachea (windpipe) to contract, which affects the velocity v of the air passing through the trachea. The velocity of the air during coughing is

$$v = k(R - r)^2, \quad 0 \leq r < R$$

where k is a constant, R is the normal radius of the trachea, and r is the radius during coughing. What radius will produce the maximum air velocity?

- AP** **102. Modeling Data** The end-of-year assets of the Medicare Hospital Insurance Trust Fund (in billions of dollars) for the years 1995 through 2006 are shown.

1995: 130.3; 1996: 124.9; 1997: 115.6; 1998: 120.4;

1999: 141.4; 2000: 177.5; 2001: 208.7; 2002: 234.8;

2003: 256.0; 2004: 269.3; 2005: 285.8; 2006: 305.4

(Source: U.S. Centers for Medicare and Medicaid Services)

- (a) Use the regression capabilities of a graphing utility to find a model of the form $M = at^4 + bt^3 + ct^2 + dt + e$ for the data. (Let $t = 5$ represent 1995.)

- (b) Use a graphing utility to plot the data and graph the model.
 (c) Find the minimum value of the model and compare the result with the actual data.

Motion Along a Line In Exercises 103–106, the function $s(t)$ describes the motion of a particle along a line. For each function, (a) find the velocity function of the particle at any time $t \geq 0$, (b) identify the time interval(s) in which the particle is moving in a positive direction, (c) identify the time interval(s) in which the particle is moving in a negative direction, and (d) identify the time(s) at which the particle changes direction.

103. $s(t) = 6t - t^2$

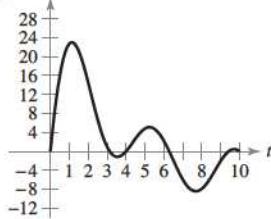
104. $s(t) = t^2 - 7t + 10$

105. $s(t) = t^3 - 5t^2 + 4t$

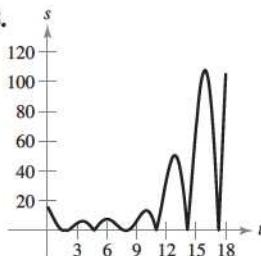
106. $s(t) = t^3 - 20t^2 + 128t - 280$

Motion Along a Line In Exercises 107 and 108, the graph shows the position of a particle moving along a line. Describe how the particle's position changes with respect to time.

107.



108.



Creating Polynomial Functions In Exercises 109–112, find a polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

that has only the specified extrema. (a) Determine the minimum degree of the function and give the criteria you used in determining the degree. (b) Using the fact that the coordinates of the extrema are solution points of the function, and that the x -coordinates are critical numbers, determine a system of linear equations whose solution yields the coefficients of the required function. (c) Use a graphing utility to solve the system of equations and determine the function. (d) Use a graphing utility to confirm your result graphically.

109. Relative minimum: $(0, 0)$; Relative maximum: $(2, 2)$
 110. Relative minimum: $(0, 0)$; Relative maximum: $(4, 1000)$
 111. Relative minima: $(0, 0), (4, 0)$; Relative maximum: $(2, 4)$
 112. Relative minimum: $(1, 2)$; Relative maxima: $(-1, 4), (3, 4)$

True or False? In Exercises 113–118, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

113. The sum of two increasing functions is increasing.
 114. The product of two increasing functions is increasing.
 115. Every n th-degree polynomial has $(n - 1)$ critical numbers.
 116. An n th-degree polynomial has at most $(n - 1)$ critical numbers.

117. There is a relative maximum or minimum at each critical number.

118. The relative maxima of the function f are $f(1) = 4$ and $f(3) = 10$. So, f has at least one minimum for some x in the interval $(1, 3)$.

119. Prove the second case of Theorem 4.5.

120. Prove the second case of Theorem 4.6.

121. Let $x > 0$ and $n > 1$ be real numbers. Prove that $(1 + x)^n > 1 + nx$.

122. Use the definitions of increasing and decreasing functions to prove that $f(x) = x^3$ is increasing on $(-\infty, \infty)$.

123. Use the definitions of increasing and decreasing functions to prove that $f(x) = 1/x$ is decreasing on $(0, \infty)$.

124. Consider $f(x) = axe^{bx^2}$. Find a and b such that the relative maximum of f is $f(4) = 2$.

PUTNAM EXAM CHALLENGE

125. Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x .

This problem was composed by the Committee on the Putnam Prize Competition.
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SECTION PROJECT

Rainbows

Rainbows are formed when light strikes raindrops and is reflected and refracted, as shown in the figure. (This figure shows a cross section of a spherical raindrop.) The Law of Refraction states that $(\sin \alpha)/(\sin \beta) = k$, where $k \approx 1.33$ (for water). The angle of deflection is given by $D = \pi + 2\alpha - 4\beta$.

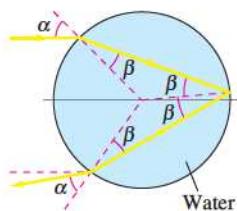
- (a) Use a graphing utility to graph

$$D = \pi + 2\alpha - 4 \sin^{-1}(1/k \sin \alpha),$$

$$0 \leq \alpha \leq \pi/2.$$

- (b) Prove that the minimum angle of deflection occurs when

$$\cos \alpha = \sqrt{\frac{k^2 - 1}{3}}.$$



For water, what is the minimum angle of deflection, D_{\min} ? (The angle $\pi - D_{\min}$ is called the *rainbow angle*.) What value of α produces this minimum angle? (A ray of sunlight that strikes a raindrop at this angle, α , is called a *rainbow ray*.)

- **FOR FURTHER INFORMATION** For more information about the mathematics of rainbows, see the article “Somewhere Within the Rainbow” by Steven Janke in *The UMAP Journal*.

4.4**Concavity and the Second Derivative Test**

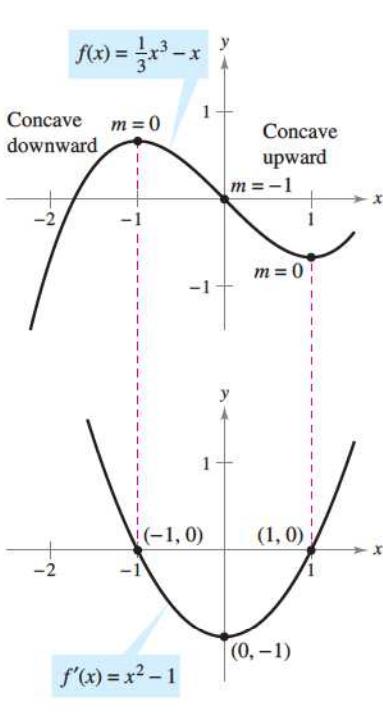
- Determine intervals on which a function is concave upward or concave downward.
- Find any points of inflection of the graph of a function.
- Apply the Second Derivative Test to find relative extrema of a function.

Concavity

You have already seen that locating the intervals on which a function f increases or decreases helps to describe its graph. In this section, you will see how locating the intervals on which f' increases or decreases can be used to determine where the graph of f is *curving upward* or *curving downward*.

DEFINITION OF CONCAVITY

Let f be differentiable on an open interval I . The graph of f is **concave upward** on I if f' is increasing on the interval and **concave downward** on I if f' is decreasing on the interval.



The concavity of f is related to the slope of the derivative.

Figure 4.25

The following graphical interpretation of concavity is useful. (See Appendix A for a proof of these results.)

1. Let f be differentiable on an open interval I . If the graph of f is concave upward on I , then the graph of f lies *above* all of its tangent lines on I . [See Figure 4.24(a).]
2. Let f be differentiable on an open interval I . If the graph of f is concave downward on I , then the graph of f lies *below* all of its tangent lines on I . [See Figure 4.24(b).]

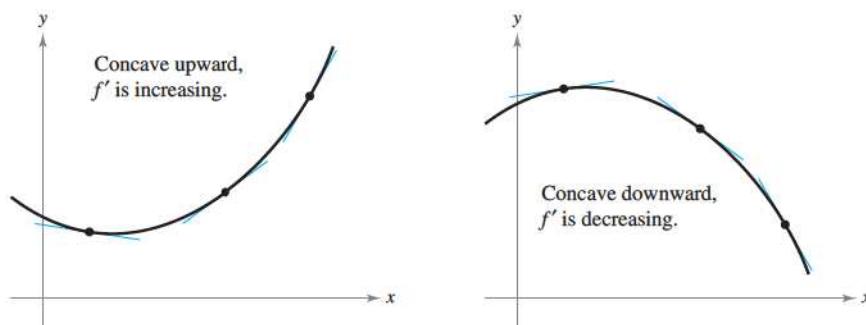


Figure 4.24

To find the open intervals on which the graph of a function f is concave upward or downward, you need to find the intervals on which f' is increasing or decreasing. For instance, the graph of

$$f(x) = \frac{1}{3}x^3 - x$$

is concave downward on the open interval $(-\infty, 0)$ because $f'(x) = x^2 - 1$ is decreasing there. (See Figure 4.25.) Similarly, the graph of f is concave upward on the interval $(0, \infty)$ because f' is increasing on $(0, \infty)$.

The following theorem shows how to use the *second* derivative of a function f to determine intervals on which the graph of f is concave upward or concave downward. A proof of this theorem (see Appendix A) follows directly from Theorem 4.5 and the definition of concavity.

THEOREM 4.7 TEST FOR CONCAVITY

Let f be a function whose second derivative exists on an open interval I .

1. If $f''(x) > 0$ for all x in I , then the graph of f is concave upward on I .
2. If $f''(x) < 0$ for all x in I , then the graph of f is concave downward on I .

Note that a third case of Theorem 4.7 could be that if $f''(x) = 0$ for all x in I , then f is linear. Note, however, that concavity is not defined for a line. In other words, a straight line is neither concave upward nor concave downward.

To apply Theorem 4.7, first locate the x -values at which $f''(x) = 0$ or f'' does not exist. Second, use these x -values to determine test intervals. Finally, test the sign of $f''(x)$ in each of the test intervals.

EXAMPLE 1 Determining Concavity

Determine the open intervals on which the graph of

$$f(x) = e^{-x^2/2}$$

is concave upward or concave downward.

Solution Begin by observing that f is continuous on the entire real number line. Next, find the second derivative of f .

$$f'(x) = -xe^{-x^2/2}$$

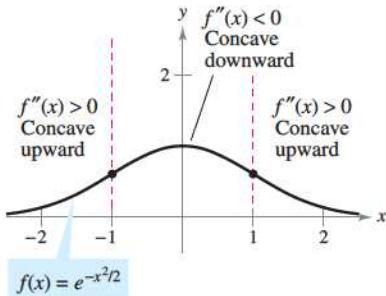
First derivative

$$f''(x) = (-x)(-x)e^{-x^2/2} + e^{-x^2/2}(-1)$$

Differentiate.

$$= e^{-x^2/2}(x^2 - 1)$$

Second derivative



From the sign of f'' you can determine the concavity of the graph of f .

Figure 4.26

NOTE The function in Example 1 is similar to the normal probability density function, whose general form is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

where σ is the standard deviation (σ is the lowercase Greek letter sigma). This “bell-shaped” curve is concave downward on the interval $(-\sigma, \sigma)$.

Interval	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Test Value	$x = -2$	$x = 0$	$x = 2$
Sign of $f''(x)$	$f''(-2) > 0$	$f''(0) < 0$	$f''(2) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

The function given in Example 1 is continuous on the entire real number line. If there are x -values at which the function is not continuous, these values should be used, along with the points at which $f''(x) = 0$ or $f''(x)$ does not exist, to form the test intervals.

EXAMPLE 2 Determining Concavity

Determine the open intervals on which the graph of $f(x) = \frac{x^2 + 1}{x^2 - 4}$ is concave upward or concave downward.

Solution Differentiating twice produces the following.

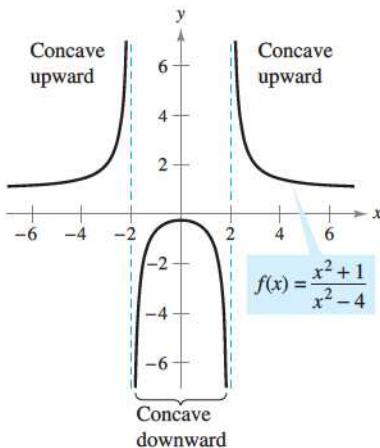
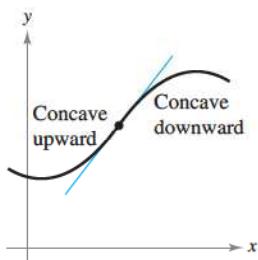


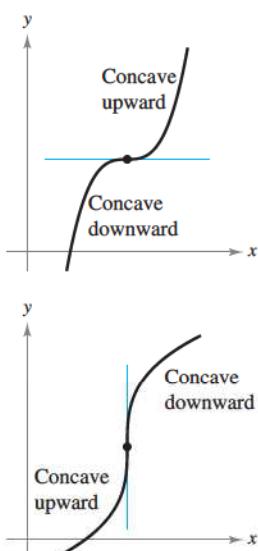
Figure 4.27

$$\begin{aligned}f(x) &= \frac{x^2 + 1}{x^2 - 4} && \text{Write original function.} \\f'(x) &= \frac{(x^2 - 4)(2x) - (x^2 + 1)(2x)}{(x^2 - 4)^2} && \text{Differentiate.} \\&= \frac{-10x}{(x^2 - 4)^2} && \text{First derivative} \\f''(x) &= \frac{(x^2 - 4)^2(-10) - (-10x)(2)(x^2 - 4)(2x)}{(x^2 - 4)^4} && \text{Differentiate.} \\&= \frac{10(3x^2 + 4)}{(x^2 - 4)^3} && \text{Second derivative}\end{aligned}$$

There are no points at which $f''(x) = 0$, but at $x = \pm 2$ the function f is not continuous, so test for concavity in the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$, as shown in the table. The graph of f is shown in Figure 4.27.



Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Test Value	$x = -3$	$x = 0$	$x = 3$
Sign of $f''(x)$	$f''(-3) > 0$	$f''(0) < 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave downward	Concave upward



The concavity of f changes at a point of inflection. Note that a graph crosses its tangent line at a point of inflection.

Figure 4.28

Points of Inflection

The graph in Figure 4.26 has two points at which the concavity changes. If the tangent line to the graph exists at such a point, that point is a **point of inflection**. Three types of points of inflection are shown in Figure 4.28.

DEFINITION OF POINT OF INFLECTION

Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a **point of inflection** of the graph of f if the concavity of f changes from upward to downward (or downward to upward) at the point.

NOTE The definition of *point of inflection* given above requires that the tangent line exists at the point of inflection. Some books do not require this. For instance, we do not consider the function

$$f(x) = \begin{cases} x^3, & x < 0 \\ x^2 + 2x, & x \geq 0 \end{cases}$$

to have a point of inflection at the origin, even though the concavity of the graph changes from concave downward to concave upward. ■

To locate *possible* points of inflection, you can determine the values of x for which $f''(x) = 0$ or $f''(x)$ does not exist. This is similar to the procedure for locating relative extrema of f .

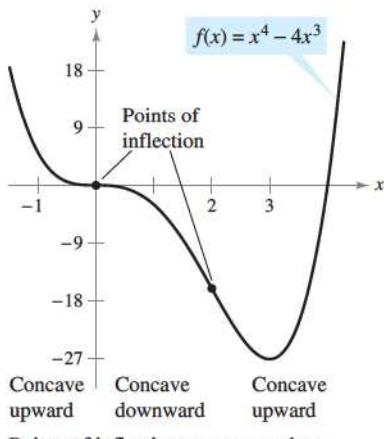
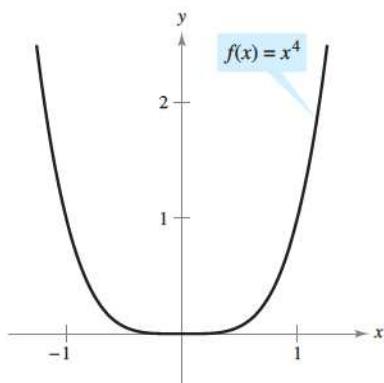


Figure 4.29



$f''(x) = 0$, but $(0, 0)$ is not a point of inflection.

Figure 4.30

THEOREM 4.8 POINTS OF INFLECTION

If $(c, f(c))$ is a point of inflection of the graph of f , then either $f''(c) = 0$ or f'' does not exist at $x = c$.

EXAMPLE 3 Finding Points of Inflection

Determine the points of inflection and discuss the concavity of the graph of

$$f(x) = x^4 - 4x^3.$$

Solution Differentiating twice produces the following.

$$f(x) = x^4 - 4x^3 \quad \text{Write original function.}$$

$$f'(x) = 4x^3 - 12x^2 \quad \text{Find first derivative.}$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2) \quad \text{Find second derivative.}$$

Setting $f''(x) = 0$, you can determine that the possible points of inflection occur at $x = 0$ and $x = 2$. By testing the intervals determined by these x -values, you can conclude that they both yield points of inflection. A summary of this testing is shown in the table, and the graph of f is shown in Figure 4.29.

Interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test Value	$x = -1$	$x = 1$	$x = 3$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(1) < 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

The converse of Theorem 4.8 is not generally true. That is, it is possible for the second derivative to be 0 at a point that is *not* a point of inflection. For instance, the graph of $f(x) = x^4$ is shown in Figure 4.30. The second derivative is 0 when $x = 0$, but the point $(0, 0)$ is not a point of inflection because the graph of f is concave upward on both intervals $-\infty < x < 0$ and $0 < x < \infty$.

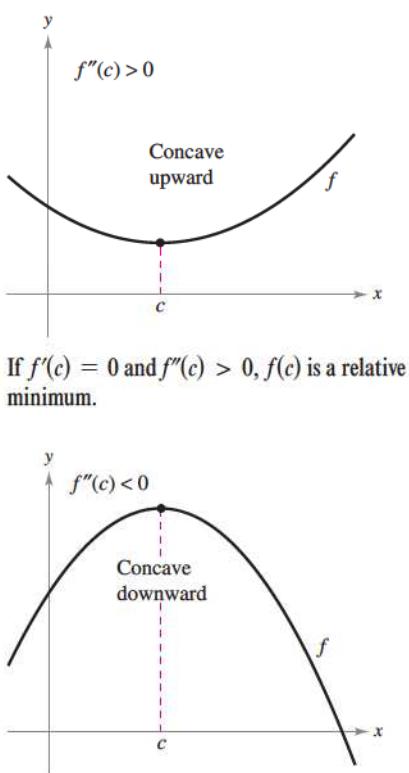
EXPLORATION

Consider a general cubic function of the form

$$f(x) = ax^3 + bx^2 + cx + d.$$

You know that the value of d has a bearing on the location of the graph but has no bearing on the value of the first derivative at given values of x .

Graphically, this is true because changes in the value of d shift the graph up or down but do not change its basic shape. Use a graphing utility to graph several cubics with different values of c . Then give a graphical explanation of why changes in c do not affect the values of the second derivative.



If $f'(c) = 0$ and $f''(c) < 0$, $f(c)$ is a relative maximum.

Figure 4.31

The Second Derivative Test

In addition to testing for concavity, the second derivative can be used to perform a simple test for relative maxima and minima. The test is based on the fact that if the graph of a function f is concave upward on an open interval containing c , and $f'(c) = 0$, $f(c)$ must be a relative minimum of f . Similarly, if the graph of a function f is concave downward on an open interval containing c , and $f'(c) = 0$, $f(c)$ must be a relative maximum of f (see Figure 4.31).

THEOREM 4.9 SECOND DERIVATIVE TEST

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c .

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.

If $f''(c) = 0$, the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

PROOF If $f'(c) = 0$ and $f''(c) > 0$, there exists an open interval I containing c for which

$$\frac{f'(x) - f'(c)}{x - c} = \frac{f'(x)}{x - c} > 0$$

for all $x \neq c$ in I . If $x < c$, then $x - c < 0$ and $f'(x) < 0$. Also, if $x > c$, then $x - c > 0$ and $f'(x) > 0$. So, $f'(x)$ changes from negative to positive at c , and the First Derivative Test implies that $f(c)$ is a relative minimum. A proof of the second case is left to you. ■

EXAMPLE 4 Using the Second Derivative Test

Find the relative extrema for $f(x) = -3x^5 + 5x^3$.

Solution Begin by finding the critical numbers of f .

$$f'(x) = -15x^4 + 15x^2 = 15x^2(1 - x^2) = 0$$

$$x = -1, 0, 1$$

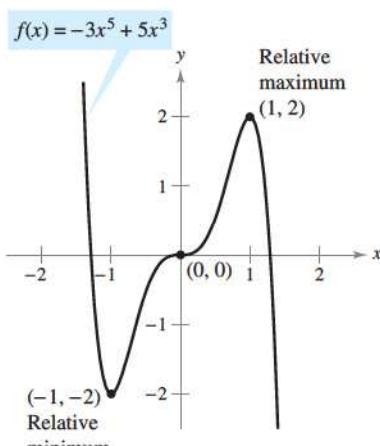
Set $f'(x)$ equal to 0.

Critical numbers

Using

$$f''(x) = -60x^3 + 30x = 30(-2x^3 + x)$$

you can apply the Second Derivative Test as shown below.



(0, 0) is neither a relative minimum nor a relative maximum.

Figure 4.32

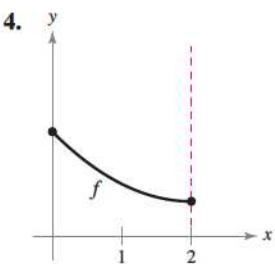
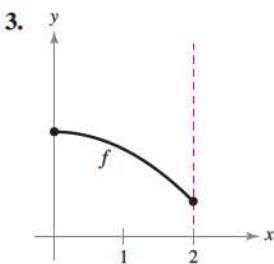
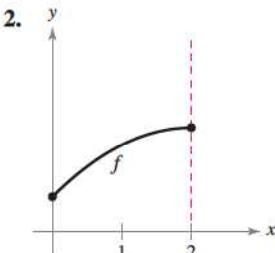
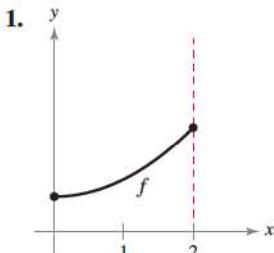
Point	$(-1, -2)$	$(1, 2)$	$(0, 0)$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(1) < 0$	$f''(0) = 0$
Conclusion	Relative minimum	Relative maximum	Test fails

Because the Second Derivative Test fails at $(0, 0)$, you can use the First Derivative Test and observe that f increases to the left and right of $x = 0$. So, $(0, 0)$ is neither a relative minimum nor a relative maximum (even though the graph has a horizontal tangent line at this point). The graph of f is shown in Figure 4.32. ■

4.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, the graph of f is shown. State the signs of f' and f'' on the interval $(0, 2)$.



In Exercises 5–18, determine the open intervals on which the graph is concave upward or concave downward.

5. $y = x^2 - x - 2$

6. $y = -x^3 + 3x^2 - 2$

7. $g(x) = 3x^2 - x^3$

8. $h(x) = x^5 - 5x + 2$

9. $f(x) = -x^3 + 6x^2 - 9x - 1$

10. $f(x) = x^5 + 5x^4 - 40x^2$

11. $f(x) = \frac{24}{x^2 + 12}$

12. $f(x) = \frac{x^2}{x^2 + 1}$

13. $f(x) = \frac{x^2 + 1}{x^2 - 1}$

14. $y = \frac{-3x^5 + 40x^3 + 135x}{270}$

15. $g(x) = \frac{x^2 + 4}{4 - x^2}$

16. $h(x) = \frac{x^2 - 1}{2x - 1}$

17. $y = 2x - \tan x, \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

18. $y = x + \frac{2}{\sin x}, \quad (-\pi, \pi)$

In Exercises 19–42, find the points of inflection and discuss the concavity of the graph of the function.

19. $f(x) = \frac{1}{2}x^4 + 2x^3$

20. $f(x) = -x^4 + 24x^2$

21. $f(x) = x^3 - 6x^2 + 12x$

22. $f(x) = 2x^3 - 3x^2 - 12x + 5$

23. $f(x) = \frac{1}{4}x^4 - 2x^2$

24. $f(x) = 2x^4 - 8x + 3$

25. $f(x) = x(x - 4)^3$

26. $f(x) = (x - 2)^3(x - 1)$

27. $f(x) = x\sqrt{x + 3}$

28. $f(x) = x\sqrt{9 - x}$

29. $f(x) = \frac{4}{x^2 + 1}$

30. $f(x) = \frac{x + 1}{\sqrt{x}}$

31. $f(x) = \sin \frac{x}{2}, \quad [0, 4\pi]$

32. $f(x) = 2 \csc \frac{3x}{2}, \quad (0, 2\pi)$

33. $f(x) = \sec\left(x - \frac{\pi}{2}\right), \quad (0, 4\pi)$

34. $f(x) = \sin x + \cos x, \quad [0, 2\pi]$

35. $f(x) = 2 \sin x + \sin 2x, \quad [0, 2\pi]$

36. $f(x) = x + 2 \cos x, \quad [0, 2\pi]$

37. $y = e^{-3/x}$

38. $y = \frac{1}{2}(e^x - e^{-x})$

39. $y = x - \ln x$

40. $y = \ln \sqrt{x^2 + 9}$

41. $f(x) = \arcsin x^{4/5}$

42. $f(x) = \arctan(x^2)$

In Exercises 43–70, find all relative extrema. Use the Second Derivative Test where applicable.

