5 Integration

In this chapter, you will study an important process of calculus that is closely related to differentiation—*integration*. You will learn new methods and rules for solving definite and indefinite integrals, including the Fundamental Theorem of Calculus. Then you will apply these rules to find such things as the position function for an object and the average value of a function.

In this chapter, you should learn the following.

- How to evaluate indefinite integrals using basic integration rules. (5.1)
- How to evaluate a sum and approximate the area of a plane region. (5.2)
- How to evaluate a definite integral using a limit. (5.3)
- How to evaluate a definite integral using the Fundamental Theorem of Calculus. (5.4)
- How to evaluate different types of definite and indefinite integrals using a variety of methods. (5.5)
- How to approximate a definite integral using the Trapezoidal Rule and Simpson's Rule. (5.6)
- How to find the antiderivative of the natural logarithmic function. (5.7)
- How to find antiderivatives of inverse trigonometric functions. (5.8)
- The properties, derivatives, and antiderivatives of hyperbolic functions. (5.9)

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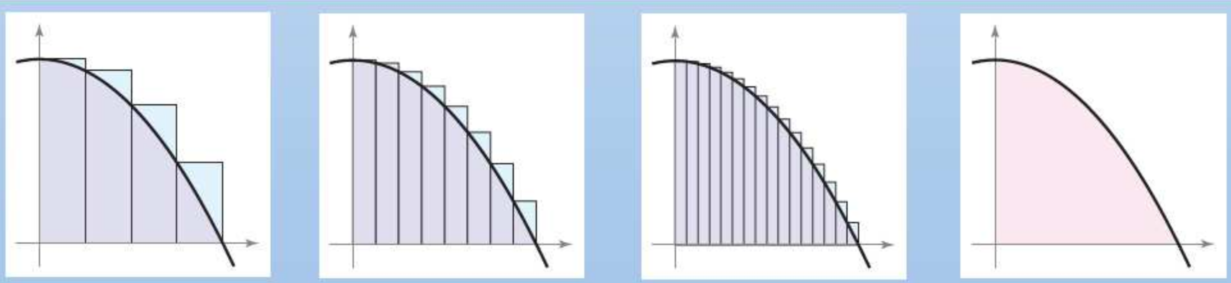
Although its official nickname is the Emerald City, Seattle is sometimes called the Rainy City due to its weather. But there are several cities, including New York and Boston, that typically get more annual precipitation. How could you use integration to calculate the normal annual precipitation for the Seattle area? (See Section 5.5, Exercise 153.)



The area of a parabolic region can be approximated as the sum of the areas of rectangles. As you increase the number of rectangles, the approximation tends to become more and more accurate. In Section 5.2, you will learn how the limit process can be used to find areas of a wide variety of regions.

5.1 Antiderivatives and Indefinite Integration

- Write the general solution of a differential equation.
- Use indefinite integral notation for antiderivatives.
- Use basic integration rules to find antiderivatives.
- Find a particular solution of a differential equation.

EXPLORATION

Finding Antiderivatives For each derivative, describe the original function F.

a. $F'(x) = 2x$ **b.** $F'(x) = x$

c. $F'(x) = x^2$ **d.** $F'(x) = \frac{1}{x^2}$

e. $F'(x) = \frac{1}{x^3}$ **f.** $F'(x) = \cos x$

What strategy did you use to find F?

Antiderivatives

Suppose you were asked to find a function F whose derivative is $f(x) = 3x^2$. From your knowledge of derivatives, you would probably say that

$$F(x) = x^3 \text{ because } \frac{d}{dx}[x^3] = 3x^2.$$

The function F is an *antiderivative* of f.

DEFINITION OF ANTIDERIVATIVE

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Note that F is called *an* antiderivative of f, rather than *the* antiderivative of f. To see why, observe that

$$F_1(x) = x^3, \quad F_2(x) = x^3 - 5, \quad \text{and} \quad F_3(x) = x^3 + 97$$

are all antiderivatives of $f(x) = 3x^2$. In fact, for any constant C, the function given by $F(x) = x^3 + C$ is an antiderivative of f.

THEOREM 5.1 REPRESENTATION OF ANTIDERIVATIVES

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$, for all x in I, where C is a constant.

PROOF The proof of Theorem 5.1 in one direction is straightforward. That is, if $G(x) = F(x) + C$, $F'(x) = f(x)$, and C is a constant, then

$$G'(x) = \frac{d}{dx}[F(x) + C] = F'(x) + 0 = f(x).$$

To prove this theorem in the other direction, assume that G is an antiderivative of f. Define a function H such that

$$H(x) = G(x) - F(x).$$

For any two points a and b $(a < b)$ in the interval, H is continuous on $[a, b]$ and differentiable on (a, b). By the Mean Value Theorem,

$$H'(c) = \frac{H(b) - H(a)}{b - a}$$

for some c in (a, b). However, $H'(c) = 0$, so $H(a) = H(b)$. Because a and b are arbitrary points in the interval, you know that H is a constant function C. So, $G(x) - F(x) = C$ and it follows that $G(x) = F(x) + C$. ■

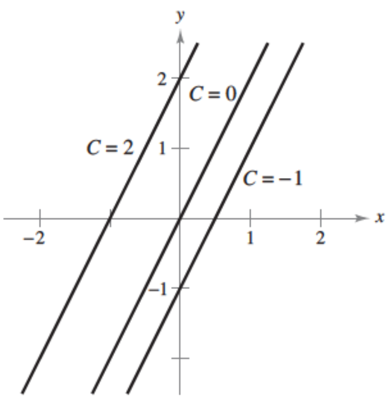
Using Theorem 5.1, you can represent the entire family of antiderivatives of a function by adding a constant to a *known* antiderivative. For example, knowing that $D_x[x^2] = 2x$, you can represent the family of *all* antiderivatives of $f(x) = 2x$ by

$$G(x) = x^2 + C \qquad \text{Family of all antiderivatives of } f(x) = 2x$$

where C is a constant. The constant C is called the **constant of integration.** The family of functions represented by G is the **general antiderivative** of f, and $G(x) = x^2 + C$ is the **general solution** of the *differential equation*

$$G'(x) = 2x. \qquad \text{Differential equation}$$

A **differential equation** in x and y is an equation that involves x, y, and derivatives of y. For instance, $y' = 3x$ and $y' = x^2 + 1$ are examples of differential equations.



Functions of the form $y = 2x + C$
Figure 5.1

EXAMPLE 1 Solving a Differential Equation

Find the general solution of the differential equation $y' = 2$.

Solution To begin, you need to find a function whose derivative is 2. One such function is

$$y = 2x. \qquad 2x \text{ is } \textit{an} \text{ antiderivative of } 2.$$

Now, you can use Theorem 5.1 to conclude that the general solution of the differential equation is

$$y = 2x + C. \qquad \text{General solution}$$

The graphs of several functions of the form $y = 2x + C$ are shown in Figure 5.1. ■

Notation for Antiderivatives

When solving a differential equation of the form

$$\frac{dy}{dx} = f(x)$$

it is convenient to write it in the equivalent differential form

$$dy = f(x)\, dx.$$

The operation of finding all solutions of this equation is called **antidifferentiation** (or **indefinite integration**) and is denoted by an integral sign $\int$. The general solution is denoted by

$$y = \int \underbrace{f(x)}_{\text{Integrand}}\, \overbrace{dx}^{\text{Variable of integration}} = \underbrace{F(x)}_{\text{An antiderivative of } f(x)} + \overbrace{C}^{\text{Constant of integration}}.$$

The expression $\int f(x)\,dx$ is read as the *antiderivative of f with respect to x*. So, the differential dx serves to identify x as the variable of integration. The term **indefinite integral** is a synonym for antiderivative.

NOTE In this text, the notation $\int f(x)\, dx = F(x) + C$ means that F is an antiderivative of f *on an interval*.

Basic Integration Rules

The inverse nature of integration and differentiation can be verified by substituting $F'(x)$ for $f(x)$ in the definition of indefinite integration to obtain

$$\int F'(x)\,dx = F(x) + C.$$

Integration is the "inverse" of differentiation.

Moreover, if $\int f(x)\,dx = F(x) + C$, then

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

Differentiation is the "inverse" of integration.

NOTE The Power Rule for Integration has the restriction that $n \neq -1$. To evaluate $\int x^{-1}\,dx$, you must use the natural log rule. (See Exercise 106.)

These two equations allow you to obtain integration formulas directly from differentiation formulas, as shown in the following summary.

BASIC INTEGRATION RULES

Differentiation Formula	*Integration Formula*	
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$	
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$	
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$	
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$	
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$	Power Rule
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$	
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\,dx = -\cos x + C$	
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$	
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$	
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$	
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$	
$\frac{d}{dx}[e^x] = e^x$	$\int e^x\,dx = e^x + C$	
$\frac{d}{dx}[a^x] = (\ln a)a^x$	$\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C$	
$\frac{d}{dx}[\ln x] = \frac{1}{x},\ x > 0$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	

EXAMPLE 2 Applying the Basic Integration Rules

Describe the antiderivatives of $3x$.

Solution

$$\int 3x\,dx = 3\int x\,dx \qquad \text{Constant Multiple Rule}$$

$$= 3\int x^1\,dx \qquad \text{Rewrite } x \text{ as } x^1.$$

$$= 3\left(\frac{x^2}{2}\right) + C \qquad \text{Power Rule } (n = 1)$$

$$= \frac{3}{2}x^2 + C \qquad \text{Simplify.}$$

■

When indefinite integrals are evaluated, a strict application of the basic integration rules tends to produce complicated constants of integration. For instance, in Example 2, you could have written

$$\int 3x\,dx = 3\int x\,dx = 3\left(\frac{x^2}{2} + C\right) = \frac{3}{2}x^2 + 3C.$$

However, because C represents *any* constant, it is both cumbersome and unnecessary to write $3C$ as the constant of integration. So, $\frac{3}{2}x^2 + 3C$ is written in the simpler form $\frac{3}{2}x^2 + C$.

In Example 2, note that the general pattern of integration is similar to that of differentiation.

Original integral ⇨ Rewrite ⇨ Integrate ⇨ Simplify

EXAMPLE 3 Rewriting Before Integrating

	Original Integral	*Rewrite*	*Integrate*	*Simplify*
a.	$\int \frac{1}{x^3}\,dx$	$\int x^{-3}\,dx$	$\frac{x^{-2}}{-2} + C$	$-\frac{1}{2x^2} + C$
b.	$\int \sqrt{x}\,dx$	$\int x^{1/2}\,dx$	$\frac{x^{3/2}}{3/2} + C$	$\frac{2}{3}x^{3/2} + C$
c.	$\int 2\sin x\,dx$	$2\int \sin x\,dx$	$2(-\cos x) + C$	$-2\cos x + C$
d.	$\int \frac{3}{x}\,dx$	$3\int \frac{1}{x}\,dx$	$3(\ln\lvert x\rvert) + C$	$3\ln\lvert x\rvert + C$

■

TECHNOLOGY Some software programs, such as *Maple*, *Mathematica*, and the *TI-89*, are capable of performing integration symbolically. If you have access to such a symbolic integration utility, try using it to evaluate the indefinite integrals in Example 3.

NOTE The properties of logarithms presented on page 53 can be used to rewrite antiderivatives in different forms. For instance, the antiderivative in Example 3(d) can be rewritten as

$$3\ln\lvert x\rvert + C = \ln\lvert x\rvert^3 + C.$$

Remember that you can check your answer to an antidifferentiation problem by differentiating. For instance, in Example 3(b), you can check that $\frac{2}{3}x^{3/2} + C$ is the correct antiderivative by differentiating the answer to obtain

$$D_x\left[\frac{2}{3}x^{3/2} + C\right] = \left(\frac{2}{3}\right)\left(\frac{3}{2}\right)x^{1/2} = \sqrt{x}. \qquad \text{Use differentiation to check antiderivative.}$$

The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

The basic integration rules listed earlier in this section allow you to integrate any polynomial function, as shown in Example 4.

EXAMPLE 4 Integrating Polynomial Functions

a. $\int dx = \int 1\,dx$ — Integrand is understood to be 1.

$= x + C$ — Integrate.

b. $\int (x+2)\,dx = \int x\,dx + \int 2\,dx$

$= \dfrac{x^2}{2} + C_1 + 2x + C_2$ — Integrate.

$= \dfrac{x^2}{2} + 2x + C$ — $C = C_1 + C_2$

The second line in the solution is usually omitted.

c. $\int (3x^4 - 5x^2 + x)\,dx = 3\left(\dfrac{x^5}{5}\right) - 5\left(\dfrac{x^3}{3}\right) + \dfrac{x^2}{2} + C$ — Integrate.

$= \dfrac{3}{5}x^5 - \dfrac{5}{3}x^3 + \dfrac{1}{2}x^2 + C$ — Simplify.

EXAMPLE 5 Rewriting Before Integrating

$\int \dfrac{x+1}{\sqrt{x}}\,dx = \int \left(\dfrac{x}{\sqrt{x}} + \dfrac{1}{\sqrt{x}}\right) dx$ — Rewrite as two fractions.

$= \int (x^{1/2} + x^{-1/2})\,dx$ — Rewrite with fractional exponents.

$= \dfrac{x^{3/2}}{3/2} + \dfrac{x^{1/2}}{1/2} + C$ — Integrate.

$= \dfrac{2}{3}x^{3/2} + 2x^{1/2} + C$ — Simplify.

$= \dfrac{2}{3}\sqrt{x}(x+3) + C$ — Factor. ■

STUDY TIP Remember that you can check your answer by differentiating.

NOTE When integrating quotients, do not integrate the numerator and denominator separately. This is no more valid in integration than it is in differentiation. For instance, in Example 5, be sure you understand that

$$\int \frac{x+1}{\sqrt{x}}\,dx = \frac{2}{3}\sqrt{x}(x+3) + C \text{ is not the same as } \frac{\int (x+1)\,dx}{\int \sqrt{x}\,dx} = \frac{\frac{1}{2}x^2 + x + C_1}{\frac{2}{3}x\sqrt{x} + C_2}.$$ ■

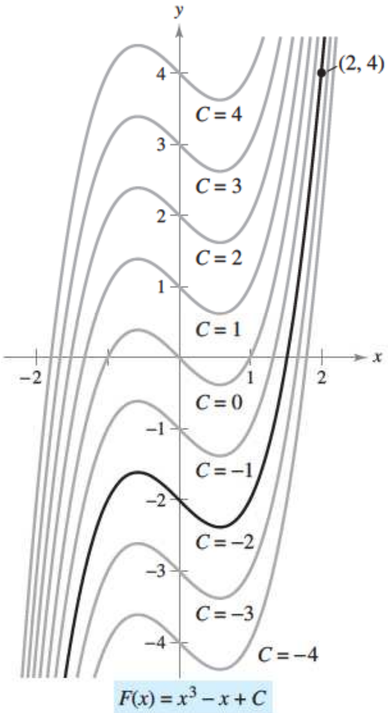
EXAMPLE 6 Rewriting Before Integrating

$\int \dfrac{\sin x}{\cos^2 x}\,dx = \int \left(\dfrac{1}{\cos x}\right)\left(\dfrac{\sin x}{\cos x}\right) dx$ — Rewrite as a product.

$= \int \sec x \tan x\,dx$ — Rewrite using trigonometric identities.

$= \sec x + C$ — Integrate. ■

Initial Conditions and Particular Solutions



The particular solution that satisfies the initial condition $F(2) = 4$ is $F(x) = x^3 - x - 2$.
Figure 5.2

You have already seen that the equation $y = \int f(x)\,dx$ has many solutions (each differing from the others by a constant). This means that the graphs of any two antiderivatives of f are vertical translations of each other. For example, Figure 5.2 shows the graphs of several antiderivatives of the form

$$y = \int (3x^2 - 1)\,dx$$

$$= x^3 - x + C \qquad \text{General solution}$$

for various integer values of C. Each of these antiderivatives is a solution of the differential equation

$$\frac{dy}{dx} = 3x^2 - 1.$$

In many applications of integration, you are given enough information to determine a **particular solution.** To do this, you need only know the value of $y = F(x)$ for one value of x. This information is called an **initial condition.** For example, in Figure 5.2, only one curve passes through the point $(2, 4)$. To find this curve, you can use the following information.

$$F(x) = x^3 - x + C \qquad \text{General solution}$$

$$F(2) = 4 \qquad \text{Initial condition}$$

By using the initial condition in the general solution, you can determine that $F(2) = 8 - 2 + C = 4$, which implies that $C = -2$. So, you obtain

$$F(x) = x^3 - x - 2. \qquad \text{Particular solution}$$

EXAMPLE 7 Finding a Particular Solution

Find the general solution of

$$F'(x) = e^x$$

and find the particular solution that satisfies the initial condition $F(0) = 3$.

Solution To find the general solution, integrate to obtain

$$F(x) = \int e^x\,dx$$

$$= e^x + C. \qquad \text{General solution}$$

Using the initial condition $F(0) = 3$, you can solve for C as follows.

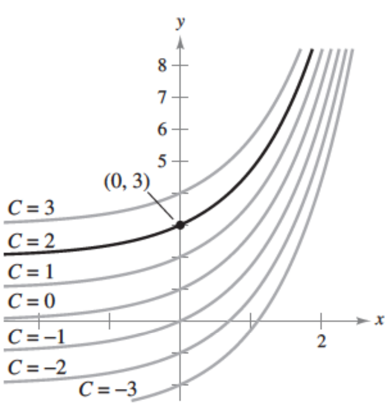
$$F(0) = e^0 + C$$

$$3 = 1 + C$$

$$2 = C$$

So, the particular solution, as shown in Figure 5.3, is

$$F(x) = e^x + 2. \qquad \text{Particular solution}$$



The particular solution that satisfies the initial condition $F(0) = 3$ is $F(x) = e^x + 2$.
Figure 5.3

So far in this section you have been using x as the variable of integration. In applications, it is often convenient to use a different variable. For instance, in the following example involving *time*, the variable of integration is t.

EXAMPLE 8 Solving a Vertical Motion Problem

A ball is thrown upward with an initial velocity of 64 feet per second from an initial height of 80 feet.

a. Find the position function giving the height s as a function of the time t.

b. When does the ball hit the ground?

Solution

a. Let $t = 0$ represent the initial time. The two given initial conditions can be written as follows.

$$s(0) = 80 \qquad \text{Initial height is 80 feet.}$$

$$s'(0) = 64 \qquad \text{Initial velocity is 64 feet per second.}$$

Using -32 feet per second per second as the acceleration due to gravity, you can write

$$s''(t) = -32$$

$$s'(t) = \int s''(t)\,dt = \int -32\,dt = -32t + C_1.$$

Using the initial velocity, you obtain $s'(0) = 64 = -32(0) + C_1$, which implies that $C_1 = 64$. Next, by integrating $s'(t)$, you obtain

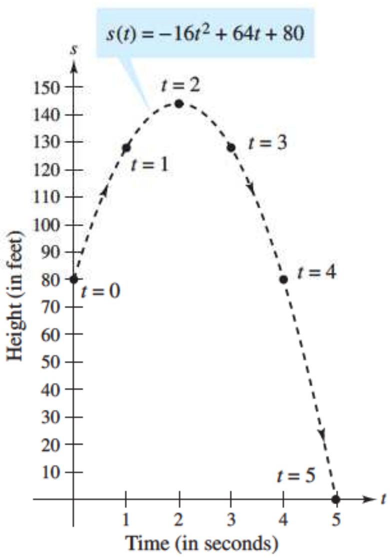
$$s(t) = \int s'(t)\,dt = \int (-32t + 64)\,dt = -16t^2 + 64t + C_2.$$

Using the initial height, you obtain

$$s(0) = 80 = -16(0^2) + 64(0) + C_2$$

which implies that $C_2 = 80$. So, the position function is

$$s(t) = -16t^2 + 64t + 80.$$ See Figure 5.4.



Height of a ball at time t
Figure 5.4

b. Using the position function found in part (a), you can find the time that the ball hits the ground by solving the equation $s(t) = 0$.

$$s(t) = -16t^2 + 64t + 80 = 0$$

$$-16(t + 1)(t - 5) = 0$$

$$t = -1, 5$$

Because t must be positive, you can conclude that the ball hits the ground 5 seconds after it is thrown. ■

NOTE In Example 8, note that the position function has the form

$$s(t) = \tfrac{1}{2}gt^2 + v_0 t + s_0$$

where $g = -32$, v_0 is the initial velocity, and s_0 is the initial height, as presented in Section 3.2.

Example 8 shows how to use calculus to analyze vertical motion problems in which the acceleration is determined by a gravitational force. You can use a similar strategy to analyze other linear motion problems (vertical or horizontal) in which the acceleration (or deceleration) is the result of some other force, as you will see in Exercises 87–94.

Before you begin the exercise set, be sure you realize that one of the most important steps in integration is *rewriting the integrand* in a form that fits the basic integration rules. To illustrate this point further, here are some additional examples.

Original Integral	*Rewrite*	*Integrate*	*Simplify*
$\int \frac{2}{\sqrt{x}}\,dx$	$2\int x^{-1/2}\,dx$	$2\left(\frac{x^{1/2}}{1/2}\right) + C$	$4x^{1/2} + C$
$\int (t^2+1)^2\,dt$	$\int (t^4 + 2t^2 + 1)\,dt$	$\frac{t^5}{5} + 2\left(\frac{t^3}{3}\right) + t + C$	$\frac{1}{5}t^5 + \frac{2}{3}t^3 + t + C$
$\int \frac{x^3+3}{x^2}\,dx$	$\int (x + 3x^{-2})\,dx$	$\frac{x^2}{2} + 3\left(\frac{x^{-1}}{-1}\right) + C$	$\frac{1}{2}x^2 - \frac{3}{x} + C$
$\int \sqrt[3]{x}(x-4)\,dx$	$\int (x^{4/3} - 4x^{1/3})\,dx$	$\frac{x^{7/3}}{7/3} - 4\left(\frac{x^{4/3}}{4/3}\right) + C$	$\frac{3}{7}x^{4/3}(x-7) + C$

5.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, verify the statement by showing that the derivative of the right side equals the integrand of the left side.

1. $\int \left(-\frac{6}{x^4}\right) dx = \frac{2}{x^3} + C$
2. $\int \left(8x^3 + \frac{1}{2x^2}\right) dx = 2x^4 - \frac{1}{2x} + C$
3. $\int (x-4)(x+4)\,dx = \frac{1}{3}x^3 - 16x + C$
4. $\int \frac{x^2-1}{x^{3/2}}\,dx = \frac{2(x^2+3)}{3\sqrt{x}} + C$

In Exercises 5–8, find the general solution of the differential equation and check the result by differentiation.

5. $\frac{dy}{dt} = 9t^2$
6. $\frac{dr}{d\theta} = \pi$
7. $\frac{dy}{dx} = x^{3/2}$
8. $\frac{dy}{dx} = 2x^{-3}$

In Exercises 9–14, complete the table using Example 3 and the examples at the top of this page as a model.

	Original Integral	*Rewrite*	*Integrate*	*Simplify*
9.	$\int \sqrt[3]{x}\,dx$			
10.	$\int \frac{1}{4x^2}\,dx$			
11.	$\int \frac{1}{x\sqrt{x}}\,dx$			
12.	$\int x(x^3+1)\,dx$			
13.	$\int \frac{1}{2x^3}\,dx$			
14.	$\int \frac{1}{(3x)^2}\,dx$			

In Exercises 15–44, find the indefinite integral and check the result by differentiation.

15. $\int (x+7)\,dx$
16. $\int (13-x)\,dx$
17. $\int (x^5+1)\,dx$
18. $\int (8x^3 - 9x^2 + 4)\,dx$
19. $\int (x^{3/2} + 2x + 1)\,dx$
20. $\int \left(\sqrt[4]{x^3} + 1\right) dx$
21. $\int \frac{1}{x^5}\,dx$
22. $\int \frac{1}{x^6}\,dx$
23. $\int \frac{x+6}{\sqrt{x}}\,dx$
24. $\int \frac{x^2+2x-3}{x^4}\,dx$
25. $\int (x+1)(3x-2)\,dx$
26. $\int (2t^2-1)^2\,dt$
27. $\int y^2\sqrt{y}\,dy$
28. $\int (1+3t)t^2\,dt$
29. $\int dx$
30. $\int 14\,dt$
31. $\int (5\cos x + 4\sin x)\,dx$
32. $\int (t^2 - \cos t)\,dt$
33. $\int (1 - \csc t \cot t)\,dt$
34. $\int (\theta^2 + \sec^2\theta)\,d\theta$
35. $\int (2\sin x - 5e^x)\,dx$
36. $\int (3x^2 + 2e^x)\,dx$
37. $\int (\sec^2\theta - \sin\theta)\,d\theta$
38. $\int \sec y(\tan y - \sec y)\,dy$
39. $\int (\tan^2 y + 1)\,dy$
40. $\int \frac{\cos x}{1-\cos^2 x}\,dx$
41. $\int (2x - 4^x)\,dx$
42. $\int (\cos x + 3^x)\,dx$
43. $\int \left(x - \frac{5}{x}\right) dx$
44. $\int \left(\frac{4}{x} + \sec^2 x\right) dx$

In Exercises 45–48, sketch the graphs of the function $g(x) = f(x) + C$ for $C = -2$, $C = 0$, and $C = 3$ on the same set of coordinate axes.

45. $f(x) = \cos x$

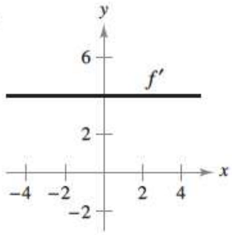
46. $f(x) = \sqrt{x}$

47. $f(x) = \ln x$

48. $f(x) = \frac{1}{2}e^x$

In Exercises 49–52, the graph of the derivative of a function is given. Sketch the graphs of *two* functions that have the given derivative. (There is more than one correct answer.) To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

49.



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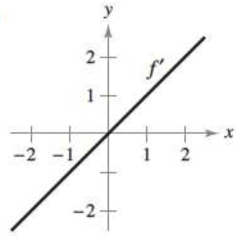

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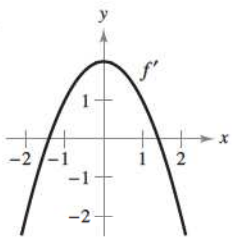
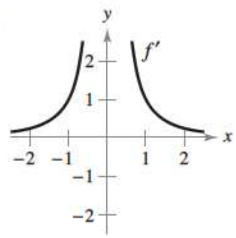
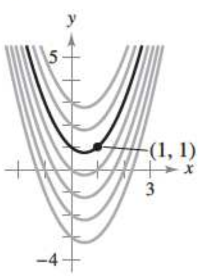


52.

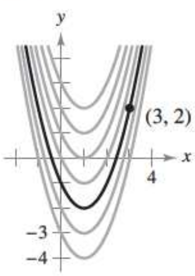


In Exercises 53–56, find the equation of y, given the derivative and the given point on the curve.

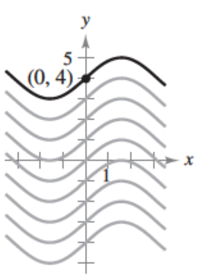
53. $\dfrac{dy}{dx} = 2x - 1$



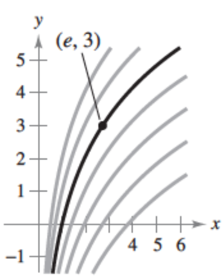
54. $\dfrac{dy}{dx} = 2(x - 1)$



55. $\dfrac{dy}{dx} = \cos x$

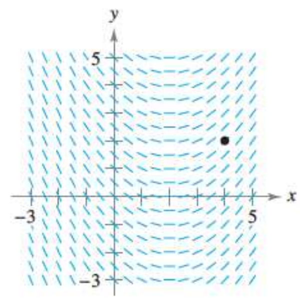


56. $\dfrac{dy}{dx} = \dfrac{3}{x}, \quad x > 0$

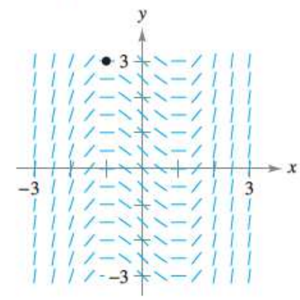


Slope Fields **In Exercises 57–60, a differential equation, a point, and a slope field are given. A *slope field* (or *direction field*) consists of line segments with slopes given by the differential equation. These line segments give a visual perspective of the slopes of the solutions of the differential equation. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.) (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a).**

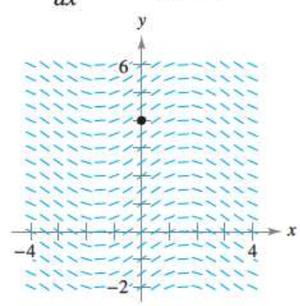
57. $\dfrac{dy}{dx} = \dfrac{1}{2}x - 1, \ (4, 2)$



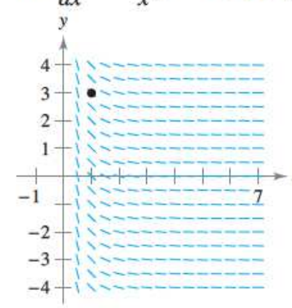
58. $\dfrac{dy}{dx} = x^2 - 1, \ (-1, 3)$



59. $\dfrac{dy}{dx} = \cos x, \ (0, 4)$



60. $\dfrac{dy}{dx} = -\dfrac{1}{x^2}, \ x > 0, \ (1, 3)$



Slope Fields **In Exercises 61 and 62, (a) use a graphing utility to graph a slope field for the differential equation, (b) use integration and the given point to find the particular solution of the differential equation, and (c) graph the solution and the slope field in the same viewing window.**

61. $\dfrac{dy}{dx} = 2x, \ (-2, -2)$

62. $\dfrac{dy}{dx} = 2\sqrt{x}, \ (4, 12)$

In Exercises 63–72, solve the differential equation.

63. $f'(x) = 6x, \ f(0) = 8$

64. $g'(x) = 6x^2, \ g(0) = -1$

65. $h'(t) = 8t^3 + 5, \ h(1) = -4$

66. $f'(s) = 10s - 12s^3, \ f(3) = 2$

67. $f''(x) = 2, \ f'(2) = 5, \ f(2) = 10$

68. $f''(x) = x^2, \ f'(0) = 8, \ f(0) = 4$

69. $f''(x) = x^{-3/2}, \ f'(4) = 2, \ f(0) = 0$

70. $f''(x) = \sin x, \ f'(0) = 1, \ f(0) = 6$

71. $f''(x) = e^x,\ f'(0) = 2,\ f(0) = 5$

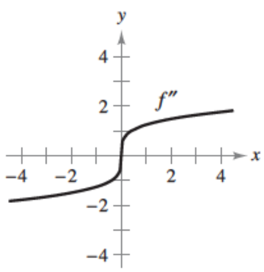
72. $f''(x) = \dfrac{2}{x^2},\ f'(1) = 4,\ f(1) = 3$

WRITING ABOUT CONCEPTS

73. What is the difference, if any, between finding the antiderivative of $f(x)$ and evaluating the integral $\int f(x)\,dx$?

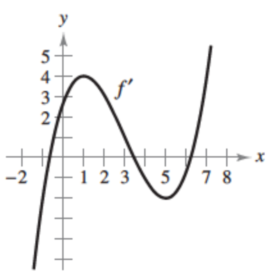
74. Consider $f(x) = \tan^2 x$ and $g(x) = \sec^2 x$. What do you notice about the derivatives of $f(x)$ and $g(x)$? What can you conclude about the relationship between $f(x)$ and $g(x)$?

75. The graphs of f and f' each pass through the origin. Use the graph of f'' shown in the figure to sketch the graphs of f and f'. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



CAPSTONE

76. Use the graph of f' shown in the figure to answer the following, given that $f(0) = -4$.



(a) Approximate the slope of f at $x = 4$. Explain.

(b) Is it possible that $f(2) = -1$? Explain.

(c) Is $f(5) - f(4) > 0$? Explain.

(d) Approximate the value of x where f is maximum. Explain.

(e) Approximate any intervals in which the graph of f is concave upward and any intervals in which it is concave downward. Approximate the x-coordinates of any points of inflection.

(f) Approximate the x-coordinate of the minimum of $f''(x)$.

(g) Sketch an approximate graph of f. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

Vertical Motion **In Exercises 77–80, use $a(t) = -32$ feet per second per second as the acceleration due to gravity. (Neglect air resistance.)**

77. A ball is thrown vertically upward from a height of 6 feet with an initial velocity of 60 feet per second. How high will the ball go?

78. Show that the height above the ground of an object thrown upward from a point s_0 feet above the ground with an initial velocity of v_0 feet per second is given by the function

$$f(t) = -16t^2 + v_0t + s_0.$$

79. With what initial velocity must an object be thrown upward (from ground level) to reach the top of the Washington Monument (approximately 550 feet)?

80. A balloon, rising vertically at a velocity of 8 feet per second, releases a sandbag at the instant it is 64 feet above the ground.

(a) How many seconds after its release will the bag strike the ground?

(b) At what velocity will it hit the ground?

Vertical Motion **In Exercises 81–84, use $a(t) = -9.8$ meters per second per second as the acceleration due to gravity. (Neglect air resistance.)**

81. Show that the height above the ground of an object thrown upward from a point s_0 meters above the ground with an initial velocity of v_0 meters per second is given by the function

$$f(t) = -4.9t^2 + v_0t + s_0.$$

82. The Grand Canyon is 1800 meters deep at its deepest point. A rock is dropped from the rim above this point. Express the height of the rock as a function of the time t in seconds. How long will it take the rock to hit the canyon floor?

83. A baseball is thrown upward from a height of 2 meters with an initial velocity of 10 meters per second. Determine its maximum height.

84. With what initial velocity must an object be thrown upward (from a height of 2 meters) to reach a maximum height of 200 meters?

85. ***Lunar Gravity*** On the moon, the acceleration due to gravity is -1.6 meters per second per second. A stone is dropped from a cliff on the moon and hits the surface of the moon 20 seconds later. How far did it fall? What was its velocity at impact?

86. ***Escape Velocity*** The minimum velocity required for an object to escape Earth's gravitational pull is obtained from the solution of the equation

$$\int v\,dv = -GM\int \frac{1}{y^2}\,dy$$

where v is the velocity of the object projected from Earth, y is the distance from the center of Earth, G is the gravitational constant, and M is the mass of Earth. Show that v and y are related by the equation

$$v^2 = v_0^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

where v_0 is the initial velocity of the object and R is the radius of Earth.

Rectilinear Motion **In Exercises 87–90, consider a particle moving along the x-axis where $x(t)$ is the position of the particle at time t, $x'(t)$ is its velocity, and $x''(t)$ is its acceleration.**

87. $x(t) = t^3 - 6t^2 + 9t - 2, \quad 0 \le t \le 5$

(a) Find the velocity and acceleration of the particle.

(b) Find the open t-intervals on which the particle is moving to the right.

(c) Find the velocity of the particle when the acceleration is 0.

88. Repeat Exercise 87 for the position function

$x(t) = (t - 1)(t - 3)^2, \quad 0 \le t \le 5.$

89. A particle moves along the x-axis at a velocity of $v(t) = 1/\sqrt{t}$, $t > 0$. At time $t = 1$, its position is $x = 4$. Find the acceleration and position functions for the particle.

90. A particle, initially at rest, moves along the x-axis such that its acceleration at time $t > 0$ is given by $a(t) = \cos t$. At the time $t = 0$, its position is $x = 3$.

(a) Find the velocity and position functions for the particle.

(b) Find the values of t for which the particle is at rest.

91. *Acceleration* The maker of an automobile advertises that it takes 13 seconds to accelerate from 25 kilometers per hour to 80 kilometers per hour. Assuming constant acceleration, compute the following.

(a) The acceleration in meters per second per second

(b) The distance the car travels during the 13 seconds

92. *Deceleration* A car traveling at 45 miles per hour is brought to a stop, at constant deceleration, 132 feet from where the brakes are applied.

(a) How far has the car moved when its speed has been reduced to 30 miles per hour?

(b) How far has the car moved when its speed has been reduced to 15 miles per hour?

(c) Draw the real number line from 0 to 132, and plot the points found in parts (a) and (b). What can you conclude?

93. *Acceleration* At the instant the traffic light turns green, a car that has been waiting at an intersection starts with a constant acceleration of 6 feet per second per second. At the same instant, a truck traveling with a constant velocity of 30 feet per second passes the car.

(a) How far beyond its starting point will the car pass the truck?

(b) How fast will the car be traveling when it passes the truck?

94. *Modeling Data* The table shows the velocities (in miles per hour) of two cars on an entrance ramp to an interstate highway. The time t is in seconds.

t	0	5	10	15	20	25	30
v_1	0	2.5	7	16	29	45	65
v_2	0	21	38	51	60	64	65

(a) Rewrite the table, converting miles per hour to feet per second.

(b) Use the regression capabilities of a graphing utility to find quadratic models for the data in part (a).

(c) Approximate the distance traveled by each car during the 30 seconds. Explain the difference in the distances.

True or False? **In Exercises 95–100, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

95. Each antiderivative of an nth-degree polynomial function is an $(n + 1)$th-degree polynomial function.

96. If $p(x)$ is a polynomial function, then p has exactly one antiderivative whose graph contains the origin.

97. If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$, then $F(x) = G(x) + C$.

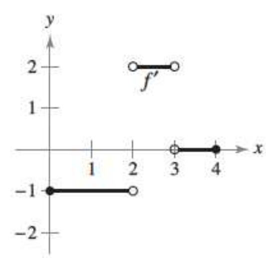
98. If $f'(x) = g(x)$, then $\int g(x)\,dx = f(x) + C$.

99. $\int f(x)g(x)\,dx = \int f(x)\,dx \int g(x)\,dx$

100. The antiderivative of $f(x)$ is unique.

101. Find a function f such that the graph of f has a horizontal tangent at $(2, 0)$ and $f''(x) = 2x$.

102. The graph of f' is shown. Sketch the graph of f given that f is continuous and $f(0) = 1$.



103. If $f'(x) = \begin{cases} 1, & 0 \le x < 2 \\ 3x, & 2 \le x \le 5 \end{cases}$, f is continuous, and $f(1) = 3$, find f. Is f differentiable at $x = 2$?

104. Let $s(x)$ and $c(x)$ be two functions satisfying $s'(x) = c(x)$ and $c'(x) = -s(x)$ for all x. If $s(0) = 0$ and $c(0) = 1$, prove that $[s(x)]^2 + [c(x)]^2 = 1$.

105. *Verification* Verify the natural log rule $\displaystyle\int \frac{1}{x}\,dx = \ln|Cx|$, $C \ne 0$, by showing that the derivative of $\ln|Cx|$ is $1/x$.

106. *Verification* Verify the natural log rule $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$ by showing that the derivative of $\ln|x| + C$ is $1/x$.

PUTNAM EXAM CHALLENGE

107. Suppose f and g are nonconstant, differentiable, real-valued functions on R. Furthermore, suppose that for each pair of real numbers x and y, $f(x + y) = f(x)f(y) - g(x)g(y)$ and $g(x + y) = f(x)g(y) + g(x)f(y)$. If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

This problem was composed by the Committee on the Putnam Prize Competition.

5.2 Area

- Use sigma notation to write and evaluate a sum.
- Understand the concept of area.
- Approximate the area of a plane region.
- Find the area of a plane region using limits.

Sigma Notation

In the preceding section, you studied antidifferentiation. In this section, you will look further into a problem introduced in Section 2.1—that of finding the area of a region in the plane. At first glance, these two ideas may seem unrelated, but you will discover in Section 5.4 that they are closely related by an extremely important theorem called the Fundamental Theorem of Calculus.

This section begins by introducing a concise notation for sums. This notation is called **sigma notation** because it uses the uppercase Greek letter sigma, written as Σ.

SIGMA NOTATION

The sum of n terms $a_1, a_2, a_3, \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the **index of summation,** a_i is the **ith term** of the sum, and the **upper and lower bounds of summation** are n and 1.

NOTE The upper and lower bounds must be constant with respect to the index of summation. However, the lower bound doesn't have to be 1. Any integer less than or equal to the upper bound is legitimate.

EXAMPLE 1 Examples of Sigma Notation

a. $\displaystyle\sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6$

b. $\displaystyle\sum_{i=0}^{5} (i+1) = 1 + 2 + 3 + 4 + 5 + 6$

c. $\displaystyle\sum_{j=3}^{7} j^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2$

d. $\displaystyle\sum_{k=1}^{n} \frac{1}{n}(k^2+1) = \frac{1}{n}(1^2+1) + \frac{1}{n}(2^2+1) + \cdots + \frac{1}{n}(n^2+1)$

e. $\displaystyle\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$

From parts (a) and (b), notice that the same sum can be represented in different ways using sigma notation.

FOR FURTHER INFORMATION For a geometric interpretation of summation formulas, see the article "Looking at $\sum_{k=1}^{n} k$ and $\sum_{k=1}^{n} k^2$ Geometrically" by Eric Hegblom in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

Although any variable can be used as the index of summation, i, j, and k are often used. Notice in Example 1 that the index of summation does not appear in the terms of the expanded sum.

