

8

Integration Techniques, L'Hôpital's Rule, and Improper Integrals

In previous chapters, you studied several basic techniques for evaluating simple integrals. In this chapter, you will study other integration techniques, such as integration by parts, that are used to evaluate more complicated integrals. You will also learn an important rule for evaluating limits called L'Hôpital's Rule. This rule can also help you evaluate improper integrals.

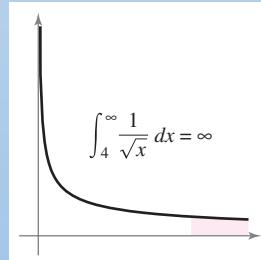
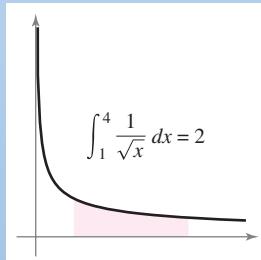
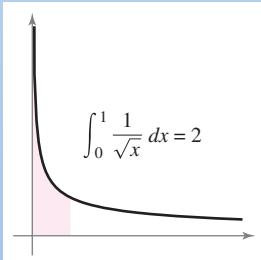
In this chapter, you should learn the following.

- How to fit an integrand to one of the basic integration rules. (8.1)
- How to find an antiderivative using integration by parts. (8.2)
- How to evaluate trigonometric integrals. (8.3)
- How to use trigonometric substitution to evaluate an integral. (8.4)
- How to use partial fraction decomposition to integrate rational functions. (8.5)
- How to evaluate an indefinite integral using a table of integrals and using reduction formulas. (8.6)
- How to apply L'Hôpital's Rule to evaluate a limit. (8.7)
- How to evaluate an improper integral. (8.8)



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Partial fraction decomposition is an integration technique that can be used to evaluate integrals involving rational functions. How can partial fraction decomposition be used to evaluate an integral that gives the average cost of removing a certain percent of a chemical from a company's waste water? (See Section 8.5, Exercise 63.)



From your studies of calculus thus far, you know that a definite integral has finite limits of integration and a continuous integrand. In Section 8.8, you will study *improper integrals*. Improper integrals have at least one infinite limit of integration or have an integrand with an infinite discontinuity. You will see that improper integrals either converge or diverge.

8.1 Basic Integration Rules

■ Review procedures for fitting an integrand to one of the basic integration rules.

Fitting Integrands to Basic Integration Rules

In this chapter, you will study several integration techniques that greatly expand the set of integrals to which the basic integration rules can be applied. These rules are reviewed on page 522. A major step in solving any integration problem is recognizing which basic integration rule to use. As shown in Example 1, slight differences in the integrand can lead to very different solution techniques.



EXAMPLE 1 A Comparison of Three Similar Integrals

EXPLORATION

A Comparison of Three Similar Integrals Which, if any, of the following integrals can be evaluated using the 20 basic integration rules? For any that can be evaluated, do so. For any that can't, explain why.

a. $\int \frac{3}{\sqrt{1-x^2}} dx$

b. $\int \frac{3x}{\sqrt{1-x^2}} dx$

c. $\int \frac{3x^2}{\sqrt{1-x^2}} dx$

Find each integral.

a. $\int \frac{4}{x^2 + 9} dx$ b. $\int \frac{4x}{x^2 + 9} dx$ c. $\int \frac{4x^2}{x^2 + 9} dx$

Solution

- a. Use the Arctangent Rule and let $u = x$ and $a = 3$.

$$\begin{aligned}\int \frac{4}{x^2 + 9} dx &= 4 \int \frac{1}{x^2 + 3^2} dx && \text{Constant Multiple Rule} \\ &= 4 \left(\frac{1}{3} \arctan \frac{x}{3} \right) + C && \text{Arctangent Rule} \\ &= \frac{4}{3} \arctan \frac{x}{3} + C && \text{Simplify.}\end{aligned}$$

- b. Here the Arctangent Rule does not apply because the numerator contains a factor of x . Consider the Log Rule and let $u = x^2 + 9$. Then $du = 2x dx$, and you have

$$\begin{aligned}\int \frac{4x}{x^2 + 9} dx &= 2 \int \frac{2x}{x^2 + 9} dx && \text{Constant Multiple Rule} \\ &= 2 \int \frac{du}{u} && \text{Substitution: } u = x^2 + 9 \\ &= 2 \ln|u| + C = 2 \ln(x^2 + 9) + C. && \text{Log Rule}\end{aligned}$$

- c. Because the degree of the numerator is equal to the degree of the denominator, you should first use division to rewrite the improper rational function as the sum of a polynomial and a proper rational function.

$$\begin{aligned}\int \frac{4x^2}{x^2 + 9} dx &= \int \left(4 - \frac{36}{x^2 + 9} \right) dx && \text{Rewrite using long division.} \\ &= \int 4 dx - 36 \int \frac{1}{x^2 + 9} dx && \text{Write as two integrals.} \\ &= 4x - 36 \left(\frac{1}{3} \arctan \frac{x}{3} \right) + C && \text{Integrate.} \\ &= 4x - 12 \arctan \frac{x}{3} + C && \text{Simplify.}\end{aligned}$$

NOTE Notice in Example 1(c) that some preliminary algebra is required before applying the rules for integration, and that subsequently more than one rule is needed to evaluate the resulting integral.

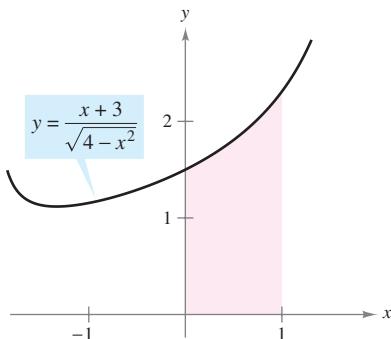
The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple and Mathematica.

EXAMPLE 2 Using Two Basic Rules to Solve a Single Integral

Evaluate $\int_0^1 \frac{x+3}{\sqrt{4-x^2}} dx$.

Solution Begin by writing the integral as the sum of two integrals. Then apply the Power Rule and the Arcsine Rule, as follows.

$$\begin{aligned}\int_0^1 \frac{x+3}{\sqrt{4-x^2}} dx &= \int_0^1 \frac{x}{\sqrt{4-x^2}} dx + \int_0^1 \frac{3}{\sqrt{4-x^2}} dx \\ &= -\frac{1}{2} \int_0^1 (4-x^2)^{-1/2}(-2x) dx + 3 \int_0^1 \frac{1}{\sqrt{2^2-x^2}} dx \\ &= \left[-(4-x^2)^{1/2} + 3 \arcsin \frac{x}{2} \right]_0^1 \\ &= \left(-\sqrt{3} + \frac{\pi}{2} \right) - (-2 + 0) \\ &\approx 1.839\end{aligned}$$



The area of the region is approximately 1.839.

Figure 8.1

See Figure 8.1. ■

TECHNOLOGY Simpson's Rule can be used to give a good approximation of the value of the integral in Example 2 (for $n = 10$, the approximation is 1.839). When using numerical integration, however, you should be aware that Simpson's Rule does not always give good approximations when one or both of the limits of integration are near a vertical asymptote. For instance, using the Fundamental Theorem of Calculus, you can obtain

$$\int_0^{1.99} \frac{x+3}{\sqrt{4-x^2}} dx \approx 6.213.$$

Applying Simpson's Rule (with $n = 10$) to this integral produces an approximation of 6.889.

EXAMPLE 3 A Substitution Involving $a^2 - u^2$

Find $\int \frac{x^2}{\sqrt{16-x^6}} dx$.

Solution Because the radical in the denominator can be written in the form

$$\sqrt{a^2 - u^2} = \sqrt{4^2 - (x^3)^2}$$

you can try the substitution $u = x^3$. Then $du = 3x^2 dx$, and you have

$$\begin{aligned}\int \frac{x^2}{\sqrt{16-x^6}} dx &= \frac{1}{3} \int \frac{3x^2 dx}{\sqrt{16-(x^3)^2}} && \text{Rewrite integral.} \\ &= \frac{1}{3} \int \frac{du}{\sqrt{4^2-u^2}} && \text{Substitution: } u = x^3 \\ &= \frac{1}{3} \arcsin \frac{u}{4} + C && \text{Arcsine Rule} \\ &= \frac{1}{3} \arcsin \frac{x^3}{4} + C. && \text{Rewrite as a function of } x.\end{aligned}$$

STUDY TIP Rules 18, 19, and 20 of the basic integration rules on the next page all have expressions involving the sum or difference of two squares:

$$a^2 - u^2$$

$$a^2 + u^2$$

$$u^2 - a^2$$

These expressions are often apparent after a u -substitution, as shown in Example 3.

Surprisingly, two of the most commonly overlooked integration rules are the Log Rule and the Power Rule. Notice in the next two examples how these two integration rules can be disguised.

EXAMPLE 4 A Disguised Form of the Log Rule

REVIEW OF BASIC INTEGRATION RULES ($a > 0$)

$$1. \int kf(u) du = k \int f(u) du$$

$$2. \int [f(u) \pm g(u)] du = \int f(u) du \pm \int g(u) du$$

$$3. \int du = u + C$$

$$4. \int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1$$

$$5. \int \frac{du}{u} = \ln|u| + C$$

$$6. \int e^u du = e^u + C$$

$$7. \int a^u du = \left(\frac{1}{\ln a} \right) a^u + C$$

$$8. \int \sin u du = -\cos u + C$$

$$9. \int \cos u du = \sin u + C$$

$$10. \int \tan u du = -\ln|\cos u| + C$$

$$11. \int \cot u du = \ln|\sin u| + C$$

$$12. \int \sec u du = \ln|\sec u + \tan u| + C$$

$$13. \int \csc u du = -\ln|\csc u + \cot u| + C$$

$$14. \int \sec^2 u du = \tan u + C$$

$$15. \int \csc^2 u du = -\cot u + C$$

$$16. \int \sec u \tan u du = \sec u + C$$

$$17. \int \csc u \cot u du = -\csc u + C$$

$$18. \int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$$

$$19. \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$$

$$20. \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$$

Find $\int \frac{1}{1 + e^x} dx$.

Solution The integral does not appear to fit any of the basic rules. However, the quotient form suggests the Log Rule. If you let $u = 1 + e^x$, then $du = e^x dx$. You can obtain the required du by adding and subtracting e^x in the numerator, as follows.

$$\begin{aligned} \int \frac{1}{1 + e^x} dx &= \int \frac{1 + e^x - e^x}{1 + e^x} dx \\ &= \int \left(\frac{1 + e^x}{1 + e^x} - \frac{e^x}{1 + e^x} \right) dx \\ &= \int dx - \int \frac{e^x dx}{1 + e^x} \\ &= x - \ln(1 + e^x) + C \end{aligned}$$

Add and subtract e^x in numerator.

Rewrite as two fractions.

Rewrite as two integrals.

Integrate. ■

NOTE There is usually more than one way to solve an integration problem. For instance, in Example 4, try integrating by multiplying the numerator and denominator by e^{-x} to obtain an integral of the form $-\int du/u$. See if you can get the same answer by this procedure. (Be careful: the answer will appear in a different form.) ■

EXAMPLE 5 A Disguised Form of the Power Rule

Find $\int (\cot x)[\ln(\sin x)] dx$.

Solution Again, the integral does not appear to fit any of the basic rules. However, considering the two primary choices for u [$u = \cot x$ and $u = \ln(\sin x)$], you can see that the second choice is the appropriate one because

$$u = \ln(\sin x) \quad \text{and} \quad du = \frac{\cos x}{\sin x} dx = \cot x dx.$$

So,

$$\begin{aligned} \int (\cot x)[\ln(\sin x)] dx &= \int u du \\ &= \frac{u^2}{2} + C \\ &= \frac{1}{2}[\ln(\sin x)]^2 + C. \end{aligned}$$

Substitution: $u = \ln(\sin x)$

Integrate.

Rewrite as a function of x . ■

NOTE In Example 5, try *checking* that the derivative of

$$\frac{1}{2}[\ln(\sin x)]^2 + C$$

is the integrand of the original integral. ■

Trigonometric identities can often be used to fit integrals to one of the basic integration rules.

EXAMPLE 6 Using Trigonometric Identities

Find $\int \tan^2 2x \, dx$.

TECHNOLOGY If you have access to a computer algebra system, try using it to evaluate the integrals in this section. Compare the forms of the antiderivatives given by the software with the forms obtained by hand. Sometimes the forms will be the same, but often they will differ. For instance, why is the antiderivative $\ln 2x + C$ equivalent to the antiderivative $\ln x + C$?

Solution Note that $\tan^2 u$ is not in the list of basic integration rules. However, $\sec^2 u$ is in the list. This suggests the trigonometric identity $\tan^2 u = \sec^2 u - 1$. If you let $u = 2x$, then $du = 2 \, dx$ and

$$\begin{aligned} \int \tan^2 2x \, dx &= \frac{1}{2} \int \tan^2 u \, du && \text{Substitution: } u = 2x \\ &= \frac{1}{2} \int (\sec^2 u - 1) \, du && \text{Trigonometric identity} \\ &= \frac{1}{2} \int \sec^2 u \, du - \frac{1}{2} \int \, du && \text{Rewrite as two integrals.} \\ &= \frac{1}{2} \tan u - \frac{u}{2} + C && \text{Integrate.} \\ &= \frac{1}{2} \tan 2x - x + C. && \text{Rewrite as a function of } x. \end{aligned}$$

This section concludes with a summary of the common procedures for fitting integrands to the basic integration rules.

PROCEDURES FOR FITTING INTEGRANDS TO BASIC INTEGRATION RULES

Technique

Expand (numerator).

Separate numerator.

Complete the square.

Divide improper rational function.

Add and subtract terms in numerator.

Use trigonometric identities.

Multiply and divide by Pythagorean conjugate.

Example

$$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$$

$$\frac{1+x}{x^2+1} = \frac{1}{x^2+1} + \frac{x}{x^2+1}$$

$$\frac{1}{\sqrt{2x-x^2}} = \frac{1}{\sqrt{1-(x-1)^2}}$$

$$\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$$

$$\frac{2x}{x^2+2x+1} = \frac{2x+2-2}{x^2+2x+1} = \frac{2x+2}{x^2+2x+1} - \frac{2}{(x+1)^2}$$

$$\cot^2 x = \csc^2 x - 1$$

$$\begin{aligned} \frac{1}{1+\sin x} &= \left(\frac{1}{1+\sin x}\right)\left(\frac{1-\sin x}{1-\sin x}\right) = \frac{1-\sin x}{1-\sin^2 x} \\ &= \frac{1-\sin x}{\cos^2 x} = \sec^2 x - \frac{\sin x}{\cos^2 x} \end{aligned}$$

NOTE Remember that you can separate numerators but not denominators. Watch out for this common error when fitting integrands to basic rules.

$$\frac{1}{x^2+1} \neq \frac{1}{x^2} + \frac{1}{1}$$

Do not separate denominators.

8.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, select the correct antiderivative.

1. $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 1}}$

- (a) $2\sqrt{x^2 + 1} + C$
 (b) $\sqrt{x^2 + 1} + C$
 (c) $\frac{1}{2}\sqrt{x^2 + 1} + C$

2. $\frac{dy}{dx} = \frac{x}{x^2 + 1}$

- (a) $\ln\sqrt{x^2 + 1} + C$
 (b) $\frac{2x}{(x^2 + 1)^2} + C$
 (c) $\arctan x + C$

3. $\frac{dy}{dx} = \frac{1}{x^2 + 1}$

- (a) $\ln\sqrt{x^2 + 1} + C$
 (b) $\frac{2x}{(x^2 + 1)^2} + C$
 (c) $\arctan x + C$

4. $\frac{dy}{dx} = x \cos(x^2 + 1)$

- (a) $2x \sin(x^2 + 1) + C$
 (b) $-\frac{1}{2} \sin(x^2 + 1) + C$
 (c) $\frac{1}{2} \sin(x^2 + 1) + C$

In Exercises 5–14, select the basic integration formula you can use to find the integral, and identify u and a when appropriate.

5. $\int (5x - 3)^4 dx$

6. $\int \frac{2t + 1}{t^2 + t - 4} dt$

7. $\int \frac{1}{\sqrt{x}(1 - 2\sqrt{x})} dx$

8. $\int \frac{2}{(2t - 1)^2 + 4} dt$

9. $\int \frac{3}{\sqrt{1 - t^2}} dt$

10. $\int \frac{-2x}{\sqrt{x^2 - 4}} dx$

11. $\int t \sin t^2 dt$

12. $\int \sec 5x \tan 5x dx$

13. $\int (\cos x)e^{\sin x} dx$

14. $\int \frac{1}{x\sqrt{x^2 - 4}} dx$

In Exercises 15–52, find the indefinite integral.

15. $\int 14(x - 5)^6 dx$

16. $\int \frac{9}{(t - 8)^2} dt$

17. $\int \frac{7}{(z - 10)^7} dz$

18. $\int t^2 \sqrt[3]{t^3 - 1} dt$

19. $\int \left[v + \frac{1}{(3v - 1)^3}\right] dv$

20. $\int \left[x - \frac{5}{(3x + 5)^2}\right] dx$

21. $\int \frac{t^2 - 3}{-t^3 + 9t + 1} dt$

22. $\int \frac{x + 1}{\sqrt{x^2 + 2x - 4}} dx$

23. $\int \frac{x^2}{x - 1} dx$

24. $\int \frac{4x}{x - 8} dx$

25. $\int \frac{e^x}{1 + e^x} dx$

26. $\int \left(\frac{1}{7x - 2} - \frac{1}{7x + 2}\right) dx$

27. $\int (5 + 4x^2)^2 dx$

29. $\int x \cos 2\pi x^2 dx$

31. $\int \csc \pi x \cot \pi x dx$

33. $\int e^{11x} dx$

35. $\int \frac{2}{e^{-x} + 1} dx$

37. $\int \frac{\ln x^2}{x} dx$

39. $\int \frac{1 + \sin x}{\cos x} dx$

41. $\int \frac{1}{\cos \theta - 1} d\theta$

43. $\int \frac{-1}{\sqrt{1 - (4t + 1)^2}} dt$

45. $\int \frac{\tan(2/t)}{t^2} dt$

47. $\int \frac{6}{\sqrt{10x - x^2}} dx$

48. $\int \frac{1}{(x - 1)\sqrt{4x^2 - 8x + 3}} dx$

49. $\int \frac{4}{4x^2 + 4x + 65} dx$

51. $\int \frac{1}{\sqrt{1 - 4x - x^2}} dx$

28. $\int x \left(1 + \frac{1}{x}\right)^3 dx$

30. $\int \sec 4x dx$

32. $\int \frac{\sin x}{\sqrt{\cos x}} dx$

34. $\int \csc^2 x e^{\cot x} dx$

36. $\int \frac{5}{3e^x - 2} dx$

38. $\int (\tan x)[\ln(\cos x)] dx$

40. $\int \frac{1 + \cos \alpha}{\sin \alpha} d\alpha$

42. $\int \frac{2}{3(\sec x - 1)} dx$

44. $\int \frac{1}{9 + 5x^2} dx$

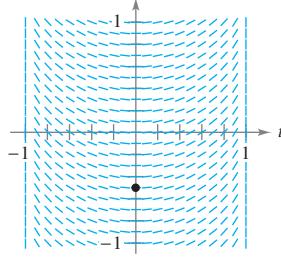
46. $\int \frac{e^{1/t}}{t^2} dt$



Slope Fields In Exercises 53–56, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

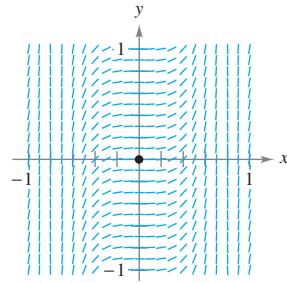
53. $\frac{ds}{dt} = \frac{t}{\sqrt{1 - t^4}}$

$$\left(0, -\frac{1}{2}\right)$$



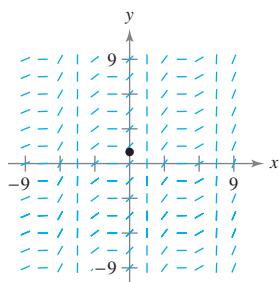
54. $\frac{dy}{dx} = \tan^2(2x)$

$$(0, 0)$$



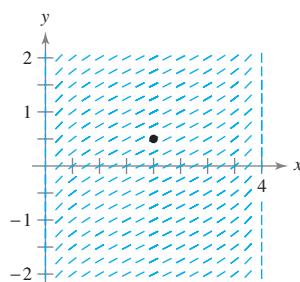
55. $\frac{dy}{dx} = (\sec x + \tan x)^2$

$(0, 1)$

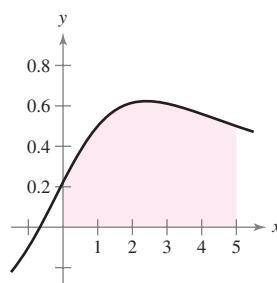


56. $\frac{dy}{dx} = \frac{1}{\sqrt{4x - x^2}}$

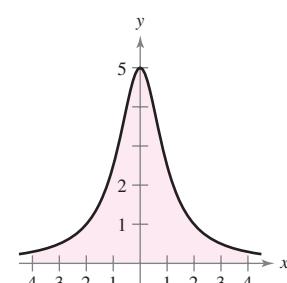
$$\left(2, \frac{1}{2}\right)$$



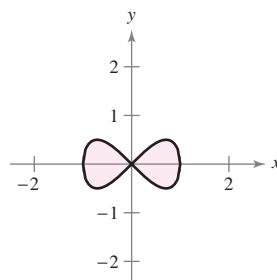
75. $y = \frac{3x + 2}{x^2 + 9}$



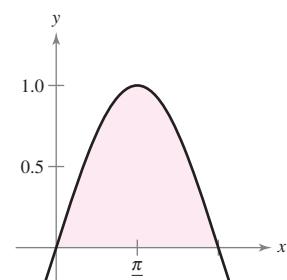
76. $y = \frac{5}{x^2 + 1}$



77. $y^2 = x^2(1 - x^2)$



78. $y = \sin 2x$



CAS Slope Fields In Exercises 57 and 58, use a computer algebra system to graph the slope field for the differential equation and graph the solution through the specified initial condition.

57. $\frac{dy}{dx} = 0.8y, y(0) = 4$

58. $\frac{dy}{dx} = 5 - y, y(0) = 1$

In Exercises 59–64, solve the differential equation.

59. $\frac{dy}{dx} = (e^x + 5)^2$

60. $\frac{dy}{dx} = (3 - e^x)^2$

61. $\frac{dr}{dt} = \frac{10e^t}{\sqrt{1 - e^{2t}}}$

62. $\frac{dr}{dt} = \frac{(1 + e^t)^2}{e^t}$

63. $(4 + \tan^2 x)y' = \sec^2 x$

64. $y' = \frac{1}{x\sqrt{4x^2 - 1}}$

In Exercises 65–72, evaluate the definite integral. Use the integration capabilities of a graphing utility to verify your result.

65. $\int_0^{\pi/4} \cos 2x \, dx$

66. $\int_0^{\pi} \sin^2 t \cos t \, dt$

67. $\int_0^1 xe^{-x^2} \, dx$

68. $\int_1^e \frac{1 - \ln x}{x} \, dx$

69. $\int_0^8 \frac{2x}{\sqrt{x^2 + 36}} \, dx$

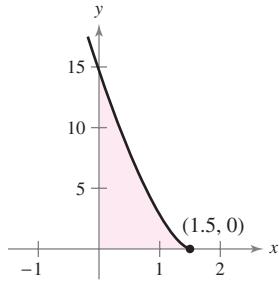
70. $\int_1^2 \frac{x - 2}{x} \, dx$

71. $\int_0^{2\sqrt{3}} \frac{1}{4 + 9x^2} \, dx$

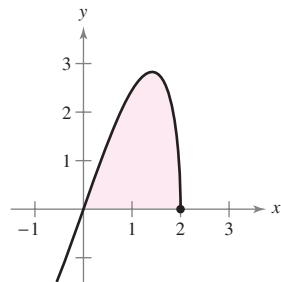
72. $\int_0^7 \frac{1}{\sqrt{100 - x^2}} \, dx$

Area In Exercises 73–78, find the area of the region.

73. $y = (-4x + 6)^{3/2}$



74. $y = x\sqrt{8 - 2x^2}$



CAS In Exercises 79–82, use a computer algebra system to find the integral. Use the computer algebra system to graph two antiderivatives. Describe the relationship between the graphs of the two antiderivatives.

79. $\int \frac{1}{x^2 + 4x + 13} \, dx$

80. $\int \frac{x - 2}{x^2 + 4x + 13} \, dx$

81. $\int \frac{1}{1 + \sin \theta} \, d\theta$

82. $\int \left(\frac{e^x + e^{-x}}{2}\right)^3 \, dx$

WRITING ABOUT CONCEPTS

In Exercises 83–86, state the integration formula you would use to perform the integration. Explain why you chose that formula. Do not integrate.

83. $\int x(x^2 + 1)^3 \, dx$

84. $\int x \sec(x^2 + 1) \tan(x^2 + 1) \, dx$

85. $\int \frac{x}{x^2 + 1} \, dx$

86. $\int \frac{1}{x^2 + 1} \, dx$

87. Determine the constants a and b such that

$$\sin x + \cos x = a \sin(x + b).$$

Use this result to integrate $\int \frac{dx}{\sin x + \cos x}$.

88. Show that $\sec x = \frac{\sin x}{\cos x} + \frac{\cos x}{1 + \sin x}$. Then use this identity to derive the basic integration rule

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C.$$

- 89. Area** The graphs of $f(x) = x$ and $g(x) = ax^2$ intersect at the points $(0, 0)$ and $(1/a, 1/a)$. Find a ($a > 0$) such that the area of the region bounded by the graphs of these two functions is $\frac{2}{3}$.

CAPSTONE

- 90.** (a) Explain why the antiderivative $y_1 = e^{x+C_1}$ is equivalent to the antiderivative $y_2 = Ce^x$.
 (b) Explain why the antiderivative $y_1 = \sec^2 x + C_1$ is equivalent to the antiderivative $y_2 = \tan^2 x + C$.



- 91. Think About It** Use a graphing utility to graph the function $f(x) = \frac{1}{5}(x^3 - 7x^2 + 10x)$. Use the graph to determine whether $\int_0^5 f(x) dx$ is positive or negative. Explain.

- 92. Think About It** When evaluating $\int_{-1}^1 x^2 dx$, is it appropriate to substitute $u = x^2$, $x = \sqrt{u}$, and $dx = \frac{du}{2\sqrt{u}}$ to obtain $\frac{1}{2} \int_1^1 \sqrt{u} du = 0$? Explain.

Approximation In Exercises 93 and 94, determine which value best approximates the area of the region between the x -axis and the function over the given interval. (Make your selection on the basis of a sketch of the region and *not* by integrating.)

93. $f(x) = \frac{4x}{x^2 + 1}$, $[0, 2]$
 (a) 3 (b) 1 (c) -8 (d) 8 (e) 10

94. $f(x) = \frac{4}{x^2 + 1}$, $[0, 2]$
 (a) 3 (b) 1 (c) -4 (d) 4 (e) 10

Interpreting Integrals In Exercises 95 and 96, (a) sketch the region whose area is given by the integral, (b) sketch the solid whose volume is given by the integral if the disk method is used, and (c) sketch the solid whose volume is given by the integral if the shell method is used. (There is more than one correct answer for each part.)

95. $\int_0^2 2\pi x^2 dx$

96. $\int_0^4 \pi y dy$

- 97. Volume** The region bounded by $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = b$ ($b > 0$) is revolved about the y -axis.

- (a) Find the volume of the solid generated if $b = 1$.
 (b) Find b such that the volume of the generated solid is $\frac{4}{3}$ cubic units.

- 98. Volume** Consider the region bounded by the graphs of $x = 0$, $y = \cos x^2$, $y = \sin x^2$, and $x = \sqrt{\pi}/2$. Find the volume of the solid generated by revolving the region about the y -axis.

- 99. Arc Length** Find the arc length of the graph of $y = \ln(\sin x)$ from $x = \pi/4$ to $x = \pi/2$.

- 100. Arc Length** Find the arc length of the graph of $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/3$.

- 101. Surface Area** Find the area of the surface formed by revolving the graph of $y = 2\sqrt{x}$ on the interval $[0, 9]$ about the x -axis.

- 102. Centroid** Find the x -coordinate of the centroid of the region bounded by the graphs of

$$y = \frac{5}{\sqrt{25 - x^2}}, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = 4.$$

In Exercises 103 and 104, find the average value of the function over the given interval.

103. $f(x) = \frac{1}{1 + x^2}$, $-3 \leq x \leq 3$

104. $f(x) = \sin nx$, $0 \leq x \leq \pi/n$, n is a positive integer.

Arc Length In Exercises 105 and 106, use the integration capabilities of a graphing utility to approximate the arc length of the curve over the given interval.

105. $y = \tan \pi x$, $[0, \frac{1}{4}]$

106. $y = x^{2/3}$, $[1, 8]$

Finding a Pattern

(a) Find $\int \cos^3 x dx$. (b) Find $\int \cos^5 x dx$.

(c) Find $\int \cos^7 x dx$.

(d) Explain how to find $\int \cos^{15} x dx$ without actually integrating.

Finding a Pattern

(a) Write $\int \tan^3 x dx$ in terms of $\int \tan x dx$. Then find $\int \tan^3 x dx$.

(b) Write $\int \tan^5 x dx$ in terms of $\int \tan^3 x dx$.

(c) Write $\int \tan^{2k+1} x dx$, where k is a positive integer, in terms of $\int \tan^{2k-1} x dx$.

(d) Explain how to find $\int \tan^{15} x dx$ without actually integrating.

- 109. Methods of Integration** Show that the following results are equivalent.

Integration by tables:

$$\int \sqrt{x^2 + 1} dx = \frac{1}{2} \left(x \sqrt{x^2 + 1} + \ln|x + \sqrt{x^2 + 1}| \right) + C$$

Integration by computer algebra system:

$$\int \sqrt{x^2 + 1} dx = \frac{1}{2} [x \sqrt{x^2 + 1} + \operatorname{arcsinh}(x)] + C$$

PUTNAM EXAM CHALLENGE

110. Evaluate $\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}} dx$.

This problem was composed by the Committee on the Putnam Prize Competition.
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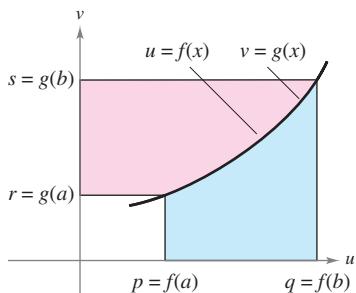
8.2 Integration by Parts

- Find an antiderivative using integration by parts.
- Use a tabular method to perform integration by parts.

Integration by Parts

EXPLORATION

Proof Without Words Here is a different approach to proving the formula for integration by parts. Exercise taken from “Proof Without Words: Integration by Parts” by Roger B. Nelsen, *Mathematics Magazine*, 64, No. 2, April 1991, p. 130, by permission of the author.



$$\text{Area } \blacksquare + \text{Area } \blacksquare = qs - pr$$

$$\int_r^s u \, dv + \int_q^p v \, du = \left[uv \right]_{(p,r)}^{(q,s)}$$

$$\int_r^s u \, dv = \left[uv \right]_{(p,r)}^{(q,s)} - \int_q^p v \, du$$

Explain how this graph proves the theorem. Which notation in this proof is unfamiliar? What do you think it means?

In this section you will study an important integration technique called **integration by parts**. This technique can be applied to a wide variety of functions and is particularly useful for integrands involving *products* of algebraic and transcendental functions. For instance, integration by parts works well with integrals such as

$$\int x \ln x \, dx, \quad \int x^2 e^x \, dx, \quad \text{and} \quad \int e^x \sin x \, dx.$$

Integration by parts is based on the formula for the derivative of a product

$$\begin{aligned} \frac{d}{dx}[uv] &= u \frac{dv}{dx} + v \frac{du}{dx} \\ &= uv' + vu' \end{aligned}$$

where both u and v are differentiable functions of x . If u' and v' are continuous, you can integrate both sides of this equation to obtain

$$\begin{aligned} uv &= \int uv' \, dx + \int vu' \, dx \\ &= \int u \, dv + \int v \, du. \end{aligned}$$

By rewriting this equation, you obtain the following theorem.

THEOREM 8.1 INTEGRATION BY PARTS

If u and v are functions of x and have continuous derivatives, then

$$\int u \, dv = uv - \int v \, du.$$

This formula expresses the original integral in terms of another integral. Depending on the choices of u and dv , it may be easier to evaluate the second integral than the original one. Because the choices of u and dv are critical in the integration by parts process, the following guidelines are provided.

GUIDELINES FOR INTEGRATION BY PARTS

1. Try letting dv be the most complicated portion of the integrand that fits a basic integration rule. Then u will be the remaining factor(s) of the integrand.
2. Try letting u be the portion of the integrand whose derivative is a function simpler than u . Then dv will be the remaining factor(s) of the integrand.

Note that dv always includes the dx of the original integrand.

EXAMPLE 1 Integration by Parts

Find $\int xe^x dx$.

Solution To apply integration by parts, you need to write the integral in the form $\int u dv$. There are several ways to do this.

$$\int \underbrace{(x)}_u \underbrace{(e^x dx)}_{dv}, \quad \int \underbrace{(e^x)}_u \underbrace{(x dx)}_{dv}, \quad \int \underbrace{(1)}_u \underbrace{(xe^x dx)}_{dv}, \quad \int \underbrace{(xe^x)}_u \underbrace{(dx)}_{dv}$$

The guidelines on page 527 suggest the first option because the derivative of $u = x$ is simpler than x , and $dv = e^x dx$ is the most complicated portion of the integrand that fits a basic integration formula.

$$\begin{aligned} dv &= e^x dx &\Rightarrow v &= \int dv = \int e^x dx = e^x \\ u &= x &\Rightarrow du &= dx \end{aligned}$$

Now, integration by parts produces

$$\begin{aligned} \int u dv &= uv - \int v du && \text{Integration by parts formula} \\ \int xe^x dx &= xe^x - \int e^x dx && \text{Substitute.} \\ &= xe^x - e^x + C. && \text{Integrate.} \end{aligned}$$

To check this, differentiate $xe^x - e^x + C$ to see that you obtain the original integrand.