43. $f(x) = (x - 5)^2$

44. $f(x) = -(x - 5)^2$

45. $f(x) = 6x - x^2$

46. $f(x) = x^2 + 3x - 8$

47. $f(x) = x^3 - 3x^2 + 3$

48. $f(x) = x^3 - 5x^2 + 7x$

49. $f(x) = x^4 - 4x^3 + 2$

50. $f(x) = -x^4 + 4x^3 + 8x^2$

51. $g(x) = x^2(6 - x)^3$

52. $g(x) = -\frac{1}{8}(x + 2)^2(x - 4)^2$

53. $f(x) = x^{2/3} - 3$

54. $f(x) = \sqrt{x^2 + 1}$

55. $f(x) = x + \frac{4}{x}$

56. $f(x) = \frac{x}{x - 1}$

57. $f(x) = \cos x - x, \quad [0, 4\pi]$

58. $f(x) = 2 \sin x + \cos 2x, \quad [0, 2\pi]$

59. $y = \frac{1}{2}x^2 - \ln x$

60. $y = x \ln x$

61. $y = \frac{x}{\ln x}$

62. $y = x^2 \ln \frac{x}{4}$

63. $f(x) = \frac{e^x + e^{-x}}{2}$

64. $g(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-3)^2/2}$

65. $f(x) = x^2 e^{-x}$

66. $f(x) = x e^{-x}$

67. $f(x) = 8x(4 - x)$

68. $y = x^2 \log_3 x$

69. $f(x) = \operatorname{arcsec} x - x$

70. $f(x) = \arcsin x - 2x$

CAS In Exercises 71–74, use a computer algebra system to analyze the function over the given interval. (a) Find the first and second derivatives of the function. (b) Find any relative extrema and points of inflection. (c) Graph f , f' , and f'' on the same set of coordinate axes and state the relationship between the behavior of f and the signs of f' and f'' .

71. $f(x) = 0.2x^2(x - 3)^3, \quad [-1, 4]$

72. $f(x) = x^2 \sqrt{6 - x^2}, \quad [-\sqrt{6}, \sqrt{6}]$

73. $f(x) = \sin x - \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x, \quad [0, \pi]$

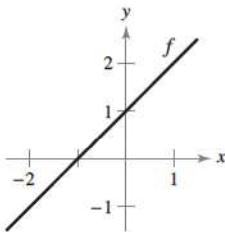
74. $f(x) = \sqrt{2x} \sin x, \quad [0, 2\pi]$

WRITING ABOUT CONCEPTS

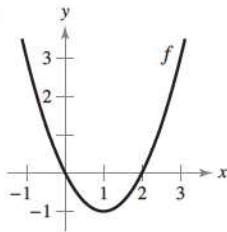
75. Consider a function f such that f' is increasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.
76. Consider a function f such that f' is decreasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.
77. Sketch the graph of a function f that does *not* have a point of inflection at $(c, f(c))$ even though $f''(c) = 0$.
78. S represents weekly sales of a product. What can be said of S' and S'' for each of the following statements?
- The rate of change of sales is increasing.
 - Sales are increasing at a slower rate.
 - The rate of change of sales is constant.
 - Sales are steady.
 - Sales are declining, but at a slower rate.
 - Sales have bottomed out and have started to rise.

In Exercises 79–82, the graph of f is shown. Graph f , f' , and f'' on the same set of coordinate axes. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

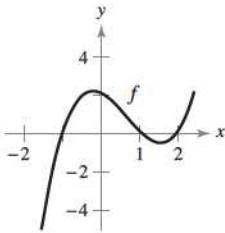
79.



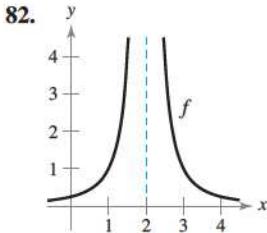
80.



81.



82.



Think About It In Exercises 83–86, sketch the graph of a function f having the given characteristics.

83. $f(2) = f(4) = 0$

$f'(x) < 0$ if $x < 3$

$f'(3)$ does not exist.

$f'(x) > 0$ if $x > 3$

$f''(x) < 0$, $x \neq 3$

85. $f(2) = f(4) = 0$

$f'(x) > 0$ if $x < 3$

$f'(3)$ does not exist.

$f'(x) < 0$ if $x > 3$

$f''(x) > 0$, $x \neq 3$

84. $f(0) = f(2) = 0$

$f'(x) > 0$ if $x < 1$

$f'(1) = 0$

$f'(x) < 0$ if $x > 1$

$f''(x) < 0$

86. $f(0) = f(2) = 0$

$f'(x) < 0$ if $x < 1$

$f'(1) = 0$

$f'(x) > 0$ if $x > 1$

$f''(x) > 0$

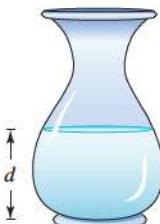
87. **Conjecture** Consider the function $f(x) = (x - 2)^n$.

- (a) Use a graphing utility to graph f for $n = 1, 2, 3$, and 4 . Use the graphs to make a conjecture about the relationship between n and any inflection points of the graph of f .
- (b) Verify your conjecture in part (a).

CAPSTONE

88. **Think About It** Water is running into the vase shown in the figure at a constant rate.

- Graph the depth d of water in the vase as a function of time.
- Does the function have any extrema? Explain.
- Interpret the inflection points of the graph of d .



In Exercises 89 and 90, find a , b , c , and d such that the cubic $f(x) = ax^3 + bx^2 + cx + d$ satisfies the given conditions.

89. Relative maximum: $(3, 3)$ 90. Relative maximum: $(2, 4)$

Relative minimum: $(5, 1)$

Inflection point: $(4, 2)$

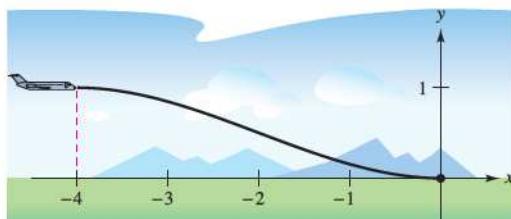
Relative minimum: $(4, 2)$

Inflection point: $(3, 3)$

91. **Aircraft Glide Path** A small aircraft starts its descent from an altitude of 1 mile, 4 miles west of the runway (see figure).

- Find the cubic $f(x) = ax^3 + bx^2 + cx + d$ on the interval $[-4, 0]$ that describes a smooth glide path for the landing.

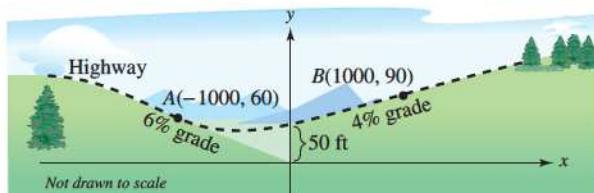
- The function in part (a) models the glide path of the plane. When would the plane be descending at the greatest rate?



FOR FURTHER INFORMATION For more information on this type of modeling, see the article "How Not to Land at Lake Tahoe!" by Richard Barshinger in *The American Mathematical Monthly*. To view this article, go to the website www.matharticles.com.

92. **Highway Design** A section of highway connecting two hillsides with grades of 6% and 4% is to be built between two points that are separated by a horizontal distance of 2000 feet (see figure on the next page). At the point where the two hillsides come together, there is a 50-foot difference in elevation.

- (a) Design a section of highway connecting the hillsides modeled by the function $f(x) = ax^3 + bx^2 + cx + d$ ($-1000 \leq x \leq 1000$). At the points A and B, the slope of the model must match the grade of the hillside.
- (b) Use a graphing utility to graph the model.
- (c) Use a graphing utility to graph the derivative of the model.
- (d) Determine the grade at the steepest part of the transitional section of the highway.



- 93. Beam Deflection** The deflection D of a beam of length L is $D = 2x^4 - 5Lx^3 + 3L^2x^2$, where x is the distance from one end of the beam. Find the value of x that yields the maximum deflection.

- 94. Specific Gravity** A model for the specific gravity of water S is

$$S = \frac{5.755}{10^8}T^3 - \frac{8.521}{10^6}T^2 + \frac{6.540}{10^5}T + 0.99987, \quad 0 < T < 25$$

where T is the water temperature in degrees Celsius.

- CAS** (a) Use a computer algebra system to find the coordinates of the maximum value of the function.
- (b) Sketch a graph of the function over the specified domain. (Use a setting in which $0.996 \leq S \leq 1.001$.)
- (c) Estimate the specific gravity of water when $T = 20^\circ$.

- 95. Average Cost** A manufacturer has determined that the total cost C of operating a factory is $C = 0.5x^2 + 15x + 5000$, where x is the number of units produced. At what level of production will the average cost per unit be minimized? (The average cost per unit is C/x .)

- 96. Modeling Data** The average typing speeds S (in words per minute) of a typing student after t weeks of lessons are shown in the table.

t	5	10	15	20	25	30
S	38	56	79	90	93	94

A model for the data is $S = \frac{100t^2}{65 + t^2}$, $t > 0$.

- (a) Use a graphing utility to plot the data and graph the model.
- (b) Use the second derivative to determine the concavity of S . Compare the result with the graph in part (a).
- (c) What is the sign of the first derivative for $t > 0$? By combining this information with the concavity of the model, what inferences can be made about the typing speed as t increases?

A *Linear and Quadratic Approximations* In Exercises 97–100, use a graphing utility to graph the function. Then graph the linear and quadratic approximations

$$P_1(x) = f(a) + f'(a)(x - a)$$

and

$$P_2(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$$

in the same viewing window. Compare the values of f , P_1 , and P_2 and their first derivatives at $x = a$. How do the approximations change as you move farther away from $x = a$?

Function	Value of a
97. $f(x) = 2(\sin x + \cos x)$	$a = \frac{\pi}{4}$
98. $f(x) = 2(\sin x + \cos x)$	$a = 0$
99. $f(x) = \arctan x$	$a = -1$
100. $f(x) = \frac{\sqrt{x}}{x - 1}$	$a = 2$

True or False? In Exercises 101–106, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

101. The graph of every cubic polynomial has precisely one point of inflection.
102. The graph of $f(x) = 1/x$ is concave downward for $x < 0$ and concave upward for $x > 0$, and thus it has a point of inflection at $x = 0$.
103. The maximum value of $y = 3 \sin x + 2 \cos x$ is 5.
104. The maximum slope of the graph of $y = \sin(bx)$ is b .
105. If $f'(c) > 0$, then f is concave upward at $x = c$.
106. If $f''(2) = 0$, then the graph of f must have a point of inflection at $x = 2$.

In Exercises 107 and 108, let f and g represent differentiable functions such that $f'' \neq 0$ and $g'' \neq 0$.

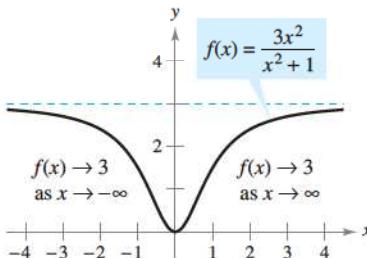
107. Show that if f and g are concave upward on the interval (a, b) , then $f + g$ is also concave upward on (a, b) .
108. Prove that if f and g are positive, increasing, and concave upward on the interval (a, b) , then fg is also concave upward on (a, b) .

- A** 109. Use a graphing utility to graph $y = x \sin(1/x)$. Show that the graph is concave downward to the right of $x = 1/\pi$.
110. (a) Graph $f(x) = \sqrt[3]{x}$ and identify the inflection point.
(b) Does $f''(x)$ exist at the inflection point? Explain.
111. Show that the point of inflection of $f(x) = x(x - 6)^2$ lies midway between the relative extrema of f .
112. Prove that every cubic function with three distinct real zeros has a point of inflection whose x -coordinate is the average of the three zeros.

4.5 Limits at Infinity

- Determine (finite) limits at infinity.
- Determine the horizontal asymptotes, if any, of the graph of a function.
- Determine infinite limits at infinity.

Limits at Infinity



The limit of $f(x)$ as x approaches $-\infty$ or ∞ is 3.

Figure 4.33

This section discusses the “end behavior” of a function on an *infinite* interval. Consider the graph of

$$f(x) = \frac{3x^2}{x^2 + 1}$$

as shown in Figure 4.33. Graphically, you can see that the values of $f(x)$ appear to approach 3 as x increases without bound or decreases without bound. You can come to the same conclusions numerically, as shown in the table.

x	$-\infty \leftarrow$	-100	-10	-1	0	1	10	100	$\rightarrow \infty$
$f(x)$	$3 \leftarrow$	2.9997	2.97	1.5	0	1.5	2.97	2.9997	$\rightarrow 3$

x decreases without bound.	x increases without bound.
$f(x)$ approaches 3.	$f(x)$ approaches 3.

The table suggests that the value of $f(x)$ approaches 3 as x increases without bound ($x \rightarrow \infty$). Similarly, $f(x)$ approaches 3 as x decreases without bound ($x \rightarrow -\infty$). These **limits at infinity** are denoted by

$$\lim_{x \rightarrow -\infty} f(x) = 3 \quad \text{Limit at negative infinity}$$

and

$$\lim_{x \rightarrow \infty} f(x) = 3. \quad \text{Limit at positive infinity}$$

To say that a statement is true as x increases *without bound* means that for some (large) real number M , the statement is true for *all* x in the interval $\{x: x > M\}$. The following definition uses this concept.

DEFINITION OF LIMITS AT INFINITY

Let L be a real number.

1. The statement $\lim_{x \rightarrow \infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $M > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > M$.
2. The statement $\lim_{x \rightarrow -\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $N < 0$ such that $|f(x) - L| < \varepsilon$ whenever $x < N$.

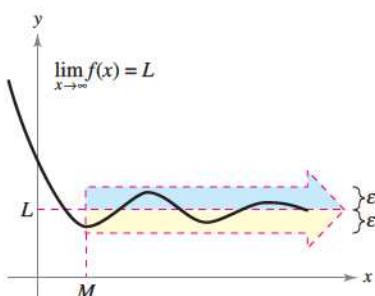


Figure 4.34

The definition of a limit at infinity is shown in Figure 4.34. In this figure, note that for a given positive number ε there exists a positive number M such that, for $x > M$, the graph of f will lie between the horizontal lines given by $y = L + \varepsilon$ and $y = L - \varepsilon$.

EXPLORATION

Use a graphing utility to graph

$$f(x) = \frac{2x^2 + 4x - 6}{3x^2 + 2x - 16}.$$

Describe all the important features of the graph. Can you find a single viewing window that shows all of these features clearly? Explain your reasoning.

What are the horizontal asymptotes of the graph? How far to the right do you have to move on the graph so that the graph is within 0.001 unit of its horizontal asymptote? Explain your reasoning.

Horizontal Asymptotes

In Figure 4.34, the graph of f approaches the line $y = L$ as x increases without bound. The line $y = L$ is called a **horizontal asymptote** of the graph of f .

DEFINITION OF HORIZONTAL ASYMPTOTE

The line $y = L$ is a **horizontal asymptote** of the graph of f if

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow \infty} f(x) = L.$$

Note that from this definition, it follows that the graph of a *function* of x can have at most two horizontal asymptotes—one to the right and one to the left.

Limits at infinity have many of the same properties of limits discussed in Section 2.3. For example, if $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow \infty} g(x)$ both exist, then

$$\lim_{x \rightarrow \infty} [f(x) + g(x)] = \lim_{x \rightarrow \infty} f(x) + \lim_{x \rightarrow \infty} g(x)$$

and

$$\lim_{x \rightarrow \infty} [f(x)g(x)] = [\lim_{x \rightarrow \infty} f(x)][\lim_{x \rightarrow \infty} g(x)].$$

Similar properties hold for limits at $-\infty$.

When evaluating limits at infinity, the following theorem is helpful. (A proof of part 1 of this theorem is given in Appendix A.)

THEOREM 4.10 LIMITS AT INFINITY

1. If r is a positive rational number and c is any real number, then

$$\lim_{x \rightarrow \infty} \frac{c}{x^r} = 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{c}{x^r} = 0.$$

The second limit is valid only if x^r is defined when $x < 0$.

2. $\lim_{x \rightarrow -\infty} e^x = 0$ and $\lim_{x \rightarrow \infty} e^{-x} = 0$

EXAMPLE 1 Evaluating a Limit at Infinity

$$\begin{aligned} \text{a. } \lim_{x \rightarrow \infty} \left(5 - \frac{2}{x^2}\right) &= \lim_{x \rightarrow \infty} 5 - \lim_{x \rightarrow \infty} \frac{2}{x^2} && \text{Property of limits} \\ &= 5 - 0 \\ &= 5 \end{aligned}$$

$$\begin{aligned} \text{b. } \lim_{x \rightarrow \infty} \frac{3}{e^x} &= \lim_{x \rightarrow \infty} 3e^{-x} && \text{Property of limits} \\ &= 3 \lim_{x \rightarrow \infty} e^{-x} \\ &= 3(0) \\ &= 0 \end{aligned}$$

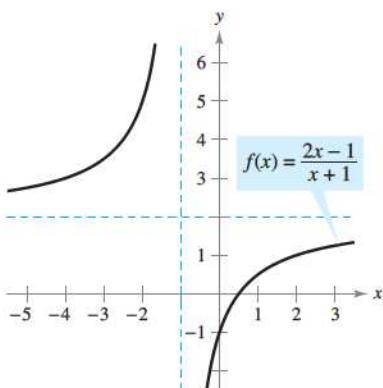
EXAMPLE 2 Evaluating a Limit at Infinity

Find the limit: $\lim_{x \rightarrow \infty} \frac{2x - 1}{x + 1}$.

Solution Note that both the numerator and the denominator approach infinity as x approaches infinity.

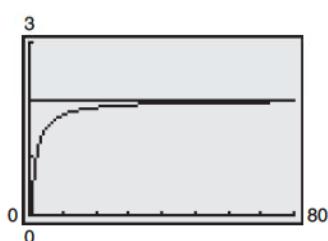
$$\lim_{x \rightarrow \infty} \frac{2x - 1}{x + 1} \quad \begin{array}{l} \nearrow \lim_{x \rightarrow \infty} (2x - 1) \rightarrow \infty \\ \searrow \lim_{x \rightarrow \infty} (x + 1) \rightarrow \infty \end{array}$$

NOTE When you encounter an indeterminate form such as the one in Example 2, you should divide the numerator and denominator by the highest power of x in the *denominator*.



$y = 2$ is a horizontal asymptote.

Figure 4.35



As x increases, the graph of f moves closer and closer to the line $y = 2$.

Figure 4.36

This results in $\frac{\infty}{\infty}$, an **indeterminate form**. To resolve this problem, you can divide both the numerator and the denominator by x . After dividing, the limit may be evaluated as follows.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{2x - 1}{x + 1} &= \lim_{x \rightarrow \infty} \frac{\frac{2x - 1}{x}}{\frac{x + 1}{x}} && \text{Divide numerator and denominator by } x. \\ &= \lim_{x \rightarrow \infty} \frac{2 - \frac{1}{x}}{1 + \frac{1}{x}} && \text{Simplify.} \\ &= \frac{\lim_{x \rightarrow \infty} 2 - \lim_{x \rightarrow \infty} \frac{1}{x}}{\lim_{x \rightarrow \infty} 1 + \lim_{x \rightarrow \infty} \frac{1}{x}} && \text{Take limits of numerator and denominator.} \\ &= \frac{2 - 0}{1 + 0} && \text{Apply Theorem 4.10.} \\ &= 2 \end{aligned}$$

So, the line $y = 2$ is a horizontal asymptote to the right. By taking the limit as $x \rightarrow -\infty$, you can see that $y = 2$ is also a horizontal asymptote to the left. The graph of the function is shown in Figure 4.35. ■

TECHNOLOGY You can test the reasonableness of the limit found in Example 2 by evaluating $f(x)$ for a few large positive values of x . For instance,

$$f(100) \approx 1.9703, \quad f(1000) \approx 1.9970, \quad \text{and} \quad f(10,000) \approx 1.9997.$$

Another way to test the reasonableness of the limit is to use a graphing utility. For instance, in Figure 4.36, the graph of f

$$f(x) = \frac{2x - 1}{x + 1}$$

is shown with the horizontal line $y = 2$. Note that as x increases, the graph of f moves closer and closer to its horizontal asymptote.

EXAMPLE 3 A Comparison of Three Rational Functions

The Granger Collection



MARIA GAETANA AGNESI (1718–1799)

Agnesi was one of a handful of women to receive credit for significant contributions to mathematics before the twentieth century. In her early twenties, she wrote the first text that included both differential and integral calculus. By age 30, she was an honorary member of the faculty at the University of Bologna.

For more information on the contributions of women to mathematics, see the article “Why Women Succeed in Mathematics” by Mona Fabricant, Sylvia Svitak, and Patricia Clark Kenshaft in *Mathematics Teacher*. To view this article, go to the website www.matharticles.com.

Find each limit.

a. $\lim_{x \rightarrow \infty} \frac{2x + 5}{3x^2 + 1}$

b. $\lim_{x \rightarrow \infty} \frac{2x^2 + 5}{3x^2 + 1}$

c. $\lim_{x \rightarrow \infty} \frac{2x^3 + 5}{3x^2 + 1}$

Solution In each case, attempting to evaluate the limit produces the indeterminate form ∞/∞ .

a. Divide both the numerator and the denominator by x^2 .

$$\lim_{x \rightarrow \infty} \frac{2x + 5}{3x^2 + 1} = \lim_{x \rightarrow \infty} \frac{(2/x) + (5/x^2)}{3 + (1/x^2)} = \frac{0 + 0}{3 + 0} = \frac{0}{3} = 0$$

b. Divide both the numerator and the denominator by x^2 .

$$\lim_{x \rightarrow \infty} \frac{2x^2 + 5}{3x^2 + 1} = \lim_{x \rightarrow \infty} \frac{2 + (5/x^2)}{3 + (1/x^2)} = \frac{2 + 0}{3 + 0} = \frac{2}{3}$$

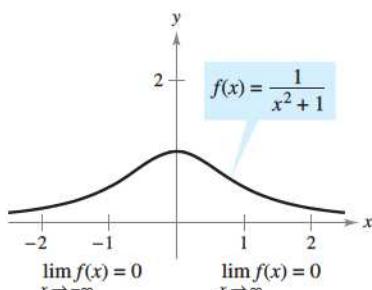
c. Divide both the numerator and the denominator by x^2 .

$$\lim_{x \rightarrow \infty} \frac{2x^3 + 5}{3x^2 + 1} = \lim_{x \rightarrow \infty} \frac{2x + (5/x^2)}{3 + (1/x^2)} = \frac{\infty}{3}$$

You can conclude that the limit *does not exist* because the numerator increases without bound while the denominator approaches 3. ■

GUIDELINES FOR FINDING LIMITS AT $\pm\infty$ OF RATIONAL FUNCTIONS

- If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is 0.
- If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
- If the degree of the numerator is *greater than* the degree of the denominator, then the limit of the rational function does not exist.



f has a horizontal asymptote at $y = 0$.
Figure 4.37

Use these guidelines to check the results in Example 3. These limits seem reasonable when you consider that for large values of x , the highest-power term of the rational function is the most “influential” in determining the limit. For instance, the limit as x approaches infinity of the function

$$f(x) = \frac{1}{x^2 + 1}$$

is 0 because the denominator overpowers the numerator as x increases or decreases without bound, as shown in Figure 4.37.

The function shown in Figure 4.37 is a special case of a type of curve studied by the Italian mathematician Maria Gaetana Agnesi. The general form of this function is

$$f(x) = \frac{8a^3}{x^2 + 4a^2} \quad \text{Witch of Agnesi}$$

and, through a mistranslation of the Italian word *vertéré*, the curve has come to be known as the Witch of Agnesi. Agnesi’s work with this curve first appeared in a comprehensive text on calculus that was published in 1748.

In Figure 4.37, you can see that the function

$$f(x) = \frac{1}{x^2 + 1}$$

approaches the same horizontal asymptote to the right and to the left. This is always true of rational functions. Functions that are not rational, however, may approach different horizontal asymptotes to the right and to the left. A common example of such a function is the **logistic function** shown in the next example.

EXAMPLE 4 A Function with Two Horizontal Asymptotes

Show that the *logistic function*

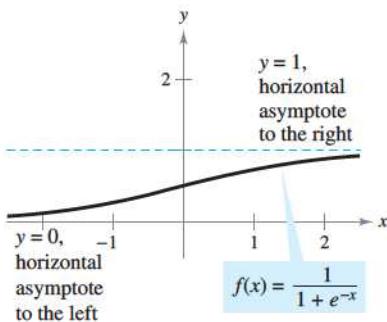
$$f(x) = \frac{1}{1 + e^{-x}}$$

has different horizontal asymptotes to the left and to the right.

Solution To begin, try using a graphing utility to graph the function. From Figure 4.38 it appears that

$$y = 0 \quad \text{and} \quad y = 1$$

are horizontal asymptotes to the left and to the right, respectively. The following table shows the same results numerically.



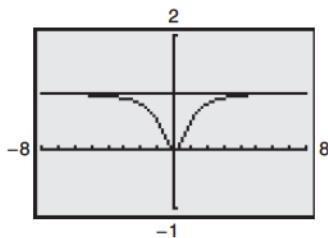
Functions that are not rational may have different right and left horizontal asymptotes.

Figure 4.38

Finally, you can obtain the same results analytically, as follows.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{1}{1 + e^{-x}} &= \frac{\lim_{x \rightarrow \infty} 1}{\lim_{x \rightarrow \infty} (1 + e^{-x})} \\ &= \frac{1}{1 + 0} \\ &= 1 \end{aligned} \quad y = 1 \text{ is a horizontal asymptote to the right.}$$

For the horizontal asymptote to the left, note that as $x \rightarrow -\infty$ the denominator of $1/(1 + e^{-x})$ approaches infinity. So, the quotient approaches 0 and thus the limit is 0. You can conclude that $y = 0$ is a horizontal asymptote to the left. ■



The horizontal asymptote appears to be the line $y = 1$ but it is actually the line $y = 2$.

Figure 4.39

TECHNOLOGY PITFALL If you use a graphing utility to help estimate a limit, be sure that you also confirm the estimate analytically—the pictures shown by a graphing utility can be misleading. For instance, Figure 4.39 shows one view of the graph of

$$y = \frac{2x^3 + 1000x^2 + x}{x^3 + 1000x^2 + x + 1000}.$$

From this view, one could be convinced that the graph has $y = 1$ as a horizontal asymptote. An analytical approach shows that the horizontal asymptote is actually $y = 2$. Confirm this by enlarging the viewing window on the graphing utility.

In Section 2.3 (Example 9), you saw how the Squeeze Theorem can be used to evaluate limits involving trigonometric functions. This theorem is also valid for limits at infinity.