THE SUM OF THE FIRST 100 INTEGERS

A teacher of Carl Friedrich Gauss (1777–1855) asked him to add all the integers from 1 to 100. When Gauss returned with the correct answer after only a few moments, the teacher could only look at him in astounded silence. This is what Gauss did:

$$\begin{array}{rcrcrcccr} 1 & + & 2 & + & 3 & + & \cdots & + & 100 \\ 100 & + & 99 & + & 98 & + & \cdots & + & 1 \\ \hline 101 & + & 101 & + & 101 & + & \cdots & + & 101 \end{array}$$

$$\frac{100 \times 101}{2} = 5050$$

This is generalized by Theorem 5.2, where

$$\sum_{i=1}^{100} i = \frac{100(101)}{2} = 5050.$$

The following properties of summation can be derived using the Associative and Commutative Properties of Addition and the Distributive Property of Multiplication over Addition. (In the first property, k is a constant.)

1. $\displaystyle\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$
2. $\displaystyle\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

The next theorem lists some useful formulas for sums of powers. A proof of this theorem is given in Appendix A.

THEOREM 5.2 SUMMATION FORMULAS

1. $\displaystyle\sum_{i=1}^{n} c = cn$
2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

EXAMPLE 2 Evaluating a Sum

Evaluate $\displaystyle\sum_{i=1}^{n} \frac{i+1}{n^2}$ for $n = 10, 100, 1000$, and $10{,}000$.

Solution Applying Theorem 5.2, you can write

$$\begin{aligned} \sum_{i=1}^{n} \frac{i+1}{n^2} &= \frac{1}{n^2}\sum_{i=1}^{n}(i+1) && \text{Factor the constant } 1/n^2 \text{ out of sum.} \\ &= \frac{1}{n^2}\left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right) && \text{Write as two sums.} \\ &= \frac{1}{n^2}\left[\frac{n(n+1)}{2} + n\right] && \text{Apply Theorem 5.2.} \\ &= \frac{1}{n^2}\left[\frac{n^2+3n}{2}\right] && \text{Simplify.} \\ &= \frac{n+3}{2n}. && \text{Simplify.} \end{aligned}$$

n	$\displaystyle\sum_{i=1}^{n} \frac{i+1}{n^2} = \frac{n+3}{2n}$
10	0.65000
100	0.51500
1000	0.50150
10,000	0.50015

Now you can evaluate the sum by substituting the appropriate values of n, as shown in the table at the left. ■

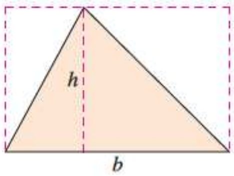
In the table, note that the sum appears to approach a limit as n increases. Although the discussion of limits at infinity in Section 4.5 applies to a variable x, where x can be any real number, many of the same results hold true for limits involving the variable n, where n is restricted to positive integer values. So, to find the limit of $(n+3)/2n$ as n approaches infinity, you can write

$$\lim_{n\to\infty} \frac{n+3}{2n} = \lim_{n\to\infty}\left(\frac{n}{2n} + \frac{3}{2n}\right) = \lim_{n\to\infty}\left(\frac{1}{2} + \frac{3}{2n}\right) = \frac{1}{2} + 0 = \frac{1}{2}.$$

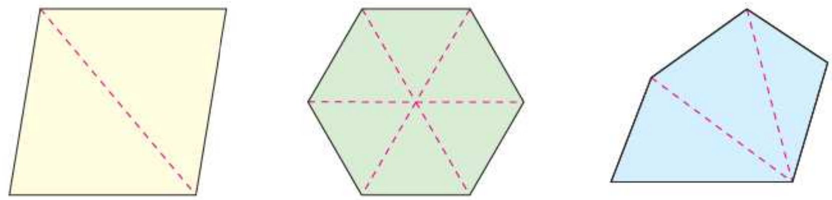
Area

In Euclidean geometry, the simplest type of plane region is a rectangle. Although people often say that the *formula* for the area of a rectangle is $A = bh$, it is actually more proper to say that this is the *definition* of the **area of a rectangle.**

From this definition, you can develop formulas for the areas of many other plane regions. For example, to determine the area of a triangle, you can form a rectangle whose area is twice that of the triangle, as shown in Figure 5.5. Once you know how to find the area of a triangle, you can determine the area of any polygon by subdividing the polygon into triangular regions, as shown in Figure 5.6.



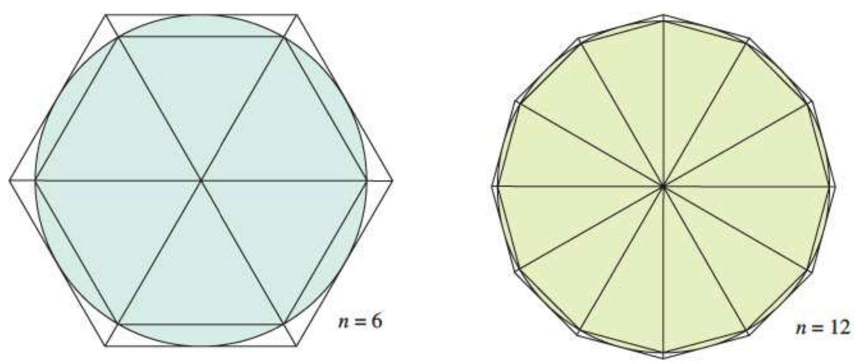
Triangle: $A = \frac{1}{2}bh$
Figure 5.5



Parallelogram Hexagon Polygon
Figure 5.6

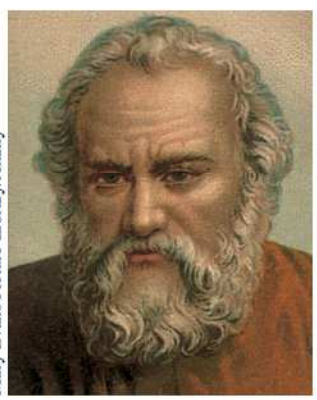
Finding the areas of regions other than polygons is more difficult. The ancient Greeks were able to determine formulas for the areas of some general regions (principally those bounded by conics) by the *exhaustion method.* The clearest description of this method was given by Archimedes. Essentially, the method is a limiting process in which the area is squeezed between two polygons—one inscribed in the region and one circumscribed about the region.

For instance, in Figure 5.7 the area of a circular region is approximated by an n-sided inscribed polygon and an n-sided circumscribed polygon. For each value of n, the area of the inscribed polygon is less than the area of the circle, and the area of the circumscribed polygon is greater than the area of the circle. Moreover, as n increases, the areas of both polygons become better and better approximations of the area of the circle.



The exhaustion method for finding the area of a circular region
Figure 5.7

A process that is similar to that used by Archimedes to determine the area of a plane region is used in the remaining examples in this section.



Mary Evans Picture Library/Alamy

ARCHIMEDES (287–212 B.C.)

Archimedes used the method of exhaustion to derive formulas for the areas of ellipses, parabolic segments, and sectors of a spiral. He is considered to have been the greatest applied mathematician of antiquity.

FOR FURTHER INFORMATION For an alternative development of the formula for the area of a circle, see the article "Proof Without Words: Area of a Disk is πR^2" by Russell Jay Hendel in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*.

The Area of a Plane Region

Recall from Section 2.1 that the origins of calculus are connected to two classic problems: the tangent line problem and the area problem. Example 3 begins the investigation of the area problem.

EXAMPLE 3 Approximating the Area of a Plane Region

Use the five rectangles in Figure 5.8(a) and (b) to find *two* approximations of the area of the region lying between the graph of

$$f(x) = -x^2 + 5$$

and the x-axis between $x = 0$ and $x = 2$.

Solution

a. The right endpoints of the five intervals are $\frac{2}{5}i$, where $i = 1, 2, 3, 4, 5$. The width of each rectangle is $\frac{2}{5}$, and the height of each rectangle can be obtained by evaluating f at the right endpoint of each interval.

$$\left[0, \frac{2}{5}\right], \left[\frac{2}{5}, \frac{4}{5}\right], \left[\frac{4}{5}, \frac{6}{5}\right], \left[\frac{6}{5}, \frac{8}{5}\right], \left[\frac{8}{5}, \frac{10}{5}\right]$$

Evaluate f at the right endpoints of these intervals.

The sum of the areas of the five rectangles is

$$\sum_{i=1}^{5} \underbrace{f\left(\frac{2i}{5}\right)}_{\text{Height}} \underbrace{\left(\frac{2}{5}\right)}_{\text{Width}} = \sum_{i=1}^{5} \left[-\left(\frac{2i}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{162}{25} = 6.48.$$

Because each of the five rectangles lies inside the parabolic region, you can conclude that the area of the parabolic region is greater than 6.48.

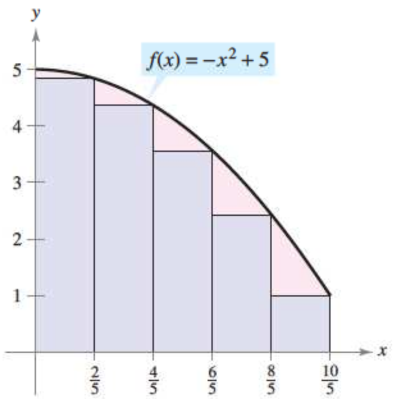
b. The left endpoints of the five intervals are $\frac{2}{5}(i - 1)$, where $i = 1, 2, 3, 4, 5$. The width of each rectangle is $\frac{2}{5}$, and the height of each rectangle can be obtained by evaluating f at the left endpoint of each interval. So, the sum is

$$\sum_{i=1}^{5} \underbrace{f\left(\frac{2i-2}{5}\right)}_{\text{Height}} \underbrace{\left(\frac{2}{5}\right)}_{\text{Width}} = \sum_{i=1}^{5} \left[-\left(\frac{2i-2}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{202}{25} = 8.08.$$

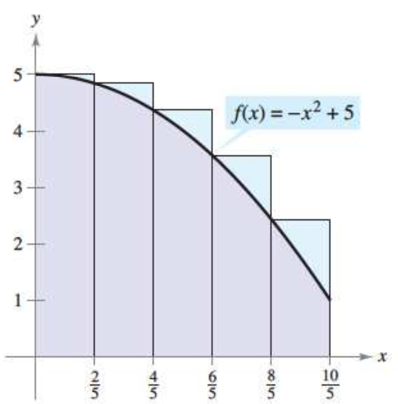
Because the parabolic region lies within the union of the five rectangular regions, you can conclude that the area of the parabolic region is less than 8.08.

By combining the results in parts (a) and (b), you can conclude that

$$6.48 < (\text{Area of region}) < 8.08.$$



(a) The area of the parabolic region is greater than the area of the rectangles.



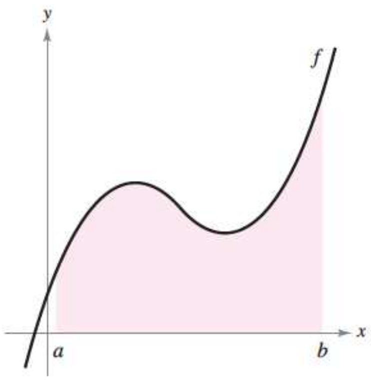
(b) The area of the parabolic region is less than the area of the rectangles.

Figure 5.8

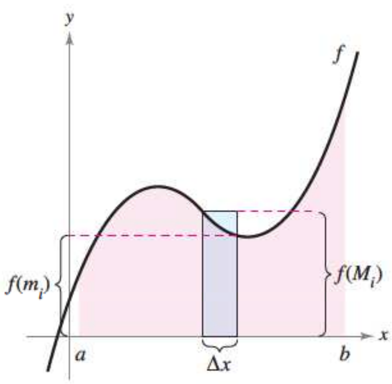
NOTE By increasing the number of rectangles used in Example 3, you can obtain closer and closer approximations of the area of the region. For instance, using 25 rectangles of width $\frac{2}{25}$ each, you can conclude that

$$7.17 < (\text{Area of region}) < 7.49.$$

Upper and Lower Sums



The region under a curve
Figure 5.9



The interval $[a, b]$ is divided into n subintervals of width $\Delta x = \dfrac{b - a}{n}$.
Figure 5.10

The procedure used in Example 3 can be generalized as follows. Consider a plane region bounded above by the graph of a nonnegative, continuous function $y = f(x)$, as shown in Figure 5.9. The region is bounded below by the x-axis, and the left and right boundaries of the region are the vertical lines $x = a$ and $x = b$.

To approximate the area of the region, begin by subdividing the interval $[a, b]$ into n subintervals, each of width

$$\Delta x = (b - a)/n$$

as shown in Figure 5.10. The endpoints of the intervals are as follows.

$$\overbrace{a + 0(\Delta x)}^{a = x_0} < \overbrace{a + 1(\Delta x)}^{x_1} < \overbrace{a + 2(\Delta x)}^{x_2} < \cdot \cdot \cdot < \overbrace{a + n(\Delta x)}^{x_n = b}$$

Because f is continuous, the Extreme Value Theorem guarantees the existence of a minimum and a maximum value of $f(x)$ in *each* subinterval.

$$f(m_i) = \text{Minimum value of } f(x) \text{ in } i\text{th subinterval}$$
$$f(M_i) = \text{Maximum value of } f(x) \text{ in } i\text{th subinterval}$$

Next, define an **inscribed rectangle** lying *inside* the ith subregion and a **circumscribed rectangle** extending *outside* the ith subregion. The height of the ith inscribed rectangle is $f(m_i)$ and the height of the ith circumscribed rectangle is $f(M_i)$. For *each* i, the area of the inscribed rectangle is less than or equal to the area of the circumscribed rectangle.

$$\begin{pmatrix}\text{Area of inscribed}\\ \text{rectangle}\end{pmatrix} = f(m_i)\,\Delta x \le f(M_i)\,\Delta x = \begin{pmatrix}\text{Area of circumscribed}\\ \text{rectangle}\end{pmatrix}$$

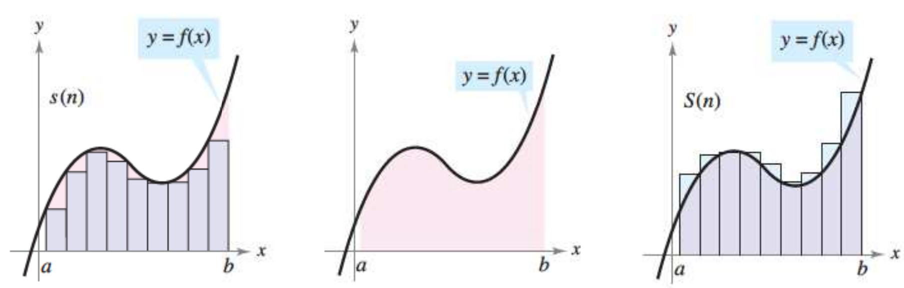
The sum of the areas of the inscribed rectangles is called a **lower sum,** and the sum of the areas of the circumscribed rectangles is called an **upper sum.**

$$\text{Lower sum} = s(n) = \sum_{i=1}^{n} f(m_i)\,\Delta x \qquad \text{Area of inscribed rectangles}$$

$$\text{Upper sum} = S(n) = \sum_{i=1}^{n} f(M_i)\,\Delta x \qquad \text{Area of circumscribed rectangles}$$

From Figure 5.11, you can see that the lower sum $s(n)$ is less than or equal to the upper sum $S(n)$. Moreover, the actual area of the region lies between these two sums.

$$s(n) \le (\text{Area of region}) \le S(n)$$



Area of inscribed rectangles is less than area of region.

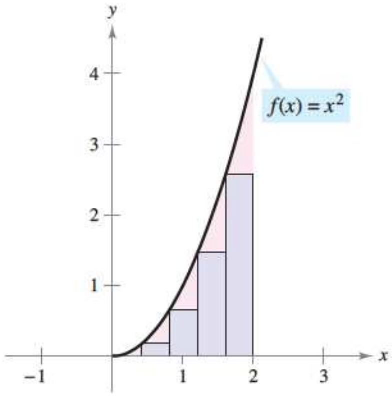
Area of region

Area of circumscribed rectangles is greater than area of region.

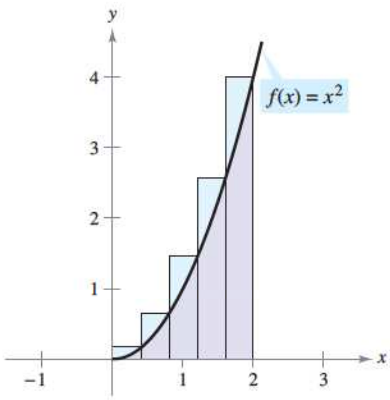
Figure 5.11

EXAMPLE 4 Finding Upper and Lower Sums for a Region

Find the upper and lower sums for the region bounded by the graph of $f(x) = x^2$ and the x-axis between $x = 0$ and $x = 2$.



Inscribed rectangles



Circumscribed rectangles
Figure 5.12

Solution To begin, partition the interval $[0, 2]$ into n subintervals, each of width

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}.$$

Figure 5.12 shows the endpoints of the subintervals and several inscribed and circumscribed rectangles. Because f is increasing on the interval $[0, 2]$, the minimum value on each subinterval occurs at the left endpoint, and the maximum value occurs at the right endpoint.

Left Endpoints	*Right Endpoints*
$m_i = 0 + (i-1)\left(\frac{2}{n}\right) = \frac{2(i-1)}{n}$	$M_i = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$

Using the left endpoints, the lower sum is

$$\begin{aligned}
s(n) = \sum_{i=1}^{n} f(m_i)\,\Delta x &= \sum_{i=1}^{n} f\left[\frac{2(i-1)}{n}\right]\left(\frac{2}{n}\right)\\
&= \sum_{i=1}^{n} \left[\frac{2(i-1)}{n}\right]^2\left(\frac{2}{n}\right)\\
&= \sum_{i=1}^{n} \left(\frac{8}{n^3}\right)(i^2 - 2i + 1)\\
&= \frac{8}{n^3}\left(\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right)\\
&= \frac{8}{n^3}\left\{\frac{n(n+1)(2n+1)}{6} - 2\left[\frac{n(n+1)}{2}\right] + n\right\}\\
&= \frac{4}{3n^3}(2n^3 - 3n^2 + n)\\
&= \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}. \qquad \text{Lower sum}
\end{aligned}$$

Using the right endpoints, the upper sum is

$$\begin{aligned}
S(n) = \sum_{i=1}^{n} f(M_i)\,\Delta x &= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)\\
&= \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2\left(\frac{2}{n}\right)\\
&= \sum_{i=1}^{n} \left(\frac{8}{n^3}\right)i^2\\
&= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\\
&= \frac{4}{3n^3}(2n^3 + 3n^2 + n)\\
&= \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}. \qquad \text{Upper sum}
\end{aligned}$$

EXPLORATION

For the region given in Example 4, evaluate the lower sum

$$s(n) = \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}$$

and the upper sum

$$S(n) = \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}$$

for $n = 10$, 100, and 1000. Use your results to determine the area of the region.

Example 4 illustrates some important things about lower and upper sums. First, notice that for any value of n, the lower sum is less than (or equal to) the upper sum.

$$s(n) = \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2} < \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} = S(n)$$

Second, the difference between these two sums lessens as n increases. In fact, if you take the limits as $n \to \infty$, both the upper sum and the lower sum approach $\frac{8}{3}$.

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty}\left(\frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}\right) = \frac{8}{3} \qquad \text{Lower sum limit}$$

$$\lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}\right) = \frac{8}{3} \qquad \text{Upper sum limit}$$

The next theorem shows that the equivalence of the limits (as $n \to \infty$) of the upper and lower sums is not mere coincidence. It is true for all functions that are continuous and nonnegative on the closed interval $[a, b]$. The proof of this theorem is best left to a course in advanced calculus.

THEOREM 5.3 LIMITS OF THE LOWER AND UPPER SUMS

Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\begin{aligned}\lim_{n\to\infty} s(n) &= \lim_{n\to\infty}\sum_{i=1}^{n} f(m_i)\,\Delta x\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f(M_i)\,\Delta x\\ &= \lim_{n\to\infty} S(n)\end{aligned}$$

where $\Delta x = (b - a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.

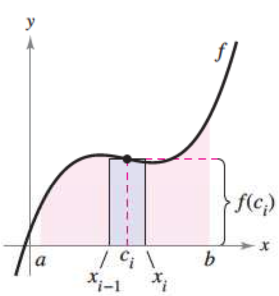
Because the same limit is attained for both the minimum value $f(m_i)$ and the maximum value $f(M_i)$, it follows from the Squeeze Theorem (Theorem 2.8) that the choice of x in the ith subinterval does not affect the limit. This means that you are free to choose an *arbitrary* x-value in the ith subinterval, as in the following *definition of the area of a region in the plane*.

DEFINITION OF THE AREA OF A REGION IN THE PLANE

Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

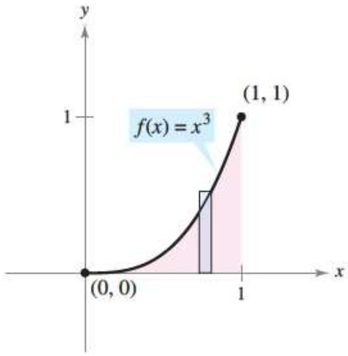
$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x, \quad x_{i-1} \le c_i \le x_i$$

where $\Delta x = (b - a)/n$ (see Figure 5.13).



The width of the ith subinterval is $\Delta x = x_i - x_{i-1}$.
Figure 5.13

EXAMPLE 5 Finding Area by the Limit Definition



The area of the region bounded by the graph of f, the x-axis, $x = 0$, and $x = 1$ is $\frac{1}{4}$.
Figure 5.14

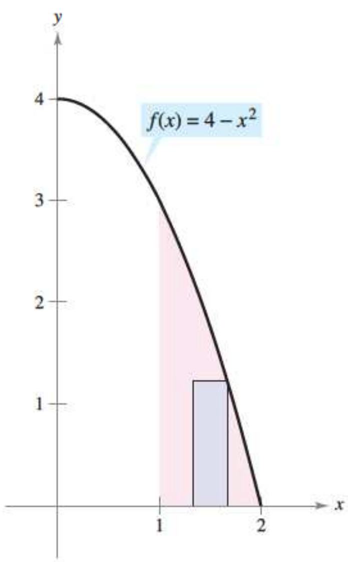
Find the area of the region bounded by the graph $f(x) = x^3$, the x-axis, and the vertical lines $x = 0$ and $x = 1$, as shown in Figure 5.14.

Solution Begin by noting that f is continuous and nonnegative on the interval $[0, 1]$. Next, partition the interval $[0, 1]$ into n subintervals, each of width $\Delta x = 1/n$. According to the definition of area, you can choose any x-value in the ith subinterval. For this example, the right endpoints $c_i = i/n$ are convenient.

$$\begin{aligned}
\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\left(\frac{1}{n}\right) && \text{Right endpoints: } c_i = \frac{i}{n}\\
&= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n} i^3\\
&= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]\\
&= \lim_{n\to\infty}\left(\frac{1}{4}+\frac{1}{2n}+\frac{1}{4n^2}\right)\\
&= \frac{1}{4}
\end{aligned}$$

The area of the region is $\frac{1}{4}$.

EXAMPLE 6 Finding Area by the Limit Definition



The area of the region bounded by the graph of f, the x-axis, $x = 1$, and $x = 2$ is $\frac{5}{3}$.
Figure 5.15

Find the area of the region bounded by the graph of $f(x) = 4 - x^2$, the x-axis, and the vertical lines $x = 1$ and $x = 2$, as shown in Figure 5.15.

Solution The function f is continuous and nonnegative on the interval $[1, 2]$, so begin by partitioning the interval into n subintervals, each of width $\Delta x = 1/n$. Choosing the right endpoint

$$c_i = a + i\Delta x = 1 + \frac{i}{n} \qquad \text{Right endpoints}$$

of each subinterval, you obtain

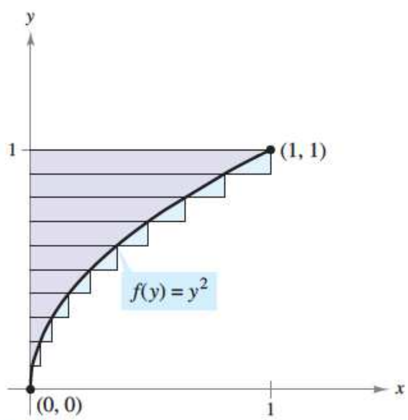
$$\begin{aligned}
\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[4-\left(1+\frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(3-\frac{2i}{n}-\frac{i^2}{n^2}\right)\left(\frac{1}{n}\right)\\
&= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}3-\frac{2}{n^2}\sum_{i=1}^{n}i-\frac{1}{n^3}\sum_{i=1}^{n}i^2\right)\\
&= \lim_{n\to\infty}\left[3-\left(1+\frac{1}{n}\right)-\left(\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right)\right]\\
&= 3-1-\frac{1}{3}\\
&= \frac{5}{3}.
\end{aligned}$$

The area of the region is $\frac{5}{3}$. ■

The last example in this section looks at a region that is bounded by the y-axis (rather than by the x-axis).

EXAMPLE 7 A Region Bounded by the y-axis

Find the area of the region bounded by the graph of $f(y) = y^2$ and the y-axis for $0 \le y \le 1$, as shown in Figure 5.16.



The area of the region bounded by the graph of f and the y-axis for $0 \le y \le 1$ is $\frac{1}{3}$.
Figure 5.16

Solution When f is a continuous, nonnegative function of y, you still can use the same basic procedure shown in Examples 5 and 6. Begin by partitioning the interval $[0, 1]$ into n subintervals, each of width $\Delta y = 1/n$. Then, using the upper endpoints $c_i = i/n$, you obtain

$$\begin{aligned}
\text{Area} &= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta y = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right) && \text{Upper endpoints: } c_i = \frac{i}{n}\\
&= \lim_{n\to\infty} \frac{1}{n^3}\sum_{i=1}^{n} i^2\\
&= \lim_{n\to\infty} \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\\
&= \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)\\
&= \frac{1}{3}.
\end{aligned}$$

The area of the region is $\frac{1}{3}$. ■

5.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the sum. Use the summation capabilities of a graphing utility to verify your result.

1. $\displaystyle\sum_{i=1}^{6} (3i + 2)$
2. $\displaystyle\sum_{k=5}^{8} k(k - 4)$
3. $\displaystyle\sum_{k=0}^{4} \frac{1}{k^2 + 1}$
4. $\displaystyle\sum_{j=4}^{7} \frac{2}{j}$
5. $\displaystyle\sum_{k=1}^{4} c$
6. $\displaystyle\sum_{i=1}^{4} [(i - 1)^2 + (i + 1)^3]$

In Exercises 7–14, use sigma notation to write the sum.

7. $\dfrac{1}{5(1)} + \dfrac{1}{5(2)} + \dfrac{1}{5(3)} + \cdots + \dfrac{1}{5(11)}$
8. $\dfrac{9}{1+1} + \dfrac{9}{1+2} + \dfrac{9}{1+3} + \cdots + \dfrac{9}{1+14}$
9. $\left[7\left(\frac{1}{6}\right) + 5\right] + \left[7\left(\frac{2}{6}\right) + 5\right] + \cdots + \left[7\left(\frac{6}{6}\right) + 5\right]$
10. $\left[1 - \left(\frac{1}{4}\right)^2\right] + \left[1 - \left(\frac{2}{4}\right)^2\right] + \cdots + \left[1 - \left(\frac{4}{4}\right)^2\right]$
11. $\left[\left(\frac{2}{n}\right)^3 - \frac{2}{n}\right]\left(\frac{2}{n}\right) + \cdots + \left[\left(\frac{2n}{n}\right)^3 - \frac{2n}{n}\right]\left(\frac{2}{n}\right)$
12. $\left[1 - \left(\frac{2}{n} - 1\right)^2\right]\left(\frac{2}{n}\right) + \cdots + \left[1 - \left(\frac{2n}{n} - 1\right)^2\right]\left(\frac{2}{n}\right)$
13. $\left[2\left(1 + \frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right) + \cdots + \left[2\left(1 + \frac{3n}{n}\right)^2\right]\left(\frac{3}{n}\right)$
14. $\left(\frac{1}{n}\right)\sqrt{1 - \left(\frac{0}{n}\right)^2} + \cdots + \left(\frac{1}{n}\right)\sqrt{1 - \left(\frac{n-1}{n}\right)^2}$

In Exercises 15–22, use the properties of summation and Theorem 5.2 to evaluate the sum. Use the summation capabilities of a graphing utility to verify your result.

15. $\displaystyle\sum_{i=1}^{12} 7$
16. $\displaystyle\sum_{i=1}^{30} -18$
17. $\displaystyle\sum_{i=1}^{24} 4i$
18. $\displaystyle\sum_{i=1}^{16} (5i - 4)$
19. $\displaystyle\sum_{i=1}^{20} (i - 1)^2$
20. $\displaystyle\sum_{i=1}^{10} (i^2 - 1)$
21. $\displaystyle\sum_{i=1}^{15} i(i - 1)^2$
22. $\displaystyle\sum_{i=1}^{10} i(i^2 + 1)$

In Exercises 23 and 24, use the summation capabilities of a graphing utility to evaluate the sum. Then use the properties of summation and Theorem 5.2 to verify the sum.

23. $\displaystyle\sum_{i=1}^{20} (i^2 + 3)$
24. $\displaystyle\sum_{i=1}^{15} (i^3 - 2i)$

25. Consider the function $f(x) = 3x + 2$.

(a) Estimate the area between the graph of f and the x-axis between $x = 0$ and $x = 3$ using six rectangles and right endpoints. Sketch the graph and the rectangles.

(b) Repeat part (a) using left endpoints.

26. Consider the function $g(x) = x^2 + x - 4$.

(a) Estimate the area between the graph of g and the x-axis between $x = 2$ and $x = 4$ using four rectangles and right endpoints. Sketch the graph and the rectangles.

(b) Repeat part (a) using left endpoints.

In Exercises 27–32, use left and right endpoints and the given number of rectangles to find two approximations of the area of the region between the graph of the function and the x-axis over the given interval.

27. $f(x) = 2x + 5$, $[0, 2]$, 4 rectangles

28. $f(x) = 9 - x$, $[2, 4]$, 6 rectangles

29. $g(x) = 2x^2 - x - 1$, $[2, 5]$, 6 rectangles

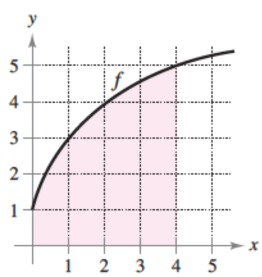
30. $g(x) = x^2 + 1$, $[1, 3]$, 8 rectangles

31. $f(x) = \cos x$, $\left[0, \dfrac{\pi}{2}\right]$, 4 rectangles

32. $g(x) = \sin x$, $[0, \pi]$, 6 rectangles

In Exercises 33–36, bound the area of the shaded region by approximating the upper and lower sums. Use rectangles of width 1.

33.



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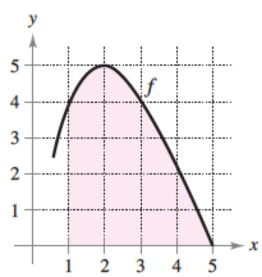

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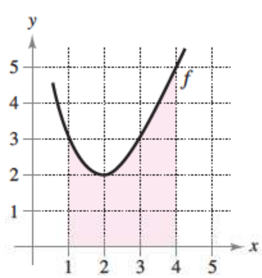
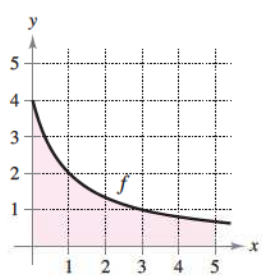


36.



In Exercises 37–40, find the limit of $s(n)$ as $n \to \infty$.

37. $s(n) = \dfrac{81}{n^4}\left[\dfrac{n^2(n+1)^2}{4}\right]$

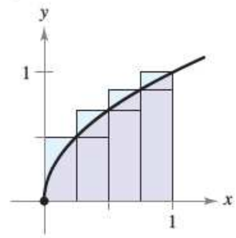
38. $s(n) = \dfrac{64}{n^3}\left[\dfrac{n(n+1)(2n+1)}{6}\right]$

39. $s(n) = \dfrac{18}{n^2}\left[\dfrac{n(n+1)}{2}\right]$

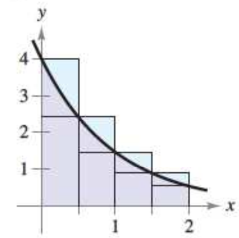
40. $s(n) = \dfrac{1}{n^2}\left[\dfrac{n(n+1)}{2}\right]$

In Exercises 41–44, use upper and lower sums to approximate the area of the region using the given number of subintervals (of equal width).

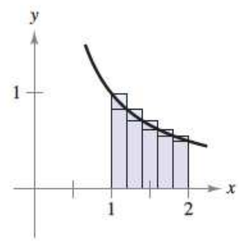
41. $y = \sqrt{x}$



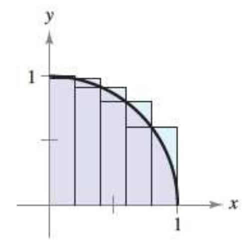
42. $y = 4e^{-x}$



43. $y = \dfrac{1}{x}$



44. $y = \sqrt{1 - x^2}$



In Exercises 45–48, use the summation formulas to rewrite the expression without the summation notation. Use the result to find the sums for $n = 10$, 100, 1000, and 10,000.

45. $\displaystyle\sum_{i=1}^{n} \frac{2i + 1}{n^2}$

46. $\displaystyle\sum_{j=1}^{n} \frac{4j + 1}{n^2}$

47. $\displaystyle\sum_{k=1}^{n} \frac{6k(k - 1)}{n^3}$

48. $\displaystyle\sum_{i=1}^{n} \frac{4i^2(i - 1)}{n^4}$

In Exercises 49–54, find a formula for the sum of n terms. Use the formula to find the limit as $n \to \infty$.

49. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{24i}{n^2}$

50. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$

51. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}(i - 1)^2$

52. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)^2\left(\frac{2}{n}\right)$

53. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)\left(\frac{2}{n}\right)$

54. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right)$

55. *Numerical Reasoning* Consider a triangle of area 2 bounded by the graphs of $y = x$, $y = 0$, and $x = 2$.

(a) Sketch the region.

(b) Divide the interval $[0, 2]$ into n subintervals of equal width and show that the endpoints are

$$0 < 1\left(\frac{2}{n}\right) < \cdots < (n - 1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right).$$

(c) Show that $s(n) = \displaystyle\sum_{i=1}^{n}\left[(i - 1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(d) Show that $S(n) = \displaystyle\sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(e) Complete the table.

n	5	10	50	100
$s(n)$				
$S(n)$				

(f) Show that $\lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n) = 2$.

56. *Numerical Reasoning* Consider a trapezoid of area 4 bounded by the graphs of $y = x$, $y = 0$, $x = 1$, and $x = 3$.

(a) Sketch the region.

(b) Divide the interval $[1, 3]$ into n subintervals of equal width and show that the endpoints are

$$1 < 1 + 1\left(\frac{2}{n}\right) < \cdots < 1 + (n-1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right).$$

(c) Show that $s(n) = \sum_{i=1}^{n} \left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(d) Show that $S(n) = \sum_{i=1}^{n} \left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(e) Complete the table.

n	5	10	50	100
$s(n)$				
$S(n)$				

(f) Show that $\lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n) = 4$.

In Exercises 57–66, use the limit process to find the area of the region between the graph of the function and the x-axis over the given interval. Sketch the region.

57. $y = -4x + 5$, $[0, 1]$

58. $y = 3x - 2$, $[2, 5]$

59. $y = x^2 + 2$, $[0, 1]$

60. $y = x^2 + 1$, $[0, 3]$

61. $y = 25 - x^2$, $[1, 4]$

62. $y = 4 - x^2$, $[-2, 2]$

63. $y = 27 - x^3$, $[1, 3]$

64. $y = 2x - x^3$, $[0, 1]$

65. $y = x^2 - x^3$, $[-1, 1]$

66. $y = x^2 - x^3$, $[-1, 0]$

In Exercises 67–72, use the limit process to find the area of the region between the graph of the function and the y-axis over the given y-interval. Sketch the region.

67. $f(y) = 4y$, $0 \le y \le 2$

68. $g(y) = \frac{1}{2}y$, $2 \le y \le 4$

69. $f(y) = y^2$, $0 \le y \le 5$

70. $f(y) = 4y - y^2$, $1 \le y \le 2$

71. $g(y) = 4y^2 - y^3$, $1 \le y \le 3$

72. $h(y) = y^3 + 1$, $1 \le y \le 2$

In Exercises 73–76, use the *Midpoint Rule*

$$\text{Area} \approx \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x$$

with $n = 4$ to approximate the area of the region bounded by the graph of the function and the x-axis over the given interval.

73. $f(x) = x^2 + 3$, $[0, 2]$

74. $f(x) = x^2 + 4x$, $[0, 4]$

75. $f(x) = \tan x$, $\left[0, \frac{\pi}{4}\right]$

76. $f(x) = \sin x$, $\left[0, \frac{\pi}{2}\right]$

Programming **Write a program for a graphing utility to approximate areas by using the Midpoint Rule. Assume that the function is positive over the given interval and that the subintervals are of equal width. In Exercises 77–82, use the program to approximate the area of the region between the graph of the function and the x-axis over the given interval, and complete the table.**

n	4	8	12	16	20
Approximate Area					

77. $f(x) = \sqrt{x}$, $[0, 4]$

78. $f(x) = \dfrac{8}{x^2 + 1}$, $[2, 6]$

79. $f(x) = \tan\left(\dfrac{\pi x}{8}\right)$, $[1, 3]$

80. $f(x) = \cos\sqrt{x}$, $[0, 2]$

81. $f(x) = \ln x$, $[1, 5]$

82. $f(x) = xe^x$, $[0, 2]$

WRITING ABOUT CONCEPTS

Approximation **In Exercises 83 and 84, determine which value best approximates the area of the region between the x-axis and the graph of the function over the given interval. (Make your selection on the basis of a sketch of the region and not by performing calculations.)**

83. $f(x) = 4 - x^2$, $[0, 2]$

(a) -2 (b) 6 (c) 10 (d) 3 (e) 8

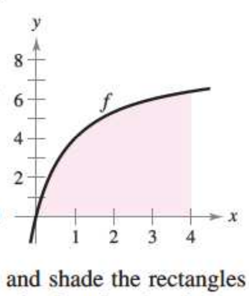
84. $f(x) = \sin\dfrac{\pi x}{4}$, $[0, 4]$

(a) 3 (b) 1 (c) -2 (d) 8 (e) 6

85. In your own words and using appropriate figures, describe the methods of upper sums and lower sums in approximating the area of a region.

86. Give the definition of the area of a region in the plane.

87. *Graphical Reasoning* Consider the region bounded by the graphs of $f(x) = 8x/(x + 1)$, $x = 0$, $x = 4$, and $y = 0$, as shown in the figure. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*



(a) Redraw the figure, and complete and shade the rectangles representing the lower sum when $n = 4$. Find this lower sum.

(b) Redraw the figure, and complete and shade the rectangles representing the upper sum when $n = 4$. Find this upper sum.

(c) Redraw the figure, and complete and shade the rectangles whose heights are determined by the functional values at the midpoint of each subinterval when $n = 4$. Find this sum using the Midpoint Rule.

(d) Verify the following formulas for approximating the area of the region using n subintervals of equal width.

Lower sum: $s(n) = \sum_{i=1}^{n} f\left[(i-1)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

Upper sum: $S(n) = \sum_{i=1}^{n} f\left[(i)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

Midpoint Rule: $M(n) = \sum_{i=1}^{n} f\left[\left(i-\frac{1}{2}\right)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

(e) Use a graphing utility and the formulas in part (d) to complete the table.

n	4	8	20	100	200
$s(n)$					
$S(n)$					
$M(n)$					

(f) Explain why $s(n)$ increases and $S(n)$ decreases for increasing values of n, as shown in the table in part (e).

CAPSTONE

88. Consider a function $f(x)$ that is increasing on the interval $[1, 4]$. The interval $[1, 4]$ is divided into 12 subintervals.

(a) What are the left endpoints of the first and last subintervals?

(b) What are the right endpoints of the first two subintervals?

(c) When using the right endpoints, will the rectangles lie above or below the graph of $f(x)$? Use a graph to explain your answer.

(d) What can you conclude about the heights of the rectangles if a function is constant on the given interval?

True or False? **In Exercises 89 and 90, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

89. The sum of the first n positive integers is $n(n+1)/2$.

90. If f is continuous and nonnegative on $[a, b]$, then the limits as $n \to \infty$ of its lower sum $s(n)$ and upper sum $S(n)$ both exist and are equal.

91. *Writing* Use the figure to write a short paragraph explaining why the formula $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ is valid for all positive integers n.

Figure for 91

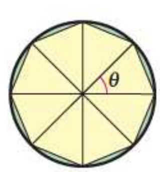


Figure for 92

92. *Graphical Reasoning* Consider an n-sided regular polygon inscribed in a circle of radius r. Join the vertices of the polygon to the center of the circle, forming n congruent triangles (see figure).

(a) Determine the central angle θ in terms of n.

(b) Show that the area of each triangle is $\frac{1}{2}r^2 \sin \theta$.