EXAMPLE 2 Integration by Parts

Find $\int x^2 \ln x dx$.

Solution In this case, x^2 is more easily integrated than $\ln x$. Furthermore, the derivative of $\ln x$ is simpler than $\ln x$. So, you should let $dv = x^2 dx$.

$$\begin{aligned} dv &= x^2 dx &\Rightarrow v &= \int x^2 dx = \frac{x^3}{3} \\ u &= \ln x &\Rightarrow du &= \frac{1}{x} dx \end{aligned}$$

Integration by parts produces

$$\begin{aligned} \int u dv &= uv - \int v du && \text{Integration by parts formula} \\ \int x^2 \ln x dx &= \frac{x^3}{3} \ln x - \int \left(\frac{x^3}{3}\right) \left(\frac{1}{x}\right) dx && \text{Substitute.} \\ &= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx && \text{Simplify.} \\ &= \frac{x^3}{3} \ln x - \frac{x^3}{9} + C. && \text{Integrate.} \end{aligned}$$

You can check this result by differentiating.

$$\frac{d}{dx} \left[\frac{x^3}{3} \ln x - \frac{x^3}{9} \right] = \frac{x^3}{3} \left(\frac{1}{x} \right) + (\ln x)(x^2) - \frac{x^2}{3} = x^2 \ln x$$

TECHNOLOGY Try graphing

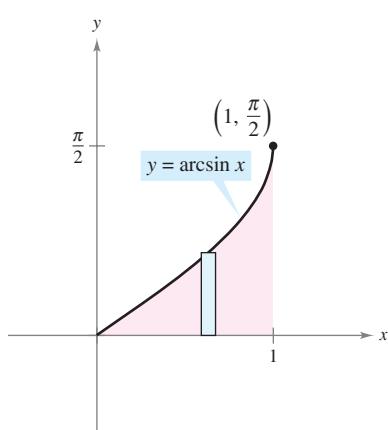
$$\int x^2 \ln x dx \quad \text{and} \quad \frac{x^3}{3} \ln x - \frac{x^3}{9}$$

on your graphing utility. Do you get the same graph? (This will take a while, so be patient.)

FOR FURTHER INFORMATION To see how integration by parts is used to prove Stirling's approximation

$$\ln(n!) = n \ln n - n$$

see the article "The Validity of Stirling's Approximation: A Physical Chemistry Project" by A. S. Wallner and K. A. Brandt in *Journal of Chemical Education*.



The area of the region is approximately 0.571.

Figure 8.2

One surprising application of integration by parts involves integrands consisting of single terms, such as $\int \ln x \, dx$ or $\int \arcsin x \, dx$. In these cases, try letting $dv = dx$, as shown in the next example.

EXAMPLE 3 An Integrand with a Single Term

Evaluate $\int_0^1 \arcsin x \, dx$.

Solution Let $dv = dx$.

$$\begin{aligned} dv &= dx & \Rightarrow v &= \int dx = x \\ u &= \arcsin x & \Rightarrow du &= \frac{1}{\sqrt{1-x^2}} dx \end{aligned}$$

Integration by parts now produces

$$\begin{aligned} \int u \, dv &= uv - \int v \, du && \text{Integration by parts formula} \\ \int \arcsin x \, dx &= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx && \text{Substitute.} \\ &= x \arcsin x + \frac{1}{2} \int (1-x^2)^{-1/2} (-2x) \, dx && \text{Rewrite.} \\ &= x \arcsin x + \sqrt{1-x^2} + C. && \text{Integrate.} \end{aligned}$$

Using this antiderivative, you can evaluate the definite integral as follows.

$$\begin{aligned} \int_0^1 \arcsin x \, dx &= \left[x \arcsin x + \sqrt{1-x^2} \right]_0^1 \\ &= \frac{\pi}{2} - 1 \\ &\approx 0.571 \end{aligned}$$

The area represented by this definite integral is shown in Figure 8.2. ■

TECHNOLOGY Remember that there are two ways to use technology to evaluate a definite integral: (1) you can use a numerical approximation such as the Trapezoidal Rule or Simpson's Rule, or (2) you can use a computer algebra system to find the antiderivative and then apply the Fundamental Theorem of Calculus. Both methods have shortcomings. To find the possible error when using a numerical method, the integrand must have a second derivative (Trapezoidal Rule) or a fourth derivative (Simpson's Rule) in the interval of integration: the integrand in Example 3 fails to meet either of these requirements. To apply the Fundamental Theorem of Calculus, the symbolic integration utility must be able to find the antiderivative.

Which method would you use to evaluate

$$\int_0^1 \arctan x \, dx?$$

Which method would you use to evaluate

$$\int_0^1 \arctan x^2 \, dx?$$

Some integrals require repeated use of the integration by parts formula.

EXAMPLE 4 Repeated Use of Integration by Parts

Find $\int x^2 \sin x \, dx$.

Solution The factors x^2 and $\sin x$ are equally easy to integrate. However, the derivative of x^2 becomes simpler, whereas the derivative of $\sin x$ does not. So, you should let $u = x^2$.

$$\begin{aligned} dv &= \sin x \, dx &\Rightarrow v &= \int \sin x \, dx = -\cos x \\ u &= x^2 &\Rightarrow du &= 2x \, dx \end{aligned}$$

Now, integration by parts produces

$$\int x^2 \sin x \, dx = -x^2 \cos x + \int 2x \cos x \, dx. \quad \text{First use of integration by parts}$$

This first use of integration by parts has succeeded in simplifying the original integral, but the integral on the right still doesn't fit a basic integration rule. To evaluate that integral, you can apply integration by parts again. This time, let $u = 2x$.

$$\begin{aligned} dv &= \cos x \, dx &\Rightarrow v &= \int \cos x \, dx = \sin x \\ u &= 2x &\Rightarrow du &= 2 \, dx \end{aligned}$$

Now, integration by parts produces

$$\begin{aligned} \int 2x \cos x \, dx &= 2x \sin x - \int 2 \sin x \, dx && \text{Second use of integration by parts} \\ &= 2x \sin x + 2 \cos x + C. \end{aligned}$$

Combining these two results, you can write

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C. \quad \blacksquare$$

When making repeated applications of integration by parts, you need to be careful not to interchange the substitutions in successive applications. For instance, in Example 4, the first substitution was $u = x^2$ and $dv = \sin x \, dx$. If, in the second application, you had switched the substitution to $u = \cos x$ and $dv = 2x$, you would have obtained

$$\begin{aligned} \int x^2 \sin x \, dx &= -x^2 \cos x + \int 2x \cos x \, dx \\ &= -x^2 \cos x + x^2 \cos x + \int x^2 \sin x \, dx = \int x^2 \sin x \, dx \end{aligned}$$

thereby undoing the previous integration and returning to the *original* integral. When making repeated applications of integration by parts, you should also watch for the appearance of a *constant multiple* of the original integral. For instance, this occurs when you use integration by parts to evaluate $\int e^x \cos 2x \, dx$, and also occurs in Example 5 on the next page.

The integral in Example 5 is an important one. In Section 8.4 (Example 5), you will see that it is used to find the arc length of a parabolic segment.

EXPLORATION

Try to find

$$\int e^x \cos 2x \, dx$$

by letting $u = \cos 2x$ and $dv = e^x \, dx$ in the first substitution. For the second substitution, let $u = \sin 2x$ and $dv = e^x \, dx$.

EXAMPLE 5 Integration by Parts

Find $\int \sec^3 x \, dx$.

Solution The most complicated portion of the integrand that can be easily integrated is $\sec^2 x$, so you should let $dv = \sec^2 x \, dx$ and $u = \sec x$.

$$\begin{aligned} dv &= \sec^2 x \, dx & \Rightarrow v &= \int \sec^2 x \, dx = \tan x \\ u &= \sec x & \Rightarrow du &= \sec x \tan x \, dx \end{aligned}$$

Integration by parts produces

$$\int u \, dv = uv - \int v \, du$$

Integration by parts formula

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

Substitute.

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

Trigonometric identity

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

Rewrite.

$$2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

Collect like integrals.

$$2 \int \sec^3 x \, dx = \sec x \tan x + \ln |\sec x + \tan x| + C$$

Integrate.

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C.$$

Divide by 2.

STUDY TIP The trigonometric identities

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

play an important role in this chapter.

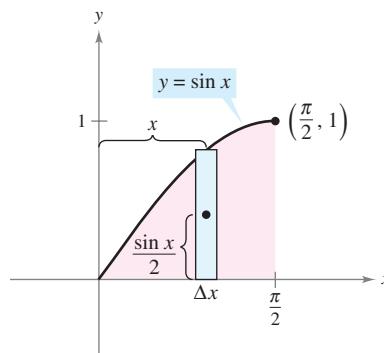


Figure 8.3

EXAMPLE 6 Finding a Centroid

A machine part is modeled by the region bounded by the graph of $y = \sin x$ and the x -axis, $0 \leq x \leq \pi/2$, as shown in Figure 8.3. Find the centroid of this region.

Solution Begin by finding the area of the region.

$$A = \int_0^{\pi/2} \sin x \, dx = \left[-\cos x \right]_0^{\pi/2} = 1$$

Now, you can find the coordinates of the centroid as follows.

$$\bar{y} = \frac{1}{A} \int_0^{\pi/2} \frac{\sin x}{2} (\sin x) \, dx = \frac{1}{4} \int_0^{\pi/2} (1 - \cos 2x) \, dx = \frac{1}{4} \left[x - \frac{\sin 2x}{2} \right]_0^{\pi/2} = \frac{\pi}{8}$$

You can evaluate the integral for \bar{x} , $(1/A) \int_0^{\pi/2} x \sin x \, dx$, with integration by parts. To do this, let $dv = \sin x \, dx$ and $u = x$. This produces $v = -\cos x$ and $du = dx$, and you can write

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C.$$

Finally, you can determine \bar{x} to be

$$\bar{x} = \frac{1}{A} \int_0^{\pi/2} x \sin x \, dx = \left[-x \cos x + \sin x \right]_0^{\pi/2} = 1.$$

So, the centroid of the region is $(1, \pi/8)$.

As you gain experience in using integration by parts, your skill in determining u and dv will increase. The following summary lists several common integrals with suggestions for the choices of u and dv .

STUDY TIP You can use the acronym LIATE as a guideline for choosing u in integration by parts. In order, check the integrand for the following.

Is there a Logarithmic part?

Is there an Inverse trigonometric part?

Is there an Algebraic part?

Is there a Trigonometric part?

Is there an Exponential part?

SUMMARY OF COMMON INTEGRALS USING INTEGRATION BY PARTS

1. For integrals of the form

$$\int x^n e^{ax} dx, \quad \int x^n \sin ax dx, \quad \text{or} \quad \int x^n \cos ax dx$$

let $u = x^n$ and let $dv = e^{ax} dx$, $\sin ax dx$, or $\cos ax dx$.

2. For integrals of the form

$$\int x^n \ln x dx, \quad \int x^n \arcsin ax dx, \quad \text{or} \quad \int x^n \arctan ax dx$$

let $u = \ln x$, $\arcsin ax$, or $\arctan ax$ and let $dv = x^n dx$.

3. For integrals of the form

$$\int e^{ax} \sin bx dx \quad \text{or} \quad \int e^{ax} \cos bx dx$$

let $u = \sin bx$ or $\cos bx$ and let $dv = e^{ax} dx$.

Tabular Method

In problems involving repeated applications of integration by parts, a tabular method, illustrated in Example 7, can help to organize the work. This method works well for integrals of the form $\int x^n \sin ax dx$, $\int x^n \cos ax dx$, and $\int x^n e^{ax} dx$.

EXAMPLE 7 Using the Tabular Method

Find $\int x^2 \sin 4x dx$.

Solution Begin as usual by letting $u = x^2$ and $dv = v' dx = \sin 4x dx$. Next, create a table consisting of three columns, as shown.

Alternate Signs	u and Its Derivatives	v' and Its Antiderivatives
+	x^2	$\sin 4x$
-	$2x$	$-\frac{1}{4} \cos 4x$
+	2	$-\frac{1}{16} \sin 4x$
-	0	$\frac{1}{64} \cos 4x$

Differentiate until you obtain 0 as a derivative.

FOR FURTHER INFORMATION

For more information on the tabular method, see the article "Tabular Integration by Parts" by David Horowitz in *The College Mathematics Journal*, and the article "More on Tabular Integration by Parts" by Leonard Gillman in *The College Mathematics Journal*. To view these articles, go to the website www.matharticles.com.

The solution is obtained by adding the signed products of the diagonal entries:

$$\int x^2 \sin 4x dx = -\frac{1}{4} x^2 \cos 4x + \frac{1}{8} x \sin 4x + \frac{1}{32} \cos 4x + C.$$

8.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, identify u and dv for finding the integral using integration by parts. (Do not evaluate the integral.)

1. $\int xe^{2x} dx$

2. $\int x^2 e^{2x} dx$

3. $\int (\ln x)^2 dx$

4. $\int \ln 5x dx$

5. $\int x \sec^2 x dx$

6. $\int x^2 \cos x dx$

In Exercises 7–10, evaluate the integral using integration by parts with the given choices of u and dv .

7. $\int x^3 \ln x dx; u = \ln x, dv = x^3 dx$

8. $\int (4x + 7)e^x dx; u = 4x + 7, dv = e^x dx$

9. $\int x \sin 3x dx; u = x, dv = \sin 3x dx$

10. $\int x \cos 4x dx; u = x, dv = \cos 4x dx$

In Exercises 11–38, find the integral. (Note: Solve by the simplest method—not all require integration by parts.)

11. $\int xe^{-4x} dx$

12. $\int \frac{4x}{e^x} dx$

13. $\int x^3 e^x dx$

14. $\int \frac{e^{1/t}}{t^2} dt$

15. $\int x^2 e^{x^3} dx$

16. $\int x^4 \ln x dx$

17. $\int t \ln(t + 1) dt$

18. $\int \frac{1}{x(\ln x)^3} dx$

19. $\int \frac{(\ln x)^2}{x} dx$

20. $\int \frac{\ln x}{x^2} dx$

21. $\int \frac{xe^{2x}}{(2x + 1)^2} dx$

22. $\int \frac{x^3 e^{x^2}}{(x^2 + 1)^2} dx$

23. $\int (x^2 - 1)e^x dx$

24. $\int \frac{\ln 2x}{x^2} dx$

25. $\int x\sqrt{x-5} dx$

26. $\int \frac{x}{\sqrt{5+4x}} dx$

27. $\int x \cos x dx$

28. $\int x \sin x dx$

29. $\int x^3 \sin x dx$

30. $\int x^2 \cos x dx$

31. $\int t \csc t \cot t dt$

32. $\int \theta \sec \theta \tan \theta d\theta$

33. $\int \arctan x dx$

34. $\int 4 \arccos x dx$

35. $\int e^{2x} \sin x dx$

36. $\int e^{-3x} \sin 5x dx$

37. $\int e^{-x} \cos 2x dx$

38. $\int e^{3x} \cos 4x dx$

In Exercises 39–44, solve the differential equation.

39. $y' = xe^{x^2}$

40. $y' = \ln x$

41. $\frac{dy}{dt} = \frac{t^2}{\sqrt{3+5t}}$

42. $\frac{dy}{dx} = x^2 \sqrt{x-3}$

43. $(\cos y)y' = 2x$

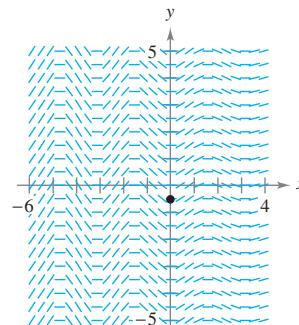
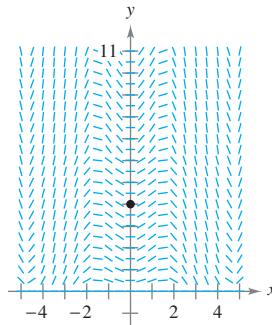
44. $y' = \arctan \frac{x}{2}$



Slope Fields In Exercises 45 and 46, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

45. $\frac{dy}{dx} = x\sqrt{y} \cos x, (0, 4)$

46. $\frac{dy}{dx} = e^{-x/3} \sin 2x, (0, -\frac{18}{37})$



CAS Slope Fields In Exercises 47 and 48, use a computer algebra system to graph the slope field for the differential equation and graph the solution through the specified initial condition.

47. $\frac{dy}{dx} = \frac{x}{y} e^{x/8}$
 $y(0) = 2$

48. $\frac{dy}{dx} = \frac{x}{y} \sin x$
 $y(0) = 4$

In Exercises 49–60, evaluate the definite integral. Use a graphing utility to confirm your result.

49. $\int_0^3 xe^{x/2} dx$

50. $\int_0^2 x^2 e^{-2x} dx$

51. $\int_0^{\pi/4} x \cos 2x dx$

52. $\int_0^\pi x \sin 2x dx$

53. $\int_0^{1/2} \arccos x dx$

54. $\int_0^1 x \arcsin x^2 dx$

55. $\int_0^1 e^x \sin x dx$

56. $\int_0^2 e^{-x} \cos x dx$

57. $\int_1^2 \sqrt{x} \ln x dx$

58. $\int_0^1 \ln(4+x^2) dx$

59. $\int_2^4 x \operatorname{arcsec} x dx$

60. $\int_0^{\pi/8} x \sec^2 2x dx$

In Exercises 61–66, use the tabular method to find the integral.

61. $\int x^2 e^{2x} dx$

62. $\int x^3 e^{-2x} dx$

63. $\int x^3 \sin x dx$

64. $\int x^3 \cos 2x dx$

65. $\int x \sec^2 x dx$

66. $\int x^2(x - 2)^{3/2} dx$

In Exercises 67–74, find or evaluate the integral using substitution first, then using integration by parts.

67. $\int \sin \sqrt{x} dx$

68. $\int \cos \sqrt{x} dx$

69. $\int_0^4 x \sqrt{4 - x} dx$

70. $\int 2x^3 \cos x^2 dx$

71. $\int x^5 e^x dx$

72. $\int_0^2 e^{\sqrt{2x}} dx$

73. $\int \cos(\ln x) dx$

74. $\int \ln(x^2 + 1) dx$

WRITING ABOUT CONCEPTS

75. Integration by parts is based on what differentiation rule? Explain.
76. In your own words, state how you determine which parts of the integrand should be u and dv .
77. When evaluating $\int x \sin x dx$, explain how letting $u = \sin x$ and $dv = x dx$ makes the solution more difficult to find.

CAPSTONE

78. State whether you would use integration by parts to evaluate each integral. If so, identify what you would use for u and dv . Explain your reasoning.

(a) $\int \frac{\ln x}{x} dx$ (b) $\int x \ln x dx$ (c) $\int x^2 e^{-3x} dx$

(d) $\int 2x e^{x^2} dx$ (e) $\int \frac{x}{\sqrt{x+1}} dx$ (f) $\int \frac{x}{\sqrt{x^2+1}} dx$

- CAS** In Exercises 79–82, use a computer algebra system to (a) find or evaluate the integral and (b) graph two antiderivatives. (c) Describe the relationship between the graphs of the antiderivatives.

79. $\int t^3 e^{-4t} dt$

80. $\int \alpha^4 \sin \pi \alpha d\alpha$

81. $\int_0^{\pi/2} e^{-2x} \sin 3x dx$

82. $\int_0^5 x^4(25 - x^2)^{3/2} dx$

83. Integrate $\int 2x \sqrt{2x - 3} dx$

(a) by parts, letting $dv = \sqrt{2x - 3} dx$.

(b) by substitution, letting $u = 2x - 3$.

84. Integrate $\int x \sqrt{9 + x} dx$

(a) by parts, letting $dv = \sqrt{9 + x} dx$.

(b) by substitution, letting $u = 9 + x$.

85. Integrate $\int \frac{x^3}{\sqrt{4 + x^2}} dx$

(a) by parts, letting $dv = (x/\sqrt{4 + x^2}) dx$.

(b) by substitution, letting $u = 4 + x^2$.

86. Integrate $\int x \sqrt{4 - x} dx$

(a) by parts, letting $dv = \sqrt{4 - x} dx$.

(b) by substitution, letting $u = 4 - x$.

- CAS** In Exercises 87 and 88, use a computer algebra system to find the integrals for $n = 0, 1, 2$, and 3 . Use the result to obtain a general rule for the integrals for any positive integer n and test your results for $n = 4$.

87. $\int x^n \ln x dx$

88. $\int x^n e^x dx$

In Exercises 89–94, use integration by parts to prove the formula. (For Exercises 89–92, assume that n is a positive integer.)

89. $\int x^n \sin x dx = -x^n \cos x + n \int x^{n-1} \cos x dx$

90. $\int x^n \cos x dx = x^n \sin x - n \int x^{n-1} \sin x dx$

91. $\int x^n \ln x dx = \frac{x^{n+1}}{(n+1)^2} [-1 + (n+1) \ln x] + C$

92. $\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$

93. $\int e^{ax} \sin bx dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + C$

94. $\int e^{ax} \cos bx dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2} + C$

In Exercises 95–98, find the integral by using the appropriate formula from Exercises 89–94.

95. $\int x^5 \ln x dx$

96. $\int x^2 \cos x dx$

97. $\int e^{2x} \cos 3x dx$

98. $\int x^3 e^{2x} dx$



Area In Exercises 99–102, use a graphing utility to graph the region bounded by the graphs of the equations, and find the area of the region.

99. $y = 2xe^{-x}$, $y = 0$, $x = 3$

100. $y = \frac{1}{16} xe^{-x/4}$, $y = 0$, $x = 0$, $x = 4$

101. $y = e^{-x} \sin \pi x$, $y = 0$, $x = 1$

102. $y = x \sin x$, $y = 0$, $x = \pi$

- 103. Area, Volume, and Centroid** Given the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$, find

- the area of the region.
- the volume of the solid generated by revolving the region about the x -axis.
- the volume of the solid generated by revolving the region about the y -axis.
- the centroid of the region.

- 104. Volume and Centroid** Given the region bounded by the graphs of $y = x \sin x$, $y = 0$, $x = 0$, and $x = \pi$, find

- the volume of the solid generated by revolving the region about the x -axis.
- the volume of the solid generated by revolving the region about the y -axis.
- the centroid of the region.

- 105. Centroid** Find the centroid of the region bounded by the graphs of $y = \arcsin x$, $x = 0$, and $y = \pi/2$. How is this problem related to Example 6 in this section?

- 106. Centroid** Find the centroid of the region bounded by the graphs of $f(x) = x^2$, $g(x) = 2^x$, $x = 2$, and $x = 4$.

- 107. Average Displacement** A damping force affects the vibration of a spring so that the displacement of the spring is given by $y = e^{-4t}(\cos 2t + 5 \sin 2t)$. Find the average value of y on the interval from $t = 0$ to $t = \pi$.

- 108. Memory Model** A model for the ability M of a child to memorize, measured on a scale from 0 to 10, is given by $M = 1 + 1.6t \ln t$, $0 < t \leq 4$, where t is the child's age in years. Find the average value of this model

- between the child's first and second birthdays.
- between the child's third and fourth birthdays.

Present Value In Exercises 109 and 110, find the present value P of a continuous income flow of $c(t)$ dollars per year if

$$P = \int_0^{t_1} c(t)e^{-rt} dt$$

where t_1 is the time in years and r is the annual interest rate compounded continuously.

109. $c(t) = 100,000 + 4000t$, $r = 5\%$, $t_1 = 10$

110. $c(t) = 30,000 + 500t$, $r = 7\%$, $t_1 = 5$

Integrals Used to Find Fourier Coefficients In Exercises 111 and 112, verify the value of the definite integral, where n is a positive integer.

111. $\int_{-\pi}^{\pi} x \sin nx dx = \begin{cases} \frac{2\pi}{n}, & n \text{ is odd} \\ -\frac{2\pi}{n}, & n \text{ is even} \end{cases}$

112. $\int_{-\pi}^{\pi} x^2 \cos nx dx = \frac{(-1)^n 4\pi}{n^2}$

- 113. Vibrating String** A string stretched between the two points $(0, 0)$ and $(2, 0)$ is plucked by displacing the string h units at its midpoint. The motion of the string is modeled by a **Fourier Sine Series** whose coefficients are given by

$$b_n = h \int_0^1 x \sin \frac{n\pi x}{2} dx + h \int_1^2 (-x + 2) \sin \frac{n\pi x}{2} dx.$$

Find b_n .

- 114.** Find the fallacy in the following argument that $0 = 1$.

$$dv = dx \quad \Rightarrow \quad v = \int dx = x$$

$$u = \frac{1}{x} \quad \Rightarrow \quad du = -\frac{1}{x^2} dx$$

$$0 + \int \frac{dx}{x} = \left(\frac{1}{x} \right)(x) - \int \left(-\frac{1}{x^2} \right)(x) dx = 1 + \int \frac{dx}{x}$$

So, $0 = 1$.

- 115.** Let $y = f(x)$ be positive and strictly increasing on the interval $0 < a \leq x \leq b$. Consider the region R bounded by the graphs of $y = f(x)$, $y = 0$, $x = a$, and $x = b$. If R is revolved about the y -axis, show that the disk method and shell method yield the same volume.

- 116. Euler's Method** Consider the differential equation $f'(x) = xe^{-x}$ with the initial condition $f(0) = 0$.

- Use integration to solve the differential equation.
- Use a graphing utility to graph the solution of the differential equation.
- Use Euler's Method with $h = 0.05$, and the recursive capabilities of a graphing utility, to generate the first 80 points of the graph of the approximate solution. Use the graphing utility to plot the points. Compare the result with the graph in part (b).
- Repeat part (c) using $h = 0.1$ and generate the first 40 points.
- Why is the result in part (c) a better approximation of the solution than the result in part (d)?

Euler's Method In Exercises 117 and 118, consider the differential equation and repeat parts (a)–(d) of Exercise 116.

117. $f'(x) = 3x \sin(2x)$

$f(0) = 0$

118. $f'(x) = \cos \sqrt{x}$

$f(0) = 1$

- 119. Think About It** Give a geometric explanation of why

$$\int_0^{\pi/2} x \sin x dx \leq \int_0^{\pi/2} x dx.$$

Verify the inequality by evaluating the integrals.

- 120. Finding a Pattern** Find the area bounded by the graphs of $y = x \sin x$ and $y = 0$ over each interval.

- $[0, \pi]$
- $[\pi, 2\pi]$
- $[2\pi, 3\pi]$

Describe any patterns that you notice. What is the area between the graphs of $y = x \sin x$ and $y = 0$ over the interval $[n\pi, (n+1)\pi]$, where n is any nonnegative integer? Explain.

8.3 Trigonometric Integrals

- Solve trigonometric integrals involving powers of sine and cosine.
- Solve trigonometric integrals involving powers of secant and tangent.
- Solve trigonometric integrals involving sine-cosine products with different angles.

Integrals Involving Powers of Sine and Cosine

SHEILA SCOTT MACINTYRE (1910–1960)

Sheila Scott Macintyre published her first paper on the asymptotic periods of integral functions in 1935. She completed her doctorate work at Aberdeen University, where she taught. In 1958 she accepted a visiting research fellowship at the University of Cincinnati.

In this section you will study techniques for evaluating integrals of the form

$$\int \sin^m x \cos^n x \, dx \quad \text{and} \quad \int \sec^m x \tan^n x \, dx$$

where either m or n is a positive integer. To find antiderivatives for these forms, try to break them into combinations of trigonometric integrals to which you can apply the Power Rule.

For instance, you can evaluate $\int \sin^5 x \cos x \, dx$ with the Power Rule by letting $u = \sin x$. Then, $du = \cos x \, dx$ and you have

$$\int \sin^5 x \cos x \, dx = \int u^5 \, du = \frac{u^6}{6} + C = \frac{\sin^6 x}{6} + C.$$

To break up $\int \sin^m x \cos^n x \, dx$ into forms to which you can apply the Power Rule, use the following identities.

$$\sin^2 x + \cos^2 x = 1 \quad \text{Pythagorean identity}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{Half-angle identity for } \sin^2 x$$

$$\cos^2 x = \frac{1 + \cos 2x}{2} \quad \text{Half-angle identity for } \cos^2 x$$

GUIDELINES FOR EVALUATING INTEGRALS INVOLVING POWERS OF SINE AND COSINE

- If the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines. Then, expand and integrate.

$$\int \sin^{2k+1} x \cos^n x \, dx \stackrel{\text{Odd}}{=} \int (\sin^2 x)^k \cos^n x \sin x \, dx \stackrel{\text{Convert to cosines}}{=} \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx \stackrel{\text{Save for } du}{=}$$

- If the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines. Then, expand and integrate.

$$\int \sin^m x \cos^{2k+1} x \, dx \stackrel{\text{Odd}}{=} \int \sin^m x (\cos^2 x)^k \cos x \, dx \stackrel{\text{Convert to sines}}{=} \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx \stackrel{\text{Save for } du}{=}$$

- If the powers of both the sine and cosine are even and nonnegative, make repeated use of the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to convert the integrand to odd powers of the cosine. Then proceed as in guideline 2.

TECHNOLOGY A computer algebra system used to find the integral in Example 1 yielded the following.

$$\int \sin^3 x \cos^4 x dx = -\cos^5 x \left(\frac{1}{7} \sin^2 x + \frac{2}{35} \right) + C$$

Is this equivalent to the result obtained in Example 1?

EXAMPLE 1 Power of Sine Is Odd and Positive

Find $\int \sin^3 x \cos^4 x dx$.

Solution Because you expect to use the Power Rule with $u = \cos x$, save one sine factor to form du and convert the remaining sine factors to cosines.

$$\begin{aligned} \int \sin^3 x \cos^4 x dx &= \int \sin^2 x \cos^4 x (\sin x) dx && \text{Rewrite.} \\ &= \int (1 - \cos^2 x) \cos^4 x \sin x dx && \text{Trigonometric identity} \\ &= \int (\cos^4 x - \cos^6 x) \sin x dx && \text{Multiply.} \\ &= \int \cos^4 x \sin x dx - \int \cos^6 x \sin x dx && \text{Rewrite.} \\ &= - \int \cos^4 x (-\sin x) dx + \int \cos^6 x (-\sin x) dx \\ &= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C && \text{Integrate.} \end{aligned}$$

■

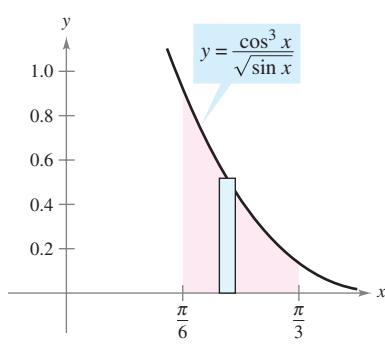
In Example 1, both of the powers m and n happened to be positive integers. However, the same strategy will work as long as either m or n is odd and positive. For instance, in the next example the power of the cosine is 3, but the power of the sine is $-\frac{1}{2}$.

EXAMPLE 2 Power of Cosine Is Odd and Positive

Evaluate $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} dx$.

Solution Because you expect to use the Power Rule with $u = \sin x$, save one cosine factor to form du and convert the remaining cosine factors to sines.

$$\begin{aligned} \int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} dx &= \int_{\pi/6}^{\pi/3} \frac{\cos^2 x \cos x}{\sqrt{\sin x}} dx \\ &= \int_{\pi/6}^{\pi/3} \frac{(1 - \sin^2 x)(\cos x)}{\sqrt{\sin x}} dx \\ &= \int_{\pi/6}^{\pi/3} [(\sin x)^{-1/2} \cos x - (\sin x)^{3/2} \cos x] dx \\ &= \left[\frac{(\sin x)^{1/2}}{1/2} - \frac{(\sin x)^{5/2}}{5/2} \right]_{\pi/6}^{\pi/3} \\ &= 2 \left(\frac{\sqrt{3}}{2} \right)^{1/2} - \frac{2}{5} \left(\frac{\sqrt{3}}{2} \right)^{5/2} - \sqrt{2} + \frac{\sqrt{32}}{80} \\ &\approx 0.239 \end{aligned}$$



The area of the region is approximately 0.239.

Figure 8.4

Figure 8.4 shows the region whose area is represented by this integral. ■

EXAMPLE 3 Power of Cosine Is Even and Nonnegative

Find $\int \cos^4 x \, dx$.

Solution Because m and n are both even and nonnegative ($m = 0$), you can replace $\cos^4 x$ by $[(1 + \cos 2x)/2]^2$.

$$\begin{aligned}\int \cos^4 x \, dx &= \int \left(\frac{1 + \cos 2x}{2} \right)^2 \, dx && \text{Half-angle identity} \\ &= \int \left(\frac{1}{4} + \frac{\cos 2x}{2} + \frac{\cos^2 2x}{4} \right) \, dx && \text{Expand.} \\ &= \int \left[\frac{1}{4} + \frac{\cos 2x}{2} + \frac{1}{4} \left(\frac{1 + \cos 4x}{2} \right) \right] \, dx && \text{Half-angle identity} \\ &= \frac{3}{8} \int dx + \frac{1}{4} \int 2 \cos 2x \, dx + \frac{1}{32} \int 4 \cos 4x \, dx && \text{Rewrite.} \\ &= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C && \text{Integrate.}\end{aligned}$$

Use a symbolic differentiation utility to verify this. Can you simplify the derivative to obtain the original integrand? ■

In Example 3, if you were to evaluate the definite integral from 0 to $\pi/2$, you would obtain

$$\begin{aligned}\int_0^{\pi/2} \cos^4 x \, dx &= \left[\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} \right]_0^{\pi/2} \\ &= \left(\frac{3\pi}{16} + 0 + 0 \right) - (0 + 0 + 0) \\ &= \frac{3\pi}{16}.\end{aligned}$$

Note that the only term that contributes to the solution is $3x/8$. This observation is generalized in the following formulas developed by John Wallis.