EXAMPLE 5 Limits Involving Trigonometric Functions

Find each limit.

a. $\lim_{x \rightarrow \infty} \sin x$ b. $\lim_{x \rightarrow \infty} \frac{\sin x}{x}$

Solution

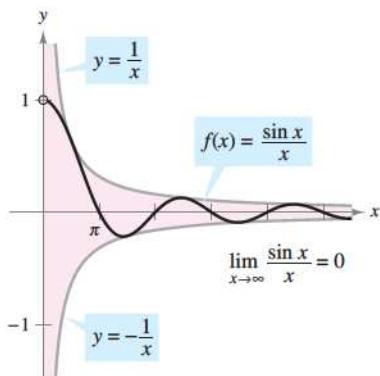
- a. As x approaches infinity, the sine function oscillates between 1 and -1 . So, this limit does not exist.
 b. Because $-1 \leq \sin x \leq 1$, it follows that for $x > 0$,

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

where $\lim_{x \rightarrow \infty} (-1/x) = 0$ and $\lim_{x \rightarrow \infty} (1/x) = 0$. So, by the Squeeze Theorem, you can obtain

$$\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$$

as shown in Figure 4.40.



As x increases without bound, $f(x)$ approaches 0.

Figure 4.40

EXAMPLE 6 Oxygen Level in a Pond

Suppose that $f(t)$ measures the level of oxygen in a pond, where $f(t) = 1$ is the normal (unpolluted) level and the time t is measured in weeks. When $t = 0$, organic waste is dumped into the pond, and as the waste material oxidizes, the level of oxygen in the pond is

$$f(t) = \frac{t^2 - t + 1}{t^2 + 1}.$$

What percent of the normal level of oxygen exists in the pond after 1 week? After 2 weeks? After 10 weeks? What is the limit as t approaches infinity?

Solution When $t = 1, 2$, and 10 , the levels of oxygen are as shown.

$$f(1) = \frac{1^2 - 1 + 1}{1^2 + 1} = \frac{1}{2} = 50\% \quad 1 \text{ week}$$

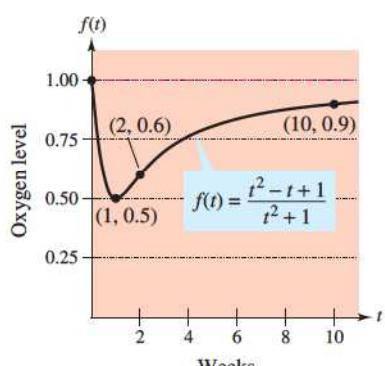
$$f(2) = \frac{2^2 - 2 + 1}{2^2 + 1} = \frac{3}{5} = 60\% \quad 2 \text{ weeks}$$

$$f(10) = \frac{10^2 - 10 + 1}{10^2 + 1} = \frac{91}{101} \approx 90.1\% \quad 10 \text{ weeks}$$

To find the limit as t approaches infinity, you can use the guidelines on page 241, or divide the numerator and the denominator by t^2 to obtain

$$\lim_{t \rightarrow \infty} \frac{t^2 - t + 1}{t^2 + 1} = \lim_{t \rightarrow \infty} \frac{1 - (1/t) + (1/t^2)}{1 + (1/t^2)} = \frac{1 - 0 + 0}{1 + 0} = 1 = 100\%.$$

See Figure 4.41. ■



The level of oxygen in a pond approaches the normal level of 1 as t approaches ∞ .

Figure 4.41

Infinite Limits at Infinity

Many functions do not approach a finite limit as x increases (or decreases) without bound. For instance, no polynomial function has a finite limit at infinity. The following definition is used to describe the behavior of polynomial and other functions at infinity.

NOTE Determining whether a function has an infinite limit at infinity is useful in analyzing the “end behavior” of its graph. You will see examples of this in Section 4.6 on curve sketching.

DEFINITION OF INFINITE LIMITS AT INFINITY

Let f be a function defined on the interval (a, ∞) .

1. The statement $\lim_{x \rightarrow \infty} f(x) = \infty$ means that for each positive number M , there is a corresponding number $N > 0$ such that $f(x) > M$ whenever $x > N$.
2. The statement $\lim_{x \rightarrow \infty} f(x) = -\infty$ means that for each negative number M , there is a corresponding number $N > 0$ such that $f(x) < M$ whenever $x > N$.

Similar definitions can be given for the statements $\lim_{x \rightarrow -\infty} f(x) = \infty$ and $\lim_{x \rightarrow -\infty} f(x) = -\infty$.

EXAMPLE 7 Finding Infinite Limits at Infinity

Find each limit.

a. $\lim_{x \rightarrow \infty} x^3$ b. $\lim_{x \rightarrow -\infty} x^3$

Solution

- a. As x increases without bound, x^3 also increases without bound. So, you can write $\lim_{x \rightarrow \infty} x^3 = \infty$.
- b. As x decreases without bound, x^3 also decreases without bound. So, you can write $\lim_{x \rightarrow -\infty} x^3 = -\infty$.

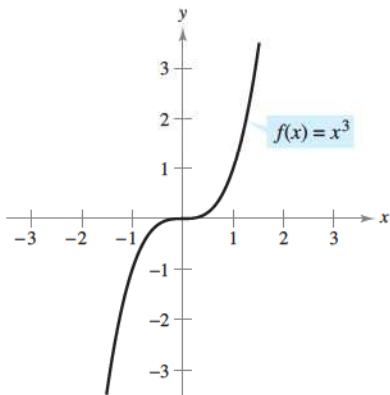


Figure 4.42

The graph of $f(x) = x^3$ in Figure 4.42 illustrates these two results. These results agree with the Leading Coefficient Test for polynomial functions as described in Section 1.3.

EXAMPLE 8 Finding Infinite Limits at Infinity

Find each limit.

a. $\lim_{x \rightarrow \infty} \frac{2x^2 - 4x}{x + 1}$ b. $\lim_{x \rightarrow -\infty} \frac{2x^2 - 4x}{x + 1}$

Solution One way to evaluate each of these limits is to use long division to rewrite the improper rational function as the sum of a polynomial and a rational function.

- a. $\lim_{x \rightarrow \infty} \frac{2x^2 - 4x}{x + 1} = \lim_{x \rightarrow \infty} \left(2x - 6 + \frac{6}{x + 1} \right) = \infty$
- b. $\lim_{x \rightarrow -\infty} \frac{2x^2 - 4x}{x + 1} = \lim_{x \rightarrow -\infty} \left(2x - 6 + \frac{6}{x + 1} \right) = -\infty$

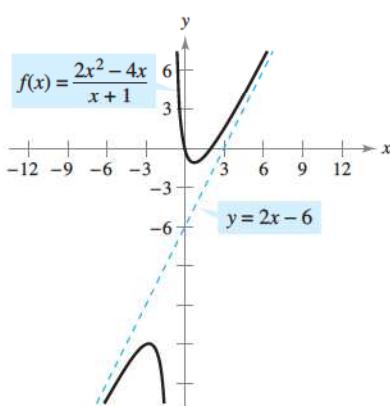


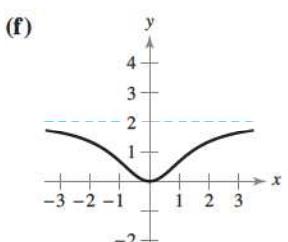
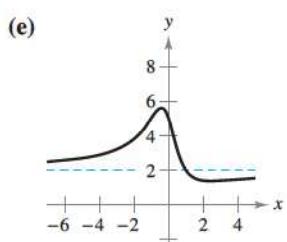
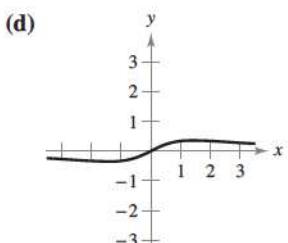
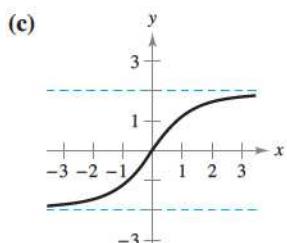
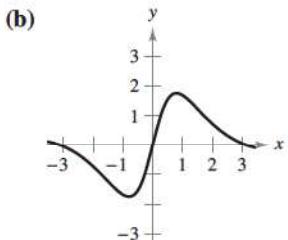
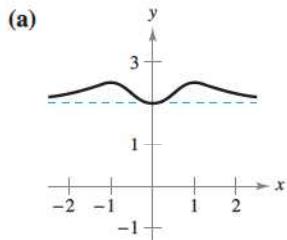
Figure 4.43

The statements above can be interpreted as saying that as x approaches $\pm\infty$, the function $f(x) = (2x^2 - 4x)/(x + 1)$ behaves like the function $g(x) = 2x - 6$. In Section 4.6, you will see that this is graphically described by saying that the line $y = 2x - 6$ is a slant asymptote of the graph of f , as shown in Figure 4.43. ■

4.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, match the function with one of the graphs [(a), (b), (c), (d), (e), or (f)] using horizontal asymptotes as an aid.



$$1. f(x) = \frac{2x^2}{x^2 + 2}$$

$$3. f(x) = \frac{x}{x^2 + 2}$$

$$5. f(x) = \frac{4 \sin x}{x^2 + 1}$$

$$2. f(x) = \frac{2x}{\sqrt{x^2 + 2}}$$

$$4. f(x) = 2 + \frac{x^2}{x^4 + 1}$$

$$6. f(x) = \frac{2x^2 - 3x + 5}{x^2 + 1}$$

Numerical and Graphical Analysis In Exercises 7–12, use a graphing utility to complete the table and estimate the limit as x approaches infinity. Then use a graphing utility to graph the function and estimate the limit graphically.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

$$7. f(x) = \frac{4x + 3}{2x - 1}$$

$$8. f(x) = \frac{2x^2}{x + 1}$$

$$9. f(x) = \frac{-6x}{\sqrt{4x^2 + 5}}$$

$$10. f(x) = \frac{20x}{\sqrt{9x^2 - 1}}$$

$$11. f(x) = 5 - \frac{1}{x^2 + 1}$$

$$12. f(x) = 4 + \frac{3}{x^2 + 2}$$

In Exercises 13 and 14, find $\lim_{x \rightarrow \infty} h(x)$, if possible.

$$13. f(x) = 5x^3 - 3x^2 + 10x \quad 14. f(x) = -4x^2 + 2x - 5$$

$$(a) h(x) = \frac{f(x)}{x^2}$$

$$(a) h(x) = \frac{f(x)}{x}$$

$$(b) h(x) = \frac{f(x)}{x^3}$$

$$(b) h(x) = \frac{f(x)}{x^2}$$

$$(c) h(x) = \frac{f(x)}{x^4}$$

$$(c) h(x) = \frac{f(x)}{x^3}$$

In Exercises 15–18, find each limit, if possible.

$$15. (a) \lim_{x \rightarrow \infty} \frac{x^2 + 2}{x^3 - 1}$$

$$16. (a) \lim_{x \rightarrow \infty} \frac{3 - 2x}{3x^3 - 1}$$

$$(b) \lim_{x \rightarrow \infty} \frac{x^2 + 2}{x^2 - 1}$$

$$(b) \lim_{x \rightarrow \infty} \frac{3 - 2x}{3x - 1}$$

$$(c) \lim_{x \rightarrow \infty} \frac{x^2 + 2}{x - 1}$$

$$(c) \lim_{x \rightarrow \infty} \frac{3 - 2x^2}{3x - 1}$$

$$17. (a) \lim_{x \rightarrow \infty} \frac{5 - 2x^{3/2}}{3x^2 - 4}$$

$$18. (a) \lim_{x \rightarrow \infty} \frac{5x^{3/2}}{4x^2 + 1}$$

$$(b) \lim_{x \rightarrow \infty} \frac{5 - 2x^{3/2}}{3x^{3/2} - 4}$$

$$(b) \lim_{x \rightarrow \infty} \frac{5x^{3/2}}{4x^{3/2} + 1}$$

$$(c) \lim_{x \rightarrow \infty} \frac{5 - 2x^{3/2}}{3x - 4}$$

$$(c) \lim_{x \rightarrow \infty} \frac{5x^{3/2}}{4\sqrt{x} + 1}$$

In Exercises 19–44, find the limit.

$$19. \lim_{x \rightarrow \infty} \left(4 + \frac{3}{x} \right)$$

$$20. \lim_{x \rightarrow -\infty} \left(\frac{5}{x} - \frac{x}{3} \right)$$

$$21. \lim_{x \rightarrow \infty} \frac{2x - 1}{3x + 2}$$

$$22. \lim_{x \rightarrow \infty} \frac{x^2 + 3}{2x^2 - 1}$$

$$23. \lim_{x \rightarrow \infty} \frac{x}{x^2 - 1}$$

$$24. \lim_{x \rightarrow \infty} \frac{5x^3 + 1}{10x^3 - 3x^2 + 7}$$

$$25. \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 - x}}$$

$$26. \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 + 1}}$$

$$27. \lim_{x \rightarrow \infty} \frac{2x + 1}{\sqrt{x^2 - x}}$$

$$28. \lim_{x \rightarrow -\infty} \frac{-3x + 1}{\sqrt{x^2 + x}}$$

$$29. \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - 1}}{2x - 1}$$

$$30. \lim_{x \rightarrow -\infty} \frac{\sqrt{x^4 - 1}}{x^3 - 1}$$

$$31. \lim_{x \rightarrow \infty} \frac{x + 1}{(x^2 + 1)^{1/3}}$$

$$32. \lim_{x \rightarrow -\infty} \frac{2x}{(x^6 - 1)^{1/3}}$$

$$33. \lim_{x \rightarrow \infty} \frac{1}{2x + \sin x}$$

$$34. \lim_{x \rightarrow \infty} \cos \frac{1}{x}$$

$$35. \lim_{x \rightarrow \infty} \frac{\sin 2x}{x}$$

$$36. \lim_{x \rightarrow \infty} \frac{x - \cos x}{x}$$

$$37. \lim_{x \rightarrow \infty} (2 - 5e^{-x})$$

$$38. \lim_{x \rightarrow -\infty} (2 + 5e^x)$$

$$39. \lim_{x \rightarrow -\infty} \frac{3}{1 + 2e^x}$$

$$40. \lim_{x \rightarrow \infty} \frac{8}{4 - 10^{-x/2}}$$

$$41. \lim_{x \rightarrow \infty} \log_{10}(1 + 10^{-x})$$

$$42. \lim_{x \rightarrow \infty} \left[\frac{5}{2} + \ln \left(\frac{x^2 + 1}{x^2} \right) \right]$$

$$43. \lim_{t \rightarrow \infty} (8t^{-1} - \arctan t)$$

$$44. \lim_{u \rightarrow \infty} \operatorname{arcsec}(u + 1)$$

In Exercises 45–48, use a graphing utility to graph the function and identify any horizontal asymptotes.

45. $f(x) = \frac{|x|}{x+1}$

46. $f(x) = \frac{|3x+2|}{x-2}$

47. $f(x) = \frac{3x}{\sqrt{x^2+2}}$

48. $f(x) = \frac{\sqrt{9x^2-2}}{2x+1}$

In Exercises 49 and 50, find the limit. (Hint: Let $x = 1/t$ and find the limit as $t \rightarrow 0^+$.)

49. $\lim_{x \rightarrow \infty} x \sin \frac{1}{x}$

50. $\lim_{x \rightarrow \infty} x \tan \frac{1}{x}$

In Exercises 51–54, find the limit. (Hint: Treat the expression as a fraction whose denominator is 1, and rationalize the numerator.) Use a graphing utility to verify your result.

51. $\lim_{x \rightarrow -\infty} (x + \sqrt{x^2 + 3})$

52. $\lim_{x \rightarrow \infty} (2x - \sqrt{4x^2 + 1})$

53. $\lim_{x \rightarrow -\infty} (3x + \sqrt{9x^2 - x})$

54. $\lim_{x \rightarrow \infty} (4x - \sqrt{16x^2 - x})$

Numerical, Graphical, and Analytic Analysis In Exercises 55–58, use a graphing utility to complete the table and estimate the limit as x approaches infinity. Then use a graphing utility to graph the function and estimate the limit. Finally, find the limit analytically and compare your results with the estimates.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

55. $f(x) = x - \sqrt{x(x-1)}$

56. $f(x) = x^2 - x\sqrt{x(x-1)}$

57. $f(x) = x \sin \frac{1}{2x}$

58. $f(x) = \frac{x+1}{x\sqrt{x}}$

WRITING ABOUT CONCEPTS

In Exercises 59 and 60, describe in your own words what the statement means.

59. $\lim_{x \rightarrow \infty} f(x) = 4$

60. $\lim_{x \rightarrow -\infty} f(x) = 2$

61. Sketch a graph of a differentiable function f that satisfies the following conditions and has $x = 2$ as its only critical number.

$$f'(x) < 0 \text{ for } x < 2 \quad f'(x) > 0 \text{ for } x > 2$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow \infty} f(x) = 6$$

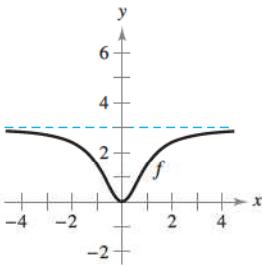
62. Is it possible to sketch a graph of a function that satisfies the conditions of Exercise 61 and has no points of inflection? Explain.

63. If f is a continuous function such that $\lim_{x \rightarrow \infty} f(x) = 5$, find, if possible, $\lim_{x \rightarrow -\infty} f(x)$ for each specified condition.

- (a) The graph of f is symmetric with respect to the y -axis.
- (b) The graph of f is symmetric with respect to the origin.

CAPSTONE

64. The graph of a function f is shown below. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.



(a) Sketch f' .

(b) Use the graphs to estimate $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow \infty} f'(x)$.

(c) Explain the answers you gave in part (b).

In Exercises 65–82, sketch the graph of the equation. Look for extrema, intercepts, symmetry, and asymptotes as necessary. Use a graphing utility to verify your result.

65. $y = \frac{x}{1-x}$

66. $y = \frac{x-4}{x-3}$

67. $y = \frac{x+1}{x^2-4}$

68. $y = \frac{2x}{9-x^2}$

69. $y = \frac{x^2}{x^2+16}$

70. $y = \frac{x^2}{x^2-16}$

71. $y = \frac{2x^2}{x^2-4}$

72. $y = \frac{2x^2}{x^2+4}$

73. $xy^2 = 9$

74. $x^2y = 9$

75. $y = \frac{3x}{1-x}$

76. $y = \frac{3x}{1-x^2}$

77. $y = 2 - \frac{3}{x^2}$

78. $y = 1 + \frac{1}{x}$

79. $y = 3 + \frac{2}{x}$

80. $y = 4\left(1 - \frac{1}{x^2}\right)$

81. $y = \frac{x^3}{\sqrt{x^2-4}}$

82. $y = \frac{x}{\sqrt{x^2-4}}$

In Exercises 83–92, use a computer algebra system to analyze the graph of the function. Label any extrema and/or asymptotes that exist.

83. $f(x) = 9 - \frac{5}{x^2}$

84. $f(x) = \frac{1}{x^2 - x - 2}$

85. $f(x) = \frac{x-2}{x^2-4x+3}$

86. $f(x) = \frac{x+1}{x^2+x+1}$

87. $f(x) = \frac{3x}{\sqrt{4x^2+1}}$

88. $g(x) = \frac{2x}{\sqrt{3x^2+1}}$

89. $g(x) = \sin\left(\frac{x}{x-2}\right), \quad x > 3$

90. $f(x) = \frac{2 \sin 2x}{x}$

91. $f(x) = 2 + (x^2 - 3)e^{-x}$

92. $f(x) = \frac{10 \ln x}{x^2 \sqrt{x}}$

In Exercises 93 and 94, (a) use a graphing utility to graph f and g in the same viewing window, (b) verify algebraically that f and g represent the same function, and (c) zoom out sufficiently far so that the graph appears as a line. What equation does this line appear to have? (Note that the points at which the function is not continuous are not readily seen when you zoom out.)

93. $f(x) = \frac{x^3 - 3x^2 + 2}{x(x-3)}$

94. $f(x) = -\frac{x^3 - 2x^2 + 2}{2x^2}$

$$g(x) = x + \frac{2}{x(x-3)}$$

$$g(x) = -\frac{1}{2}x + 1 - \frac{1}{x^2}$$

95. **Engine Efficiency** The efficiency of an internal combustion engine is

$$\text{Efficiency (\%)} = 100 \left[1 - \frac{1}{(v_1/v_2)^c} \right]$$

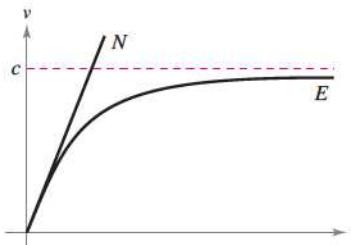
where v_1/v_2 is the ratio of the uncompressed gas to the compressed gas and c is a positive constant dependent on the engine design. Find the limit of the efficiency as the compression ratio approaches infinity.

96. **Average Cost** A business has a cost of $C = 0.5x + 500$ for producing x units. The average cost per unit is

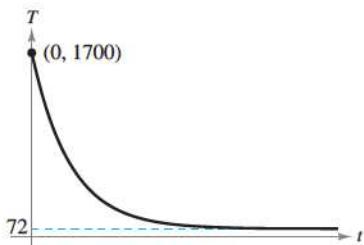
$$\bar{C} = \frac{C}{x}$$

Find the limit of \bar{C} as x approaches infinity.

97. **Physics** Newton's First Law of Motion and Einstein's Special Theory of Relativity differ concerning a particle's behavior as its velocity approaches the speed of light c . In the graph, functions N and E represent the velocity v , with respect to time t , of a particle accelerated by a constant force as predicted by Newton and Einstein. Write limit statements that describe these two theories.



98. **Temperature** The graph shows the temperature T , in degrees Fahrenheit, of molten glass t seconds after it is removed from a kiln.



(a) Find $\lim_{t \rightarrow 0^+} T$. What does this limit represent?

(b) Find $\lim_{t \rightarrow \infty} T$. What does this limit represent?

(c) Will the temperature of the glass ever actually reach room temperature? Why?

99. **Modeling Data** A heat probe is attached to the heat exchanger of a heating system. The temperature T (in degrees Celsius) is recorded t seconds after the furnace is started. The results for the first 2 minutes are recorded in the table.

<i>t</i>	0	15	30	45	60
<i>T</i>	25.2°	36.9°	45.5°	51.4°	56.0°

<i>t</i>	75	90	105	120
<i>T</i>	59.6°	62.0°	64.0°	65.2°

(a) Use the regression capabilities of a graphing utility to find a model of the form $T_1 = at^2 + bt + c$ for the data.

(b) Use a graphing utility to graph T_1 .

(c) A rational model for the data is $T_2 = \frac{1451 + 86t}{58 + t}$. Use a graphing utility to graph T_2 .

(d) Find $T_1(0)$ and $T_2(0)$.

(e) Find $\lim_{t \rightarrow \infty} T_2$.

(f) Interpret the result in part (e) in the context of the problem. Is it possible to do this type of analysis using T_1 ? Explain.

100. **Modeling Data** A container holds 5 liters of a 25% brine solution. The table shows the concentrations C of the mixture after adding x liters of a 75% brine solution to the container.

<i>x</i>	0	0.5	1	1.5	2
<i>C</i>	0.25	0.295	0.333	0.365	0.393

<i>x</i>	2.5	3	3.5	4
<i>C</i>	0.417	0.438	0.456	0.472

(a) Use the regression features of a graphing utility to find a model of the form $C_1 = ax^2 + bx + c$ for the data.

(b) Use a graphing utility to graph C_1 .

(c) A rational model for these data is $C_2 = \frac{5 + 3x}{20 + 4x}$. Use a graphing utility to graph C_2 .

(d) Find $\lim_{x \rightarrow \infty} C_1$ and $\lim_{x \rightarrow \infty} C_2$. Which model do you think best represents the concentration of the mixture? Explain.

(e) What is the limiting concentration?

101. **Timber Yield** The yield V (in millions of cubic feet per acre) for a stand of timber at age t (in years) is $V = 7.1e^{(-48.1)/t}$.

(a) Find the limiting volume of wood per acre as t approaches infinity.

(b) Find the rates at which the yield is changing when $t = 20$ years and $t = 60$ years.

- 102. Learning Theory** In a group project in learning theory, a mathematical model for the proportion P of correct responses after n trials was found to be

$$P = \frac{0.83}{1 + e^{-0.2n}}.$$

- (a) Find the limiting proportion of correct responses as n approaches infinity.
 (b) Find the rates at which P is changing after $n = 3$ trials and $n = 10$ trials.

- 103. Writing** Consider the function $f(x) = \frac{2}{1 + e^{1/x}}$.

- (a) Use a graphing utility to graph f .
 (b) Write a short paragraph explaining why the graph has a horizontal asymptote at $y = 1$ and why the function has a nonremovable discontinuity at $x = 0$.

- 104. Writing** In your own words, state the guidelines for finding the limit of a rational function. Give examples.

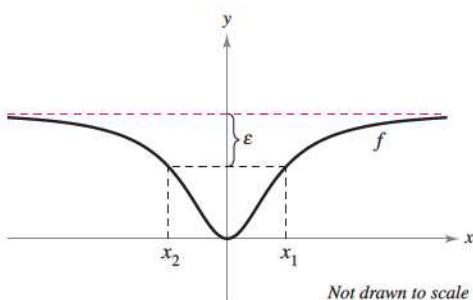
- 105. A line with slope m** passes through the point $(0, 4)$.

- (a) Write the distance d between the line and the point $(3, 1)$ as a function of m .
 (b) Use a graphing utility to graph the equation in part (a).
 (c) Find $\lim_{m \rightarrow \infty} d(m)$ and $\lim_{m \rightarrow -\infty} d(m)$. Interpret the results geometrically.

- 106. A line with slope m** passes through the point $(0, -2)$.

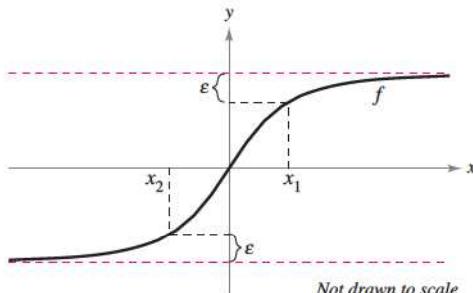
- (a) Write the distance d between the line and the point $(4, 2)$ as a function of m .
 (b) Use a graphing utility to graph the equation in part (a).
 (c) Find $\lim_{m \rightarrow \infty} d(m)$ and $\lim_{m \rightarrow -\infty} d(m)$. Interpret the results geometrically.