(c) Let A_n be the sum of the areas of the n triangles. Find $\lim_{n\to\infty} A_n$.

93. *Modeling Data* The table lists the measurements of a lot bounded by a stream and two straight roads that meet at right angles, where x and y are measured in feet (see figure).

x	0	50	100	150	200	250	300
y	450	362	305	268	245	156	0

(a) Use the regression capabilities of a graphing utility to find a model of the form $y = ax^3 + bx^2 + cx + d$ for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the model in part (a) to estimate the area of the lot.

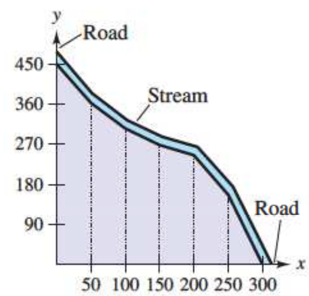


Figure for 93

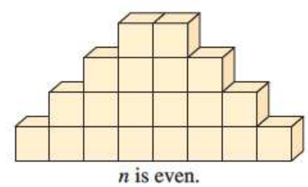


Figure for 94

94. *Building Blocks* A child places n cubic building blocks in a row to form the base of a triangular design (see figure). Each successive row contains two fewer blocks than the preceding row. Find a formula for the number of blocks used in the design. (*Hint:* The number of building blocks in the design depends on whether n is odd or even.)

95. Prove each formula by mathematical induction. (You may need to review the method of proof by induction from a precalculus text.)

(a) $\sum_{i=1}^{n} 2i = n(n+1)$ (b) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

PUTNAM EXAM CHALLENGE

96. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Write your answer in the form $(a\sqrt{b} + c)/d$, where a, b, c, and d are positive integers.

This problem was composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.

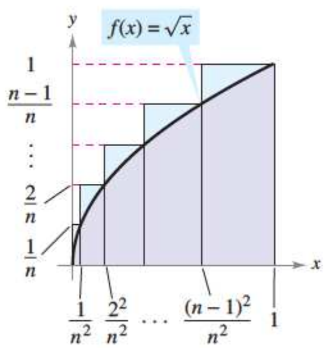
5.3 Riemann Sums and Definite Integrals

- Understand the definition of a Riemann sum.
- Evaluate a definite integral using limits.
- Evaluate a definite integral using properties of definite integrals.

Riemann Sums

In the definition of area given in Section 5.2, the partitions have subintervals of *equal width*. This was done only for computational convenience. The following example shows that it is not necessary to have subintervals of equal width.

EXAMPLE 1 A Partition with Subintervals of Unequal Widths



The subintervals do not have equal widths.
Figure 5.17

Consider the region bounded by the graph of $f(x) = \sqrt{x}$ and the x-axis for $0 \leq x \leq 1$, as shown in Figure 5.17. Evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

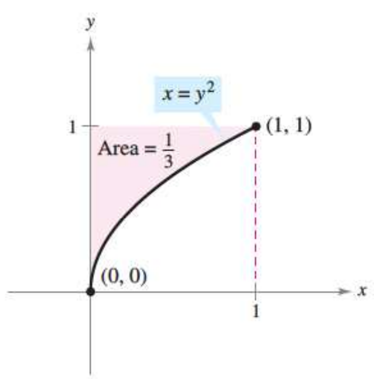
where c_i is the right endpoint of the partition given by $c_i = i^2/n^2$ and Δx_i is the width of the ith interval.

Solution The width of the ith interval is given by

$$\begin{aligned}\Delta x_i &= \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2}\\ &= \frac{i^2 - i^2 + 2i - 1}{n^2}\\ &= \frac{2i-1}{n^2}.\end{aligned}$$

So, the limit is

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{i^2}{n^2}}\left(\frac{2i-1}{n^2}\right)\\ &= \lim_{n\to\infty} \frac{1}{n^3}\sum_{i=1}^{n}(2i^2 - i)\\ &= \lim_{n\to\infty} \frac{1}{n^3}\left[2\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{n(n+1)}{2}\right]\\ &= \lim_{n\to\infty} \frac{4n^3 + 3n^2 - n}{6n^3}\\ &= \frac{2}{3}.\end{aligned}$$



The area of the region bounded by the graph of $x = y^2$ and the y-axis for $0 \leq y \leq 1$ is $\frac{1}{3}$.
Figure 5.18

From Example 7 in Section 5.2, you know that the region shown in Figure 5.18 has an area of $\frac{1}{3}$. Because the square bounded by $0 \leq x \leq 1$ and $0 \leq y \leq 1$ has an area of 1, you can conclude that the area of the region shown in Figure 5.17 has an area of $\frac{2}{3}$. This agrees with the limit found in Example 1, even though that example used a partition having subintervals of unequal widths. The reason this particular partition gave the proper area is that as n increases, the *width of the largest subinterval approaches zero*. This is a key feature of the development of definite integrals.


The Granger Collection

GEORG FRIEDRICH BERNHARD RIEMANN (1826–1866)

German mathematician Riemann did his most famous work in the areas of non-Euclidean geometry, differential equations, and number theory. It was Riemann's results in physics and mathematics that formed the structure on which Einstein's General Theory of Relativity is based.

In the preceding section, the limit of a sum was used to define the area of a region in the plane. Finding area by this means is only one of *many* applications involving the limit of a sum. A similar approach can be used to determine quantities as diverse as arc lengths, average values, centroids, volumes, work, and surface areas. The following definition is named after Georg Friedrich Bernhard Riemann. Although the definite integral had been defined and used long before the time of Riemann, he generalized the concept to cover a broader category of functions.

In the following definition of a Riemann sum, note that the function f has no restrictions other than being defined on the interval $[a, b]$. (In the preceding section, the function f was assumed to be continuous and nonnegative because we were dealing with the area under a curve.)

DEFINITION OF RIEMANN SUM

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval $[x_{i-1}, x_i]$. If c_i is *any* point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of f for the partition Δ.

NOTE The sums in Section 5.2 are examples of Riemann sums, but there are more general Riemann sums than those covered there.

The width of the largest subinterval of a partition Δ is the **norm** of the partition and is denoted by $\|\Delta\|$. If every subinterval is of equal width, the partition is **regular** and the norm is denoted by

$$\|\Delta\| = \Delta x = \frac{b-a}{n}. \qquad \text{Regular partition}$$

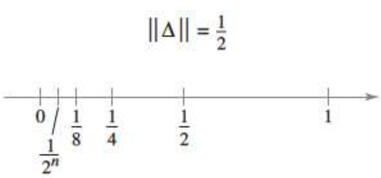
For a general partition, the norm is related to the number of subintervals of $[a, b]$ in the following way.

$$\frac{b-a}{\|\Delta\|} \le n \qquad \text{General partition}$$

So, the number of subintervals in a partition approaches infinity as the norm of the partition approaches 0. That is, $\|\Delta\| \to 0$ implies that $n \to \infty$.

The converse of this statement is not true. For example, let Δ_n be the partition of the interval $[0, 1]$ given by

$$0 < \frac{1}{2^n} < \frac{1}{2^{n-1}} < \cdot \cdot \cdot < \frac{1}{8} < \frac{1}{4} < \frac{1}{2} < 1.$$



$n \to \infty$ does not imply that $\|\Delta\| \to 0$.
Figure 5.19

As shown in Figure 5.19, for any positive value of n, the norm of the partition Δ_n is $\frac{1}{2}$. So, letting n approach infinity does not force $\|\Delta\|$ to approach 0. In a regular partition, however, the statements $\|\Delta\| \to 0$ and $n \to \infty$ are equivalent.

Definite Integrals

To define the definite integral, consider the following limit.

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = L$$

To say that this limit exists means there exists a real number L such that for each $\varepsilon > 0$ there exists a $\delta > 0$ so that for every partition with $\|\Delta\| < \delta$ it follows that

$$\left| L - \sum_{i=1}^{n} f(c_i)\,\Delta x_i \right| < \varepsilon$$

regardless of the choice of c_i in the ith subinterval of each partition Δ.

■ **FOR FURTHER INFORMATION** For insight into the history of the definite integral, see the article "The Evolution of Integration" by A. Shenitzer and J. Steprāns in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

DEFINITION OF DEFINITE INTEGRAL

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

exists (as described above), then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

It is not a coincidence that the notation used for definite integrals is similar to that used for indefinite integrals. You will see why in the next section when the Fundamental Theorem of Calculus is introduced. For now it is important to see that definite integrals and indefinite integrals are different concepts. A definite integral is a *number*, whereas an indefinite integral is a *family of functions*.

Though Riemann sums were defined for functions with very few restrictions, a sufficient condition for a function f to be integrable on $[a, b]$ is that it is continuous on $[a, b]$. A proof of this theorem is beyond the scope of this text.

STUDY TIP Later in this chapter, you will learn convenient methods for calculating $\int_a^b f(x)\,dx$ for continuous functions. For now, you must use the limit definition.

THEOREM 5.4 CONTINUITY IMPLIES INTEGRABILITY

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\,dx$ exists.

EXPLORATION

The Converse of Theorem 5.4 Is the converse of Theorem 5.4 true? That is, if a function is integrable, does it have to be continuous? Explain your reasoning and give examples.

Describe the relationships among continuity, differentiability, and integrability. Which is the strongest condition? Which is the weakest? Which conditions imply other conditions?

EXAMPLE 2 Evaluating a Definite Integral as a Limit

Evaluate the definite integral $\int_{-2}^{1} 2x\,dx$.

Solution The function $f(x) = 2x$ is integrable on the interval $[-2, 1]$ because it is continuous on $[-2, 1]$. Moreover, the definition of integrability implies that any partition whose norm approaches 0 can be used to determine the limit. For computational convenience, define Δ by subdividing $[-2, 1]$ into n subintervals of equal width

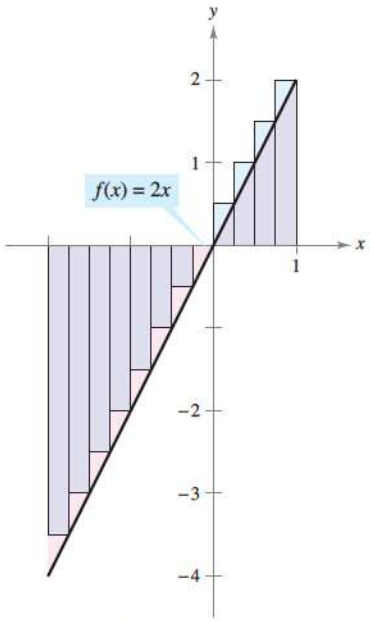
$$\Delta x_i = \Delta x = \frac{b-a}{n} = \frac{3}{n}.$$

Choosing c_i as the right endpoint of each subinterval produces

$$c_i = a + i(\Delta x) = -2 + \frac{3i}{n}.$$

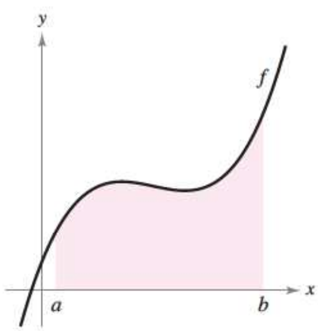
So, the definite integral is given by

$$\begin{aligned}
\int_{-2}^{1} 2x\,dx &= \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} 2\left(-2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} \left(-2 + \frac{3i}{n}\right) \\
&= \lim_{n\to\infty} \frac{6}{n} \left\{-2n + \frac{3}{n}\left[\frac{n(n+1)}{2}\right]\right\} \\
&= \lim_{n\to\infty} \left(-12 + 9 + \frac{9}{n}\right) \\
&= -3.
\end{aligned}$$



Because the definite integral is negative, it does not represent the area of the region.
Figure 5.20

Because the definite integral in Example 2 is negative, it *does not* represent the area of the region shown in Figure 5.20. Definite integrals can be positive, negative, or zero. For a definite integral to be interpreted as an area (as defined in Section 5.2), the function f must be continuous and nonnegative on $[a, b]$, as stated in the following theorem. The proof of this theorem is straightforward—you simply use the definition of area given in Section 5.2, because it is a Riemann sum.



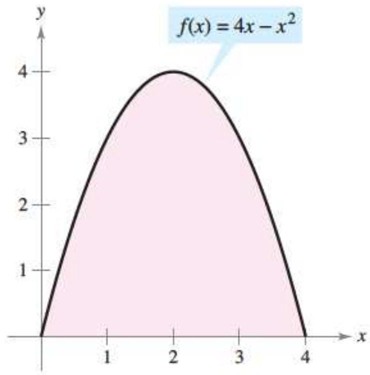
You can use a definite integral to find the area of the region bounded by the graph of f, the x-axis, $x = a$, and $x = b$.
Figure 5.21

THEOREM 5.5 THE DEFINITE INTEGRAL AS THE AREA OF A REGION

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x)\,dx.$$

(See Figure 5.21.)



Area $= \int_0^4 (4x - x^2)\,dx$

Figure 5.22

As an example of Theorem 5.5, consider the region bounded by the graph of

$$f(x) = 4x - x^2$$

and the x-axis, as shown in Figure 5.22. Because f is continuous and nonnegative on the closed interval $[0, 4]$, the area of the region is

$$\text{Area} = \int_0^4 (4x - x^2)\,dx.$$

A straightforward technique for evaluating a definite integral such as this will be discussed in Section 5.4. For now, however, you can evaluate a definite integral in two ways—you can use the limit definition *or* you can check to see whether the definite integral represents the area of a common geometric region such as a rectangle, triangle, or semicircle.

EXAMPLE 3 Areas of Common Geometric Figures

Sketch the region corresponding to each definite integral. Then evaluate each integral using a geometric formula.

a. $\displaystyle\int_1^3 4\,dx$ **b.** $\displaystyle\int_0^3 (x + 2)\,dx$ **c.** $\displaystyle\int_{-2}^2 \sqrt{4 - x^2}\,dx$

Solution A sketch of each region is shown in Figure 5.23.

a. This region is a rectangle of height 4 and width 2.

$$\int_1^3 4\,dx = (\text{Area of rectangle}) = 4(2) = 8$$

b. This region is a trapezoid with an altitude of 3 and parallel bases of lengths 2 and 5. The formula for the area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$.

$$\int_0^3 (x + 2)\,dx = (\text{Area of trapezoid}) = \frac{1}{2}(3)(2 + 5) = \frac{21}{2}$$

c. This region is a semicircle of radius 2. The formula for the area of a semicircle is $\frac{1}{2}\pi r^2$.

$$\int_{-2}^2 \sqrt{4 - x^2}\,dx = (\text{Area of semicircle}) = \frac{1}{2}\pi(2^2) = 2\pi$$

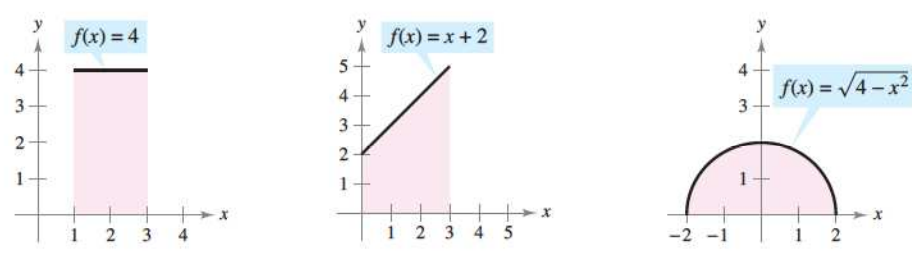


Figure 5.23

NOTE The variable of integration in a definite integral is sometimes called a *dummy variable* because it can be replaced by any other variable without changing the value of the integral. For instance, the definite integrals

$$\int_0^3 (x + 2)\,dx$$

and

$$\int_0^3 (t + 2)\,dt$$

have the same value.

Properties of Definite Integrals

The definition of the definite integral of f on the interval $[a, b]$ specifies that $a < b$. Now, however, it is convenient to extend the definition to cover cases in which $a = b$ or $a > b$. Geometrically, the following two definitions seem reasonable. For instance, it makes sense to define the area of a region of zero width and finite height to be 0.

DEFINITIONS OF TWO SPECIAL DEFINITE INTEGRALS

1. If f is defined at $x = a$, then we define $\int_a^a f(x)\,dx = 0$.

2. If f is integrable on $[a, b]$, then we define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

EXAMPLE 4 Evaluating Definite Integrals

a. Because the sine function is defined at $x = \pi$, and the upper and lower limits of integration are equal, you can write

$$\int_\pi^\pi \sin x\,dx = 0.$$

b. The integral $\int_3^0 (x + 2)\,dx$ is the same as that given in Example 3(b) except that the upper and lower limits are interchanged. Because the integral in Example 3(b) has a value of $\frac{21}{2}$, you can write

$$\int_3^0 (x + 2)\,dx = -\int_0^3 (x + 2)\,dx = -\frac{21}{2}.$$

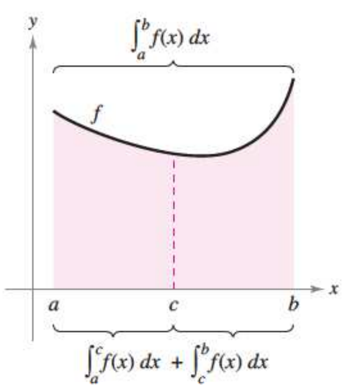


Figure 5.24

In Figure 5.24, the shaded region can be divided at $x = c$ into two subregions whose intersection is a line segment. Because the line segment has zero area, it follows that the area of the entire shaded region is equal to the sum of the areas of the two subregions.

THEOREM 5.6 ADDITIVE INTERVAL PROPERTY

If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

EXAMPLE 5 Using the Additive Interval Property

$$\begin{aligned}\int_{-1}^1 |x|\,dx &= \int_{-1}^0 -x\,dx + \int_0^1 x\,dx && \text{Theorem 5.6}\\ &= \frac{1}{2} + \frac{1}{2} && \text{Area of a triangle}\\ &= 1\end{aligned}$$

Because the definite integral is defined as the limit of a sum, it inherits the properties of summation given at the top of page 296.

THEOREM 5.7 PROPERTIES OF DEFINITE INTEGRALS

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

2. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

Note that Property 2 of Theorem 5.7 can be extended to cover any finite number of functions. For example,

$$\int_a^b [f(x) + g(x) + h(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx + \int_a^b h(x)\,dx.$$

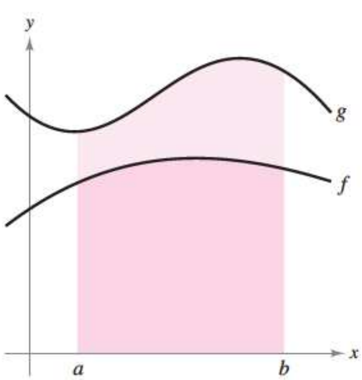
EXAMPLE 6 Evaluation of a Definite Integral

Evaluate $\displaystyle\int_1^3 (-x^2 + 4x - 3)\,dx$ using each of the following values.

$$\int_1^3 x^2\,dx = \frac{26}{3}, \qquad \int_1^3 x\,dx = 4, \qquad \int_1^3 dx = 2$$

Solution

$$\begin{aligned}\int_1^3 (-x^2 + 4x - 3)\,dx &= \int_1^3 (-x^2)\,dx + \int_1^3 4x\,dx + \int_1^3 (-3)\,dx \\ &= -\int_1^3 x^2\,dx + 4\int_1^3 x\,dx - 3\int_1^3 dx \\ &= -\left(\frac{26}{3}\right) + 4(4) - 3(2) \\ &= \frac{4}{3}\end{aligned}$$



$\displaystyle\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

Figure 5.25

If f and g are continuous on the closed interval $[a, b]$ and

$0 \le f(x) \le g(x)$

for $a \le x \le b$, the following properties are true. First, the area of the region bounded by the graph of f and the x-axis (between a and b) must be nonnegative. Second, this area must be less than or equal to the area of the region bounded by the graph of g and the x-axis (between a and b), as shown in Figure 5.25. These two properties are generalized in Theorem 5.8. (A proof of this theorem is given in Appendix A.)

THEOREM 5.8 PRESERVATION OF INEQUALITY

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \le \int_a^b f(x)\,dx.$$

2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \le g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

5.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use Example 1 as a model to evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

over the region bounded by the graphs of the equations.

1. $f(x) = \sqrt{x}, \quad y = 0, \quad x = 0, \quad x = 3$
   (*Hint:* Let $c_i = 3i^2/n^2$.)
2. $f(x) = 2\sqrt[3]{x}, \quad y = 0, \quad x = 0, \quad x = 1$
   (*Hint:* Let $c_i = i^3/n^3$.)

In Exercises 3–8, evaluate the definite integral by the limit definition.

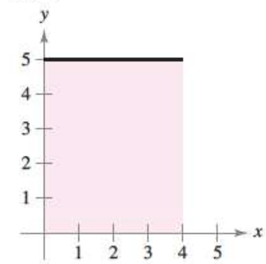
3. $\displaystyle\int_2^6 8\,dx$
4. $\displaystyle\int_{-2}^3 x\,dx$
5. $\displaystyle\int_{-1}^1 x^3\,dx$
6. $\displaystyle\int_1^4 4x^2\,dx$
7. $\displaystyle\int_1^2 (x^2 + 1)\,dx$
8. $\displaystyle\int_{-2}^1 (2x^2 + 3)\,dx$

In Exercises 9–14, write the limit as a definite integral on the interval $[a, b]$, where c_i is any point in the ith subinterval.

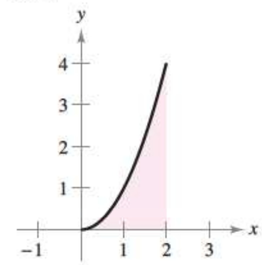
	Limit	*Interval*
9.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (3c_i + 10)\,\Delta x_i$	$[-1, 5]$
10.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 6c_i(4 - c_i)^2\,\Delta x_i$	$[0, 4]$
11.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \sqrt{c_i^2 + 4}\,\Delta x_i$	$[0, 3]$
12.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \left(\frac{3}{c_i^2}\right)\Delta x_i$	$[1, 3]$
13.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \left(1 + \frac{3}{c_i}\right)\Delta x_i$	$[1, 5]$
14.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (2^{-c_i}\sin c_i)\,\Delta x_i$	$[0, \pi]$

In Exercises 15–22, set up a definite integral that yields the area of the region. (Do not evaluate the integral.)

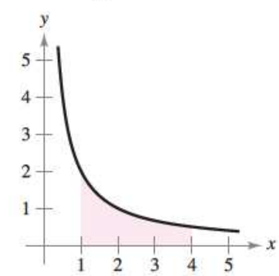
15. $f(x) = 5$



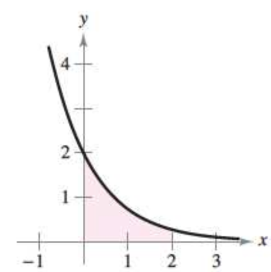
16. $f(x) = x^2$



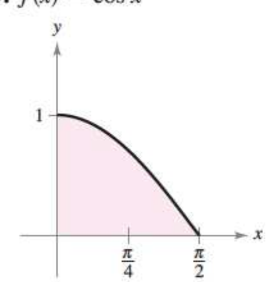
17. $f(x) = \dfrac{2}{x}$



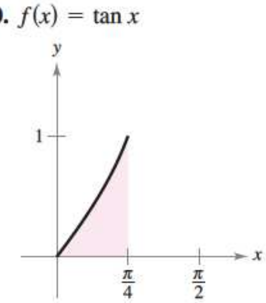
18. $f(x) = 2e^{-x}$



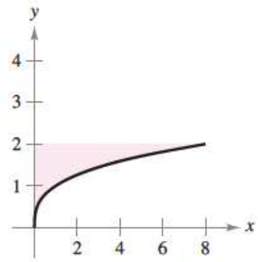
19. $f(x) = \cos x$



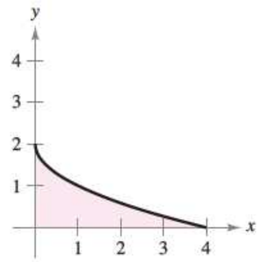
20. $f(x) = \tan x$



21. $g(y) = y^3$



22. $f(y) = (y - 2)^2$



In Exercises 23–32, sketch the region whose area is given by the definite integral. Then use a geometric formula to evaluate the integral ($a > 0, r > 0$).

23. $\int_0^3 4\,dx$

24. $\int_{-a}^a 4\,dx$

25. $\int_0^4 x\,dx$

26. $\int_0^4 \frac{x}{2}\,dx$

27. $\int_0^2 (3x + 4)\,dx$

28. $\int_0^6 (6 - x)\,dx$

29. $\int_{-1}^1 (1 - |x|)\,dx$

30. $\int_{-a}^a (a - |x|)\,dx$

31. $\int_{-7}^7 \sqrt{49 - x^2}\,dx$

32. $\int_{-r}^r \sqrt{r^2 - x^2}\,dx$

In Exercises 33–40, evaluate the integral using the following values.

$$\int_2^4 x^3\,dx = 60, \qquad \int_2^4 x\,dx = 6, \qquad \int_2^4 dx = 2$$

33. $\int_4^2 x\,dx$

34. $\int_2^2 x^3\,dx$

35. $\int_2^4 8x\,dx$

36. $\int_2^4 25\,dx$

37. $\int_2^4 (x - 9)\,dx$

38. $\int_2^4 (x^3 + 4)\,dx$

39. $\int_2^4 \left(\frac{1}{2}x^3 - 3x + 2\right) dx$

40. $\int_2^4 (10 + 4x - 3x^3)\,dx$

41. Given $\int_0^5 f(x)\,dx = 10$ and $\int_5^7 f(x)\,dx = 3$, evaluate

(a) $\int_0^7 f(x)\,dx$. (b) $\int_5^0 f(x)\,dx$.

(c) $\int_5^5 f(x)\,dx$. (d) $\int_0^5 3f(x)\,dx$.

42. Given $\int_0^3 f(x)\,dx = 4$ and $\int_3^6 f(x)\,dx = -1$, evaluate

(a) $\int_0^6 f(x)\,dx$. (b) $\int_6^3 f(x)\,dx$.

(c) $\int_3^3 f(x)\,dx$. (d) $\int_3^6 -5f(x)\,dx$.

43. Given $\int_2^6 f(x)\,dx = 10$ and $\int_2^6 g(x)\,dx = -2$, evaluate

(a) $\int_2^6 [f(x) + g(x)]\,dx$. (b) $\int_2^6 [g(x) - f(x)]\,dx$.

(c) $\int_2^6 2g(x)\,dx$. (d) $\int_2^6 3f(x)\,dx$.

44. Given $\int_{-1}^1 f(x)\,dx = 0$ and $\int_0^1 f(x)\,dx = 5$, evaluate

(a) $\int_{-1}^0 f(x)\,dx$. (b) $\int_0^1 f(x)\,dx - \int_{-1}^0 f(x)\,dx$.

(c) $\int_{-1}^1 3f(x)\,dx$. (d) $\int_0^1 3f(x)\,dx$.

45. Use the table of values to find lower and upper estimates of

$$\int_0^{10} f(x)\,dx.$$

Assume that f is a decreasing function.

x	0	2	4	6	8	10
$f(x)$	32	24	12	-4	-20	-36

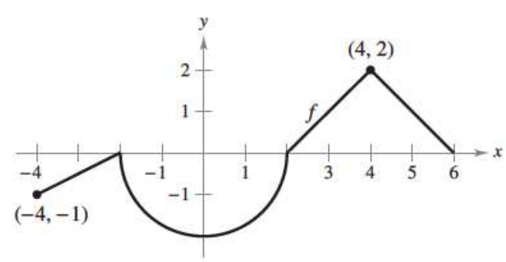
46. Use the table of values to estimate

$$\int_0^6 f(x)\,dx.$$

Use three equal subintervals and (a) the left endpoints, (b) the right endpoints, and (c) the midpoints. If f is an increasing function, how does each estimate compare with the actual value? Explain your reasoning.

x	0	1	2	3	4	5	6
$f(x)$	-6	0	8	18	30	50	80

47. ***Think About It*** The graph of f consists of line segments and a semicircle, as shown in the figure. Evaluate each definite integral by using geometric formulas.

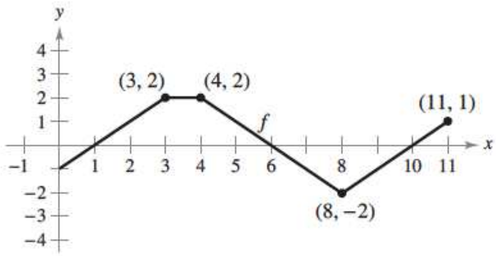


(a) $\int_0^2 f(x)\,dx$ (b) $\int_2^6 f(x)\,dx$

(c) $\int_{-4}^2 f(x)\,dx$ (d) $\int_{-4}^6 f(x)\,dx$

(e) $\int_{-4}^6 |f(x)|\,dx$ (f) $\int_{-4}^6 [f(x) + 2]\,dx$

48. ***Think About It*** The graph of f consists of line segments, as shown in the figure. Evaluate each definite integral by using geometric formulas.



(a) $\int_0^1 -f(x)\,dx$ (b) $\int_3^4 3f(x)\,dx$

(c) $\int_0^7 f(x)\,dx$ (d) $\int_5^{11} f(x)\,dx$

(e) $\int_0^{11} f(x)\,dx$ (f) $\int_4^{10} f(x)\,dx$

49. ***Think About It*** Consider the function f that is continuous on the interval $[-5, 5]$ and for which

$$\int_0^5 f(x)\,dx = 4.$$

Evaluate each integral.

(a) $\int_0^5 [f(x) + 2]\,dx$ (b) $\int_{-2}^{3} f(x + 2)\,dx$

(c) $\int_{-5}^{5} f(x)\,dx$ (f is even.) (d) $\int_{-5}^{5} f(x)\,dx$ (f is odd.)

50. ***Think About It*** A function f is defined below. Use geometric formulas to find $\int_0^8 f(x)\,dx$.

$$f(x) = \begin{cases} 4, & x < 4 \\ x, & x \geq 4 \end{cases}$$

51. ***Think About It*** A function f is defined below. Use geometric formulas to find $\int_0^{12} f(x)\,dx$.

$$f(x) = \begin{cases} 6, & x > 6 \\ -\frac{1}{2}x + 9, & x \leq 6 \end{cases}$$

CAPSTONE

52. Find possible values of a and b that make the statement true. If possible, use a graph to support your answer. (There may be more than one correct answer.)

(a) $\int_{-2}^{1} f(x)\,dx + \int_1^5 f(x)\,dx = \int_a^b f(x)\,dx$

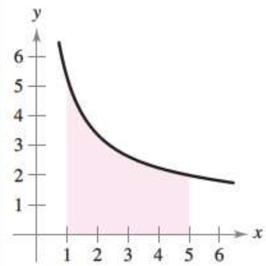
(b) $\int_{-3}^{3} f(x)\,dx + \int_3^6 f(x)\,dx - \int_a^b f(x)\,dx = \int_{-1}^{6} f(x)\,dx$

(c) $\int_a^b \sin x\,dx < 0$

(d) $\int_a^b \cos x\,dx = 0$

WRITING ABOUT CONCEPTS

In Exercises 53 and 54, use the figure to fill in the blank with the symbol <, >, or =.



53. The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the left endpoint of the ith subinterval.

$$\sum_{i=1}^{n} f(x_i)\,\Delta x \quad \square \quad \int_1^5 f(x)\,dx$$

54. The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the right endpoint of the ith subinterval.

$$\sum_{i=1}^{n} f(x_i)\,\Delta x \quad \square \quad \int_1^5 f(x)\,dx$$

55. Determine whether the function $f(x) = \dfrac{1}{x - 4}$ is integrable on the interval $[3, 5]$. Explain.

56. Give an example of a function that is integrable on the interval $[-1, 1]$, but not continuous on $[-1, 1]$.

In Exercises 57–62, determine which value best approximates the definite integral. Make your selection on the basis of a sketch.

57. $\int_0^4 \sqrt{x}\,dx$

(a) 5 (b) -3 (c) 10 (d) 2 (e) 8

58. $\int_0^{1/2} 4\cos \pi x\,dx$

(a) 4 (b) $\frac{4}{3}$ (c) 16 (d) 2π (e) -6

59. $\int_0^1 2\sin \pi x\,dx$

(a) 6 (b) $\frac{1}{2}$ (c) 4 (d) $\frac{5}{4}$

60. $\int_0^9 (1 + \sqrt{x})\,dx$

(a) -3 (b) 9 (c) 27 (d) 3

61. $\int_0^2 2e^{-x^2}\,dx$

(a) $\frac{1}{3}$ (b) 6 (c) 2 (d) 4

62. $\int_1^2 \ln x\,dx$

(a) $\frac{1}{3}$ (b) 1 (c) 4 (d) 3

Programming **Write a program for your graphing utility to approximate a definite integral using the Riemann sum**

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

where the subintervals are of equal width. The output should give three approximations of the integral, where c_i is the left-hand endpoint $L(n)$, the midpoint $M(n)$, and the right-hand endpoint $R(n)$ of each subinterval. In Exercises 63–68, use the program to approximate the definite integral and complete the table.

n	4	8	12	16	20
$L(n)$					
$M(n)$					
$R(n)$					

63. $\displaystyle\int_0^3 x\sqrt{3-x}\,dx$

64. $\displaystyle\int_0^3 \frac{5}{x^2+1}\,dx$

65. $\displaystyle\int_1^3 \frac{1}{x}\,dx$

66. $\displaystyle\int_0^4 e^x\,dx$

67. $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx$

68. $\displaystyle\int_0^3 x\sin x\,dx$

True or False? **In Exercises 69–74, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

69. $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

70. $\displaystyle\int_a^b f(x)g(x)\,dx = \left[\int_a^b f(x)\,dx\right]\left[\int_a^b g(x)\,dx\right]$

71. If the norm of a partition approaches zero, then the number of subintervals approaches infinity.

72. If f is increasing on $[a, b]$, then the minimum value of $f(x)$ on $[a, b]$ is $f(a)$.

73. The value of $\int_a^b f(x)\,dx$ must be positive.

74. The value of $\int_2^2 \sin(x^2)\,dx$ is 0.

75. Find the Riemann sum for $f(x) = x^2 + 3x$ over the interval $[0, 8]$ (see figure), where $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 7$, and $x_4 = 8$, and where $c_1 = 1$, $c_2 = 2$, $c_3 = 5$, and $c_4 = 8$.

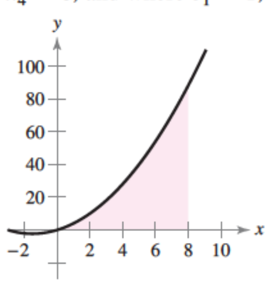


Figure for 75

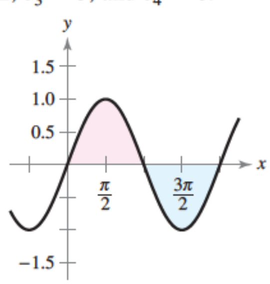


Figure for 76

76. Find the Riemann sum for $f(x) = \sin x$ over the interval $[0, 2\pi]$ (see figure), where $x_0 = 0$, $x_1 = \pi/4$, $x_2 = \pi/3$, $x_3 = \pi$, and $x_4 = 2\pi$, and where $c_1 = \pi/6$, $c_2 = \pi/3$, $c_3 = 2\pi/3$, and $c_4 = 3\pi/2$.

77. Prove that $\displaystyle\int_a^b x\,dx = \frac{b^2-a^2}{2}$.

78. Prove that $\displaystyle\int_a^b x^2\,dx = \frac{b^3-a^3}{3}$.

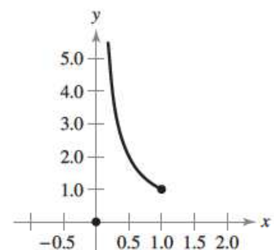
79. ***Think About It*** Determine whether the Dirichlet function

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

is integrable on the interval $[0, 1]$. Explain.

80. Suppose the function f is defined on $[0, 1]$, as shown in the figure.

$$f(x) = \begin{cases} 0, & x = 0 \\ \dfrac{1}{x}, & 0 < x \le 1 \end{cases}$$



Show that $\int_0^1 f(x)\,dx$ does not exist. Why doesn't this contradict Theorem 5.4?

81. Find the constants a and b that maximize the value of

$$\int_a^b (1-x^2)\,dx.$$

Explain your reasoning.

82. Evaluate, if possible, the integral $\displaystyle\int_0^2 [\![x]\!]\,dx$.

83. Determine

$$\lim_{n\to\infty} \frac{1}{n^3}[1^2 + 2^2 + 3^2 + \cdots + n^2]$$

by using an appropriate Riemann sum.

PUTNAM EXAM CHALLENGE

84. For each continuous function $f: [0, 1] \to R$, let $I(f) = \int_0^1 x^2 f(x)\,dx$ and $J(x) = \int_0^1 x(f(x))^2\,dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

5.4 The Fundamental Theorem of Calculus

- Evaluate a definite integral using the Fundamental Theorem of Calculus.
- Understand and use the Mean Value Theorem for Integrals.
- Find the average value of a function over a closed interval.
- Understand and use the Second Fundamental Theorem of Calculus.
- Understand and use the Net Change Theorem.

EXPLORATION

Integration and Antidifferentiation Throughout this chapter, you have been using the integral symbol to denote an antiderivative (a family of functions) and a definite integral (a number).

Antidifferentiation: $\int f(x)\,dx$

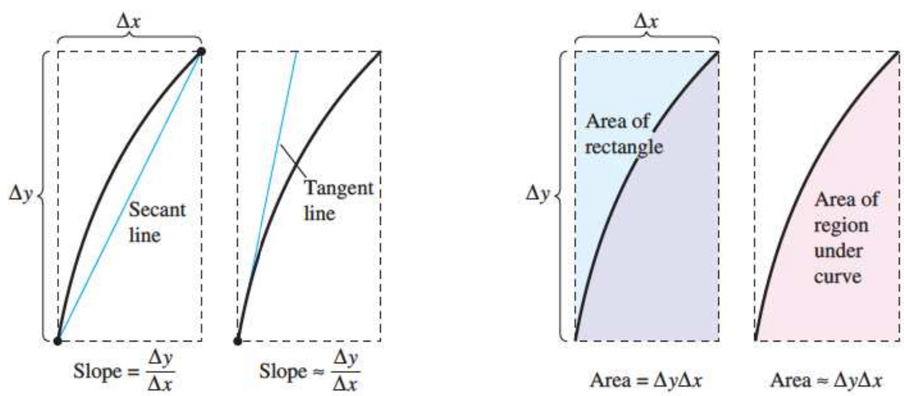
Definite integration: $\int_a^b f(x)\,dx$

The use of this same symbol for both operations makes it appear that they are related. In the early work with calculus, however, it was not known that the two operations were related. Do you think the symbol $\int$ was first applied to antidifferentiation or to definite integration? Explain your reasoning. (*Hint:* The symbol was first used by Leibniz and was derived from the letter S.)

The Fundamental Theorem of Calculus

You have now been introduced to the two major branches of calculus: differential calculus (introduced with the tangent line problem) and integral calculus (introduced with the area problem). At this point, these two problems might seem unrelated—but there is a very close connection. The connection was discovered independently by Isaac Newton and Gottfried Leibniz and is stated in a theorem that is appropriately called the **Fundamental Theorem of Calculus.**

Informally, the theorem states that differentiation and (definite) integration are inverse operations, in the same sense that division and multiplication are inverse operations. To see how Newton and Leibniz might have anticipated this relationship, consider the approximations shown in Figure 5.26. The slope of the tangent line was defined using the *quotient* $\Delta y/\Delta x$ (the slope of the secant line). Similarly, the area of a region under a curve was defined using the *product* $\Delta y \Delta x$ (the area of a rectangle). So, at least in the primitive approximation stage, the operations of differentiation and definite integration appear to have an inverse relationship in the same sense that division and multiplication are inverse operations. The Fundamental Theorem of Calculus states that the limit processes (used to define the derivative and definite integral) preserve this inverse relationship.



(a) Differentiation **(b)** Definite integration

Differentiation and definite integration have an "inverse" relationship.

Figure 5.26

THEOREM 5.9 THE FUNDAMENTAL THEOREM OF CALCULUS

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

PROOF The key to the proof is in writing the difference $F(b) - F(a)$ in a convenient form. Let Δ be any partition of $[a, b]$.

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

By pairwise subtraction and addition of like terms, you can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \cdots - F(x_1) + F(x_1) - F(x_0) \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]. \end{aligned}$$

By the Mean Value Theorem, you know that there exists a number c_i in the ith subinterval such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}.$$

Because $F'(c_i) = f(c_i)$, you can let $\Delta x_i = x_i - x_{i-1}$ and obtain

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i.$$

This important equation tells you that by repeatedly applying the Mean Value Theorem, you can always find a collection of c_i's such that the *constant* $F(b) - F(a)$ is a Riemann sum of f on $[a, b]$ for any partition. Theorem 5.4 guarantees that the limit of Riemann sums over the partition with $\|\Delta\| \to 0$ exists. So, taking the limit (as $\|\Delta\| \to 0$) produces

$$F(b) - F(a) = \int_a^b f(x)\,dx.$$

The following guidelines can help you understand the use of the Fundamental Theorem of Calculus.