Bettmann/Corbis



JOHN WALLIS (1616–1703)

Wallis did much of his work in calculus prior to Newton and Leibniz, and he influenced the thinking of both of these men. Wallis is also credited with introducing the present symbol (∞) for infinity.

WALLIS'S FORMULAS

1. If n is odd ($n \geq 3$), then

$$\int_0^{\pi/2} \cos^n x \, dx = \left(\frac{2}{3} \right) \left(\frac{4}{5} \right) \left(\frac{6}{7} \right) \cdots \left(\frac{n-1}{n} \right).$$

2. If n is even ($n \geq 2$), then

$$\int_0^{\pi/2} \cos^n x \, dx = \left(\frac{1}{2} \right) \left(\frac{3}{4} \right) \left(\frac{5}{6} \right) \cdots \left(\frac{n-1}{n} \right) \left(\frac{\pi}{2} \right).$$

These formulas are also valid if $\cos^n x$ is replaced by $\sin^n x$. (You are asked to prove both formulas in Exercise 108.)

Integrals Involving Powers of Secant and Tangent

The following guidelines can help you evaluate integrals of the form

$$\int \sec^m x \tan^n x dx.$$

GUIDELINES FOR EVALUATING INTEGRALS INVOLVING POWERS OF SECANT AND TANGENT

- If the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then expand and integrate.

$$\int \sec^{2k} x \tan^n x dx \stackrel{\text{Even}}{=} \int \overbrace{(\sec^2 x)^{k-1}}^{\text{Convert to tangents}} \tan^n x \sec^2 x dx \stackrel{\text{Save for } du}{=} \int (1 + \tan^2 x)^{k-1} \tan^n x \sec^2 x dx$$

- If the power of the tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then expand and integrate.

$$\int \sec^m x \tan^{2k+1} x dx \stackrel{\text{Odd}}{=} \int \sec^{m-1} x \overbrace{(\tan^2 x)^k}^{\text{Convert to secants}} \sec x \tan x dx = \int \sec^{m-1} x (\sec^2 x - 1)^k \sec x \tan x dx$$

- If there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.

$$\int \tan^n x dx \stackrel{\text{Convert to secants}}{=} \int \tan^{n-2} x \overbrace{(\tan^2 x)}^{\text{Convert to secants}} dx = \int \tan^{n-2} x (\sec^2 x - 1) dx$$

- If the integral is of the form $\int \sec^m x dx$, where m is odd and positive, use integration by parts, as illustrated in Example 5 in the preceding section.
- If none of the first four guidelines applies, try converting to sines and cosines.

EXAMPLE 4 Power of Tangent Is Odd and Positive

Find $\int \frac{\tan^3 x}{\sqrt{\sec x}} dx$.

Solution Because you expect to use the Power Rule with $u = \sec x$, save a factor of $(\sec x \tan x)$ to form du and convert the remaining tangent factors to secants.

$$\begin{aligned} \int \frac{\tan^3 x}{\sqrt{\sec x}} dx &= \int (\sec x)^{-1/2} \tan^3 x dx \\ &= \int (\sec x)^{-3/2} (\tan^2 x) (\sec x \tan x) dx \\ &= \int (\sec x)^{-3/2} (\sec^2 x - 1) (\sec x \tan x) dx \\ &= \int [(\sec x)^{1/2} - (\sec x)^{-3/2}] (\sec x \tan x) dx \\ &= \frac{2}{3} (\sec x)^{3/2} + 2 (\sec x)^{-1/2} + C \end{aligned}$$

NOTE In Example 5, the power of the tangent is odd and positive. So, you could also find the integral using the procedure described in guideline 2 on page 539. In Exercise 89, you are asked to show that the results obtained by these two procedures differ only by a constant.

EXAMPLE 5 Power of Secant Is Even and Positive

Find $\int \sec^4 3x \tan^3 3x \, dx$.

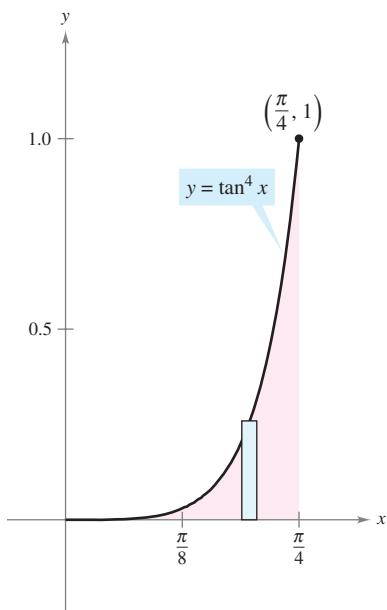
Solution Let $u = \tan 3x$, then $du = 3 \sec^2 3x \, dx$ and you can write

$$\begin{aligned}\int \sec^4 3x \tan^3 3x \, dx &= \int \sec^2 3x \tan^3 3x (\sec^2 3x) \, dx \\&= \int (1 + \tan^2 3x) \tan^3 3x (\sec^2 3x) \, dx \\&= \frac{1}{3} \int (\tan^3 3x + \tan^5 3x)(3 \sec^2 3x) \, dx \\&= \frac{1}{3} \left(\frac{\tan^4 3x}{4} + \frac{\tan^6 3x}{6} \right) + C \\&= \frac{\tan^4 3x}{12} + \frac{\tan^6 3x}{18} + C.\end{aligned}$$

EXAMPLE 6 Power of Tangent Is Even

Evaluate $\int_0^{\pi/4} \tan^4 x \, dx$.

Solution Because there are no secant factors, you can begin by converting a tangent-squared factor to a secant-squared factor.



The area of the region is approximately 0.119.

Figure 8.5

$$\begin{aligned}\int \tan^4 x \, dx &= \int \tan^2 x (\tan^2 x) \, dx \\&= \int \tan^2 x (\sec^2 x - 1) \, dx \\&= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\&= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx \\&= \frac{\tan^3 x}{3} - \tan x + x + C\end{aligned}$$

You can evaluate the definite integral as follows.

$$\begin{aligned}\int_0^{\pi/4} \tan^4 x \, dx &= \left[\frac{\tan^3 x}{3} - \tan x + x \right]_0^{\pi/4} \\&= \frac{\pi}{4} - \frac{2}{3} \\&\approx 0.119\end{aligned}$$

The area represented by the definite integral is shown in Figure 8.5. Try using Simpson's Rule to approximate this integral. With $n = 18$, you should obtain an approximation that is within 0.00001 of the actual value. ■

For integrals involving powers of cotangents and cosecants, you can follow a strategy similar to that used for powers of tangents and secants. Also, when integrating trigonometric functions, remember that it sometimes helps to convert the entire integrand to powers of sines and cosines.

EXAMPLE 7 Converting to Sines and Cosines

Find $\int \frac{\sec x}{\tan^2 x} dx$.

Solution Because the first four guidelines on page 539 do not apply, try converting the integrand to sines and cosines. In this case, you are able to integrate the resulting powers of sine and cosine as follows.

$$\begin{aligned}\int \frac{\sec x}{\tan^2 x} dx &= \int \left(\frac{1}{\cos x} \right) \left(\frac{\cos x}{\sin x} \right)^2 dx \\ &= \int (\sin x)^{-2} (\cos x) dx \\ &= -(\sin x)^{-1} + C \\ &= -\csc x + C\end{aligned}$$

■

Integrals Involving Sine-Cosine Products with Different Angles

FOR FURTHER INFORMATION

To learn more about integrals involving sine-cosine products with different angles, see the article “Integrals of Products of Sine and Cosine with Different Arguments” by Sherrie J. Nicol in *The College Mathematics Journal*. To view this article, go to the website www.matharticles.com.

Integrals involving the products of sines and cosines of two *different* angles occur in many applications. In such instances you can use the following product-to-sum identities.

$$\begin{aligned}\sin mx \sin nx &= \frac{1}{2}(\cos[(m-n)x] - \cos[(m+n)x]) \\ \sin mx \cos nx &= \frac{1}{2}(\sin[(m-n)x] + \sin[(m+n)x]) \\ \cos mx \cos nx &= \frac{1}{2}(\cos[(m-n)x] + \cos[(m+n)x])\end{aligned}$$

EXAMPLE 8 Using Product-to-Sum Identities

Find $\int \sin 5x \cos 4x dx$.

Solution Considering the second product-to-sum identity above, you can write

$$\begin{aligned}\int \sin 5x \cos 4x dx &= \frac{1}{2} \int (\sin x + \sin 9x) dx \\ &= \frac{1}{2} \left(-\cos x - \frac{\cos 9x}{9} \right) + C \\ &= -\frac{\cos x}{2} - \frac{\cos 9x}{18} + C.\end{aligned}$$

■

8.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, use differentiation to match the antiderivative with the correct integral. [Integrals are labeled (a), (b), (c), and (d).]

- | | |
|----------------------------------|-----------------------------|
| (a) $\int \sin x \tan^2 x \, dx$ | (b) $8 \int \cos^4 x \, dx$ |
| (c) $\int \sin x \sec^2 x \, dx$ | (d) $\int \tan^4 x \, dx$ |

1. $y = \sec x$
2. $y = \cos x + \sec x$
3. $y = x - \tan x + \frac{1}{3} \tan^3 x$
4. $y = 3x + 2 \sin x \cos^3 x + 3 \sin x \cos x$

In Exercises 5–18, find the integral.

- | | |
|--|---|
| 5. $\int \cos^5 x \sin x \, dx$ | 6. $\int \cos^3 x \sin^4 x \, dx$ |
| 7. $\int \sin^7 2x \cos 2x \, dx$ | 8. $\int \sin^3 x \, dx$ |
| 9. $\int \sin^3 x \cos^2 x \, dx$ | 10. $\int \cos^3 \frac{x}{3} \, dx$ |
| 11. $\int \sin^3 2\theta \sqrt{\cos 2\theta} \, d\theta$ | 12. $\int \frac{\cos^5 t}{\sqrt{\sin t}} \, dt$ |
| 13. $\int \cos^2 3x \, dx$ | 14. $\int \sin^2 5x \, dx$ |
| 15. $\int \cos^4 3\alpha \, d\alpha$ | 16. $\int \sin^4 6\theta \, d\theta$ |
| 17. $\int x \sin^2 x \, dx$ | 18. $\int x^2 \sin^2 x \, dx$ |

In Exercises 19–24, use Wallis's Formulas to evaluate the integral.

- | | |
|--|-------------------------------------|
| 19. $\int_0^{\pi/2} \cos^7 x \, dx$ | 20. $\int_0^{\pi/2} \cos^9 x \, dx$ |
| 21. $\int_0^{\pi/2} \cos^{10} x \, dx$ | 22. $\int_0^{\pi/2} \sin^5 x \, dx$ |
| 23. $\int_0^{\pi/2} \sin^6 x \, dx$ | 24. $\int_0^{\pi/2} \sin^8 x \, dx$ |

In Exercises 25–42, find the integral involving secant and tangent.

- | | |
|-------------------------------------|--|
| 25. $\int \sec 7x \, dx$ | 26. $\int \sec^2(2x - 1) \, dx$ |
| 27. $\int \sec^4 5x \, dx$ | 28. $\int \sec^6 3x \, dx$ |
| 29. $\int \sec^3 \pi x \, dx$ | 30. $\int \tan^5 x \, dx$ |
| 31. $\int \tan^5 \frac{x}{2} \, dx$ | 32. $\int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2} \, dx$ |

- | | |
|--|--|
| 33. $\int \sec^2 x \tan x \, dx$ | 34. $\int \tan^3 2t \sec^3 2t \, dt$ |
| 35. $\int \tan^2 x \sec^4 x \, dx$ | 36. $\int \tan^5 2x \sec^4 2x \, dx$ |
| 37. $\int \sec^6 4x \tan 4x \, dx$ | 38. $\int \sec^2 \frac{x}{2} \tan \frac{x}{2} \, dx$ |
| 39. $\int \sec^5 x \tan^3 x \, dx$ | 40. $\int \tan^3 3x \, dx$ |
| 41. $\int \frac{\tan^2 x}{\sec x} \, dx$ | 42. $\int \frac{\tan^2 x}{\sec^5 x} \, dx$ |

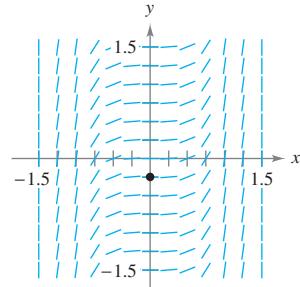
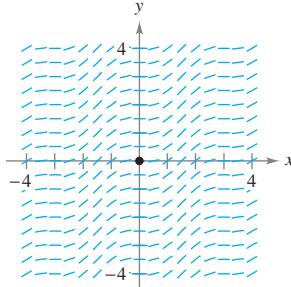
In Exercises 43–46, solve the differential equation.

- | | |
|---|--|
| 43. $\frac{dr}{d\theta} = \sin^4 \pi\theta$ | 44. $\frac{ds}{d\alpha} = \sin^2 \frac{\alpha}{2} \cos^2 \frac{\alpha}{2}$ |
| 45. $y' = \tan^3 3x \sec 3x$ | 46. $y' = \sqrt{\tan x} \sec^4 x$ |



Slope Fields In Exercises 47 and 48, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

- | | |
|--|--|
| 47. $\frac{dy}{dx} = \sin^2 x, (0, 0)$ | 48. $\frac{dy}{dx} = \sec^2 x \tan^2 x, (0, -\frac{1}{4})$ |
|--|--|



CAS Slope Fields In Exercises 49 and 50, use a computer algebra system to graph the slope field for the differential equation, and graph the solution through the specified initial condition.

- | | |
|--|--|
| 49. $\frac{dy}{dx} = \frac{3 \sin x}{y}, y(0) = 2$ | 50. $\frac{dy}{dx} = 3\sqrt{y} \tan^2 x, y(0) = 3$ |
|--|--|

In Exercises 51–56, find the integral.

- | | |
|--|---|
| 51. $\int \cos 2x \cos 6x \, dx$ | 52. $\int \cos 4\theta \cos(-3\theta) \, d\theta$ |
| 53. $\int \sin 2x \cos 4x \, dx$ | 54. $\int \sin(-4x) \cos 3x \, dx$ |
| 55. $\int \sin \theta \sin 3\theta \, d\theta$ | 56. $\int \sin 5x \sin 4x \, dx$ |

In Exercises 57–66, find the integral. Use a computer algebra system to confirm your result.

57. $\int \cot^3 2x \, dx$

58. $\int \tan^4 \frac{x}{2} \sec^4 \frac{x}{2} \, dx$

59. $\int \csc^4 2x \, dx$

60. $\int \cot^3 x \csc^3 x \, dx$

61. $\int \frac{\cot^2 t}{\csc t} \, dt$

62. $\int \frac{\cot^3 t}{\csc t} \, dt$

63. $\int \frac{1}{\sec x \tan x} \, dx$

64. $\int \frac{\sin^2 x - \cos^2 x}{\cos x} \, dx$

65. $\int (\tan^4 t - \sec^4 t) \, dt$

66. $\int \frac{1 - \sec t}{\cos t - 1} \, dt$

In Exercises 67–74, evaluate the definite integral.

67. $\int_{-\pi}^{\pi} \sin^2 x \, dx$

68. $\int_0^{\pi/3} \tan^2 x \, dx$

69. $\int_0^{\pi/4} 6 \tan^3 x \, dx$

70. $\int_0^{\pi/4} \sec^2 t \sqrt{\tan t} \, dt$

71. $\int_0^{\pi/2} \frac{\cos t}{1 + \sin t} \, dt$

72. $\int_{-\pi}^{\pi} \sin 5x \cos 3x \, dx$

73. $\int_{-\pi/2}^{\pi/2} 3 \cos^3 x \, dx$

74. $\int_{-\pi/2}^{\pi/2} (\sin^2 x + 1) \, dx$

CAS In Exercises 75–80, use a computer algebra system to find the integral. Graph the antiderivatives for two different values of the constant of integration.

75. $\int \cos^4 \frac{x}{2} \, dx$

76. $\int \sin^2 x \cos^2 x \, dx$

77. $\int \sec^5 \pi x \, dx$

78. $\int \tan^3(1 - x) \, dx$

79. $\int \sec^5 \pi x \tan \pi x \, dx$

80. $\int \sec^4(1 - x) \tan(1 - x) \, dx$

CAS In Exercises 81–84, use a computer algebra system to evaluate the definite integral.

81. $\int_0^{\pi/4} \sin 3\theta \sin 4\theta \, d\theta$

82. $\int_0^{\pi/2} (1 - \cos \theta)^2 \, d\theta$

83. $\int_0^{\pi/2} \sin^4 x \, dx$

84. $\int_0^{\pi/2} \sin^{12} x \, dx$

WRITING ABOUT CONCEPTS

85. In your own words, describe how you would integrate $\int \sin^m x \cos^n x \, dx$ for each condition.

- (a) m is positive and odd.
- (b) n is positive and odd.
- (c) m and n are both positive and even.

86. In your own words, describe how you would integrate $\int \sec^m x \tan^n x \, dx$ for each condition.

- (a) m is positive and even.
- (b) n is positive and odd.
- (c) n is positive and even, and there are no secant factors.
- (d) m is positive and odd, and there are no tangent factors.

87. Evaluate $\int \sin x \cos x \, dx$ using the given method. Explain how your answers differ for each method.

- (a) Substitution where $u = \sin x$
- (b) Substitution where $u = \cos x$
- (c) Integration by parts
- (d) Using the identity $\sin 2x = 2 \sin x \cos x$

CAPSTONE

88. For each pair of integrals, determine which one is more difficult to evaluate. Explain your reasoning.

- (a) $\int \sin^{372} x \cos x \, dx$, $\int \sin^4 x \cos^4 x \, dx$
- (b) $\int \tan^{400} x \sec^2 x \, dx$, $\int \tan^{400} x \sec x \, dx$



In Exercises 89 and 90, (a) find the indefinite integral in two different ways. (b) Use a graphing utility to graph the antiderivative (without the constant of integration) obtained by each method to show that the results differ only by a constant. (c) Verify analytically that the results differ only by a constant.

89. $\int \sec^4 3x \tan^3 3x \, dx$

90. $\int \sec^2 x \tan x \, dx$

Area In Exercises 91–94, find the area of the region bounded by the graphs of the equations.

91. $y = \sin x$, $y = \sin^3 x$, $x = 0$, $x = \pi/2$

92. $y = \sin^2 \pi x$, $y = 0$, $x = 0$, $x = 1$

93. $y = \cos^2 x$, $y = \sin^2 x$, $x = -\pi/4$, $x = \pi/4$

94. $y = \cos^2 x$, $y = \sin x \cos x$, $x = -\pi/2$, $x = \pi/4$

Volume In Exercises 95 and 96, find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the x -axis.

95. $y = \tan x$, $y = 0$, $x = -\pi/4$, $x = \pi/4$

96. $y = \cos \frac{x}{2}$, $y = \sin \frac{x}{2}$, $x = 0$, $x = \pi/2$

Volume and Centroid In Exercises 97 and 98, for the region bounded by the graphs of the equations, find (a) the volume of the solid formed by revolving the region about the x -axis and (b) the centroid of the region.

97. $y = \sin x$, $y = 0$, $x = 0$, $x = \pi$

98. $y = \cos x$, $y = 0$, $x = 0$, $x = \pi/2$

In Exercises 99–102, use integration by parts to verify the reduction formula.

99. $\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$

100. $\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$

101. $\int \cos^m x \sin^n x \, dx = -\frac{\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x \, dx$

102. $\int \sec^n x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$

In Exercises 103–106, use the results of Exercises 99–102 to find the integral.

103. $\int \sin^5 x \, dx$

104. $\int \cos^4 x \, dx$

105. $\int \sec^4(2\pi x/5) \, dx$

106. $\int \sin^4 x \cos^2 x \, dx$

- 107. Modeling Data** The table shows the normal maximum (high) and minimum (low) temperatures (in degrees Fahrenheit) in Erie, Pennsylvania for each month of the year. (Source: NOAA)

Month	Jan	Feb	Mar	Apr	May	Jun
Max	33.5	35.4	44.7	55.6	67.4	76.2
Min	20.3	20.9	28.2	37.9	48.7	58.5

Month	Jul	Aug	Sep	Oct	Nov	Dec
Max	80.4	79.0	72.0	61.0	49.3	38.6
Min	63.7	62.7	55.9	45.5	36.4	26.8

The maximum and minimum temperatures can be modeled by $f(t) = a_0 + a_1 \cos(\pi t/6) + b_1 \sin(\pi t/6)$, where $t = 0$ corresponds to January 1 and a_0 , a_1 , and b_1 are as follows.

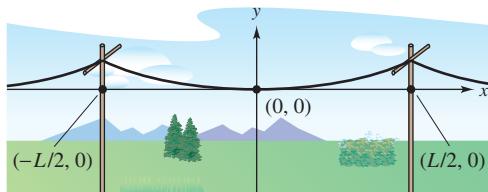
$$a_0 = \frac{1}{12} \int_0^{12} f(t) \, dt \quad a_1 = \frac{1}{6} \int_0^{12} f(t) \cos \frac{\pi t}{6} \, dt$$

$$b_1 = \frac{1}{6} \int_0^{12} f(t) \sin \frac{\pi t}{6} \, dt$$

SECTION PROJECT

Power Lines

Power lines are constructed by stringing wire between supports and adjusting the tension on each span. The wire hangs between supports in the shape of a catenary, as shown in the figure.



Let T be the tension (in pounds) on a span of wire, let u be the density (in pounds per foot), let $g \approx 32.2$ be the acceleration due to gravity (in feet per second per second), and let L be the distance (in feet) between the supports. Then the equation of the catenary is $y = \frac{T}{ug} \left(\cosh \frac{ugx}{T} - 1 \right)$, where x and y are measured in feet.

- (a) Approximate the model $H(t)$ for the maximum temperatures. (Hint: Use Simpson's Rule to approximate the integrals and use the January data twice.)

- (b) Repeat part (a) for a model $L(t)$ for the minimum temperature data.

- (c) Use a graphing utility to graph each model. During what part of the year is the difference between the maximum and minimum temperatures greatest?

- 108. Wallis's Formulas** Use the result of Exercise 100 to prove the following versions of Wallis's Formulas.

- (a) If n is odd ($n \geq 3$), then

$$\int_0^{\pi/2} \cos^n x \, dx = \left(\frac{2}{3} \right) \left(\frac{4}{5} \right) \left(\frac{6}{7} \right) \cdots \left(\frac{n-1}{n} \right).$$

- (b) If n is even ($n \geq 2$), then

$$\int_0^{\pi/2} \cos^n x \, dx = \left(\frac{1}{2} \right) \left(\frac{3}{4} \right) \left(\frac{5}{6} \right) \cdots \left(\frac{n-1}{n} \right) \left(\frac{\pi}{2} \right).$$

- 109.** The **inner product** of two functions f and g on $[a, b]$ is given by $\langle f, g \rangle = \int_a^b f(x)g(x) \, dx$. Two distinct functions f and g are said to be **orthogonal** if $\langle f, g \rangle = 0$. Show that the following set of functions is orthogonal on $[-\pi, \pi]$.

$$\{\sin x, \sin 2x, \sin 3x, \dots, \cos x, \cos 2x, \cos 3x, \dots\}$$

- 110. Fourier Series** The following sum is a *finite Fourier series*.

$$\begin{aligned} f(x) &= \sum_{i=1}^N a_i \sin ix \\ &= a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots + a_N \sin Nx \end{aligned}$$

- (a) Use Exercise 109 to show that the n th coefficient a_n is given by $a_n = (1/\pi) \int_{-\pi}^{\pi} f(x) \sin nx \, dx$.

- (b) Let $f(x) = x$. Find a_1 , a_2 , and a_3 .

- (a) Find the length of the wire between two spans.

- (b) To measure the tension in a span, power line workers use the *return wave method*. The wire is struck at one support, creating a wave in the line, and the time t (in seconds) it takes for the wave to make a round trip is measured. The velocity v (in feet per second) is given by $v = \sqrt{T/u}$. How long does it take the wave to make a round trip between supports?

- (c) The sag s (in inches) can be obtained by evaluating y when $x = L/2$ in the equation for the catenary (and multiplying by 12). In practice, however, power line workers use the "lineman's equation" given by $s \approx 12.075t^2$. Use the fact that $[\cosh(ugL/2T) + 1] \approx 2$ to derive this equation.

- FOR FURTHER INFORMATION** To learn more about the mathematics of power lines, see the article "Constructing Power Lines" by Thomas O'Neil in *The UMAP Journal*.

8.4 Trigonometric Substitution

- Use trigonometric substitution to solve an integral.
- Use integrals to model and solve real-life applications.

Trigonometric Substitution

EXPLORATION

Integrating a Radical Function

Up to this point in the text, you have not evaluated the following integral.

$$\int_{-1}^1 \sqrt{1 - x^2} dx$$

From geometry, you should be able to find the exact value of this integral—what is it? Using numerical integration, with Simpson's Rule or the Trapezoidal Rule, you can't be sure of the accuracy of the approximation. Why?

Try finding the exact value using the substitution

$$x = \sin \theta \text{ and } dx = \cos \theta d\theta.$$

Does your answer agree with the value you obtained using geometry?

Now that you can evaluate integrals involving powers of trigonometric functions, you can use **trigonometric substitution** to evaluate integrals involving the radicals

$$\sqrt{a^2 - u^2}, \quad \sqrt{a^2 + u^2}, \quad \text{and} \quad \sqrt{u^2 - a^2}.$$

The objective with trigonometric substitution is to eliminate the radical in the integrand. You do this by using the Pythagorean identities

$$\cos^2 \theta = 1 - \sin^2 \theta, \quad \sec^2 \theta = 1 + \tan^2 \theta, \quad \text{and} \quad \tan^2 \theta = \sec^2 \theta - 1.$$

For example, if $a > 0$, let $u = a \sin \theta$, where $-\pi/2 \leq \theta \leq \pi/2$. Then

$$\begin{aligned}\sqrt{a^2 - u^2} &= \sqrt{a^2 - a^2 \sin^2 \theta} \\ &= \sqrt{a^2(1 - \sin^2 \theta)} \\ &= \sqrt{a^2 \cos^2 \theta} \\ &= a \cos \theta.\end{aligned}$$

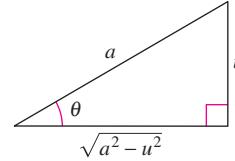
Note that $\cos \theta \geq 0$, because $-\pi/2 \leq \theta \leq \pi/2$.

TRIGONOMETRIC SUBSTITUTION ($a > 0$)

1. For integrals involving $\sqrt{a^2 - u^2}$, let

$$u = a \sin \theta.$$

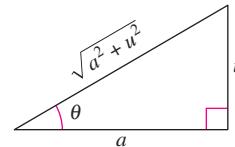
Then $\sqrt{a^2 - u^2} = a \cos \theta$, where $-\pi/2 \leq \theta \leq \pi/2$.



2. For integrals involving $\sqrt{a^2 + u^2}$, let

$$u = a \tan \theta.$$

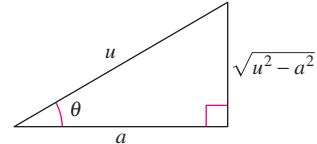
Then $\sqrt{a^2 + u^2} = a \sec \theta$, where $-\pi/2 < \theta < \pi/2$.



3. For integrals involving $\sqrt{u^2 - a^2}$, let

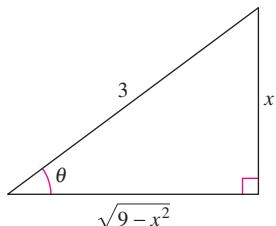
$$u = a \sec \theta.$$

Then



$$\sqrt{u^2 - a^2} = \begin{cases} a \tan \theta & \text{if } u > a, \text{ where } 0 \leq \theta < \pi/2 \\ -a \tan \theta & \text{if } u < -a, \text{ where } \pi/2 < \theta \leq \pi \end{cases}$$

NOTE The restrictions on θ ensure that the function that defines the substitution is one-to-one. In fact, these are the same intervals over which the arcsine, arctangent, and arcsecant are defined.



$$\sin \theta = \frac{x}{3}, \cot \theta = \frac{\sqrt{9 - x^2}}{x}$$

Figure 8.6

EXAMPLE 1 Trigonometric Substitution: $u = a \sin \theta$

Find $\int \frac{dx}{x^2 \sqrt{9 - x^2}}$.

Solution First, note that none of the basic integration rules applies. To use trigonometric substitution, you should observe that $\sqrt{9 - x^2}$ is of the form $\sqrt{a^2 - u^2}$. So, you can use the substitution

$$x = a \sin \theta = 3 \sin \theta.$$

Using differentiation and the triangle shown in Figure 8.6, you obtain

$$dx = 3 \cos \theta d\theta, \quad \sqrt{9 - x^2} = 3 \cos \theta, \quad \text{and} \quad x^2 = 9 \sin^2 \theta.$$

So, trigonometric substitution yields

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{9 - x^2}} &= \int \frac{3 \cos \theta d\theta}{(9 \sin^2 \theta)(3 \cos \theta)} && \text{Substitute.} \\ &= \frac{1}{9} \int \frac{d\theta}{\sin^2 \theta} && \text{Simplify.} \\ &= \frac{1}{9} \int \csc^2 \theta d\theta && \text{Trigonometric identity} \\ &= -\frac{1}{9} \cot \theta + C && \text{Apply Cosecant Rule.} \\ &= -\frac{1}{9} \left(\frac{\sqrt{9 - x^2}}{x} \right) + C && \text{Substitute for } \cot \theta. \\ &= -\frac{\sqrt{9 - x^2}}{9x} + C. \end{aligned}$$

Note that the triangle in Figure 8.6 can be used to convert the θ 's back to x 's, as follows.

$$\begin{aligned} \cot \theta &= \frac{\text{adj.}}{\text{opp.}} \\ &= \frac{\sqrt{9 - x^2}}{x} \end{aligned}$$

■

TECHNOLOGY Use a computer algebra system to find each indefinite integral.

$$\int \frac{dx}{\sqrt{9 - x^2}}, \quad \int \frac{dx}{x \sqrt{9 - x^2}}, \quad \int \frac{dx}{x^2 \sqrt{9 - x^2}}, \quad \int \frac{dx}{x^3 \sqrt{9 - x^2}}$$

Then use trigonometric substitution to duplicate the results obtained with the computer algebra system.

In an earlier chapter, you saw how the inverse hyperbolic functions can be used to evaluate the integrals

$$\int \frac{du}{\sqrt{u^2 \pm a^2}}, \quad \int \frac{du}{a^2 - u^2}, \quad \text{and} \quad \int \frac{du}{u \sqrt{a^2 \pm u^2}}.$$

You can also evaluate these integrals using trigonometric substitution. This is shown in the next example.

EXAMPLE 2 Trigonometric Substitution: $u = a \tan \theta$

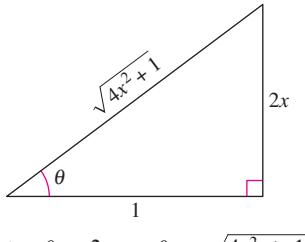
Find $\int \frac{dx}{\sqrt{4x^2 + 1}}$.

Solution Let $u = 2x$, $a = 1$, and $2x = \tan \theta$, as shown in Figure 8.7. Then,

$$dx = \frac{1}{2} \sec^2 \theta d\theta \quad \text{and} \quad \sqrt{4x^2 + 1} = \sec \theta.$$

Trigonometric substitution produces

$$\begin{aligned} \int \frac{1}{\sqrt{4x^2 + 1}} dx &= \frac{1}{2} \int \frac{\sec^2 \theta d\theta}{\sec \theta} && \text{Substitute.} \\ &= \frac{1}{2} \int \sec \theta d\theta && \text{Simplify.} \\ &= \frac{1}{2} \ln |\sec \theta + \tan \theta| + C && \text{Apply Secant Rule.} \\ &= \frac{1}{2} \ln |\sqrt{4x^2 + 1} + 2x| + C. && \text{Back-substitute.} \end{aligned}$$



$$\tan \theta = 2x, \sec \theta = \sqrt{4x^2 + 1}$$

Figure 8.7

Try checking this result with a computer algebra system. Is the result given in this form or in the form of an inverse hyperbolic function? ■

You can extend the use of trigonometric substitution to cover integrals involving expressions such as $(a^2 - u^2)^{n/2}$ by writing the expression as

$$(a^2 - u^2)^{n/2} = (\sqrt{a^2 - u^2})^n.$$

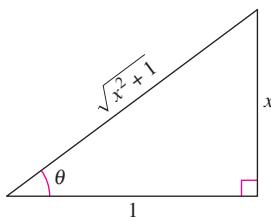
EXAMPLE 3 Trigonometric Substitution: Rational Powers

Find $\int \frac{dx}{(x^2 + 1)^{3/2}}$.

Solution Begin by writing $(x^2 + 1)^{3/2}$ as $(\sqrt{x^2 + 1})^3$. Then, let $a = 1$ and $u = x = \tan \theta$, as shown in Figure 8.8. Using

$$dx = \sec^2 \theta d\theta \quad \text{and} \quad \sqrt{x^2 + 1} = \sec \theta$$

you can apply trigonometric substitution, as follows.



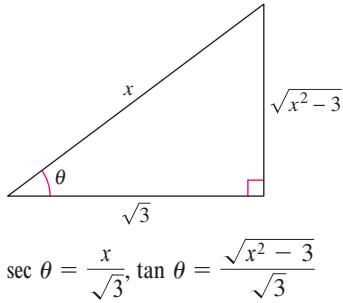
$$\tan \theta = x, \sin \theta = \frac{x}{\sqrt{x^2 + 1}}$$

Figure 8.8

$$\begin{aligned} \int \frac{dx}{(x^2 + 1)^{3/2}} &= \int \frac{dx}{(\sqrt{x^2 + 1})^3} && \text{Rewrite denominator.} \\ &= \int \frac{\sec^2 \theta d\theta}{\sec^3 \theta} && \text{Substitute.} \\ &= \int \frac{d\theta}{\sec \theta} && \text{Simplify.} \\ &= \int \cos \theta d\theta && \text{Trigonometric identity} \\ &= \sin \theta + C && \text{Apply Cosine Rule.} \\ &= \frac{x}{\sqrt{x^2 + 1}} + C && \text{Back-substitute.} \end{aligned}$$

For definite integrals, it is often convenient to determine the integration limits for θ that avoid converting back to x . You might want to review this procedure in Section 5.5, Examples 8 and 9.