- 107. The graph of $f(x) = \frac{2x^2}{x^2 + 2}$** is shown.



- (a) Find $L = \lim_{x \rightarrow \infty} f(x)$.
 (b) Determine x_1 and x_2 in terms of ε .
 (c) Determine M , where $M > 0$, such that $|f(x) - L| < \varepsilon$ for $x > M$.
 (d) Determine N , where $N < 0$, such that $|f(x) - L| < \varepsilon$ for $x < N$.

- 108.** The graph of $f(x) = \frac{6x}{\sqrt{x^2 + 2}}$ is shown.



- (a) Find $L = \lim_{x \rightarrow \infty} f(x)$ and $K = \lim_{x \rightarrow -\infty} f(x)$.
 (b) Determine x_1 and x_2 in terms of ε .
 (c) Determine M , where $M > 0$, such that $|f(x) - L| < \varepsilon$ for $x > M$.
 (d) Determine N , where $N < 0$, such that $|f(x) - K| < \varepsilon$ for $x < N$.

- 109.** Consider $\lim_{x \rightarrow \infty} \frac{3x}{\sqrt{x^2 + 3}}$. Use the definition of limits at infinity to find values of M that correspond to (a) $\varepsilon = 0.5$ and (b) $\varepsilon = 0.1$.

- 110.** Consider $\lim_{x \rightarrow -\infty} \frac{3x}{\sqrt{x^2 + 3}}$. Use the definition of limits at infinity to find values of N that correspond to (a) $\varepsilon = 0.5$ and (b) $\varepsilon = 0.1$.

In Exercises 111–114, use the definition of limits at infinity to prove the limit.

$$\lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$$

$$\lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{1}{x^3} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{1}{x-2} = 0$$

- 115.** Prove that if $p(x) = a_n x^n + \dots + a_1 x + a_0$ and $q(x) = b_m x^m + \dots + b_1 x + b_0$ ($a_n \neq 0, b_m \neq 0$), then

$$\lim_{x \rightarrow \infty} \frac{p(x)}{q(x)} = \begin{cases} 0, & n < m \\ \frac{a_n}{b_m}, & n = m \\ \pm\infty, & n > m \end{cases}$$

- 116.** Use the definition of infinite limits at infinity to prove that $\lim_{x \rightarrow \infty} x^3 = \infty$.

True or False? In Exercises 117 and 118, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

- 117.** If $f'(x) > 0$ for all real numbers x , then f increases without bound.

- 118.** If $f''(x) < 0$ for all real numbers x , then f decreases without bound.

4.6 A Summary of Curve Sketching

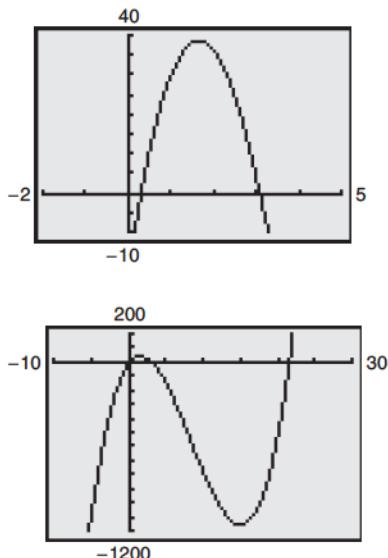
- Analyze and sketch the graph of a function.

Analyzing the Graph of a Function

It would be difficult to overstate the importance of using graphs in mathematics. Descartes's introduction of analytic geometry contributed significantly to the rapid advances in calculus that began during the mid-seventeenth century. In the words of Lagrange, "As long as algebra and geometry traveled separate paths, their advance was slow and their applications limited. But when these two sciences joined company, they drew from each other fresh vitality and thenceforth marched on at a rapid pace toward perfection."

So far, you have studied several concepts that are useful in analyzing the graph of a function.

- x -intercepts and y -intercepts (Section 1.1)
- Symmetry (Section 1.1)
- Domain and range (Section 1.3)
- Continuity (Section 2.4)
- Vertical asymptotes (Section 2.5)
- Differentiability (Section 3.1)
- Relative extrema (Section 4.1)
- Concavity (Section 4.4)
- Points of inflection (Section 4.4)
- Horizontal asymptotes (Section 4.5)
- Infinite limits at infinity (Section 4.5)



Different viewing windows for the graph of $f(x) = x^3 - 25x^2 + 74x - 20$

Figure 4.44

When you are sketching the graph of a function, either by hand or with a graphing utility, remember that normally you cannot show the *entire* graph. The decision as to which part of the graph you choose to show is often crucial. For instance, which of the viewing windows in Figure 4.44 better represents the graph of

$$f(x) = x^3 - 25x^2 + 74x - 20?$$

By seeing both views, it is clear that the second viewing window gives a more complete representation of the graph. But would a third viewing window reveal other interesting portions of the graph? To answer this, you need to use calculus to interpret the first and second derivatives. Here are some guidelines for determining a good viewing window for the graph of a function.

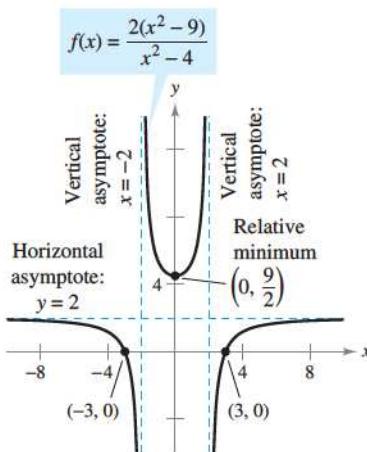
GUIDELINES FOR ANALYZING THE GRAPH OF A FUNCTION

1. Determine the domain and range of the function.
2. Determine the intercepts, asymptotes, and symmetry of the graph.
3. Locate the x -values for which $f'(x)$ and $f''(x)$ either are zero or do not exist. Use the results to determine relative extrema and points of inflection.

NOTE In these guidelines, note the importance of *algebra* (as well as calculus) for solving the equations $f(x) = 0$, $f'(x) = 0$, and $f''(x) = 0$.

EXAMPLE 1 Sketching the Graph of a Rational Function

Analyze and sketch the graph of $f(x) = \frac{2(x^2 - 9)}{x^2 - 4}$.

Solution

Using calculus, you can be certain that you have determined all characteristics of the graph of f .

Figure 4.45

FOR FURTHER INFORMATION For more information on the use of technology to graph rational functions, see the article "Graphs of Rational Functions for Computer Assisted Calculus" by Stan Byrd and Terry Walters in *The College Mathematics Journal*. To view this article, go to the website www.matharticles.com.

First derivative: $f'(x) = \frac{20x}{(x^2 - 4)^2}$

Second derivative: $f''(x) = \frac{-20(3x^2 + 4)}{(x^2 - 4)^3}$

x -intercepts: $(-3, 0), (3, 0)$

y -intercept: $(0, \frac{9}{2})$

Vertical asymptotes: $x = -2, x = 2$

Horizontal asymptote: $y = 2$

Critical number: $x = 0$

Possible points of inflection: None

Domain: All real numbers except $x = \pm 2$

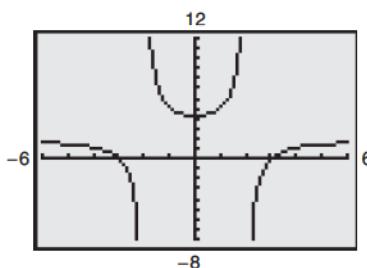
Symmetry: With respect to y -axis

Test intervals: $(-\infty, -2), (-2, 0), (0, 2), (2, \infty)$

The table shows how the test intervals are used to determine several characteristics of the graph. The graph of f is shown in Figure 4.45.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < -2$		—	—	Decreasing, concave downward
$x = -2$	Undef.	Undef.	Undef.	Vertical asymptote
$-2 < x < 0$		—	+	Decreasing, concave upward
$x = 0$	$\frac{9}{2}$	0	+	Relative minimum
$0 < x < 2$		+	+	Increasing, concave upward
$x = 2$	Undef.	Undef.	Undef.	Vertical asymptote
$2 < x < \infty$		+	—	Increasing, concave downward

Be sure you understand all of the implications of creating a table such as that shown in Example 1. By using calculus, you can be sure that the graph has no relative extrema or points of inflection other than those shown in Figure 4.45.



By not using calculus you may overlook important characteristics of the graph of g .

TECHNOLOGY PITFALL Without using the type of analysis outlined in Example 1, it is easy to obtain an incomplete view of a graph's basic characteristics. For instance, Figure 4.46 shows a view of the graph of

$$g(x) = \frac{2(x^2 - 9)(x - 20)}{(x^2 - 4)(x - 21)}$$

From this view, it appears that the graph of g is about the same as the graph of f shown in Figure 4.45. The graphs of these two functions, however, differ significantly. Try enlarging the viewing window to see the differences.

EXAMPLE 2 Sketching the Graph of a Rational Function

Analyze and sketch the graph of $f(x) = \frac{x^2 - 2x + 4}{x - 2}$.

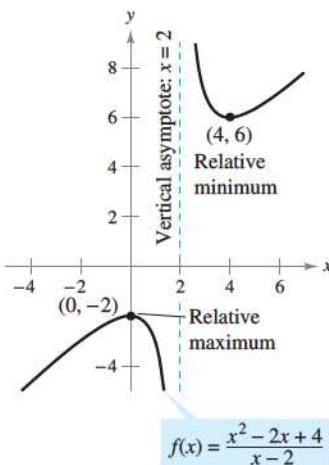
Solution

Figure 4.47

$$\text{First derivative: } f'(x) = \frac{x(x - 4)}{(x - 2)^2}$$

$$\text{Second derivative: } f''(x) = \frac{8}{(x - 2)^3}$$

x-intercepts: None

y-intercept: $(0, -2)$

Vertical asymptote: $x = 2$

Horizontal asymptotes: None

End behavior: $\lim_{x \rightarrow -\infty} f(x) = -\infty, \lim_{x \rightarrow \infty} f(x) = \infty$

Critical numbers: $x = 0, x = 4$

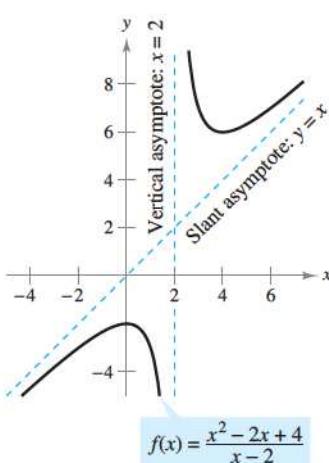
Possible points of inflection: None

Domain: All real numbers except $x = 2$

Test intervals: $(-\infty, 0), (0, 2), (2, 4), (4, \infty)$

The analysis of the graph of f is shown in the table, and the graph is shown in Figure 4.47.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		+	-	Increasing, concave downward
$x = 0$	-2	0	-	Relative maximum
$0 < x < 2$		-	-	Decreasing, concave downward
$x = 2$	Undef.	Undef.	Undef.	Vertical asymptote
$2 < x < 4$		-	+	Decreasing, concave upward
$x = 4$	6	0	+	Relative minimum
$4 < x < \infty$		+	+	Increasing, concave upward



A slant asymptote

Figure 4.48

Although the graph of the function in Example 2 has no horizontal asymptote, it does have a slant asymptote. The graph of a rational function (having no common factors and whose denominator is of degree 1 or greater) has a **slant asymptote** if the degree of the numerator exceeds the degree of the denominator by exactly 1. To find the slant asymptote, use long division to rewrite the rational function as the sum of a first-degree polynomial and another rational function.

$$\begin{aligned} f(x) &= \frac{x^2 - 2x + 4}{x - 2} && \text{Write original equation.} \\ &= x + \frac{4}{x - 2} && \text{Rewrite using long division.} \end{aligned}$$

In Figure 4.48, note that the graph of f approaches the slant asymptote $y = x$ as x approaches $-\infty$ or ∞ .

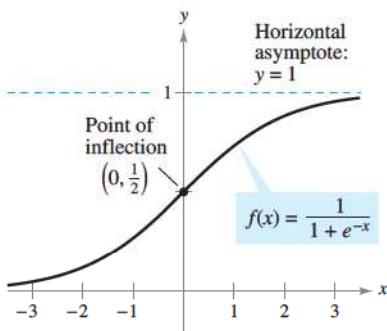


Figure 4.49

EXAMPLE 3 Sketching the Graph of a Logistic Function

Analyze and sketch the graph of the *logistic function* $f(x) = \frac{1}{1+e^{-x}}$.

Solution

$$f'(x) = \frac{e^{-x}}{(1+e^{-x})^2} \quad \text{and} \quad f''(x) = -\frac{e^{-x}(e^{-x}-1)}{(1+e^{-x})^3}$$

The graph has only one intercept, $(0, \frac{1}{2})$. It has no vertical asymptotes, but it has two horizontal asymptotes: $y = 1$ (to the right) and $y = 0$ (to the left). The function has no critical numbers and one possible point of inflection (at $x = 0$). The domain of the function is all real numbers. The analysis of the graph of f is shown in the table, and the graph is shown in Figure 4.49.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		+	+	Increasing, concave upward
$x = 0$	$\frac{1}{2}$	$\frac{1}{4}$	0	Point of inflection
$0 < x < \infty$		+	-	Increasing, concave downward

EXAMPLE 4 Sketching the Graph of a Radical Function

Analyze and sketch the graph of $f(x) = 2x^{5/3} - 5x^{4/3}$.

Solution

$$f'(x) = \frac{10}{3}x^{1/3}(x^{1/3} - 2) \quad f''(x) = \frac{20(x^{1/3} - 1)}{9x^{2/3}}$$

The function has two intercepts: $(0, 0)$ and $(\frac{125}{8}, 0)$. There are no horizontal or vertical asymptotes. The function has two critical numbers ($x = 0$ and $x = 8$) and two possible points of inflection ($x = 0$ and $x = 1$). The domain is all real numbers. The analysis of the graph of f is shown in the table, and the graph is shown in Figure 4.50.

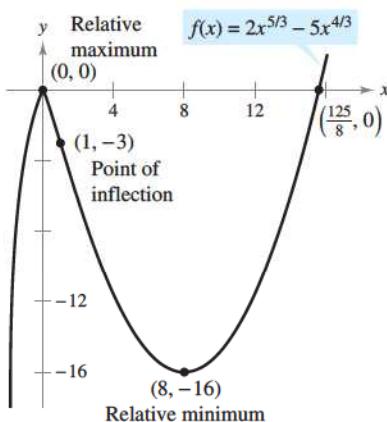


Figure 4.50

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		+	-	Increasing, concave downward
$x = 0$	0	0	Undef.	Relative maximum
$0 < x < 1$		-	-	Decreasing, concave downward
$x = 1$	-3	-	0	Point of inflection
$1 < x < 8$		-	+	Decreasing, concave upward
$x = 8$	-16	0	+	Relative minimum
$8 < x < \infty$		+	+	Increasing, concave upward

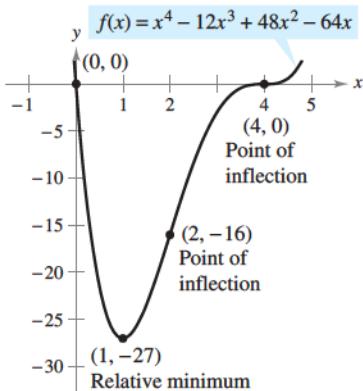
EXAMPLE 5 Sketching the Graph of a Polynomial Function

Analyze and sketch the graph of $f(x) = x^4 - 12x^3 + 48x^2 - 64x$.

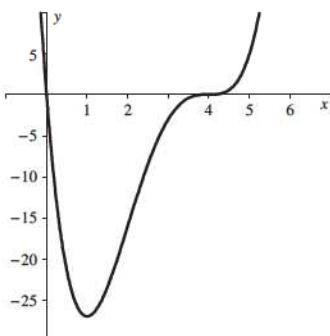
Solution Begin by factoring to obtain

$$\begin{aligned} f(x) &= x^4 - 12x^3 + 48x^2 - 64x \\ &= x(x-4)^3. \end{aligned}$$

Then, using the factored form of $f(x)$, you can perform the following analysis.



(a)



Generated by Maple

(b)

A polynomial function of even degree must have at least one relative extremum.

Figure 4.51

First derivative:	$f'(x) = 4(x-1)(x-4)^2$
Second derivative:	$f''(x) = 12(x-4)(x-2)$
x-intercepts:	(0, 0), (4, 0)
y-intercept:	(0, 0)
Vertical asymptotes:	None
Horizontal asymptotes:	None
End behavior:	$\lim_{x \rightarrow -\infty} f(x) = \infty, \lim_{x \rightarrow \infty} f(x) = \infty$
Critical numbers:	$x = 1, x = 4$
Possible points of inflection:	$x = 2, x = 4$
Domain:	All real numbers
Test intervals:	$(-\infty, 1), (1, 2), (2, 4), (4, \infty)$

The analysis of the graph of f is shown in the table, and the graph is shown in Figure 4.51(a). Using a computer algebra system such as *Maple* [see Figure 4.51(b)] can help you verify your analysis.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 1$		-	+	Decreasing, concave upward
$x = 1$	-27	0	+	Relative minimum
$1 < x < 2$		+	+	Increasing, concave upward
$x = 2$	-16	+	0	Point of inflection
$2 < x < 4$		+	-	Increasing, concave downward
$x = 4$	0	0	0	Point of inflection
$4 < x < \infty$		+	+	Increasing, concave upward

The fourth-degree polynomial function in Example 5 has one relative minimum and no relative maxima. In general, a polynomial function of degree n can have *at most* $n - 1$ relative extrema, and *at most* $n - 2$ points of inflection. Moreover, polynomial functions of even degree must have *at least* one relative extremum.

Remember from the Leading Coefficient Test described in Section 1.3 that the “end behavior” of the graph of a polynomial function is determined by its leading coefficient and its degree. For instance, because the polynomial in Example 5 has a positive leading coefficient, the graph rises to the right. Moreover, because the degree is even, the graph also rises to the left.

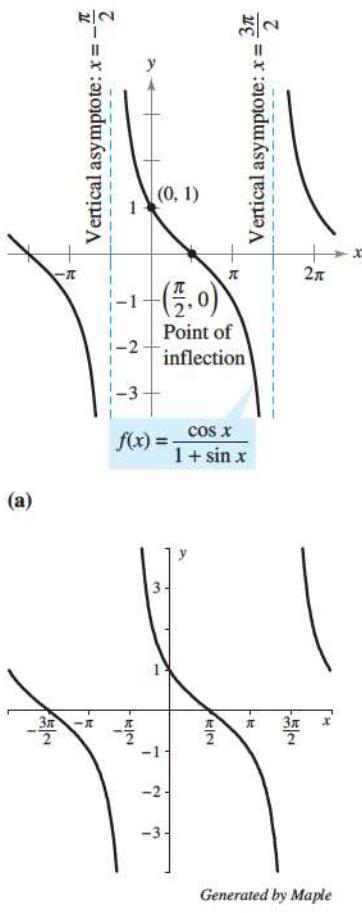


Figure 4.52

EXAMPLE 6 Sketching the Graph of a Trigonometric Function

Analyze and sketch the graph of $f(x) = \frac{\cos x}{1 + \sin x}$.

Solution Because the function has a period of 2π , you can restrict the analysis of the graph to any interval of length 2π . For convenience, choose $(-\pi/2, 3\pi/2)$.

$$\text{First derivative: } f'(x) = -\frac{1}{1 + \sin x}$$

$$\text{Second derivative: } f''(x) = \frac{\cos x}{(1 + \sin x)^2}$$

$$\text{Period: } 2\pi$$

$$x\text{-intercept: } \left(\frac{\pi}{2}, 0\right)$$

$$y\text{-intercept: } (0, 1)$$

$$\text{Vertical asymptotes: } x = -\frac{\pi}{2}, x = \frac{3\pi}{2} \quad \text{See Note below.}$$

$$\text{Horizontal asymptotes: } \text{None}$$

$$\text{Critical numbers: } \text{None}$$

$$\text{Possible points of inflection: } x = \frac{\pi}{2}$$

$$\text{Domain: } \text{All real numbers except } x = \frac{3 + 4n}{2}\pi$$

$$\text{Test intervals: } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

The analysis of the graph of f on the interval $(-\pi/2, 3\pi/2)$ is shown in the table, and the graph is shown in Figure 4.52(a). Compare this with the graph generated by the computer algebra system *Maple* in Figure 4.52(b).

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$x = -\frac{\pi}{2}$	Undefined.	Undefined.	Undefined.	Vertical asymptote
$-\frac{\pi}{2} < x < \frac{\pi}{2}$		-	+	Decreasing, concave upward
$x = \frac{\pi}{2}$	0	$-\frac{1}{2}$	0	Point of inflection
$\frac{\pi}{2} < x < \frac{3\pi}{2}$		-	-	Decreasing, concave downward
$x = \frac{3\pi}{2}$	Undefined.	Undefined.	Undefined.	Vertical asymptote

NOTE By substituting $-\pi/2$ or $3\pi/2$ into the function, you obtain the form $0/0$. This is called an indeterminate form, which you will study in Section 8.7. To determine that the function has vertical asymptotes at these two values, you can rewrite the function as follows.

$$f(x) = \frac{\cos x}{1 + \sin x} = \frac{(\cos x)(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} = \frac{(\cos x)(1 - \sin x)}{\cos^2 x} = \frac{1 - \sin x}{\cos x}$$

In this form, it is clear that the graph of f has vertical asymptotes at $x = -\pi/2$ and $3\pi/2$. ■

EXAMPLE 7 Analyzing an Inverse Trigonometric Graph

Analyze the graph of $y = (\arctan x)^2$.

Solution From the derivative

$$\begin{aligned}y' &= 2(\arctan x) \left(\frac{1}{1+x^2} \right) \\&= \frac{2 \arctan x}{1+x^2}\end{aligned}$$

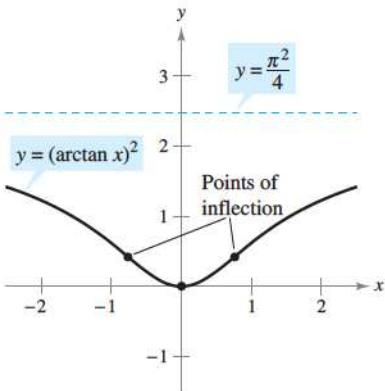
you can see that the only critical number is $x = 0$. By the First Derivative Test, this value corresponds to a relative minimum. From the second derivative

$$\begin{aligned}y'' &= \frac{(1+x^2)\left(\frac{2}{1+x^2}\right) - (2 \arctan x)(2x)}{(1+x^2)^2} \\&= \frac{2(1-2x \arctan x)}{(1+x^2)^2}\end{aligned}$$

it follows that points of inflection occur when $2x \arctan x = 1$. Using Newton's Method, these points occur when $x \approx \pm 0.765$. Finally, because

$$\lim_{x \rightarrow \pm\infty} (\arctan x)^2 = \frac{\pi^2}{4}$$

it follows that the graph has a horizontal asymptote at $y = \pi^2/4$. The graph is shown in Figure 4.53.



The graph of $y = (\arctan x)^2$ has a horizontal asymptote at $y = \pi^2/4$.

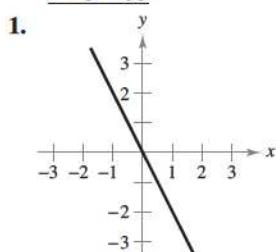
Figure 4.53

4.6 Exercises

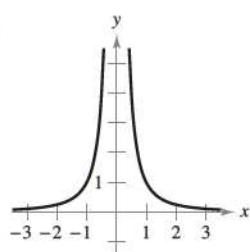
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, match the graph of f in the left column with that of its derivative in the right column.

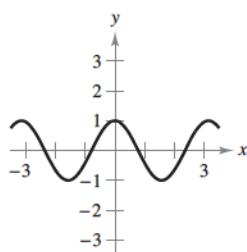
Graph of f



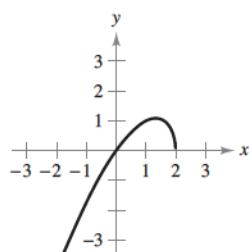
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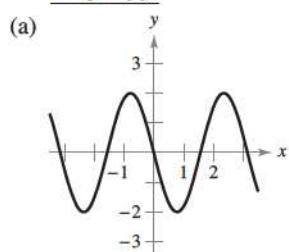
3.



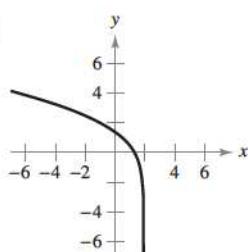
4.



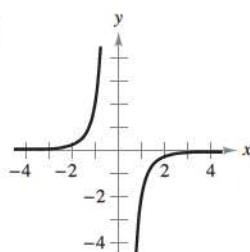
Graph of f'



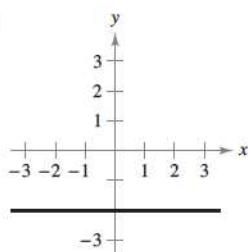
(b)



(c)



(d)



In Exercises 5–48, analyze and sketch a graph of the function. Label any intercepts, relative extrema, points of inflection, and asymptotes. Use a graphing utility to verify your results.