GUIDELINES FOR USING THE FUNDAMENTAL THEOREM OF CALCULUS

1. *Provided you can find* an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the following notation is convenient.

$$\begin{aligned} \int_a^b f(x)\,dx &= F(x)\Big]_a^b \\ &= F(b) - F(a) \end{aligned}$$

For instance, to evaluate $\int_1^3 x^3\,dx$, you can write

$$\int_1^3 x^3\,dx = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

3. It is not necessary to include a constant of integration C in the antiderivative because

$$\begin{aligned} \int_a^b f(x)\,dx &= \Big[F(x) + C\Big]_a^b \\ &= [F(b) + C] - [F(a) + C] \\ &= F(b) - F(a). \end{aligned}$$

EXAMPLE 1 Evaluating a Definite Integral

Evaluate each definite integral.

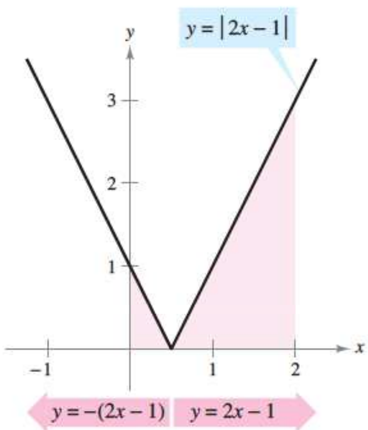
a. $\int_1^2 (x^2 - 3)\,dx$ **b.** $\int_1^4 3\sqrt{x}\,dx$ **c.** $\int_0^{\pi/4} \sec^2 x\,dx$

Solution

a. $\int_1^2 (x^2 - 3)\,dx = \left[\frac{x^3}{3} - 3x\right]_1^2 = \left(\frac{8}{3} - 6\right) - \left(\frac{1}{3} - 3\right) = -\frac{2}{3}$

b. $\int_1^4 3\sqrt{x}\,dx = 3\int_1^4 x^{1/2}\,dx = 3\left[\frac{x^{3/2}}{3/2}\right]_1^4 = 2(4)^{3/2} - 2(1)^{3/2} = 14$

c. $\int_0^{\pi/4} \sec^2 x\,dx = \tan x\Big]_0^{\pi/4} = 1 - 0 = 1$



The definite integral of y on $[0, 2]$ is $\frac{5}{2}$.
Figure 5.27

EXAMPLE 2 A Definite Integral Involving Absolute Value

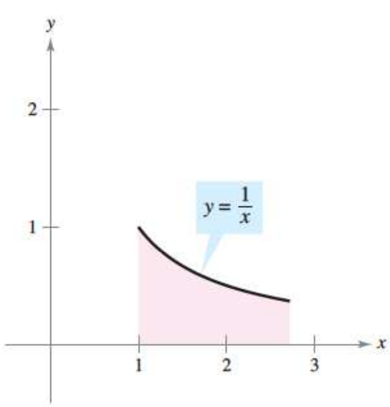
Evaluate $\int_0^2 |2x - 1|\,dx$.

Solution Using Figure 5.27 and the definition of absolute value, you can rewrite the integrand as shown.

$$|2x - 1| = \begin{cases} -(2x - 1), & x < \frac{1}{2} \\ 2x - 1, & x \geq \frac{1}{2} \end{cases}$$

From this, you can rewrite the integral in two parts.

$$\begin{aligned}
\int_0^2 |2x - 1|\,dx &= \int_0^{1/2} -(2x - 1)\,dx + \int_{1/2}^2 (2x - 1)\,dx \\
&= \Big[-x^2 + x\Big]_0^{1/2} + \Big[x^2 - x\Big]_{1/2}^2 \\
&= \left(-\frac{1}{4} + \frac{1}{2}\right) - (0 + 0) + (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right) \\
&= \frac{5}{2}
\end{aligned}$$



The area of the region bounded by the graph of $y = 1/x$, the x-axis, $x = 1$, and $x = e$ is 1.
Figure 5.28

EXAMPLE 3 Using the Fundamental Theorem to Find Area

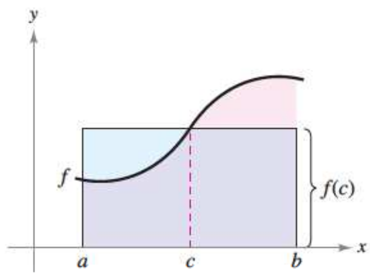
Find the area of the region bounded by the graph of $y = 1/x$, the x-axis, and the vertical lines $x = 1$ and $x = e$, as shown in Figure 5.28.

Solution Note that $y > 0$ on the interval $[1, e]$.

$$\begin{aligned}
\text{Area} &= \int_1^e \frac{1}{x}\,dx && \text{Integrate between } x = 1 \text{ and } x = e. \\
&= \Big[\ln x\Big]_1^e && \text{Find antiderivative.} \\
&= (\ln e) - (\ln 1) && \text{Apply Fundamental Theorem of Calculus.} \\
&= 1 && \text{Simplify.}
\end{aligned}$$

■

The Mean Value Theorem for Integrals



Mean value rectangle:

$f(c)(b-a) = \int_a^b f(x)\,dx$

Figure 5.29

In Section 5.2, you saw that the area of a region under a curve is greater than the area of an inscribed rectangle and less than the area of a circumscribed rectangle. The Mean Value Theorem for Integrals states that somewhere "between" the inscribed and circumscribed rectangles there is a rectangle whose area is precisely equal to the area of the region under the curve, as shown in Figure 5.29.

> **THEOREM 5.10 MEAN VALUE THEOREM FOR INTEGRALS**
>
> If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that
>
> $$\int_a^b f(x)\,dx = f(c)(b-a).$$

PROOF

Case 1: If f is constant on the interval $[a, b]$, the theorem is clearly valid because c can be any point in $[a, b]$.

Case 2: If f is not constant on $[a, b]$, then, by the Extreme Value Theorem, you can choose $f(m)$ and $f(M)$ to be the minimum and maximum values of f on $[a, b]$. Because $f(m) \le f(x) \le f(M)$ for all x in $[a, b]$, you can apply Theorem 5.8 to write the following.

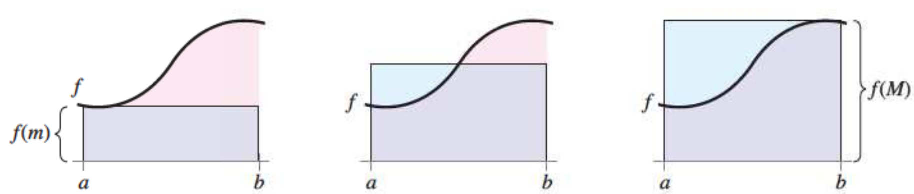
$$\int_a^b f(m)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(M)\,dx \qquad \text{See Figure 5.30.}$$

$$f(m)(b-a) \le \int_a^b f(x)\,dx \le f(M)(b-a)$$

$$f(m) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(M)$$

From the third inequality, you can apply the Intermediate Value Theorem to conclude that there exists some c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \qquad \text{or} \qquad f(c)(b-a) = \int_a^b f(x)\,dx.$$



Inscribed rectangle (less than actual area)

$\int_a^b f(m)\,dx = f(m)(b-a)$

Mean value rectangle (equal to actual area)

$\int_a^b f(x)\,dx$

Circumscribed rectangle (greater than actual area)

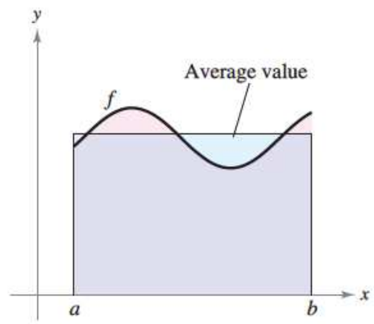
$\int_a^b f(M)\,dx = f(M)(b-a)$

Figure 5.30 ■

NOTE Notice that Theorem 5.10 does not specify how to determine c. It merely guarantees the existence of at least one number c in the interval. ■

Average Value of a Function

The value of $f(c)$ given in the Mean Value Theorem for Integrals is called the **average value** of f on the interval $[a, b]$.



Average value $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

Figure 5.31

NOTE Notice in Figure 5.31 that the area of the region under the graph of f is equal to the area of the rectangle whose height is the average value.

DEFINITION OF THE AVERAGE VALUE OF A FUNCTION ON AN INTERVAL

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

To see why the average value of f is defined in this way, suppose that you partition $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. If c_i is any point in the ith subinterval, the arithmetic average (or mean) of the function values at the c_i's is given by

$$a_n = \frac{1}{n}[f(c_1) + f(c_2) + \cdot\cdot\cdot + f(c_n)]. \qquad \text{Average of } f(c_1), \ldots, f(c_n)$$

By multiplying and dividing by $(b-a)$, you can write the average as

$$\begin{aligned} a_n &= \frac{1}{n}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{b-a}\right) = \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{n}\right) \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\,\Delta x. \end{aligned}$$

Finally, taking the limit as $n \to \infty$ produces the average value of f on the interval $[a, b]$, as given in the definition above.

This development of the average value of a function on an interval is only one of many practical uses of definite integrals to represent summation processes. In Chapter 7, you will study other applications, such as volume, arc length, centers of mass, and work.

EXAMPLE 4 Finding the Average Value of a Function

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

Solution The average value is given by

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\,dx &= \frac{1}{4-1}\int_1^4 (3x^2 - 2x)\,dx \\ &= \frac{1}{3}\Big[x^3 - x^2\Big]_1^4 \\ &= \frac{1}{3}[64 - 16 - (1 - 1)] \\ &= \frac{48}{3} \\ &= 16. \end{aligned}$$

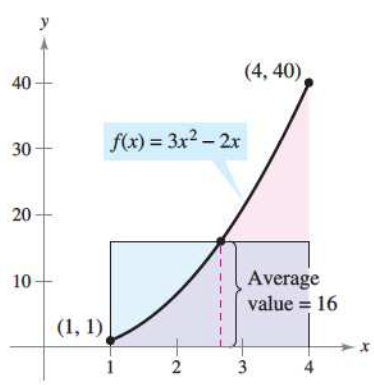


Figure 5.32

(See Figure 5.32.) ■


George Hall/Corbis

The first person to fly at a speed greater than the speed of sound was Charles Yeager. On October 14, 1947, Yeager was clocked at 295.9 meters per second at an altitude of 12.2 kilometers. If Yeager had been flying at an altitude below 11.275 kilometers, this speed would not have "broken the sound barrier." The photo above shows an F-14 *Tomcat*, a supersonic, twin-engine strike fighter. Currently, the *Tomcat* can reach heights of 15.24 kilometers and speeds up to 2 mach (707.78 meters per second).

EXAMPLE 5 The Speed of Sound

At different altitudes in Earth's atmosphere, sound travels at different speeds. The speed of sound $s(x)$ (in meters per second) can be modeled by

$$s(x) = \begin{cases} -4x + 341, & 0 \le x < 11.5 \\ 295, & 11.5 \le x < 22 \\ \frac{3}{4}x + 278.5, & 22 \le x < 32 \\ \frac{3}{2}x + 254.5, & 32 \le x < 50 \\ -\frac{3}{2}x + 404.5, & 50 \le x \le 80 \end{cases}$$

where x is the altitude in kilometers (see Figure 5.33). What is the average speed of sound over the interval $[0, 80]$?

Solution Begin by integrating $s(x)$ over the interval $[0, 80]$. To do this, you can break the integral into five parts.

$$\int_0^{11.5} s(x)\,dx = \int_0^{11.5} (-4x + 341)\,dx = \Big[-2x^2 + 341x\Big]_0^{11.5} = 3657$$

$$\int_{11.5}^{22} s(x)\,dx = \int_{11.5}^{22} (295)\,dx = \Big[295x\Big]_{11.5}^{22} = 3097.5$$

$$\int_{22}^{32} s(x)\,dx = \int_{22}^{32} \left(\tfrac{3}{4}x + 278.5\right)dx = \Big[\tfrac{3}{8}x^2 + 278.5x\Big]_{22}^{32} = 2987.5$$

$$\int_{32}^{50} s(x)\,dx = \int_{32}^{50} \left(\tfrac{3}{2}x + 254.5\right)dx = \Big[\tfrac{3}{4}x^2 + 254.5x\Big]_{32}^{50} = 5688$$

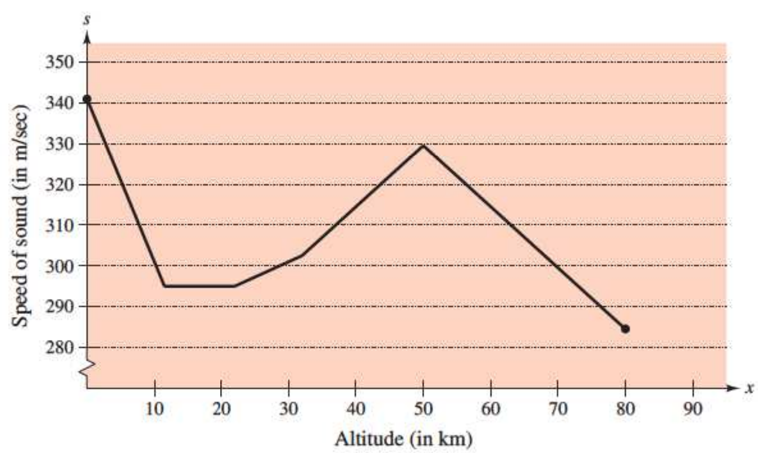
$$\int_{50}^{80} s(x)\,dx = \int_{50}^{80} \left(-\tfrac{3}{2}x + 404.5\right)dx = \Big[-\tfrac{3}{4}x^2 + 404.5x\Big]_{50}^{80} = 9210$$

By adding the values of the five integrals, you have

$$\int_0^{80} s(x)\,dx = 24{,}640.$$

So, the average speed of sound from an altitude of 0 kilometers to an altitude of 80 kilometers is

$$\text{Average speed} = \frac{1}{80}\int_0^{80} s(x)\,dx = \frac{24{,}640}{80} = 308 \text{ meters per second.}$$



Speed of sound depends on altitude.
Figure 5.33

The Second Fundamental Theorem of Calculus

Earlier you saw that the definite integral of f on the interval $[a, b]$ was defined using the constant b as the upper limit of integration and x as the variable of integration. However, a slightly different situation may arise in which the variable x is used as the upper limit of integration. To avoid the confusion of using x in two different ways, t is temporarily used as the variable of integration. (Remember that the definite integral is *not* a function of its variable of integration.)

The Definite Integral as a Number

Constant

$$\int_a^b f(x)\,dx$$

Constant

f is a function of x.

The Definite Integral as a Function of x

F is a function of x.

$$F(x) = \int_a^x f(t)\,dt$$

Constant

f is a function of t.

EXPLORATION

Use a graphing utility to graph the function

$$F(x) = \int_0^x \cos t\,dt$$

for $0 \le x \le \pi$. Do you recognize this graph? Explain.

EXAMPLE 6 The Definite Integral as a Function

Evaluate the function

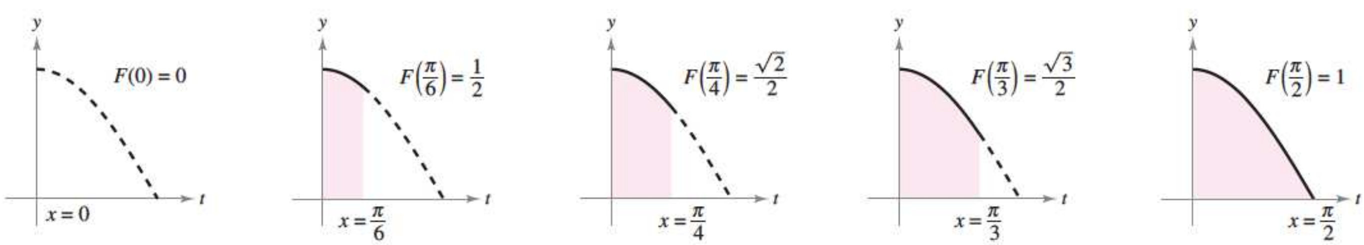
$$F(x) = \int_0^x \cos t\,dt$$

at $x = 0, \pi/6, \pi/4, \pi/3$, and $\pi/2$.

Solution You could evaluate five different definite integrals, one for each of the given upper limits. However, it is much simpler to fix x (as a constant) temporarily and apply the Fundamental Theorem once, to obtain

$$\int_0^x \cos t\,dt = \sin t\Big]_0^x = \sin x - \sin 0 = \sin x.$$

Now, using $F(x) = \sin x$, you can obtain the results shown in Figure 5.34.



$F(x) = \int_0^x \cos t\,dt$ is the area under the curve $f(t) = \cos t$ from 0 to x.

Figure 5.34

You can think of the function $F(x)$ as *accumulating* the area under the curve $f(t) = \cos t$ from $t = 0$ to $t = x$. For $x = 0$, the area is 0 and $F(0) = 0$. For $x = \pi/2$, $F(\pi/2) = 1$ gives the accumulated area under the cosine curve on the entire interval $[0, \pi/2]$. This interpretation of an integral as an **accumulation function** is used often in applications of integration.

In Example 6, note that the derivative of F is the original integrand (with only the variable changed). That is,

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}[\sin x] = \frac{d}{dx}\left[\int_0^x \cos t\,dt\right] = \cos x.$$

This result is generalized in the following theorem, called the **Second Fundamental Theorem of Calculus.**

THEOREM 5.11 THE SECOND FUNDAMENTAL THEOREM OF CALCULUS

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

PROOF Begin by defining F as

$$F(x) = \int_a^x f(t)\,dt.$$

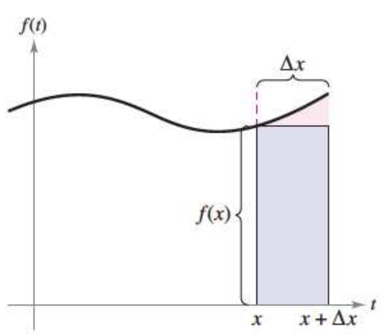
Then, by the definition of the derivative, you can write

$$\begin{aligned}
F'(x) &= \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt\right] \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)\,dt + \int_x^a f(t)\,dt\right] \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_x^{x+\Delta x} f(t)\,dt\right].
\end{aligned}$$

From the Mean Value Theorem for Integrals (assuming $\Delta x > 0$), you know there exists a number c in the interval $[x, x+\Delta x]$ such that the integral in the expression above is equal to $f(c)\,\Delta x$. Moreover, because $x \le c \le x + \Delta x$, it follows that $c \to x$ as $\Delta x \to 0$. So, you obtain

$$\begin{aligned}
F'(x) &= \lim_{\Delta x \to 0}\left[\frac{1}{\Delta x} f(c)\,\Delta x\right] \\
&= \lim_{\Delta x \to 0} f(c) \\
&= f(x).
\end{aligned}$$

A similar argument can be made for $\Delta x < 0$. ■



$$f(x)\,\Delta x \approx \int_x^{x+\Delta x} f(t)\,dt$$

Figure 5.35

NOTE Using the area model for definite integrals, you can view the approximation

$$f(x)\,\Delta x \approx \int_x^{x+\Delta x} f(t)\,dt$$

as saying that the area of the rectangle of height $f(x)$ and width Δx is approximately equal to the area of the region lying between the graph of f and the x-axis on the interval $[x, x+\Delta x]$, as shown in Figure 5.35. ■

Note that the Second Fundamental Theorem of Calculus tells you that if a function is continuous, you can be sure that it has an antiderivative. This antiderivative need not, however, be an elementary function. (Recall the discussion of elementary functions in Section 1.3.)

EXAMPLE 7 Using the Second Fundamental Theorem of Calculus

Evaluate $\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right]$.

Solution Note that $f(t) = \sqrt{t^2+1}$ is continuous on the entire real number line. So, using the Second Fundamental Theorem of Calculus, you can write

$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right] = \sqrt{x^2+1}.$$

The differentiation shown in Example 7 is a straightforward application of the Second Fundamental Theorem of Calculus. The next example shows how this theorem can be combined with the Chain Rule to find the derivative of a function.

EXAMPLE 8 Using the Second Fundamental Theorem of Calculus

Find the derivative of $F(x) = \int_{\pi/2}^{x^3} \cos t\,dt$.

Solution Using $u = x^3$, you can apply the Second Fundamental Theorem of Calculus with the Chain Rule, as shown.

$$F'(x) = \frac{dF}{du}\frac{du}{dx}$$ Chain Rule

$$= \frac{d}{du}[F(x)]\frac{du}{dx}$$ Definition of $\frac{dF}{du}$

$$= \frac{d}{du}\left[\int_{\pi/2}^{x^3} \cos t\,dt\right]\frac{du}{dx}$$ Substitute $\int_{\pi/2}^{x^3} \cos t\,dt$ for $F(x)$.

$$= \frac{d}{du}\left[\int_{\pi/2}^{u} \cos t\,dt\right]\frac{du}{dx}$$ Substitute u for x^3.

$$= (\cos u)(3x^2)$$ Apply Second Fundamental Theorem of Calculus.

$$= (\cos x^3)(3x^2)$$ Rewrite as function of x.

Because the integrand in Example 8 is easily integrated, you can verify the derivative as follows.

$$F(x) = \int_{\pi/2}^{x^3} \cos t\,dt = \sin t\Big]_{\pi/2}^{x^3} = \sin x^3 - \sin\frac{\pi}{2} = (\sin x^3) - 1$$

In this form, you can apply the Power Rule to verify that the derivative is the same as that obtained in Example 8.

$$F'(x) = (\cos x^3)(3x^2)$$

Net Change Theorem

The Fundamental Theorem of Calculus (Theorem 5.9) states that if f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

But because $F'(x) = f(x)$, this statement can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

where the quantity $F(b) - F(a)$ represents the *net change of F* on the interval $[a, b]$.

THEOREM 5.12 THE NET CHANGE THEOREM

The definite integral of the rate of change of a quantity $F'(x)$ gives the total change, or **net change,** in that quantity on the interval $[a, b]$.

$$\int_a^b F'(x)\,dx = F(b) - F(a) \qquad \text{Net change of } F$$

EXAMPLE 9 Using the Net Change Theorem

A chemical flows into a storage tank at a rate of $180 + 3t$ liters per minute at time t (in minutes), where $0 \leq t \leq 60$. Find the amount of the chemical that flows into the tank during the first 20 minutes.

Solution Let $c(t)$ be the amount of the chemical in the tank at time t. Then $c'(t)$ represents the rate at which the chemical flows into the tank at time t. During the first 20 minutes, the amount that flows into the tank is

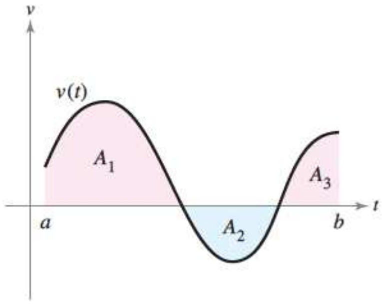
$$\begin{aligned}\int_0^{20} c'(t)\,dt &= \int_0^{20} (180 + 3t)\,dt \\ &= \left[180t + \frac{3}{2}t^2\right]_0^{20} \\ &= 3600 + 600 = 4200.\end{aligned}$$

So, the amount that flows into the tank during the first 20 minutes is 4200 liters. ■

Another way to illustrate the Net Change Theorem is to examine the velocity of a particle moving along a straight line, where $s(t)$ is the position at time t. Then the particle's velocity is $v(t) = s'(t)$ and

$$\int_a^b v(t)\,dt = s(b) - s(a).$$

This definite integral represents the net change in position, or the **displacement,** of the particle.



A_1, A_2, and A_3 are the areas of the shaded regions.
Figure 5.36

When calculating the *total* distance traveled by the particle, you must consider the intervals where $v(t) \le 0$ and the intervals where $v(t) \ge 0$. When $v(t) \le 0$, the particle moves to the left, and when $v(t) \ge 0$, the particle moves to the right. To calculate the total distance traveled, integrate the absolute value of velocity $|v(t)|$. So, the displacement of a particle and the total distance traveled by a particle over $[a, b]$ can be written as

$$\textbf{Displacement on } [a, b] = \int_a^b v(t)\,dt = A_1 - A_2 + A_3$$

$$\textbf{Total distance traveled on } [a, b] = \int_a^b |v(t)|\,dt = A_1 + A_2 + A_3$$

(see Figure 5.36).

EXAMPLE 10 Solving a Particle Motion Problem

A particle is moving along a line so that its velocity is $v(t) = t^3 - 10t^2 + 29t - 20$ feet per second at time t.

a. What is the displacement of the particle on the time interval $1 \le t \le 5$?

b. What is the total distance traveled by the particle on the time interval $1 \le t \le 5$?

Solution

a. By definition, you know that the displacement is

$$\begin{aligned}\int_1^5 v(t)\,dt &= \int_1^5 (t^3 - 10t^2 + 29t - 20)\,dt \\ &= \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_1^5 \\ &= \frac{25}{12} - \left(-\frac{103}{12}\right) \\ &= \frac{128}{12} \\ &= \frac{32}{3}.\end{aligned}$$

So, the particle moves $\frac{32}{3}$ feet to the right.

b. To find the total distance traveled, calculate $\int_1^5 |v(t)|\,dt$. Using Figure 5.37 and the fact that $v(t)$ can be factored as $(t - 1)(t - 4)(t - 5)$, you can determine that $v(t) \ge 0$ on $[1, 4]$ and $v(t) \le 0$ on $[4, 5]$. So, the total distance traveled is

$$\begin{aligned}\int_1^5 |v(t)|\,dt &= \int_1^4 v(t)\,dt - \int_4^5 v(t)\,dt \\ &= \int_1^4 (t^3 - 10t^2 + 29t - 20)\,dt - \int_4^5 (t^3 - 10t^2 + 29t - 20)\,dt \\ &= \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_1^4 - \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_4^5 \\ &= \frac{45}{4} - \left(-\frac{7}{12}\right) \\ &= \frac{71}{6}\text{ feet.}\end{aligned}$$

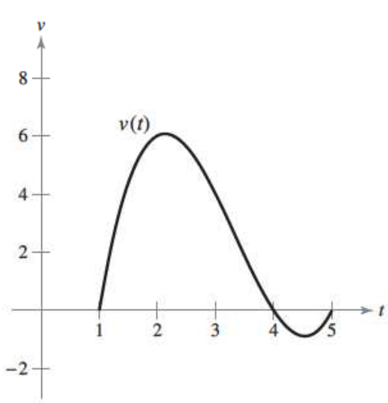


Figure 5.37

5.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Graphical Reasoning **In Exercises 1–4, use a graphing utility to graph the integrand. Use the graph to determine whether the definite integral is positive, negative, or zero.**

1. $\int_0^{\pi} \frac{4}{x^2+1}\,dx$

2. $\int_0^{\pi} \cos x\,dx$

3. $\int_{-2}^{2} x\sqrt{x^2+1}\,dx$

4. $\int_{-2}^{2} x\sqrt{2-x}\,dx$

In Exercises 5–26, evaluate the definite integral of the algebraic function. Use a graphing utility to verify your result.

5. $\int_0^2 6x\,dx$

6. $\int_4^9 5\,dv$

7. $\int_{-1}^{0} (2x-1)\,dx$

8. $\int_2^5 (-3v+4)\,dv$

9. $\int_{-1}^{1} (t^2-2)\,dt$

10. $\int_1^7 (6x^2+2x-3)\,dx$

11. $\int_0^1 (2t-1)^2\,dt$

12. $\int_{-1}^{1} (t^3-9t)\,dt$

13. $\int_1^2 \left(\frac{3}{x^2}-1\right)dx$

14. $\int_{-2}^{-1} \left(u-\frac{1}{u^2}\right)du$

15. $\int_1^4 \frac{u-2}{\sqrt{u}}\,du$

16. $\int_{-3}^{3} v^{1/3}\,dv$

17. $\int_{-1}^{1} \left(\sqrt[3]{t}-2\right)dt$

18. $\int_1^8 \sqrt{\frac{2}{x}}\,dx$

19. $\int_0^1 \frac{x-\sqrt{x}}{3}\,dx$

20. $\int_0^2 (2-t)\sqrt{t}\,dt$

21. $\int_{-1}^{0} \left(t^{1/3}-t^{2/3}\right)dt$

22. $\int_{-8}^{-1} \frac{x-x^2}{2\sqrt[3]{x}}\,dx$

23. $\int_0^5 |2x-5|\,dx$

24. $\int_2^5 (3-|x-4|)\,dx$

25. $\int_0^4 |x^2-9|\,dx$

26. $\int_0^4 |x^2-4x+3|\,dx$

In Exercises 27–38, evaluate the definite integral of the transcendental function. Use a graphing utility to verify your result.

27. $\int_0^{\pi} (1+\sin x)\,dx$

28. $\int_0^{\pi/4} \frac{1-\sin^2\theta}{\cos^2\theta}\,d\theta$

29. $\int_{-\pi/6}^{\pi/6} \sec^2 x\,dx$

30. $\int_{\pi/4}^{\pi/2} (2-\csc^2 x)\,dx$

31. $\int_1^e \left(2x-\frac{1}{x}\right)dx$

32. $\int_1^5 \frac{x+1}{x}\,dx$

33. $\int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\,d\theta$

34. $\int_{-\pi/2}^{\pi/2} (2t+\cos t)\,dt$

35. $\int_0^2 (2^x+6)\,dx$

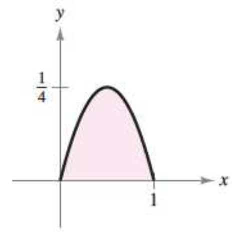
36. $\int_0^3 (t-5^t)\,dt$

37. $\int_{-1}^{1} (e^{\theta}+\sin\theta)\,d\theta$

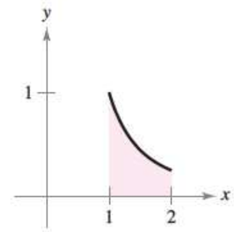
38. $\int_e^{2e} \left(\cos x-\frac{1}{x}\right)dx$

In Exercises 39–42, determine the area of the given region.

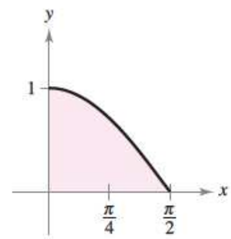
39. $y = x - x^2$



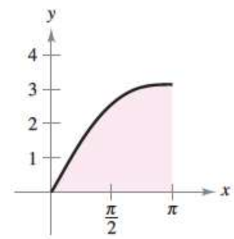
40. $y = \frac{1}{x^2}$



41. $y = \cos x$



42. $y = x + \sin x$



In Exercises 43–50, find the area of the region bounded by the graphs of the equations.

43. $y = 5x^2+2,\quad x=0,\quad x=2,\quad y=0$

44. $y = x^3+x,\quad x=2,\quad y=0$

45. $y = 1+\sqrt[3]{x},\quad x=0,\quad x=8,\quad y=0$

46. $y = (3-x)\sqrt{x},\quad y=0$

47. $y = -x^2+4x,\quad y=0$

48. $y = 1-x^4,\quad y=0$

49. $y = \frac{4}{x},\quad x=1,\quad x=e,\quad y=0$

50. $y = e^x,\quad x=0,\quad x=2,\quad y=0$

In Exercises 51–58, find the value(s) of c guaranteed by the Mean Value Theorem for Integrals for the function over the given interval.

51. $f(x) = x^3,\quad [0, 3]$

52. $f(x) = \frac{9}{x^3},\quad [1, 3]$

53. $f(x) = \sqrt{x},\quad [4, 9]$

54. $f(x) = x - 2\sqrt{x},\quad [0, 2]$

55. $f(x) = 2\sec^2 x,\quad [-\pi/4, \pi/4]$

56. $f(x) = \cos x,\quad [-\pi/3, \pi/3]$

57. $f(x) = 5 - \frac{1}{x},\quad [1, 4]$

58. $f(x) = 10 - 2^x,\quad [0, 3]$

In Exercises 59–66, find the average value of the function over the given interval and every (all) value(s) of x in the interval for which the function equals its average value.

59. $f(x) = 9 - x^2, \quad [-3, 3]$

60. $f(x) = \dfrac{4(x^2 + 1)}{x^2}, \quad [1, 3]$

61. $f(x) = x^3, \quad [0, 1]$

62. $f(x) = 4x^3 - 3x^2, \quad [0, 1]$

63. $f(x) = 2e^x, \quad [-1, 1]$

64. $f(x) = \dfrac{1}{2x}, \quad [1, 4]$

65. $f(x) = \sin x, \quad [0, \pi]$

66. $f(x) = \cos x, \quad [0, \pi/2]$

67. ***Velocity*** The graph shows the velocity, in feet per second, of a car accelerating from rest. Use the graph to estimate the distance the car travels in 8 seconds.

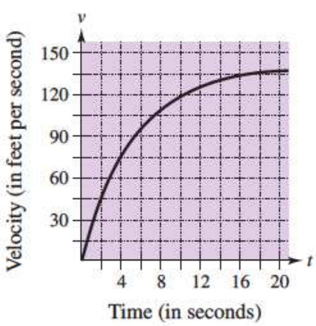


Figure for 67

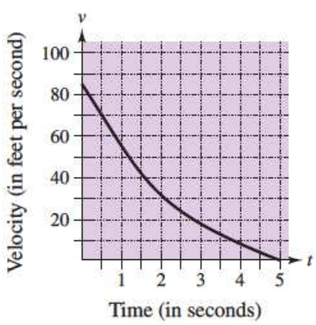
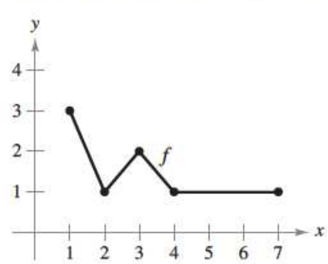


Figure for 68

68. ***Velocity*** The graph shows the velocity, in feet per second, of a decelerating car after the driver applies the brakes. Use the graph to estimate how far the car travels before it comes to a stop.

WRITING ABOUT CONCEPTS

69. The graph of f is shown in the figure.



(a) Evaluate $\int_1^7 f(x)\,dx$.

(b) Determine the average value of f on the interval $[1, 7]$.

(c) Determine the answers to parts (a) and (b) if the graph is translated two units upward.

70. If $r'(t)$ represents the rate of growth of a dog in pounds per year, what does $r(t)$ represent? What does $\int_2^6 r'(t)\,dt$ represent about the dog?

71. ***Force*** The force F (in newtons) of a hydraulic cylinder in a press is proportional to the square of $\sec x$, where x is the distance (in meters) that the cylinder is extended in its cycle. The domain of F is $[0, \pi/3]$, and $F(0) = 500$.

(a) Find F as a function of x.

(b) Find the average force exerted by the press over the interval $[0, \pi/3]$.

72. ***Blood Flow*** The velocity v of the flow of blood at a distance r from the central axis of an artery of radius R is

$$v = k(R^2 - r^2)$$

where k is the constant of proportionality. Find the average rate of flow of blood along a radius of the artery. (Use 0 and R as the limits of integration.)

73. ***Respiratory Cycle*** The volume V (in liters) of air in the lungs during a five-second respiratory cycle is approximated by the model

$$V = 0.1729t + 0.1522t^2 - 0.0374t^3$$

where t is the time in seconds. Approximate the average volume of air in the lungs during one cycle.

74. ***Average Sales*** A company fits a model to the monthly sales data for a seasonal product. The model is

$$S(t) = \frac{t}{4} + 1.8 + 0.5\sin\left(\frac{\pi t}{6}\right), \quad 0 \le t \le 24$$

where S is sales (in thousands) and t is time in months.

(a) Use a graphing utility to graph $f(t) = 0.5\sin(\pi t/6)$ for $0 \le t \le 24$. Use the graph to explain why the average value of $f(t)$ is 0 over the interval.

(b) Use a graphing utility to graph $S(t)$ and the line $g(t) = t/4 + 1.8$ in the same viewing window. Use the graph and the result of part (a) to explain why g is called the *trend line*.

75. ***Modeling Data*** An experimental vehicle is tested on a straight track. It starts from rest, and its velocity v (in meters per second) is recorded every 10 seconds for 1 minute (see table).

t	0	10	20	30	40	50	60
v	0	5	21	40	62	78	83

(a) Use a graphing utility to find a model of the form $v = at^3 + bt^2 + ct + d$ for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the Fundamental Theorem of Calculus to approximate the distance traveled by the vehicle during the test.

CAPSTONE

76. The graph of f is shown in the figure. The shaded region A has an area of 1.5, and $\int_0^6 f(x)\,dx = 3.5$. Use this information to fill in the blanks.

(a) $\displaystyle\int_0^2 f(x)\,dx = \square$

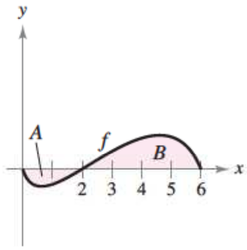
(b) $\displaystyle\int_2^6 f(x)\,dx = \square$

(c) $\displaystyle\int_0^6 |f(x)|\,dx = \square$

(d) $\displaystyle\int_0^2 -2f(x)\,dx = \square$

(e) $\displaystyle\int_0^6 [2 + f(x)]\,dx = \square$

(f) The average value of f over the interval $[0, 6]$ is $\square$.



In Exercises 77–82, find F as a function of x and evaluate it at $x = 2$, $x = 5$, and $x = 8$.

77. $F(x) = \displaystyle\int_0^x (4t - 7)\,dt$

78. $F(x) = \displaystyle\int_2^x (t^3 + 2t - 2)\,dt$

79. $F(x) = \displaystyle\int_1^x \frac{20}{v^2}\,dv$

80. $F(x) = \displaystyle\int_2^x -\frac{2}{t^3}\,dt$

81. $F(x) = \displaystyle\int_1^x \cos\theta\,d\theta$

82. $F(x) = \displaystyle\int_0^x \sin\theta\,d\theta$

83. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown in the figure.

(a) Estimate $g(0)$, $g(2)$, $g(4)$, $g(6)$, and $g(8)$.

(b) Find the largest open interval on which g is increasing. Find the largest open interval on which g is decreasing.

(c) Identify any extrema of g.

(d) Sketch a rough graph of g.

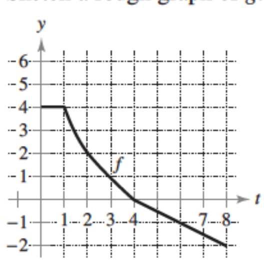


Figure for 83

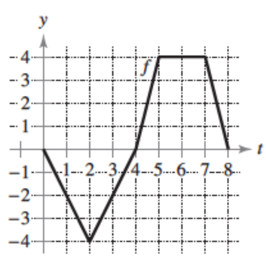


Figure for 84

84. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown in the figure.

(a) Estimate $g(0)$, $g(2)$, $g(4)$, $g(6)$, and $g(8)$.

(b) Find the largest open interval on which g is increasing. Find the largest open interval on which g is decreasing.

(c) Identify any extrema of g.

(d) Sketch a rough graph of g.

In Exercises 85–92, (a) integrate to find F as a function of x and (b) demonstrate the Second Fundamental Theorem of Calculus by differentiating the result in part (a).

85. $F(x) = \displaystyle\int_0^x (t + 2)\,dt$

86. $F(x) = \displaystyle\int_0^x t(t^2 + 1)\,dt$

87. $F(x) = \displaystyle\int_8^x \sqrt[3]{t}\,dt$

88. $F(x) = \displaystyle\int_4^x \sqrt{t}\,dt$

89. $F(x) = \displaystyle\int_{\pi/4}^x \sec^2 t\,dt$

90. $F(x) = \displaystyle\int_{\pi/3}^x \sec t \tan t\,dt$

91. $F(x) = \displaystyle\int_{-1}^x e^t\,dt$

92. $F(x) = \displaystyle\int_1^x \frac{1}{t}\,dt$

In Exercises 93–98, use the Second Fundamental Theorem of Calculus to find $F'(x)$.

93. $F(x) = \displaystyle\int_{-2}^x (t^2 - 2t)\,dt$

94. $F(x) = \displaystyle\int_1^x \frac{t^2}{t^2 + 1}\,dt$

95. $F(x) = \displaystyle\int_{-1}^x \sqrt{t^4 + 1}\,dt$

96. $F(x) = \displaystyle\int_1^x \sqrt[4]{t}\,dt$

97. $F(x) = \displaystyle\int_0^x t\cos t\,dt$

98. $F(x) = \displaystyle\int_0^x \sec^3 t\,dt$

In Exercises 99–104, find $F'(x)$.