EXAMPLE 4 Converting the Limits of Integration



$$\sec \theta = \frac{x}{\sqrt{3}}, \tan \theta = \frac{\sqrt{x^2 - 3}}{\sqrt{3}}$$

Figure 8.9

Evaluate $\int_{\sqrt{3}}^2 \frac{\sqrt{x^2 - 3}}{x} dx$.

Solution Because $\sqrt{x^2 - 3}$ has the form $\sqrt{u^2 - a^2}$, you can consider

$$u = x, \quad a = \sqrt{3}, \quad \text{and} \quad x = \sqrt{3} \sec \theta$$

as shown in Figure 8.9. Then,

$$dx = \sqrt{3} \sec \theta \tan \theta d\theta \quad \text{and} \quad \sqrt{x^2 - 3} = \sqrt{3} \tan \theta.$$

To determine the upper and lower limits of integration, use the substitution $x = \sqrt{3} \sec \theta$, as follows.

Lower Limit

$$\begin{aligned} \text{When } x = \sqrt{3}, \sec \theta &= 1 \\ \text{and } \theta &= 0. \end{aligned}$$

Upper Limit

$$\begin{aligned} \text{When } x = 2, \sec \theta &= \frac{2}{\sqrt{3}} \\ \text{and } \theta &= \frac{\pi}{6}. \end{aligned}$$

So, you have

$$\begin{aligned} \text{Integration limits for } x &\quad \text{Integration limits for } \theta \\ \downarrow &\quad \downarrow \\ \int_{\sqrt{3}}^2 \frac{\sqrt{x^2 - 3}}{x} dx &= \int_0^{\pi/6} \frac{(\sqrt{3} \tan \theta)(\sqrt{3} \sec \theta \tan \theta)}{\sqrt{3} \sec \theta} d\theta \\ &= \int_0^{\pi/6} \sqrt{3} \tan^2 \theta d\theta \\ &= \sqrt{3} \int_0^{\pi/6} (\sec^2 \theta - 1) d\theta \\ &= \sqrt{3} \left[\tan \theta - \theta \right]_0^{\pi/6} \\ &= \sqrt{3} \left(\frac{1}{\sqrt{3}} - \frac{\pi}{6} \right) \\ &= 1 - \frac{\sqrt{3}\pi}{6} \\ &\approx 0.0931. \end{aligned}$$

In Example 4, try converting back to the variable x and evaluating the antiderivative at the original limits of integration. You should obtain

$$\int_{\sqrt{3}}^2 \frac{\sqrt{x^2 - 3}}{x} dx = \sqrt{3} \frac{\sqrt{x^2 - 3}}{\sqrt{3}} - \operatorname{arcsec} \frac{x}{\sqrt{3}} \Big|_{\sqrt{3}}^2.$$

When using trigonometric substitution to evaluate definite integrals, you must be careful to check that the values of θ lie in the intervals discussed at the beginning of this section. For instance, if in Example 4 you had been asked to evaluate the definite integral

$$\int_{-2}^{-\sqrt{3}} \frac{\sqrt{x^2 - 3}}{x} dx$$

then using $u = x$ and $a = \sqrt{3}$ in the interval $[-2, -\sqrt{3}]$ would imply that $u < -a$. So, when determining the upper and lower limits of integration, you would have to choose θ such that $\pi/2 < \theta \leq \pi$. In this case the integral would be evaluated as follows.

$$\begin{aligned} \int_{-2}^{-\sqrt{3}} \frac{\sqrt{x^2 - 3}}{x} dx &= \int_{5\pi/6}^{\pi} \frac{(-\sqrt{3} \tan \theta)(\sqrt{3} \sec \theta \tan \theta)}{\sqrt{3} \sec \theta} d\theta \\ &= \int_{5\pi/6}^{\pi} -\sqrt{3} \tan^2 \theta d\theta \\ &= -\sqrt{3} \int_{5\pi/6}^{\pi} (\sec^2 \theta - 1) d\theta \\ &= -\sqrt{3} \left[\tan \theta - \theta \right]_{5\pi/6}^{\pi} \\ &= -\sqrt{3} \left[(0 - \pi) - \left(-\frac{1}{\sqrt{3}} - \frac{5\pi}{6} \right) \right] \\ &= -1 + \frac{\sqrt{3}\pi}{6} \\ &\approx -0.0931 \end{aligned}$$

Trigonometric substitution can be used with completing the square. For instance, try evaluating the following integral.

$$\int \sqrt{x^2 - 2x} dx$$

To begin, you could complete the square and write the integral as

$$\int \sqrt{(x - 1)^2 - 1^2} dx.$$

Trigonometric substitution can be used to evaluate the three integrals listed in the following theorem. These integrals will be encountered several times in the remainder of the text. When this happens, we will simply refer to this theorem. (In Exercise 85, you are asked to verify the formulas given in the theorem.)

THEOREM 8.2 SPECIAL INTEGRATION FORMULAS ($a > 0$)

1. $\int \sqrt{a^2 - u^2} du = \frac{1}{2} \left(a^2 \arcsin \frac{u}{a} + u \sqrt{a^2 - u^2} \right) + C$
2. $\int \sqrt{u^2 - a^2} du = \frac{1}{2} \left(u \sqrt{u^2 - a^2} - a^2 \ln |u + \sqrt{u^2 - a^2}| \right) + C, \quad u > a$
3. $\int \sqrt{u^2 + a^2} du = \frac{1}{2} \left(u \sqrt{u^2 + a^2} + a^2 \ln |u + \sqrt{u^2 + a^2}| \right) + C$

Applications

EXAMPLE 5 Finding Arc Length

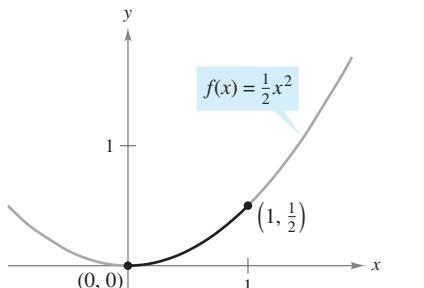


Figure 8.10

Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$ (see Figure 8.10).

Solution Refer to the arc length formula in Section 7.4.

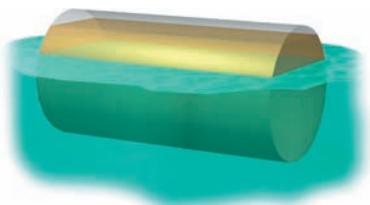
$$\begin{aligned}s &= \int_0^1 \sqrt{1 + [f'(x)]^2} dx \\&= \int_0^1 \sqrt{1 + x^2} dx \\&= \int_0^{\pi/4} \sec^3 \theta d\theta \\&= \frac{1}{2} \left[\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right]_0^{\pi/4} \\&= \frac{1}{2} [\sqrt{2} + \ln(\sqrt{2} + 1)] \approx 1.148\end{aligned}$$

Formula for arc length

$$f'(x) = x$$

Let $a = 1$ and $x = \tan \theta$.

Example 5, Section 8.2



The barrel is not quite full of oil—the top 0.2 foot of the barrel is empty.

Figure 8.11

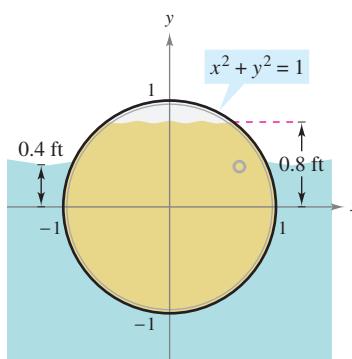
EXAMPLE 6 Comparing Two Fluid Forces

A sealed barrel of oil (weighing 48 pounds per cubic foot) is floating in seawater (weighing 64 pounds per cubic foot), as shown in Figures 8.11 and 8.12. (The barrel is not completely full of oil. With the barrel lying on its side, the top 0.2 foot of the barrel is empty.) Compare the fluid forces against one end of the barrel from the inside and from the outside.

Solution In Figure 8.12, locate the coordinate system with the origin at the center of the circle given by $x^2 + y^2 = 1$. To find the fluid force against an end of the barrel *from the inside*, integrate between -1 and 0.8 (using a weight of $w = 48$).

$$\begin{aligned}F &= w \int_c^d h(y)L(y) dy \\F_{\text{inside}} &= 48 \int_{-1}^{0.8} (0.8 - y)(2)\sqrt{1 - y^2} dy \\&= 76.8 \int_{-1}^{0.8} \sqrt{1 - y^2} dy - 96 \int_{-1}^{0.8} y\sqrt{1 - y^2} dy\end{aligned}$$

General equation (see Section 7.7)



To find the fluid force *from the outside*, integrate between -1 and 0.4 (using a weight of $w = 64$).

$$\begin{aligned}F_{\text{outside}} &= 64 \int_{-1}^{0.4} (0.4 - y)(2)\sqrt{1 - y^2} dy \\&= 51.2 \int_{-1}^{0.4} \sqrt{1 - y^2} dy - 128 \int_{-1}^{0.4} y\sqrt{1 - y^2} dy\end{aligned}$$

The details of integration are left for you to complete in Exercise 84. Intuitively, would you say that the force from the oil (the inside) or the force from the seawater (the outside) is greater? By evaluating these two integrals, you can determine that

$$F_{\text{inside}} \approx 121.3 \text{ pounds} \quad \text{and} \quad F_{\text{outside}} \approx 93.0 \text{ pounds.}$$

Figure 8.12

8.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, state the trigonometric substitution you would use to find the integral. Do not integrate.

1. $\int (9 + x^2)^{-2} dx$

2. $\int \sqrt{4 - x^2} dx$

3. $\int \frac{x^2}{\sqrt{16 - x^2}} dx$

4. $\int x^2(x^2 - 25)^{3/2} dx$

In Exercises 5–8, find the indefinite integral using the substitution $x = 4 \sin \theta$.

5. $\int \frac{1}{(16 - x^2)^{3/2}} dx$

6. $\int \frac{4}{x^2 \sqrt{16 - x^2}} dx$

7. $\int \frac{\sqrt{16 - x^2}}{x} dx$

8. $\int \frac{x^2}{\sqrt{16 - x^2}} dx$

In Exercises 9–12, find the indefinite integral using the substitution $x = 5 \sec \theta$.

9. $\int \frac{1}{\sqrt{x^2 - 25}} dx$

10. $\int \frac{\sqrt{x^2 - 25}}{x} dx$

11. $\int x^3 \sqrt{x^2 - 25} dx$

12. $\int \frac{x^3}{\sqrt{x^2 - 25}} dx$

In Exercises 13–16, find the indefinite integral using the substitution $x = \tan \theta$.

13. $\int x \sqrt{1 + x^2} dx$

14. $\int \frac{9x^3}{\sqrt{1 + x^2}} dx$

15. $\int \frac{1}{(1 + x^2)^2} dx$

16. $\int \frac{x^2}{(1 + x^2)^2} dx$

In Exercises 17–20, use the Special Integration Formulas (Theorem 8.2) to find the integral.

17. $\int \sqrt{9 + 16x^2} dx$

18. $\int \sqrt{4 + x^2} dx$

19. $\int \sqrt{25 - 4x^2} dx$

20. $\int \sqrt{5x^2 - 1} dx$

In Exercises 21–42, find the integral.

21. $\int \frac{x}{\sqrt{x^2 + 36}} dx$

22. $\int \frac{x}{\sqrt{36 - x^2}} dx$

23. $\int \frac{1}{\sqrt{16 - x^2}} dx$

24. $\int \frac{1}{\sqrt{49 - x^2}} dx$

25. $\int \sqrt{16 - 4x^2} dx$

26. $\int x \sqrt{16 - 4x^2} dx$

27. $\int \frac{1}{\sqrt{x^2 - 4}} dx$

28. $\int \frac{t}{(4 - t^2)^{3/2}} dt$

29. $\int \frac{\sqrt{1 - x^2}}{x^4} dx$

30. $\int \frac{\sqrt{4x^2 + 9}}{x^4} dx$

31. $\int \frac{1}{x \sqrt{4x^2 + 9}} dx$

32. $\int \frac{1}{x \sqrt{4x^2 + 16}} dx$

33. $\int \frac{-3x}{(x^2 + 3)^{3/2}} dx$

34. $\int \frac{1}{(x^2 + 5)^{3/2}} dx$

35. $\int e^{2x} \sqrt{1 + e^{2x}} dx$

36. $\int (x + 1) \sqrt{x^2 + 2x + 2} dx$

37. $\int e^x \sqrt{1 - e^{2x}} dx$

38. $\int \frac{\sqrt{1 - x}}{\sqrt{x}} dx$

39. $\int \frac{1}{4 + 4x^2 + x^4} dx$

40. $\int \frac{x^3 + x + 1}{x^4 + 2x^2 + 1} dx$

41. $\int \operatorname{arcsec} 2x dx, \quad x > \frac{1}{2}$

42. $\int x \arcsin x dx$

In Exercises 43–46, complete the square and find the integral.

43. $\int \frac{1}{\sqrt{4x - x^2}} dx$

44. $\int \frac{x^2}{\sqrt{2x - x^2}} dx$

45. $\int \frac{x}{\sqrt{x^2 + 6x + 12}} dx$

46. $\int \frac{x}{\sqrt{x^2 - 6x + 5}} dx$

In Exercises 47–52, evaluate the integral using (a) the given integration limits and (b) the limits obtained by trigonometric substitution.

47. $\int_0^{\sqrt{3}/2} \frac{t^2}{(1 - t^2)^{3/2}} dt$

48. $\int_0^{\sqrt{3}/2} \frac{1}{(1 - t^2)^{5/2}} dt$

49. $\int_0^3 \frac{x^3}{\sqrt{x^2 + 9}} dx$

50. $\int_0^{3/5} \sqrt{9 - 25x^2} dx$

51. $\int_4^6 \frac{x^2}{\sqrt{x^2 - 9}} dx$

52. $\int_3^6 \frac{\sqrt{x^2 - 9}}{x^2} dx$

In Exercises 53 and 54, find the particular solution of the differential equation.

53. $x \frac{dy}{dx} = \sqrt{x^2 - 9}, \quad x \geq 3, \quad y(3) = 1$

54. $\sqrt{x^2 + 4} \frac{dy}{dx} = 1, \quad x \geq -2, \quad y(0) = 4$

CAS In Exercises 55–58, use a computer algebra system to find the integral. Verify the result by differentiation.

55. $\int \frac{x^2}{\sqrt{x^2 + 10x + 9}} dx$

56. $\int (x^2 + 2x + 11)^{3/2} dx$

57. $\int \frac{x^2}{\sqrt{x^2 - 1}} dx$

58. $\int x^2 \sqrt{x^2 - 4} dx$

WRITING ABOUT CONCEPTS

59. State the substitution you would make if you used trigonometric substitution and the integral involving the given radical, where $a > 0$. Explain your reasoning.

- (a) $\sqrt{a^2 - u^2}$ (b) $\sqrt{a^2 + u^2}$ (c) $\sqrt{u^2 - a^2}$

WRITING ABOUT CONCEPTS (continued)

In Exercises 60 and 61, state the method of integration you would use to perform each integration. Explain why you chose that method. Do not integrate.

60. $\int x\sqrt{x^2 + 1} dx$

61. $\int x^2 \sqrt{x^2 - 1} dx$

CAPSTONE

62. (a) Evaluate the integral $\int \frac{x}{x^2 + 9} dx$ using u -substitution.

Then evaluate using trigonometric substitution. Discuss the results.

- (b) Evaluate the integral $\int \frac{x^2}{x^2 + 9} dx$ algebraically using $x^2 = (x^2 + 9) - 9$. Then evaluate using trigonometric substitution. Discuss the results.

- (c) Evaluate the integral $\int \frac{4}{4 - x^2} dx$ using trigonometric substitution. Then evaluate using the identity $\frac{4}{4 - x^2} = \left(\frac{1}{x+2} - \frac{1}{x-2}\right)$. Discuss the results.

True or False? In Exercises 63–66, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

63. If $x = \sin \theta$, then $\int \frac{dx}{\sqrt{1 - x^2}} = \int d\theta$.

64. If $x = \sec \theta$, then $\int \frac{\sqrt{x^2 - 1}}{x} dx = \int \sec \theta \tan \theta d\theta$.

65. If $x = \tan \theta$, then $\int_0^{\sqrt{3}} \frac{dx}{(1 + x^2)^{3/2}} = \int_0^{4\pi/3} \cos \theta d\theta$.

66. If $x = \sin \theta$, then $\int_{-1}^1 x^2 \sqrt{1 - x^2} dx = 2 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta$.

67. **Area** Find the area enclosed by the ellipse shown in the figure.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

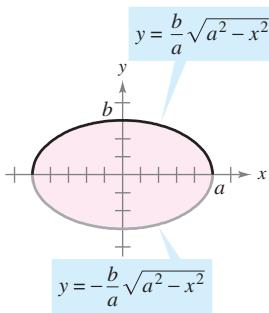


Figure for 67

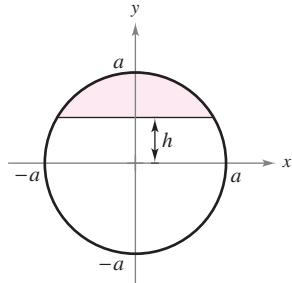


Figure for 68

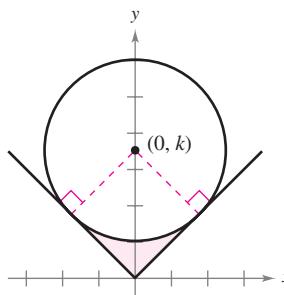
68. **Area** Find the area of the shaded region of the circle of radius a , if the chord is h units ($0 < h < a$) from the center of the circle (see figure).

69. **Mechanical Design** The surface of a machine part is the region between the graphs of $y = |x|$ and $x^2 + (y - k)^2 = 25$ (see figure).

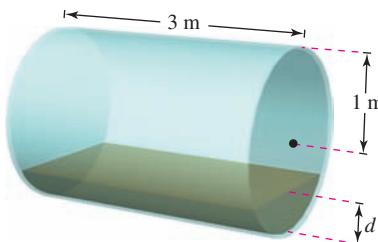
- (a) Find k if the circle is tangent to the graph of $y = |x|$.

- (b) Find the area of the surface of the machine part.

- (c) Find the area of the surface of the machine part as a function of the radius r of the circle.



70. **Volume** The axis of a storage tank in the form of a right circular cylinder is horizontal (see figure). The radius and length of the tank are 1 meter and 3 meters, respectively.



- (a) Determine the volume of fluid in the tank as a function of its depth d .

- (b) Use a graphing utility to graph the function in part (a).

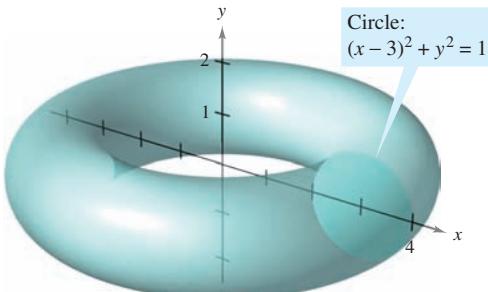
- (c) Design a dip stick for the tank with markings of $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$.

- (d) Fluid is entering the tank at a rate of $\frac{1}{4}$ cubic meter per minute. Determine the rate of change of the depth of the fluid as a function of its depth d .

- (e) Use a graphing utility to graph the function in part (d). When will the rate of change of the depth be minimum? Does this agree with your intuition? Explain.

Volume of a Torus In Exercises 71 and 72, find the volume of the torus generated by revolving the region bounded by the graph of the circle about the y -axis.

71. $(x - 3)^2 + y^2 = 1$ (see figure)



72. $(x - h)^2 + y^2 = r^2$, $h > r$

Arc Length In Exercises 73 and 74, find the arc length of the curve over the given interval.

73. $y = \ln x$, $[1, 5]$

74. $y = \frac{1}{2}x^2$, $[0, 4]$

75. Arc Length Show that the length of one arch of the sine curve is equal to the length of one arch of the cosine curve.

76. Conjecture

- Find formulas for the distances between $(0, 0)$ and (a, a^2) along the line between these points and along the parabola $y = x^2$.
- Use the formulas from part (a) to find the distances for $a = 1$ and $a = 10$.
- Make a conjecture about the difference between the two distances as a increases.

 **Projectile Motion** In Exercises 77 and 78, (a) use a graphing utility to graph the path of a projectile that follows the path given by the graph of the equation, (b) determine the range of the projectile, and (c) use the integration capabilities of a graphing utility to determine the distance the projectile travels.

77. $y = x - 0.005x^2$

78. $y = x - \frac{x^2}{72}$

Centroid In Exercises 79 and 80, find the centroid of the region determined by the graphs of the inequalities.

79. $y \leq 3/\sqrt{x^2 + 9}$, $y \geq 0$, $x \geq -4$, $x \leq 4$

80. $y \leq \frac{1}{4}x^2$, $(x - 4)^2 + y^2 \leq 16$, $y \geq 0$

81. Surface Area Find the surface area of the solid generated by revolving the region bounded by the graphs of $y = x^2$, $y = 0$, $x = 0$, and $x = \sqrt{2}$ about the x -axis.

82. Field Strength The field strength H of a magnet of length $2L$ on a particle r units from the center of the magnet is

$$H = \frac{2mL}{(r^2 + L^2)^{3/2}}$$

where $\pm m$ are the poles of the magnet (see figure). Find the average field strength as the particle moves from 0 to R units from the center by evaluating the integral

$$\frac{1}{R} \int_0^R \frac{2mL}{(r^2 + L^2)^{3/2}} dr.$$

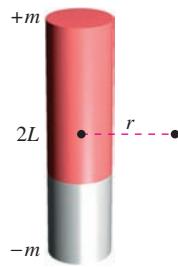


Figure for 82

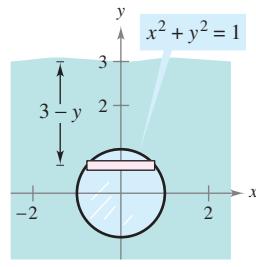


Figure for 83

83. Fluid Force Find the fluid force on a circular observation window of radius 1 foot in a vertical wall of a large water-filled tank at a fish hatchery when the center of the window is (a) 3 feet and (b) d feet ($d > 1$) below the water's surface (see figure). Use trigonometric substitution to evaluate the one integral. (Recall that in Section 7.7 in a similar problem, you evaluated one integral by a geometric formula and the other by observing that the integrand was odd.)

84. Fluid Force Evaluate the following two integrals, which yield the fluid forces given in Example 6.

(a) $F_{\text{inside}} = 48 \int_{-1}^{0.8} (0.8 - y)(2) \sqrt{1 - y^2} dy$

(b) $F_{\text{outside}} = 64 \int_{-1}^{0.4} (0.4 - y)(2) \sqrt{1 - y^2} dy$

85. Use trigonometric substitution to verify the integration formulas given in Theorem 8.2.

86. Arc Length Show that the arc length of the graph of $y = \sin x$ on the interval $[0, 2\pi]$ is equal to the circumference of the ellipse $x^2 + 2y^2 = 2$ (see figure).

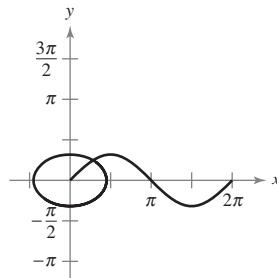


Figure for 86

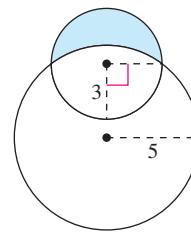
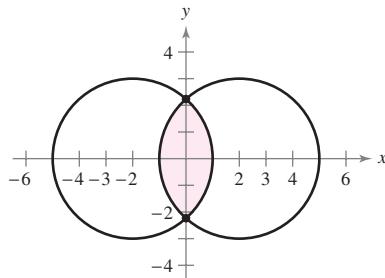


Figure for 87

87. Area of a Lune The crescent-shaped region bounded by two circles forms a *lune* (see figure). Find the area of the lune given that the radius of the smaller circle is 3 and the radius of the larger circle is 5.

88. Area Two circles of radius 3, with centers at $(-2, 0)$ and $(2, 0)$, intersect as shown in the figure. Find the area of the shaded region.



PUTNAM EXAM CHALLENGE

89. Evaluate

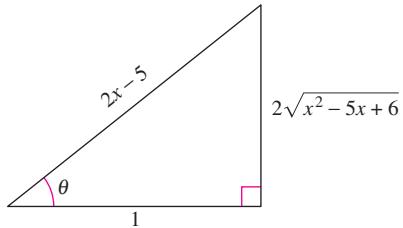
$$\int_0^1 \frac{\ln(x+1)}{x^2+1} dx.$$

This problem was composed by the Committee on the Putnam Prize Competition.
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8.5 Partial Fractions

- Understand the concept of partial fraction decomposition.
- Use partial fraction decomposition with linear factors to integrate rational functions.
- Use partial fraction decomposition with quadratic factors to integrate rational functions.

Partial Fractions



$$\sec \theta = 2x - 5$$

Figure 8.13

This section examines a procedure for decomposing a rational function into simpler rational functions to which you can apply the basic integration formulas. This procedure is called the **method of partial fractions**. To see the benefit of the method of partial fractions, consider the integral

$$\int \frac{1}{x^2 - 5x + 6} dx.$$

To evaluate this integral *without* partial fractions, you can complete the square and use trigonometric substitution (see Figure 8.13) to obtain

$$\begin{aligned} \int \frac{1}{x^2 - 5x + 6} dx &= \int \frac{dx}{(x - 5/2)^2 - (1/2)^2} && a = \frac{1}{2}, x - \frac{5}{2} = \frac{1}{2} \sec \theta \\ &= \int \frac{(1/2) \sec \theta \tan \theta d\theta}{(1/4) \tan^2 \theta} && dx = \frac{1}{2} \sec \theta \tan \theta d\theta \\ &= 2 \int \csc \theta d\theta \\ &= 2 \ln|\csc \theta - \cot \theta| + C \\ &= 2 \ln \left| \frac{2x - 5}{2\sqrt{x^2 - 5x + 6}} - \frac{1}{2\sqrt{x^2 - 5x + 6}} \right| + C \\ &= 2 \ln \left| \frac{x - 3}{\sqrt{x^2 - 5x + 6}} \right| + C \\ &= 2 \ln \left| \frac{\sqrt{x - 3}}{\sqrt{x - 2}} \right| + C \\ &= \ln \left| \frac{x - 3}{x - 2} \right| + C \\ &= \ln|x - 3| - \ln|x - 2| + C. \end{aligned}$$

Now, suppose you had observed that

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{x - 3} - \frac{1}{x - 2}.$$

Partial fraction decomposition

Then you could evaluate the integral easily, as follows.

$$\begin{aligned} \int \frac{1}{x^2 - 5x + 6} dx &= \int \left(\frac{1}{x - 3} - \frac{1}{x - 2} \right) dx \\ &= \ln|x - 3| - \ln|x - 2| + C \end{aligned}$$

This method is clearly preferable to trigonometric substitution. However, its use depends on the ability to factor the denominator, $x^2 - 5x + 6$, and to find the **partial fractions**

$$\frac{1}{x - 3} \quad \text{and} \quad -\frac{1}{x - 2}.$$

In this section, you will study techniques for finding partial fraction decompositions.



The Granger Collection

JOHN BERNOULLI (1667–1748)

The method of partial fractions was introduced by John Bernoulli, a Swiss mathematician who was instrumental in the early development of calculus. John Bernoulli was a professor at the University of Basel and taught many outstanding students, the most famous of whom was Leonhard Euler.

STUDY TIP In precalculus you learned how to combine functions such as

$$\frac{1}{x-2} + \frac{-1}{x+3} = \frac{5}{(x-2)(x+3)}.$$

The method of partial fractions shows you how to reverse this process.

$$\frac{5}{(x-2)(x+3)} = \frac{?}{x-2} + \frac{?}{x+3}$$

Recall from algebra that every polynomial with real coefficients can be factored into linear and irreducible quadratic factors.* For instance, the polynomial

$$x^5 + x^4 - x - 1$$

can be written as

$$\begin{aligned} x^5 + x^4 - x - 1 &= x^4(x+1) - (x+1) \\ &= (x^4 - 1)(x+1) \\ &= (x^2 + 1)(x^2 - 1)(x+1) \\ &= (x^2 + 1)(x+1)(x-1)(x+1) \\ &= (x-1)(x+1)^2(x^2 + 1) \end{aligned}$$

where $(x-1)$ is a linear factor, $(x+1)^2$ is a repeated linear factor, and $(x^2 + 1)$ is an irreducible quadratic factor. Using this factorization, you can write the partial fraction decomposition of the rational expression

$$\frac{N(x)}{x^5 + x^4 - x - 1}$$

where $N(x)$ is a polynomial of degree less than 5, as follows.

$$\frac{N(x)}{(x-1)(x+1)^2(x^2 + 1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{Dx+E}{x^2 + 1}$$

DECOMPOSITION OF $N(x)/D(x)$ INTO PARTIAL FRACTIONS

- Divide if improper:** If $N(x)/D(x)$ is an improper fraction (that is, if the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $N_1(x)/D(x)$.

- Factor denominator:** Completely factor the denominator into factors of the form

$$(px+q)^m \quad \text{and} \quad (ax^2+bx+c)^n$$

where ax^2+bx+c is irreducible.

- Linear factors:** For each factor of the form $(px+q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px+q)} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m}$$

- Quadratic factors:** For each factor of the form $(ax^2+bx+c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

* For a review of factorization techniques, see Precalculus, 8th edition, by Larson (© 2011, Brooks/Cole, Cengage Learning) or Precalculus: A Graphing Approach, 5th edition, by Larson/Hostetler/Edwards (© 2008, Brooks/Cole, Cengage Learning).

Linear Factors

Algebraic techniques for determining the constants in the numerators of a partial fraction decomposition with linear or repeated linear factors are shown in Examples 1 and 2.

EXAMPLE 1 Distinct Linear Factors

Write the partial fraction decomposition for $\frac{1}{x^2 - 5x + 6}$.

Solution Because $x^2 - 5x + 6 = (x - 3)(x - 2)$, you should include one partial fraction for each factor and write

$$\frac{1}{x^2 - 5x + 6} = \frac{A}{x - 3} + \frac{B}{x - 2}$$

where A and B are to be determined. Multiplying this equation by the least common denominator $(x - 3)(x - 2)$ yields the **basic equation**

$$1 = A(x - 2) + B(x - 3). \quad \text{Basic equation}$$

Because this equation is to be true for all x , you can substitute any *convenient* values for x to obtain equations in A and B . The most convenient values are the ones that make particular factors equal to 0.

NOTE Note that the substitutions for x in Example 1 are chosen for their convenience in determining values for A and B ; $x = 2$ is chosen to eliminate the term $A(x - 2)$, and $x = 3$ is chosen to eliminate the term $B(x - 3)$. The goal is to make *convenient* substitutions whenever possible.

To solve for A , let $x = 3$ and obtain

$$\begin{aligned} 1 &= A(3 - 2) + B(3 - 3) && \text{Let } x = 3 \text{ in basic equation.} \\ 1 &= A(1) + B(0) \\ A &= 1. \end{aligned}$$

To solve for B , let $x = 2$ and obtain

$$\begin{aligned} 1 &= A(2 - 2) + B(2 - 3) && \text{Let } x = 2 \text{ in basic equation.} \\ 1 &= A(0) + B(-1) \\ B &= -1. \end{aligned}$$

So, the decomposition is

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{x - 3} - \frac{1}{x - 2}$$

as shown at the beginning of this section. ■

FOR FURTHER INFORMATION To learn a different method for finding partial fraction decompositions, called the Heavyside Method, see the article “Calculus to Algebra Connections in Partial Fraction Decomposition” by Joseph Wiener and Will Watkins in *The AMATYC Review*.

Be sure you see that the method of partial fractions is practical only for integrals of rational functions whose denominators factor “nicely.” For instance, if the denominator in Example 1 were changed to $x^2 - 5x + 5$, its factorization as

$$x^2 - 5x + 5 = \left[x + \frac{5 + \sqrt{5}}{2} \right] \left[x - \frac{5 - \sqrt{5}}{2} \right]$$

would be too cumbersome to use with partial fractions. In such cases, you should use completing the square or a computer algebra system to perform the integration. If you do this, you should obtain

$$\int \frac{1}{x^2 - 5x + 5} dx = \frac{\sqrt{5}}{5} \ln|2x - \sqrt{5} - 5| - \frac{\sqrt{5}}{5} \ln|2x + \sqrt{5} - 5| + C.$$

EXAMPLE 2 Repeated Linear Factors

Find $\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} dx$.