5. $y = \frac{1}{x-2} - 3$

6. $y = \frac{x}{x^2+1}$

7. $y = \frac{x^2}{x^2+3}$

8. $y = \frac{x^2+1}{x^2-4}$

9. $y = \frac{3x}{x^2-1}$

10. $f(x) = \frac{x-3}{x}$

11. $g(x) = x - \frac{8}{x^2}$

12. $f(x) = x + \frac{32}{x^2}$

13. $f(x) = \frac{x^2+1}{x}$

14. $f(x) = \frac{x^3}{x^2-9}$

15. $y = \frac{x^2-6x+12}{x-4}$

16. $y = \frac{2x^2-5x+5}{x-2}$

17. $y = x\sqrt{4-x}$

18. $g(x) = x\sqrt{9-x}$

19. $h(x) = x\sqrt{4-x^2}$

20. $g(x) = x\sqrt{9-x^2}$

21. $y = 3x^{2/3} - 2x$

22. $y = 3(x-1)^{2/3} - (x-1)^2$

23. $y = x^3 - 3x^2 + 3$

24. $y = -\frac{1}{3}(x^3 - 3x + 2)$

25. $y = 2 - x - x^3$

26. $f(x) = \frac{1}{3}(x-1)^3 + 2$

27. $f(x) = 3x^3 - 9x + 1$

28. $f(x) = (x+1)(x-2)(x-5)$

29. $y = 3x^4 + 4x^3$

30. $y = 3x^4 - 6x^2 + \frac{5}{3}$

31. $f(x) = x^4 - 4x^3 + 16x$

32. $f(x) = x^4 - 8x^3 + 18x^2 - 16x + 5$

33. $y = x^5 - 5x$

34. $y = (x-1)^5$

35. $y = |2x-3|$

36. $y = |x^2 - 6x + 5|$

37. $f(x) = e^{3x}(2-x)$

38. $f(x) = -2 + e^{3x}(4-2x)$

39. $g(t) = \frac{10}{1+4e^{-t}}$

40. $h(x) = \frac{8}{2+3e^{-x/2}}$

41. $y = (x-1)\ln(x-1)$

42. $y = \frac{1}{24}x^3 - \ln x$

43. $g(x) = 6 \arcsin\left(\frac{x-2}{2}\right)^2$

44. $h(x) = 7 \arctan(x+1) - \ln(x^2+2x+2)$

45. $f(x) = \frac{x}{3^{x-3}}$

46. $g(t) = (5-t)5^t$

47. $g(x) = \log_4(x-x^2)$

48. $f(x) = \log_2|x^2 - 4x|$

CAS In Exercises 49–54, use a computer algebra system to analyze and graph the function. Identify any relative extrema, points of inflection, and asymptotes.

49. $f(x) = \frac{20x}{x^2+1} - \frac{1}{x}$

50. $f(x) = x + \frac{4}{x^2+1}$

51. $f(x) = \frac{-2x}{\sqrt{x^2+7}}$

52. $f(x) = \frac{4x}{\sqrt{x^2+15}}$

53. $y = \frac{x}{2} + \ln\left(\frac{x}{x+3}\right)$

54. $y = \frac{3x}{2}(1+4e^{-x/3})$

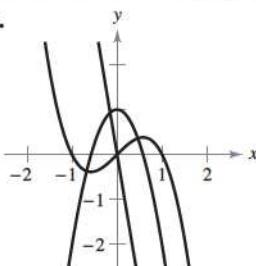
In Exercises 55–64, sketch a graph of the function over the given interval. Use a graphing utility to verify your graph.

Function	Interval
55. $f(x) = 2x - 4 \sin x$	$0 \leq x \leq 2\pi$
56. $f(x) = -x + 2 \cos x$	$0 \leq x \leq 2\pi$
57. $y = \sin x - \frac{1}{18} \sin 3x$	$0 \leq x \leq 2\pi$
58. $y = \cos x - \frac{1}{4} \cos 2x$	$0 \leq x \leq 2\pi$
59. $y = 2x - \tan x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
60. $y = 2(x-2) + \cot x$	$0 < x < \pi$
61. $y = 2(\csc x + \sec x)$	$0 < x < \frac{\pi}{2}$
62. $y = \sec^2\left(\frac{\pi x}{8}\right) - 2 \tan\left(\frac{\pi x}{8}\right) - 1$	$-3 < x < 3$
63. $g(x) = x \tan x$	$-\frac{3\pi}{2} < x < \frac{3\pi}{2}$
64. $g(x) = x \cot x$	$-2\pi < x < 2\pi$

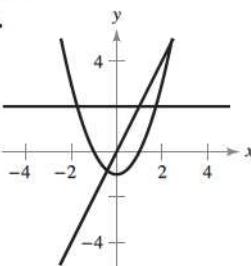
WRITING ABOUT CONCEPTS

In Exercises 65 and 66, the graphs of f , f' , and f'' are shown on the same set of coordinate axes. Which is which? Explain your reasoning. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

65.



66.



67. Suppose $f'(t) < 0$ for all t in the interval $(2, 8)$. Explain why $f(3) > f(5)$.

68. Suppose $f(0) = 3$ and $2 \leq f'(x) \leq 4$ for all x in the interval $[-5, 5]$. Determine the greatest and least possible values of $f(2)$.



In Exercises 69–72, use a graphing utility to graph the function. Use the graph to determine whether it is possible for the graph of a function to cross its horizontal asymptote. Do you think it is possible for the graph of a function to cross its vertical asymptote? Why or why not?

69. $f(x) = \frac{4(x-1)^2}{x^2-4x+5}$

70. $g(x) = \frac{3x^4-5x+3}{x^4+1}$

71. $h(x) = \frac{\sin 2x}{x}$

72. $f(x) = \frac{\cos 3x}{4x}$

WRITING ABOUT CONCEPTS (continued)

In Exercises 73 and 74, use a graphing utility to graph the function. Explain why there is no vertical asymptote when a superficial examination of the function may indicate that there should be one.

73. $h(x) = \frac{6 - 2x}{3 - x}$

74. $g(x) = \frac{x^2 + x - 2}{x - 1}$



In Exercises 75–78, use a graphing utility to graph the function and determine the slant asymptote of the graph. Zoom out repeatedly and describe how the graph on the display appears to change. Why does this occur?

75. $f(x) = -\frac{x^2 - 3x - 1}{x - 2}$

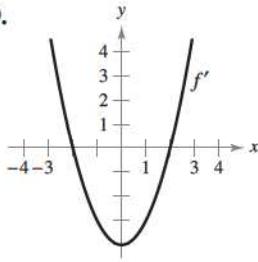
76. $g(x) = \frac{2x^2 - 8x - 15}{x - 5}$

77. $f(x) = \frac{x^3}{x^2 + 1}$

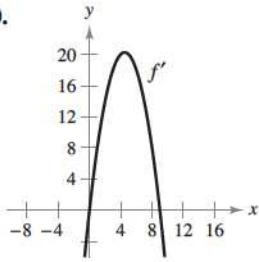
78. $h(x) = \frac{-x^3 + x^2 + 4}{x^2}$

Graphical Reasoning In Exercises 79–82, use the graph of f' to sketch a graph of f and the graph of f'' . To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

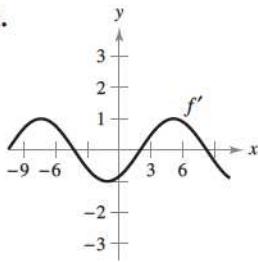
79.



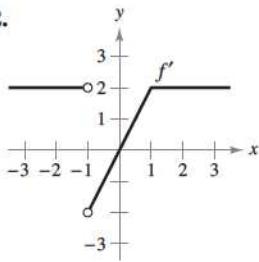
80.



81.



82.



(Submitted by Bill Fox, Moberly Area Community College, Moberly, MO)

CAS 83. **Graphical Reasoning** Consider the function

$$f(x) = \frac{\cos^2 \pi x}{\sqrt{x^2 + 1}}, \quad 0 < x < 4.$$

- (a) Use a computer algebra system to graph the function and use the graph to approximate the critical numbers visually.
- (b) Use a computer algebra system to find f' and approximate the critical numbers. Are the results the same as the visual approximation in part (a)? Explain.



84. **Graphical Reasoning** Consider the function

$$f(x) = \tan(\sin \pi x).$$

- (a) Use a graphing utility to graph the function.
- (b) Identify any symmetry of the graph.
- (c) Is the function periodic? If so, what is the period?
- (d) Identify any extrema on $(-1, 1)$.
- (e) Use a graphing utility to determine the concavity of the graph on $(0, 1)$.

Think About It In Exercises 85–88, create a function whose graph has the given characteristics. (There is more than one correct answer.)

85. Vertical asymptote: $x = 3$

Horizontal asymptote: $y = 0$

86. Vertical asymptote: $x = -5$

Horizontal asymptote: None

87. Vertical asymptote: $x = 3$

Slant asymptote: $y = 3x + 2$

88. Vertical asymptote: $x = 2$

Slant asymptote: $y = -x$

89. **Graphical Reasoning** The graph of f is shown in the figure.

(a) For which values of x is $f'(x)$ zero? Positive? Negative?

(b) For which values of x is $f''(x)$ zero? Positive? Negative?

(c) On what interval is f' an increasing function?

(d) For which value of x is $f'(x)$ minimum? For this value of x , how does the rate of change of f compare with the rates of change of f for other values of x ? Explain.

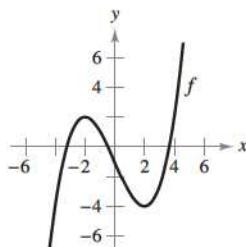


Figure for 89

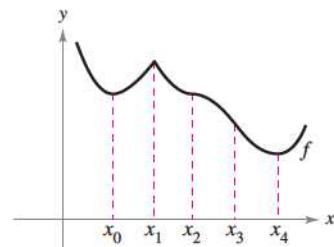


Figure for 90

CAPSTONE

90. **Graphical Reasoning** Identify the real numbers x_0, x_1, x_2, x_3 , and x_4 in the figure such that each of the following is true.

- | | |
|------------------------------------|---------------------------------|
| (a) $f'(x) = 0$ | (b) $f''(x) = 0$ |
| (c) $f'(x)$ does not exist. | (d) f has a relative maximum. |
| (e) f has a point of inflection. | |

- 91. Graphical Reasoning** Consider the function

$$f(x) = \frac{ax}{(x-b)^2}$$

Determine the effect on the graph of f as a and b are changed. Consider cases where a and b are both positive or both negative, and cases where a and b have opposite signs.

- 92.** Consider the function $f(x) = \frac{1}{2}(ax)^2 - ax$, $a \neq 0$.

- (a) Determine the changes (if any) in the intercepts, extrema, and concavity of the graph of f when a is varied.
A (b) In the same viewing window, use a graphing utility to graph the function for four different values of a .

- 93. Investigation** Consider the function $f(x) = \frac{2x^n}{x^4 + 1}$ for nonnegative integer values of n .

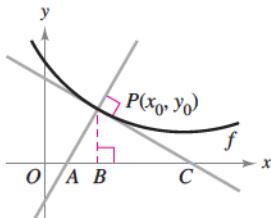
- (a) Discuss the relationship between the value of n and the symmetry of the graph.
(b) For which values of n will the x -axis be the horizontal asymptote?
(c) For which value of n will $y = 2$ be the horizontal asymptote?
(d) What is the asymptote of the graph when $n = 5$?

- A** (e) Use a graphing utility to graph f for the given values of n in the table. Use the graph to determine the number of extrema M and the number of inflection points N of the graph.

n	0	1	2	3	4	5
M						
N						

- 94. Investigation** Let $P(x_0, y_0)$ be an arbitrary point on the graph of f such that $f'(x_0) \neq 0$, as shown in the figure. Verify each statement.

- (a) The x -intercept of the tangent line is $\left(x_0 - \frac{f(x_0)}{f'(x_0)}, 0\right)$.
(b) The y -intercept of the tangent line is $(0, f(x_0) - x_0 f'(x_0))$.
(c) The x -intercept of the normal line is $(x_0 + f(x_0) f'(x_0), 0)$.
(d) The y -intercept of the normal line is $\left(0, y_0 + \frac{x_0}{f'(x_0)}\right)$.
(e) $|BC| = \left|\frac{f(x_0)}{f'(x_0)}\right|$ (f) $|PC| = \left|\frac{f(x_0) \sqrt{1 + [f'(x_0)]^2}}{f'(x_0)}\right|$
(g) $|AB| = |f(x_0) f'(x_0)|$
(h) $|AP| = |f(x_0)| \sqrt{1 + [f'(x_0)]^2}$



- 95. Modeling Data** A meteorologist measures the atmospheric pressure P (in kilograms per square meter) at altitude h (in kilometers). The data are shown below.

h	0	5	10	15	20
P	10,332	5583	2376	1240	517

- (a) Use a graphing utility to plot the points $(h, \ln P)$. Use the regression capabilities of the graphing utility to find a linear model for the revised data points.

- (b) The line in part (a) has the form

$$\ln P = ah + b.$$

Write the equation in exponential form.

- (c) Use a graphing utility to plot the original data and graph the exponential model in part (b).
(d) Find the rate of change of the pressure when $h = 5$ and $h = 18$.

- 96.** Let f be a function that is positive and differentiable on the entire real number line. Let $g(x) = \ln f(x)$.

- (a) If g is increasing, must f be increasing? Explain.

- (b) If the graph of f is concave upward, must the graph of g be concave upward? Explain.

- A** **97. Conjecture** Use a graphing utility to graph f and g in the same viewing window and determine which is increasing at the faster rate for “large” values of x . What can you conclude about the rate of growth of the natural logarithmic function?

$$(a) f(x) = \ln x, g(x) = \sqrt{x}$$

$$(b) f(x) = \ln x, g(x) = \sqrt[4]{x}$$

Slant Asymptotes In Exercises 98 and 99, the graph of the function has two slant asymptotes. Identify each slant asymptote. Then graph the function and its asymptotes.

$$98. y = \sqrt{4 + 16x^2}$$

$$99. y = \sqrt{x^2 + 6x}$$

PUTNAM EXAM CHALLENGE

- 100.** Let $f(x)$ be defined for $a \leq x \leq b$. Assuming appropriate properties of continuity and derivability, prove for $a < x < b$ that

$$\frac{f(x) - f(a)}{x - a} - \frac{f(b) - f(a)}{b - a} = \frac{1}{2} f''(\beta)$$

where β is some number between a and b .

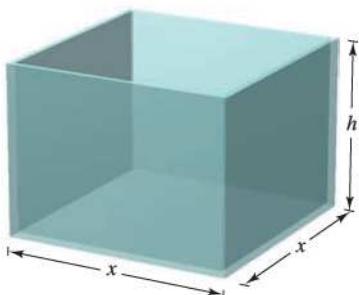
This problem was composed by the Committee on the Putnam Prize Competition.
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4.7**Optimization Problems**

- Solve applied minimum and maximum problems.

Applied Minimum and Maximum Problems

One of the most common applications of calculus involves the determination of minimum and maximum values. Consider how frequently you hear or read terms such as greatest profit, least cost, least time, greatest voltage, optimum size, least size, greatest strength, and greatest distance. Before outlining a general problem-solving strategy for such problems, consider the next example.

EXAMPLE 1 Finding Maximum Volume

Open box with square base:
 $S = x^2 + 4xh = 108$

Figure 4.54

A manufacturer wants to design an open box having a square base and a surface area of 108 square inches, as shown in Figure 4.54. What dimensions will produce a box with maximum volume?

Solution Because the box has a square base, its volume is

$$V = x^2h.$$

Primary equation

This equation is called the **primary equation** because it gives a formula for the quantity to be optimized. The surface area of the box is

$$S = (\text{area of base}) + (\text{area of four sides})$$

$$S = x^2 + 4xh = 108.$$

Secondary equation

Because V is to be maximized, you want to write V as a function of just one variable. To do this, you can solve the equation $x^2 + 4xh = 108$ for h in terms of x to obtain $h = (108 - x^2)/(4x)$. Substituting into the primary equation produces

$$\begin{aligned} V &= x^2h && \text{Function of two variables} \\ &= x^2\left(\frac{108 - x^2}{4x}\right) && \text{Substitute for } h. \\ &= 27x - \frac{x^3}{4}. && \text{Function of one variable} \end{aligned}$$

Before finding which x -value will yield a maximum value of V , you should determine the **feasible domain**. That is, what values of x make sense in this problem? You know that $V \geq 0$. You also know that x must be nonnegative and that the area of the base ($A = x^2$) is at most 108. So, the feasible domain is

$$0 \leq x \leq \sqrt{108}.$$

Feasible domain

To maximize V , find the critical numbers of the volume function on the interval $(0, \sqrt{108})$.

$$\begin{aligned} \frac{dV}{dx} &= 27 - \frac{3x^2}{4} = 0 && \text{Set derivative equal to 0.} \\ 3x^2 &= 108 && \text{Simplify.} \\ x &= \pm 6 && \text{Critical numbers} \end{aligned}$$

So, the critical numbers are $x = \pm 6$. You do not need to consider $x = -6$ because it is outside the domain. Evaluating V at the critical number $x = 6$ and at the endpoints of the domain produces $V(0) = 0$, $V(6) = 108$, and $V(\sqrt{108}) = 0$. So, V is maximum when $x = 6$ and the dimensions of the box are $6 \times 6 \times 3$ inches. ■

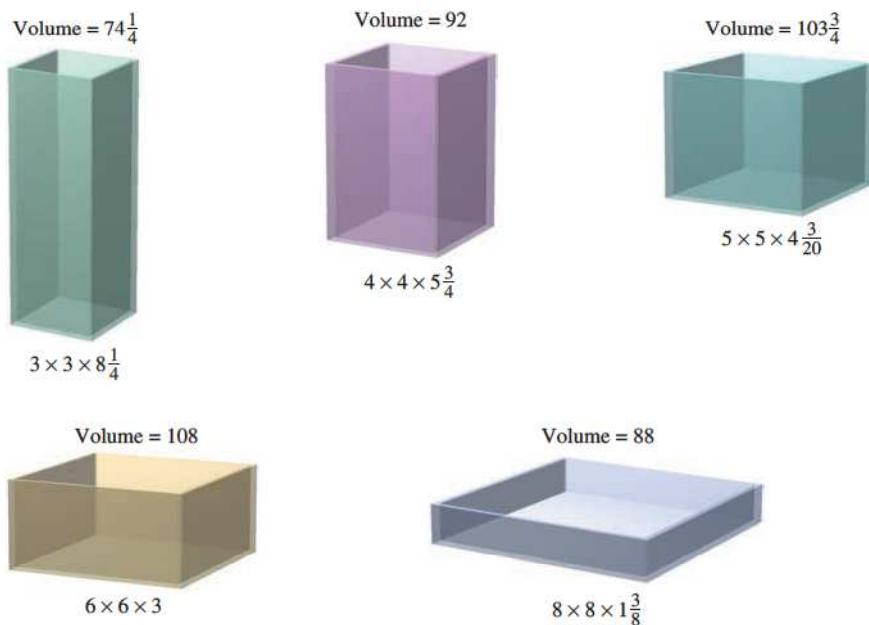
TECHNOLOGY You can verify your answer in Example 1 by using a graphing utility to graph the volume function

$$V = 27x - \frac{x^3}{4}.$$

Use a viewing window in which $0 \leq x \leq \sqrt{108} \approx 10.4$ and $0 \leq y \leq 120$ and use the *trace* feature to determine the maximum value of V .

In Example 1, you should realize that there are infinitely many open boxes having 108 square inches of surface area. To begin solving the problem, you might ask yourself which basic shape would seem to yield a maximum volume. Should the box be tall, squat, or nearly cubical?

You might even try calculating a few volumes, as shown in Figure 4.55, to see if you can get a better feeling for what the optimum dimensions should be. Remember that you are not ready to begin solving a problem until you have clearly identified what the problem is.



Which box has the greatest volume?

Figure 4.55

Example 1 illustrates the following guidelines for solving applied minimum and maximum problems.

GUIDELINES FOR SOLVING APPLIED MINIMUM AND MAXIMUM PROBLEMS

1. Identify all *given* quantities and all quantities *to be determined*. If possible, make a sketch.
2. Write a **primary equation** for the quantity that is to be maximized or minimized. (A review of several useful formulas from geometry is presented inside the back cover.)
3. Reduce the primary equation to one having a *single independent variable*. This may involve the use of **secondary equations** relating the independent variables of the primary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by the calculus techniques discussed in Sections 4.1 through 4.4.

NOTE When performing Step 5, recall that to determine the maximum or minimum value of a continuous function f on a closed interval, you should compare the values of f at its critical numbers with the values of f at the endpoints of the interval.

EXAMPLE 2 Finding Minimum Distance

Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$?

Solution Figure 4.56 shows that there are two points at a minimum distance from the point $(0, 2)$. The distance between the point $(0, 2)$ and a point (x, y) on the graph of $y = 4 - x^2$ is given by

$$d = \sqrt{(x - 0)^2 + (y - 2)^2}.$$

Primary equation

Using the secondary equation $y = 4 - x^2$, you can rewrite the primary equation as

$$d = \sqrt{x^2 + (4 - x^2 - 2)^2} = \sqrt{x^4 - 3x^2 + 4}.$$

Because d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of $f(x) = x^4 - 3x^2 + 4$. Note that the domain of f is the entire real number line. So, there are no endpoints of the domain to consider. Moreover, setting $f'(x)$ equal to 0 yields

$$f'(x) = 4x^3 - 6x = 2x(2x^2 - 3) = 0$$

$$x = 0, \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}.$$

The First Derivative Test verifies that $x = 0$ yields a relative maximum, whereas both $x = \sqrt{3/2}$ and $x = -\sqrt{3/2}$ yield a minimum distance. So, the closest points are $(\sqrt{3/2}, 5/2)$ and $(-\sqrt{3/2}, 5/2)$.

EXAMPLE 3 Finding Minimum Area

A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be $1\frac{1}{2}$ inches, and the margins on the left and right are to be 1 inch (see Figure 4.57). What should the dimensions of the page be so that the least amount of paper is used?

Solution Let A be the area to be minimized.

$$A = (x + 3)(y + 2)$$

Primary equation

The printed area inside the margins is given by

$$24 = xy.$$

Secondary equation

Solving this equation for y produces $y = 24/x$. Substitution into the primary equation produces

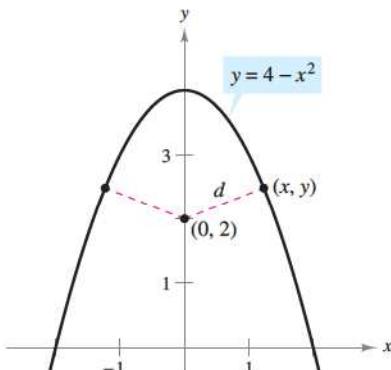
$$A = (x + 3)\left(\frac{24}{x} + 2\right) = 30 + 2x + \frac{72}{x}.$$

Function of one variable

Because x must be positive, you are interested only in values of A for $x > 0$. To find the critical numbers, differentiate with respect to x .

$$\frac{dA}{dx} = 2 - \frac{72}{x^2} = 0 \quad \Rightarrow \quad x^2 = 36$$

So, the critical numbers are $x = \pm 6$. You do not have to consider $x = -6$ because it is outside the domain. The First Derivative Test confirms that A is a minimum when $x = 6$. So, $y = \frac{24}{6} = 4$ and the dimensions of the page should be $x + 3 = 9$ inches by $y + 2 = 6$ inches. ■



The quantity to be minimized is distance:
 $d = \sqrt{(x - 0)^2 + (y - 2)^2}$.

Figure 4.56

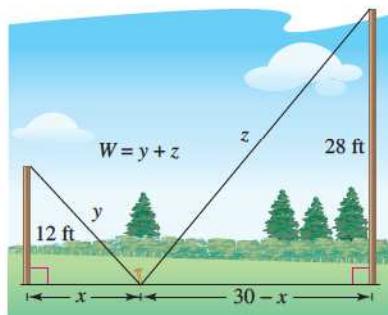


The quantity to be minimized is area:
 $A = (x + 3)(y + 2)$.

Figure 4.57

EXAMPLE 4 Finding Minimum Length

Two posts, one 12 feet high and the other 28 feet high, stand 30 feet apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. Where should the stake be placed to use the least amount of wire?



The quantity to be minimized is length. From the diagram, you can see that x varies between 0 and 30.

Figure 4.58

Solution Let W be the wire length to be minimized. Using Figure 4.58, you can write

$$W = y + z. \quad \text{Primary equation}$$

In this problem, rather than solving for y in terms of z (or vice versa), you can solve for both y and z in terms of a third variable x , as shown in Figure 4.58. From the Pythagorean Theorem, you obtain

$$\begin{aligned} x^2 + 12^2 &= y^2 \\ (30 - x)^2 + 28^2 &= z^2 \end{aligned}$$

which implies that

$$\begin{aligned} y &= \sqrt{x^2 + 144} \\ z &= \sqrt{x^2 - 60x + 1684}. \end{aligned}$$

So, W is given by

$$\begin{aligned} W &= y + z \\ &= \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}, \quad 0 \leq x \leq 30. \end{aligned}$$

Differentiating W with respect to x yields

$$\frac{dW}{dx} = \frac{x}{\sqrt{x^2 + 144}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1684}}.$$

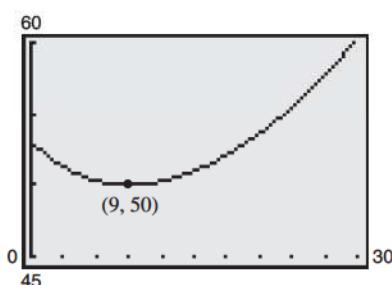
By letting $dW/dx = 0$, you obtain

$$\begin{aligned} \frac{x}{\sqrt{x^2 + 144}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1684}} &= 0 \\ x\sqrt{x^2 - 60x + 1684} &= (30 - x)\sqrt{x^2 + 144} \\ x^2(x^2 - 60x + 1684) &= (30 - x)^2(x^2 + 144) \\ x^4 - 60x^3 + 1684x^2 &= x^4 - 60x^3 + 1044x^2 - 8640x + 129,600 \\ 640x^2 + 8640x - 129,600 &= 0 \\ 320(x - 9)(2x + 45) &= 0 \\ x &= 9, -22.5. \end{aligned}$$

Because $x = -22.5$ is not in the domain and

$$W(0) \approx 53.04, \quad W(9) = 50, \quad \text{and} \quad W(30) \approx 60.31$$

you can conclude that the wire should be staked at 9 feet from the 12-foot pole. ■



You can confirm the minimum value of W with a graphing utility.

Figure 4.59

TECHNOLOGY From Example 4, you can see that applied optimization problems can involve a lot of algebra. If you have access to a graphing utility, you can confirm that $x = 9$ yields a minimum value of W by graphing

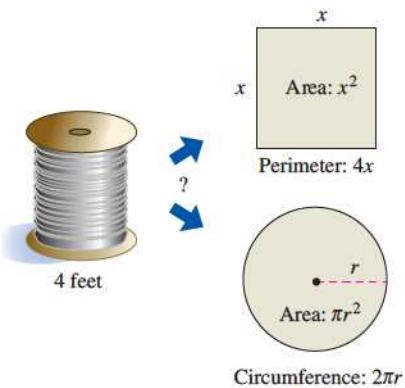
$$W = \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}$$

as shown in Figure 4.59.