99. $F(x) = \displaystyle\int_x^{x+2} (4t + 1)\,dt$

100. $F(x) = \displaystyle\int_{-x}^x t^3\,dt$

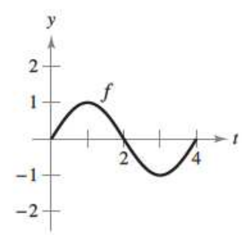
101. $F(x) = \displaystyle\int_0^{\sin x} \sqrt{t}\,dt$

102. $F(x) = \displaystyle\int_2^{x^2} \frac{1}{t^3}\,dt$

103. $F(x) = \displaystyle\int_0^{x^3} \sin t^2\,dt$

104. $F(x) = \displaystyle\int_0^{x^2} \sin\theta^2\,d\theta$

105. ***Graphical Analysis*** Sketch an approximate graph of g on the interval $0 \le x \le 4$, where $g(x) = \int_0^x f(t)\,dt$. Identify the x-coordinate of an extremum of g. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



106. ***Area*** The area A between the graph of the function $g(t) = 4 - 4/t^2$ and the t-axis over the interval $[1, x]$ is

$$A(x) = \int_1^x \left(4 - \frac{4}{t^2}\right) dt.$$

(a) Find the horizontal asymptote of the graph of g.

(b) Integrate to find A as a function of x. Does the graph of A have a horizontal asymptote? Explain.

In Exercises 107–112, the velocity function, in feet per second, is given for a particle moving along a straight line. Find (a) the displacement and (b) the total distance that the particle travels over the given interval.

107. $v(t) = 5t - 7, \quad 0 \le t \le 3$

108. $v(t) = t^2 - t - 12, \quad 1 \le t \le 5$

109. $v(t) = t^3 - 10t^2 + 27t - 18, \quad 1 \le t \le 7$

110. $v(t) = t^3 - 8t^2 + 15t, \quad 0 \le t \le 5$

111. $v(t) = \dfrac{1}{\sqrt{t}}, \quad 1 \le t \le 4$ **112.** $v(t) = \cos t, \quad 0 \le t \le 3\pi$

113. A particle is moving along the x-axis. The position of the particle at time t is given by $x(t) = t^3 - 6t^2 + 9t - 2$, $0 \le t \le 5$. Find the total distance the particle travels in 5 units of time.

114. Repeat Exercise 113 for the position function given by $x(t) = (t-1)(t-3)^2, 0 \le t \le 5$.

115. ***Water Flow*** Water flows from a storage tank at a rate of $500 - 5t$ liters per minute at time t (in minutes). Find the amount of water that flows out of the tank during the first 18 minutes.

116. ***Oil Leak*** At 1:00 P.M., oil begins leaking from a tank at a rate of $4 + 0.75t$ gallons per hour at time t (in hours).

(a) How much oil is lost from 1:00 P.M. to 4:00 P.M.?

(b) How much oil is lost from 4:00 P.M. to 7:00 P.M.?

(c) Compare your answers from parts (a) and (b). What do you notice?

In Exercises 117–120, describe why the statement is incorrect.

117. $\displaystyle\int_{-1}^{1} x^{-2}\,dx = \left[-x^{-1}\right]_{-1}^{1} = (-1) - 1 = -2$

118. $\displaystyle\int_{-2}^{1} \frac{2}{x^3}\,dx = \left[-\frac{1}{x^2}\right]_{-2}^{1} = -\frac{3}{4}$

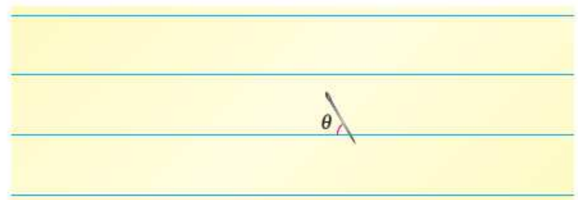
119. $\displaystyle\int_{\pi/4}^{3\pi/4} \sec^2 x\,dx = \Big[\tan x\Big]_{\pi/4}^{3\pi/4} = -2$

120. $\displaystyle\int_{\pi/2}^{3\pi/2} \csc x \cot x\,dx = \Big[-\csc x\Big]_{\pi/2}^{3\pi/2} = 2$

121. ***Buffon's Needle Experiment*** A horizontal plane is ruled with parallel lines 2 inches apart (see figure). A two-inch needle is tossed randomly onto the plane. The probability that the needle will touch a line is

$$P = \frac{2}{\pi}\int_0^{\pi/2} \sin\theta\,d\theta$$

where θ is the acute angle between the needle and any one of the parallel lines. Find this probability.



122. Prove that $\dfrac{d}{dx}\left[\displaystyle\int_{u(x)}^{v(x)} f(t)\,dt\right] = f(v(x))v'(x) - f(u(x))u'(x).$

True or False? **In Exercises 123 and 124, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

123. If $F'(x) = G'(x)$ on the interval $[a, b]$, then $F(b) - F(a) = G(b) - G(a)$.

124. If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

125. Show that the function

$$f(x) = \int_0^{1/x} \frac{1}{t^2+1}\,dt + \int_0^x \frac{1}{t^2+1}\,dt$$

is constant for $x > 0$.

126. Find the function $f(x)$ and all values of c such that

$$\int_c^x f(t)\,dt = x^2 + x - 2.$$

127. Let $G(x) = \displaystyle\int_0^x \left[s\int_0^s f(t)dt\right] ds$, where f is continuous for all real t. Find (a) $G(0)$, (b) $G'(0)$, (c) $G''(x)$, and (d) $G''(0)$.

SECTION PROJECT

Demonstrating the Fundamental Theorem

Use a graphing utility to graph the function $y_1 = \sin^2 t$ on the interval $0 \le t \le \pi$. Let $F(x)$ be the following function of x.

$$F(x) = \int_0^x \sin^2 t\,dt$$

(a) Complete the table. Explain why the values of F are increasing.

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$F(x)$							

(b) Use the integration capabilities of a graphing utility to graph F.

(c) Use the differentiation capabilities of a graphing utility to graph $F'(x)$. How is this graph related to the graph in part (b)?

(d) Verify that the derivative of $y = (1/2)t - (\sin 2t)/4$ is $\sin^2 t$. Graph y and write a short paragraph about how this graph is related to those in parts (b) and (c).

5.5 Integration by Substitution

- Use pattern recognition to find an indefinite integral.
- Use a change of variables to find an indefinite integral.
- Use the General Power Rule for Integration to find an indefinite integral.
- Use a change of variables to evaluate a definite integral.
- Evaluate a definite integral involving an even or odd function.

Pattern Recognition

In this section you will study techniques for integrating composite functions. The discussion is split into two parts—*pattern recognition* and *change of variables*. Both techniques involve a ***u*-substitution.** With pattern recognition you perform the substitution mentally, and with change of variables you write the substitution steps.

The role of substitution in integration is comparable to the role of the Chain Rule in differentiation. Recall that for differentiable functions given by $y = F(u)$ and $u = g(x)$, the Chain Rule states that

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x).$$

From the definition of an antiderivative, it follows that

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

These results are summarized in the following theorem.

NOTE The statement of Theorem 5.13 doesn't tell how to distinguish between $f(g(x))$ and $g'(x)$ in the integrand. As you become more experienced at integration, your skill in doing this will increase. Of course, part of the key is familiarity with derivatives.

THEOREM 5.13 ANTIDIFFERENTIATION OF A COMPOSITE FUNCTION

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C.$$

Examples 1 and 2 show how to apply Theorem 5.13 *directly*, by recognizing the presence of $f(g(x))$ and $g'(x)$. Note that the composite function in the integrand has an *outside function* f and an *inside function* g. Moreover, the derivative $g'(x)$ is present as a factor of the integrand.

Outside function

$$\int f(g(x))\underbrace{g'(x)}_{\text{Derivative of inside function}}\,dx = F(g(x)) + C$$

Inside function

Derivative of inside function

EXAMPLE 1 Recognizing the $f(g(x))g'(x)$ Pattern

Find $\int (x^2 + 1)^2(2x)\, dx$.

Solution Letting $g(x) = x^2 + 1$, you obtain

$$g'(x) = 2x$$

and

$$f(g(x)) = f(x^2 + 1) = (x^2 + 1)^2.$$

From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern. Using the Power Rule for Integration and Theorem 5.13, you can write

$$\int \overbrace{(x^2 + 1)^2}^{f(g(x))}\overbrace{(2x)}^{g'(x)}\, dx = \frac{1}{3}(x^2 + 1)^3 + C.$$

Try using the Chain Rule to check that the derivative of $\frac{1}{3}(x^2 + 1)^3 + C$ is the integrand of the original integral.

EXAMPLE 2 Recognizing the $f(g(x))g'(x)$ Pattern

Find $\int 5e^{5x}\, dx$.

Solution Letting $g(x) = 5x$, you obtain

$$g'(x) = 5$$

and

$$f(g(x)) = f(5x) = e^{5x}.$$

TECHNOLOGY Try using a computer algebra system, such as *Maple*, *Mathematica*, or the *TI-89*, to solve the integrals given in Examples 1 and 2. Do you obtain the same antiderivatives that are listed in the examples?

From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern. Using the Exponential Rule for Integration and Theorem 5.13, you can write

$$\int \underbrace{e^{5x}}_{f(g(x))}\underbrace{(5)}_{g'(x)}\, dx = e^{5x} + C.$$

You can check this by differentiating $e^{5x} + C$ to obtain the original integrand. ■

EXPLORATION

Recognizing Patterns The integrand in each of the following integrals fits the pattern $f(g(x))g'(x)$. Identify the pattern and use the result to evaluate the integral.

a. $\int 2x(x^2 + 1)^4\, dx$ **b.** $\int 3x^2\sqrt{x^3 + 1}\, dx$ **c.** $\int \sec^2 x(\tan x + 3)\, dx$

The next three integrals are similar to the first three. Show how you can multiply and divide by a constant to evaluate these integrals.

d. $\int x(x^2 + 1)^4\, dx$ **e.** $\int x^2\sqrt{x^3 + 1}\, dx$ **f.** $\int 2\sec^2 x(\tan x + 3)\, dx$

The integrands in Examples 1 and 2 fit the $f(g(x))g'(x)$ pattern exactly—you only had to recognize the pattern. You can extend this technique considerably with the Constant Multiple Rule

$$\int kf(x)\,dx = k\int f(x)\,dx.$$

Many integrands contain the essential part (the variable part) of $g'(x)$ but are missing a constant multiple. In such cases, you can multiply and divide by the necessary constant multiple, as shown in Example 3.

EXAMPLE 3 Multiplying and Dividing by a Constant

Find $\int x(x^2+1)^2\,dx$.

Solution This is similar to the integral given in Example 1, except that the integrand is missing a factor of 2. Recognizing that $2x$ is the derivative of x^2+1, you can let $g(x) = x^2+1$ and supply the $2x$ as follows.

$$\begin{aligned}
\int x(x^2+1)^2\,dx &= \int (x^2+1)^2\left(\frac{1}{2}\right)(2x)\,dx && \text{Multiply and divide by 2.}\\
&= \frac{1}{2}\int \overbrace{(x^2+1)^2}^{f(g(x))}\,\overbrace{(2x)}^{g'(x)}\,dx && \text{Constant Multiple Rule}\\
&= \frac{1}{2}\left[\frac{(x^2+1)^3}{3}\right] + C && \text{Integrate.}\\
&= \frac{1}{6}(x^2+1)^3 + C && \text{Simplify.}
\end{aligned}$$

In practice, most people would not write as many steps as are shown in Example 3. For instance, you could evaluate the integral by simply writing

$$\begin{aligned}
\int x(x^2+1)^2\,dx &= \frac{1}{2}\int (x^2+1)^2\,2x\,dx\\
&= \frac{1}{2}\left[\frac{(x^2+1)^3}{3}\right] + C\\
&= \frac{1}{6}(x^2+1)^3 + C.
\end{aligned}$$

NOTE Be sure you see that the *Constant* Multiple Rule applies only to *constants*. You cannot multiply and divide by a variable and then move the variable outside the integral sign. For instance,

$$\int (x^2+1)^2\,dx \neq \frac{1}{2x}\int (x^2+1)^2(2x)\,dx.$$

After all, if it were legitimate to move variable quantities outside the integral sign, you could move the entire integrand out and simplify the whole process. But the result would be incorrect.

Change of Variables

With a formal **change of variables,** you completely rewrite the integral in terms of u and du (or any other convenient variable). Although this procedure can involve more written steps than the pattern recognition illustrated in Examples 1 to 3, it is useful for complicated integrands. The change of variables technique uses the Leibniz notation for the differential. That is, if $u = g(x)$, then $du = g'(x)\,dx$, and the integral in Theorem 5.13 takes the form

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C.$$

EXAMPLE 4 Change of Variables

Find $\int \sqrt{2x - 1}\,dx$.

Solution First, let u be the inner function, $u = 2x - 1$. Then calculate the differential du to be $du = 2\,dx$. Now, using $\sqrt{2x-1} = \sqrt{u}$ and $dx = du/2$, substitute to obtain

$$\begin{aligned}
\int \sqrt{2x-1}\,dx &= \int \sqrt{u}\left(\frac{du}{2}\right) && \text{Integral in terms of } u\\
&= \frac{1}{2}\int u^{1/2}\,du && \text{Constant Multiple Rule}\\
&= \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C && \text{Antiderivative in terms of } u\\
&= \frac{1}{3}u^{3/2} + C && \text{Simplify.}\\
&= \frac{1}{3}(2x-1)^{3/2} + C. && \text{Antiderivative in terms of } x
\end{aligned}$$

STUDY TIP Because integration is usually more difficult than differentiation, you should always check your answer to an integration problem by differentiating. For instance, in Example 4, you should differentiate $\frac{1}{3}(2x-1)^{3/2} + C$ to verify that you obtain the original integrand.

EXAMPLE 5 Change of Variables

Find $\int x\sqrt{2x-1}\,dx$.

Solution As in Example 4, let $u = 2x - 1$ and obtain $dx = du/2$. Because the integrand contains a factor of x, you must also solve for x in terms of u, as shown.

$$u = 2x - 1 \quad \Rightarrow \quad x = (u+1)/2 \qquad \text{Solve for } x \text{ in terms of } u.$$

Now, using substitution, you obtain

$$\begin{aligned}
\int x\sqrt{2x-1}\,dx &= \int \left(\frac{u+1}{2}\right)u^{1/2}\left(\frac{du}{2}\right)\\
&= \frac{1}{4}\int (u^{3/2} + u^{1/2})\,du\\
&= \frac{1}{4}\left(\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + C\\
&= \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C.
\end{aligned}$$

■

To complete the change of variables in Example 5, you solved for x in terms of u. Sometimes this is very difficult. Fortunately, it is not always necessary, as shown in the next example.

EXAMPLE 6 Change of Variables

Find $\int \sin^2 3x \cos 3x \, dx$.

Solution Because $\sin^2 3x = (\sin 3x)^2$, you can let $u = \sin 3x$. Then

$$du = (\cos 3x)(3)\, dx.$$

Now, because $\cos 3x\, dx$ is part of the original integral, you can write

$$\frac{du}{3} = \cos 3x\, dx.$$

Substituting u and $du/3$ in the original integral yields

$$\begin{aligned} \int \sin^2 3x \cos 3x \, dx &= \int u^2 \frac{du}{3} \\ &= \frac{1}{3}\int u^2\, du \\ &= \frac{1}{3}\left(\frac{u^3}{3}\right) + C \\ &= \frac{1}{9}\sin^3 3x + C. \end{aligned}$$

You can check this by differentiating.

$$\begin{aligned} \frac{d}{dx}\left[\frac{1}{9}\sin^3 3x\right] &= \left(\frac{1}{9}\right)(3)(\sin 3x)^2(\cos 3x)(3) \\ &= \sin^2 3x \cos 3x \end{aligned}$$

Because differentiation produces the original integrand, you know that you have obtained the correct antiderivative. ■

STUDY TIP When making a change of variables, be sure that your answer is written using the same variables as in the original integrand. For instance, in Example 6, you should not leave your answer as

$$\tfrac{1}{9}u^3 + C$$

but rather, replace u by $\sin 3x$.

The steps used for integration by substitution are summarized in the following guidelines.

GUIDELINES FOR MAKING A CHANGE OF VARIABLES

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)\, dx$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

The General Power Rule for Integration

One of the most common u-substitutions involves quantities in the integrand that are raised to a power. Because of the importance of this type of substitution, it is given a special name—the **General Power Rule for Integration.** A proof of this rule follows directly from the (simple) Power Rule for Integration, together with Theorem 5.13.

THEOREM 5.14 THE GENERAL POWER RULE FOR INTEGRATION

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

EXAMPLE 7 Substitution and the General Power Rule

a. $$\int 3(3x-1)^4\,dx = \int \underbrace{(3x-1)^4}_{u^4}\underbrace{(3)\,dx}_{du} = \underbrace{\frac{(3x-1)^5}{5}}_{u^5/5} + C$$

b. $$\int (e^x+1)(e^x+x)dx = \int \underbrace{(e^x+x)}_{u^1}\underbrace{(e^x+1)dx}_{du} = \underbrace{\frac{(e^x+x)^2}{2}}_{u^2/2} + C$$

c. $$\int 3x^2\sqrt{x^3-2}\,dx = \int \underbrace{(x^3-2)^{1/2}}_{u^{1/2}}\underbrace{(3x^2)\,dx}_{du} = \underbrace{\frac{(x^3-2)^{3/2}}{3/2}}_{u^{3/2}/(3/2)} + C = \frac{2}{3}(x^3-2)^{3/2} + C$$

d. $$\int \frac{-4x}{(1-2x^2)^2}\,dx = \int \underbrace{(1-2x^2)^{-2}}_{u^{-2}}\underbrace{(-4x)\,dx}_{du} = \underbrace{\frac{(1-2x^2)^{-1}}{-1}}_{u^{-1}/(-1)} + C = -\frac{1}{1-2x^2} + C$$

e. $$\int \cos^2 x \sin x\,dx = -\int \underbrace{(\cos x)^2}_{u^2}\underbrace{(-\sin x)\,dx}_{du} = -\underbrace{\frac{(\cos x)^3}{3}}_{u^3/3} + C$$ ■

EXPLORATION

Suppose you were asked to find one of the following integrals. Which one would you choose? Explain your reasoning.

a. $\int \sqrt{x^3+1}\,dx$ or

$\int x^2\sqrt{x^3+1}\,dx$

b. $\int \tan(3x)\sec^2(3x)\,dx$ or

$\int \tan(3x)\,dx$

Some integrals whose integrands involve quantities raised to powers cannot be found by the General Power Rule. Consider the two integrals

$$\int x(x^2+1)^2\,dx \quad \text{and} \quad \int (x^2+1)^2\,dx.$$

The substitution $u = x^2 + 1$ works in the first integral but not in the second. In the second, the substitution fails because the integrand lacks the factor x needed for du. Fortunately, *for this particular integral*, you can expand the integrand as $(x^2+1)^2 = x^4 + 2x^2 + 1$ and use the (simple) Power Rule to integrate each term.

Change of Variables for Definite Integrals

When using u-substitution with a definite integral, it is often convenient to determine the limits of integration for the variable u rather than to convert the antiderivative back to the variable x and evaluate at the original limits. This change of variables is stated explicitly in the next theorem. The proof follows from Theorem 5.13 combined with the Fundamental Theorem of Calculus.

THEOREM 5.15 CHANGE OF VARIABLES FOR DEFINITE INTEGRALS

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

EXAMPLE 8 Change of Variables

Evaluate $\displaystyle\int_0^1 x(x^2 + 1)^3\,dx.$

Solution To evaluate this integral, let $u = x^2 + 1$. Then, you obtain

$$u = x^2 + 1 \implies du = 2x\,dx.$$

Before substituting, determine the new upper and lower limits of integration.

Lower Limit	*Upper Limit*
When $x = 0$, $u = 0^2 + 1 = 1$.	When $x = 1$, $u = 1^2 + 1 = 2$.

Now, you can substitute to obtain

$$\int_0^1 x(x^2 + 1)^3\,dx = \frac{1}{2}\int_0^1 (x^2 + 1)^3(2x)\,dx$$

Integration limits for x

$$= \frac{1}{2}\int_1^2 u^3\,du$$

Integration limits for u

$$= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2$$

$$= \frac{1}{2}\left(4 - \frac{1}{4}\right)$$

$$= \frac{15}{8}.$$

Try rewriting the antiderivative $\frac{1}{2}(u^4/4)$ in terms of the variable x and evaluate the definite integral at the original limits of integration, as shown.

$$\frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 = \frac{1}{2}\left[\frac{(x^2 + 1)^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}$$

Notice that you obtain the same result. ■

EXAMPLE 9 Change of Variables

Evaluate $A = \int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$.

Solution To evaluate this integral, let $u = \sqrt{2x-1}$. Then, you obtain

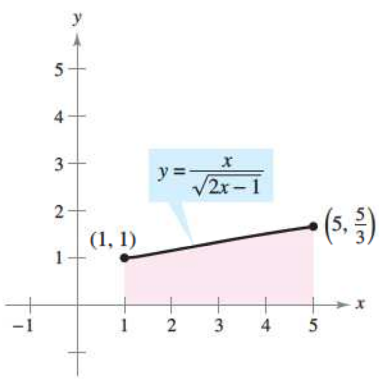
$$u^2 = 2x - 1$$
$$u^2 + 1 = 2x$$
$$\frac{u^2+1}{2} = x$$
$$u\,du = dx. \qquad \text{Differentiate each side.}$$

Before substituting, determine the new upper and lower limits of integration.

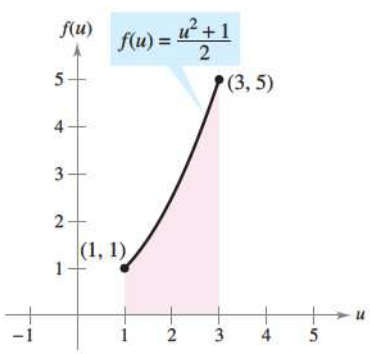
Lower Limit	*Upper Limit*
When $x = 1$, $u = \sqrt{2-1} = 1$.	When $x = 5$, $u = \sqrt{10-1} = 3$.

Now, substitute to obtain

$$\begin{aligned}\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx &= \int_1^3 \frac{1}{u}\left(\frac{u^2+1}{2}\right)u\,du \\ &= \frac{1}{2}\int_1^3 (u^2+1)\,du \\ &= \frac{1}{2}\left[\frac{u^3}{3} + u\right]_1^3 \\ &= \frac{1}{2}\left(9 + 3 - \frac{1}{3} - 1\right) \\ &= \frac{16}{3}.\end{aligned}$$



The region before substitution has an area of $\frac{16}{3}$.
Figure 5.38



The region after substitution has an area of $\frac{16}{3}$.
Figure 5.39

Geometrically, you can interpret the equation

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \int_1^3 \frac{u^2+1}{2}\,du$$

to mean that the two *different* regions shown in Figures 5.38 and 5.39 have the *same* area.

When evaluating definite integrals by substitution, it is possible for the upper limit of integration of the u-variable form to be smaller than the lower limit. If this happens, don't rearrange the limits. Simply evaluate as usual. For example, after substituting $u = \sqrt{1-x}$ in the integral

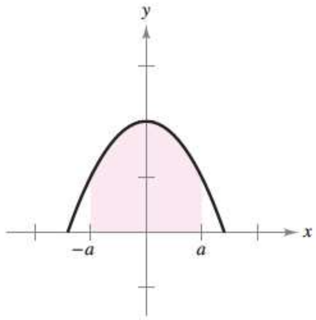
$$\int_0^1 x^2(1-x)^{1/2}\,dx$$

you obtain $u = \sqrt{1-1} = 0$ when $x = 1$, and $u = \sqrt{1-0} = 1$ when $x = 0$. So, the correct u-variable form of this integral is

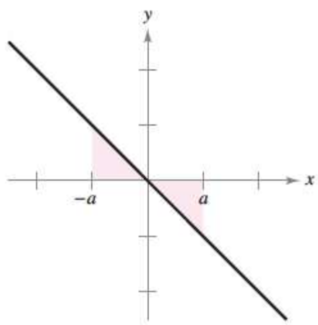
$$-2\int_1^0 (1-u^2)^2 u^2\,du.$$

Integration of Even and Odd Functions

Even with a change of variables, integration can be difficult. Occasionally, you can simplify the evaluation of a definite integral over an interval that is symmetric about the y-axis or about the origin by recognizing the integrand to be an even or odd function (see Figure 5.40).



Even function



Odd function
Figure 5.40

THEOREM 5.16 INTEGRATION OF EVEN AND ODD FUNCTIONS

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an *even* function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

2. If f is an *odd* function, then $\int_{-a}^{a} f(x)\,dx = 0$.

PROOF Because f is even, you know that $f(x) = f(-x)$. Using Theorem 5.13 with the substitution $u = -x$ produces

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-u)(-du) = -\int_{a}^{0} f(u)\,du = \int_{0}^{a} f(u)\,du = \int_{0}^{a} f(x)\,dx.$$

Finally, using Theorem 5.6, you obtain

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx\\ &= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.\end{aligned}$$

This proves the first property. The proof of the second property is left to you (see Exercise 173). ■

EXAMPLE 10 Integration of an Odd Function

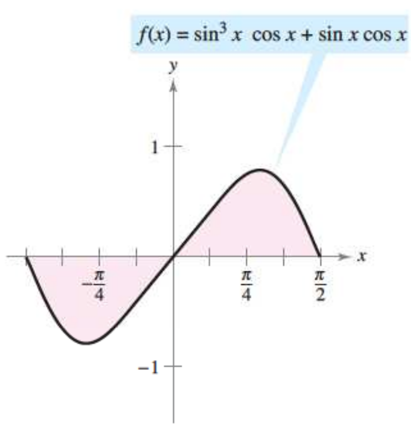
Evaluate $\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\,dx$.

Solution Letting $f(x) = \sin^3 x \cos x + \sin x \cos x$ produces

$$\begin{aligned}f(-x) &= \sin^3(-x)\cos(-x) + \sin(-x)\cos(-x)\\ &= -\sin^3 x \cos x - \sin x \cos x = -f(x).\end{aligned}$$

So, f is an odd function, and because f is symmetric about the origin over $[-\pi/2, \pi/2]$, you can apply Theorem 5.16 to conclude that

$$\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\,dx = 0.$$ ■



Because f is an odd function, $\int_{-\pi/2}^{\pi/2} f(x)\,dx = 0$.
Figure 5.41

NOTE From Figure 5.41 you can see that the two regions on either side of the y-axis have the same area. However, because one lies below the x-axis and one lies above it, integration produces a cancellation effect. (More will be said about this in Section 7.1.) ■

5.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, complete the table by identifying u and du for the integral.

$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
1. $\int (8x^2+1)^2(16x)\,dx$		
2. $\int x^2\sqrt{x^3+1}\,dx$		
3. $\int \frac{x}{\sqrt{x^2+1}}\,dx$		
4. $\int \sec 2x \tan 2x\,dx$		
5. $\int \tan^2 x \sec^2 x\,dx$		
6. $\int \frac{\cos x}{\sin^3 x}\,dx$		

In Exercises 7–10, determine whether it is necessary to use substitution to evaluate the integral. (Do not evaluate the integral.)

7. $\int \sqrt{x}(6-x)\,dx$

8. $\int x\sqrt{x+4}\,dx$

9. $\int x\sqrt[3]{1+x^2}\,dx$

10. $\int x\cos x^2\,dx$

In Exercises 11–38, find the indefinite integral and check the result by differentiation.

11. $\int (1+6x)^4(6)\,dx$

12. $\int (x^2-9)^3(2x)\,dx$

13. $\int \sqrt{25-x^2}\,(-2x)\,dx$

14. $\int \sqrt[3]{3-4x^2}(-8x)\,dx$

15. $\int x^3(x^4+3)^2\,dx$

16. $\int x^2(x^3+5)^4\,dx$

17. $\int x^2(x^3-1)^4\,dx$

18. $\int x(5x^2+4)^3\,dx$

19. $\int t\sqrt{t^2+2}\,dt$

20. $\int t^3\sqrt{t^4+5}\,dt$

21. $\int 5x\sqrt[3]{1-x^2}\,dx$

22. $\int u^2\sqrt{u^3+2}\,du$

23. $\int \frac{x}{(1-x^2)^3}\,dx$

24. $\int \frac{x^3}{(1+x^4)^2}\,dx$

25. $\int \frac{x^2}{(1+x^3)^2}\,dx$

26. $\int \frac{x^2}{(16-x^3)^2}\,dx$

27. $\int \frac{x}{\sqrt{1-x^2}}\,dx$

28. $\int \frac{x^3}{\sqrt{1+x^4}}\,dx$

29. $\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt$

30. $\int \left[x^2+\frac{1}{(3x)^2}\right]dx$

31. $\int \frac{1}{\sqrt{2x}}\,dx$

32. $\int \frac{1}{2\sqrt{x}}\,dx$

33. $\int \frac{x^2+5x-8}{\sqrt{x}}\,dx$

34. $\int \frac{t-9t^2}{\sqrt{t}}\,dt$

35. $\int t^2\left(t-\frac{8}{t}\right)dt$

36. $\int \left(\frac{t^3}{3}+\frac{1}{4t^2}\right)dt$

37. $\int (9-y)\sqrt{y}\,dy$

38. $\int 4\pi y(6+y^{3/2})\,dy$

In Exercises 39–42, solve the differential equation.

39. $\frac{dy}{dx} = 4x + \frac{4x}{\sqrt{16-x^2}}$

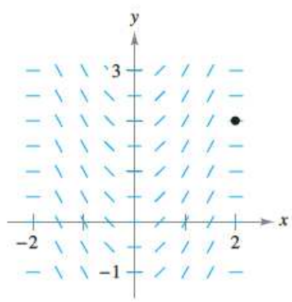
40. $\frac{dy}{dx} = \frac{10x^2}{\sqrt{1+x^3}}$

41. $\frac{dy}{dx} = \frac{x+1}{(x^2+2x-3)^2}$

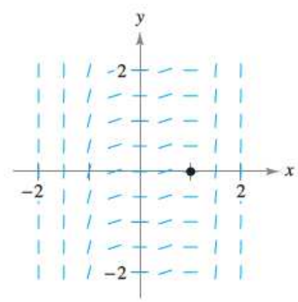
42. $\frac{dy}{dx} = \frac{x-4}{\sqrt{x^2-8x+1}}$

Slope Fields **In Exercises 43–48, a differential equation, a point, and a slope field are given. A** ***slope field*** **(or** ***direction field*****) consists of line segments with slopes given by the differential equation. These line segments give a visual perspective of the slopes of the solutions of the differential equation. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (To print an enlarged copy of the graph, go to the website** ***www.mathgraphs.com*****.) (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a).**

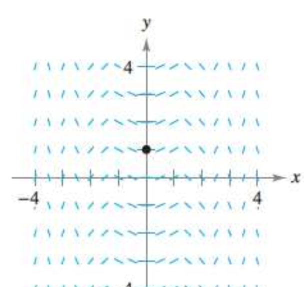
43. $\frac{dy}{dx} = x\sqrt{4-x^2}$, $(2, 2)$



44. $\frac{dy}{dx} = x^2(x^3-1)^2$, $(1, 0)$

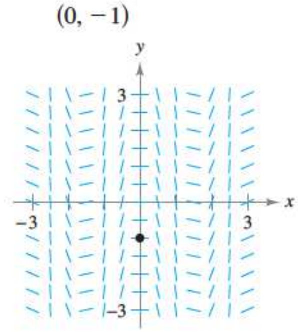


45. $\frac{dy}{dx} = x\cos x^2$, $(0, 1)$

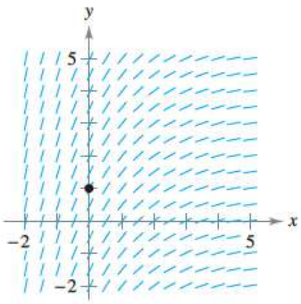


46. $\frac{dy}{dx} = -2\sec(2x)\tan(2x)$

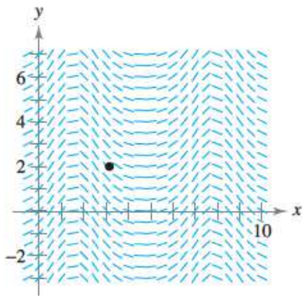
$(0, -1)$



47. $\dfrac{dy}{dx} = 2e^{-x/2}$, $(0, 1)$



48. $\dfrac{dy}{dx} = e^{\sin x} \cos x$, $(\pi, 2)$



In Exercises 49–82, find the indefinite integral.

49. $\displaystyle\int \pi \sin \pi x \, dx$

50. $\displaystyle\int 4x^3 \sin x^4 \, dx$

51. $\displaystyle\int \sin 4x \, dx$

52. $\displaystyle\int \cos 8x \, dx$

53. $\displaystyle\int \frac{1}{\theta^2} \cos \frac{1}{\theta} \, d\theta$

54. $\displaystyle\int x \sin x^2 \, dx$

55. $\displaystyle\int e^{7x}(7) \, dx$

56. $\displaystyle\int e^{x/3}\left(\frac{1}{3}\right) dx$

57. $\displaystyle\int e^{x^2}(2x) \, dx$

58. $\displaystyle\int e^{-x^3}(-3x^2) \, dx$

59. $\displaystyle\int x^2 e^{-x^3} \, dx$

60. $\displaystyle\int (x + 1)e^{x^2+2x} \, dx$

61. $\displaystyle\int \sin 2x \cos 2x \, dx$

62. $\displaystyle\int \sec(2 - x) \tan(2 - x) \, dx$

63. $\displaystyle\int \tan^4 x \sec^2 x \, dx$

64. $\displaystyle\int \sqrt{\tan x} \sec^2 x \, dx$

65. $\displaystyle\int \frac{\csc^2 x}{\cot^3 x} \, dx$

66. $\displaystyle\int \frac{\sin x}{\cos^3 x} \, dx$

67. $\displaystyle\int \cot^2 x \, dx$

68. $\displaystyle\int \csc^2\left(\frac{x}{2}\right) dx$

69. $\displaystyle\int e^x(e^x + 1)^2 \, dx$

70. $\displaystyle\int e^x(1 - 3e^x) \, dx$

71. $\displaystyle\int e^x\sqrt{1 - e^x} \, dx$

72. $\displaystyle\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2} \, dx$

73. $\displaystyle\int \frac{5 - e^x}{e^{2x}} \, dx$

74. $\displaystyle\int \frac{e^{2x} + 2e^x + 1}{e^x} \, dx$

75. $\displaystyle\int e^{\sin \pi x} \cos \pi x \, dx$

76. $\displaystyle\int e^{\tan 2x} \sec^2 2x \, dx$

77. $\displaystyle\int e^{-x} \sec^2(e^{-x}) \, dx$

78. $\displaystyle\int \ln(e^{2x-1}) \, dx$

79. $\displaystyle\int 3^{x/2} \, dx$

80. $\displaystyle\int 4^{-x} \, dx$

81. $\displaystyle\int x5^{-x^2} \, dx$

82. $\displaystyle\int (3 - x)7^{(3-x)^2} \, dx$

In Exercises 83–88, find an equation for the function f that has the given derivative and whose graph passes through the given point.

	Derivative	*Point*
83.	$f'(x) = x\sqrt{4 - x^2}$	$(2, 2)$
84.	$f'(x) = 0.4^{x/3}$	$\left(0, \frac{1}{2}\right)$
85.	$f'(x) = -\sin \dfrac{x}{2}$	$(0, 6)$
86.	$f'(x) = \pi \sec \pi x \tan \pi x$	$\left(\frac{1}{3}, 1\right)$
87.	$f'(x) = 2e^{-x/4}$	$(0, 1)$
88.	$f'(x) = x^2 e^{-0.2x^3}$	$\left(0, \frac{3}{2}\right)$

In Exercises 89 and 90, find the particular solution of the differential equation that satisfies the initial conditions.

89. $f''(x) = \frac{1}{2}(e^x + e^{-x})$, $f(0) = 1, f'(0) = 0$

90. $f''(x) = \sin x + e^{2x}$, $f(0) = \frac{1}{4}, f'(0) = \frac{1}{2}$

In Exercises 91–98, find the indefinite integral by the method shown in Example 5.

91. $\displaystyle\int x\sqrt{x + 6} \, dx$

92. $\displaystyle\int x\sqrt{4x + 1} \, dx$

93. $\displaystyle\int x^2\sqrt{1 - x} \, dx$

94. $\displaystyle\int (x + 1)\sqrt{2 - x} \, dx$

95. $\displaystyle\int \frac{x^2 - 1}{\sqrt{2x - 1}} \, dx$

96. $\displaystyle\int \frac{2x + 1}{\sqrt{x + 4}} \, dx$

97. $\displaystyle\int \frac{-x}{(x + 1) - \sqrt{x + 1}} \, dx$

98. $\displaystyle\int t \sqrt[3]{t + 10} \, dt$

In Exercises 99–116, evaluate the definite integral. Use a graphing utility to verify your result.

99. $\displaystyle\int_{-1}^{1} x(x^2 + 1)^3 \, dx$

100. $\displaystyle\int_{-2}^{4} x^2(x^3 + 8)^2 \, dx$

101. $\displaystyle\int_{1}^{2} 2x^2\sqrt{x^3 + 1} \, dx$

102. $\displaystyle\int_{0}^{2} x\sqrt{4 - x^2} \, dx$

103. $\displaystyle\int_{0}^{4} \frac{1}{\sqrt{2x + 1}} \, dx$

104. $\displaystyle\int_{0}^{2} \frac{x}{\sqrt{1 + 2x^2}} \, dx$

105. $\displaystyle\int_{0}^{1} e^{-2x} \, dx$

106. $\displaystyle\int_{1}^{2} e^{1-x} \, dx$

107. $\displaystyle\int_{1}^{3} \frac{e^{3/x}}{x^2} \, dx$

108. $\displaystyle\int_{0}^{\sqrt{2}} xe^{-(x^2/2)} \, dx$

109. $\displaystyle\int_{1}^{9} \frac{1}{\sqrt{x}\left(1 + \sqrt{x}\right)^2} \, dx$

110. $\displaystyle\int_{0}^{2} x\sqrt[3]{4 + x^2} \, dx$

111. $\displaystyle\int_{1}^{2} (x - 1)\sqrt{2 - x} \, dx$

112. $\displaystyle\int_{1}^{5} \frac{x}{\sqrt{2x - 1}} \, dx$

113. $\displaystyle\int_{0}^{\pi/2} \cos\left(\frac{2x}{3}\right) dx$

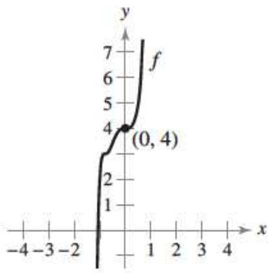
114. $\displaystyle\int_{\pi/3}^{\pi/2} (x + \cos x) \, dx$

115. $\displaystyle\int_{-1}^{2} 2^x \, dx$

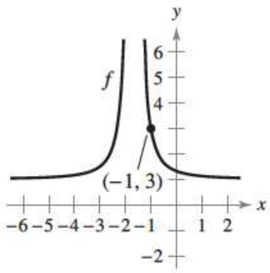
116. $\displaystyle\int_{-2}^{0} (3^3 - 5^2) \, dx$

Differential Equations **In Exercises 117–120, the graph of a function f is shown. Use the differential equation and the given point to find an equation of the function.**

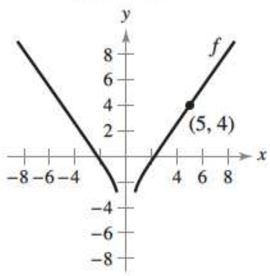
117. $\dfrac{dy}{dx} = 18x^2(2x^3 + 1)^2$



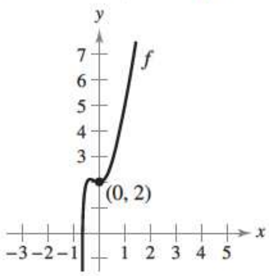
118. $\dfrac{dy}{dx} = \dfrac{-48}{(3x + 5)^3}$



119. $\dfrac{dy}{dx} = \dfrac{2x}{\sqrt{2x^2 - 1}}$

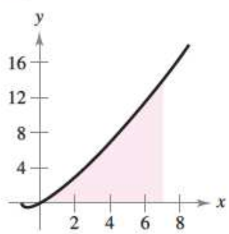


120. $\dfrac{dy}{dx} = 4x + \dfrac{9x^2}{(3x^3 + 1)^{(3/2)}}$

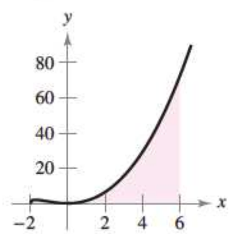


In Exercises 121–126, find the area of the region. Use a graphing utility to verify your result.