Solution Because

$$\begin{aligned}x^3 + 2x^2 + x &= x(x^2 + 2x + 1) \\&= x(x + 1)^2\end{aligned}$$

you should include one fraction for *each power* of x and $(x + 1)$ and write

$$\frac{5x^2 + 20x + 6}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}.$$

Multiplying by the least common denominator $x(x + 1)^2$ yields the *basic equation*

$$5x^2 + 20x + 6 = A(x + 1)^2 + Bx(x + 1) + Cx. \quad \text{Basic equation}$$

To solve for A , let $x = 0$. This eliminates the B and C terms and yields

$$6 = A(1) + 0 + 0$$

$$A = 6.$$

To solve for C , let $x = -1$. This eliminates the A and B terms and yields

$$5 - 20 + 6 = 0 + 0 - C$$

$$C = 9.$$

The most convenient choices for x have been used, so to find the value of B , you can use *any other value* of x along with the calculated values of A and C . Using $x = 1$, $A = 6$, and $C = 9$ produces

$$5 + 20 + 6 = A(4) + B(2) + C$$

$$31 = 6(4) + 2B + 9$$

$$-2 = 2B$$

$$B = -1.$$

So, it follows that

$$\begin{aligned}\int \frac{5x^2 + 20x + 6}{x(x + 1)^2} dx &= \int \left(\frac{6}{x} - \frac{1}{x + 1} + \frac{9}{(x + 1)^2} \right) dx \\&= 6 \ln|x| - \ln|x + 1| + 9 \frac{(x + 1)^{-1}}{-1} + C \\&= \ln \left| \frac{x^6}{x + 1} \right| - \frac{9}{x + 1} + C.\end{aligned}$$

Try checking this result by differentiating. Include algebra in your check, simplifying the derivative until you have obtained the original integrand. ■

TECHNOLOGY Most computer algebra systems, such as *Maple*, *Mathematica*, and the *TI-89*, can be used to convert a rational function to its partial fraction decomposition. For instance, using *Maple*, you obtain the following.

```
> convert((5x^2 + 20x + 6)/(x^3 + 2x^2 + x), parfrac, x)
6
x + --- - 1
(x + 1)^2
```

NOTE It is necessary to make as many substitutions for x as there are unknowns (A , B , C , . . .) to be determined. For instance, in Example 2, three substitutions ($x = 0$, $x = -1$, and $x = 1$) were made to solve for A , B , and C . ■

Quadratic Factors

When using the method of partial fractions with *linear* factors, a convenient choice of x immediately yields a value for one of the coefficients. With *quadratic* factors, a system of linear equations usually has to be solved, regardless of the choice of x .

EXAMPLE 3 Distinct Linear and Quadratic Factors

$$\text{Find } \int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)} dx.$$

Solution Because

$$(x^2 - x)(x^2 + 4) = x(x - 1)(x^2 + 4)$$

you should include one partial fraction for each factor and write

$$\frac{2x^3 - 4x - 8}{x(x - 1)(x^2 + 4)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{Cx + D}{x^2 + 4}.$$

Multiplying by the least common denominator $x(x - 1)(x^2 + 4)$ yields the *basic equation*

$$2x^3 - 4x - 8 = A(x - 1)(x^2 + 4) + Bx(x^2 + 4) + (Cx + D)x(x - 1).$$

To solve for A , let $x = 0$ and obtain

$$-8 = A(-1)(4) + 0 + 0 \quad \Rightarrow \quad 2 = A.$$

To solve for B , let $x = 1$ and obtain

$$-10 = 0 + B(5) + 0 \quad \Rightarrow \quad -2 = B.$$

At this point, C and D are yet to be determined. You can find these remaining constants by choosing two other values for x and solving the resulting system of linear equations. If $x = -1$, then, using $A = 2$ and $B = -2$, you can write

$$\begin{aligned} -6 &= (2)(-2)(5) + (-2)(-1)(5) + (-C + D)(-1)(-2) \\ 2 &= -C + D. \end{aligned}$$

If $x = 2$, you have

$$\begin{aligned} 0 &= (2)(1)(8) + (-2)(2)(8) + (2C + D)(2)(1) \\ 8 &= 2C + D. \end{aligned}$$

Solving the linear system by subtracting the first equation from the second

$$\begin{aligned} -C + D &= 2 \\ 2C + D &= 8 \end{aligned}$$

yields $C = 2$. Consequently, $D = 4$, and it follows that

$$\begin{aligned} \int \frac{2x^3 - 4x - 8}{x(x - 1)(x^2 + 4)} dx &= \int \left(\frac{2}{x} - \frac{2}{x - 1} + \frac{2x}{x^2 + 4} + \frac{4}{x^2 + 4} \right) dx \\ &= 2 \ln|x| - 2 \ln|x - 1| + \ln(x^2 + 4) + 2 \arctan \frac{x}{2} + C. \end{aligned}$$

In Examples 1, 2, and 3, the solution of the basic equation began with substituting values of x that made the linear factors equal to 0. This method works well when the partial fraction decomposition involves linear factors. However, if the decomposition involves only quadratic factors, an alternative procedure is often more convenient.

EXAMPLE 4 Repeated Quadratic Factors

Find $\int \frac{8x^3 + 13x}{(x^2 + 2)^2} dx$.

Solution Include one partial fraction for each power of $(x^2 + 2)$ and write

$$\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2}.$$

Multiplying by the least common denominator $(x^2 + 2)^2$ yields the *basic equation*

$$8x^3 + 13x = (Ax + B)(x^2 + 2) + Cx + D.$$

Expanding the basic equation and collecting like terms produces

$$8x^3 + 13x = Ax^3 + 2Ax + Bx^2 + 2B + Cx + D$$

$$8x^3 + 13x = Ax^3 + Bx^2 + (2A + C)x + (2B + D).$$

Now, you can equate the coefficients of like terms on opposite sides of the equation.

$$\begin{array}{c} 8 = A \\ 0 = B \\ 13 = 2A + C \\ 0 = 2B + D \end{array}$$

$$8x^3 + 0x^2 + 13x + 0 = Ax^3 + Bx^2 + (2A + C)x + (2B + D)$$

Using the known values $A = 8$ and $B = 0$, you can write

$$13 = 2A + C = 2(8) + C \Rightarrow C = -3$$

$$0 = 2B + D = 2(0) + D \Rightarrow D = 0.$$

Finally, you can conclude that

$$\begin{aligned} \int \frac{8x^3 + 13x}{(x^2 + 2)^2} dx &= \int \left(\frac{8x}{x^2 + 2} + \frac{-3x}{(x^2 + 2)^2} \right) dx \\ &= 4 \ln(x^2 + 2) + \frac{3}{2(x^2 + 2)} + C. \end{aligned}$$

TECHNOLOGY Use a computer algebra system to evaluate the integral in Example 4—you might find that the form of the antiderivative is different. For instance, when you use a computer algebra system to work Example 4, you obtain

$$\int \frac{8x^3 + 13x}{(x^2 + 2)^2} dx = \ln(x^8 + 8x^6 + 24x^4 + 32x^2 + 16) + \frac{3}{2(x^2 + 2)} + C.$$

Is this result equivalent to that obtained in Example 4?

When integrating rational expressions, keep in mind that for *improper* rational expressions such as

$$\frac{N(x)}{D(x)} = \frac{2x^3 + x^2 - 7x + 7}{x^2 + x - 2}$$

you must first divide to obtain

$$\frac{N(x)}{D(x)} = 2x - 1 + \frac{-2x + 5}{x^2 + x - 2}.$$

The proper rational expression is then decomposed into its partial fractions by the usual methods. Here are some guidelines for solving the basic equation that is obtained in a partial fraction decomposition.

GUIDELINES FOR SOLVING THE BASIC EQUATION

Linear Factors

1. Substitute the roots of the distinct linear factors in the basic equation.
2. For repeated linear factors, use the coefficients determined in guideline 1 to rewrite the basic equation. Then substitute other convenient values of x and solve for the remaining coefficients.

Quadratic Factors

1. Expand the basic equation.
2. Collect terms according to powers of x .
3. Equate the coefficients of like powers to obtain a system of linear equations involving A , B , C , and so on.
4. Solve the system of linear equations.

Before concluding this section, here are a few things you should remember. First, it is not necessary to use the partial fractions technique on all rational functions. For instance, the following integral is evaluated more easily by the Log Rule.

$$\begin{aligned}\int \frac{x^2 + 1}{x^3 + 3x - 4} dx &= \frac{1}{3} \int \frac{3x^2 + 3}{x^3 + 3x - 4} dx \\ &= \frac{1}{3} \ln|x^3 + 3x - 4| + C\end{aligned}$$

Second, if the integrand is not in reduced form, reducing it may eliminate the need for partial fractions, as shown in the following integral.

$$\begin{aligned}\int \frac{x^2 - x - 2}{x^3 - 2x - 4} dx &= \int \frac{(x+1)(x-2)}{(x-2)(x^2 + 2x + 2)} dx \\ &= \int \frac{x+1}{x^2 + 2x + 2} dx \\ &= \frac{1}{2} \ln|x^2 + 2x + 2| + C\end{aligned}$$

Finally, partial fractions can be used with some quotients involving transcendental functions. For instance, the substitution $u = \sin x$ allows you to write

$$\int \frac{\cos x}{\sin x(\sin x - 1)} dx = \int \frac{du}{u(u-1)}. \quad u = \sin x, du = \cos x dx$$

8.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, write the form of the partial fraction decomposition of the rational expression. Do not solve for the constants.

1. $\frac{4}{x^2 - 8x}$

3. $\frac{2x - 3}{x^3 + 10x}$

5. $\frac{x - 9}{x^2 - 6x}$

2. $\frac{2x^2 + 1}{(x - 3)^3}$

4. $\frac{x - 4}{x^2 + 6x + 5}$

6. $\frac{2x - 1}{x(x^2 + 1)^2}$

In Exercises 7–28, use partial fractions to find the integral.

7. $\int \frac{1}{x^2 - 9} dx$

9. $\int \frac{5}{x^2 + 3x - 4} dx$

11. $\int \frac{5 - x}{2x^2 + x - 1} dx$

13. $\int \frac{x^2 + 12x + 12}{x^3 - 4x} dx$

15. $\int \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8} dx$

17. $\int \frac{4x^2 + 2x - 1}{x^3 + x^2} dx$

19. $\int \frac{x^2 + 3x - 4}{x^3 - 4x^2 + 4x} dx$

21. $\int \frac{x^2 - 1}{x^3 + x} dx$

23. $\int \frac{x^2}{x^4 - 2x^2 - 8} dx$

25. $\int \frac{x}{16x^4 - 1} dx$

27. $\int \frac{x^2 + 5}{x^3 - x^2 + x + 3} dx$

8. $\int \frac{1}{4x^2 - 1} dx$

10. $\int \frac{x + 2}{x^2 + 11x + 18} dx$

12. $\int \frac{5x^2 - 12x - 12}{x^3 - 4x} dx$

14. $\int \frac{x^3 - x + 3}{x^2 + x - 2} dx$

16. $\int \frac{x + 2}{x^2 - 4x} dx$

18. $\int \frac{3x - 4}{(x - 1)^2} dx$

20. $\int \frac{4x^2}{x^3 + x^2 - x - 1} dx$

22. $\int \frac{6x}{x^3 - 8} dx$

24. $\int \frac{x^2 - x + 9}{(x^2 + 9)^2} dx$

26. $\int \frac{x^2 - 4x + 7}{x^3 - x^2 + x + 3} dx$

28. $\int \frac{x^2 + x + 3}{x^4 + 6x^2 + 9} dx$

In Exercises 29–32, evaluate the definite integral. Use a graphing utility to verify your result.

29. $\int_0^2 \frac{3}{4x^2 + 5x + 1} dx$

31. $\int_1^2 \frac{x + 1}{x(x^2 + 1)} dx$

30. $\int_1^5 \frac{x - 1}{x^2(x + 1)} dx$

32. $\int_0^1 \frac{x^2 - x}{x^2 + x + 1} dx$

37. $\int \frac{2x^2 - 2x + 3}{x^3 - x^2 - x - 2} dx, (3, 10)$

38. $\int \frac{x(2x - 9)}{x^3 - 6x^2 + 12x - 8} dx, (3, 2)$

39. $\int \frac{1}{x^2 - 25} dx, (7, 2)$

40. $\int \frac{x^2 - x + 2}{x^3 - x^2 + x - 1} dx, (2, 6)$

In Exercises 41–50, use substitution to find the integral.

41. $\int \frac{\sin x}{\cos x(\cos x - 1)} dx$

42. $\int \frac{\sin x}{\cos x + \cos^2 x} dx$

43. $\int \frac{\cos x}{\sin x + \sin^2 x} dx$

44. $\int \frac{5 \cos x}{\sin^2 x + 3 \sin x - 4} dx$

45. $\int \frac{\sec^2 x}{\tan^2 x + 5 \tan x + 6} dx$

46. $\int \frac{\sec^2 x}{\tan x(\tan x + 1)} dx$

47. $\int \frac{e^x}{(e^x - 1)(e^x + 4)} dx$

48. $\int \frac{e^x}{(e^{2x} + 1)(e^x - 1)} dx$

49. $\int \frac{\sqrt{x}}{x - 4} dx$

50. $\int \frac{1}{\sqrt{x} - \sqrt[3]{x}} dx$

In Exercises 51–54, use the method of partial fractions to verify the integration formula.

51. $\int \frac{1}{x(a + bx)} dx = \frac{1}{a} \ln \left| \frac{x}{a + bx} \right| + C$

52. $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right| + C$

53. $\int \frac{x}{(a + bx)^2} dx = \frac{1}{b^2} \left(\frac{a}{a + bx} + \ln |a + bx| \right) + C$

54. $\int \frac{1}{x^2(a + bx)} dx = -\frac{1}{ax} - \frac{b}{a^2} \ln \left| \frac{x}{a + bx} \right| + C$

CAS **Slope Fields** In Exercises 55 and 56, use a computer algebra system to graph the slope field for the differential equation and graph the solution through the given initial condition.

55. $\frac{dy}{dx} = \frac{6}{4 - x^2}$

$y(0) = 3$

56. $\frac{dy}{dx} = \frac{4}{x^2 - 2x - 3}$

$y(0) = 5$

WRITING ABOUT CONCEPTS

57. What is the first step when integrating $\int \frac{x^3}{x - 5} dx$? Explain.

58. Describe the decomposition of the proper rational function $N(x)/D(x)$ (a) if $D(x) = (px + q)^m$ and (b) if $D(x) = (ax^2 + bx + c)^n$, where $ax^2 + bx + c$ is irreducible. Explain why you chose that method.

CAS In Exercises 33–40, use a computer algebra system to determine the antiderivative that passes through the given point. Use the system to graph the resulting antiderivative.

33. $\int \frac{5x}{x^2 - 10x + 25} dx, (6, 0)$

34. $\int \frac{6x^2 + 1}{x^2(x - 1)^3} dx, (2, 1)$

35. $\int \frac{x^2 + x + 2}{(x^2 + 2)^2} dx, (0, 1)$

36. $\int \frac{x^3}{(x^2 - 4)^2} dx, (3, 4)$

- 59. Area** Find the area of the region bounded by the graphs of $y = 12/(x^2 + 5x + 6)$, $y = 0$, $x = 0$, and $x = 1$.
- 60. Area** Find the area of the region bounded by the graphs of $y = 15/(x^2 + 7x + 12)$, $y = 0$, $x = 0$, and $x = 2$.
- 61. Area** Find the area of the region bounded by the graphs of $y = 7/(16 - x^2)$ and $y = 1$.

CAPSTONE

- 62.** State the method you would use to evaluate each integral. Explain why you chose that method. Do not integrate.

$$(a) \int \frac{x+1}{x^2+2x-8} dx \quad (b) \int \frac{7x+4}{x^2+2x-8} dx$$

$$(c) \int \frac{4}{x^2+2x+5} dx$$

- 63. Modeling Data** The predicted cost C (in hundreds of thousands of dollars) for a company to remove $p\%$ of a chemical from its waste water is shown in the table.

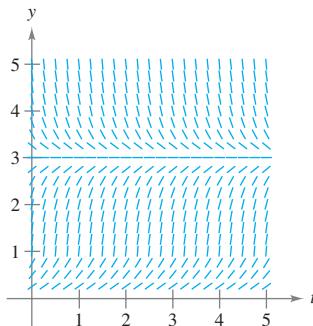
P	0	10	20	30	40	50	60	70	80	90
C	0	0.7	1.0	1.3	1.7	2.0	2.7	3.6	5.5	11.2

A model for the data is given by $C = \frac{124p}{(10+p)(100-p)}$, $0 \leq p < 100$. Use the model to find the average cost of removing between 75% and 80% of the chemical.

- 64. Logistic Growth** In Chapter 6, the exponential growth equation was derived from the assumption that the rate of growth was proportional to the existing quantity. In practice, there often exists some upper limit L past which growth cannot occur. In such cases, you assume the rate of growth to be proportional not only to the existing quantity, but also to the difference between the existing quantity y and the upper limit L . That is, $dy/dt = ky(L-y)$. In integral form, you can write this relationship as

$$\int \frac{dy}{y(L-y)} = \int k dt.$$

- (a) A slope field for the differential equation $dy/dt = y(3-y)$ is shown. Draw a possible solution to the differential equation if $y(0) = 5$, and another if $y(0) = \frac{1}{2}$. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.



- (b) Where $y(0)$ is greater than 3, what is the sign of the slope of the solution?

- (c) For $y > 0$, find $\lim_{t \rightarrow \infty} y(t)$.

- (d) Evaluate the two given integrals and solve for y as a function of t , where y_0 is the initial quantity.

- (e) Use the result of part (d) to find and graph the solutions in part (a). Use a graphing utility to graph the solutions and compare the results with the solutions in part (a).

- (f) The graph of the function y is a **logistic curve**. Show that the rate of growth is maximum at the point of inflection, and that this occurs when $y = L/2$.

- 65. Volume and Centroid** Consider the region bounded by the graphs of $y = 2x/(x^2 + 1)$, $y = 0$, $x = 0$, and $x = 3$. Find the volume of the solid generated by revolving this region about the x -axis. Find the centroid of the region.

- 66. Volume** Consider the region bounded by the graph of $y^2 = (2-x)^2/(1+x)^2$ on the interval $[0, 1]$. Find the volume of the solid generated by revolving this region about the x -axis.

- 67. Epidemic Model** A single infected individual enters a community of n susceptible individuals. Let x be the number of newly infected individuals at time t . The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. So, $dx/dt = k(x+1)(n-x)$ and you obtain

$$\int \frac{1}{(x+1)(n-x)} dx = \int k dt.$$

Solve for x as a function of t .

- 68. Chemical Reactions** In a chemical reaction, one unit of compound Y and one unit of compound Z are converted into a single unit of compound X. x is the amount of compound X formed, and the rate of formation of X is proportional to the product of the amounts of unconverted compounds Y and Z. So, $dx/dt = k(y_0 - x)(z_0 - x)$, where y_0 and z_0 are the initial amounts of compounds Y and Z. From this equation you obtain

$$\int \frac{1}{(y_0 - x)(z_0 - x)} dx = \int k dt.$$

- (a) Perform the two integrations and solve for x in terms of t .
 (b) Use the result of part (a) to find x as $t \rightarrow \infty$ if (1) $y_0 < z_0$, (2) $y_0 > z_0$, and (3) $y_0 = z_0$.

- 69. Evaluate**

$$\int_0^1 \frac{x}{1+x^4} dx$$

in two different ways, one of which is partial fractions.

PUTNAM EXAM CHALLENGE

- 70.** Prove $\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx$.

This problem was composed by the Committee on the Putnam Prize Competition.
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8.6**Integration by Tables and Other Integration Techniques**

- Evaluate an indefinite integral using a table of integrals.
- Evaluate an indefinite integral using reduction formulas.
- Evaluate an indefinite integral involving rational functions of sine and cosine.

Integration by Tables

So far in this chapter you have studied several integration techniques that can be used with the basic integration rules. But merely knowing *how* to use the various techniques is not enough. You also need to know *when* to use them. Integration is first and foremost a problem of recognition. That is, you must recognize which rule or technique to apply to obtain an antiderivative. Frequently, a slight alteration of an integrand will require a different integration technique (or produce a function whose antiderivative is not an elementary function), as shown below.

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$$

Integration by parts

$$\int \frac{\ln x}{x} \, dx = \frac{(\ln x)^2}{2} + C$$

Power Rule

$$\int \frac{1}{x \ln x} \, dx = \ln|\ln x| + C$$

Log Rule

$$\int \frac{x}{\ln x} \, dx = ?$$

Not an elementary function

TECHNOLOGY A computer algebra system consists, in part, of a database of integration formulas. The primary difference between using a computer algebra system and using tables of integrals is that with a computer algebra system the computer searches through the database to find a fit. With integration tables, *you* must do the searching.

Many people find tables of integrals to be a valuable supplement to the integration techniques discussed in this chapter. Tables of common integrals can be found in Appendix B. **Integration by tables** is not a “cure-all” for all of the difficulties that can accompany integration—using tables of integrals requires considerable thought and insight and often involves substitution.

Each integration formula in Appendix B can be developed using one or more of the techniques in this chapter. You should try to verify several of the formulas. For instance, Formula 4

$$\int \frac{u}{(a + bu)^2} \, du = \frac{1}{b^2} \left(\frac{a}{a + bu} + \ln|a + bu| \right) + C$$

Formula 4

can be verified using the method of partial fractions, and Formula 19

$$\int \frac{\sqrt{a + bu}}{u} \, du = 2\sqrt{a + bu} + a \int \frac{du}{u\sqrt{a + bu}}$$

Formula 19

can be verified using integration by parts. Note that the integrals in Appendix B are classified according to forms involving the following.

 u^n $(a + bu)$ $(a + bu + cu^2)$ $\sqrt{a + bu}$ $(a^2 \pm u^2)$ $\sqrt{u^2 \pm a^2}$ $\sqrt{a^2 - u^2}$

Trigonometric functions

Inverse trigonometric functions

Exponential functions

Logarithmic functions

EXPLORATION

Use the tables of integrals in Appendix B and the substitution

$$u = \sqrt{x - 1}$$

to evaluate the integral in Example 1. If you do this, you should obtain

$$\int \frac{dx}{x\sqrt{x-1}} = \int \frac{2 du}{u^2 + 1}.$$

Does this produce the same result as that obtained in Example 1?

EXAMPLE 1 Integration by Tables

Find $\int \frac{dx}{x\sqrt{x-1}}$.

Solution Because the expression inside the radical is linear, you should consider forms involving $\sqrt{a+bu}$.

$$\int \frac{du}{u\sqrt{a+bu}} = \frac{2}{\sqrt{-a}} \arctan \sqrt{\frac{a+bu}{-a}} + C \quad \text{Formula 17 } (a < 0)$$

Let $a = -1$, $b = 1$, and $u = x$. Then $du = dx$, and you can write

$$\int \frac{dx}{x\sqrt{x-1}} = 2 \arctan \sqrt{x-1} + C.$$

 **EXAMPLE 2** Integration by Tables

Find $\int x\sqrt{x^4 - 9} dx$.

Solution Because the radical has the form $\sqrt{u^2 - a^2}$, you should consider Formula 26.

$$\int \sqrt{u^2 - a^2} du = \frac{1}{2} \left(u \sqrt{u^2 - a^2} - a^2 \ln|u + \sqrt{u^2 - a^2}| \right) + C$$

Let $u = x^2$ and $a = 3$. Then $du = 2x dx$, and you have

$$\begin{aligned} \int x\sqrt{x^4 - 9} dx &= \frac{1}{2} \int \sqrt{(x^2)^2 - 3^2} (2x) dx \\ &= \frac{1}{4} \left(x^2 \sqrt{x^4 - 9} - 9 \ln|x^2 + \sqrt{x^4 - 9}| \right) + C. \end{aligned}$$

EXAMPLE 3 Integration by Tables

Find $\int \frac{x}{1 + e^{-x^2}} dx$.

Solution Of the forms involving e^u , consider the following formula.

$$\int \frac{du}{1 + e^u} = u - \ln(1 + e^u) + C \quad \text{Formula 84}$$

Let $u = -x^2$. Then $du = -2x dx$, and you have

$$\begin{aligned} \int \frac{x}{1 + e^{-x^2}} dx &= -\frac{1}{2} \int \frac{-2x dx}{1 + e^{-x^2}} \\ &= -\frac{1}{2} [-x^2 - \ln(1 + e^{-x^2})] + C \\ &= \frac{1}{2} [x^2 + \ln(1 + e^{-x^2})] + C. \end{aligned}$$

■

TECHNOLOGY Example 3 shows the importance of having several solution techniques at your disposal. This integral is not difficult to solve with a table, but when it was entered into a well-known computer algebra system, the utility was unable to find the antiderivative.

Reduction Formulas

Several of the integrals in the integration tables have the form $\int f(x) dx = g(x) + \int h(x) dx$. Such integration formulas are called **reduction formulas** because they reduce a given integral to the sum of a function and a simpler integral.

EXAMPLE 4 Using a Reduction Formula

Find $\int x^3 \sin x dx$.

Solution Consider the following three formulas.

$$\int u \sin u du = \sin u - u \cos u + C \quad \text{Formula 52}$$

$$\int u^n \sin u du = -u^n \cos u + n \int u^{n-1} \cos u du \quad \text{Formula 54}$$

$$\int u^n \cos u du = u^n \sin u - n \int u^{n-1} \sin u du \quad \text{Formula 55}$$

Using Formula 54, Formula 55, and then Formula 52 produces

$$\begin{aligned} \int x^3 \sin x dx &= -x^3 \cos x + 3 \int x^2 \cos x dx \\ &= -x^3 \cos x + 3 \left(x^2 \sin x - 2 \int x \sin x dx \right) \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C. \end{aligned}$$

EXAMPLE 5 Using a Reduction Formula

Find $\int \frac{\sqrt{3-5x}}{2x} dx$.

Solution Consider the following two formulas.

$$\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}} \ln \left| \frac{\sqrt{a+bu} - \sqrt{a}}{\sqrt{a+bu} + \sqrt{a}} \right| + C \quad \text{Formula 17 } (a > 0)$$

$$\int \frac{\sqrt{a+bu}}{u} du = 2\sqrt{a+bu} + a \int \frac{du}{u\sqrt{a+bu}} \quad \text{Formula 19}$$

Using Formula 19, with $a = 3$, $b = -5$, and $u = x$, produces

$$\begin{aligned} \frac{1}{2} \int \frac{\sqrt{3-5x}}{x} dx &= \frac{1}{2} \left(2\sqrt{3-5x} + 3 \int \frac{dx}{x\sqrt{3-5x}} \right) \\ &= \sqrt{3-5x} + \frac{3}{2} \int \frac{dx}{x\sqrt{3-5x}}. \end{aligned}$$

Using Formula 17, with $a = 3$, $b = -5$, and $u = x$, produces

$$\begin{aligned} \int \frac{\sqrt{3-5x}}{2x} dx &= \sqrt{3-5x} + \frac{3}{2} \left(\frac{1}{\sqrt{3}} \ln \left| \frac{\sqrt{3-5x} - \sqrt{3}}{\sqrt{3-5x} + \sqrt{3}} \right| \right) + C \\ &= \sqrt{3-5x} + \frac{\sqrt{3}}{2} \ln \left| \frac{\sqrt{3-5x} - \sqrt{3}}{\sqrt{3-5x} + \sqrt{3}} \right| + C. \end{aligned}$$

TECHNOLOGY Sometimes when you use computer algebra systems you obtain results that look very different, but are actually equivalent. Here is how two different systems evaluated the integral in Example 5.

Maple

$$\begin{aligned} &\sqrt{3-5x} - \\ &\sqrt{3} \operatorname{arctanh}\left(\frac{1}{3}\sqrt{3-5x}\sqrt{3}\right) \end{aligned}$$

Mathematica

$$\begin{aligned} &\operatorname{Sqrt}[3-5x] - \\ &\operatorname{Sqrt}[3] \operatorname{ArcTanh}\left[\frac{\operatorname{Sqrt}[3-5x]}{\operatorname{Sqrt}[3]}\right] \end{aligned}$$

Notice that computer algebra systems do not include a constant of integration.

Rational Functions of Sine and Cosine

EXAMPLE 6 Integration by Tables

Find $\int \frac{\sin 2x}{2 + \cos x} dx$.

Solution Substituting $2 \sin x \cos x$ for $\sin 2x$ produces

$$\int \frac{\sin 2x}{2 + \cos x} dx = 2 \int \frac{\sin x \cos x}{2 + \cos x} dx.$$

A check of the forms involving $\sin u$ or $\cos u$ in Appendix B shows that none of those listed applies. So, you can consider forms involving $a + bu$. For example,

$$\int \frac{u du}{a + bu} = \frac{1}{b^2} (bu - a \ln|a + bu|) + C. \quad \text{Formula 3}$$

Let $a = 2$, $b = 1$, and $u = \cos x$. Then $du = -\sin x dx$, and you have

$$\begin{aligned} 2 \int \frac{\sin x \cos x}{2 + \cos x} dx &= -2 \int \frac{\cos x (-\sin x dx)}{2 + \cos x} \\ &= -2(\cos x - 2 \ln|2 + \cos x|) + C \\ &= -2 \cos x + 4 \ln|2 + \cos x| + C. \end{aligned}$$

■

Example 6 involves a rational expression of $\sin x$ and $\cos x$. If you are unable to find an integral of this form in the integration tables, try using the following special substitution to convert the trigonometric expression to a standard rational expression.

SUBSTITUTION FOR RATIONAL FUNCTIONS OF SINE AND COSINE

For integrals involving rational functions of sine and cosine, the substitution

$$u = \frac{\sin x}{1 + \cos x} = \tan \frac{x}{2}$$

yields

$$\cos x = \frac{1 - u^2}{1 + u^2}, \quad \sin x = \frac{2u}{1 + u^2}, \quad \text{and} \quad dx = \frac{2 du}{1 + u^2}.$$

PROOF From the substitution for u , it follows that

$$u^2 = \frac{\sin^2 x}{(1 + \cos x)^2} = \frac{1 - \cos^2 x}{(1 + \cos x)^2} = \frac{1 - \cos x}{1 + \cos x}.$$

Solving for $\cos x$ produces $\cos x = (1 - u^2)/(1 + u^2)$. To find $\sin x$, write $u = \sin x/(1 + \cos x)$ as

$$\sin x = u(1 + \cos x) = u \left(1 + \frac{1 - u^2}{1 + u^2}\right) = \frac{2u}{1 + u^2}.$$

Finally, to find dx , consider $u = \tan(x/2)$. Then you have $\arctan u = x/2$ and $dx = (2 du)/(1 + u^2)$.

8.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use a table of integrals with forms involving $a + bu$ to find the integral.

1. $\int \frac{x^2}{5+x} dx$

2. $\int \frac{2}{3x^2(2x-5)^2} dx$

In Exercises 3 and 4, use a table of integrals with forms involving $\sqrt{u^2 \pm a^2}$ to find the integral.

3. $\int e^x \sqrt{1+e^{2x}} dx$

4. $\int \frac{\sqrt{x^2-36}}{6x} dx$

In Exercises 5 and 6, use a table of integrals with forms involving $\sqrt{a^2 - u^2}$ to find the integral.

5. $\int \frac{1}{x^2\sqrt{1-x^2}} dx$

6. $\int \frac{x}{\sqrt{100-x^4}} dx$

In Exercises 7–10, use a table of integrals with forms involving the trigonometric functions to find the integral.

7. $\int \cos^4 3x dx$

8. $\int \frac{\sin^3 \sqrt{x}}{\sqrt{x}} dx$

9. $\int \frac{1}{\sqrt{x}(1-\cos \sqrt{x})} dx$

10. $\int \frac{1}{1-\tan 5x} dx$

In Exercises 11 and 12, use a table of integrals with forms involving e^u to find the integral.

11. $\int \frac{1}{1+e^{2x}} dx$

12. $\int e^{-x/2} \sin 2x dx$

In Exercises 13 and 14, use a table of integrals with forms involving $\ln u$ to find the integral.

13. $\int x^7 \ln x dx$

14. $\int (\ln x)^3 dx$

In Exercises 15–18, find the indefinite integral (a) using integration tables and (b) using the given method.

Integral	Method
$\int x^2 e^{3x} dx$	Integration by parts

In Exercises 19–42, use integration tables to find the integral.

19. $\int x \operatorname{arccsc}(x^2+1) dx$

20. $\int \operatorname{arcsec} 2x dx$

21. $\int \frac{1}{x^2\sqrt{x^2-4}} dx$

22. $\int \frac{1}{x^2+4x+8} dx$

23. $\int \frac{4x}{(2-5x)^2} dx$

24. $\int \frac{\theta^2}{1-\sin \theta^3} d\theta$

25. $\int e^x \arccos e^x dx$

26. $\int \frac{e^x}{1-\tan e^x} dx$

27. $\int \frac{x}{1-\sec x^2} dx$

28. $\int \frac{1}{t[1+(\ln t)^2]} dt$

29. $\int \frac{\cos \theta}{3+2\sin \theta+\sin^2 \theta} d\theta$

30. $\int x^2 \sqrt{2+9x^2} dx$

31. $\int \frac{1}{x^2\sqrt{2+9x^2}} dx$

32. $\int \sqrt{x} \arctan x^{3/2} dx$

33. $\int \frac{\ln x}{x(3+2\ln x)} dx$

34. $\int \frac{e^x}{(1-e^{2x})^{3/2}} dx$

35. $\int \frac{x}{(x^2-6x+10)^2} dx$

36. $\int (2x-3)^2 \sqrt{(2x-3)^2+4} dx$

37. $\int \frac{x}{\sqrt{x^4-6x^2+5}} dx$

38. $\int \frac{\cos x}{\sqrt{\sin^2 x+1}} dx$

39. $\int \frac{x^3}{\sqrt{4-x^2}} dx$

40. $\int \sqrt{\frac{5-x}{5+x}} dx$

41. $\int \frac{e^{3x}}{(1+e^x)^3} dx$

42. $\int \cot^4 \theta d\theta$

In Exercises 43–50, use integration tables to evaluate the integral.

43. $\int_0^1 xe^{x^2} dx$

44. $\int_0^7 \frac{x}{\sqrt{9+x}} dx$

45. $\int_1^2 x^4 \ln x dx$

46. $\int_0^{\pi/2} x \cos x dx$

47. $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+\sin^2 x} dx$

48. $\int_4^6 \frac{x^2}{(2x-7)^2} dx$

49. $\int_0^{\pi/2} t^3 \cos t dt$

50. $\int_0^1 \sqrt{3+x^2} dx$

CAS In Exercises 57–62, use a computer algebra system to determine the antiderivative that passes through the given point. Use the system to graph the resulting antiderivative.

57. $\int \frac{1}{x^{3/2}\sqrt{1-x}} dx, (\frac{1}{2}, 5)$

58. $\int x\sqrt{x^2+2x} dx, (0, 0)$

59. $\int \frac{1}{(x^2-6x+10)^2} dx, (3, 0)$

60. $\int \frac{\sqrt{2-2x-x^2}}{x+1} dx, (0, \sqrt{2})$

61. $\int \frac{1}{\sin \theta \tan \theta} d\theta, (\frac{\pi}{4}, 2)$

62. $\int \frac{\sin \theta}{(\cos \theta)(1+\sin \theta)} d\theta, (0, 1)$

In Exercises 63–70, find or evaluate the integral.