In each of the first four examples, the extreme value occurred at a critical number. Although this happens often, remember that an extreme value can also occur at an endpoint of an interval, as shown in Example 5.

EXAMPLE 5 An Endpoint Maximum

Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?



The quantity to be maximized is area:

$$A = x^2 + \pi r^2.$$

Figure 4.60

Solution The total area (see Figure 4.60) is given by

$$A = (\text{area of square}) + (\text{area of circle})$$

$$A = x^2 + \pi r^2.$$

Primary equation

Because the total length of wire is 4 feet, you obtain

$$4 = (\text{perimeter of square}) + (\text{circumference of circle})$$

$$4 = 4x + 2\pi r.$$

So, $r = 2(1 - x)/\pi$, and by substituting into the primary equation you have

$$A = x^2 + \pi \left[\frac{2(1-x)}{\pi} \right]^2$$

$$= x^2 + \frac{4(1-x)^2}{\pi}$$

$$= \frac{1}{\pi} [(\pi + 4)x^2 - 8x + 4].$$

You know that x must be nonnegative and that the perimeter of the square ($P = 4x$) is at most 4. So, the feasible domain is $0 \leq x \leq 1$. Because

$$\frac{dA}{dx} = \frac{2(\pi + 4)x - 8}{\pi}$$

the only critical number in $(0, 1)$ is $x = 4/(\pi + 4) \approx 0.56$. So, using

$$A(0) \approx 1.273, \quad A(0.56) \approx 0.56, \quad \text{and} \quad A(1) = 1$$

you can conclude that the maximum area occurs when $x = 0$. That is, *all* the wire is used for the circle. ■

EXPLORATION

What would the answer be if Example 5 asked for the dimensions needed to enclose the *minimum* total area?

Let's review the primary equations developed in the first five examples. As applications go, these five examples are fairly simple, and yet the resulting primary equations are quite complicated.

$$V = 27x - \frac{x^3}{4}$$

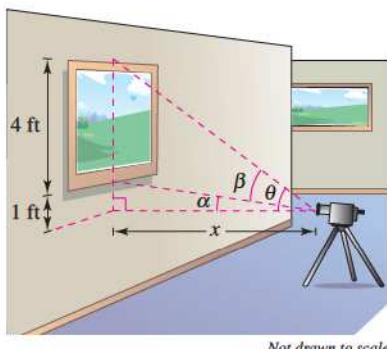
$$W = \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}$$

$$d = \sqrt{x^4 - 3x^2 + 4}$$

$$A = \frac{1}{\pi} [(\pi + 4)x^2 - 8x + 4]$$

$$A = 30 + 2x + \frac{72}{x}$$

You must expect that real-life applications often involve equations that are *at least as complicated* as these five. Remember that one of the main goals of this course is to learn to use calculus to analyze equations that initially seem formidable.


EXAMPLE 6 Maximizing an Angle


The camera should be 2.236 feet from the painting to maximize the angle β .

Figure 4.61

A photographer is taking a picture of a 4-foot painting hung in an art gallery. The camera lens is 1 foot below the lower edge of the painting, as shown in Figure 4.61. How far should the camera be from the painting to maximize the angle subtended by the camera lens?

Solution In Figure 4.61, let β be the angle to be maximized.

$$\beta = \theta - \alpha$$

Primary equation

From Figure 4.61, you can see that $\cot \theta = \frac{x}{5}$ and $\cot \alpha = \frac{x}{1}$. Therefore, $\theta = \operatorname{arccot} \frac{x}{5}$ and $\alpha = \operatorname{arccot} x$. So,

$$\beta = \operatorname{arccot} \frac{x}{5} - \operatorname{arccot} x.$$

Differentiating β with respect to x produces

$$\begin{aligned}\frac{d\beta}{dx} &= \frac{-1/5}{1 + (x^2/25)} - \frac{-1}{1 + x^2} \\ &= \frac{-5}{25 + x^2} + \frac{1}{1 + x^2} \\ &= \frac{4(5 - x^2)}{(25 + x^2)(1 + x^2)}.\end{aligned}$$

Because $d\beta/dx = 0$ when $x = \sqrt{5}$, you can conclude from the First Derivative Test that this distance yields a maximum value of β . So, the distance is $x \approx 2.236$ feet and the angle is $\beta \approx 0.7297$ radian $\approx 41.81^\circ$.

EXAMPLE 7 Finding a Maximum Revenue

The demand function for a product is modeled by

$$p = 56e^{-0.000012x} \quad \text{Demand function}$$

where p is the price per unit (in dollars) and x is the number of units. What price will yield a maximum revenue?

Solution Substituting for p (from the demand function) produces the revenue function

$$R = xp = 56xe^{-0.000012x}. \quad \text{Primary equation}$$

The rate of change of revenue R with respect to the number of units sold x is called the *marginal revenue*. Setting the marginal revenue equal to zero,

$$\frac{dR}{dx} = 56x(e^{-0.000012x})(-0.000012) + e^{-0.000012x}(56) = 0$$

yields $x \approx 83,333$ units. From this, you can conclude that the maximum revenue occurs when the price is

$$p = 56e^{-0.000012(83,333)} \approx \$20.60.$$

So, a price of about \$20.60 will yield a maximum revenue (see Figure 4.62). ■

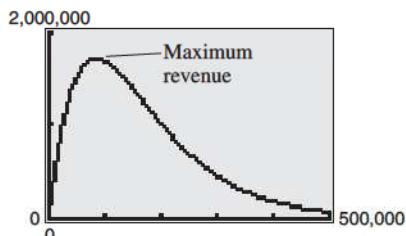


Figure 4.62

4.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

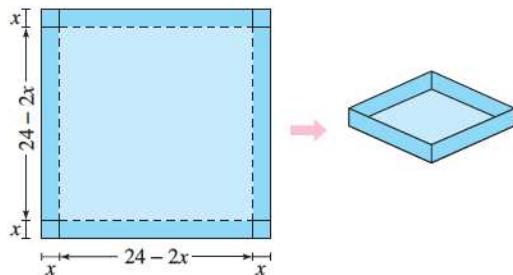
1. **Numerical, Graphical, and Analytic Analysis** Find two positive numbers whose sum is 110 and whose product is a maximum.

- (a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

First Number x	Second Number	Product P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$

- (b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the solution. (*Hint:* Use the *table* feature of the graphing utility.)
 (c) Write the product P as a function of x .
 (d) Use a graphing utility to graph the function in part (c) and estimate the solution from the graph.
 (e) Use calculus to find the critical number of the function in part (c). Then find the two numbers.

2. **Numerical, Graphical, and Analytic Analysis** An open box of maximum volume is to be made from a square piece of material, 24 inches on a side, by cutting equal squares from the corners and turning up the sides (see figure).



- (a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.) Use the table to guess the maximum volume.

Height x	Length and Width	Volume V
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$

- (b) Write the volume V as a function of x .
 (c) Use calculus to find the critical number of the function in part (b) and find the maximum value.
 (d) Use a graphing utility to graph the function in part (b) and verify the maximum volume from the graph.

In Exercises 3–8, find two positive numbers that satisfy the given requirements.

3. The sum is S and the product is a maximum.
 4. The product is 185 and the sum is a minimum.
 5. The product is 147 and the sum of the first number plus three times the second number is a minimum.
 6. The second number is the reciprocal of the first number and the sum is a minimum.
 7. The sum of the first number and twice the second number is 108 and the product is a maximum.
 8. The sum of the first number squared and the second number is 54 and the product is a maximum.

In Exercises 9 and 10, find the length and width of a rectangle that has the given perimeter and a maximum area.

9. Perimeter: 80 meters 10. Perimeter: P units

In Exercises 11 and 12, find the length and width of a rectangle that has the given area and a minimum perimeter.

11. Area: 32 square feet 12. Area: A square centimeters

In Exercises 13–16, find the point on the graph of the function that is closest to the given point.

Function	Point	Function	Point
13. $f(x) = x^2$	$(2, \frac{1}{2})$	14. $f(x) = (x - 1)^2$	$(-5, 3)$
15. $f(x) = \sqrt{x}$	$(4, 0)$	16. $f(x) = \sqrt{x - 8}$	$(12, 0)$

17. **Area** A rectangular page is to contain 30 square inches of print. The margins on each side are 1 inch. Find the dimensions of the page such that the least amount of paper is used.

18. **Area** A rectangular page is to contain 36 square inches of print. The margins on each side are $1\frac{1}{2}$ inches. Find the dimensions of the page such that the least amount of paper is used.

19. **Chemical Reaction** In an autocatalytic chemical reaction, the product formed is a catalyst for the reaction. If Q_0 is the amount of the original substance and x is the amount of catalyst formed, the rate of chemical reaction is

$$\frac{dQ}{dx} = kx(Q_0 - x).$$

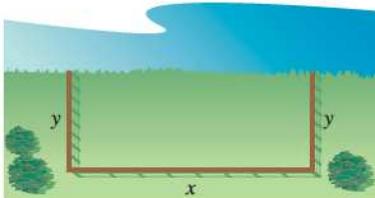
For what value of x will the rate of chemical reaction be greatest?

20. **Traffic Control** On a given day, the flow rate F (in cars per hour) on a congested roadway is

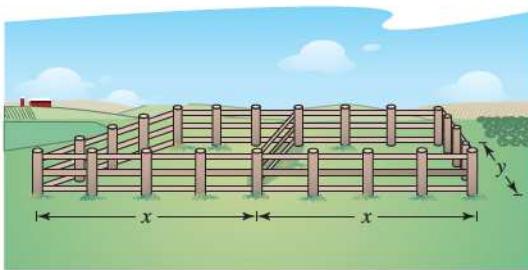
$$F = \frac{v}{22 + 0.02v^2}$$

where v is the speed of the traffic in miles per hour. What speed will maximize the flow rate on the road?

- 21. Area** A farmer plans to fence a rectangular pasture adjacent to a river (see figure). The pasture must contain 245,000 square meters in order to provide enough grass for the herd. What dimensions will require the least amount of fencing if no fencing is needed along the river?

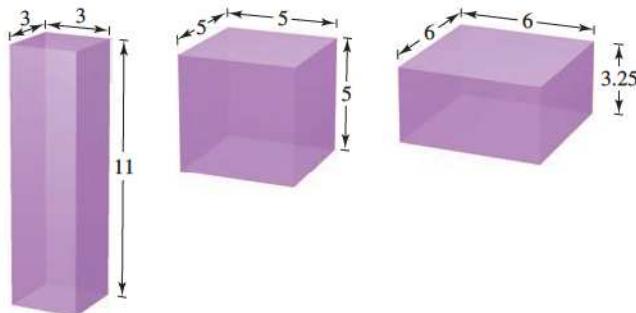


- 22. Maximum Area** A rancher has 400 feet of fencing with which to enclose two adjacent rectangular corrals (see figure). What dimensions should be used so that the enclosed area will be a maximum?



23. Maximum Volume

- Verify that each of the rectangular solids shown in the figure has a surface area of 150 square inches.
- Find the volume of each solid.
- Determine the dimensions of a rectangular solid (with a square base) of maximum volume if its surface area is 150 square inches.



- 24. Maximum Volume** Determine the dimensions of a rectangular solid (with a square base) with maximum volume if its surface area is 337.5 square centimeters.

- 25. Maximum Area** A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window (see figure). Find the dimensions of a Norman window of maximum area if the total perimeter is 16 feet.



Figure for 25

- 26. Maximum Area** A rectangle is bounded by the x - and y -axes and the graph of $y = (6 - x)/2$ (see figure). What length and width should the rectangle have so that its area is a maximum?

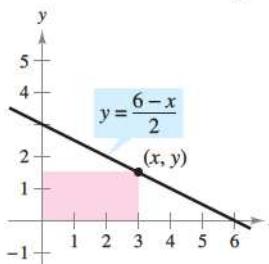


Figure for 26

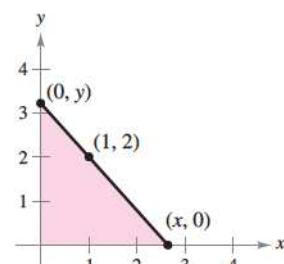


Figure for 27

- 27. Minimum Length** A right triangle is formed in the first quadrant by the x - and y -axes and a line through the point $(1, 2)$ (see figure).

- Write the length L of the hypotenuse as a function of x .
 - Use a graphing utility to approximate x graphically such that the length of the hypotenuse is a minimum.
 - Find the vertices of the triangle such that its area is a minimum.
- 28. Maximum Area** Find the area of the largest isosceles triangle that can be inscribed in a circle of radius 6 (see figure).
- Solve by writing the area as a function of h .
 - Solve by writing the area as a function of α .
 - Identify the type of triangle of maximum area.

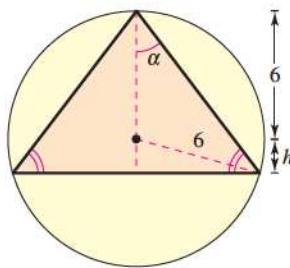


Figure for 28

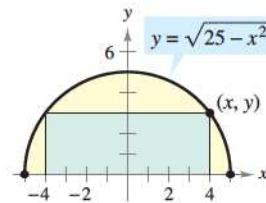


Figure for 29

- 29. Maximum Area** A rectangle is bounded by the x -axis and the semicircle $y = \sqrt{25 - x^2}$ (see figure). What length and width should the rectangle have so that its area is a maximum?

- 30. Area** Find the dimensions of the largest rectangle that can be inscribed in a semicircle of radius r (see Exercise 29).

- 31. Numerical, Graphical, and Analytic Analysis** An exercise room consists of a rectangle with a semicircle on each end. A 200-meter running track runs around the outside of the room.

- Draw a figure to represent the problem. Let x and y represent the length and width of the rectangle.
- Analytically complete six rows of a table such as the one below. (The first two rows are shown.) Use the table to guess the maximum area of the rectangular region.

Length x	Width y	Area A
10	$\frac{2}{\pi}(100 - 10)$	$(10)\frac{2}{\pi}(100 - 10) \approx 573$
20	$\frac{2}{\pi}(100 - 20)$	$(20)\frac{2}{\pi}(100 - 20) \approx 1019$

- Write the area A as a function of x .
- Use calculus to find the critical number of the function in part (c) and find the maximum value.
- Use a graphing utility to graph the function in part (c) and verify the maximum area from the graph.

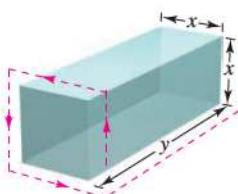
- 32. Numerical, Graphical, and Analytic Analysis** A right circular cylinder is to be designed to hold 22 cubic inches of a soft drink (approximately 12 fluid ounces).

- Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

Radius r	Height	Surface Area S
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$

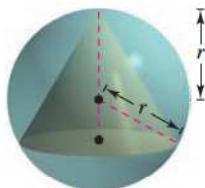
- Use a graphing utility to generate additional rows of the table. Use the table to estimate the minimum surface area. (Hint: Use the *table* feature of the graphing utility.)
- Write the surface area S as a function of r .
- Use a graphing utility to graph the function in part (c) and estimate the minimum surface area from the graph.
- Use calculus to find the critical number of the function in part (c) and find dimensions that will yield the minimum surface area.

- 33. Maximum Volume** A rectangular package to be sent by a postal service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches (see figure). Find the dimensions of the package of maximum volume that can be sent. (Assume the cross section is square.)



- 34. Maximum Volume** Rework Exercise 33 for a cylindrical package. (The cross section is circular.)

- 35. Maximum Volume** Find the volume of the largest right circular cone that can be inscribed in a sphere of radius r .



- 36. Maximum Volume** Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius r .

WRITING ABOUT CONCEPTS

37. A shampoo bottle is a right circular cylinder. Because the surface area of the bottle does not change when it is squeezed, is it true that the volume remains the same? Explain.

CAPSTONE

38. The perimeter of a rectangle is 20 feet. Of all possible dimensions, the maximum area is 25 square feet when its length and width are both 5 feet. Are there dimensions that yield a minimum area? Explain.

- 39. Minimum Surface Area** A solid is formed by adjoining two hemispheres to the ends of a right circular cylinder. The total volume of the solid is 14 cubic centimeters. Find the radius of the cylinder that produces the minimum surface area.

- 40. Minimum Cost** An industrial tank of the shape described in Exercise 39 must have a volume of 4000 cubic feet. The hemispherical ends cost twice as much per square foot of surface area as the sides. Find the dimensions that will minimize cost.

- 41. Minimum Area** The sum of the perimeters of an equilateral triangle and a square is 10. Find the dimensions of the triangle and the square that produce a minimum total area.

- 42. Maximum Area** Twenty feet of wire is to be used to form two figures. In each of the following cases, how much wire should be used for each figure so that the total enclosed area is a maximum?

(a) Equilateral triangle and square

(b) Square and regular pentagon

(c) Regular pentagon and regular hexagon

(d) Regular hexagon and circle

What can you conclude from this pattern? (Hint: The area of a regular polygon with n sides of length x is $A = (n/4)[\cot(\pi/n)]x^2$.)

- 43. Beam Strength** A wooden beam has a rectangular cross section of height h and width w (see figure on the next page). The strength S of the beam is directly proportional to the width and the square of the height. What are the dimensions of the strongest beam that can be cut from a round log of diameter 20 inches? (Hint: $S = kh^2w$, where k is the proportionality constant.)

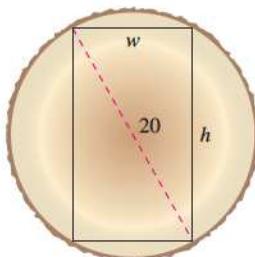


Figure for 43

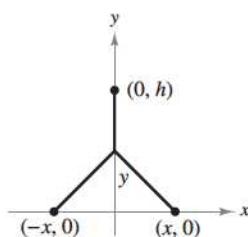


Figure for 44

- 44. Minimum Length** Two factories are located at the coordinates $(-x, 0)$ and $(x, 0)$, and their power supply is at $(0, h)$ (see figure). Find y such that the total length of power line from the power supply to the factories is a minimum.

- 45. Projectile Range** The range R of a projectile fired with an initial velocity v_0 at an angle θ with the horizontal is

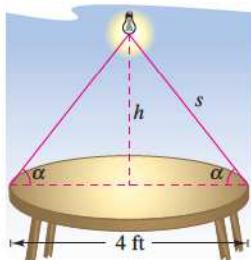
$$R = \frac{v_0^2 \sin 2\theta}{g}$$

where g is the acceleration due to gravity. Find the angle θ such that the range is a maximum.

- 46. Conjecture** Consider the functions $f(x) = \frac{1}{2}x^2$ and $g(x) = \frac{1}{16}x^4 - \frac{1}{2}x^2$ on the domain $[0, 4]$.

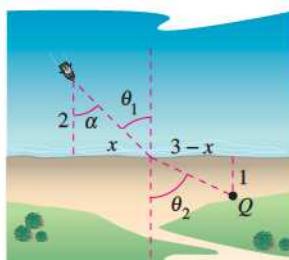
- (a) Use a graphing utility to graph the functions on the specified domain.
 (b) Write the vertical distance d between the functions as a function of x and use calculus to find the value of x for which d is maximum.
 (c) Find the equations of the tangent lines to the graphs of f and g at the critical number found in part (b). Graph the tangent lines. What is the relationship between the lines?
 (d) Make a conjecture about the relationship between tangent lines to the graphs of two functions at the value of x at which the vertical distance between the functions is greatest, and prove your conjecture.

- 47. Illumination** A light source is located over the center of a circular table of diameter 4 feet (see figure). Find the height h of the light source such that the illumination I at the perimeter of the table is maximum if $I = k(\sin \alpha)/s^2$, where s is the slant height, α is the angle at which the light strikes the table, and k is a constant.



- 48. Illumination** The illumination from a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. Two light sources of intensities I_1 and I_2 are d units apart. What point on the line segment joining the two sources has the least illumination?

- 49. Minimum Time** A man is in a boat 2 miles from the nearest point on the coast. He is to go to a point Q , located 3 miles down the coast and 1 mile inland (see figure). He can row at 2 miles per hour and walk at 4 miles per hour. Toward what point on the coast should he row in order to reach point Q in the least time?



- 50. Minimum Time** Consider Exercise 49 if the point Q is on the shoreline rather than 1 mile inland.

- (a) Write the travel time T as a function of α .
 (b) Use the result of part (a) to find the minimum time to reach Q .
 (c) The man can row at v_1 miles per hour and walk at v_2 miles per hour. Write the time T as a function of α . Show that the critical number of T depends only on v_1 and v_2 and not on the distances. Explain how this result would be more beneficial to the man than the result of Exercise 49.
 (d) Describe how to apply the result of part (c) to minimizing the cost of constructing a power transmission cable that costs c_1 dollars per mile under water and c_2 dollars per mile over land.

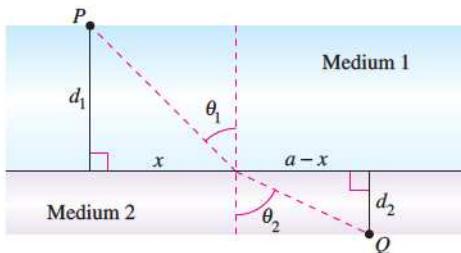
- 51. Minimum Time** The conditions are the same as in Exercise 49 except that the man can row at v_1 miles per hour and walk at v_2 miles per hour. If θ_1 and θ_2 are the magnitudes of the angles, show that the man will reach point Q in the least time when

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.$$

- 52. Minimum Time** When light waves traveling in a transparent medium strike the surface of a second transparent medium, they change direction. This change of direction is called *refraction* and is defined by **Snell's Law of Refraction**,

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

where θ_1 and θ_2 are the magnitudes of the angles shown in the figure and v_1 and v_2 are the velocities of light in the two media. Show that this problem is equivalent to that in Exercise 51, and that light waves traveling from P to Q follow the path of minimum time.



- A** 53. Sketch the graph of $f(x) = 2 - 2 \sin x$ on the interval $[0, \pi/2]$.
- Find the distance from the origin to the y -intercept and the distance from the origin to the x -intercept.
 - Write the distance d from the origin to a point on the graph of f as a function of x . Use your graphing utility to graph d and find the minimum distance.
 - Use calculus and the *zero* or *root* feature of a graphing utility to find the value of x that minimizes the function d on the interval $[0, \pi/2]$. What is the minimum distance?
- (Submitted by Tim Chapell, Penn Valley Community College, Kansas City, MO)
- 54. Minimum Cost** An offshore oil well is 2 kilometers off the coast. The refinery is 4 kilometers down the coast. Laying pipe in the ocean is twice as expensive as on land. What path should the pipe follow in order to minimize the cost?
- 55. Maximum Volume** A sector with central angle θ is cut from a circle of radius 12 inches (see figure), and the edges of the sector are brought together to form a cone. Find the magnitude of θ such that the volume of the cone is a maximum.

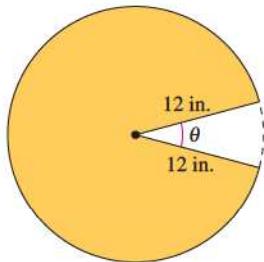


Figure for 55

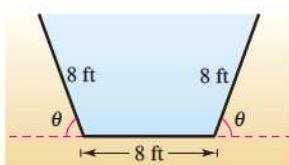


Figure for 56

- A** 56. *Numerical, Graphical, and Analytic Analysis* The cross sections of an irrigation canal are isosceles trapezoids of which three sides are 8 feet long (see figure). Determine the angle of elevation θ of the sides such that the area of the cross sections is a maximum by completing the following.
- Analytically complete six rows of a table such as the one below. (The first two rows are shown.)
- | Base 1 | Base 2 | Altitude | Area |
|--------|------------------------|-------------------|----------------|
| 8 | $8 + 16 \cos 10^\circ$ | $8 \sin 10^\circ$ | ≈ 22.1 |
| 8 | $8 + 16 \cos 20^\circ$ | $8 \sin 20^\circ$ | ≈ 42.5 |
- Use a graphing utility to generate additional rows of the table and estimate the maximum cross-sectional area. (Hint: Use the *table* feature of the graphing utility.)
 - Write the cross-sectional area A as a function of θ .
 - Use calculus to find the critical number of the function in part (c) and find the angle that will yield the maximum cross-sectional area.
 - Use a graphing utility to graph the function in part (c) and verify the maximum cross-sectional area.
- A** 57. *Minimum Cost* The ordering and transportation cost C of the components used in manufacturing a product is
- $$C = 100\left(\frac{200}{x^2} + \frac{x}{x+30}\right), \quad x \geq 1$$
- where C is measured in thousands of dollars and x is the order size in hundreds. Find the order size that minimizes the cost. (Hint: Use the *root* feature of a graphing utility.)
- 58. Diminishing Returns** The profit P (in thousands of dollars) for a company spending an amount s (in thousands of dollars) on advertising is
- $$P = -\frac{1}{10}s^3 + 6s^2 + 400.$$
- Find the amount of money the company should spend on advertising in order to yield a maximum profit.
 - The *point of diminishing returns* is the point at which the rate of growth of the profit function begins to decline. Find the point of diminishing returns.
- 59. Area** Find the area of the largest rectangle that can be inscribed under the curve $y = e^{-x^2}$ in the first and second quadrants.
- 60. Population Growth** Fifty elk are introduced into a game preserve. It is estimated that their population will increase according to the model $p(t) = 250/(1 + 4e^{-t/3})$, where t is measured in years. At what rate is the population increasing when $t = 2$? After how many years is the population increasing most rapidly?
- 61. Verify that the function**

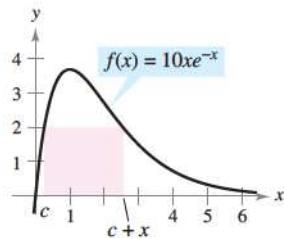
$$y = \frac{L}{1 + ae^{-bx}}, \quad a > 0, b > 0, L > 0$$

increases at the maximum rate when $y = L/2$.

- A** 62. *Area* Perform the following steps to find the maximum area of the rectangle shown in the figure.
- Solve for c in the equation $f(c) = f(c+x)$.
 - Use the result in part (a) to write the area A as a function of x . [Hint: $A = xf(c)$]
 - Use a graphing utility to graph the area function. Use the graph to approximate the dimensions of the rectangle of maximum area. Determine the required area.
 - Use a graphing utility to graph the expression for c found in part (a). Use the graph to approximate

$$\lim_{x \rightarrow 0^+} c \quad \text{and} \quad \lim_{x \rightarrow \infty} c.$$

Use this result to describe the changes in the dimensions and position of the rectangle for $0 < x < \infty$.