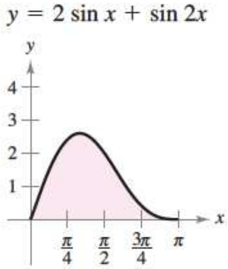
121. $\displaystyle\int_0^7 x\sqrt[3]{x + 1}\,dx$



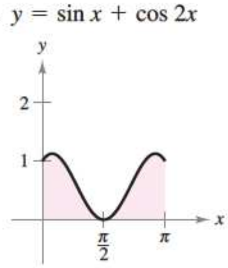
122. $\displaystyle\int_{-2}^6 x^2\sqrt[3]{x + 2}\,dx$



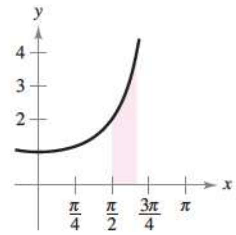
123. $y = 2\sin x + \sin 2x$



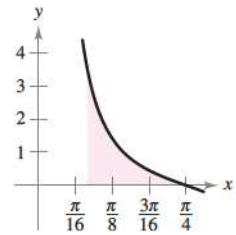
124. $y = \sin x + \cos 2x$



125. $\displaystyle\int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right) dx$



126. $\displaystyle\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x\,dx$



Area **In Exercises 127–130, find the area of the region bounded by the graphs of the equations. Use a graphing utility to graph the region and verify your result.**

127. $y = e^x, y = 0, x = 0, x = 5$

128. $y = e^{-x}, y = 0, x = a, x = b$

129. $y = xe^{-x^2/4}, y = 0, x = 0, x = \sqrt{6}$

130. $y = e^{-2x} + 2, y = 0, x = 0, x = 2$

In Exercises 131–138, use a graphing utility to evaluate the integral. Graph the region whose area is given by the definite integral.

131. $\displaystyle\int_0^6 \frac{x}{\sqrt{4x + 1}}\,dx$

132. $\displaystyle\int_0^2 x^3\sqrt{2x + 3}\,dx$

133. $\displaystyle\int_3^7 x\sqrt{x - 3}\,dx$

134. $\displaystyle\int_1^5 x^2\sqrt{x - 1}\,dx$

135. $\displaystyle\int_1^4 \left(\theta + \sin\frac{\theta}{4}\right) d\theta$

136. $\displaystyle\int_0^{\pi/6} \cos 3x\,dx$

137. $\displaystyle\int_0^{\sqrt{2}} xe^{-x^2/2}\,dx$

138. $\displaystyle\int_0^2 (e^{-2x} + 2)\,dx$

In Exercises 139–142, evaluate the integral using the properties of even and odd functions as an aid.

139. $\displaystyle\int_{-2}^2 x^2(x^2 + 1)\,dx$

140. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\,dx$

141. $\displaystyle\int_{-2}^2 x(x^2 + 1)^3\,dx$

142. $\displaystyle\int_{-\pi/4}^{\pi/4} \sin x \cos x\,dx$

143. Use $\int_0^4 x^2\,dx = \frac{64}{3}$ to evaluate each definite integral without using the Fundamental Theorem of Calculus.

(a) $\displaystyle\int_{-4}^0 x^2\,dx$ (b) $\displaystyle\int_{-4}^4 x^2\,dx$

(c) $\displaystyle\int_0^4 -x^2\,dx$ (d) $\displaystyle\int_{-4}^0 3x^2\,dx$

144. Use the symmetry of the graphs of the sine and cosine functions as an aid in evaluating each definite integral.

(a) $\displaystyle\int_{-\pi/4}^{\pi/4} \sin x\,dx$ (b) $\displaystyle\int_{-\pi/4}^{\pi/4} \cos x\,dx$

(c) $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x\,dx$ (d) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x \cos x\,dx$

In Exercises 145 and 146, write the integral as the sum of the integral of an odd function and the integral of an even function. Use this simplification to evaluate the integral.

145. $\int_{-3}^{3} (x^3 + 4x^2 - 3x - 6)\, dx$ **146.** $\int_{-\pi/2}^{\pi/2} (\sin 4x + \cos 4x)\, dx$

WRITING ABOUT CONCEPTS

147. Describe why

$$\int x(5 - x^2)^3\, dx \neq \int u^3\, du$$

where $u = 5 - x^2$.

148. Without integrating, explain why $\int_{-2}^{2} x(x^2 + 1)^2\, dx = 0$.

149. If f is continuous and $\int_0^8 f(x)\, dx = 32$, find $\int_0^4 f(2x)\, dx$.

CAPSTONE

150. *Writing* Find the indefinite integral in two ways. Explain any difference in the forms of the answers.

(a) $\int (2x - 1)^2\, dx$ (b) $\int \sin x \cos x\, dx$

(c) $\int \tan x \sec^2 x\, dx$

151. *Cash Flow* The rate of disbursement dQ/dt of a 4-million-dollar federal grant is proportional to the square of $100 - t$. Time t is measured in days $(0 \le t \le 100)$, and Q is the amount that remains to be disbursed. Find the amount that remains to be disbursed after 50 days. Assume that all the money will be disbursed in 100 days.

152. *Depreciation* The rate of depreciation dV/dt of a machine is inversely proportional to the square of $t + 1$, where V is the value of the machine t years after it was purchased. The initial value of the machine was \$500,000, and its value decreased \$100,000 in the first year. Estimate its value after 4 years.

153. *Precipitation* The normal monthly precipitation at the Seattle-Tacoma airport can be approximated by the model

$$R = 2.876 + 2.202 \sin(0.576t + 0.847)$$

where R is measured in inches and t is the time in months, with $t = 0$ corresponding to January 1. (*Source: U.S. National Oceanic and Atmospheric Administration*)

(a) Determine the extrema of the function over a one-year period.

(b) Use integration to approximate the normal annual precipitation. (*Hint:* Integrate over the interval $[0, 12]$.)

(c) Approximate the average monthly precipitation during the months of October, November, and December.

154. *Sales* The sales S (in thousands of units) of a seasonal product are given by the model

$$S = 74.50 + 43.75 \sin \frac{\pi t}{6}$$

where t is the time in months, with $t = 1$ corresponding to January. Find the average sales for each time period.

(a) The first quarter $(0 \le t \le 3)$

(b) The second quarter $(3 \le t \le 6)$

(c) The entire year $(0 \le t \le 12)$

155. *Water Supply* A model for the flow rate of water at a pumping station on a given day is

$$R(t) = 53 + 7 \sin\left(\frac{\pi t}{6} + 3.6\right) + 9 \cos\left(\frac{\pi t}{12} + 8.9\right)$$

where $0 \le t \le 24$. R is the flow rate in thousands of gallons per hour, and t is the time in hours.

(a) Use a graphing utility to graph the rate function and approximate the maximum flow rate at the pumping station.

(b) Approximate the total volume of water pumped in 1 day.

156. *Electricity* The oscillating current in an electrical circuit is

$$I = 2 \sin(60\pi t) + \cos(120\pi t)$$

where I is measured in amperes and t is measured in seconds. Find the average current for each time interval.

(a) $0 \le t \le \frac{1}{60}$

(b) $0 \le t \le \frac{1}{240}$

(c) $0 \le t \le \frac{1}{30}$

Probability **In Exercises 157 and 158, the function**

$$f(x) = kx^n(1 - x)^m, \quad 0 \le x \le 1$$

where $n > 0$, $m > 0$, and k is a constant, can be used to represent various probability distributions. If k is chosen such that

$$\int_0^1 f(x)\, dx = 1$$

the probability that x will fall between a and b $(0 \le a \le b \le 1)$ is

$$P_{a,\,b} = \int_a^b f(x)\, dx.$$

157. The probability that a person will remember between $100a\%$ and $100b\%$ of material learned in an experiment is

$$P_{a,\,b} = \int_a^b \frac{15}{4} x\sqrt{1 - x}\, dx$$

where x represents the proportion remembered. (See figure on the next page.)

(a) What is the probability that a randomly chosen individual will recall between 50% and 75% of the material?

(b) What is the median percent recall? That is, for what value of b is it true that the probability of recalling 0 to b is 0.5?

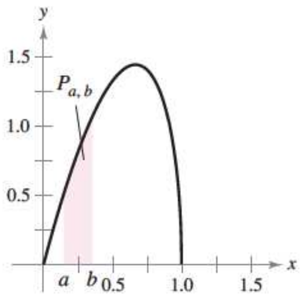


Figure for 157

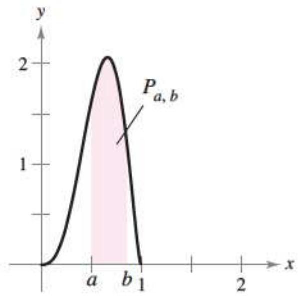


Figure for 158

158. The probability that ore samples taken from a region contain between $100a\%$ and $100b\%$ iron is

$$P_{a,\,b} = \int_a^b \frac{1155}{32} x^3(1-x)^{3/2}\,dx$$

where x represents the proportion of iron. (See figure.) What is the probability that a sample will contain between

(a) 0% and 25% iron?

(b) 50% and 100% iron?

159. ***Probability*** A car battery has an average lifetime of 48 months with a standard deviation of 6 months. The battery lives are normally distributed. The probability that a given battery will last between 48 months and 60 months is

$$0.0665 \int_{48}^{60} e^{-0.0139(t-48)^2}\,dt.$$

Use the integration capabilities of a graphing utility to approximate the integral. Interpret the resulting probability.

160. Given $e^x \geq 1$ for $x \geq 0$, it follows that

$$\int_0^x e^t\,dt \geq \int_0^x 1\,dt.$$

Perform this integration to derive the inequality $e^x \geq 1 + x$ for $x \geq 0$.

161. ***Graphical Analysis*** Consider the functions f and g, where

$$f(x) = 6 \sin x \cos^2 x \quad \text{and} \quad g(t) = \int_0^t f(x)\,dx.$$

(a) Use a graphing utility to graph f and g in the same viewing window.

(b) Explain why g is nonnegative.

(c) Identify the points on the graph of g that correspond to the extrema of f.

(d) Does each of the zeros of f correspond to an extremum of g? Explain.

(e) Consider the function $h(t) = \int_{\pi/2}^t f(x)\,dx$. Use a graphing utility to graph h. What is the relationship between g and h? Verify your conjecture.

162. Find $\displaystyle\lim_{n\to+\infty} \sum_{i=1}^{n} \frac{\sin(i\pi/n)}{n}$ by evaluating an appropriate definite integral over the interval $[0, 1]$.

163. (a) Show that $\int_0^1 x^2(1-x)^5\,dx = \int_0^1 x^5(1-x)^2\,dx$.

(b) Show that $\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx$.

164. (a) Show that $\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx$.

(b) Show that $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$, where n is a positive integer.

True or False? **In Exercises 165–170, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

165. $\displaystyle\int (2x+1)^2\,dx = \frac{1}{3}(2x+1)^3 + C$

166. $\displaystyle\int x(x^2+1)\,dx = \frac{1}{2}x^2\left(\frac{1}{3}x^3 + x\right) + C$

167. $\displaystyle\int_{-10}^{10} (ax^3 + bx^2 + cx + d)\,dx = 2\int_0^{10} (bx^2 + d)\,dx$

168. $\displaystyle\int_a^b \sin x\,dx = \int_a^{b+2\pi} \sin x\,dx$

169. $\displaystyle 4\int \sin x \cos x\,dx = -\cos 2x + C$

170. $\displaystyle\int \sin^2 2x \cos 2x\,dx = \frac{1}{3}\sin^3 2x + C$

171. Assume that f is continuous everywhere and that c is a constant. Show that

$$\int_{ca}^{cb} f(x)\,dx = c\int_a^b f(cx)\,dx.$$

172. (a) Verify that $\sin u - u\cos u + C = \int u \sin u\,du$.

(b) Use part (a) to show that $\int_0^{\pi^2} \sin\sqrt{x}\,dx = 2\pi$.

173. Complete the proof of Theorem 5.16.

174. Show that if f is continuous on the entire real number line, then

$$\int_a^b f(x+h)\,dx = \int_{a+h}^{b+h} f(x)\,dx.$$

PUTNAM EXAM CHALLENGE

175. If $a_0, a_1, \ldots, a_n$ are real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$$

show that the equation

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$$

has at least one real zero.

176. Find all the continuous positive functions $f(x)$, for $0 \leq x \leq 1$, such that

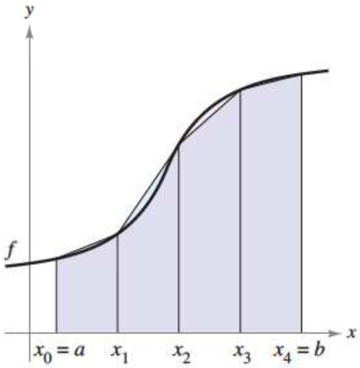
$$\int_0^1 f(x)\,dx = 1, \quad \int_0^1 f(x)x\,dx = \alpha, \text{ and } \int_0^1 f(x)x^2\,dx = \alpha^2$$

where α is a real number.

5.6 Numerical Integration

- Approximate a definite integral using the Trapezoidal Rule.
- Approximate a definite integral using Simpson's Rule.
- Analyze the approximate errors in the Trapezoidal Rule and Simpson's Rule.

The Trapezoidal Rule



The area of the region can be approximated using four trapezoids.
Figure 5.42

Some elementary functions simply do not have antiderivatives that are elementary functions. For example, there is no elementary function that has any of the following functions as its derivative.

$$\sqrt[3]{x}\sqrt{1-x}, \qquad \sqrt{x}\cos x, \qquad \frac{\cos x}{x}, \qquad \sqrt{1-x^3}, \qquad \sin x^2$$

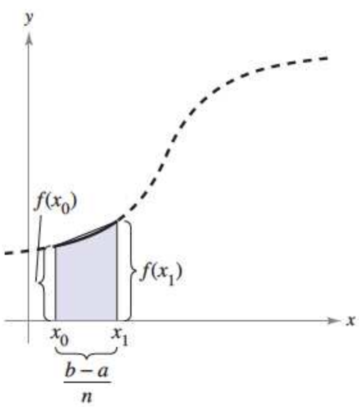
If you need to evaluate a definite integral involving a function whose antiderivative cannot be found, then while the Fundamental Theorem of Calculus is still true, it cannot be easily applied. In this case, it is easier to resort to an approximation technique. Two such techniques are described in this section.

One way to approximate a definite integral is to use n trapezoids, as shown in Figure 5.42. In the development of this method, assume that f is continuous and positive on the interval $[a, b]$. So, the definite integral

$$\int_a^b f(x)\,dx$$

represents the area of the region bounded by the graph of f and the x-axis, from $x = a$ to $x = b$. First, partition the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$, such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$



The area of the first trapezoid is $\left[\frac{f(x_0) + f(x_1)}{2}\right]\left(\frac{b-a}{n}\right)$.
Figure 5.43

Then form a trapezoid for each subinterval (see Figure 5.43). The area of the ith trapezoid is

$$\text{Area of } i\text{th trapezoid} = \left[\frac{f(x_{i-1}) + f(x_i)}{2}\right]\left(\frac{b-a}{n}\right).$$

This implies that the sum of the areas of the n trapezoids is

$$\begin{aligned}
\text{Area} &= \left(\frac{b-a}{n}\right)\left[\frac{f(x_0) + f(x_1)}{2} + \cdots + \frac{f(x_{n-1}) + f(x_n)}{2}\right] \\
&= \left(\frac{b-a}{2n}\right)[f(x_0) + f(x_1) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n)] \\
&= \left(\frac{b-a}{2n}\right)[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)].
\end{aligned}$$

Letting $\Delta x = (b - a)/n$, you can take the limit as $n \to \infty$ to obtain

$$\begin{aligned}
&\lim_{n\to\infty} \left(\frac{b-a}{2n}\right)[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)] \\
&= \lim_{n\to\infty} \left[\frac{[f(a) - f(b)]\,\Delta x}{2} + \sum_{i=1}^{n} f(x_i)\,\Delta x\right] \\
&= \lim_{n\to\infty} \frac{[f(a) - f(b)](b-a)}{2n} + \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \\
&= 0 + \int_a^b f(x)\,dx.
\end{aligned}$$

The result is summarized in the following theorem.

THEOREM 5.17 THE TRAPEZOIDAL RULE

Let f be continuous on $[a, b]$. The Trapezoidal Rule for approximating $\int_a^b f(x)\,dx$ is given by

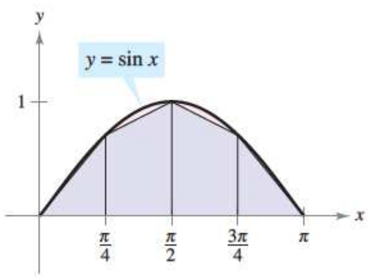
$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

Moreover, as $n \to \infty$, the right-hand side approaches $\int_a^b f(x)\,dx$.

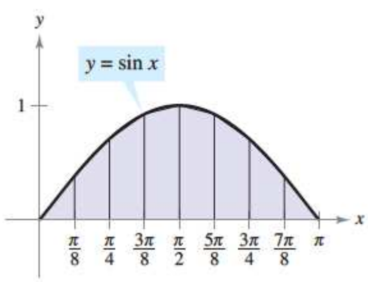
NOTE Observe that the coefficients in the Trapezoidal Rule have the following pattern.

1 2 2 2 . . . 2 2 1

EXAMPLE 1 Approximation with the Trapezoidal Rule



Four subintervals



Eight subintervals

Trapezoidal approximations
Figure 5.44

Use the Trapezoidal Rule to approximate

$$\int_0^{\pi} \sin x\,dx.$$

Compare the results for $n = 4$ and $n = 8$, as shown in Figure 5.44.

Solution When $n = 4$, $\Delta x = \pi/4$, and you obtain

$$\int_0^{\pi} \sin x\,dx \approx \frac{\pi}{8}\left(\sin 0 + 2\sin\frac{\pi}{4} + 2\sin\frac{\pi}{2} + 2\sin\frac{3\pi}{4} + \sin\pi\right)$$

$$= \frac{\pi}{8}\left(0 + \sqrt{2} + 2 + \sqrt{2} + 0\right) = \frac{\pi\left(1 + \sqrt{2}\right)}{4} \approx 1.896.$$

When $n = 8$, $\Delta x = \pi/8$, and you obtain

$$\int_0^{\pi} \sin x\,dx \approx \frac{\pi}{16}\left(\sin 0 + 2\sin\frac{\pi}{8} + 2\sin\frac{\pi}{4} + 2\sin\frac{3\pi}{8} + 2\sin\frac{\pi}{2}\right.$$

$$\left. + 2\sin\frac{5\pi}{8} + 2\sin\frac{3\pi}{4} + 2\sin\frac{7\pi}{8} + \sin\pi\right)$$

$$= \frac{\pi}{16}\left(2 + 2\sqrt{2} + 4\sin\frac{\pi}{8} + 4\sin\frac{3\pi}{8}\right) \approx 1.974.$$

For this particular integral, you could have found an antiderivative and determined that the exact area of the region is 2.

TECHNOLOGY Most graphing utilities and computer algebra systems have built-in programs that can be used to approximate the value of a definite integral. Try using such a program to approximate the integral in Example 1. How close is your approximation?

When you use such a program, you need to be aware of its limitations. Often, you are given no indication of the degree of accuracy of the approximation. Other times, you may be given an approximation that is completely wrong. For instance, try using a built-in numerical integration program to evaluate

$$\int_{-1}^{2} \frac{1}{x}\,dx.$$

Your calculator should give an error message. Does it?

It is interesting to compare the Trapezoidal Rule with the Midpoint Rule given in Section 5.2 (Exercises 73–76). For the Trapezoidal Rule, you average the function values at the endpoints of the subintervals, but for the Midpoint Rule you take the function values of the subinterval midpoints.

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x \qquad \text{Midpoint Rule}$$

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \left(\frac{f(x_i) + f(x_{i-1})}{2}\right)\Delta x \qquad \text{Trapezoidal Rule}$$

NOTE There are two important points that should be made concerning the Trapezoidal Rule (or the Midpoint Rule). First, the approximation tends to become more accurate as n increases. For instance, in Example 1, if $n = 16$, the Trapezoidal Rule yields an approximation of 1.994. Second, although you could have used the Fundamental Theorem to evaluate the integral in Example 1, this theorem cannot be used to evaluate an integral as simple as $\int_0^{\pi} \sin x^2\,dx$ because $\sin x^2$ has no elementary antiderivative. Yet, the Trapezoidal Rule can be applied easily to estimate this integral. ■

Simpson's Rule

One way to view the trapezoidal approximation of a definite integral is to say that on each subinterval you approximate f by a *first*-degree polynomial. In Simpson's Rule, named after the English mathematician Thomas Simpson (1710–1761), you take this procedure one step further and approximate f by *second*-degree polynomials.

Before presenting Simpson's Rule, a theorem for evaluating integrals of polynomials of degree 2 (or less) is listed.

THEOREM 5.18 INTEGRAL OF $p(x) = Ax^2 + Bx + C$

If $p(x) = Ax^2 + Bx + C$, then

$$\int_a^b p(x)\,dx = \left(\frac{b-a}{6}\right)\left[p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right].$$

PROOF

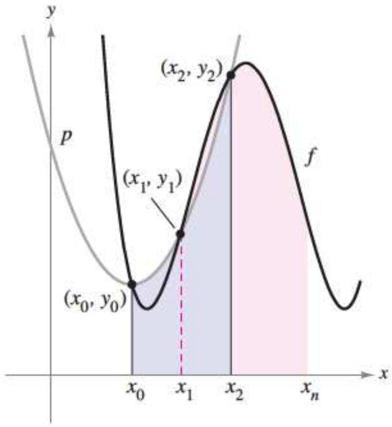
$$\begin{aligned}
\int_a^b p(x)\,dx &= \int_a^b (Ax^2 + Bx + C)\,dx \\
&= \left[\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right]_a^b \\
&= \frac{A(b^3 - a^3)}{3} + \frac{B(b^2 - a^2)}{2} + C(b - a) \\
&= \left(\frac{b-a}{6}\right)[2A(a^2 + ab + b^2) + 3B(b + a) + 6C]
\end{aligned}$$

By expansion and collection of terms, the expression inside the brackets becomes

$$\underbrace{(Aa^2 + Ba + C)}_{p(a)} + \underbrace{4\left[A\left(\frac{b+a}{2}\right)^2 + B\left(\frac{b+a}{2}\right) + C\right]}_{4p\left(\frac{a+b}{2}\right)} + \underbrace{(Ab^2 + Bb + C)}_{p(b)}$$

and you can write

$$\int_a^b p(x)\,dx = \left(\frac{b-a}{6}\right)\left[p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right].$$ ■



$\int_{x_0}^{x_2} p(x)\,dx \approx \int_{x_0}^{x_2} f(x)\,dx$

Figure 5.45

To develop Simpson's Rule for approximating a definite integral, you again partition the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$. This time, however, n is required to be even, and the subintervals are grouped in pairs such that

$$a = \underbrace{x_0 < x_1 < x_2}_{[x_0,\, x_2]} < \underbrace{x_3 < x_4}_{[x_2,\, x_4]} < \cdot \cdot \cdot < \underbrace{x_{n-2} < x_{n-1} < x_n}_{[x_{n-2},\, x_n]} = b.$$

On each (double) subinterval $[x_{i-2}, x_i]$, you can approximate f by a polynomial p of degree less than or equal to 2. (See Exercise 57.) For example, on the subinterval $[x_0, x_2]$, choose the polynomial of least degree passing through the points (x_0, y_0), (x_1, y_1), and (x_2, y_2), as shown in Figure 5.45. Now, using p as an approximation of f on this subinterval, you have, by Theorem 5.18,

$$\begin{aligned}\int_{x_0}^{x_2} f(x)\,dx \approx \int_{x_0}^{x_2} p(x)\,dx &= \frac{x_2 - x_0}{6}\left[p(x_0) + 4p\left(\frac{x_0 + x_2}{2}\right) + p(x_2)\right] \\ &= \frac{2[(b-a)/n]}{6}[p(x_0) + 4p(x_1) + p(x_2)] \\ &= \frac{b-a}{3n}[f(x_0) + 4f(x_1) + f(x_2)].\end{aligned}$$

Repeating this procedure on the entire interval $[a, b]$ produces the following theorem.

NOTE Observe that the coefficients in Simpson's Rule have the following pattern.

1 4 2 4 2 4 . . . 4 2 4 1

> **THEOREM 5.19 SIMPSON'S RULE**
>
> Let f be continuous on $[a, b]$ and let n be an even integer. Simpson's Rule for approximating $\int_a^b f(x)\,dx$ is
>
> $$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdot \cdot \cdot + 4f(x_{n-1}) + f(x_n)].$$
>
> Moreover, as $n \to \infty$, the right-hand side approaches $\int_a^b f(x)\,dx$.

In Example 1, the Trapezoidal Rule was used to estimate $\int_0^{\pi} \sin x\,dx$. In the next example, Simpson's Rule is applied to the same integral.

EXAMPLE 2 Approximation with Simpson's Rule

NOTE In Example 1, the Trapezoidal Rule with $n = 8$ approximates $\int_0^{\pi} \sin x\,dx$ as 1.974. In Example 2, Simpson's Rule with $n = 8$ gives an approximation of 2.0003. The antiderivative would produce the true value of 2.

Use Simpson's Rule to approximate

$$\int_0^{\pi} \sin x\,dx.$$

Compare the results for $n = 4$ and $n = 8$.

Solution When $n = 4$, you have

$$\begin{aligned}\int_0^{\pi} \sin x\,dx &\approx \frac{\pi}{12}\left(\sin 0 + 4\sin\frac{\pi}{4} + 2\sin\frac{\pi}{2} + 4\sin\frac{3\pi}{4} + \sin\pi\right) \\ &\approx 2.005\end{aligned}$$

When $n = 8$, you have $\int_0^{\pi} \sin x\,dx \approx 2.0003.$ ■

Error Analysis

If you must use an approximation technique, it is important to know how accurate you can expect the approximation to be. The following theorem, which is listed without proof, gives the formulas for estimating the errors involved in the use of Simpson's Rule and the Trapezoidal Rule. In general, when using an approximation, you can think of the error E as the difference between $\int_a^b f(x)\,dx$ and the approximation.

NOTE In Theorem 5.20, $\max|f''(x)|$ is the least upper bound of the absolute value of the second derivative on $[a, b]$, and $\max|f^{(4)}(x)|$ is the least upper bound of the absolute value of the fourth derivative on $[a, b]$.

THEOREM 5.20 ERRORS IN THE TRAPEZOIDAL RULE AND SIMPSON'S RULE

If f has a continuous second derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x)\,dx$ by the Trapezoidal Rule is

$$|E| \le \frac{(b-a)^3}{12n^2}\left[\max|f''(x)|\right], \quad a \le x \le b. \qquad \text{Trapezoidal Rule}$$

Moreover, if f has a continuous fourth derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x)\,dx$ by Simpson's Rule is

$$|E| \le \frac{(b-a)^5}{180n^4}\left[\max|f^{(4)}(x)|\right], \quad a \le x \le b. \qquad \text{Simpson's Rule}$$

TECHNOLOGY If you have access to a computer algebra system, use it to evaluate the definite integral in Example 3. You should obtain a value of

$$\int_0^1 \sqrt{1+x^2}\,dx = \tfrac{1}{2}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right] \approx 1.14779.$$

Theorem 5.20 states that the errors generated by the Trapezoidal Rule and Simpson's Rule have upper bounds dependent on the extreme values of $f''(x)$ and $f^{(4)}(x)$ in the interval $[a, b]$. Furthermore, these errors can be made arbitrarily small by *increasing n*, provided that f'' and $f^{(4)}$ are continuous and therefore bounded in $[a, b]$.

EXAMPLE 3 The Approximate Error in the Trapezoidal Rule

Determine a value of n such that the Trapezoidal Rule will approximate the value of $\int_0^1 \sqrt{1+x^2}\,dx$ with an error that is less than 0.01.

Solution Begin by letting $f(x) = \sqrt{1+x^2}$ and finding the second derivative of f.

$$f'(x) = x(1+x^2)^{-1/2} \quad \text{and} \quad f''(x) = (1+x^2)^{-3/2}$$

The maximum value of $|f''(x)|$ on the interval $[0, 1]$ is $|f''(0)| = 1$. So, by Theorem 5.20, you can write

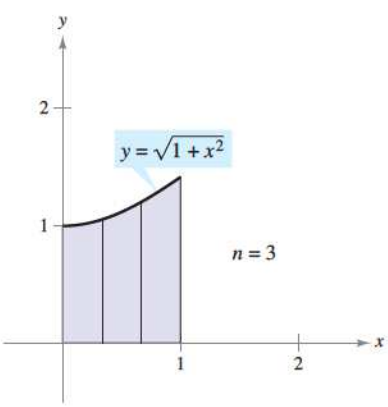
$$|E| \le \frac{(b-a)^3}{12n^2}|f''(0)| = \frac{1}{12n^2}(1) = \frac{1}{12n^2}.$$

To obtain an error E that is less than 0.01, you must choose n such that $1/(12n^2) \le 1/100$.

$$100 \le 12n^2 \quad \Longrightarrow \quad n \ge \sqrt{\tfrac{100}{12}} \approx 2.89$$

So, you can choose $n = 3$ (because n must be greater than or equal to 2.89) and apply the Trapezoidal Rule, as shown in Figure 5.46, to obtain

$$\int_0^1 \sqrt{1+x^2}\,dx \approx \frac{1}{6}\left[\sqrt{1+0^2} + 2\sqrt{1+\left(\tfrac{1}{3}\right)^2} + 2\sqrt{1+\left(\tfrac{2}{3}\right)^2} + \sqrt{1+1^2}\right] \approx 1.154.$$



$1.144 \le \int_0^1 \sqrt{1+x^2}\,dx \le 1.164$

Figure 5.46

So, by adding and subtracting the error from this estimate, you know that

$$1.144 \le \int_0^1 \sqrt{1+x^2}\,dx \le 1.164.$$

5.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, use the Trapezoidal Rule and Simpson's Rule to approximate the value of the definite integral for the given value of n. Round your answer to four decimal places and compare the results with the exact value of the definite integral.

1. $\int_0^2 x^2\,dx,\quad n = 4$
2. $\int_1^2 \left(\frac{x^2}{4} + 1\right) dx,\quad n = 4$
3. $\int_0^2 x^3\,dx,\quad n = 4$
4. $\int_2^3 \frac{2}{x^2}\,dx,\quad n = 4$
5. $\int_1^3 x^3\,dx,\quad n = 6$
6. $\int_0^8 \sqrt[3]{x}\,dx,\quad n = 8$
7. $\int_4^9 \sqrt{x}\,dx,\quad n = 8$
8. $\int_1^4 (4 - x^2)\,dx,\quad n = 6$
9. $\int_0^1 \frac{2}{(x+2)^2}\,dx,\quad n = 4$
10. $\int_0^2 x\sqrt{x^2+1}\,dx,\quad n = 4$

In Exercises 11–24, approximate the definite integral using the Trapezoidal Rule and Simpson's Rule with $n = 4$. Compare these results with the approximation of the integral using a graphing utility.

11. $\int_0^2 \sqrt{1 + x^3}\,dx$
12. $\int_0^2 \frac{1}{\sqrt{1 + x^3}}\,dx$
13. $\int_0^1 \sqrt{x}\,\sqrt{1 - x}\,dx$
14. $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx$
15. $\int_0^{\sqrt{\pi/2}} \sin x^2\,dx$
16. $\int_0^{\sqrt{\pi/4}} \tan x^2\,dx$
17. $\int_3^{3.1} \cos x^2\,dx$
18. $\int_0^{\pi/2} \sqrt{1 + \sin^2 x}\,dx$
19. $\int_0^2 x\ln(x+1)\,dx$
20. $\int_1^3 \ln x\,dx$
21. $\int_0^{\pi/4} x\tan x\,dx$
22. $\int_0^{\pi} f(x)\,dx,\quad f(x) = \begin{cases} \frac{\sin x}{x}, & x > 0 \\ 1, & x = 0 \end{cases}$
23. $\int_0^4 \sqrt{x}\,e^x\,dx$
24. $\int_0^2 xe^{-x}\,dx$

WRITING ABOUT CONCEPTS

25. The Trapezoidal Rule and Simpson's Rule yield approximations of a definite integral $\int_a^b f(x)\,dx$ based on polynomial approximations of f. What is the degree of the polynomial used for each?
26. Describe the size of the error when the Trapezoidal Rule is used to approximate $\int_a^b f(x)\,dx$ when $f(x)$ is a linear function. Use a graph to explain your answer.

In Exercises 27–30, use the error formulas in Theorem 5.20 to estimate the errors in approximating the integral, with $n = 4$, using (a) the Trapezoidal Rule and (b) Simpson's Rule.

27. $\int_1^3 2x^3\,dx$
28. $\int_3^5 (5x + 2)\,dx$
29. $\int_0^{\pi} \cos x\,dx$
30. $\int_0^1 \sin(\pi x)\,dx$

In Exercises 31–34, use the error formulas in Theorem 5.20 to find n such that the error in the approximation of the definite integral is less than 0.00001 using (a) the Trapezoidal Rule and (b) Simpson's Rule.

31. $\int_0^2 \sqrt{x+2}\,dx$
32. $\int_1^3 \frac{1}{\sqrt{x}}\,dx$
33. $\int_0^1 \cos(\pi x)\,dx$
34. $\int_0^{\pi/2} \sin x\,dx$

CAS **In Exercises 35–38, use a computer algebra system and the error formulas to find n such that the error in the approximation of the definite integral is less than 0.00001 using (a) the Trapezoidal Rule and (b) Simpson's Rule.**

35. $\int_0^2 \sqrt{1+x}\,dx$
36. $\int_0^2 (x+1)^{2/3}\,dx$
37. $\int_0^1 \tan x^2\,dx$
38. $\int_0^1 \sin x^2\,dx$

39. Approximate the area of the shaded region using (a) the Trapezoidal Rule and (b) Simpson's Rule with $n = 4$.

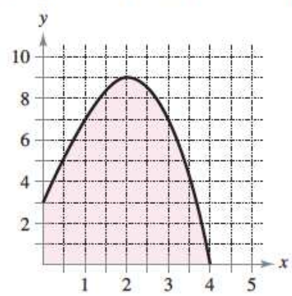


Figure for 39

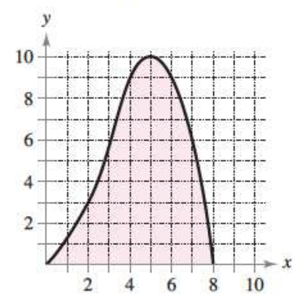


Figure for 40

40. Approximate the area of the shaded region using (a) the Trapezoidal Rule and (b) Simpson's Rule with $n = 8$.
41. *Programming* Write a program for a graphing utility to approximate a definite integral using the Trapezoidal Rule and Simpson's Rule. Start with the program written in Section 5.3, Exercises 63–68, and note that the Trapezoidal Rule can be written as $T(n) = \frac{1}{2}[L(n) + R(n)]$ and Simpson's Rule can be written as $S(n) = \frac{1}{3}[T(n/2) + 2M(n/2)]$. [Recall that $L(n)$, $M(n)$, and $R(n)$ represent the Riemann sums using the left-hand endpoints, midpoints, and right-hand endpoints of subintervals of equal width.]

Programming **In Exercises 42–47, use the program in Exercise 41 to approximate the definite integral and complete the table.**

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4					
8					
10					
12					
16					
20					

42. $\displaystyle\int_0^4 \sqrt{2 + 3x^2}\, dx$

43. $\displaystyle\int_0^1 \sqrt{1 - x^2}\, dx$

44. $\displaystyle\int_0^4 \sin \sqrt{x}\, dx$

45. $\displaystyle\int_1^2 \frac{\sin x}{x}\, dx$

46. $\displaystyle\int_0^2 6e^{-x^2/2}\, dx$

47. $\displaystyle\int_0^3 \sqrt{x} \ln(x + 1)\, dx$

CAPSTONE

48. Consider a function $f(x)$ that is concave upward on the interval $[0, 2]$ and a function $g(x)$ that is concave downward on $[0, 2]$.

(a) Using the Trapezoidal Rule, which integral would be overestimated? Which integral would be underestimated? Assume $n = 4$. Use graphs to explain your answer.

(b) Which rule would you use for more accurate approximations of $\int_0^2 f(x)\, dx$ and $\int_0^2 g(x)\, dx$, the Trapezoidal Rule or Simpson's Rule? Explain your reasoning.

49. *Area* Use Simpson's Rule with $n = 14$ to approximate the area of the region bounded by the graphs of $y = \sqrt{x} \cos x$, $y = 0$, $x = 0$, and $x = \pi/2$.

50. The table lists several measurements gathered in an experiment to approximate an unknown continuous function $y = f(x)$.

x	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
y	4.32	4.36	4.58	5.79	6.14	7.25	7.64	8.08	8.14

(a) Approximate the integral $\int_0^2 f(x)\, dx$ using the Trapezoidal Rule and Simpson's Rule.

(b) Use a graphing utility to find a model of the form $y = ax^3 + bx^2 + cx + d$ for the data. Integrate the resulting polynomial over $[0, 2]$ and compare your result with your results in part (a).

Approximation of Pi **In Exercises 51 and 52, use Simpson's Rule with $n = 6$ to approximate π using the given equation. (In Section 5.8, you will be able to evaluate the integral using inverse trigonometric functions.)**

51. $\displaystyle\pi = \int_0^{1/2} \frac{6}{\sqrt{1 - x^2}}\, dx$

52. $\displaystyle\pi = \int_0^1 \frac{4}{1 + x^2}\, dx$

Area **In Exercises 53 and 54, use the Trapezoidal Rule to estimate the number of square meters of land in a lot, where x and y are measured in meters, as shown in the figures. The land is bounded by a stream and two straight roads that meet at right angles.**

53.

x	0	100	200	300	400	500	600	700	800	900	1000
y	125	125	120	112	90	90	95	88	75	35	0

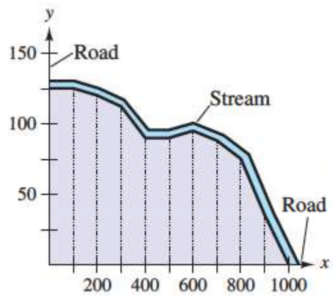


Figure for 53

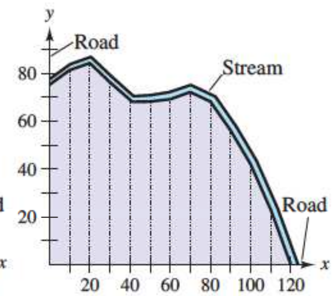


Figure for 54

54.

x	0	10	20	30	40	50	60	70	80	90	100	110	120
y	75	81	84	76	67	68	69	72	68	56	42	23	0

55. Prove that Simpson's Rule is exact when approximating the integral of a cubic polynomial function, and demonstrate the result for $\int_0^1 x^3\, dx$, $n = 2$.

CAS **56.** Use Simpson's Rule with $n = 10$ and a computer algebra system to approximate t in the integral equation

$$\int_0^t \sin \sqrt{x}\, dx = 2.$$

57. Prove that you can find a polynomial $p(x) = Ax^2 + Bx + C$ that passes through any three points (x_1, y_1), (x_2, y_2), and (x_3, y_3), where the x_i's are distinct.

5.7 The Natural Logarithmic Function: Integration

- Use the Log Rule for Integration to integrate a rational function.
- Integrate trigonometric functions.

> **EXPLORATION**
>
> ***Integrating Rational Functions*** Earlier in this chapter, you learned rules that allowed you to integrate *any* polynomial function. The Log Rule presented in this section goes a long way toward enabling you to integrate rational functions. For instance, each of the following functions can be integrated with the Log Rule.
>
> $\frac{1}{2x}$ Example 1
>
> $\frac{1}{4x-1}$ Example 2
>
> $\frac{x}{x^2+1}$ Example 3
>
> $\frac{3x^2+1}{x^3+x}$ Example 4(a)
>
> $\frac{x+1}{x^2+2x}$ Example 4(c)
>
> $\frac{1}{3x+2}$ Example 4(d)
>
> $\frac{x^2+x+1}{x^2+1}$ Example 5
>
> $\frac{2x}{(x+1)^2}$ Example 6
>
> There are still some rational functions that cannot be integrated using the Log Rule. Give examples of these functions, and explain your reasoning.

Log Rule for Integration

In Chapter 3 you studied two differentiation rules for logarithms. The differentiation rule $d/dx[\ln x] = 1/x$ produces the Log Rule for Integration that you learned in Section 5.1. The differentiation rule $d/dx[\ln u] = u'/u$ produces the integration rule $\int 1/u = \ln|u| + C$. These rules are summarized below. (See Exercise 115.)

> **THEOREM 5.21 LOG RULE FOR INTEGRATION**
>
> Let u be a differentiable function of x.
>
> **1.** $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$ **2.** $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C$

Because $du = u'\,dx$, the second formula can also be written as

$$\int \frac{u'}{u}\,dx = \ln|u| + C. \qquad \text{Alternative form of Log Rule}$$

EXAMPLE 1 Using the Log Rule for Integration

To find $\int 1/(2x)\,dx$, let $u = 2x$. Then $du = 2\,dx$.

$$\begin{aligned}
\int \frac{1}{2x}\,dx &= \frac{1}{2}\int\left(\frac{1}{2x}\right)2\,dx && \text{Multiply and divide by 2.}\\
&= \frac{1}{2}\int \frac{1}{u}\,du && \text{Substitute: } u = 2x.\\
&= \frac{1}{2}\ln|u| + C && \text{Apply Log Rule.}\\
&= \frac{1}{2}\ln|2x| + C && \text{Back-substitute.}
\end{aligned}$$

EXAMPLE 2 Using the Log Rule with a Change of Variables

To find $\int 1/(4x-1)\,dx$, let $u = 4x - 1$. Then $du = 4\,dx$.

$$\begin{aligned}
\int \frac{1}{4x-1}\,dx &= \frac{1}{4}\int\left(\frac{1}{4x-1}\right)4\,dx && \text{Multiply and divide by 4.}\\
&= \frac{1}{4}\int \frac{1}{u}\,du && \text{Substitute: } u = 4x-1.\\
&= \frac{1}{4}\ln|u| + C && \text{Apply Log Rule.}\\
&= \frac{1}{4}\ln|4x-1| + C && \text{Back-substitute.}
\end{aligned}$$

Example 3 uses the alternative form of the Log Rule. To apply this rule, look for quotients in which the numerator is the derivative of the denominator.