63. $\int \frac{1}{2-3\sin \theta} d\theta$

64. $\int \frac{\sin \theta}{1+\cos^2 \theta} d\theta$

65. $\int_0^{\pi/2} \frac{1}{1+\sin \theta + \cos \theta} d\theta$

66. $\int_0^{\pi/2} \frac{1}{3-2\cos \theta} d\theta$

67. $\int \frac{\sin \theta}{3-2\cos \theta} d\theta$

68. $\int \frac{\cos \theta}{1+\cos \theta} d\theta$

69. $\int \frac{\sin \sqrt{\theta}}{\sqrt{\theta}} d\theta$

70. $\int \frac{4}{\csc \theta - \cot \theta} d\theta$

Area In Exercises 71 and 72, find the area of the region bounded by the graphs of the equations.

71. $y = \frac{x}{\sqrt{x+3}}, y = 0, x = 6$

72. $y = \frac{x}{1+e^{x^2}}, y = 0, x = 2$

WRITING ABOUT CONCEPTS

73. (a) Evaluate $\int x^n \ln x dx$ for $n = 1, 2$, and 3 . Describe any patterns you notice.

(b) Write a general rule for evaluating the integral in part (a), for an integer $n \geq 1$.

74. Describe what is meant by a reduction formula. Give an example.

True or False? In Exercises 75 and 76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

75. To use a table of integrals, the integral you are evaluating must appear in the table.

76. When using a table of integrals, you may have to make substitutions to rewrite your integral in the form in which it appears in the table.

77. **Volume** Consider the region bounded by the graphs of $y = x\sqrt{16-x^2}$, $y = 0$, $x = 0$, and $x = 4$. Find the volume of the solid generated by revolving the region about the y -axis.

CAPSTONE

78. State (if possible) the method or integration formula you would use to find the antiderivative. Explain why you chose that method or formula. Do not integrate.

(a) $\int \frac{e^x}{e^{2x}+1} dx$

(b) $\int \frac{e^x}{e^x+1} dx$

(c) $\int x e^{x^2} dx$

(d) $\int x e^x dx$

(e) $\int e^{x^2} dx$

(f) $\int e^{2x} \sqrt{e^{2x}+1} dx$

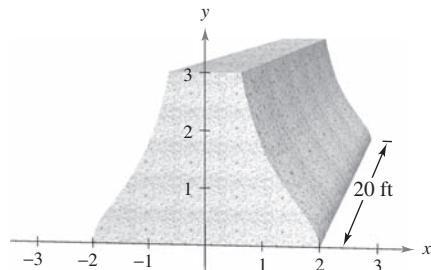
79. **Work** A hydraulic cylinder on an industrial machine pushes a steel block a distance of x feet ($0 \leq x \leq 5$), where the variable force required is $F(x) = 2000xe^{-x}$ pounds. Find the work done in pushing the block the full 5 feet through the machine.

80. **Work** Repeat Exercise 79, using $F(x) = \frac{500x}{\sqrt{26-x^2}}$ pounds.

81. **Building Design** The cross section of a precast concrete beam for a building is bounded by the graphs of the equations

$$x = \frac{2}{\sqrt{1+y^2}}, x = \frac{-2}{\sqrt{1+y^2}}, y = 0, \text{ and } y = 3$$

where x and y are measured in feet. The length of the beam is 20 feet (see figure). (a) Find the volume V and the weight W of the beam. Assume the concrete weighs 148 pounds per cubic foot. (b) Then find the centroid of a cross section of the beam.



82. **Population** A population is growing according to the logistic model $N = \frac{5000}{1+e^{4.8-1.9t}}$, where t is the time in days. Find the average population over the interval $[0, 2]$.



In Exercises 83 and 84, use a graphing utility to (a) solve the integral equation for the constant k and (b) graph the region whose area is given by the integral.

83. $\int_0^4 \frac{k}{2+3x} dx = 10$

84. $\int_0^k 6x^2 e^{-x/2} dx = 50$

PUTNAM EXAM CHALLENGE

85. Evaluate $\int_0^{\pi/2} \frac{dx}{1+(\tan x)^{\sqrt{2}}}.$

This problem was composed by the Committee on the Putnam Prize Competition.
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8.7**Indeterminate Forms and L'Hôpital's Rule**

- Recognize limits that produce indeterminate forms.
- Apply L'Hôpital's Rule to evaluate a limit.

Indeterminate Forms

Recall that the forms $0/0$ and ∞/∞ are called *indeterminate* because they do not guarantee that a limit exists, nor do they indicate what the limit is, if one does exist. When you encountered one of these indeterminate forms earlier in the text, you attempted to rewrite the expression by using various algebraic techniques.

*Indeterminate Form**Limit**Algebraic Technique*

$$\frac{0}{0}$$

$$\lim_{x \rightarrow -1} \frac{2x^2 - 2}{x + 1} = \lim_{x \rightarrow -1} 2(x - 1) \\ = -4$$

Divide numerator and denominator by $(x + 1)$.

$$\frac{\infty}{\infty}$$

$$\lim_{x \rightarrow \infty} \frac{3x^2 - 1}{2x^2 + 1} = \lim_{x \rightarrow \infty} \frac{3 - (1/x^2)}{2 + (1/x^2)} \\ = \frac{3}{2}$$

Divide numerator and denominator by x^2 .

Occasionally, you can extend these algebraic techniques to find limits of transcendental functions. For instance, the limit

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{e^x - 1}$$

produces the indeterminate form $0/0$. Factoring and then dividing produces

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{e^x - 1} = \lim_{x \rightarrow 0} \frac{(e^x + 1)(e^x - 1)}{e^x - 1} = \lim_{x \rightarrow 0} (e^x + 1) = 2.$$

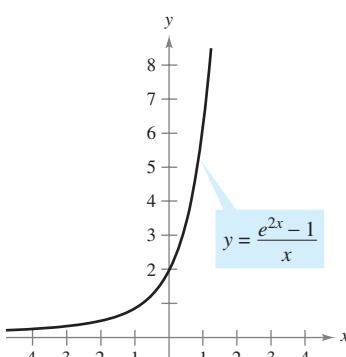
However, not all indeterminate forms can be evaluated by algebraic manipulation. This is often true when *both* algebraic and transcendental functions are involved. For instance, the limit

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x}$$

produces the indeterminate form $0/0$. Rewriting the expression to obtain

$$\lim_{x \rightarrow 0} \left(\frac{e^{2x}}{x} - \frac{1}{x} \right)$$

merely produces another indeterminate form, $\infty - \infty$. Of course, you could use technology to estimate the limit, as shown in the table and in Figure 8.14. From the table and the graph, the limit appears to be 2. (This limit will be verified in Example 1.)



The limit as x approaches 0 appears to be 2.

Figure 8.14

x	-1	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	1
$\frac{e^{2x} - 1}{x}$	0.865	1.813	1.980	1.998	?	2.002	2.020	2.214	6.389

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GUILLAUME L'HÔPITAL (1661–1704)

L'Hôpital's Rule is named after the French mathematician Guillaume François Antoine de L'Hôpital. L'Hôpital is credited with writing the first text on differential calculus (in 1696) in which the rule publicly appeared. It was recently discovered that the rule and its proof were written in a letter from John Bernoulli to L'Hôpital. "... I acknowledge that I owe very much to the bright minds of the Bernoulli brothers. ... I have made free use of their discoveries ...," said L'Hôpital.

L'Hôpital's Rule

To find the limit illustrated in Figure 8.14, you can use a theorem called **L'Hôpital's Rule**. This theorem states that under certain conditions the limit of the quotient $f(x)/g(x)$ is determined by the limit of the quotient of the derivatives

$$\frac{f'(x)}{g'(x)}.$$

To prove this theorem, you can use a more general result called the **Extended Mean Value Theorem**.

THEOREM 8.3 THE EXTENDED MEAN VALUE THEOREM

If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \neq 0$ for any x in (a, b) , then there exists a point c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

NOTE To see why this is called the Extended Mean Value Theorem, consider the special case in which $g(x) = x$. For this case, you obtain the “standard” Mean Value Theorem as presented in Section 4.2. ■

The Extended Mean Value Theorem and L'Hôpital's Rule are both proved in Appendix A.

THEOREM 8.4 L'HÔPITAL'S RULE

Let f and g be functions that are differentiable on an open interval (a, b) containing c , except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b) , except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies if the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞ , $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

NOTE People occasionally use L'Hôpital's Rule incorrectly by applying the Quotient Rule to $f(x)/g(x)$. Be sure you see that the rule involves $f'(x)/g'(x)$, not the derivative of $f(x)/g(x)$. ■

FOR FURTHER INFORMATION

To enhance your understanding of the necessity of the restriction that $g'(x)$ be nonzero for all x in (a, b) , except possibly at c , see the article “Counterexamples to L'Hôpital's Rule” by R. P. Boas in *The American Mathematical Monthly*. To view this article, go to the website www.matharticles.com.

L'Hôpital's Rule can also be applied to one-sided limits. For instance, if the limit of $f(x)/g(x)$ as x approaches c from the right produces the indeterminate form $0/0$, then

$$\lim_{x \rightarrow c^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c^+} \frac{f'(x)}{g'(x)}$$

provided the limit exists (or is infinite).

TECHNOLOGY Numerical and Graphical Approaches Use a numerical or a graphical approach to approximate each limit.

a. $\lim_{x \rightarrow 0} \frac{2^{2x} - 1}{x}$

b. $\lim_{x \rightarrow 0} \frac{3^{2x} - 1}{x}$

c. $\lim_{x \rightarrow 0} \frac{4^{2x} - 1}{x}$

d. $\lim_{x \rightarrow 0} \frac{5^{2x} - 1}{x}$

What pattern do you observe? Does an analytic approach have an advantage for these limits? If so, explain your reasoning.

EXAMPLE 1 Indeterminate Form 0/0

Evaluate $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x}$.

Solution Because direct substitution results in the indeterminate form 0/0

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x} &\quad \begin{matrix} \lim_{x \rightarrow 0} (e^{2x} - 1) = 0 \\ \lim_{x \rightarrow 0} x = 0 \end{matrix} \\ & \end{aligned}$$

you can apply L'Hôpital's Rule, as shown below.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}[e^{2x} - 1]}{\frac{d}{dx}[x]} && \text{Apply L'Hôpital's Rule.} \\ &= \lim_{x \rightarrow 0} \frac{2e^{2x}}{1} && \text{Differentiate numerator and denominator.} \\ &= 2 && \text{Evaluate the limit.} \end{aligned}$$

NOTE In writing the string of equations in Example 1, you actually do not know that the first limit is equal to the second until you have shown that the second limit exists. In other words, if the second limit had not existed, it would not have been permissible to apply L'Hôpital's Rule.

Another form of L'Hôpital's Rule states that if the limit of $f(x)/g(x)$ as x approaches ∞ (or $-\infty$) produces the indeterminate form 0/0 or ∞/∞ , then

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

EXAMPLE 2 Indeterminate Form ∞/∞

Evaluate $\lim_{x \rightarrow \infty} \frac{\ln x}{x}$.

Solution Because direct substitution results in the indeterminate form ∞/∞ , you can apply L'Hôpital's Rule to obtain

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\ln x}{x} &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}[\ln x]}{\frac{d}{dx}[x]} && \text{Apply L'Hôpital's Rule.} \\ &= \lim_{x \rightarrow \infty} \frac{1}{x} && \text{Differentiate numerator and denominator.} \\ &= 0. && \text{Evaluate the limit.} \end{aligned}$$

NOTE Try graphing $y_1 = \ln x$ and $y_2 = x$ in the same viewing window. Which function grows faster as x approaches ∞ ? How is this observation related to Example 2?

Occasionally it is necessary to apply L'Hôpital's Rule more than once to remove an indeterminate form, as shown in Example 3.

EXAMPLE 3 Applying L'Hôpital's Rule More Than Once

Evaluate $\lim_{x \rightarrow -\infty} \frac{x^2}{e^{-x}}$.

Solution Because direct substitution results in the indeterminate form ∞/∞ , you can apply L'Hôpital's Rule.

$$\lim_{x \rightarrow -\infty} \frac{x^2}{e^{-x}} = \lim_{x \rightarrow -\infty} \frac{\frac{d}{dx}[x^2]}{\frac{d}{dx}[e^{-x}]} = \lim_{x \rightarrow -\infty} \frac{2x}{-e^{-x}}$$

This limit yields the indeterminate form $(-\infty)/(-\infty)$, so you can apply L'Hôpital's Rule again to obtain

$$\lim_{x \rightarrow -\infty} \frac{2x}{-e^{-x}} = \lim_{x \rightarrow -\infty} \frac{\frac{d}{dx}[2x]}{\frac{d}{dx}[-e^{-x}]} = \lim_{x \rightarrow -\infty} \frac{2}{e^{-x}} = 0. \quad \blacksquare$$

In addition to the forms $0/0$ and ∞/∞ , there are other indeterminate forms such as $0 \cdot \infty$, 1^∞ , ∞^0 , 0^0 , and $\infty - \infty$. For example, consider the following four limits that lead to the indeterminate form $0 \cdot \infty$.

$$\underbrace{\lim_{x \rightarrow 0} (x)\left(\frac{1}{x}\right)}_{\text{Limit is } 1.}, \quad \underbrace{\lim_{x \rightarrow 0} (x)\left(\frac{2}{x}\right)}_{\text{Limit is } 2.}, \quad \underbrace{\lim_{x \rightarrow \infty} (x)\left(\frac{1}{e^x}\right)}_{\text{Limit is } 0.}, \quad \underbrace{\lim_{x \rightarrow \infty} (e^x)\left(\frac{1}{x}\right)}_{\text{Limit is } \infty.}$$

Because each limit is different, it is clear that the form $0 \cdot \infty$ is indeterminate in the sense that it does not determine the value (or even the existence) of the limit. The following examples indicate methods for evaluating these forms. Basically, you attempt to convert each of these forms to $0/0$ or ∞/∞ so that L'Hôpital's Rule can be applied.

EXAMPLE 4 Indeterminate Form $0 \cdot \infty$

Evaluate $\lim_{x \rightarrow \infty} e^{-x} \sqrt{x}$.

Solution Because direct substitution produces the indeterminate form $0 \cdot \infty$, you should try to rewrite the limit to fit the form $0/0$ or ∞/∞ . In this case, you can rewrite the limit to fit the second form.

$$\lim_{x \rightarrow \infty} e^{-x} \sqrt{x} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{e^x}$$

Now, by L'Hôpital's Rule, you have

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{e^x} = \lim_{x \rightarrow \infty} \frac{1/(2\sqrt{x})}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x} e^x} = 0. \quad \blacksquare$$

If rewriting a limit in one of the forms $0/0$ or ∞/∞ does not seem to work, try the other form. For instance, in Example 4 you can write the limit as

$$\lim_{x \rightarrow \infty} e^{-x} \sqrt{x} = \lim_{x \rightarrow \infty} \frac{e^{-x}}{x^{1/2}}$$

which yields the indeterminate form $0/0$. As it happens, applying L'Hôpital's Rule to this limit produces

$$\lim_{x \rightarrow \infty} \frac{e^{-x}}{x^{1/2}} = \lim_{x \rightarrow \infty} \frac{-e^{-x}}{-1/(2x^{3/2})}$$

which also yields the indeterminate form $0/0$.

The indeterminate forms 1^∞ , ∞^0 , and 0^0 arise from limits of functions that have variable bases and variable exponents. When you previously encountered this type of function, you used logarithmic differentiation to find the derivative. You can use a similar procedure when taking limits, as shown in the next example.

EXAMPLE 5 Indeterminate Form 1^∞

Evaluate $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$.

Solution Because direct substitution yields the indeterminate form 1^∞ , you can proceed as follows. To begin, assume that the limit exists and is equal to y .

$$y = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$$

Taking the natural logarithm of each side produces

$$\ln y = \ln \left[\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \right].$$

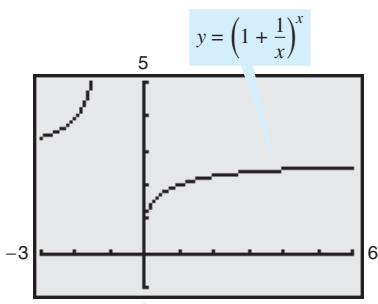
Because the natural logarithmic function is continuous, you can write

$$\begin{aligned} \ln y &= \lim_{x \rightarrow \infty} \left[x \ln \left(1 + \frac{1}{x}\right) \right] && \text{Indeterminate form } \infty \cdot 0 \\ &= \lim_{x \rightarrow \infty} \left(\frac{\ln[1 + (1/x)]}{1/x} \right) && \text{Indeterminate form } 0/0 \\ &= \lim_{x \rightarrow \infty} \left(\frac{(-1/x^2)\{1/[1 + (1/x)]\}}{-1/x^2} \right) && \text{L'Hôpital's Rule} \\ &= \lim_{x \rightarrow \infty} \frac{1}{1 + (1/x)} \\ &= 1. \end{aligned}$$

Now, because you have shown that $\ln y = 1$, you can conclude that $y = e$ and obtain

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e.$$

You can use a graphing utility to confirm this result, as shown in Figure 8.15. ■



The limit of $[1 + (1/x)]^x$ as x approaches infinity is e .

Figure 8.15

L'Hôpital's Rule can also be applied to one-sided limits, as demonstrated in Examples 6 and 7.

EXAMPLE 6 Indeterminate Form 0^0

Find $\lim_{x \rightarrow 0^+} (\sin x)^x$.

Solution Because direct substitution produces the indeterminate form 0^0 , you can proceed as shown below. To begin, assume that the limit exists and is equal to y .

$$\begin{aligned} y &= \lim_{x \rightarrow 0^+} (\sin x)^x && \text{Indeterminate form } 0^0 \\ \ln y &= \ln \left[\lim_{x \rightarrow 0^+} (\sin x)^x \right] && \text{Take natural log of each side.} \\ &= \lim_{x \rightarrow 0^+} [\ln(\sin x)^x] && \text{Continuity} \\ &= \lim_{x \rightarrow 0^+} [x \ln(\sin x)] && \text{Indeterminate form } 0 \cdot (-\infty) \\ &= \lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{1/x} && \text{Indeterminate form } -\infty/\infty \\ &= \lim_{x \rightarrow 0^+} \frac{\cot x}{-1/x^2} && \text{L'Hôpital's Rule} \\ &= \lim_{x \rightarrow 0^+} \frac{-x^2}{\tan x} && \text{Indeterminate form } 0/0 \\ &= \lim_{x \rightarrow 0^+} \frac{-2x}{\sec^2 x} = 0 && \text{L'Hôpital's Rule} \end{aligned}$$

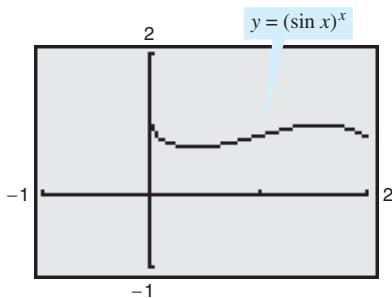
Now, because $\ln y = 0$, you can conclude that $y = e^0 = 1$, and it follows that

$$\lim_{x \rightarrow 0^+} (\sin x)^x = 1.$$

■

TECHNOLOGY When evaluating complicated limits such as the one in Example 6, it is helpful to check the reasonableness of the solution with a computer or with a graphing utility. For instance, the calculations in the following table and the graph in Figure 8.16 are consistent with the conclusion that $(\sin x)^x$ approaches 1 as x approaches 0 from the right.

x	1.0	0.1	0.01	0.001	0.0001	0.00001
$(\sin x)^x$	0.8415	0.7942	0.9550	0.9931	0.9991	0.9999



The limit of $(\sin x)^x$ is 1 as x approaches 0 from the right.

Figure 8.16

Use a computer algebra system or graphing utility to estimate the following limits:

$$\lim_{x \rightarrow 0} (1 - \cos x)^x$$

and

$$\lim_{x \rightarrow 0^+} (\tan x)^x.$$

Then see if you can verify your estimates analytically.

STUDY TIP In each of the examples presented in this section, L'Hôpital's Rule is used to find a limit that exists. It can also be used to conclude that a limit is infinite. For instance, try using L'Hôpital's Rule to show that

$$\lim_{x \rightarrow \infty} \frac{e^x}{x} = \infty.$$

EXAMPLE 7 Indeterminate Form $\infty - \infty$

Evaluate $\lim_{x \rightarrow 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$.

Solution Because direct substitution yields the indeterminate form $\infty - \infty$, you should try to rewrite the expression to produce a form to which you can apply L'Hôpital's Rule. In this case, you can combine the two fractions to obtain

$$\lim_{x \rightarrow 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) = \lim_{x \rightarrow 1^+} \left[\frac{x-1 - \ln x}{(x-1)\ln x} \right].$$

Now, because direct substitution produces the indeterminate form $0/0$, you can apply L'Hôpital's Rule to obtain

$$\begin{aligned} \lim_{x \rightarrow 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) &= \lim_{x \rightarrow 1^+} \frac{\frac{d}{dx}[x-1 - \ln x]}{\frac{d}{dx}[(x-1)\ln x]} \\ &= \lim_{x \rightarrow 1^+} \left[\frac{1 - (1/x)}{(x-1)(1/x) + \ln x} \right] \\ &= \lim_{x \rightarrow 1^+} \left(\frac{x-1}{x-1 + x \ln x} \right). \end{aligned}$$

This limit also yields the indeterminate form $0/0$, so you can apply L'Hôpital's Rule again to obtain

$$\begin{aligned} \lim_{x \rightarrow 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) &= \lim_{x \rightarrow 1^+} \left[\frac{1}{1 + x(1/x) + \ln x} \right] \\ &= \frac{1}{2}. \end{aligned}$$
■

The forms $0/0$, ∞/∞ , $\infty - \infty$, $0 \cdot \infty$, 0^0 , 1^∞ , and ∞^0 have been identified as *indeterminate*. There are similar forms that you should recognize as “determinate.”

$\infty + \infty \rightarrow \infty$ Limit is positive infinity.

$-\infty - \infty \rightarrow -\infty$ Limit is negative infinity.

$0^\infty \rightarrow 0$ Limit is zero.

$0^{-\infty} \rightarrow \infty$ Limit is positive infinity.

(You are asked to verify two of these in Exercises 116 and 117.)

As a final comment, remember that L'Hôpital's Rule can be applied only to quotients leading to the indeterminate forms $0/0$ and ∞/∞ . For instance, the following application of L'Hôpital's Rule is *incorrect*.

$$\lim_{x \rightarrow 0} \frac{e^x}{x} \stackrel{?}{=} \lim_{x \rightarrow 0} \frac{e^x}{1} = 1$$

Incorrect use of L'Hôpital's Rule

The reason this application is incorrect is that, even though the limit of the denominator is 0, the limit of the numerator is 1, which means that the hypotheses of L'Hôpital's Rule have not been satisfied.

8.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Numerical and Graphical Analysis In Exercises 1–4, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to support your result.

1. $\lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 3x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

2. $\lim_{x \rightarrow 0} \frac{1 - e^x}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

3. $\lim_{x \rightarrow \infty} x^5 e^{-x/100}$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$						

4. $\lim_{x \rightarrow \infty} \frac{6x}{\sqrt{3x^2 - 2x}}$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$						

In Exercises 5–10, evaluate the limit (a) using techniques from Chapters 2 and 4 and (b) using L'Hôpital's Rule.

5. $\lim_{x \rightarrow 4} \frac{3(x - 4)}{x^2 - 16}$

6. $\lim_{x \rightarrow -2} \frac{2x^2 + x - 6}{x + 2}$

7. $\lim_{x \rightarrow 6} \frac{\sqrt{x + 10} - 4}{x - 6}$

8. $\lim_{x \rightarrow 0} \frac{\sin 6x}{4x}$

9. $\lim_{x \rightarrow \infty} \frac{5x^2 - 3x + 1}{3x^2 - 5}$

10. $\lim_{x \rightarrow \infty} \frac{2x + 1}{4x^2 + x}$

In Exercises 11–44, evaluate the limit, using L'Hôpital's Rule if necessary. (In Exercise 18, n is a positive integer.)

11. $\lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x - 3}$

12. $\lim_{x \rightarrow -1} \frac{x^2 - 2x - 3}{x + 1}$

13. $\lim_{x \rightarrow 0} \frac{\sqrt{25 - x^2} - 5}{x}$

14. $\lim_{x \rightarrow 5^-} \frac{\sqrt{25 - x^2}}{x - 5}$

15. $\lim_{x \rightarrow 0} \frac{e^x - (1 - x)}{x}$

16. $\lim_{x \rightarrow 1} \frac{\ln x^2}{x^2 - 1}$

17. $\lim_{x \rightarrow 0^+} \frac{e^x - (1 + x)}{x^3}$

18. $\lim_{x \rightarrow 0^+} \frac{e^x - (1 + x)}{x^n}$

19. $\lim_{x \rightarrow 1} \frac{x^{11} - 1}{x^4 - 1}$

21. $\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 5x}$

23. $\lim_{x \rightarrow 0} \frac{\arcsin x}{x}$

25. $\lim_{x \rightarrow \infty} \frac{5x^2 + 3x - 1}{4x^2 + 5}$

27. $\lim_{x \rightarrow \infty} \frac{x^2 + 4x + 7}{x - 6}$

29. $\lim_{x \rightarrow \infty} \frac{x^3}{e^{x/2}}$

31. $\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}}$

33. $\lim_{x \rightarrow \infty} \frac{\cos x}{x}$

35. $\lim_{x \rightarrow \infty} \frac{\ln x}{x^2}$

37. $\lim_{x \rightarrow \infty} \frac{e^x}{x^4}$

39. $\lim_{x \rightarrow 0} \frac{\sin 5x}{\tan 9x}$

41. $\lim_{x \rightarrow 0} \frac{\arctan x}{\sin x}$

43. $\lim_{x \rightarrow \infty} \frac{\int_1^x \ln(e^{4t-1}) dt}{x}$

20. $\lim_{x \rightarrow 1} \frac{x^a - 1}{x^b - 1}$, where $a, b \neq 0$

22. $\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}$, where $a, b \neq 0$

24. $\lim_{x \rightarrow 1} \frac{\arctan x - (\pi/4)}{x - 1}$

26. $\lim_{x \rightarrow \infty} \frac{x - 6}{x^2 + 4x + 7}$

28. $\lim_{x \rightarrow \infty} \frac{x^3}{x + 2}$

30. $\lim_{x \rightarrow \infty} \frac{x^3}{e^{x^2}}$

32. $\lim_{x \rightarrow \infty} \frac{x^2}{\sqrt{x^2 + 1}}$

34. $\lim_{x \rightarrow \infty} \frac{\sin x}{x - \pi}$

36. $\lim_{x \rightarrow \infty} \frac{\ln x^4}{x^3}$

38. $\lim_{x \rightarrow \infty} \frac{e^{x/2}}{x}$

40. $\lim_{x \rightarrow 1} \frac{\ln x}{\sin \pi x}$

42. $\lim_{x \rightarrow 0} \frac{x}{\arctan 2x}$

44. $\lim_{x \rightarrow 1^+} \frac{\int_1^x \cos \theta d\theta}{x - 1}$

In Exercises 45–62, (a) describe the type of indeterminate form (if any) that is obtained by direct substitution. (b) Evaluate the limit, using L'Hôpital's Rule if necessary. (c) Use a graphing utility to graph the function and verify the result in part (b).

45. $\lim_{x \rightarrow \infty} x \ln x$

46. $\lim_{x \rightarrow 0^+} x^3 \cot x$

47. $\lim_{x \rightarrow \infty} \left(x \sin \frac{1}{x} \right)$

48. $\lim_{x \rightarrow \infty} x \tan \frac{1}{x}$

49. $\lim_{x \rightarrow 0^+} x^{1/x}$

50. $\lim_{x \rightarrow 0^+} (e^x + x)^{2/x}$

51. $\lim_{x \rightarrow \infty} x^{1/x}$

52. $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x$

53. $\lim_{x \rightarrow 0^+} (1 + x)^{1/x}$

54. $\lim_{x \rightarrow \infty} (1 + x)^{1/x}$

55. $\lim_{x \rightarrow 0^+} [3(x)]^{x/2}$

56. $\lim_{x \rightarrow 4^+} [3(x - 4)]^{x-4}$

57. $\lim_{x \rightarrow 1^+} (\ln x)^{x-1}$

58. $\lim_{x \rightarrow 0^+} \left[\cos \left(\frac{\pi}{2} - x \right) \right]^x$

59. $\lim_{x \rightarrow 2^+} \left(\frac{8}{x^2 - 4} - \frac{x}{x - 2} \right)$

60. $\lim_{x \rightarrow 2^+} \left(\frac{1}{x^2 - 4} - \frac{\sqrt{x-1}}{x^2 - 4} \right)$

61. $\lim_{x \rightarrow 1^+} \left(\frac{3}{\ln x} - \frac{2}{x-1} \right)$

62. $\lim_{x \rightarrow 0^+} \left(\frac{10}{x} - \frac{3}{x^2} \right)$

In Exercises 63–66, use a graphing utility to (a) graph the function and (b) find the required limit (if it exists).

63. $\lim_{x \rightarrow 3} \frac{x-3}{\ln(2x-5)}$

64. $\lim_{x \rightarrow 0^+} (\sin x)^x$

65. $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x + 2} - x)$

66. $\lim_{x \rightarrow \infty} \frac{x^3}{e^{2x}}$

WRITING ABOUT CONCEPTS

67. List six different indeterminate forms.

68. State L'Hôpital's Rule.

69. Find differentiable functions f and g that satisfy the specified condition such that $\lim_{x \rightarrow 5} f(x) = 0$ and $\lim_{x \rightarrow 5} g(x) = 0$.

Explain how you obtained your answers. (Note: There are many correct answers.)

(a) $\lim_{x \rightarrow 5} \frac{f(x)}{g(x)} = 10$ (b) $\lim_{x \rightarrow 5} \frac{f(x)}{g(x)} = 0$

(c) $\lim_{x \rightarrow 5} \frac{f(x)}{g(x)} = \infty$

70. Find differentiable functions f and g such that

$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = \infty$ and

$\lim_{x \rightarrow \infty} [f(x) - g(x)] = 25$.

Explain how you obtained your answers. (Note: There are many correct answers.)

71. **Numerical Approach** Complete the table to show that x eventually “overpowers” $(\ln x)^4$.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\frac{(\ln x)^4}{x}$						

72. **Numerical Approach** Complete the table to show that e^x eventually “overpowers” x^5 .

x	1	5	10	20	30	40	50	100
$\frac{e^x}{x^5}$								

Comparing Functions In Exercises 73–78, use L'Hôpital's Rule to determine the comparative rates of increase of the functions $f(x) = x^m$, $g(x) = e^{nx}$, and $h(x) = (\ln x)^n$, where $n > 0$, $m > 0$, and $x \rightarrow \infty$.

73. $\lim_{x \rightarrow \infty} \frac{x^2}{e^{5x}}$

74. $\lim_{x \rightarrow \infty} \frac{x^3}{e^{2x}}$

75. $\lim_{x \rightarrow \infty} \frac{(\ln x)^3}{x}$

76. $\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x^3}$

77. $\lim_{x \rightarrow \infty} \frac{(\ln x)^n}{x^m}$

78. $\lim_{x \rightarrow \infty} \frac{x^m}{e^{nx}}$

In Exercises 79–82, find any asymptotes and relative extrema that may exist and use a graphing utility to graph the function. (Hint: Some of the limits required in finding asymptotes have been found in previous exercises.)

79. $y = x^{1/x}$, $x > 0$

80. $y = x^x$, $x > 0$

81. $y = 2xe^{-x}$

82. $y = \frac{\ln x}{x}$

Think About It In Exercises 83–87, L'Hôpital's Rule is used incorrectly. Describe the error.

83. $\lim_{x \rightarrow 2} \frac{3x^2 + 4x + 1}{x^2 - x - 2} = \lim_{x \rightarrow 2} \frac{6x + 4}{2x - 1} = \lim_{x \rightarrow 2} \frac{6}{2} = 3$

84. $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{e^x} = \lim_{x \rightarrow 0} \frac{2e^{2x}}{e^x} = \lim_{x \rightarrow 0} 2e^x = 2$

85. $\lim_{x \rightarrow 0} \frac{\sin \pi x - 1}{x} = \lim_{x \rightarrow 0} \frac{\pi \cos \pi x}{1} = \pi$

86. $\lim_{x \rightarrow \infty} x \cos \frac{1}{x} = \lim_{x \rightarrow \infty} \frac{\cos(1/x)}{1/x}$
 $= \lim_{x \rightarrow \infty} \frac{[-\sin(1/x)](1/x^2)}{-1/x^2}$
 $= 0$

87. $\lim_{x \rightarrow \infty} \frac{e^{-x}}{1 + e^{-x}} = \lim_{x \rightarrow \infty} \frac{-e^{-x}}{-e^{-x}}$
 $= \lim_{x \rightarrow \infty} 1$
 $= 1$

CAPSTONE

88. Determine which of the following limits can be evaluated using L'Hôpital's Rule. Explain your reasoning. Do not evaluate the limit.

(a) $\lim_{x \rightarrow 2} \frac{x-2}{x^3-x-6}$

(b) $\lim_{x \rightarrow 0} \frac{x^2-4x}{2x-1}$

(c) $\lim_{x \rightarrow \infty} \frac{x^3}{e^x}$

(d) $\lim_{x \rightarrow 3} \frac{e^{x^2}-e^9}{x-3}$

(e) $\lim_{x \rightarrow 1} \frac{\cos \pi x}{\ln x}$

(f) $\lim_{x \rightarrow 1} \frac{1+x(\ln x-1)}{\ln x(x-1)}$

Analytical Approach In Exercises 89 and 90, (a) explain why L'Hôpital's Rule cannot be used to find the limit, (b) find the limit analytically, and (c) use a graphing utility to graph the function and approximate the limit from the graph. Compare the result with that in part (b).