Minimum Distance In Exercises 63–65, consider a fuel distribution center located at the origin of the rectangular coordinate system (units in miles; see figures). The center supplies three factories with coordinates $(4, 1)$, $(5, 6)$, and $(10, 3)$. A trunk line will run from the distribution center along the line $y = mx$, and feeder lines will run to the three factories. The objective is to find m such that the lengths of the feeder lines are minimized.

63. Minimize the sum of the squares of the lengths of the vertical feeder lines given by

$$S_1 = (4m - 1)^2 + (5m - 6)^2 + (10m - 3)^2.$$

Find the equation of the trunk line by this method and then determine the sum of the lengths of the feeder lines.

-  64. Minimize the sum of the absolute values of the lengths of the vertical feeder lines given by

$$S_2 = |4m - 1| + |5m - 6| + |10m - 3|.$$

Find the equation of the trunk line by this method and then determine the sum of the lengths of the feeder lines. (*Hint:* Use a graphing utility to graph the function S_2 and approximate the required critical number.)

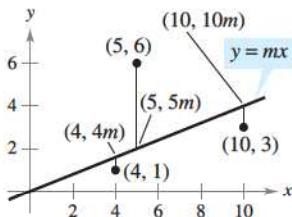


Figure for 63 and 64

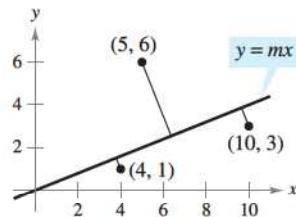


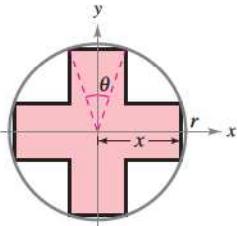
Figure for 65

-  65. Minimize the sum of the perpendicular distances (see Exercises 87–92 in Section 1.2) from the trunk line to the factories given by

$$S_3 = \frac{|4m - 1|}{\sqrt{m^2 + 1}} + \frac{|5m - 6|}{\sqrt{m^2 + 1}} + \frac{|10m - 3|}{\sqrt{m^2 + 1}}.$$

Find the equation of the trunk line by this method and then determine the sum of the lengths of the feeder lines. (*Hint:* Use a graphing utility to graph the function S_3 and approximate the required critical number.)

66. **Maximum Area** Consider a symmetric cross inscribed in a circle of radius r (see figure).



- (a) Write the area A of the cross as a function of x and find the value of x that maximizes the area.
 (b) Write the area A of the cross as a function of θ and find the value of θ that maximizes the area.
 (c) Show that the critical numbers of parts (a) and (b) yield the same maximum area. What is that area?

PUTNAM EXAM CHALLENGE

67. Find the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$. Explain your reasoning.

68. Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)} \text{ for } x > 0.$$

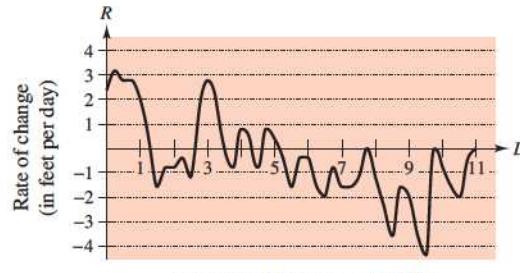
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SECTION PROJECT

Connecticut River

Whenever the Connecticut River reaches a level of 105 feet above sea level, two Northampton, Massachusetts flood control station operators begin a round-the-clock river watch. Every 2 hours, they check the height of the river, using a scale marked off in tenths of a foot, and record the data in a log book. In the spring of 1996, the flood watch lasted from April 4, when the river reached 105 feet and was rising at 0.2 foot per hour, until April 25, when the level subsided again to 105 feet. Between those dates, their log shows that the river rose and fell several times, at one point coming close to the 115-foot mark. If the river had reached 115 feet, the city would have closed down Mount Tom Road (Route 5, south of Northampton).

The graph below shows the rate of change of the level of the river during one portion of the flood watch. Use the graph to answer each question.



Day (0 ↔ 12:01 A.M. April 14)

- On what date was the river rising most rapidly? How do you know?
- On what date was the river falling most rapidly? How do you know?
- There were two dates in a row on which the river rose, then fell, then rose again during the course of the day. On which days did this occur, and how do you know?
- At 1 minute past midnight, April 14, the river level was 111.0 feet. Estimate its height 24 hours later and 48 hours later. Explain how you made your estimates.
- The river crested at 114.4 feet. On what date do you think this occurred?

(Submitted by Mary Murphy, Smith College, Northampton, MA)

4.8 Differentials

- Understand the concept of a tangent line approximation.
- Compare the value of the differential, dy , with the actual change in y , Δy .
- Estimate a propagated error using a differential.
- Find the differential of a function using differentiation formulas.

EXPLORATION

Tangent Line Approximation

Use a graphing utility to graph

$$f(x) = x^2.$$

In the same viewing window, graph the tangent line to the graph of f at the point $(1, 1)$. Zoom in twice on the point of tangency. Does your graphing utility distinguish between the two graphs? Use the *trace* feature to compare the two graphs. As the x -values get closer to 1, what can you say about the y -values?

Tangent Line Approximations

Newton's Method (Section 3.8) is an example of the use of a tangent line to a graph to approximate the graph. In this section, you will study other situations in which the graph of a function can be approximated by a straight line.

To begin, consider a function f that is differentiable at c . The equation for the tangent line at the point $(c, f(c))$ is given by

$$y - f(c) = f'(c)(x - c)$$

$$y = f(c) + f'(c)(x - c)$$

and is called the **tangent line approximation** (or **linear approximation**) of f at c . Because c is a constant, y is a linear function of x . Moreover, by restricting the values of x to those sufficiently close to c , the values of y can be used as approximations (to any desired degree of accuracy) of the values of the function f . In other words, as $x \rightarrow c$, the limit of y is $f(c)$.

EXAMPLE 1 Using a Tangent Line Approximation

Find the tangent line approximation of

$$f(x) = 1 + \sin x$$

at the point $(0, 1)$. Then use a table to compare the y -values of the linear function with those of $f(x)$ on an open interval containing $x = 0$.

Solution The derivative of f is

$$f'(x) = \cos x. \quad \text{First derivative}$$

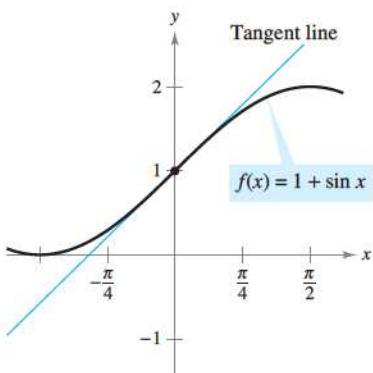
So, the equation of the tangent line to the graph of f at the point $(0, 1)$ is

$$y - f(0) = f'(0)(x - 0)$$

$$y - 1 = (1)(x - 0)$$

$$y = 1 + x. \quad \text{Tangent line approximation}$$

The table compares the values of y given by this linear approximation with the values of $f(x)$ near $x = 0$. Notice that the closer x is to 0, the better the approximation is. This conclusion is reinforced by the graph shown in Figure 4.63.

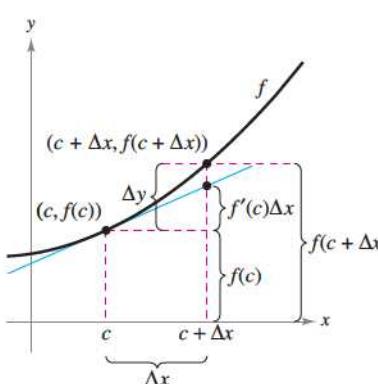


The tangent line approximation of f at the point $(0, 1)$

Figure 4.63

x	-0.5	-0.1	-0.01	0	0.01	0.1	0.5
$f(x) = 1 + \sin x$	0.521	0.9002	0.9900002	1	1.0099998	1.0998	1.479
$y = 1 + x$	0.5	0.9	0.99	1	1.01	1.1	1.5

NOTE Be sure you see that this linear approximation of $f(x) = 1 + \sin x$ depends on the point of tangency. At a different point on the graph of f , you would obtain a different tangent line approximation.



When Δx is small, $\Delta y = f(c + \Delta x) - f(c)$ is approximated by $f'(c)\Delta x$.

Figure 4.64

Differentials

When the tangent line to the graph of f at the point $(c, f(c))$

$$y = f(c) + f'(c)(x - c)$$

Tangent line at $(c, f(c))$

is used as an approximation of the graph of f , the quantity $x - c$ is called the change in x , and is denoted by Δx , as shown in Figure 4.64. When Δx is small, the change in y (denoted by Δy) can be approximated as shown.

$$\begin{aligned}\Delta y &= f(c + \Delta x) - f(c) \\ &\approx f'(c)\Delta x\end{aligned}$$

Actual change in y

Approximate change in y

For such an approximation, the quantity Δx is traditionally denoted by dx , and is called the **differential of x** . The expression $f'(x)dx$ is denoted by dy , and is called the **differential of y** .

DEFINITION OF DIFFERENTIALS

Let $y = f(x)$ represent a function that is differentiable on an open interval containing x . The **differential of x** (denoted by dx) is any nonzero real number. The **differential of y** (denoted by dy) is

$$dy = f'(x) dx.$$

In many types of applications, the differential of y can be used as an approximation of the change in y . That is, for small values of $\Delta x = dx$,

$$\Delta y \approx dy \quad \text{or} \quad \Delta y \approx f'(x)dx.$$

EXAMPLE 2 Comparing Δy and dy

Let $y = x^2$. Find dy when $x = 1$ and $dx = 0.01$. Compare this value with Δy for $x = 1$ and $\Delta x = 0.01$.

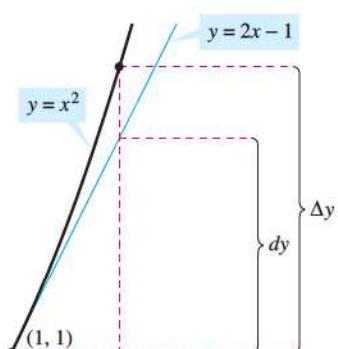
Solution Because $y = f(x) = x^2$, you have $f'(x) = 2x$, and the differential dy is given by

$$dy = f'(x) dx = f'(1)(0.01) = 2(0.01) = 0.02. \quad \text{Differential of } y$$

Now, using $\Delta x = 0.01$, the change in y is

$$\Delta y = f(x + \Delta x) - f(x) = f(1.01) - f(1) = (1.01)^2 - 1^2 = 0.0201.$$

Figure 4.65 shows the geometric comparison of dy and Δy . Try comparing other values of dy and Δy . You will see that the values become closer to each other as dx (or Δx) approaches 0. ■



The change in y , Δy , is approximated by the differential of y , dy .

Figure 4.65

In Example 2, the tangent line to the graph of $f(x) = x^2$ at $x = 1$ is

$$y = 2x - 1 \quad \text{or} \quad g(x) = 2x - 1.$$

Tangent line to the graph of f at $x = 1$

For x -values near 1, this line is close to the graph of f , as shown in Figure 4.65. For instance,

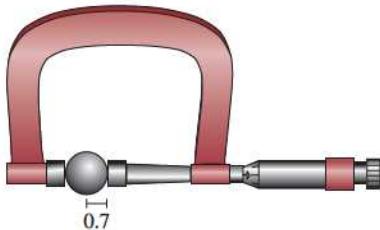
$$f(1.01) = 1.01^2 = 1.0201 \quad \text{and} \quad g(1.01) = 2(1.01) - 1 = 1.02.$$

Error Propagation

Physicists and engineers tend to make liberal use of the approximation of Δy by dy . One way this occurs in practice is in the estimation of errors propagated by physical measuring devices. For example, if you let x represent the measured value of a variable and let $x + \Delta x$ represent the exact value, then Δx is the *error in measurement*. Finally, if the measured value x is used to compute another value $f(x)$, the difference between $f(x + \Delta x)$ and $f(x)$ is the **propagated error**.

$$\begin{array}{ccc} \text{Measurement} & & \text{Propagated} \\ \text{error} & & \text{error} \\ f(x + \Delta x) - f(x) = \Delta y & & \\ \underbrace{f(x)}_{\text{Exact}} & & \underbrace{f(x + \Delta x)}_{\text{Measured}} \\ & & \text{value} \end{array}$$

EXAMPLE 3 Estimation of Error



Ball bearing with measured radius that is correct to within 0.01 inch

Figure 4.66

The measured radius of a ball bearing is 0.7 inch, as shown in Figure 4.66. If the measurement is correct to within 0.01 inch, estimate the propagated error in the volume V of the ball bearing.

Solution The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere. So, you can write

$$r = 0.7$$

Measured radius

and

$$-0.01 \leq \Delta r \leq 0.01.$$

Possible error

To approximate the propagated error in the volume, differentiate V to obtain $dV/dr = 4\pi r^2$ and write

$$\begin{aligned} \Delta V &\approx dV && \text{Approximate } \Delta V \text{ by } dV. \\ &= 4\pi r^2 dr \\ &= 4\pi(0.7)^2(\pm 0.01) && \text{Substitute for } r \text{ and } dr. \\ &\approx \pm 0.06158 \text{ cubic inch.} \end{aligned}$$

So, the volume has a propagated error of about 0.06 cubic inch. ■

Would you say that the propagated error in Example 3 is large or small? The answer is best given in *relative* terms by comparing dV with V . The ratio

$$\begin{aligned} \frac{dV}{V} &= \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} && \text{Ratio of } dV \text{ to } V \\ &= \frac{3 dr}{r} && \text{Simplify.} \\ &\approx \frac{3}{0.7}(\pm 0.01) && \text{Substitute for } dr \text{ and } r. \\ &\approx \pm 0.0429 \end{aligned}$$

is called the **relative error**. The corresponding **percent error** is approximately 4.29%.

Calculating Differentials

Each of the differentiation rules that you studied in Chapter 3 can be written in **differential form**. For example, suppose u and v are differentiable functions of x . By the definition of differentials, you have

$$du = u' dx \quad \text{and} \quad dv = v' dx.$$

So, you can write the differential form of the Product Rule as shown below.

$$\begin{aligned} d[uv] &= \frac{d}{dx}[uv] dx && \text{Differential of } uv \\ &= [uv' + vu'] dx && \text{Product Rule} \\ &= uv' dx + vu' dx \\ &= u dv + v du \end{aligned}$$

DIFFERENTIAL FORMULAS

Let u and v be differentiable functions of x .

Constant multiple: $d[cu] = c du$

Sum or difference: $d[u \pm v] = du \pm dv$

Product: $d[uv] = u dv + v du$

Quotient: $d\left[\frac{u}{v}\right] = \frac{v du - u dv}{v^2}$

EXAMPLE 4 Finding Differentials

Mary Evans Picture Library



GOTTFRIED WILHELM LEIBNIZ (1646–1716)

Both Leibniz and Newton are credited with creating calculus. It was Leibniz, however, who tried to broaden calculus by developing rules and formal notation. He often spent days choosing an appropriate notation for a new concept.

Function	Derivative	Differential
a. $y = x^2$	$\frac{dy}{dx} = 2x$	$dy = 2x dx$
b. $y = 2 \sin x$	$\frac{dy}{dx} = 2 \cos x$	$dy = 2 \cos x dx$
c. $y = xe^x$	$\frac{dy}{dx} = e^x(x + 1)$	$dy = e^x(x + 1) dx$
d. $y = \frac{1}{x}$	$\frac{dy}{dx} = -\frac{1}{x^2}$	$dy = -\frac{dx}{x^2}$

The notation in Example 4 is called the **Leibniz notation** for derivatives and differentials, named after the German mathematician Gottfried Wilhelm Leibniz. The beauty of this notation is that it provides an easy way to remember several important calculus formulas by making it seem as though the formulas were derived from algebraic manipulations of differentials. For instance, in Leibniz notation, the *Chain Rule*

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

would appear to be true because the du 's divide out. Even though this reasoning is *incorrect*, the notation does help one remember the Chain Rule.

EXAMPLE 5 Finding the Differential of a Composite Function

$$y = f(x) = \sin 3x \quad \text{Original function}$$

$f'(x) = 3 \cos 3x \quad \text{Apply Chain Rule.}$

$$dy = f'(x) dx = 3 \cos 3x dx \quad \text{Differential form}$$

EXAMPLE 6 Finding the Differential of a Composite Function

$$y = f(x) = (x^2 + 1)^{1/2} \quad \text{Original function}$$

$$f'(x) = \frac{1}{2}(x^2 + 1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + 1}} \quad \text{Apply Chain Rule.}$$

$$dy = f'(x) dx = \frac{x}{\sqrt{x^2 + 1}} dx \quad \text{Differential form}$$

Differentials can be used to approximate function values. To do this for the function given by $y = f(x)$, use the formula

$$f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x) dx$$

which is derived from the approximation $\Delta y = f(x + \Delta x) - f(x) \approx dy$. The key to using this formula is to choose a value for x that makes the calculations easier, as shown in Example 7. (This formula is equivalent to the tangent line approximation given earlier in this section.)

EXAMPLE 7 Approximating Function Values

Use differentials to approximate $\sqrt{16.5}$.

Solution Using $f(x) = \sqrt{x}$, you can write

$$f(x + \Delta x) \approx f(x) + f'(x) dx = \sqrt{x} + \frac{1}{2\sqrt{x}} dx.$$

Now, choosing $x = 16$ and $dx = 0.5$, you obtain the following approximation.

$$f(x + \Delta x) = \sqrt{16.5} \approx \sqrt{16} + \frac{1}{2\sqrt{16}}(0.5) = 4 + \left(\frac{1}{8}\right)\left(\frac{1}{2}\right) = 4.0625$$

The tangent line approximation to $f(x) = \sqrt{x}$ at $x = 16$ is the line $g(x) = \frac{1}{8}x + 2$. For x -values near 16, the graphs of f and g are close together, as shown in Figure 4.67. For instance,

$$f(16.5) = \sqrt{16.5} \approx 4.0620 \quad \text{and} \quad g(16.5) = \frac{1}{8}(16.5) + 2 = 4.0625.$$

In fact, if you use a graphing utility to zoom in near the point of tangency $(16, 4)$, you will see that the two graphs appear to coincide. Notice also that as you move farther away from the point of tangency, the linear approximation becomes less accurate.

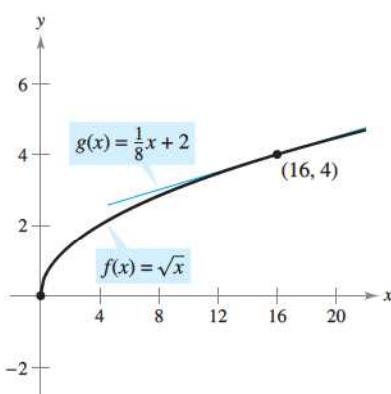


Figure 4.67

4.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the equation of the tangent line T to the graph of f at the given point. Use this linear approximation to complete the table.

x	1.9	1.99	2	2.01	2.1
$f(x)$					
$T(x)$					

- | Function | Point |
|---------------------------|-------------------------------|
| 1. $f(x) = x^2$ | (2, 4) |
| 2. $f(x) = \frac{6}{x^2}$ | $\left(2, \frac{3}{2}\right)$ |
| 3. $f(x) = x^5$ | (2, 32) |
| 4. $f(x) = \sqrt{x}$ | $(2, \sqrt{2})$ |
| 5. $f(x) = \sin x$ | $(2, \sin 2)$ |
| 6. $f(x) = \log_2 x$ | (2, 1) |

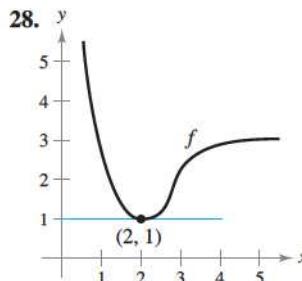
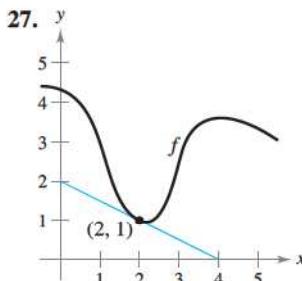
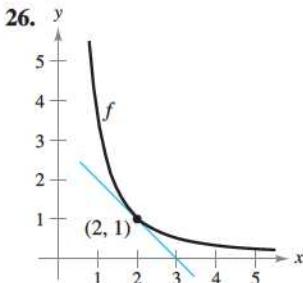
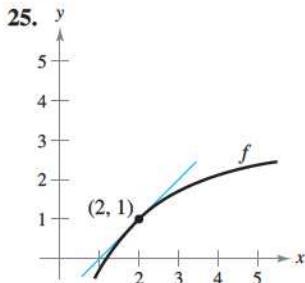
In Exercises 7–10, use the information to evaluate and compare Δy and dy .

- | | | |
|-------------------|----------|------------------------|
| 7. $y = x^3$ | $x = 1$ | $\Delta x = dx = 0.1$ |
| 8. $y = 1 - 2x^2$ | $x = 0$ | $\Delta x = dx = -0.1$ |
| 9. $y = x^4 + 1$ | $x = -1$ | $\Delta x = dx = 0.01$ |
| 10. $y = 2 - x^4$ | $x = 2$ | $\Delta x = dx = 0.01$ |

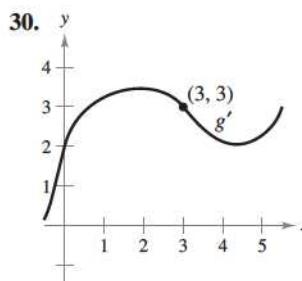
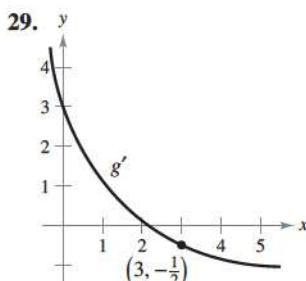
In Exercises 11–24, find the differential dy of the given function.

- | | |
|---|---|
| 11. $y = 3x^2 - 4$ | 12. $y = 3x^{2/3}$ |
| 13. $y = \frac{x+1}{2x-1}$ | 14. $y = \sqrt{9-x^2}$ |
| 15. $y = x\sqrt{1-x^2}$ | 16. $y = \sqrt{x} + \frac{1}{\sqrt{x}}$ |
| 17. $y = \ln\sqrt{4-x^2}$ | 18. $y = e^{-0.5x} \cos 4x$ |
| 19. $y = 3x - \sin^2 x$ | 20. $y = x \cos x$ |
| 21. $y = \frac{1}{3} \cos\left(\frac{6\pi x - 1}{2}\right)$ | 22. $y = \frac{\sec^2 x}{x^2 + 1}$ |
| 23. $y = x \arcsin x$ | 24. $y = \arctan(x-2)$ |

In Exercises 25–28, use differentials and the graph of f to approximate (a) $f(1.9)$ and (b) $f(2.04)$. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.



In Exercises 29 and 30, use differentials and the graph of g' to approximate (a) $g(2.93)$ and (b) $g(3.1)$ given that $g(3) = 8$.



31. **Area** The measurement of the side of a square floor tile is 10 inches, with a possible error of $\frac{1}{32}$ inch. Use differentials to approximate the possible propagated error in computing the area of the square.

32. **Area** The measurements of the base and altitude of a triangle are found to be 36 and 50 centimeters, respectively. The possible error in each measurement is 0.25 centimeter. Use differentials to approximate the possible propagated error in computing the area of the triangle.

33. **Area** The measurement of the radius of the end of a log is found to be 16 inches, with a possible error of $\frac{1}{4}$ inch. Use differentials to approximate the possible propagated error in computing the area of the end of the log.

34. **Volume and Surface Area** The measurement of the edge of a cube is found to be 15 inches, with a possible error of 0.03 inch. Use differentials to approximate the maximum possible propagated error in computing (a) the volume of the cube and (b) the surface area of the cube.

35. **Area** The measurement of a side of a square is found to be 12 centimeters, with a possible error of 0.05 centimeter.

- (a) Approximate the percent error in computing the area of the square.
 (b) Estimate the maximum allowable percent error in measuring the side if the error in computing the area cannot exceed 2.5%.

36. **Circumference** The measurement of the circumference of a circle is found to be 64 centimeters, with a possible error of 0.9 centimeter.

- (a) Approximate the percent error in computing the area of the circle.

- (b) Estimate the maximum allowable percent error in measuring the circumference if the error in computing the area cannot exceed 3%.
- 37. Volume and Surface Area** The radius of a spherical balloon is measured as 8 inches, with a possible error of 0.02 inch. Use differentials to approximate the maximum possible error in calculating (a) the volume of the sphere, (b) the surface area of the sphere, and (c) the relative errors in parts (a) and (b).
- 38. Stopping Distance** The total stopping distance T of a vehicle is

$$T = 2.5x + 0.5x^2$$

where T is in feet and x is the speed in miles per hour. Approximate the change and percent change in total stopping distance as speed increases from $x = 25$ to $x = 26$ miles per hour.

- 39. Profit** The profit P for a company is $P = 100xe^{-x/400}$, where x is sales. Approximate the change and percent change in profit as sales increase from $x = 115$ to $x = 120$ units.
- 40. Relative Humidity** When the dewpoint is 65° Fahrenheit, the relative humidity H is modeled by

$$H = \frac{4347}{400,000,000} e^{369.444/(50t + 19.793)}$$

where t is the air temperature in degrees Fahrenheit. Use differentials to approximate the change in relative humidity at $t = 72$ for a 1-degree change in the air temperature.

In Exercises 41 and 42, the thickness of the shell is 0.2 centimeter. Use differentials to approximate the volume of the shell.