EXAMPLE 3 Finding Area with the Log Rule

Find the area of the region bounded by the graph of

$$y = \frac{x}{x^2 + 1},$$

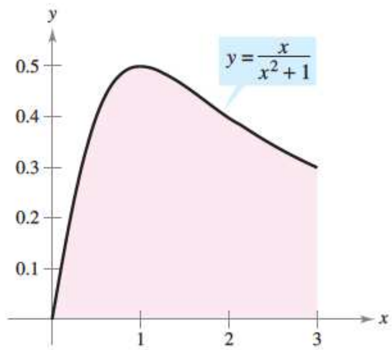
the x-axis, and the line $x = 3$.

Solution In Figure 5.47, you can see that the area of the region is given by the definite integral

$$\int_0^3 \frac{x}{x^2 + 1}\, dx.$$

If you let $u = x^2 + 1$, then $u' = 2x$. To apply the Log Rule, multiply and divide by 2 as shown.

$$\begin{aligned}
\int_0^3 \frac{x}{x^2 + 1}\, dx &= \frac{1}{2}\int_0^3 \frac{2x}{x^2 + 1}\, dx && \text{Multiply and divide by 2.} \\
&= \frac{1}{2}\Big[\ln(x^2 + 1)\Big]_0^3 && \int \frac{u'}{u}\, dx = \ln|u| + C \\
&= \frac{1}{2}(\ln 10 - \ln 1) \\
&= \frac{1}{2}\ln 10 && \ln 1 = 0 \\
&\approx 1.151
\end{aligned}$$



$$\text{Area} = \int_0^3 \frac{x}{x^2 + 1}\, dx$$

The area of the region bounded by the graph of y, the x-axis, and $x = 3$ is $\frac{1}{2}\ln 10$.
Figure 5.47

EXAMPLE 4 Recognizing Quotient Forms of the Log Rule

a. $\displaystyle\int \frac{3x^2 + 1}{x^3 + x}\, dx = \ln|x^3 + x| + C$ $\qquad u = x^3 + x$

b. $\displaystyle\int \frac{\sec^2 x}{\tan x}\, dx = \ln|\tan x| + C$ $\qquad u = \tan x$

c. $\displaystyle\int \frac{x + 1}{x^2 + 2x}\, dx = \frac{1}{2}\int \frac{2x + 2}{x^2 + 2x}\, dx$ $\qquad u = x^2 + 2x$

$$= \frac{1}{2}\ln|x^2 + 2x| + C$$

d. $\displaystyle\int \frac{1}{3x + 2}\, dx = \frac{1}{3}\int \frac{3}{3x + 2}\, dx$ $\qquad u = 3x + 2$

$$= \frac{1}{3}\ln|3x + 2| + C$$

■

With antiderivatives involving logarithms, it is easy to obtain forms that look quite different but are still equivalent. For instance, both of the following are equivalent to the antiderivative listed in Example 4(d).

$$\ln|(3x + 2)^{1/3}| + C \qquad \text{and} \qquad \ln|3x + 2|^{1/3} + C$$

Integrals to which the Log Rule can be applied often appear in disguised form. For instance, if a rational function has a *numerator of degree greater than or equal to that of the denominator*, division may reveal a form to which you can apply the Log Rule. This is illustrated in Example 5.

EXAMPLE 5 Using Long Division Before Integrating

Find $\int \frac{x^2+x+1}{x^2+1}\,dx.$

Solution Begin by using long division to rewrite the integrand.

$$\frac{x^2+x+1}{x^2+1} \quad \Rightarrow \quad \begin{array}{r} 1 \\ x^2+1\,\overline{)\,x^2+x+1} \\ \underline{x^2\quad\;\;+1} \\ x \end{array} \quad \Rightarrow \quad 1+\frac{x}{x^2+1}$$

Now, you can integrate to obtain

$$\int \frac{x^2+x+1}{x^2+1}\,dx = \int\left(1+\frac{x}{x^2+1}\right)dx \qquad \text{Rewrite using long division.}$$

$$= \int dx + \frac{1}{2}\int\frac{2x}{x^2+1}\,dx \qquad \text{Rewrite as two integrals.}$$

$$= x + \frac{1}{2}\ln(x^2+1) + C. \qquad \text{Integrate.}$$

Check this result by differentiating to obtain the original integrand. ■

The next example presents another instance in which the use of the Log Rule is disguised. In this case, a change of variables helps you recognize the Log Rule.

EXAMPLE 6 Change of Variables with the Log Rule

Find $\int \frac{2x}{(x+1)^2}\,dx.$

Solution If you let $u = x+1$, then $du = dx$ and $x = u-1$.

$$\int \frac{2x}{(x+1)^2}\,dx = \int\frac{2(u-1)}{u^2}\,du \qquad \text{Substitute.}$$

$$= 2\int\left(\frac{u}{u^2}-\frac{1}{u^2}\right)du \qquad \text{Rewrite as two fractions.}$$

$$= 2\int\frac{du}{u} - 2\int u^{-2}\,du \qquad \text{Rewrite as two integrals.}$$

$$= 2\ln|u| - 2\left(\frac{u^{-1}}{-1}\right) + C \qquad \text{Integrate.}$$

$$= 2\ln|u| + \frac{2}{u} + C \qquad \text{Simplify.}$$

$$= 2\ln|x+1| + \frac{2}{x+1} + C \qquad \text{Back-substitute.}$$

Check this result by differentiating to obtain the original integrand. ■

TECHNOLOGY If you have access to a computer algebra system, try using it to find the indefinite integrals in Examples 5 and 6. How do the forms of the antiderivatives that it gives you compare with those given in Examples 5 and 6?

As you study the methods shown in Examples 5 and 6, be aware that both methods involve rewriting a disguised integrand so that it fits one or more of the basic integration formulas. Throughout the remaining sections of Chapter 5 and in Chapter 8, much time will be devoted to integration techniques. To master these techniques, you must recognize the "form-fitting" nature of integration. In this sense, integration is not nearly as straightforward as differentiation. Differentiation takes the form

"Here is the question; what is the answer?"

Integration is more like

"Here is the answer; what is the question?"

The following are guidelines you can use for integration.

GUIDELINES FOR INTEGRATION

1. Learn a basic list of integration formulas. (By the end of Section 5.8, this list will have expanded to 20 basic rules.)
2. Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice of u that will make the integrand conform to the formula.
3. If you cannot find a u-substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, addition and subtraction of the same quantity, or long division. Be creative.
4. If you have access to computer software that will find antiderivatives symbolically, use it.

EXAMPLE 7 *u*-Substitution and the Log Rule

Solve the differential equation

$$\frac{dy}{dx} = \frac{1}{x \ln x}.$$

Solution The solution can be written as an indefinite integral.

$$y = \int \frac{1}{x \ln x}\, dx$$

Because the integrand is a quotient whose denominator is raised to the first power, you should try the Log Rule. There are three basic choices for u. The choices $u = x$ and $u = x \ln x$ fail to fit the u'/u form of the Log Rule. However, the third choice does fit. Letting $u = \ln x$ produces $u' = 1/x$, and you obtain the following.

$$\int \frac{1}{x \ln x}\, dx = \int \frac{1/x}{\ln x}\, dx \qquad \text{Divide numerator and denominator by } x.$$

$$= \int \frac{u'}{u}\, dx \qquad \text{Substitute: } u = \ln x.$$

$$= \ln|u| + C \qquad \text{Apply Log Rule.}$$

$$= \ln|\ln x| + C \qquad \text{Back-substitute.}$$

So, the solution is $y = \ln|\ln x| + C$. ■

STUDY TIP Keep in mind that you can check your answer to an integration problem by differentiating the answer. For instance, in Example 7, the derivative of $y = \ln|\ln x| + C$ is $y' = 1/(x \ln x)$.

Integrals of Trigonometric Functions

In Section 5.1, you looked at six trigonometric integration rules—the six that correspond directly to differentiation rules. With the Log Rule, you can now complete the set of basic trigonometric integration formulas.

EXAMPLE 8 Using a Trigonometric Identity

Find $\int \tan x\, dx$.

Solution This integral does not seem to fit any formulas on our basic list. However, by using a trigonometric identity, you obtain the following.

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx.$$

Knowing that $D_x[\cos x] = -\sin x$, you can let $u = \cos x$ and write

$$\begin{aligned}
\int \tan x\, dx &= -\int \frac{-\sin x}{\cos x}\, dx && \text{Trigonometric identity} \\
&= -\int \frac{u'}{u}\, dx && \text{Substitute: } u = \cos x. \\
&= -\ln|u| + C && \text{Apply Log Rule.} \\
&= -\ln|\cos x| + C. && \text{Back-substitute.}
\end{aligned}$$

Example 8 uses a trigonometric identity to derive an integration rule for the tangent function. The next example takes a rather unusual step (multiplying and dividing by the same quantity) to derive an integration rule for the secant function.

EXAMPLE 9 Derivation of the Secant Formula

Find $\int \sec x\, dx$.

Solution Consider the following procedure.

$$\begin{aligned}
\int \sec x\, dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx
\end{aligned}$$

Letting u be the denominator of this quotient produces

$$u = \sec x + \tan x \quad \Rightarrow \quad u' = \sec x \tan x + \sec^2 x.$$

Therefore, you can conclude that

$$\begin{aligned}
\int \sec x\, dx &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx && \text{Rewrite integrand.} \\
&= \int \frac{u'}{u}\, dx && \text{Substitute: } u = \sec x + \tan x. \\
&= \ln|u| + C && \text{Apply Log Rule.} \\
&= \ln|\sec x + \tan x| + C. && \text{Back-substitute.}
\end{aligned}$$

With the results of Examples 8 and 9, you now have integration formulas for $\sin x$, $\cos x$, $\tan x$, and $\sec x$. All six trigonometric rules are summarized below. (For proofs of $\cot u$ and $\csc u$, see Exercises 93 and 94.)

NOTE Using trigonometric identities and properties of logarithms, you could rewrite these six integration rules in other forms. For instance, you could write

$$\int \csc u\, du = \ln|\csc u - \cot u| + C.$$

(See Exercises 95–98.)

INTEGRALS OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

$$\int \sin u\, du = -\cos u + C \qquad \int \cos u\, du = \sin u + C$$

$$\int \tan u\, du = -\ln|\cos u| + C \qquad \int \cot u\, du = \ln|\sin u| + C$$

$$\int \sec u\, du = \ln|\sec u + \tan u| + C \qquad \int \csc u\, du = -\ln|\csc u + \cot u| + C$$

EXAMPLE 10 Integrating Trigonometric Functions

Evaluate $\displaystyle\int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx$.

Solution Using $1 + \tan^2 x = \sec^2 x$, you can write

$$\begin{aligned}
\int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx &= \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx \\
&= \int_0^{\pi/4} \sec x\, dx && \sec x \ge 0 \text{ for } 0 \le x \le \frac{\pi}{4}. \\
&= \ln|\sec x + \tan x|\Big]_0^{\pi/4} \\
&= \ln\left(\sqrt{2} + 1\right) - \ln 1 \\
&\approx 0.881.
\end{aligned}$$

EXAMPLE 11 Finding an Average Value

Find the average value of $f(x) = \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$.

Solution

$$\begin{aligned}
\text{Average value} &= \frac{1}{(\pi/4) - 0}\int_0^{\pi/4} \tan x\, dx && \text{Average value} = \frac{1}{b-a}\int_a^b f(x)\, dx \\
&= \frac{4}{\pi}\int_0^{\pi/4} \tan x\, dx && \text{Simplify.} \\
&= \frac{4}{\pi}\Big[-\ln|\cos x|\Big]_0^{\pi/4} && \text{Integrate.} \\
&= -\frac{4}{\pi}\left[\ln\left(\frac{\sqrt{2}}{2}\right) - \ln(1)\right] \\
&= -\frac{4}{\pi}\ln\left(\frac{\sqrt{2}}{2}\right) \\
&\approx 0.441
\end{aligned}$$

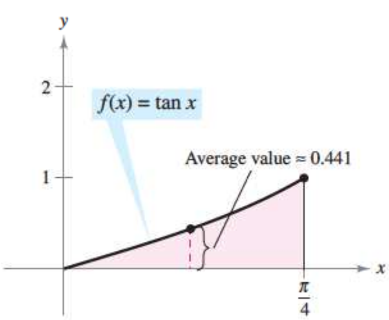


Figure 5.48

The average value is about 0.441, as shown in Figure 5.48. ■

5.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–26, find the indefinite integral.

1. $\int \frac{5}{x}\,dx$
2. $\int \frac{10}{x}\,dx$
3. $\int \frac{1}{x+1}\,dx$
4. $\int \frac{1}{x-5}\,dx$
5. $\int \frac{1}{2x+5}\,dx$
6. $\int \frac{1}{4-3x}\,dx$
7. $\int \frac{x}{x^2-3}\,dx$
8. $\int \frac{x^2}{5-x^3}\,dx$
9. $\int \frac{4x^3+3}{x^4+3x}\,dx$
10. $\int \frac{x^2-2x}{x^3-3x^2}\,dx$
11. $\int \frac{x^2-4}{x}\,dx$
12. $\int \frac{x}{\sqrt{9-x^2}}\,dx$
13. $\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx$
14. $\int \frac{x(x+2)}{x^3+3x^2-4}\,dx$
15. $\int \frac{x^2-3x+2}{x+1}\,dx$
16. $\int \frac{2x^2+7x-3}{x-2}\,dx$
17. $\int \frac{x^3-3x^2+5}{x-3}\,dx$
18. $\int \frac{x^3-6x-20}{x+5}\,dx$
19. $\int \frac{x^4+x-4}{x^2+2}\,dx$
20. $\int \frac{x^3-3x^2+4x-9}{x^2+3}\,dx$
21. $\int \frac{(\ln x)^2}{x}\,dx$
22. $\int \frac{1}{x\ln x^3}\,dx$
23. $\int \frac{1}{\sqrt{x}+1}\,dx$
24. $\int \frac{1}{x^{2/3}(1+x^{1/3})}\,dx$
25. $\int \frac{2x}{(x-1)^2}\,dx$
26. $\int \frac{x(x-2)}{(x-1)^3}\,dx$

In Exercises 27–30, find the indefinite integral by *u*-substitution. (*Hint:* Let *u* be the denominator of the integrand.)

27. $\int \frac{1}{1+\sqrt{2x}}\,dx$
28. $\int \frac{1}{1+\sqrt{3x}}\,dx$
29. $\int \frac{\sqrt{x}}{\sqrt{x}-3}\,dx$
30. $\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\,dx$

In Exercises 31–42, find the indefinite integral.

31. $\int \cot\frac{\theta}{3}\,d\theta$
32. $\int \tan 5\theta\,d\theta$
33. $\int \csc 2x\,dx$
34. $\int \sec\frac{x}{2}\,dx$
35. $\int (\cos 3\theta - 1)\,d\theta$
36. $\int \left(2-\tan\frac{\theta}{4}\right)d\theta$
37. $\int \frac{\cos t}{1+\sin t}\,dt$
38. $\int \frac{\csc^2 t}{\cot t}\,dt$
39. $\int \frac{\sec x\tan x}{\sec x - 1}\,dx$
40. $\int (\sec 2x + \tan 2x)\,dx$
41. $\int e^{-x}\tan(e^{-x})\,dx$
42. $\int \sec t(\sec t + \tan t)\,dt$

In Exercises 43–48, solve the differential equation. Use a graphing utility to graph three solutions, one of which passes through the given point.

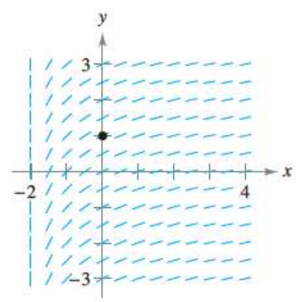
43. $\frac{dy}{dx} = \frac{4}{x}$, $(1, 2)$
44. $\frac{dy}{dx} = \frac{x-2}{x}$, $(-1, 0)$
45. $\frac{dy}{dx} = \frac{3}{2-x}$, $(1, 0)$
46. $\frac{dy}{dx} = \frac{2x}{x^2-9}$, $(0, 4)$
47. $\frac{ds}{d\theta} = \tan 2\theta$, $(0, 2)$
48. $\frac{dr}{dt} = \frac{\sec^2 t}{\tan t + 1}$, $(\pi, 4)$

49. Determine the function f if $f''(x) = \frac{2}{x^2}$, $f(1) = 1$, and $f'(1) = 1$, $x > 0$.

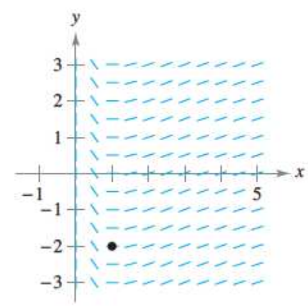
50. Determine the function f if $f''(x) = -\frac{4}{(x-1)^2} - 2$, $f(2) = 3$, and $f'(2) = 0$, $x > 1$.

Slope Fields **In Exercises 51–54, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

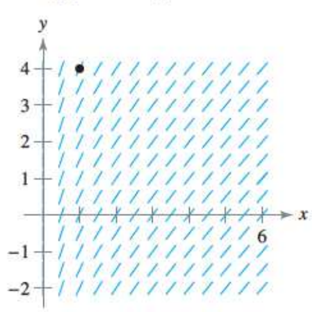
51. $\frac{dy}{dx} = \frac{1}{x+2}$, $(0, 1)$



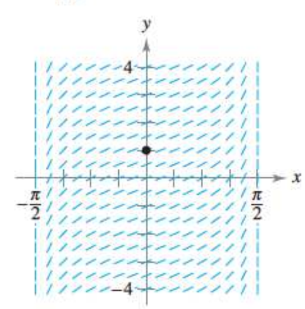
52. $\frac{dy}{dx} = \frac{\ln x}{x}$, $(1, -2)$



53. $\frac{dy}{dx} = 1 + \frac{1}{x}$, $(1, 4)$



54. $\frac{dy}{dx} = \sec x$, $(0, 1)$



In Exercises 55–62, evaluate the definite integral. Use a graphing utility to verify your result.

55. $\displaystyle\int_0^4 \frac{5}{3x+1}\,dx$

56. $\displaystyle\int_{-1}^1 \frac{1}{2x+3}\,dx$

57. $\displaystyle\int_1^e \frac{(1+\ln x)^2}{x}\,dx$

58. $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx$

59. $\displaystyle\int_0^2 \frac{x^2-2}{x+1}\,dx$

60. $\displaystyle\int_0^1 \frac{x-1}{x+1}\,dx$

61. $\displaystyle\int_1^2 \frac{1-\cos\theta}{\theta-\sin\theta}\,d\theta$

62. $\displaystyle\int_{0.1}^{0.2} (\csc 2\theta - \cot 2\theta)^2\,d\theta$

CAS **In Exercises 63–68, use a computer algebra system to find or evaluate the integral.**

63. $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$

64. $\displaystyle\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$

65. $\displaystyle\int \frac{\sqrt{x}}{x-1}\,dx$

66. $\displaystyle\int \frac{x^2}{x-1}\,dx$

67. $\displaystyle\int_{\pi/4}^{\pi/2} (\csc x - \sin x)\,dx$

68. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx$

In Exercises 69–72, find $F'(x)$.

69. $\displaystyle F(x) = \int_1^x \frac{1}{t}\,dt$

70. $\displaystyle F(x) = \int_0^x \tan t\,dt$

71. $\displaystyle F(x) = \int_1^{3x} \frac{1}{t}\,dt$

72. $\displaystyle F(x) = \int_1^{x^2} \frac{1}{t}\,dt$

Approximation **In Exercises 73 and 74, determine which value best approximates the area of the region between the x-axis and the graph of the function over the given interval. (Make your selection on the basis of a sketch of the region and not by performing any calculations.)**

73. $f(x) = \sec x,\quad [0, 1]$

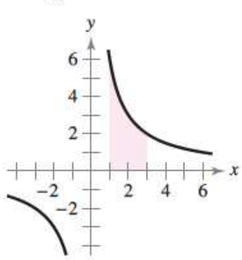
(a) 6 (b) -6 (c) $\frac{1}{2}$ (d) 1.25 (e) 3

74. $f(x) = \dfrac{2x}{x^2+1},\quad [0, 4]$

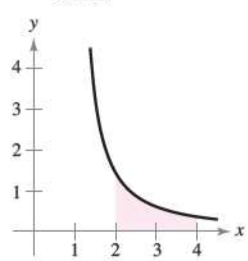
(a) 3 (b) 7 (c) -2 (d) 5 (e) 1

Area **In Exercises 75–78, find the area of the given region. Use a graphing utility to verify your result.**

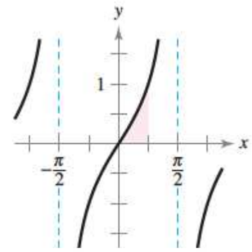
75. $y = \dfrac{6}{x}$



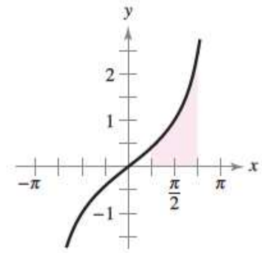
76. $y = \dfrac{2}{x\ln x}$



77. $y = \tan x$



78. $y = \dfrac{\sin x}{1+\cos x}$



Area **In Exercises 79–82, find the area of the region bounded by the graphs of the equations. Use a graphing utility to verify your result.**

79. $y = \dfrac{x^2+4}{x},\ x = 1,\ x = 4,\ y = 0$

80. $y = \dfrac{x+6}{x},\ x = 1,\ x = 5,\ y = 0$

81. $y = 2\sec\dfrac{\pi x}{6},\ x = 0,\ x = 2,\ y = 0$

82. $y = 2x - \tan 0.3x,\ x = 1,\ x = 4,\ y = 0$

Numerical Integration **In Exercises 83–86, use the Trapezoidal Rule and Simpson's Rule to approximate the value of the definite integral. Let $n = 4$ and round your answers to four decimal places. Use a graphing utility to verify your result.**

83. $\displaystyle\int_1^5 \frac{12}{x}\,dx$

84. $\displaystyle\int_0^4 \frac{8x}{x^2+4}\,dx$

85. $\displaystyle\int_2^6 \ln x\,dx$

86. $\displaystyle\int_{-\pi/3}^{\pi/3} \sec x\,dx$

WRITING ABOUT CONCEPTS

In Exercises 87–90, state the integration formula you would use to perform the integration. Do not integrate.

87. $\displaystyle\int \sqrt[3]{x}\,dx$

88. $\displaystyle\int \frac{x}{(x^2+4)^3}\,dx$

89. $\displaystyle\int \frac{x}{x^2+4}\,dx$

90. $\displaystyle\int \frac{\sec^2 x}{\tan x}\,dx$

91. Find a value of x such that $\displaystyle\int_1^x \frac{3}{t}\,dt = \int_{1/4}^x \frac{1}{t}\,dt.$

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92. Find a value of x such that

$$\int_1^x \frac{1}{t}\,dt$$

is equal to (a) ln 5 and (b) 1.

93. Show that $\displaystyle\int \cot u\,du = \ln|\sin u| + C$.

94. Show that $\displaystyle\int \csc u\,du = -\ln|\csc u + \cot u| + C$.

In Exercises 95–98, show that the two formulas are equivalent.

95. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C$

$\displaystyle\int \tan x\,dx = \ln|\sec x| + C$

96. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$

$\displaystyle\int \cot x\,dx = -\ln|\csc x| + C$

97. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$

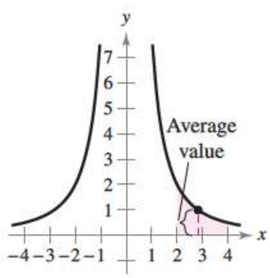
$\displaystyle\int \sec x\,dx = -\ln|\sec x - \tan x| + C$

98. $\displaystyle\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

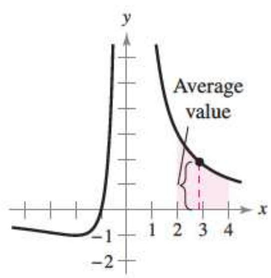
$\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x| + C$

In Exercises 99–102, find the average value of the function over the given interval.

99. $f(x) = \dfrac{8}{x^2}$, $[2, 4]$



100. $f(x) = \dfrac{4(x+1)}{x^2}$, $[2, 4]$



101. $f(x) = \dfrac{2\ln x}{x}$, $[1, e]$

102. $f(x) = \sec\dfrac{\pi x}{6}$, $[0, 2]$

103. ***Population Growth*** A population P of bacteria is changing at a rate of

$$\frac{dP}{dt} = \frac{3000}{1 + 0.25t}$$

where t is the time in days. The initial population (when $t = 0$) is 1000. Write an equation that gives the population at any time t, and find the population when $t = 3$ days.

104. ***Heat Transfer*** Find the time required for an object to cool from 300°F to 250°F by evaluating

$$t = \frac{10}{\ln 2}\int_{250}^{300} \frac{1}{T - 100}\,dT$$

where t is time in minutes and T is the temperature.

105. ***Average Price*** The demand equation for x units of a product is

$$p = \frac{90{,}000}{400 + 3x}.$$

Find the *average* price p on the interval $40 \le x \le 50$.

106. ***Sales*** The rate of change in sales S is inversely proportional to time t $(t > 1)$ measured in weeks. Find S as a function of t if sales after 2 and 4 weeks are 200 and 300 units, respectively.

True or False? **In Exercises 107–110, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

107. $(\ln x)^{1/2} = \frac{1}{2}(\ln x)$

108. $\int \ln x\,dx = (1/x) + C$

109. $\displaystyle\int \frac{1}{x}\,dx = \ln|cx|,\quad c \ne 0$

110. $\displaystyle\int_{-1}^{2} \frac{1}{x}\,dx = \Big[\ln|x|\Big]_{-1}^{2} = \ln 2 - \ln 1 = \ln 2$

111. ***Orthogonal Trajectory***

(a) Use a graphing utility to graph the equation $2x^2 - y^2 = 8$.

(b) Evaluate the integral to find y^2 in terms of x.

$$y^2 = e^{-\int(1/x)\,dx}$$

For a particular value of the constant of integration, graph the result in the same viewing window used in part (a).

(c) Verify that the tangents to the graphs in parts (a) and (b) are perpendicular at the points of intersection.

112. Graph the function

$$f(x) = \frac{x}{1 + x^2}$$

on the interval $[0, \infty)$.

(a) Find the area bounded by the graph of f and the line $y = \frac{1}{2}x$.

(b) Determine the values of the slope m such that the line $y = mx$ and the graph of f enclose a finite region.

(c) Calculate the area of this region as a function of m.

113. ***Napier's Inequality*** For $0 < x < y$, show that

$$\frac{1}{y} < \frac{\ln y - \ln x}{y - x} < \frac{1}{x}.$$

114. Prove that the function

$$F(x) = \int_{x}^{2x} \frac{1}{t}\,dt$$

is constant on the interval $(0, \infty)$.

115. Prove Theorem 5.21.

5.8 Inverse Trigonometric Functions: Integration

- Integrate functions whose antiderivatives involve inverse trigonometric functions.
- Use the method of completing the square to integrate a function.
- Review the basic integration rules involving elementary functions.

Integrals Involving Inverse Trigonometric Functions

The derivatives of the six inverse trigonometric functions fall into three pairs. In each pair, the derivative of one function is the negative of the other. For example,

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

and

$$\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}.$$

When listing the *antiderivative* that corresponds to each of the inverse trigonometric functions, you need to use only one member from each pair. It is conventional to use $\arcsin x$ as the antiderivative of $1/\sqrt{1-x^2}$, rather than $-\arccos x$. The next theorem gives one antiderivative formula for each of the three pairs. The proofs of these integration rules are left to you (see Exercises 87–89).

■ **FOR FURTHER INFORMATION** For a proof of rule 2 of Theorem 5.22, see the article "A Direct Proof of the Integral Formula for Arctangent" by Arnold J. Insel in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

THEOREM 5.22 INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x, and let $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$
2. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$
3. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\,\text{arcsec}\,\frac{|u|}{a} + C$

EXAMPLE 1 Integration with Inverse Trigonometric Functions

a. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = \arcsin\frac{x}{2} + C$

b. $\displaystyle\int \frac{dx}{2+9x^2} = \frac{1}{3}\int \frac{3\,dx}{(\sqrt{2})^2+(3x)^2}$ $\quad u = 3x,\ a = \sqrt{2}$

$$= \frac{1}{3\sqrt{2}}\arctan\frac{3x}{\sqrt{2}} + C$$

c. $\displaystyle\int \frac{dx}{x\sqrt{4x^2-9}} = \int \frac{2\,dx}{2x\sqrt{(2x)^2-3^2}}$ $\quad u = 2x,\ a = 3$

$$= \frac{1}{3}\,\text{arcsec}\,\frac{|2x|}{3} + C$$

■

The integrals in Example 1 are fairly straightforward applications of integration formulas. Unfortunately, this is not typical. The integration formulas for inverse trigonometric functions can be disguised in many ways.

TECHNOLOGY PITFALL

Computer software that can perform symbolic integration is useful for integrating functions such as the one in Example 2. When using such software, however, you must remember that it can fail to find an antiderivative for two reasons. First, some elementary functions simply do not have antiderivatives that are elementary functions. Second, every symbolic integration utility has limitations—you might have entered a function that the software was not programmed to handle. You should also remember that antiderivatives involving trigonometric functions or logarithmic functions can be written in many different forms. For instance, one symbolic integration utility found the integral in Example 2 to be

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \arctan \sqrt{e^{2x}-1} + C.$$

Try showing that this antiderivative is equivalent to that obtained in Example 2.

EXAMPLE 2 Integration by Substitution

Find $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$.

Solution As it stands, this integral doesn't fit any of the three inverse trigonometric formulas. Using the substitution $u = e^x$, however, produces

$$u = e^x \quad \Longrightarrow \quad du = e^x\,dx \quad \Longrightarrow \quad dx = \frac{du}{e^x} = \frac{du}{u}.$$

With this substitution, you can integrate as follows.

$$\begin{aligned}
\int \frac{dx}{\sqrt{e^{2x}-1}} &= \int \frac{dx}{\sqrt{(e^x)^2-1}} && \text{Write } e^{2x} \text{ as } (e^x)^2. \\
&= \int \frac{du/u}{\sqrt{u^2-1}} && \text{Substitute.} \\
&= \int \frac{du}{u\sqrt{u^2-1}} && \text{Rewrite to fit Arcsecant Rule.} \\
&= \operatorname{arcsec} \frac{|u|}{1} + C && \text{Apply Arcsecant Rule.} \\
&= \operatorname{arcsec} e^x + C && \text{Back-substitute.}
\end{aligned}$$

EXAMPLE 3 Rewriting as the Sum of Two Quotients

Find $\displaystyle\int \frac{x+2}{\sqrt{4-x^2}}\,dx$.

Solution This integral does not appear to fit any of the basic integration formulas. By splitting the integrand into two parts, however, you can see that the first part can be found with the Power Rule and the second part yields an inverse sine function.

$$\begin{aligned}
\int \frac{x+2}{\sqrt{4-x^2}}\,dx &= \int \frac{x}{\sqrt{4-x^2}}\,dx + \int \frac{2}{\sqrt{4-x^2}}\,dx \\
&= -\frac{1}{2}\int (4-x^2)^{-1/2}(-2x)\,dx + 2\int \frac{1}{\sqrt{4-x^2}}\,dx \\
&= -\frac{1}{2}\left[\frac{(4-x^2)^{1/2}}{1/2}\right] + 2\arcsin\frac{x}{2} + C \\
&= -\sqrt{4-x^2} + 2\arcsin\frac{x}{2} + C
\end{aligned}$$

Completing the Square

Completing the square helps when quadratic functions are involved in the integrand. For example, the quadratic $x^2 + bx + c$ can be written as the difference of two squares by adding and subtracting $(b/2)^2$.

$$\begin{aligned}
x^2 + bx + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\
&= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c
\end{aligned}$$

EXAMPLE 4 Completing the Square

Find $\displaystyle\int \frac{dx}{x^2 - 4x + 7}$.

Solution You can write the denominator as the sum of two squares, as follows.

$$\begin{aligned} x^2 - 4x + 7 &= (x^2 - 4x + 4) - 4 + 7 \\ &= (x - 2)^2 + 3 = u^2 + a^2 \end{aligned}$$

Now, in this completed square form, let $u = x - 2$ and $a = \sqrt{3}$.

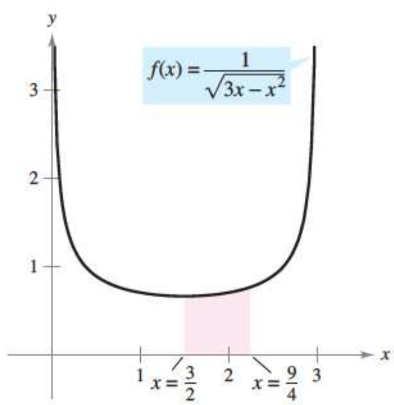
$$\int \frac{dx}{x^2 - 4x + 7} = \int \frac{dx}{(x-2)^2 + 3} = \frac{1}{\sqrt{3}} \arctan \frac{x - 2}{\sqrt{3}} + C$$ ■

If the leading coefficient is not 1, it helps to factor before completing the square. For instance, you can complete the square of $2x^2 - 8x + 10$ by factoring first.

$$\begin{aligned} 2x^2 - 8x + 10 &= 2(x^2 - 4x + 5) \\ &= 2(x^2 - 4x + 4 - 4 + 5) \\ &= 2[(x - 2)^2 + 1] \end{aligned}$$

To complete the square when the coefficient of x^2 is negative, use the same factoring process shown above. For instance, you can complete the square for $3x - x^2$ as shown.

$$\begin{aligned} 3x - x^2 &= -(x^2 - 3x) \\ &= -\left[x^2 - 3x + \left(\tfrac{3}{2}\right)^2 - \left(\tfrac{3}{2}\right)^2\right] \\ &= \left(\tfrac{3}{2}\right)^2 - \left(x - \tfrac{3}{2}\right)^2 \end{aligned}$$



The area of the region bounded by the graph of f, the x-axis, $x = \frac{3}{2}$, and $x = \frac{9}{4}$ is $\pi/6$.
Figure 5.49

EXAMPLE 5 Completing the Square (Negative Leading Coefficient)

Find the area of the region bounded by the graph of

$$f(x) = \frac{1}{\sqrt{3x - x^2}},$$

the x-axis, and the lines $x = \frac{3}{2}$ and $x = \frac{9}{4}$.

Solution In Figure 5.49, you can see that the area is given by

$$\text{Area} = \int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}}\, dx.$$

Using the completed square form derived above, you can integrate as shown.

$$\begin{aligned} \int_{3/2}^{9/4} \frac{dx}{\sqrt{3x - x^2}} &= \int_{3/2}^{9/4} \frac{dx}{\sqrt{(3/2)^2 - [x - (3/2)]^2}} \\ &= \arcsin \frac{x - (3/2)}{3/2} \bigg]_{3/2}^{9/4} \\ &= \arcsin \frac{1}{2} - \arcsin 0 \\ &= \frac{\pi}{6} \\ &\approx 0.524 \end{aligned}$$ ■

TECHNOLOGY With definite integrals such as the one given in Example 5, remember that you can resort to a numerical solution. For instance, applying Simpson's Rule (with $n = 12$) to the integral in the example, you obtain

$$\int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}}\, dx \approx 0.523599.$$

This differs from the exact value of the integral ($\pi/6 \approx 0.5235988$) by less than one millionth.

Review of Basic Integration Rules

You have now completed the introduction of the **basic integration rules.** To be efficient at applying these rules, you should have practiced enough so that each rule is committed to memory.

BASIC INTEGRATION RULES ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$
2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$
3. $\int du = u + C$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$
5. $\int \dfrac{du}{u} = \ln|u| + C$
6. $\int e^u\,du = e^u + C$
7. $\int a^u\,du = \left(\dfrac{1}{\ln a}\right)a^u + C$
8. $\int \sin u\,du = -\cos u + C$
9. $\int \cos u\,du = \sin u + C$
10. $\int \tan u\,du = -\ln|\cos u| + C$
11. $\int \cot u\,du = \ln|\sin u| + C$
12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$
14. $\int \sec^2 u\,du = \tan u + C$
15. $\int \csc^2 u\,du = -\cot u + C$
16. $\int \sec u \tan u\,du = \sec u + C$
17. $\int \csc u \cot u\,du = -\csc u + C$
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin\dfrac{u}{a} + C$
19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\arctan\dfrac{u}{a} + C$
20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\operatorname{arcsec}\dfrac{|u|}{a} + C$

You can learn a lot about the nature of integration by comparing this list with the summary of differentiation rules given in Section 3.6. For differentiation, you now have rules that allow you to differentiate *any* elementary function. For integration, this is far from true.

The integration rules listed above are primarily those that were happened on during the development of differentiation rules. So far, you have not learned any rules or techniques for finding the antiderivative of a general product or quotient, the natural logarithmic function, or the inverse trigonometric functions. More importantly, you cannot apply any of the rules in this list unless you can create the proper du corresponding to the u in the formula. The point is that you need to work more on integration techniques, which you will do in Chapter 8. The next two examples should give you a better feeling for the integration problems that you *can* and *cannot* do with the techniques and rules you now know.

EXAMPLE 6 Comparing Integration Problems

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - 1}}$ **b.** $\displaystyle\int \frac{x\,dx}{\sqrt{x^2 - 1}}$ **c.** $\displaystyle\int \frac{dx}{\sqrt{x^2 - 1}}$

Solution

a. You *can* find this integral (it fits the Arcsecant Rule).

$$\int \frac{dx}{x\sqrt{x^2 - 1}} = \operatorname{arcsec}|x| + C$$

b. You *can* find this integral (it fits the Power Rule).

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{x^2 - 1}} &= \frac{1}{2}\int (x^2 - 1)^{-1/2}(2x)\,dx \\ &= \frac{1}{2}\left[\frac{(x^2 - 1)^{1/2}}{1/2}\right] + C \\ &= \sqrt{x^2 - 1} + C\end{aligned}$$

c. You *cannot* find this integral using the techniques you have studied so far. (You should scan the list of basic integration rules to verify this conclusion.)

EXAMPLE 7 Comparing Integration Problems

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\displaystyle\int \frac{dx}{x \ln x}$ **b.** $\displaystyle\int \frac{\ln x\,dx}{x}$ **c.** $\displaystyle\int \ln x\,dx$

Solution

a. You *can* find this integral (it fits the Log Rule).

$$\begin{aligned}\int \frac{dx}{x \ln x} &= \int \frac{1/x}{\ln x}\,dx \\ &= \ln|\ln x| + C\end{aligned}$$

b. You *can* find this integral (it fits the Power Rule).

$$\begin{aligned}\int \frac{\ln x\,dx}{x} &= \int \left(\frac{1}{x}\right)(\ln x)^1\,dx \\ &= \frac{(\ln x)^2}{2} + C\end{aligned}$$

c. You *cannot* find this integral using the techniques you have studied so far. ■

NOTE Note in Examples 6 and 7 that the *simplest* functions are the ones that you cannot yet integrate. ■

5.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–24, find the integral.

1. $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$

2. $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}}$

3. $\displaystyle\int \frac{11}{36+x^2}\,dx$

4. $\displaystyle\int \frac{12}{1+9x^2}\,dx$

5. $\displaystyle\int \frac{1}{x\sqrt{4x^2-1}}\,dx$

6. $\displaystyle\int \frac{1}{4+(x-3)^2}\,dx$

7. $\displaystyle\int \frac{1}{\sqrt{1-(x+1)^2}}\,dx$

8. $\displaystyle\int \frac{t}{t^4+16}\,dt$

9. $\displaystyle\int \frac{t}{\sqrt{1-t^4}}\,dt$

10. $\displaystyle\int \frac{1}{x\sqrt{x^4-4}}\,dx$

11. $\displaystyle\int \frac{t}{t^4+25}\,dt$

12. $\displaystyle\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx$

13. $\displaystyle\int \frac{e^{2x}}{4+e^{4x}}\,dx$

14. $\displaystyle\int \frac{1}{3+(x-2)^2}\,dx$

15. $\displaystyle\int \frac{\sec^2 x}{\sqrt{25-\tan^2 x}}\,dx$

16. $\displaystyle\int \frac{\sin x}{7+\cos^2 x}\,dx$

17. $\displaystyle\int \frac{x^3}{x^2+1}\,dx$

18. $\displaystyle\int \frac{x^4-1}{x^2+1}\,dx$

19. $\displaystyle\int \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx$

20. $\displaystyle\int \frac{3}{2\sqrt{x}(1+x)}\,dx$

21. $\displaystyle\int \frac{x-3}{x^2+1}\,dx$

22. $\displaystyle\int \frac{4x+3}{\sqrt{1-x^2}}\,dx$

23. $\displaystyle\int \frac{x+5}{\sqrt{9-(x-3)^2}}\,dx$

24. $\displaystyle\int \frac{x-2}{(x+1)^2+4}\,dx$

In Exercises 25–38, evaluate the integral.