89. $\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}}$

90. $\lim_{x \rightarrow \pi/2^-} \frac{\tan x}{\sec x}$

Graphical Analysis In Exercises 91 and 92, graph $f(x)/g(x)$ and $f'(x)/g'(x)$ near $x = 0$. What do you notice about these ratios as $x \rightarrow 0$? How does this illustrate L'Hôpital's Rule?

91. $f(x) = \sin 3x$, $g(x) = \sin 4x$

92. $f(x) = e^{3x} - 1$, $g(x) = x$

- 93. Velocity in a Resisting Medium** The velocity v of an object falling through a resisting medium such as air or water is given by

$$v = \frac{32}{k} \left(1 - e^{-kt} + \frac{v_0 k e^{-kt}}{32} \right)$$

where v_0 is the initial velocity, t is the time in seconds, and k is the resistance constant of the medium. Use L'Hôpital's Rule to find the formula for the velocity of a falling body in a vacuum by fixing v_0 and t and letting k approach zero. (Assume that the downward direction is positive.)

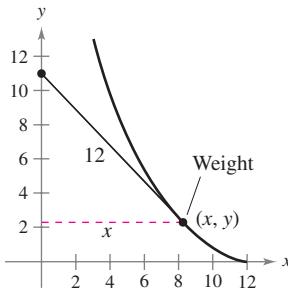
- 94. Compound Interest** The formula for the amount A in a savings account compounded n times per year for t years at an interest rate r and an initial deposit of P is given by

$$A = P \left(1 + \frac{r}{n} \right)^{nt}.$$

Use L'Hôpital's Rule to show that the limiting formula as the number of compoundings per year approaches infinity is given by $A = Pe^{rt}$.

- 95. The Gamma Function** The Gamma Function $\Gamma(n)$ is defined in terms of the integral of the function given by $f(x) = x^{n-1}e^{-x}$, $n > 0$. Show that for any fixed value of n , the limit of $f(x)$ as x approaches infinity is zero.

-  **96. Tractrix** A person moves from the origin along the positive y -axis pulling a weight at the end of a 12-meter rope (see figure). Initially, the weight is located at the point $(12, 0)$.



- (a) Show that the slope of the tangent line of the path of the weight is

$$\frac{dy}{dx} = -\frac{\sqrt{144 - x^2}}{x}.$$

- (b) Use the result of part (a) to find the equation of the path of the weight. Use a graphing utility to graph the path and compare it with the figure.

- (c) Find any vertical asymptotes of the graph in part (b).

- (d) When the person has reached the point $(0, 12)$, how far has the weight moved?

In Exercises 97–100, apply the Extended Mean Value Theorem to the functions f and g on the given interval. Find all values c in the interval (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Functions

97. $f(x) = x^3$, $g(x) = x^2 + 1$

[0, 1]

98. $f(x) = \frac{1}{x}$, $g(x) = x^2 - 4$

[1, 2]

99. $f(x) = \sin x$, $g(x) = \cos x$

$\left[0, \frac{\pi}{2}\right]$

100. $f(x) = \ln x$, $g(x) = x^3$

[1, 4]

True or False? In Exercises 101–104, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

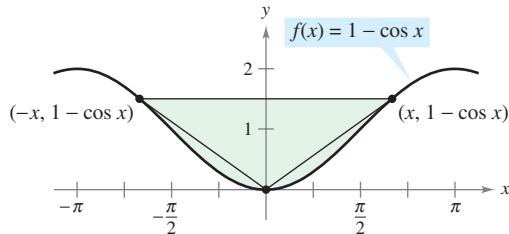
101. $\lim_{x \rightarrow 0} \left[\frac{x^2 + x + 1}{x} \right] = \lim_{x \rightarrow 0} \left[\frac{2x + 1}{1} \right] = 1$

102. If $y = e^x/x^2$, then $y' = e^x/2x$.

103. If $p(x)$ is a polynomial, then $\lim_{x \rightarrow \infty} [p(x)/e^x] = 0$.

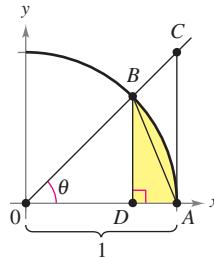
104. If $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$, then $\lim_{x \rightarrow \infty} [f(x) - g(x)] = 0$.

- 105. Area** Find the limit, as x approaches 0, of the ratio of the area of the triangle to the total shaded area in the figure.



106. In Section 2.3, a geometric argument (see figure) was used to prove that

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$



- (a) Write the area of $\triangle ABD$ in terms of θ .

- (b) Write the area of the shaded region in terms of θ .

- (c) Write the ratio R of the area of $\triangle ABD$ to that of the shaded region.

- (d) Find $\lim_{\theta \rightarrow 0} R$.

Continuous Functions In Exercises 107 and 108, find the value of c that makes the function continuous at $x = 0$.

107. $f(x) = \begin{cases} \frac{4x - 2 \sin 2x}{2x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$

108. $f(x) = \begin{cases} (e^x + x)^{1/x}, & x \neq 0 \\ c, & x = 0 \end{cases}$

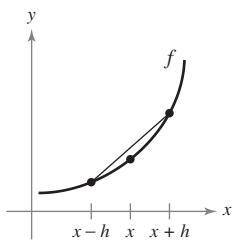
109. Find the values of a and b such that $\lim_{x \rightarrow 0} \frac{a - \cos bx}{x^2} = 2$.

110. Show that $\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0$ for any integer $n > 0$.

111. (a) Let $f'(x)$ be continuous. Show that

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x).$$

(b) Explain the result of part (a) graphically.



112. Let $f''(x)$ be continuous. Show that

$$\lim_{h \rightarrow 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x).$$

113. Sketch the graph of

$$g(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and determine $g'(0)$.

114. Use a graphing utility to graph

$$f(x) = \frac{x^k - 1}{k}$$

for $k = 1, 0.1$, and 0.01 . Then evaluate the limit

$$\lim_{x \rightarrow 0^+} \frac{x^k - 1}{k}.$$

115. Consider the limit $\lim_{x \rightarrow 0^+} (-x \ln x)$.

(a) Describe the type of indeterminate form that is obtained by direct substitution.

(b) Evaluate the limit. Use a graphing utility to verify the result.

FOR FURTHER INFORMATION For a geometric approach to this exercise, see the article "A Geometric Proof of $\lim_{d \rightarrow 0^+} (-d \ln d) = 0$ " by John H. Mathews in the *College Mathematics Journal*. To view this article, go to the website www.matharticles.com.

116. Prove that if $f(x) \geq 0$, $\lim_{x \rightarrow a} f(x) = 0$, and $\lim_{x \rightarrow a} g(x) = \infty$, then

$$\lim_{x \rightarrow a} f(x)^{g(x)} = 0.$$

117. Prove that if $f(x) \geq 0$, $\lim_{x \rightarrow a} f(x) = 0$, and $\lim_{x \rightarrow a} g(x) = -\infty$, then $\lim_{x \rightarrow a} f(x)^{g(x)} = \infty$.

118. Prove the following generalization of the Mean Value Theorem. If f is twice differentiable on the closed interval $[a, b]$, then

$$f(b) - f(a) = f'(a)(b-a) - \int_a^b f''(t)(t-b) dt.$$

119. **Indeterminate Forms** Show that the indeterminate forms 0^0 , ∞^0 , and 1^∞ do not always have a value of 1 by evaluating each limit.

(a) $\lim_{x \rightarrow 0^+} x^{\ln 2/(1+\ln x)}$

(b) $\lim_{x \rightarrow \infty} x^{\ln 2/(1+\ln x)}$

(c) $\lim_{x \rightarrow 0} (x+1)^{(\ln 2)/x}$

120. **Calculus History** In L'Hôpital's 1696 calculus textbook, he illustrated his rule using the limit of the function

$$f(x) = \frac{\sqrt[3]{2a^3 x - x^4} - a \sqrt[3]{a^2 x}}{a - \sqrt[4]{ax^3}}$$

as x approaches a , $a > 0$. Find this limit.

121. Consider the function

$$h(x) = \frac{x + \sin x}{x}.$$

(a) Use a graphing utility to graph the function. Then use the *zoom* and *trace* features to investigate $\lim_{x \rightarrow \infty} h(x)$.

(b) Find $\lim_{x \rightarrow \infty} h(x)$ analytically by writing

$$h(x) = \frac{x}{x} + \frac{\sin x}{x}.$$

(c) Can you use L'Hôpital's Rule to find $\lim_{x \rightarrow \infty} h(x)$? Explain your reasoning.

122. Let $f(x) = x + x \sin x$ and $g(x) = x^2 - 4$.

(a) Show that $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$.

(b) Show that $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow \infty} g(x) = \infty$.

(c) Evaluate the limit $\lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$. What do you notice?

(d) Do your answers to parts (a) through (c) contradict L'Hôpital's Rule? Explain your reasoning.

PUTNAM EXAM CHALLENGE

123. Evaluate $\lim_{x \rightarrow \infty} \left[\frac{1}{x} \cdot \frac{a^x - 1}{a - 1} \right]^{1/x}$ where $a > 0$, $a \neq 1$.

This problem was composed by the Committee on the Putnam Prize Competition.
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8.8 Improper Integrals

- Evaluate an improper integral that has an infinite limit of integration.
- Evaluate an improper integral that has an infinite discontinuity.

Improper Integrals with Infinite Limits of Integration

The definition of a definite integral

$$\int_a^b f(x) dx$$

requires that the interval $[a, b]$ be finite. Furthermore, the Fundamental Theorem of Calculus, by which you have been evaluating definite integrals, requires that f be continuous on $[a, b]$. In this section you will study a procedure for evaluating integrals that do not satisfy these requirements—usually because either one or both of the limits of integration are infinite, or f has a finite number of infinite discontinuities in the interval $[a, b]$. Integrals that possess either property are **improper integrals**. Note that a function f is said to have an **infinite discontinuity** at c if, from the right or left,

$$\lim_{x \rightarrow c} f(x) = \infty \quad \text{or} \quad \lim_{x \rightarrow c} f(x) = -\infty.$$

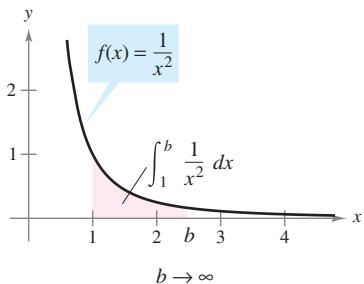
To get an idea of how to evaluate an improper integral, consider the integral

$$\int_1^b \frac{dx}{x^2} = -\frac{1}{x} \Big|_1^b = -\frac{1}{b} + 1 = 1 - \frac{1}{b}$$

which can be interpreted as the area of the shaded region shown in Figure 8.17. Taking the limit as $b \rightarrow \infty$ produces

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b \rightarrow \infty} \left(\int_1^b \frac{dx}{x^2} \right) = \lim_{b \rightarrow \infty} \left(1 - \frac{1}{b} \right) = 1.$$

This improper integral can be interpreted as the area of the *unbounded* region between the graph of $f(x) = 1/x^2$ and the x -axis (to the right of $x = 1$).



The unbounded region has an area of 1.

Figure 8.17

DEFINITION OF IMPROPER INTEGRALS WITH INFINITE INTEGRATION LIMITS

1. If f is continuous on the interval $[a, \infty)$, then

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$$

2. If f is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx.$$

3. If f is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^\infty f(x) dx$$

where c is any real number (see Exercise 120).

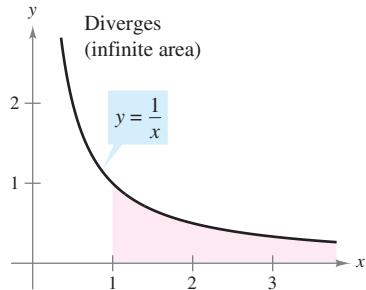
In the first two cases, the improper integral **converges** if the limit exists—otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

EXAMPLE 1 An Improper Integral That Diverges

Evaluate $\int_1^\infty \frac{dx}{x}$.

Solution

$$\begin{aligned}\int_1^\infty \frac{dx}{x} &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x} && \text{Take limit as } b \rightarrow \infty. \\ &= \lim_{b \rightarrow \infty} \left[\ln x \right]_1^b && \text{Apply Log Rule.} \\ &= \lim_{b \rightarrow \infty} (\ln b - 0) && \text{Apply Fundamental Theorem of Calculus.} \\ &= \infty && \text{Evaluate limit.}\end{aligned}$$



This unbounded region has an infinite area.

Figure 8.18

See Figure 8.18. ■

NOTE Try comparing the regions shown in Figures 8.17 and 8.18. They look similar, yet the region in Figure 8.17 has a finite area of 1 and the region in Figure 8.18 has an infinite area. ■

EXAMPLE 2 Improper Integrals That Converge

Evaluate each improper integral.

a. $\int_0^\infty e^{-x} dx$

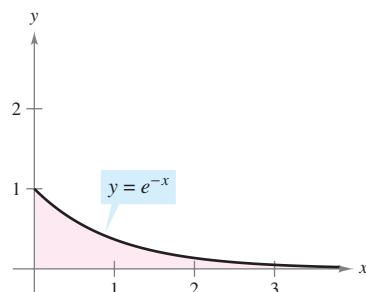
b. $\int_0^\infty \frac{1}{x^2 + 1} dx$

Solution

$$\begin{aligned}\text{a. } \int_0^\infty e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx \\ &= \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_0^b \\ &= \lim_{b \rightarrow \infty} (-e^{-b} + 1) \\ &= 1\end{aligned}$$

$$\begin{aligned}\text{b. } \int_0^\infty \frac{1}{x^2 + 1} dx &= \lim_{b \rightarrow \infty} \int_0^b \frac{1}{x^2 + 1} dx \\ &= \lim_{b \rightarrow \infty} \left[\arctan x \right]_0^b \\ &= \lim_{b \rightarrow \infty} \arctan b \\ &= \frac{\pi}{2}\end{aligned}$$

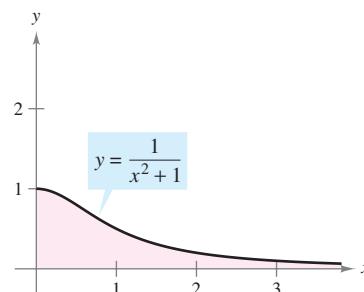
See Figure 8.19.



The area of the unbounded region is 1.

Figure 8.19

See Figure 8.20.



The area of the unbounded region is π/2.

Figure 8.20

In the following example, note how L'Hôpital's Rule can be used to evaluate an improper integral.

EXAMPLE 3 Using L'Hôpital's Rule with an Improper Integral

Evaluate $\int_1^\infty (1-x)e^{-x} dx$.

Solution Use integration by parts, with $dv = e^{-x} dx$ and $u = (1-x)$.

$$\begin{aligned}\int (1-x)e^{-x} dx &= -e^{-x}(1-x) - \int e^{-x} dx \\ &= -e^{-x} + xe^{-x} + e^{-x} + C \\ &= xe^{-x} + C\end{aligned}$$

Now, apply the definition of an improper integral.

$$\begin{aligned}\int_1^\infty (1-x)e^{-x} dx &= \lim_{b \rightarrow \infty} \left[xe^{-x} \right]_1^b \\ &= \left(\lim_{b \rightarrow \infty} \frac{b}{e^b} \right) - \frac{1}{e}\end{aligned}$$

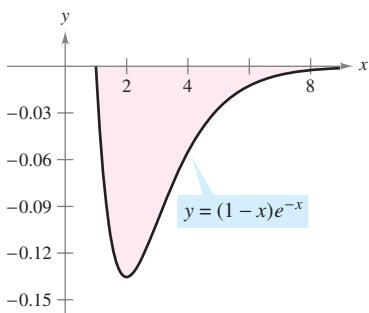
Finally, using L'Hôpital's Rule on the right-hand limit produces

$$\lim_{b \rightarrow \infty} \frac{b}{e^b} = \lim_{b \rightarrow \infty} \frac{1}{e^b} = 0$$

from which you can conclude that

$$\int_1^\infty (1-x)e^{-x} dx = -\frac{1}{e}.$$

See Figure 8.21.



The area of the unbounded region is $| -1/e |$.

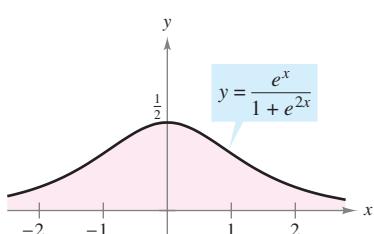
Figure 8.21

EXAMPLE 4 Infinite Upper and Lower Limits of Integration

Evaluate $\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}} dx$.

Solution Note that the integrand is continuous on $(-\infty, \infty)$. To evaluate the integral, you can break it into two parts, choosing $c = 0$ as a convenient value.

$$\begin{aligned}\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}} dx &= \int_{-\infty}^0 \frac{e^x}{1+e^{2x}} dx + \int_0^\infty \frac{e^x}{1+e^{2x}} dx \\ &= \lim_{b \rightarrow -\infty} \left[\arctan e^x \right]_b^0 + \lim_{b \rightarrow \infty} \left[\arctan e^x \right]_0^b \\ &= \lim_{b \rightarrow -\infty} \left(\frac{\pi}{4} - \arctan e^b \right) + \lim_{b \rightarrow \infty} \left(\arctan e^b - \frac{\pi}{4} \right) \\ &= \frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{4} \\ &= \frac{\pi}{2}\end{aligned}$$



The area of the unbounded region is $\pi/2$.

Figure 8.22

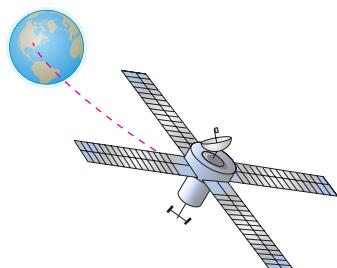
See Figure 8.22.

EXAMPLE 5 Sending a Space Module into Orbit

In Example 3 in Section 7.5, you found that it would require 10,000 mile-tones of work to propel a 15-metric-ton space module to a height of 800 miles above Earth. How much work is required to propel the module an unlimited distance away from Earth's surface?

Solution At first you might think that an infinite amount of work would be required. But if this were the case, it would be impossible to send rockets into outer space. Because this has been done, the work required must be finite. You can determine the work in the following manner. Using the integral in Example 3, Section 7.5, replace the upper bound of 4800 miles by ∞ and write

$$\begin{aligned} W &= \int_{4000}^{\infty} \frac{240,000,000}{x^2} dx \\ &= \lim_{b \rightarrow \infty} \left[-\frac{240,000,000}{x} \right]_{4000}^b \\ &= \lim_{b \rightarrow \infty} \left(-\frac{240,000,000}{b} + \frac{240,000,000}{4000} \right) \\ &= 60,000 \text{ mile-tones} \\ &\approx 6.984 \times 10^{11} \text{ foot-pounds.} \end{aligned}$$



The work required to move a space module an unlimited distance away from Earth is approximately 6.984×10^{11} foot-pounds.

Figure 8.23

See Figure 8.23. ■

Improper Integrals with Infinite Discontinuities

The second basic type of improper integral is one that has an infinite discontinuity at or between the limits of integration.

DEFINITION OF IMPROPER INTEGRALS WITH INFINITE DISCONTINUITIES

- If f is continuous on the interval $[a, b)$ and has an infinite discontinuity at b , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx.$$

- If f is continuous on the interval $(a, b]$ and has an infinite discontinuity at a , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx.$$

- If f is continuous on the interval $[a, b]$, except for some c in (a, b) at which f has an infinite discontinuity, then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

In the first two cases, the improper integral **converges** if the limit exists—otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

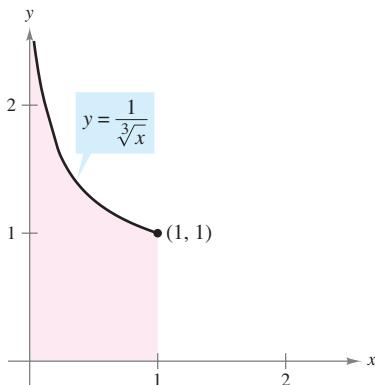


Figure 8.24

EXAMPLE 6 An Improper Integral with an Infinite Discontinuity

Evaluate $\int_0^1 \frac{dx}{\sqrt[3]{x}}$.

Solution The integrand has an infinite discontinuity at $x = 0$, as shown in Figure 8.24. You can evaluate this integral as shown below.

$$\begin{aligned}\int_0^1 x^{-1/3} dx &= \lim_{b \rightarrow 0^+} \left[\frac{x^{2/3}}{2/3} \right]_b^1 \\ &= \lim_{b \rightarrow 0^+} \frac{3}{2} (1 - b^{2/3}) \\ &= \frac{3}{2}\end{aligned}$$

EXAMPLE 7 An Improper Integral That Diverges

Evaluate $\int_0^2 \frac{dx}{x^3}$.

Solution Because the integrand has an infinite discontinuity at $x = 0$, you can write

$$\begin{aligned}\int_0^2 \frac{dx}{x^3} &= \lim_{b \rightarrow 0^+} \left[-\frac{1}{2x^2} \right]_b^2 \\ &= \lim_{b \rightarrow 0^+} \left(-\frac{1}{8} + \frac{1}{2b^2} \right) \\ &= \infty.\end{aligned}$$

So, you can conclude that the improper integral diverges.

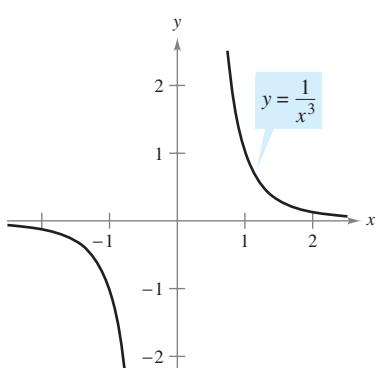
EXAMPLE 8 An Improper Integral with an Interior Discontinuity

Evaluate $\int_{-1}^2 \frac{dx}{x^3}$.

Solution This integral is improper because the integrand has an infinite discontinuity at the interior point $x = 0$, as shown in Figure 8.25. So, you can write

$$\int_{-1}^2 \frac{dx}{x^3} = \int_{-1}^0 \frac{dx}{x^3} + \int_0^2 \frac{dx}{x^3}.$$

From Example 7 you know that the second integral diverges. So, the original improper integral also diverges. ■



The improper integral $\int_{-1}^2 1/x^3 dx$ diverges.

Figure 8.25

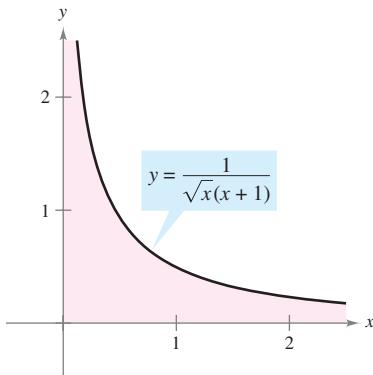
NOTE Remember to check for infinite discontinuities at interior points as well as at endpoints when determining whether an integral is improper. For instance, if you had not recognized that the integral in Example 8 was improper, you would have obtained the *incorrect* result

$$\int_{-1}^2 \frac{dx}{x^3} \stackrel{\text{Incorrect}}{=} \left[-\frac{1}{2x^2} \right]_{-1}^2 = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}.$$

Incorrect evaluation ■

The integral in the next example is improper for *two* reasons. One limit of integration is infinite, and the integrand has an infinite discontinuity at the outer limit of integration.

EXAMPLE 9 A Doubly Improper Integral



The area of the unbounded region is π .

Figure 8.26

Solution To evaluate this integral, split it at a convenient point (say, $x = 1$) and write

$$\begin{aligned}\int_0^\infty \frac{dx}{\sqrt{x(x+1)}} &= \int_0^1 \frac{dx}{\sqrt{x(x+1)}} + \int_1^\infty \frac{dx}{\sqrt{x(x+1)}} \\ &= \lim_{b \rightarrow 0^+} \left[2 \arctan \sqrt{x} \right]_b^1 + \lim_{c \rightarrow \infty} \left[2 \arctan \sqrt{x} \right]_1^c \\ &= 2\left(\frac{\pi}{4}\right) - 0 + 2\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{4}\right) \\ &= \pi.\end{aligned}$$

See Figure 8.26.

EXAMPLE 10 An Application Involving Arc Length

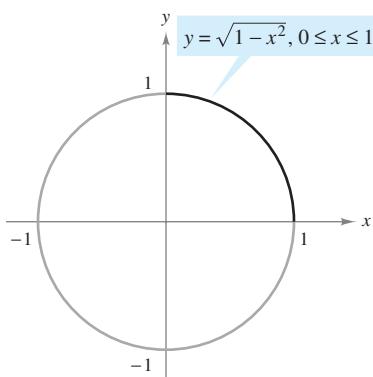
Use the formula for arc length to show that the circumference of the circle $x^2 + y^2 = 1$ is 2π .

Solution To simplify the work, consider the quarter circle given by $y = \sqrt{1 - x^2}$, where $0 \leq x \leq 1$. The function y is differentiable for any x in this interval except $x = 1$. Therefore, the arc length of the quarter circle is given by the improper integral

$$\begin{aligned}s &= \int_0^1 \sqrt{1 + (y')^2} dx \\ &= \int_0^1 \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2} dx \\ &= \int_0^1 \frac{dx}{\sqrt{1-x^2}}.\end{aligned}$$

This integral is improper because it has an infinite discontinuity at $x = 1$. So, you can write

$$\begin{aligned}s &= \int_0^1 \frac{dx}{\sqrt{1-x^2}} \\ &= \lim_{b \rightarrow 1^-} \left[\arcsin x \right]_0^b \\ &= \frac{\pi}{2} - 0 \\ &= \frac{\pi}{2}.\end{aligned}$$



The circumference of the circle is 2π .

Figure 8.27

Finally, multiplying by 4, you can conclude that the circumference of the circle is $4s = 2\pi$, as shown in Figure 8.27. ■

This section concludes with a useful theorem describing the convergence or divergence of a common type of improper integral. The proof of this theorem is left as an exercise (see Exercise 55).

THEOREM 8.5 A SPECIAL TYPE OF IMPROPER INTEGRAL

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1 \\ \text{diverges,} & \text{if } p \leq 1 \end{cases}$$

EXAMPLE 11 An Application Involving A Solid of Revolution

FOR FURTHER INFORMATION For further investigation of solids that have finite volumes and infinite surface areas, see the article “Supersolids: Solids Having Finite Volume and Infinite Surfaces” by William P. Love in *Mathematics Teacher*. To view this article, go to the website www.matharticles.com.

The solid formed by revolving (about the x -axis) the *unbounded* region lying between the graph of $f(x) = 1/x$ and the x -axis ($x \geq 1$) is called **Gabriel's Horn**. (See Figure 8.28.) Show that this solid has a finite volume and an infinite surface area.

Solution Using the disk method and Theorem 8.5, you can determine the volume to be

$$\begin{aligned} V &= \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx && \text{Theorem 8.5, } p = 2 > 1 \\ &= \pi \left(\frac{1}{2-1}\right) = \pi. \end{aligned}$$

The surface area is given by

$$S = 2\pi \int_1^\infty f(x) \sqrt{1 + [f'(x)]^2} dx = 2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx.$$

Because

$$\sqrt{1 + \frac{1}{x^4}} > 1$$

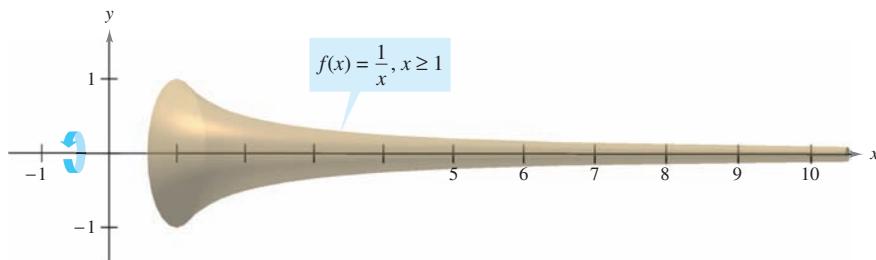
on the interval $[1, \infty)$, and the improper integral

$$\int_1^\infty \frac{1}{x} dx$$

diverges, you can conclude that the improper integral

$$\int_1^\infty \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx$$

also diverges. (See Exercise 58.) So, the surface area is infinite.



Gabriel's Horn has a finite volume and an infinite surface area.

Figure 8.28

8.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, decide whether the integral is improper. Explain your reasoning.

1. $\int_0^1 \frac{dx}{5x - 3}$

3. $\int_0^1 \frac{2x - 5}{x^2 - 5x + 6} dx$

5. $\int_0^2 e^{-x} dx$

7. $\int_{-\infty}^{\infty} \frac{\sin x}{4 + x^2} dx$

2. $\int_1^2 \frac{dx}{x^3}$

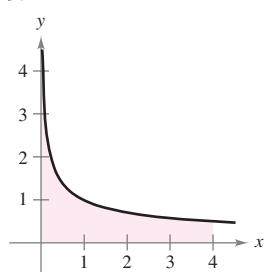
4. $\int_1^{\infty} \ln(x^2) dx$

6. $\int_0^{\infty} \cos x dx$

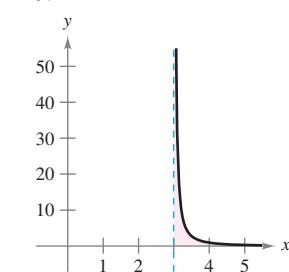
8. $\int_0^{\pi/4} \csc x dx$

In Exercises 9–14, explain why the integral is improper and determine whether it diverges or converges. Evaluate the integral if it converges.

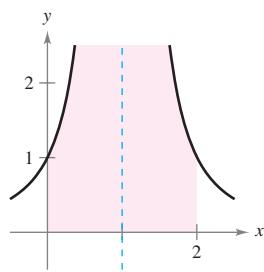
9. $\int_0^4 \frac{1}{\sqrt{x}} dx$



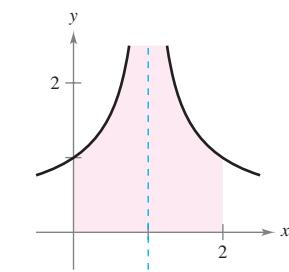
10. $\int_3^4 \frac{1}{(x-3)^{3/2}} dx$



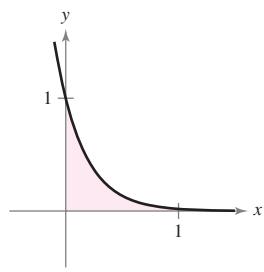
11. $\int_0^2 \frac{1}{(x-1)^2} dx$



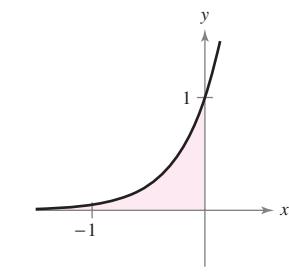
12. $\int_0^2 \frac{1}{(x-1)^{2/3}} dx$



13. $\int_0^{\infty} e^{-4x} dx$



14. $\int_{-\infty}^0 e^{3x} dx$



Writing In Exercises 15–18, explain why the evaluation of the integral is *incorrect*. Use the integration capabilities of a graphing utility to attempt to evaluate the integral. Determine whether the utility gives the correct answer.

15. $\int_{-1}^1 \frac{1}{x^2} dx = -2$

17. $\int_0^{\infty} e^{-x} dx = 0$

16. $\int_{-2}^2 \frac{-2}{(x-1)^3} dx = \frac{8}{9}$

18. $\int_0^{\pi} \sec x dx = 0$

In Exercises 19–36, determine whether the improper integral diverges or converges. Evaluate the integral if it converges.

19. $\int_1^{\infty} \frac{1}{x^3} dx$

20. $\int_1^{\infty} \frac{3}{x^5} dx$

21. $\int_1^{\infty} \frac{3}{\sqrt[3]{x}} dx$

22. $\int_1^{\infty} \frac{4}{\sqrt[4]{x}} dx$

23. $\int_{-\infty}^0 xe^{-4x} dx$

24. $\int_0^{\infty} xe^{-x/4} dx$

25. $\int_0^{\infty} x^2 e^{-x} dx$

26. $\int_0^{\infty} (x-1)e^{-x} dx$

27. $\int_0^{\infty} e^{-x} \cos x dx$

28. $\int_0^{\infty} e^{-ax} \sin bx dx, a > 0$

29. $\int_4^{\infty} \frac{1}{x(\ln x)^3} dx$

30. $\int_1^{\infty} \frac{\ln x}{x} dx$

31. $\int_{-\infty}^{\infty} \frac{4}{16+x^2} dx$

32. $\int_0^{\infty} \frac{x^3}{(x^2+1)^2} dx$

33. $\int_0^{\infty} \frac{1}{e^x + e^{-x}} dx$

34. $\int_0^{\infty} \frac{e^x}{1+e^x} dx$

35. $\int_0^{\infty} \cos \pi x dx$

36. $\int_0^{\infty} \sin \frac{x}{2} dx$

In Exercises 37–54, determine whether the improper integral diverges or converges. Evaluate the integral if it converges, and check your results with the results obtained by using the integration capabilities of a graphing utility.

37. $\int_0^1 \frac{1}{x^2} dx$

38. $\int_0^5 \frac{10}{x} dx$

39. $\int_0^8 \frac{1}{\sqrt[3]{8-x}} dx$

40. $\int_0^{12} \frac{9}{\sqrt{12-x}} dx$

41. $\int_0^1 x \ln x dx$

42. $\int_0^e \ln x^2 dx$

43. $\int_0^{\pi/2} \tan \theta d\theta$

44. $\int_0^{\pi/2} \sec \theta d\theta$

45. $\int_2^4 \frac{2}{x\sqrt{x^2-4}} dx$

46. $\int_0^5 \frac{1}{\sqrt{25-x^2}} dx$

47. $\int_2^4 \frac{1}{\sqrt{x^2-4}} dx$

48. $\int_0^5 \frac{1}{25-x^2} dx$

49. $\int_0^2 \frac{1}{\sqrt[3]{x-1}} dx$

50. $\int_1^3 \frac{2}{(x-2)^{8/3}} dx$

51. $\int_3^\infty \frac{1}{x\sqrt{x^2 - 9}} dx$

53. $\int_0^\infty \frac{4}{\sqrt{x}(x+6)} dx$

52. $\int_5^\infty \frac{1}{x\sqrt{x^2 - 25}} dx$

54. $\int_1^\infty \frac{1}{x \ln x} dx$

In Exercises 55 and 56, determine all values of p for which the improper integral converges.