- 41.** A cylindrical shell with height 40 centimeters and radius 5 centimeters
- 42.** A spherical shell of radius 100 centimeters

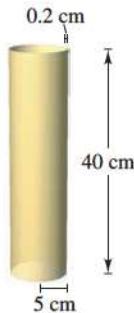


Figure for 41

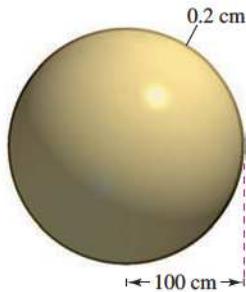


Figure for 42

- 43. Triangle Measurements** The measurement of one side of a right triangle is found to be 9.5 inches, and the angle opposite that side is $26^\circ 45'$ with a possible error of $15'$.
- (a) Approximate the percent error in computing the length of the hypotenuse.
- (b) Estimate the maximum allowable percent error in measuring the angle if the error in computing the length of the hypotenuse cannot exceed 2%.

- 44. Ohm's Law** A current of I amperes passes through a resistor of R ohms. **Ohm's Law** states that the voltage E applied to the resistor is $E = IR$. If the voltage is constant, show that the magnitude of the relative error in R caused by a change in I is equal in magnitude to the relative error in I .

- 45. Projectile Motion** The range R of a projectile is

$$R = \frac{v_0^2}{32} (\sin 2\theta)$$

where v_0 is the initial velocity in feet per second and θ is the angle of elevation. If $v_0 = 2500$ feet per second and θ is changed from 10° to 11° , use differentials to approximate the change in the range.

- 46. Surveying** A surveyor standing 50 feet from the base of a large tree measures the angle of elevation to the top of the tree as 71.5° . How accurately must the angle be measured if the percent error in estimating the height of the tree is to be less than 6%?

In Exercises 47–50, use differentials to approximate the value of the expression. Compare your answer with that of a calculator.

47. $\sqrt{99.4}$ 48. $\sqrt[3]{26}$ 49. $\sqrt[4]{624}$ 50. $(2.99)^3$



In Exercises 51 and 52, verify the tangent line approximation of the function at the given point. Then use a graphing utility to graph the function and its approximation in the same viewing window.

Function	Approximation	Point
51. $f(x) = \sqrt{x+4}$	$y = 2 + \frac{x}{4}$	$(0, 2)$
52. $f(x) = \tan x$	$y = x$	$(0, 0)$

WRITING ABOUT CONCEPTS

53. Describe the change in the accuracy of dy as an approximation for Δy when Δx is decreased.
54. When using differentials, what is meant by the terms *propagated error*, *relative error*, and *percent error*?
55. Give a short explanation of why the approximation is valid.
- $\sqrt{4.02} \approx 2 + \frac{1}{4}(0.02)$
 - $\tan 0.05 \approx 0 + 1(0.05)$

CAPSTONE

56. Would you use $y = x$ to approximate $f(x) = \sin x$ near $x = 0$? Why or why not?

True or False? In Exercises 57–60, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

57. If $y = x + c$, then $dy = dx$.
58. If $y = ax + b$, then $\Delta y/\Delta x = dy/dx$.
59. If y is differentiable, then $\lim_{\Delta x \rightarrow 0} (\Delta y - dy) = 0$.
60. If $y = f(x)$, f is increasing and differentiable, and $\Delta x > 0$, then $\Delta y \geq dy$.

4

REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

- Give the definition of a critical number, and graph a function showing the different types of critical numbers.
- Consider the odd function f that is continuous and differentiable and has the functional values shown in the table.

x	-5	-4	-1	0	2	3	6
$f(x)$	1	3	2	0	-1	-4	0

- Determine $f(4)$.
- Determine $f(-3)$.
- Plot the points and make a possible sketch of the graph of f on the interval $[-6, 6]$. What is the smallest number of critical points in the interval? Explain.
- Does there exist at least one real number c in the interval $(-6, 6)$ where $f'(c) = -1$? Explain.
- Is it possible that $\lim_{x \rightarrow 0} f(x)$ does not exist? Explain.
- Is it necessary that $f'(x)$ exists at $x = 2$? Explain.

In Exercises 3–6, find the absolute extrema of the function on the closed interval. Use a graphing utility to graph the function over the given interval to confirm your results.

- $f(x) = x^2 + 5x$, $[-4, 0]$
- $h(x) = 3\sqrt{x} - x$, $[0, 9]$
- $g(x) = 2x + 5 \cos x$, $[0, 2\pi]$
- $f(x) = \frac{x}{\sqrt{x^2 + 1}}$, $[0, 2]$

In Exercises 7–10, determine whether Rolle's Theorem can be applied to f on the closed interval $[a, b]$. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$. If Rolle's Theorem cannot be applied, explain why not.

- $f(x) = 2x^2 - 7$, $[0, 4]$
- $f(x) = (x - 2)(x + 3)^2$, $[-3, 2]$
- $f(x) = \frac{x^2}{1 - x^2}$, $[-2, 2]$
- $f(x) = |x - 2| - 2$, $[0, 4]$

- Consider the function $f(x) = 3 - |x - 4|$.
 - Graph the function and verify that $f(1) = f(7)$.
 - Note that $f'(x)$ is not equal to zero for any x in $[1, 7]$. Explain why this does not contradict Rolle's Theorem.
- Can the Mean Value Theorem be applied to the function $f(x) = 1/x^2$ on the interval $[-2, 1]$? Explain.

In Exercises 13–18, determine whether the Mean Value Theorem can be applied to f on the closed interval $[a, b]$. If the Mean Value Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$. If the Mean Value Theorem cannot be applied, explain why not.

- $f(x) = x^{2/3}$, $[1, 8]$
- $f(x) = \frac{1}{x}$, $[1, 4]$

- $f(x) = |5 - x|$, $[2, 6]$
- $f(x) = 2x - 3\sqrt{x}$, $[-1, 1]$
- $f(x) = x - \cos x$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $f(x) = x \log_2 x$, $[1, 2]$

- For the function $f(x) = Ax^2 + Bx + C$, determine the value of c guaranteed by the Mean Value Theorem on the interval $[x_1, x_2]$.
- Demonstrate the result of Exercise 19 for $f(x) = 2x^2 - 3x + 1$ on the interval $[0, 4]$.

In Exercises 21–28, find the critical numbers (if any) and the open intervals on which the function is increasing or decreasing.

- $f(x) = x^2 + 3x - 12$
- $h(x) = (x + 2)^{1/3} + 8$
- $f(x) = (x - 1)^2(x - 3)$
- $g(x) = (x + 1)^3$
- $h(x) = \sqrt{x}(x - 3)$, $x > 0$
- $f(x) = \sin x + \cos x$, $[0, 2\pi]$
- $f(t) = (2 - t)2^t$
- $g(x) = 2x \ln x$

In Exercises 29–32, use the First Derivative Test to find any relative extrema of the function. Use a graphing utility to confirm your results.

- $f(x) = 4x^3 - 5x$
- $g(x) = \frac{x^3 - 8x}{4}$
- $h(t) = \frac{1}{4}t^4 - 8t$
- $g(x) = \frac{3}{2} \sin\left(\frac{\pi x}{2} - 1\right)$, $[0, 4]$

- Harmonic Motion** The height of an object attached to a spring is given by the harmonic equation

$$y = \frac{1}{3} \cos 12t - \frac{1}{4} \sin 12t$$

where y is measured in inches and t is measured in seconds.

- Calculate the height and velocity of the object when $t = \pi/8$ second.
- Show that the maximum displacement of the object is $\frac{5}{12}$ inch.
- Find the period P of y . Also, find the frequency f (number of oscillations per second) if $f = 1/P$.

- Writing** The general equation giving the height of an oscillating object attached to a spring is

$$y = A \sin \sqrt{\frac{k}{m}} t + B \cos \sqrt{\frac{k}{m}} t$$

where k is the spring constant and m is the mass of the object.

- Show that the maximum displacement of the object is $\sqrt{A^2 + B^2}$.
- Show that the object oscillates with a frequency of

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

In Exercises 35–38, determine the points of inflection of the function and discuss the concavity of the graph of the function.

35. $f(x) = x^3 - 9x^2$

36. $g(x) = x\sqrt{x+5}$

37. $f(x) = x + \cos x, [0, 2\pi]$

38. $f(x) = (x+2)^2(x-4)$

In Exercises 39–42, use the Second Derivative Test to find all relative extrema.

39. $f(x) = (x+9)^2$

40. $h(x) = x - 2 \cos x, [0, 4\pi]$

41. $g(x) = 2x^2(1-x^2)$

42. $h(t) = t - 4\sqrt{t+1}$

Think About It In Exercises 43 and 44, sketch the graph of a function f having the given characteristics.

43. $f(0) = f(6) = 0$

44. $f(0) = 4, f(6) = 0$

$f'(3) = f'(5) = 0$

$f'(x) < 0$ if $x < 2$ or $x > 4$

$f'(x) > 0$ if $x < 3$

$f'(2)$ does not exist.

$f'(x) > 0$ if $3 < x < 5$

$f'(4) = 0$

$f'(x) < 0$ if $x > 5$

$f'(x) > 0$ if $2 < x < 4$

$f''(x) < 0$ if $x < 3$ or $x > 4$

$f''(x) < 0, x \neq 2$

$f''(x) > 0, 3 < x < 4$

45. **Writing** A newspaper headline states that “The rate of growth of the national deficit is decreasing.” What does this mean? What does it imply about the graph of the deficit as a function of time?

46. **Inventory Cost** The cost of inventory depends on the ordering and storage costs according to the inventory model

$$C = \left(\frac{Q}{x}\right)s + \left(\frac{x}{2}\right)r.$$

Determine the order size that will minimize the cost, assuming that sales occur at a constant rate, Q is the number of units sold per year, r is the cost of storing one unit for 1 year, s is the cost of placing an order, and x is the number of units per order.

47. **Modeling Data** Outlays for national defense D (in billions of dollars) for selected years from 1970 through 2005 are shown in the table, where t is time in years, with $t = 0$ corresponding to 1970. (Source: U.S. Office of Management and Budget)

t	0	5	10	15	20
D	81.7	86.5	134.0	252.7	299.3

t	25	30	35
D	272.1	294.5	495.3

- (a) Use the regression capabilities of a graphing utility to fit a model of the form $D = at^4 + bt^3 + ct^2 + dt + e$ to the data.
 (b) Use a graphing utility to plot the data and graph the model.

(c) For the years shown in the table, when does the model indicate that the outlay for national defense was at a maximum? When was it at a minimum?

(d) For the years shown in the table, when does the model indicate that the outlay for national defense was increasing at the greatest rate?

48. **Climb Rate** The time t (in minutes) for a small plane to climb to an altitude of h feet is

$$t = 50 \log_{10} \frac{18,000}{18,000 - h}$$

where 18,000 feet is the plane’s absolute ceiling.

(a) Determine the domain of the function appropriate for the context of the problem.

(b) Use a graphing utility to graph the time function and identify any asymptotes.

(c) Find the time when the altitude is increasing at the greatest rate.

In Exercises 49–58, find the limit.

49. $\lim_{x \rightarrow \infty} \left(8 + \frac{1}{x}\right)$

50. $\lim_{x \rightarrow \infty} \frac{3-x}{2x+5}$

51. $\lim_{x \rightarrow \infty} \frac{2x^2}{3x^2+5}$

52. $\lim_{x \rightarrow \infty} \frac{2x}{3x^2+5}$

53. $\lim_{x \rightarrow -\infty} \frac{3x^2}{x+5}$

54. $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+x}}{-2x}$

55. $\lim_{x \rightarrow \infty} \frac{5 \cos x}{x}$

56. $\lim_{x \rightarrow \infty} \frac{3x}{\sqrt{x^2+4}}$

57. $\lim_{x \rightarrow -\infty} \frac{6x}{x+\cos x}$

58. $\lim_{x \rightarrow -\infty} \frac{x}{2 \sin x}$

In Exercises 59–66, find any vertical and horizontal asymptotes of the graph of the function. Use a graphing utility to verify your results.

59. $f(x) = \frac{3}{x} - 2$

60. $g(x) = \frac{5x^2}{x^2+2}$

61. $h(x) = \frac{2x+3}{x-4}$

62. $f(x) = \frac{3x}{\sqrt{x^2+2}}$

63. $f(x) = \frac{5}{3+2e^{-x}}$

64. $g(x) = 30xe^{-2x}$

65. $g(x) = 3 \ln(1 + e^{-x/4})$

66. $h(x) = 10 \ln\left(\frac{x}{x+1}\right)$

In Exercises 67–70, use a graphing utility to graph the function. Use the graph to approximate any relative extrema or asymptotes.

67. $f(x) = x^3 + \frac{243}{x}$

68. $f(x) = |x^3 - 3x^2 + 2x|$

69. $f(x) = \frac{x-1}{1+3x^2}$

70. $g(x) = \frac{\pi^2}{3} - 4 \cos x + \cos 2x$

In Exercises 71–96, analyze and sketch the graph of the function.

71. $f(x) = 4x - x^2$

73. $f(x) = x\sqrt{16 - x^2}$

75. $f(x) = (x - 1)^3(x - 3)^2$

77. $f(x) = x^{1/3}(x + 3)^{2/3}$

79. $f(x) = \frac{5 - 3x}{x - 2}$

81. $f(x) = \frac{7}{1 + x^2}$

83. $f(x) = x^3 + x + \frac{2}{x}$

85. $f(x) = |x^2 - 9|$

86. $f(x) = |x - 1| + |x - 3|$

87. $h(x) = (1 - x)e^x$

88. $g(x) = 5xe^{-x^2}$

89. $g(x) = (x + 3) \ln(x + 3)$

90. $h(t) = \frac{\ln t}{t^2}$

91. $f(x) = \frac{10 \log_4 x}{x}$

92. $g(x) = 100x(3^{-x})$

93. $f(x) = x + \cos x, \quad 0 \leq x \leq 2\pi$

94. $f(x) = \frac{1}{\pi}(2 \sin \pi x - \sin 2\pi x), \quad -1 \leq x \leq 1$

95. $y = 4x - 6 \arctan x$

96. $y = \frac{1}{2}x^2 - \arcsin \frac{x}{2}$

97. Find the maximum and minimum points on the graph of

$x^2 + 4y^2 - 2x - 16y + 13 = 0$

(a) without using calculus.

(b) using calculus.

98. Consider the function $f(x) = x^n$ for positive integer values of n .

(a) For what values of n does the function have a relative minimum at the origin?

(b) For what values of n does the function have a point of inflection at the origin?

99. **Minimum Distance** At noon, ship A is 100 kilometers due east of ship B. Ship A is sailing west at 12 kilometers per hour, and ship B is sailing south at 10 kilometers per hour. At what time will the ships be nearest to each other, and what will this distance be?

100. **Maximum Area** Find the dimensions of the rectangle of maximum area, with sides parallel to the coordinate axes, that can be inscribed in the ellipse given by

$\frac{x^2}{144} + \frac{y^2}{16} = 1$

101. **Minimum Length** A right triangle in the first quadrant has the coordinate axes as sides, and the hypotenuse passes through the point $(1, 8)$. Find the vertices of the triangle such that the length of the hypotenuse is minimum.

102. **Minimum Length** The wall of a building is to be braced by a beam that must pass over a parallel fence 5 feet high and 4 feet from the building. Find the length of the shortest beam that can be used.

103. **Maximum Area** Three sides of a trapezoid have the same length s . Of all such possible trapezoids, show that the one of maximum area has a fourth side of length $2s$.

104. **Maximum Area** Show that the greatest area of any rectangle inscribed in a triangle is one-half the area of the triangle.

105. **Maximum Length** Find the length of the longest pipe that can be carried level around a right-angle corner at the intersection of two corridors of widths 4 feet and 6 feet. (Do not use trigonometry.)

106. **Maximum Length** Rework Exercise 105, given corridors of widths a meters and b meters.

107. **Maximum Length** A hallway of width 6 feet meets a hallway of width 9 feet at right angles. Find the length of the longest pipe that can be carried level around this corner. [Hint: If L is the length of the pipe, show that

$$L = 6 \csc \theta + 9 \csc\left(\frac{\pi}{2} - \theta\right)$$

where θ is the angle between the pipe and the wall of the narrower hallway.]

108. **Maximum Length** Rework Exercise 107, given that one hallway is of width a meters and the other is of width b meters. Show that the result is the same as in Exercise 106.

Minimum Cost In Exercises 109 and 110, find the speed v (in miles per hour) that will minimize costs on a 110-mile delivery trip. The cost per hour for fuel is C dollars, and the driver is paid W dollars per hour. (Assume there are no costs other than wages and fuel.)

109. Fuel cost: $C = \frac{v^2}{600}$

Driver: $W = \$5$

110. Fuel cost: $C = \frac{v^2}{500}$

Driver: $W = \$7.50$

In Exercises 111 and 112, find the differential dy .

111. $y = x(1 - \cos x)$

112. $y = \sqrt{36 - x^2}$

113. **Surface Area and Volume** The diameter of a sphere is measured as 18 centimeters, with a maximum possible error of 0.05 centimeter. Use differentials to approximate the possible propagated error and percent error in calculating the surface area and the volume of the sphere.

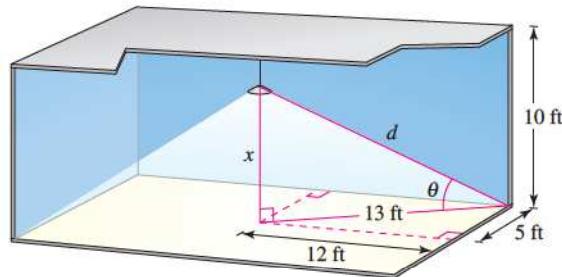
114. **Demand Function** A company finds that the demand for its commodity is $p = 75 - \frac{1}{4}x$. If x changes from 7 to 8, find and compare the values of Δp and dp .

P.S. PROBLEM SOLVING

- Graph the fourth-degree polynomial $p(x) = x^4 + ax^2 + 1$ for various values of the constant a .
 - Determine the values of a for which p has exactly one relative minimum.
 - Determine the values of a for which p has exactly one relative maximum.
 - Determine the values of a for which p has exactly two relative minima.
 - Show that the graph of p cannot have exactly two relative extrema.
- (a) Graph the fourth-degree polynomial $p(x) = ax^4 - 6x^2$ for $a = -3, -2, -1, 0, 1, 2$, and 3. For what values of the constant a does p have a relative minimum or relative maximum?
 - Show that p has a relative maximum for all values of the constant a .
 - Determine analytically the values of a for which p has a relative minimum.
 - Let $(x, y) = (x, p(x))$ be a relative extremum of p . Show that (x, y) lies on the graph of $y = -3x^2$. Verify this result graphically by graphing $y = -3x^2$ together with the seven curves from part (a).
- Let $f(x) = \frac{c}{x} + x^2$. Determine all values of the constant c such that f has a relative minimum, but no relative maximum.
- (a) Let $f(x) = ax^2 + bx + c$, $a \neq 0$, be a quadratic polynomial. How many points of inflection does the graph of f have?
 - Let $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, be a cubic polynomial. How many points of inflection does the graph of f have?
 - Suppose the function $y = f(x)$ satisfies the equation $\frac{dy}{dx} = ky\left(1 - \frac{y}{L}\right)$, where k and L are positive constants. Show that the graph of f has a point of inflection at the point where $y = \frac{L}{2}$. (This equation is called the **logistic differential equation**.)
- Prove Darboux's Theorem: Let f be differentiable on the closed interval $[a, b]$ such that $f'(a) = y_1$ and $f'(b) = y_2$. If d lies between y_1 and y_2 , then there exists c in (a, b) such that $f'(c) = d$.
- Let f and g be functions that are continuous on $[a, b]$ and differentiable on (a, b) . Prove that if $f(a) = g(a)$ and $g'(x) > f'(x)$ for all x in (a, b) , then $g(b) > f(b)$.
- Prove the following **Extended Mean Value Theorem**. If f and f' are continuous on the closed interval $[a, b]$, and if f'' exists on the open interval (a, b) , then there exists a number c in (a, b) such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{1}{2}f''(c)(b - a)^2.$$

- (a) Let $V = x^3$. Find dV and ΔV . Show that for small values of x , the difference $\Delta V - dV$ is very small in the sense that there exists ε such that $\Delta V - dV = \varepsilon \Delta x$, where $\varepsilon \rightarrow 0$ as $\Delta x \rightarrow 0$.
 - Generalize this result by showing that if $y = f(x)$ is a differentiable function, then $\Delta y - dy = \varepsilon \Delta x$, where $\varepsilon \rightarrow 0$ as $\Delta x \rightarrow 0$.
- The amount of illumination of a surface is proportional to the intensity of the light source, inversely proportional to the square of the distance from the light source, and proportional to $\sin \theta$, where θ is the angle at which the light strikes the surface. A rectangular room measures 10 feet by 24 feet, with a 10-foot ceiling. Determine the height at which the light should be placed to allow the corners of the floor to receive as much light as possible.



- Consider a room in the shape of a cube, 4 meters on each side. A bug at point P wants to walk to point Q at the opposite corner, as shown in the figure. Use calculus to determine the shortest path. Can you solve the problem without calculus?

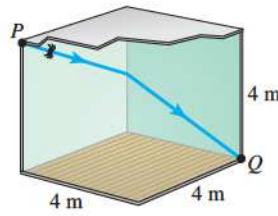


Figure for 10

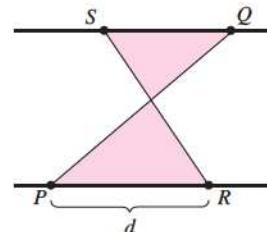
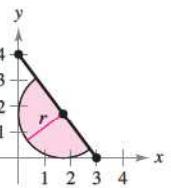
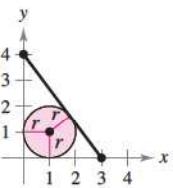
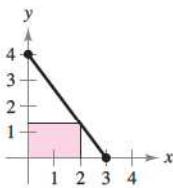


Figure for 11

- The line joining P and Q crosses two parallel lines, as shown in the figure. The point R is d units from P . How far from Q should the point S be positioned so that the sum of the areas of the two shaded triangles is a minimum? So that the sum is a maximum?

- A** 12. (a) Let x be a positive number. Use the *table* feature of a graphing utility to verify that $\sin x < x$.
- (b) Use the Mean Value Theorem to prove that $\sin x < x$ for all positive real numbers x .
- A** 13. (a) Let x be a positive number. Use the *table* feature of a graphing utility to verify that $\sqrt{1+x} < \frac{1}{2}x + 1$.
- (b) Use the Mean Value Theorem to prove that $\sqrt{1+x} < \frac{1}{2}x + 1$ for all positive real numbers x .

14. The figures show a rectangle, a circle, and a semicircle inscribed in a triangle bounded by the coordinate axes and the first-quadrant portion of the line with intercepts $(3, 0)$ and $(0, 4)$. Find the dimensions of each inscribed figure such that its area is maximum. State whether calculus was helpful in finding the required dimensions. Explain your reasoning.



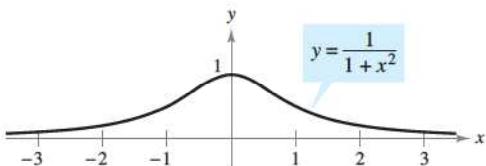
15. (a) Prove that $\lim_{x \rightarrow \infty} x^2 = \infty$.

$$(b) \text{Prove that } \lim_{x \rightarrow \infty} \left(\frac{1}{x^2}\right) = 0.$$

- (c) Let L be a real number. Prove that if $\lim_{x \rightarrow \infty} f(x) = L$, then

$$\lim_{y \rightarrow 0^+} f\left(\frac{1}{y}\right) = L.$$

16. Find the point on the graph of $y = \frac{1}{1+x^2}$ (see figure) where the tangent line has the greatest slope, and the point where the tangent line has the least slope.



17. The police department must determine the speed limit on a bridge such that the flow rate of cars is maximized per unit time. The greater the speed limit, the farther apart the cars must be in order to keep a safe stopping distance. Experimental data on the stopping distances d (in meters) for various speeds v (in kilometers per hour) are shown in the table.

v	20	40	60	80	100
d	5.1	13.7	27.2	44.2	66.4

- (a) Convert the speeds v in the table to speeds s in meters per second. Use the regression capabilities of a graphing utility to find a model of the form $d(s) = as^2 + bs + c$ for the data.

- (b) Consider two consecutive vehicles of average length 5.5 meters, traveling at a safe speed on the bridge. Let T be the difference between the times (in seconds) when the front bumpers of the two vehicles pass a given point on the bridge. Verify that this difference in times is given by

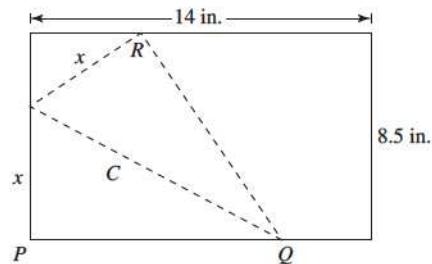
$$T = \frac{d(s)}{s} + \frac{5.5}{s}.$$

- (c) Use a graphing utility to graph the function T and estimate the speed s that minimizes the time between vehicles.

- (d) Use calculus to determine the speed that minimizes T . What is the minimum value of T ? Convert the required speed to kilometers per hour.

- (e) Find the optimal distance between vehicles for the posted speed limit determined in part (d).

18. A legal-sized sheet of paper (8.5 inches by 14 inches) is folded so that corner P touches the opposite 14-inch edge at R (see figure). (Note: $PQ = \sqrt{C^2 - x^2}$.)



$$(a) \text{Show that } C^2 = \frac{2x^3}{2x - 8.5}.$$

- (b) What is the domain of C ?

- (c) Determine the x -value that minimizes C .

- (d) Determine the minimum length of C .

19. Let $f(x) = \sin(\ln x)$.

- (a) Determine the domain of the function f .

- (b) Find two values of x satisfying $f(x) = 1$.

- (c) Find two values of x satisfying $f(x) = -1$.

- (d) What is the range of the function f ?

- (e) Calculate $f'(x)$ and use calculus to find the maximum value of f on the interval $[1, 10]$.

- (f) Use a graphing utility to graph f in the viewing window $[0, 5] \times [-2, 2]$ and estimate $\lim_{x \rightarrow 0^+} f(x)$, if it exists.

- (g) Determine $\lim_{x \rightarrow 0^+} f(x)$ analytically, if it exists.

20. Show that the cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ has exactly one point of inflection (x_0, y_0) , where

$$x_0 = \frac{-b}{3a} \quad \text{and} \quad y_0 = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d.$$

Use this formula to find the point of inflection of

$$p(x) = x^3 - 3x^2 + 2.$$