55. $\int_1^\infty \frac{1}{x^p} dx$

56. $\int_0^1 \frac{1}{x^p} dx$

57. Use mathematical induction to verify that the following integral converges for any positive integer n .

$$\int_0^\infty x^n e^{-x} dx$$

58. Comparison Test for Improper Integrals In some cases, it is impossible to find the exact value of an improper integral but it is important to determine whether the integral converges or diverges. Suppose the functions f and g are continuous and $0 \leq g(x) \leq f(x)$ on the interval $[a, \infty)$. It can be shown that if $\int_a^\infty f(x) dx$ converges, then $\int_a^\infty g(x) dx$ also converges, and if $\int_a^\infty g(x) dx$ diverges, then $\int_a^\infty f(x) dx$ also diverges. This is known as the Comparison Test for improper integrals.

(a) Use the Comparison Test to determine if $\int_1^\infty e^{-x^2} dx$ converges or diverges. (Hint: Use the fact that $e^{-x^2} \leq e^{-x}$ for $x \geq 1$.)

(b) Use the Comparison Test to determine if $\int_1^\infty \frac{1}{x^5 + 1} dx$ converges or diverges. (Hint: Use the fact that $\frac{1}{x^5 + 1} \leq \frac{1}{x^5}$ for $x \geq 1$.)

In Exercises 59–70, use the results of Exercises 55–58 to determine whether the improper integral converges or diverges.

59. $\int_0^1 \frac{1}{x^5} dx$

61. $\int_1^\infty \frac{1}{x^5} dx$

63. $\int_1^\infty \frac{1}{x^2 + 5} dx$

65. $\int_2^\infty \frac{1}{\sqrt[3]{x(x-1)}} dx$

67. $\int_1^\infty \frac{2 + e^{-x}}{x} dx$

69. $\int_1^\infty \frac{1 - \sin x}{x^2} dx$

60. $\int_0^1 \frac{1}{\sqrt[5]{x}} dx$

62. $\int_0^\infty x^4 e^{-x} dx$

64. $\int_2^\infty \frac{1}{\sqrt{x-1}} dx$

66. $\int_1^\infty \frac{1}{\sqrt{x(x+1)}} dx$

68. $\int_0^\infty \frac{1}{e^x + x} dx$

70. $\int_2^\infty \frac{1}{\sqrt{x} \ln x} dx$

WRITING ABOUT CONCEPTS

71. Describe the different types of improper integrals.
72. Define the terms *converges* and *diverges* when working with improper integrals.

WRITING ABOUT CONCEPTS (continued)

73. Explain why $\int_{-1}^1 \frac{1}{x^3} dx \neq 0$.

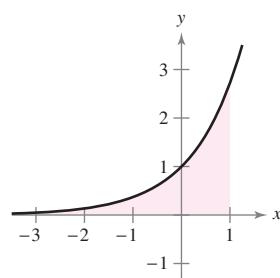
74. Consider the integral

$$\int_0^3 \frac{10}{x^2 - 2x} dx.$$

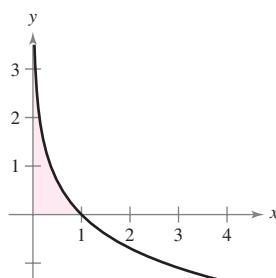
To determine the convergence or divergence of the integral, how many improper integrals must be analyzed? What must be true of each of these integrals if the given integral converges?

Area In Exercises 75–78, find the area of the unbounded shaded region.

75. $y = e^x, -\infty < x \leq 1$

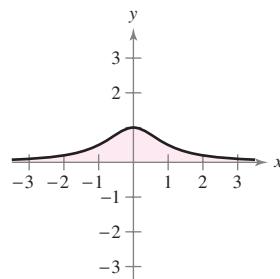


76. $y = -\ln x$



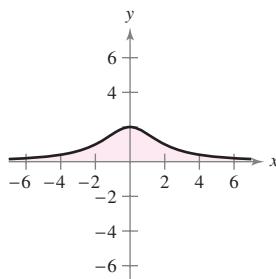
77. Witch of Agnesi:

$$y = \frac{1}{x^2 + 1}$$



78. Witch of Agnesi:

$$y = \frac{8}{x^2 + 4}$$



Area and Volume In Exercises 79 and 80, consider the region satisfying the inequalities. (a) Find the area of the region. (b) Find the volume of the solid generated by revolving the region about the x -axis. (c) Find the volume of the solid generated by revolving the region about the y -axis.

79. $y \leq e^{-x}, y \geq 0, x \geq 0$

80. $y \leq \frac{1}{x^2}, y \geq 0, x \geq 1$

81. **Arc Length** Sketch the graph of the hypocycloid of four cusps $x^{2/3} + y^{2/3} = 4$ and find its perimeter.

82. **Arc Length** Find the arc length of the graph of $y = \sqrt{16 - x^2}$ over the interval $[0, 4]$.

83. **Surface Area** The region bounded by $(x-2)^2 + y^2 = 1$ is revolved about the y -axis to form a torus. Find the surface area of the torus.

- 84. Surface Area** Find the area of the surface formed by revolving the graph of $y = 2e^{-x}$ on the interval $[0, \infty)$ about the x -axis.

Propulsion In Exercises 85 and 86, use the weight of the rocket to answer each question. (Use 4000 miles as the radius of Earth and do not consider the effect of air resistance.)

- (a) How much work is required to propel the rocket an unlimited distance away from Earth's surface?
 (b) How far has the rocket traveled when half the total work has occurred?

85. 5-ton rocket

86. 10-ton rocket

Probability A nonnegative function f is called a *probability density function* if

$$\int_{-\infty}^{\infty} f(t) dt = 1.$$

The probability that x lies between a and b is given by

$$P(a \leq x \leq b) = \int_a^b f(t) dt.$$

The expected value of x is given by

$$E(x) = \int_{-\infty}^{\infty} tf(t) dt.$$

In Exercises 87 and 88, (a) show that the nonnegative function is a probability density function, (b) find $P(0 \leq x \leq 4)$, and (c) find $E(x)$.

$$87. f(t) = \begin{cases} \frac{1}{7}e^{-t/7}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$88. f(t) = \begin{cases} \frac{2}{5}e^{-2t/5}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Capitalized Cost In Exercises 89 and 90, find the capitalized cost C of an asset (a) for $n = 5$ years, (b) for $n = 10$ years, and (c) forever. The capitalized cost is given by

$$C = C_0 + \int_0^n c(t)e^{-rt} dt$$

where C_0 is the original investment, t is the time in years, r is the annual interest rate compounded continuously, and $c(t)$ is the annual cost of maintenance.

$$89. C_0 = \$650,000$$

$$c(t) = \$25,000$$

$$r = 0.06$$

$$90. C_0 = \$650,000$$

$$c(t) = \$25,000(1 + 0.08t)$$

$$r = 0.06$$

91. Electromagnetic Theory The magnetic potential P at a point on the axis of a circular coil is given by

$$P = \frac{2\pi NIr}{k} \int_c^{\infty} \frac{1}{(r^2 + x^2)^{3/2}} dx$$

where N, I, r, k , and c are constants. Find P .

- 92. Gravitational Force** A "semi-infinite" uniform rod occupies the nonnegative x -axis. The rod has a linear density δ which means that a segment of length dx has a mass of δdx . A particle of mass M is located at the point $(-a, 0)$. The gravitational force F that the rod exerts on the mass is given by $F = \int_0^{\infty} \frac{GM\delta}{(a+x)^2} dx$, where G is the gravitational constant. Find F .

True or False? In Exercises 93–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

93. If f is continuous on $[0, \infty)$ and $\lim_{x \rightarrow \infty} f(x) = 0$, then $\int_0^{\infty} f(x) dx$ converges.

94. If f is continuous on $[0, \infty)$ and $\int_0^{\infty} f(x) dx$ diverges, then $\lim_{x \rightarrow \infty} f(x) \neq 0$.

95. If f' is continuous on $[0, \infty)$ and $\lim_{x \rightarrow \infty} f(x) = 0$, then $\int_0^{\infty} f'(x) dx = -f(0)$.

96. If the graph of f is symmetric with respect to the origin or the y -axis, then $\int_0^{\infty} f(x) dx$ converges if and only if $\int_{-\infty}^{\infty} f(x) dx$ converges.

97. (a) Show that $\int_{-\infty}^{\infty} \sin x dx$ diverges.

- (b) Show that $\lim_{a \rightarrow \infty} \int_{-a}^a \sin x dx = 0$.

- (c) What do parts (a) and (b) show about the definition of improper integrals?

CAPSTONE

98. For each integral, find a nonnegative real number b that makes the integral improper. Explain your reasoning.

(a) $\int_0^b \frac{1}{x^2 - 9} dx$	(b) $\int_0^b \frac{1}{\sqrt{4-x}} dx$
(c) $\int_0^b \frac{x}{x^2 - 7x + 12} dx$	(d) $\int_b^{10} \ln x dx$
(e) $\int_0^b \tan 2x dx$	(f) $\int_0^b \frac{\cos x}{1 - \sin x} dx$

99. **Writing**

- (a) The improper integrals

$$\int_1^{\infty} \frac{1}{x} dx \quad \text{and} \quad \int_1^{\infty} \frac{1}{x^2} dx$$

diverge and converge, respectively. Describe the essential differences between the integrands that cause one integral to converge and the other to diverge.

- (b) Sketch a graph of the function $y = \sin x/x$ over the interval $(1, \infty)$. Use your knowledge of the definite integral to make an inference as to whether the integral

$$\int_1^{\infty} \frac{\sin x}{x} dx$$

converges. Give reasons for your answer.

- (c) Use one iteration of integration by parts on the integral in part (b) to determine its divergence or convergence.

100. Exploration Consider the integral

$$\int_0^{\pi/2} \frac{4}{1 + (\tan x)^n} dx$$

where n is a positive integer.

- (a) Is the integral improper? Explain.
- (b) Use a graphing utility to graph the integrand for $n = 2, 4, 8$, and 12 .
- (c) Use the graphs to approximate the integral as $n \rightarrow \infty$.
- CAS** (d) Use a computer algebra system to evaluate the integral for the values of n in part (b). Make a conjecture about the value of the integral for any positive integer n . Compare your results with your answer in part (c).

101. The Gamma Function The Gamma Function $\Gamma(n)$ is defined by

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx, \quad n > 0.$$

- (a) Find $\Gamma(1)$, $\Gamma(2)$, and $\Gamma(3)$.
 - (b) Use integration by parts to show that $\Gamma(n+1) = n\Gamma(n)$.
 - (c) Write $\Gamma(n)$ using factorial notation where n is a positive integer.
- 102.** Prove that $I_n = \binom{n-1}{n+2} I_{n-1}$, where

$$I_n = \int_0^\infty \frac{x^{2n-1}}{(x^2 + 1)^{n+3}} dx, \quad n \geq 1.$$

Then evaluate each integral.

- (a) $\int_0^\infty \frac{x}{(x^2 + 1)^4} dx$
- (b) $\int_0^\infty \frac{x^3}{(x^2 + 1)^5} dx$
- (c) $\int_0^\infty \frac{x^5}{(x^2 + 1)^6} dx$

Laplace Transforms Let $f(t)$ be a function defined for all positive values of t . The Laplace Transform of $f(t)$ is defined by

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

if the improper integral exists. Laplace Transforms are used to solve differential equations. In Exercises 103–110, find the Laplace Transform of the function.

- | | |
|-------------------------------|-------------------------------|
| 103. $f(t) = 1$ | 104. $f(t) = t$ |
| 105. $f(t) = t^2$ | 106. $f(t) = e^{at}$ |
| 107. $f(t) = \cos at$ | 108. $f(t) = \sin at$ |
| 109. $f(t) = \cosh at$ | 110. $f(t) = \sinh at$ |

111. Normal Probability The mean height of American men between 20 and 29 years old is 70 inches, and the standard deviation is 3 inches. A 20- to 29-year-old man is chosen at random from the population. The probability that he is 6 feet tall or taller is

$$P(72 \leq x < \infty) = \int_{72}^\infty \frac{1}{3\sqrt{2\pi}} e^{-(x-70)^2/18} dx.$$

(Source: National Center for Health Statistics)

- (a) Use a graphing utility to graph the integrand. Use the graphing utility to convince yourself that the area between the x -axis and the integrand is 1.
- (b) Use a graphing utility to approximate $P(72 \leq x < \infty)$.
- (c) Approximate $0.5 - P(70 \leq x \leq 72)$ using a graphing utility. Use the graph in part (a) to explain why this result is the same as the answer in part (b).

- 112.** (a) Sketch the semicircle $y = \sqrt{4 - x^2}$.

(b) Explain why

$$\int_{-2}^2 \frac{2 dx}{\sqrt{4 - x^2}} = \int_{-2}^2 \sqrt{4 - x^2} dx$$

without evaluating either integral.

- 113.** For what value of c does the integral

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2 + 1}} - \frac{c}{x+1} \right) dx$$

converge? Evaluate the integral for this value of c .

- 114.** For what value of c does the integral

$$\int_1^\infty \left(\frac{cx}{x^2 + 2} - \frac{1}{3x} \right) dx$$

converge? Evaluate the integral for this value of c .

- 115. Volume** Find the volume of the solid generated by revolving the region bounded by the graph of f about the x -axis.

$$f(x) = \begin{cases} x \ln x, & 0 < x \leq 2 \\ 0, & x = 0 \end{cases}$$

- 116. Volume** Find the volume of the solid generated by revolving the unbounded region lying between $y = -\ln x$ and the y -axis ($y \geq 0$) about the x -axis.

***u*-Substitution** In Exercises 117 and 118, rewrite the improper integral as a proper integral using the given u -substitution. Then use the Trapezoidal Rule with $n = 5$ to approximate the integral.

- 117.** $\int_0^1 \frac{\sin x}{\sqrt{x}} dx, \quad u = \sqrt{x}$

- 118.** $\int_0^1 \frac{\cos x}{\sqrt{1-x}} dx, \quad u = \sqrt{1-x}$

- 119.** (a) Use a graphing utility to graph the function $y = e^{-x^2}$.

- (b) Show that $\int_0^\infty e^{-x^2} dx = \int_0^1 \sqrt{-\ln y} dy$.

- 120.** Let $\int_{-\infty}^\infty f(x) dx$ be convergent and let a and b be real numbers where $a \neq b$. Show that

$$\int_{-\infty}^a f(x) dx + \int_a^\infty f(x) dx = \int_{-\infty}^b f(x) dx + \int_b^\infty f(x) dx.$$

8 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, use the basic integration rules to find or evaluate the integral.

1. $\int x\sqrt{x^2 - 36} dx$

2. $\int xe^{x^2-1} dx$

3. $\int \frac{x}{x^2 - 49} dx$

4. $\int \frac{x}{\sqrt{25 - x^2}} dx$

5. $\int_1^e \frac{\ln(2x)}{x} dx$

6. $\int_{3/2}^2 2x\sqrt{2x - 3} dx$

7. $\int \frac{100}{\sqrt{100 - x^2}} dx$

8. $\int \frac{x^4 + 2x^2 + x + 1}{(x^2 + 1)^2} dx$

In Exercises 9–18, use integration by parts to find the integral.

9. $\int xe^{3x} dx$

10. $\int x^3 e^x dx$

11. $\int e^{2x} \sin 3x dx$

12. $\int (x^2 - 3x)e^x dx$

13. $\int x\sqrt{x-1} dx$

14. $\int \arctan 2x dx$

15. $\int x^2 \sin 2x dx$

16. $\int \ln \sqrt{x^2 - 4} dx$

17. $\int x \arcsin 2x dx$

18. $\int e^x \arctan e^x dx$

In Exercises 19–24, find the trigonometric integral.

19. $\int \cos^3(\pi x - 1) dx$

20. $\int \sin^2 \frac{\pi x}{2} dx$

21. $\int \sec^4 \frac{x}{2} dx$

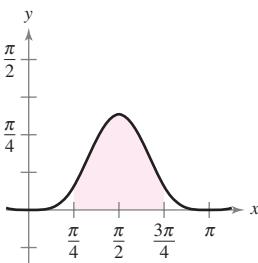
22. $\int \tan \theta \sec^4 \theta d\theta$

23. $\int \frac{1}{1 - \sin \theta} d\theta$

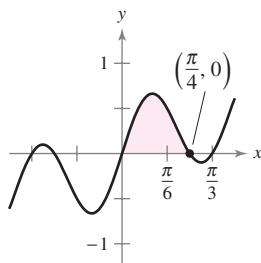
24. $\int \cos 2\theta (\sin \theta + \cos \theta)^2 d\theta$

Area In Exercises 25 and 26, find the area of the region.

25. $y = \sin^4 x$



26. $y = \sin 3x \cos 2x$



In Exercises 27–32, use trigonometric substitution to find or evaluate the integral.

27. $\int \frac{-12}{x^2\sqrt{4-x^2}} dx$

28. $\int \frac{\sqrt{x^2-9}}{x} dx, \quad x > 3$

29. $\int \frac{x^3}{\sqrt{4+x^2}} dx$

30. $\int \sqrt{25 - 9x^2} dx$

31. $\int_{-2}^0 \sqrt{4 - x^2} dx$

32. $\int_0^{\pi/2} \frac{\sin \theta}{1 + 2 \cos^2 \theta} d\theta$

In Exercises 33 and 34, find the integral using each method.

33. $\int \frac{x^3}{\sqrt{4+x^2}} dx$

(a) Trigonometric substitution

(b) Substitution: $u^2 = 4 + x^2$

(c) Integration by parts: $dv = (x/\sqrt{4+x^2}) dx$

34. $\int x\sqrt{4+x} dx$

(a) Trigonometric substitution

(b) Substitution: $u^2 = 4 + x$

(c) Substitution: $u = 4 + x$

(d) Integration by parts: $dv = \sqrt{4+x} dx$

In Exercises 35–40, use partial fractions to find the integral.

35. $\int \frac{x - 39}{x^2 - x - 12} dx$

36. $\int \frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x} dx$

37. $\int \frac{x^2 + 2x}{x^3 - x^2 + x - 1} dx$

38. $\int \frac{4x - 2}{3(x - 1)^2} dx$

39. $\int \frac{x^2}{x^2 + 5x - 24} dx$

40. $\int \frac{\sec^2 \theta}{\tan \theta (\tan \theta - 1)} d\theta$

In Exercises 41–48, use integration tables to find or evaluate the integral.

41. $\int \frac{x}{(4+5x)^2} dx$

42. $\int \frac{x}{\sqrt{4+5x}} dx$

43. $\int_0^{\sqrt{\pi}/2} \frac{x}{1 + \sin x^2} dx$

44. $\int_0^1 \frac{x}{1 + e^{x^2}} dx$

45. $\int \frac{x}{x^2 + 4x + 8} dx$

46. $\int \frac{3}{2x\sqrt{9x^2 - 1}} dx, \quad x > \frac{1}{3}$

47. $\int \frac{1}{\sin \pi x \cos \pi x} dx$

48. $\int \frac{1}{1 + \tan \pi x} dx$

49. Verify the reduction formula

$$\int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx.$$

50. Verify the reduction formula

$$\int \tan^n x dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx.$$

In Exercises 51–58, find the integral using any method.

51. $\int \theta \sin \theta \cos \theta d\theta$

52. $\int \frac{\csc \sqrt{2x}}{\sqrt{x}} dx$

53. $\int \frac{x^{1/4}}{1+x^{1/2}} dx$

54. $\int \sqrt{1+\sqrt{x}} dx$

55. $\int \sqrt{1+\cos x} dx$

56. $\int \frac{3x^3+4x}{(x^2+1)^2} dx$

57. $\int \cos x \ln(\sin x) dx$

58. $\int (\sin \theta + \cos \theta)^2 d\theta$

In Exercises 59–62, solve the differential equation using any method.

59. $\frac{dy}{dx} = \frac{25}{x^2 - 25}$

60. $\frac{dy}{dx} = \frac{\sqrt{4-x^2}}{2x}$

61. $y' = \ln(x^2 + x)$

62. $y' = \sqrt{1 - \cos \theta}$

In Exercises 63–68, evaluate the definite integral using any method. Use a graphing utility to verify your result.

63. $\int_2^{\sqrt{5}} x(x^2 - 4)^{3/2} dx$

64. $\int_0^1 \frac{x}{(x-2)(x-4)} dx$

65. $\int_1^4 \frac{\ln x}{x} dx$

66. $\int_0^2 xe^{3x} dx$

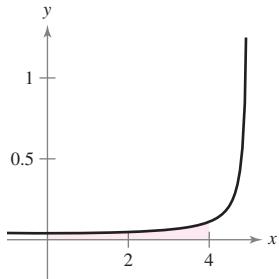
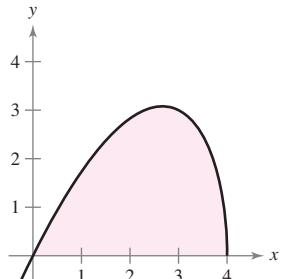
67. $\int_0^{\pi} x \sin x dx$

68. $\int_0^5 \frac{x}{\sqrt{4+x}} dx$

Area In Exercises 69 and 70, find the area of the region.

69. $y = x\sqrt{4-x^2}$

70. $y = \frac{1}{25-x^2}$



Centroid In Exercises 71 and 72, find the centroid of the region bounded by the graphs of the equations.

71. $y = \sqrt{1-x^2}, \quad y = 0$

72. $(x-1)^2 + y^2 = 1, \quad (x-4)^2 + y^2 = 4$

Arc Length In Exercises 73 and 74, approximate to two decimal places the arc length of the curve over the given interval.

Function

73. $y = \sin x$

74. $y = \sin^2 x$

Interval

[0, π]

[0, π]

In Exercises 75–82, use L'Hôpital's Rule to evaluate the limit.

75. $\lim_{x \rightarrow 1} \frac{(\ln x)^2}{x-1}$

76. $\lim_{x \rightarrow 0} \frac{\sin \pi x}{\sin 5\pi x}$

77. $\lim_{x \rightarrow \infty} \frac{e^{2x}}{x^2}$

78. $\lim_{x \rightarrow \infty} x e^{-x^2}$

79. $\lim_{x \rightarrow \infty} (\ln x)^{2/x}$

80. $\lim_{x \rightarrow 1^+} (x-1)^{\ln x}$

81. $\lim_{n \rightarrow \infty} 1000 \left(1 + \frac{0.09}{n}\right)^n$

82. $\lim_{x \rightarrow 1^+} \left(\frac{2}{\ln x} - \frac{2}{x-1} \right)$

In Exercises 83–90, determine whether the improper integral diverges or converges. Evaluate the integral if it converges.

83. $\int_0^{16} \frac{1}{\sqrt[4]{x}} dx$

84. $\int_0^2 \frac{7}{x-2} dx$

85. $\int_1^\infty x^2 \ln x dx$

86. $\int_0^\infty \frac{e^{-1/x}}{x^2} dx$

87. $\int_1^\infty \frac{\ln x}{x^2} dx$

88. $\int_1^\infty \frac{1}{\sqrt[4]{x}} dx$

89. $\int_2^\infty \frac{1}{x\sqrt{x^2-4}} dx$

90. $\int_0^\infty \frac{2}{\sqrt{x}(x+4)} dx$

91. Present Value The board of directors of a corporation is calculating the price to pay for a business that is forecast to yield a continuous flow of profit of \$500,000 per year. If money will earn a nominal rate of 5% per year compounded continuously, what is the present value of the business

(a) for 20 years?

(b) forever (in perpetuity)?

(Note: The present value for t_0 years is $\int_0^{t_0} 500,000 e^{-0.05t} dt$.)

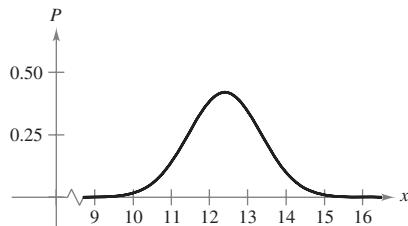
92. Volume Find the volume of the solid generated by revolving the region bounded by the graphs of $y = xe^{-x}$, $y = 0$, and $x = 0$ about the x -axis.



93. Probability The average lengths (from beak to tail) of different species of warblers in the eastern United States are approximately normally distributed with a mean of 12.9 centimeters and a standard deviation of 0.95 centimeter (see figure). The probability that a randomly selected warbler has a length between a and b centimeters is

$$P(a \leq x \leq b) = \frac{1}{0.95\sqrt{2\pi}} \int_a^b e^{-(x-12.9)^2/2(0.95)^2} dx.$$

Use a graphing utility to approximate the probability that a randomly selected warbler has a length of (a) 13 centimeters or greater and (b) 15 centimeters or greater. (Source: Peterson's Field Guide: Eastern Birds)



P.S. PROBLEM SOLVING

1. (a) Evaluate the integrals

$$\int_{-1}^1 (1 - x^2) dx \quad \text{and} \quad \int_{-1}^1 (1 - x^2)^2 dx.$$

- (b) Use Wallis's Formulas to prove that

$$\int_{-1}^1 (1 - x^2)^n dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!}$$

for all positive integers n .

2. (a) Evaluate the integrals $\int_0^1 \ln x dx$ and $\int_0^1 (\ln x)^2 dx$.

- (b) Prove that

$$\int_0^1 (\ln x)^n dx = (-1)^n n!$$

for all positive integers n .

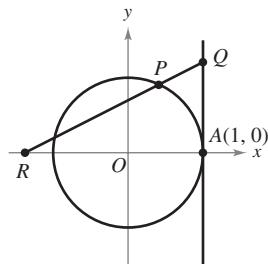
3. Find the value of the positive constant c such that

$$\lim_{x \rightarrow \infty} \left(\frac{x+c}{x-c} \right)^x = 9.$$

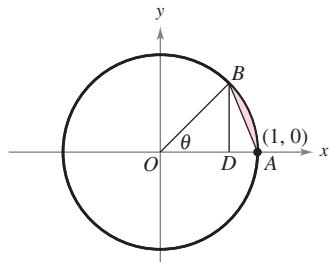
4. Find the value of the positive constant c such that

$$\lim_{x \rightarrow \infty} \left(\frac{x-c}{x+c} \right)^x = \frac{1}{4}.$$

5. The line $x = 1$ is tangent to the unit circle at A . The length of segment QA equals the length of the circular arc \widehat{PA} (see figure). Show that the length of segment OR approaches 2 as P approaches A .



6. The segment BD is the height of $\triangle OAB$. Let R be the ratio of the area of $\triangle DAB$ to that of the shaded region formed by deleting $\triangle OAB$ from the circular sector subtended by angle θ (see figure). Find $\lim_{\theta \rightarrow 0^+} R$.



7. Consider the problem of finding the area of the region bounded by the x -axis, the line $x = 4$, and the curve

$$y = \frac{x^2}{(x^2 + 9)^{3/2}}.$$

- (a) Use a graphing utility to graph the region and approximate its area.

- (b) Use an appropriate trigonometric substitution to find the exact area.

- (c) Use the substitution $x = 3 \sinh u$ to find the exact area and verify that you obtain the same answer as in part (b).

8. Use the substitution $u = \tan \frac{x}{2}$ to find the area of the shaded region under the graph of $y = \frac{1}{2 + \cos x}$, $0 \leq x \leq \pi/2$ (see figure).

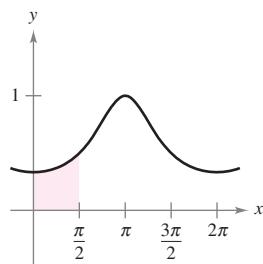


Figure for 8

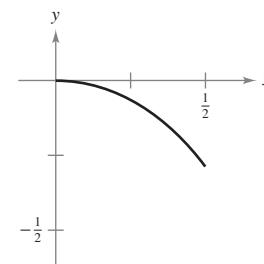
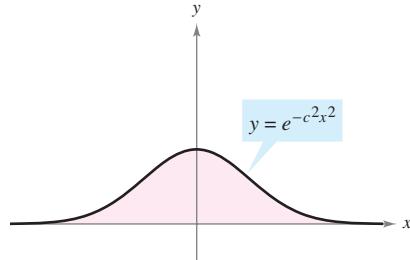


Figure for 9

9. Find the arc length of the graph of the function $y = \ln(1 - x^2)$ on the interval $0 \leq x \leq \frac{1}{2}$ (see figure).

10. Find the centroid of the region above the x -axis and bounded above by the curve $y = e^{-c^2 x^2}$, where c is a positive constant (see figure).

(Hint: Show that $\int_0^\infty e^{-c^2 x^2} dx = \frac{1}{c} \int_0^\infty e^{-x^2} dx$)



11. Some elementary functions, such as $f(x) = \sin(x^2)$, do not have antiderivatives that are elementary functions. Joseph Liouville proved that

$$\int \frac{e^x}{x} dx$$

does not have an elementary antiderivative. Use this fact to prove that

$$\int \frac{1}{\ln x} dx$$

is not elementary.

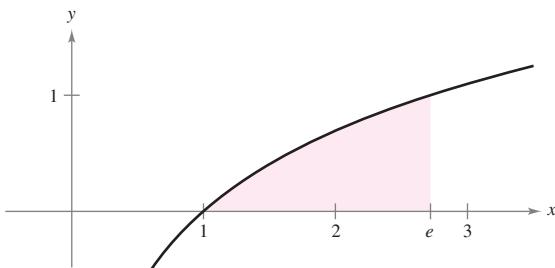
12. (a) Let $y = f^{-1}(x)$ be the inverse function of f . Use integration by parts to derive the formula

$$\int f^{-1}(x) dx = xf^{-1}(x) - \int f(y) dy.$$

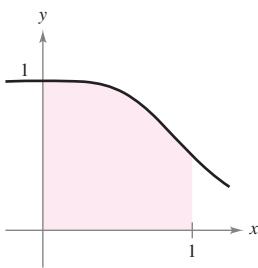
- (b) Use the formula in part (a) to find the integral

$$\int \arcsin x dx.$$

- (c) Use the formula in part (a) to find the area under the graph of $y = \ln x$, $1 \leq x \leq e$ (see figure).



13. Factor the polynomial $p(x) = x^4 + 1$ and then find the area under the graph of $y = \frac{1}{x^4 + 1}$, $0 \leq x \leq 1$ (see figure).



14. (a) Use the substitution $u = \frac{\pi}{2} - x$ to evaluate the integral

$$\int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x} dx.$$

- (b) Let n be a positive integer. Evaluate the integral

$$\int_0^{\pi/2} \frac{\sin^n x}{\cos^n x + \sin^n x} dx.$$



15. Use a graphing utility to estimate each limit. Then calculate each limit using L'Hôpital's Rule. What can you conclude about the form $0 \cdot \infty$?

$$(a) \lim_{x \rightarrow 0^+} \left(\cot x + \frac{1}{x} \right) \quad (b) \lim_{x \rightarrow 0^+} \left(\cot x - \frac{1}{x} \right)$$

$$(c) \lim_{x \rightarrow 0^+} \left[\left(\cot x + \frac{1}{x} \right) \left(\cot x - \frac{1}{x} \right) \right]$$

16. Suppose the denominator of a rational function can be factored into distinct linear factors

$$D(x) = (x - c_1)(x - c_2) \cdots (x - c_n)$$

for a positive integer n and distinct real numbers c_1, c_2, \dots, c_n . If N is a polynomial of degree less than n , show that

$$\frac{N(x)}{D(x)} = \frac{P_1}{x - c_1} + \frac{P_2}{x - c_2} + \cdots + \frac{P_n}{x - c_n}$$

where $P_k = N(c_k)/D'(c_k)$ for $k = 1, 2, \dots, n$. Note that this is the partial fraction decomposition of $N(x)/D(x)$.

17. Use the result of Exercise 16 to find the partial fraction decomposition of

$$\frac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x}.$$

18. The velocity v (in feet per second) of a rocket whose initial mass (including fuel) is m is given by

$$v = gt + u \ln \frac{m}{m - rt}, \quad t < \frac{m}{r}$$

where u is the expulsion speed of the fuel, r is the rate at which the fuel is consumed, and $g = -32$ feet per second per second is the acceleration due to gravity. Find the position equation for a rocket for which $m = 50,000$ pounds, $u = 12,000$ feet per second, and $r = 400$ pounds per second. What is the height of the rocket when $t = 100$ seconds? (Assume that the rocket was fired from ground level and is moving straight upward.)

19. Suppose that $f(a) = f(b) = g(a) = g(b) = 0$ and the second derivatives of f and g are continuous on the closed interval $[a, b]$. Prove that

$$\int_a^b f(x)g''(x) dx = \int_a^b f''(x)g(x) dx.$$

20. Suppose that $f(a) = f(b) = 0$ and the second derivatives of f exist on the closed interval $[a, b]$. Prove that

$$\int_a^b (x - a)(x - b)f''(x) dx = 2 \int_a^b f(x) dx.$$

21. Using the inequality

$$\frac{1}{x^5} + \frac{1}{x^{10}} + \frac{1}{x^{15}} < \frac{1}{x^5 - 1} < \frac{1}{x^5} + \frac{1}{x^{10}} + \frac{2}{x^{15}}$$

for $x \geq 2$, approximate $\int_2^\infty \frac{1}{x^5 - 1} dx$.

22. Consider the shaded region between the graph of $y = \sin x$, where $0 \leq x \leq \pi$, and the line $y = c$, where $0 \leq c \leq 1$ (see figure). A solid is formed by revolving the region about the line $y = c$.

- (a) For what value of c does the solid have minimum volume?
 (b) For what value of c does the solid have maximum volume?