25. $\displaystyle\int_0^{1/6} \frac{3}{\sqrt{1-9x^2}}\,dx$

26. $\displaystyle\int_0^{1} \frac{dx}{\sqrt{4-x^2}}$

27. $\displaystyle\int_0^{\sqrt{3}/2} \frac{1}{1+4x^2}\,dx$

28. $\displaystyle\int_{\sqrt{3}}^{3} \frac{6}{9+x^2}\,dx$

29. $\displaystyle\int_{-1/2}^{0} \frac{x}{\sqrt{1-x^2}}\,dx$

30. $\displaystyle\int_{-\sqrt{3}}^{0} \frac{x}{1+x^2}\,dx$

31. $\displaystyle\int_3^{6} \frac{1}{25+(x-3)^2}\,dx$

32. $\displaystyle\int_1^{4} \frac{1}{x\sqrt{16x^2-5}}\,dx$

33. $\displaystyle\int_0^{\ln 5} \frac{e^x}{1+e^{2x}}\,dx$

34. $\displaystyle\int_{\ln 2}^{\ln 4} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx$

35. $\displaystyle\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$

36. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx$

37. $\displaystyle\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

38. $\displaystyle\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}}\,dx$

In Exercises 39–50, find or evaluate the integral. (Complete the square, if necessary.)

39. $\displaystyle\int_0^{2} \frac{dx}{x^2-2x+2}$

40. $\displaystyle\int_{-2}^{2} \frac{dx}{x^2+4x+13}$

41. $\displaystyle\int \frac{2x}{x^2+6x+13}\,dx$

42. $\displaystyle\int \frac{2x-5}{x^2+2x+2}\,dx$

43. $\displaystyle\int \frac{1}{\sqrt{-x^2-4x}}\,dx$

44. $\displaystyle\int \frac{2}{\sqrt{-x^2+4x}}\,dx$

45. $\displaystyle\int \frac{x+2}{\sqrt{-x^2-4x}}\,dx$

46. $\displaystyle\int \frac{x-1}{\sqrt{x^2-2x}}\,dx$

47. $\displaystyle\int_2^{3} \frac{2x-3}{\sqrt{4x-x^2}}\,dx$

48. $\displaystyle\int \frac{1}{(x-1)\sqrt{x^2-2x}}\,dx$

49. $\displaystyle\int \frac{x}{x^4+2x^2+2}\,dx$

50. $\displaystyle\int \frac{x}{\sqrt{9+8x^2-x^4}}\,dx$

In Exercises 51–54, use the specified substitution to find or evaluate the integral.

51. $\displaystyle\int \sqrt{e^t-3}\,dt$
$u=\sqrt{e^t-3}$

52. $\displaystyle\int \frac{\sqrt{x-2}}{x+1}\,dx$
$u=\sqrt{x-2}$

53. $\displaystyle\int_1^{3} \frac{dx}{\sqrt{x}(1+x)}$
$u=\sqrt{x}$

54. $\displaystyle\int_0^{1} \frac{dx}{2\sqrt{3-x}\sqrt{x+1}}$
$u=\sqrt{x+1}$

WRITING ABOUT CONCEPTS

In Exercises 55–57, determine which of the integrals can be found using the basic integration formulas you have studied so far in the text.

55. (a) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$ (b) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$ (c) $\displaystyle\int \frac{1}{x\sqrt{1-x^2}}\,dx$

56. (a) $\displaystyle\int e^{x^2}\,dx$ (b) $\displaystyle\int xe^{x^2}\,dx$ (c) $\displaystyle\int \frac{1}{x^2}e^{1/x}\,dx$

57. (a) $\displaystyle\int \sqrt{x-1}\,dx$ (b) $\displaystyle\int x\sqrt{x-1}\,dx$ (c) $\displaystyle\int \frac{x}{\sqrt{x-1}}\,dx$

58. Determine which value best approximates the area of the region between the x-axis and the function

$$f(x)=\frac{1}{\sqrt{1-x^2}}$$

over the interval $[-0.5, 0.5]$. (Make your selection on the basis of a sketch of the region and *not* by performing any calculations.)

(a) 4 (b) -3 (c) 1 (d) 2 (e) 3

59. Decide whether you can find the integral

$$\int \frac{2\,dx}{\sqrt{x^2+4}}$$

using the formulas and techniques you have studied so far. Explain your reasoning.

CAPSTONE

60. Determine which of the integrals can be found using the basic integration formulas you have studied so far in the text.

(a) $\displaystyle\int \frac{1}{1+x^4}\,dx$ (b) $\displaystyle\int \frac{x}{1+x^4}\,dx$ (c) $\displaystyle\int \frac{x^3}{1+x^4}\,dx$

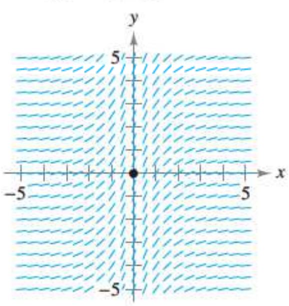
Differential Equations **In Exercises 61 and 62, use the differential equation and the specified initial condition to find *y*.**

61. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{4-x^2}}$, $y(0) = \pi$

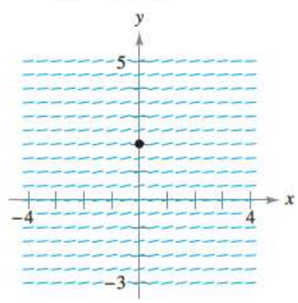
62. $\dfrac{dy}{dx} = \dfrac{1}{4+x^2}$, $y(2) = \pi$

Slope Fields **In Exercises 63–66, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

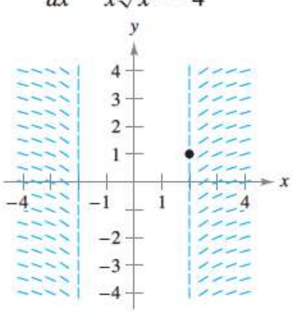
63. $\dfrac{dy}{dx} = \dfrac{3}{1+x^2}$, $(0, 0)$



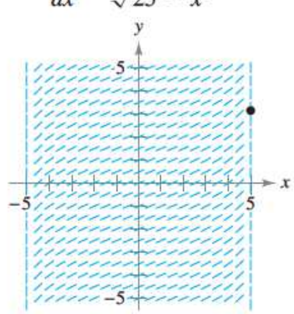
64. $\dfrac{dy}{dx} = \dfrac{2}{9+x^2}$, $(0, 2)$



65. $\dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-4}}$, $(2, 1)$



66. $\dfrac{dy}{dx} = \dfrac{2}{\sqrt{25-x^2}}$, $(5, \pi)$



CAS *Slope Fields* **In Exercises 67–70, use a computer algebra system to graph the slope field for the differential equation and graph the solution satisfying the specified initial condition.**

67. $\dfrac{dy}{dx} = \dfrac{10}{x\sqrt{x^2-1}}$, $y(3) = 0$

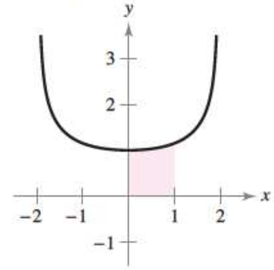
68. $\dfrac{dy}{dx} = \dfrac{1}{12+x^2}$, $y(4) = 2$

69. $\dfrac{dy}{dx} = \dfrac{2y}{\sqrt{16-x^2}}$, $y(0) = 2$

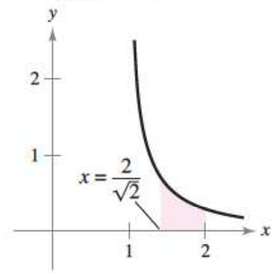
70. $\dfrac{dy}{dx} = \dfrac{\sqrt{y}}{1+x^2}$, $y(0) = 4$

Area **In Exercises 71–76, find the area of the region.**

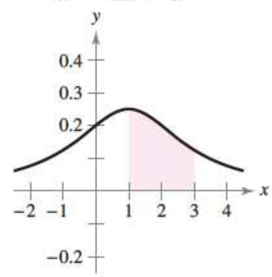
71. $y = \dfrac{2}{\sqrt{4-x^2}}$



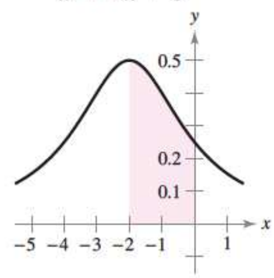
72. $y = \dfrac{1}{x\sqrt{x^2-1}}$



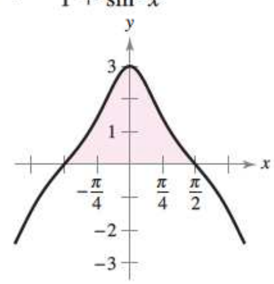
73. $y = \dfrac{1}{x^2-2x+5}$



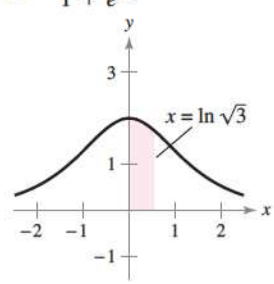
74. $y = \dfrac{2}{x^2+4x+8}$



75. $y = \dfrac{3\cos x}{1+\sin^2 x}$



76. $y = \dfrac{4e^x}{1+e^{2x}}$



In Exercises 77 and 78, (a) verify the integration formula, then (b) use it to find the area of the region.

77. $\displaystyle\int \frac{\arctan x}{x^2}\,dx = \ln x - \frac{1}{2}\ln(1+x^2) - \frac{\arctan x}{x} + C$

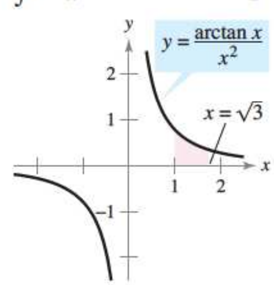


Figure for 77

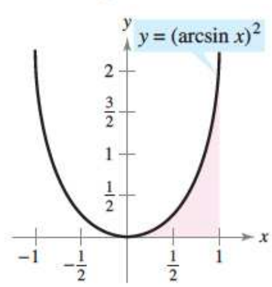


Figure for 78

78. $\displaystyle\int (\arcsin x)^2\,dx$

$= x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2}\arcsin x + C$

79. (a) Sketch the region whose area is represented by

$$\int_0^1 \arcsin x\, dx.$$

(b) Use the integration capabilities of a graphing utility to approximate the area.

(c) Find the exact area analytically.

80. (a) Show that $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx = \pi$.

(b) Approximate the number π using Simpson's Rule (with $n = 6$) and the integral in part (a).

(c) Approximate the number π by using the integration capabilities of a graphing utility.

81. *Investigation* Consider the function $F(x) = \dfrac{1}{2}\displaystyle\int_x^{x+2} \frac{2}{t^2+1}\,dt$.

(a) Write a short paragraph giving a geometric interpretation of the function $F(x)$ relative to the function $f(x) = \dfrac{2}{x^2+1}$. Use what you have written to guess the value of x that will make F maximum.

(b) Perform the specified integration to find an alternative form of $F(x)$. Use calculus to locate the value of x that will make F maximum and compare the result with your guess in part (a).

82. Consider the integral $\displaystyle\int \frac{1}{\sqrt{6x - x^2}}\,dx$.

(a) Find the integral by completing the square of the radicand.

(b) Find the integral by making the substitution $u = \sqrt{x}$.

(c) The antiderivatives in parts (a) and (b) appear to be significantly different. Use a graphing utility to graph each antiderivative in the same viewing window and determine the relationship between them. Find the domain of each.

True or False? **In Exercises 83–86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

83. $\displaystyle\int \frac{dx}{3x\sqrt{9x^2-16}} = \frac{1}{4}\operatorname{arcsec}\frac{3x}{4} + C$

84. $\displaystyle\int \frac{dx}{25+x^2} = \frac{1}{25}\arctan\frac{x}{25} + C$

85. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = -\arccos\frac{x}{2} + C$

86. One way to find $\displaystyle\int \frac{2e^{2x}}{\sqrt{9-e^{2x}}}\,dx$ is to use the Arcsine Rule.

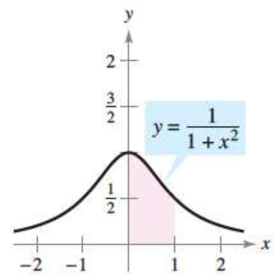
Verifying Integration Rules **In Exercises 87–89, verify each rule by differentiating. Let $a > 0$.**

87. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$

88. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

89. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

90. *Numerical Integration* (a) Write an integral that represents the area of the region in the figure. (b) Then use the Trapezoidal Rule with $n = 8$ to estimate the area of the region. (c) Explain how you can use the results of parts (a) and (b) to estimate π.



91. *Vertical Motion* An object is projected upward from ground level with an initial velocity of 500 feet per second. In this exercise, the goal is to analyze the motion of the object during its upward flight.

(a) If air resistance is neglected, find the velocity of the object as a function of time. Use a graphing utility to graph this function.

(b) Use the result of part (a) to find the position function and determine the maximum height attained by the object.

(c) If the air resistance is proportional to the square of the velocity, you obtain the equation

$$\frac{dv}{dt} = -(32 + kv^2)$$

where -32 feet per second per second is the acceleration due to gravity and k is a constant. Find the velocity as a function of time by solving the equation

$$\int \frac{dv}{32+kv^2} = -\int dt.$$

(d) Use a graphing utility to graph the velocity function $v(t)$ in part (c) for $k = 0.001$. Use the graph to approximate the time t_0 at which the object reaches its maximum height.

(e) Use the integration capabilities of a graphing utility to approximate the integral

$$\int_0^{t_0} v(t)\,dt$$

where $v(t)$ and t_0 are those found in part (d). This is the approximation of the maximum height of the object.

(f) Explain the difference between the results in parts (b) and (e).

■ **FOR FURTHER INFORMATION** For more information on this topic, see "What Goes Up Must Come Down; Will Air Resistance Make It Return Sooner, or Later?" by John Lekner in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*.

92. Graph $y_1 = \dfrac{x}{1+x^2}$, $y_2 = \arctan x$, and $y_3 = x$ on $[0, 10]$.

Prove that $\dfrac{x}{1+x^2} < \arctan x < x$ for $x > 0$.

5.9 Hyperbolic Functions

- **Develop properties of hyperbolic functions.**
- **Differentiate and integrate hyperbolic functions.**
- **Develop properties of inverse hyperbolic functions.**
- **Differentiate and integrate functions involving inverse hyperbolic functions.**

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JOHANN HEINRICH LAMBERT (1728–1777)

The first person to publish a comprehensive study on hyperbolic functions was Johann Heinrich Lambert, a Swiss-German mathematician and colleague of Euler.

Hyperbolic Functions

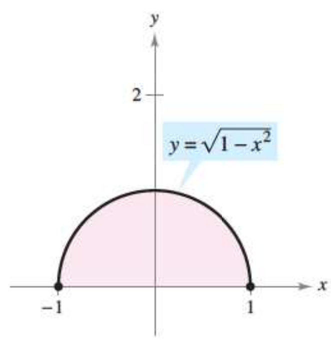
In this section you will look briefly at a special class of exponential functions called **hyperbolic functions.** The name *hyperbolic function* arose from comparison of the area of a semicircular region, as shown in Figure 5.50, with the area of a region under a hyperbola, as shown in Figure 5.51. The integral for the semicircular region involves an inverse trigonometric (circular) function:

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{1}{2}\left[x\sqrt{1-x^2} + \arcsin x\right]_{-1}^{1} = \frac{\pi}{2} \approx 1.571.$$

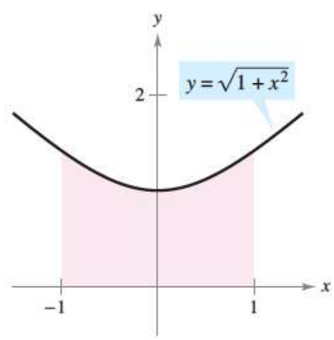
The integral for the hyperbolic region involves an inverse hyperbolic function:

$$\int_{-1}^{1} \sqrt{1+x^2}\,dx = \frac{1}{2}\left[x\sqrt{1+x^2} + \sinh^{-1} x\right]_{-1}^{1} \approx 2.296.$$

This is only one of many ways in which the hyperbolic functions are similar to the trigonometric functions.



Circle: $x^2 + y^2 = 1$
Figure 5.50



Hyperbola: $-x^2 + y^2 = 1$
Figure 5.51

FOR FURTHER INFORMATION For more information on the development of hyperbolic functions, see the article "An Introduction to Hyperbolic Functions in Elementary Calculus" by Jerome Rosenthal in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

DEFINITIONS OF THE HYPERBOLIC FUNCTIONS

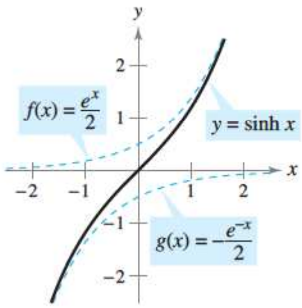
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}, \quad x \neq 0$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

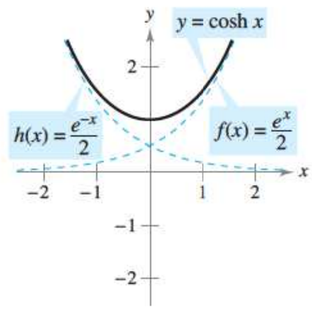
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}, \quad x \neq 0$$

NOTE $\sinh x$ is read as "the hyperbolic sine of x," $\cosh x$ as "the hyperbolic cosine of x," and so on.

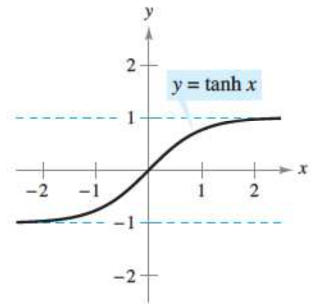
The graphs of the six hyperbolic functions and their domains and ranges are shown in Figure 5.52. Note that the graph of $\sinh x$ can be obtained by adding the corresponding y-coordinates of the exponential functions $f(x) = \frac{1}{2}e^x$ and $g(x) = -\frac{1}{2}e^{-x}$. Likewise, the graph of $\cosh x$ can be obtained by adding the corresponding y-coordinates of the exponential functions $f(x) = \frac{1}{2}e^x$ and $h(x) = \frac{1}{2}e^{-x}$.



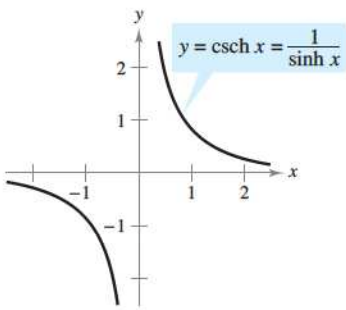
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



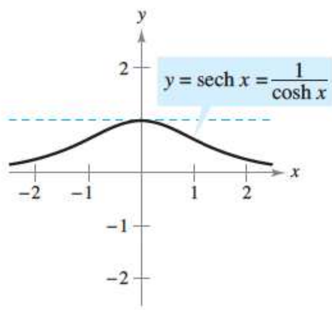
Domain: $(-\infty, \infty)$
Range: $[1, \infty)$



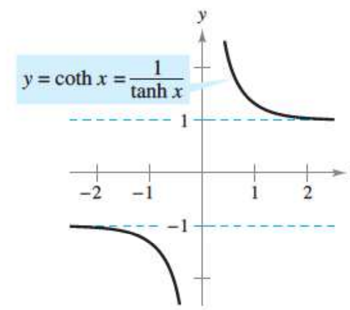
Domain: $(-\infty, \infty)$
Range: $(-1, 1)$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$



Domain: $(-\infty, \infty)$
Range: $(0, 1]$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, -1) \cup (1, \infty)$

Figure 5.52

Many of the trigonometric identities have corresponding *hyperbolic identities*. For instance,

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\
&= \frac{4}{4} \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
2 \sinh x \cosh x &= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\
&= \frac{e^{2x} - e^{-2x}}{2} \\
&= \sinh 2x.
\end{aligned}$$

■ **FOR FURTHER INFORMATION** To understand geometrically the relationship between the hyperbolic and exponential functions, see the article "A Short Proof Linking the Hyperbolic and Exponential Functions" by Michael J. Seery in *The AMATYC Review*.

HYPERBOLIC IDENTITIES

$\cosh^2 x - \sinh^2 x = 1$

$\tanh^2 x + \operatorname{sech}^2 x = 1$

$\coth^2 x - \operatorname{csch}^2 x = 1$

$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$

$\sinh^2 x = \dfrac{-1 + \cosh 2x}{2}$

$\cosh^2 x = \dfrac{1 + \cosh 2x}{2}$

$\sinh 2x = 2 \sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

Differentiation and Integration of Hyperbolic Functions

Because the hyperbolic functions are written in terms of e^x and e^{-x}, you can easily derive rules for their derivatives. The following theorem lists these derivatives with the corresponding integration rules.

THEOREM 5.23 DERIVATIVES AND INTEGRALS OF HYPERBOLIC FUNCTIONS

Let u be a differentiable function of x.

$$\frac{d}{dx}[\sinh u] = (\cosh u)u' \qquad \int \cosh u\, du = \sinh u + C$$

$$\frac{d}{dx}[\cosh u] = (\sinh u)u' \qquad \int \sinh u\, du = \cosh u + C$$

$$\frac{d}{dx}[\tanh u] = (\operatorname{sech}^2 u)u' \qquad \int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\frac{d}{dx}[\coth u] = -(\operatorname{csch}^2 u)u' \qquad \int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\frac{d}{dx}[\operatorname{sech} u] = -(\operatorname{sech} u \tanh u)u' \qquad \int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\frac{d}{dx}[\operatorname{csch} u] = -(\operatorname{csch} u \coth u)u' \qquad \int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

PROOF

$$\begin{aligned}
\frac{d}{dx}[\sinh x] &= \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] \\
&= \frac{e^x + e^{-x}}{2} = \cosh x \\
\frac{d}{dx}[\tanh x] &= \frac{d}{dx}\left[\frac{\sinh x}{\cosh x}\right] \\
&= \frac{\cosh x(\cosh x) - \sinh x(\sinh x)}{\cosh^2 x} \\
&= \frac{1}{\cosh^2 x} \\
&= \operatorname{sech}^2 x
\end{aligned}$$

In Exercises 122–124, you are asked to prove some of the other differentiation rules.

EXAMPLE 1 Differentiation of Hyperbolic Functions

a. $\dfrac{d}{dx}[\sinh(x^2 - 3)] = 2x\cosh(x^2 - 3)$ **b.** $\dfrac{d}{dx}[\ln(\cosh x)] = \dfrac{\sinh x}{\cosh x} = \tanh x$

c. $\dfrac{d}{dx}[x\sinh x - \cosh x] = x\cosh x + \sinh x - \sinh x = x\cosh x$

EXAMPLE 2 Finding Relative Extrema

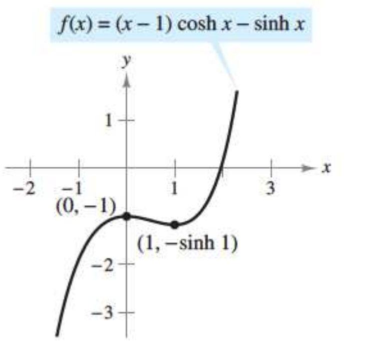
Find the relative extrema of $f(x) = (x - 1)\cosh x - \sinh x$.

Solution Begin by setting the first derivative of f equal to 0.

$$f'(x) = (x - 1)\sinh x + \cosh x - \cosh x = 0$$
$$(x - 1)\sinh x = 0$$

So, the critical numbers are $x = 1$ and $x = 0$. Using the Second Derivative Test, you can verify that the point $(0, -1)$ yields a relative maximum and the point $(1, -\sinh 1)$ yields a relative minimum, as shown in Figure 5.53. Try using a graphing utility to confirm this result. If your graphing utility does not have hyperbolic functions, you can use exponential functions, as follows.

$$\begin{aligned} f(x) &= (x - 1)\left(\tfrac{1}{2}\right)(e^x + e^{-x}) - \tfrac{1}{2}(e^x - e^{-x}) \\ &= \tfrac{1}{2}(xe^x + xe^{-x} - e^x - e^{-x} - e^x + e^{-x}) \\ &= \tfrac{1}{2}(xe^x + xe^{-x} - 2e^x) \end{aligned}$$



$f''(0) < 0$, so $(0, -1)$ is a relative maximum. $f''(1) > 0$, so $(1, -\sinh 1)$ is a relative minimum.
Figure 5.53

When a uniform flexible cable, such as a telephone wire, is suspended from two points, it takes the shape of a *catenary*, as discussed in Example 3.

EXAMPLE 3 Hanging Power Cables

Power cables are suspended between two towers, forming the catenary shown in Figure 5.54. The equation for this catenary is

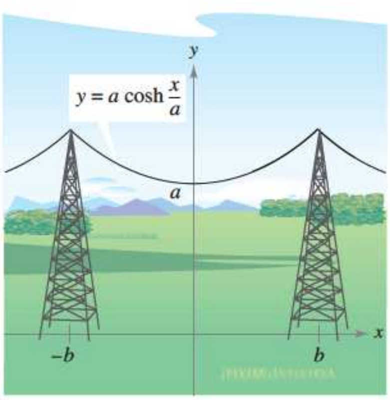
$$y = a\cosh\frac{x}{a}.$$

The distance between the two towers is $2b$. Find the slope of the catenary at the point where the cable meets the right-hand tower.

Solution Differentiating produces

$$y' = a\left(\frac{1}{a}\right)\sinh\frac{x}{a} = \sinh\frac{x}{a}.$$

At the point $(b, a\cosh(b/a))$, the slope (from the left) is given by $m = \sinh\dfrac{b}{a}$.



Catenary
Figure 5.54

FOR FURTHER INFORMATION In Example 3, the cable is a catenary between two supports at the same height. To learn about the shape of a cable hanging between supports of different heights, see the article "Reexamining the Catenary" by Paul Cella in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

EXAMPLE 4 Integrating a Hyperbolic Function

Find $\int \cosh 2x \sinh^2 2x \, dx$.

Solution

$$\begin{aligned}\int \cosh 2x \sinh^2 2x \, dx &= \frac{1}{2}\int (\sinh 2x)^2(2\cosh 2x)\, dx \qquad u = \sinh 2x\\ &= \frac{1}{2}\left[\frac{(\sinh 2x)^3}{3}\right] + C\\ &= \frac{\sinh^3 2x}{6} + C\end{aligned}$$

Inverse Hyperbolic Functions

Unlike trigonometric functions, hyperbolic functions are not periodic. In fact, by looking back at Figure 5.52, you can see that four of the six hyperbolic functions are actually one-to-one (the hyperbolic sine, tangent, cosecant, and cotangent). So, you can conclude that these four functions have inverse functions. The other two (the hyperbolic cosine and secant) are one-to-one if their domains are restricted to the positive real numbers, and for this restricted domain they also have inverse functions. Because the hyperbolic functions are defined in terms of exponential functions, it is not surprising to find that the inverse hyperbolic functions can be written in terms of logarithmic functions, as shown in Theorem 5.24.

THEOREM 5.24 INVERSE HYPERBOLIC FUNCTIONS

Function	*Domain*
$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$	$(-\infty, \infty)$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$	$[1, \infty)$
$\tanh^{-1} x = \dfrac{1}{2}\ln\dfrac{1+x}{1-x}$	$(-1, 1)$
$\coth^{-1} x = \dfrac{1}{2}\ln\dfrac{x+1}{x-1}$	$(-\infty, -1) \cup (1, \infty)$
$\operatorname{sech}^{-1} x = \ln\dfrac{1+\sqrt{1-x^2}}{x}$	$(0, 1]$
$\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$	$(-\infty, 0) \cup (0, \infty)$

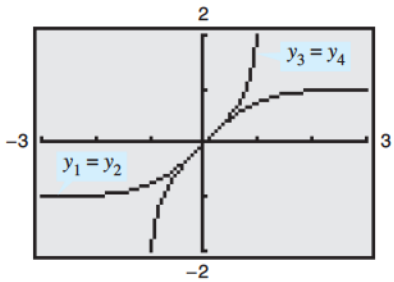
PROOF The proof of this theorem is a straightforward application of the properties of the exponential and logarithmic functions. For example, if

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$$

and

$$g(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

you can show that $f(g(x)) = x$ and $g(f(x)) = x$, which implies that g is the inverse function of f.



Graphs of the hyperbolic tangent function and the inverse hyperbolic tangent function
Figure 5.55

TECHNOLOGY You can use a graphing utility to confirm graphically the results of Theorem 5.24. For instance, graph the following functions.

$y_1 = \tanh x$ — Hyperbolic tangent

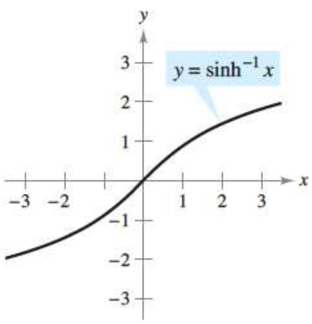
$y_2 = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ — Definition of hyperbolic tangent

$y_3 = \tanh^{-1} x$ — Inverse hyperbolic tangent

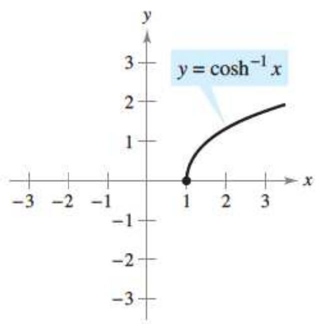
$y_4 = \dfrac{1}{2} \ln \dfrac{1 + x}{1 - x}$ — Definition of inverse hyperbolic tangent

The resulting display is shown in Figure 5.55. As you watch the graphs being traced out, notice that $y_1 = y_2$ and $y_3 = y_4$. Also notice that the graph of y_1 is the reflection of the graph of y_3 in the line $y = x$.

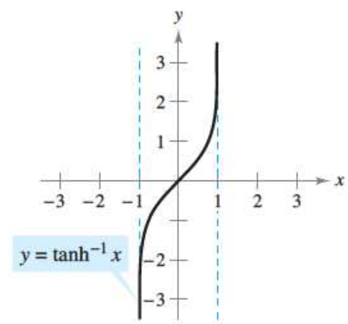
The graphs of the inverse hyperbolic functions are shown in Figure 5.56.



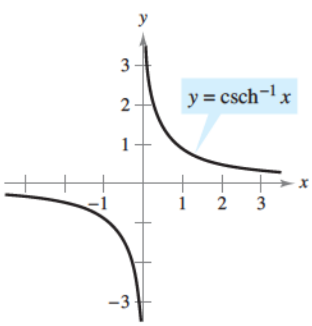
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



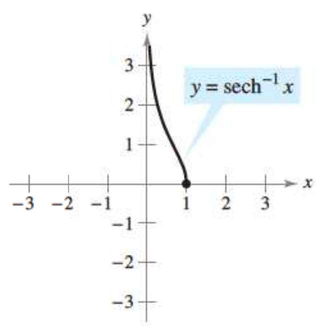
Domain: $[1, \infty)$
Range: $[0, \infty)$



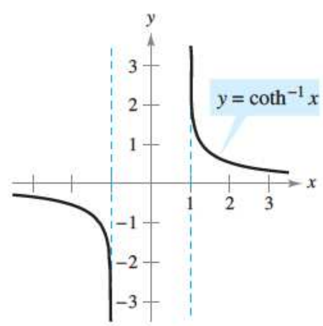
Domain: $(-1, 1)$
Range: $(-\infty, \infty)$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$



Domain: $(0, 1]$
Range: $[0, \infty)$



Domain: $(-\infty, -1) \cup (1, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Figure 5.56

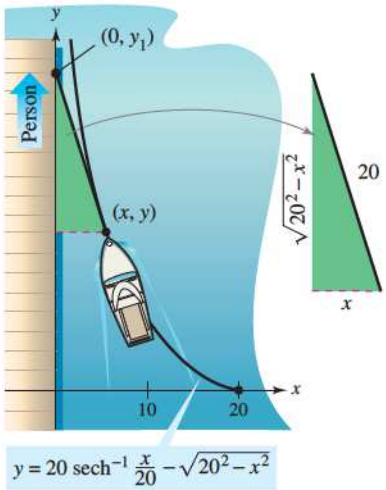
The inverse hyperbolic secant can be used to define a curve called a *tractrix* or *pursuit curve*, as discussed in Example 5.

EXAMPLE 5 A Tractrix

A person is holding a rope that is tied to a boat, as shown in Figure 5.57. As the person walks along the dock, the boat travels along a **tractrix,** given by the equation

$$y = a \operatorname{sech}^{-1}\frac{x}{a} - \sqrt{a^2 - x^2}$$

where a is the length of the rope. If $a = 20$ feet, find the distance the person must walk to bring the boat to a position 5 feet from the dock.



A person must walk 41.27 feet to bring the boat to a position 5 feet from the dock.
Figure 5.57

Solution In Figure 5.57, notice that the distance the person has walked is given by

$$\begin{aligned} y_1 = y + \sqrt{20^2 - x^2} &= \left(20 \operatorname{sech}^{-1}\frac{x}{20} - \sqrt{20^2 - x^2}\right) + \sqrt{20^2 - x^2} \\ &= 20 \operatorname{sech}^{-1}\frac{x}{20}. \end{aligned}$$

When $x = 5$, this distance is

$$\begin{aligned} y_1 = 20 \operatorname{sech}^{-1}\frac{5}{20} &= 20 \ln \frac{1 + \sqrt{1 - (1/4)^2}}{1/4} \\ &= 20 \ln\left(4 + \sqrt{15}\right) \\ &\approx 41.27 \text{ feet.} \end{aligned}$$

Differentiation and Integration of Inverse Hyperbolic Functions

The derivatives of the inverse hyperbolic functions, which resemble the derivatives of the inverse trigonometric functions, are listed in Theorem 5.25 with the corresponding integration formulas (in logarithmic form). You can verify each of these formulas by applying the logarithmic definitions of the inverse hyperbolic functions. (See Exercises 119–121.)

THEOREM 5.25 DIFFERENTIATION AND INTEGRATION INVOLVING INVERSE HYPERBOLIC FUNCTIONS

Let u be a differentiable function of x.

$$\frac{d}{dx}[\sinh^{-1} u] = \frac{u'}{\sqrt{u^2 + 1}} \qquad \frac{d}{dx}[\cosh^{-1} u] = \frac{u'}{\sqrt{u^2 - 1}}$$

$$\frac{d}{dx}[\tanh^{-1} u] = \frac{u'}{1 - u^2} \qquad \frac{d}{dx}[\coth^{-1} u] = \frac{u'}{1 - u^2}$$

$$\frac{d}{dx}[\operatorname{sech}^{-1} u] = \frac{-u'}{u\sqrt{1 - u^2}} \qquad \frac{d}{dx}[\operatorname{csch}^{-1} u] = \frac{-u'}{|u|\sqrt{1 + u^2}}$$

$$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left(u + \sqrt{u^2 \pm a^2}\right) + C$$

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{a + u}{a - u}\right| + C$$

$$\int \frac{du}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a}\ln\frac{a + \sqrt{a^2 \pm u^2}}{|u|} + C$$

EXAMPLE 6 More About a Tractrix

For the tractrix given in Example 5, show that the boat is always pointing toward the person.

Solution For a point (x, y) on a tractrix, the slope of the graph gives the direction of the boat, as shown in Figure 5.57.

$$\begin{aligned} y' &= \frac{d}{dx}\left[20 \operatorname{sech}^{-1}\frac{x}{20} - \sqrt{20^2 - x^2}\right] \\ &= -20\left(\frac{1}{20}\right)\left[\frac{1}{(x/20)\sqrt{1-(x/20)^2}}\right] - \left(\frac{1}{2}\right)\left(\frac{-2x}{\sqrt{20^2 - x^2}}\right) \\ &= \frac{-20^2}{x\sqrt{20^2 - x^2}} + \frac{x}{\sqrt{20^2 - x^2}} \\ &= -\frac{\sqrt{20^2 - x^2}}{x} \end{aligned}$$

However, from Figure 5.57, you can see that the slope of the line segment connecting the point $(0, y_1)$ with the point (x, y) is also

$$m = -\frac{\sqrt{20^2 - x^2}}{x}.$$

So, the boat is always pointing toward the person. (It is because of this property that a tractrix is called a *pursuit curve*.)

EXAMPLE 7 Integration Using Inverse Hyperbolic Functions

Find $\displaystyle\int \frac{dx}{x\sqrt{4 - 9x^2}}$.

Solution Let $a = 2$ and $u = 3x$.

$$\begin{aligned} \int \frac{dx}{x\sqrt{4 - 9x^2}} &= \int \frac{3\,dx}{(3x)\sqrt{4 - 9x^2}} & \int \frac{du}{u\sqrt{a^2 - u^2}} \\ &= -\frac{1}{2}\ln\frac{2 + \sqrt{4 - 9x^2}}{|3x|} + C & -\frac{1}{a}\ln\frac{a + \sqrt{a^2 - u^2}}{|u|} + C \end{aligned}$$

EXAMPLE 8 Integration Using Inverse Hyperbolic Functions

Find $\displaystyle\int \frac{dx}{5 - 4x^2}$.

Solution Let $a = \sqrt{5}$ and $u = 2x$.

$$\begin{aligned} \int \frac{dx}{5 - 4x^2} &= \frac{1}{2}\int \frac{2\,dx}{(\sqrt{5})^2 - (2x)^2} & \int \frac{du}{a^2 - u^2} \\ &= \frac{1}{2}\left(\frac{1}{2\sqrt{5}}\ln\left|\frac{\sqrt{5} + 2x}{\sqrt{5} - 2x}\right|\right) + C & \frac{1}{2a}\ln\left|\frac{a + u}{a - u}\right| + C \\ &= \frac{1}{4\sqrt{5}}\ln\left|\frac{\sqrt{5} + 2x}{\sqrt{5} - 2x}\right| + C \end{aligned}$$

5.9 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, evaluate the function. If the value is not a rational number, give the answer to three-decimal-place accuracy.

1. (a) $\sinh 3$
(b) $\tanh(-2)$

2. (a) $\cosh 0$
(b) $\operatorname{sech} 1$

3. (a) $\operatorname{csch}(\ln 2)$
(b) $\coth(\ln 5)$

4. (a) $\sinh^{-1} 0$
(b) $\tanh^{-1} 0$

5. (a) $\cosh^{-1} 2$
(b) $\operatorname{sech}^{-1} \frac{2}{3}$

6. (a) $\operatorname{csch}^{-1} 2$
(b) $\coth^{-1} 3$

In Exercises 7–16, verify the identity.

7. $e^x = \sinh x + \cosh x$

8. $e^{2x} = \sinh 2x + \cosh 2x$

9. $\tanh^2 x + \operatorname{sech}^2 x = 1$

10. $\coth^2 x - \operatorname{csch}^2 x = 1$

11. $\cosh^2 x = \dfrac{1 + \cosh 2x}{2}$

12. $\sinh^2 x = \dfrac{-1 + \cosh 2x}{2}$

13. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

14. $\sinh 2x = 2 \sinh x \cosh x$

15. $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$

16. $\cosh x + \cosh y = 2 \cosh \dfrac{x + y}{2} \cosh \dfrac{x - y}{2}$

In Exercises 17 and 18, use the value of the given hyperbolic function to find the values of the other hyperbolic functions at x.

17. $\sinh x = \dfrac{3}{2}$

18. $\tanh x = \dfrac{1}{2}$

In Exercises 19–30, find the derivative of the function.

19. $f(x) = \sinh 3x$

20. $f(x) = \cosh(x - 2)$

21. $y = \operatorname{sech}(5x^2)$

22. $y = \tanh(3x^2 - 1)$

23. $f(x) = \ln(\sinh x)$

24. $g(x) = \ln(\cosh x)$

25. $y = \ln\left(\tanh \dfrac{x}{2}\right)$

26. $y = x \cosh x - \sinh x$

27. $h(x) = \dfrac{1}{4} \sinh 2x - \dfrac{x}{2}$

28. $h(t) = t - \coth t$

29. $f(t) = \arctan(\sinh t)$

30. $g(x) = \operatorname{sech}^2 3x$

In Exercises 31–34, find an equation of the tangent line to the graph of the function at the given point.

31. $y = \sinh(1 - x^2)$, $(1, 0)$

32. $y = x^{\cosh x}$, $(1, 1)$

33. $y = (\cosh x - \sinh x)^2$, $(0, 1)$

34. $y = e^{\sinh x}$, $(0, 1)$

In Exercises 35–38, find any relative extrema of the function. Use a graphing utility to confirm your result.

35. $f(x) = \sin x \sinh x - \cos x \cosh x$, $-4 \le x \le 4$

36. $f(x) = x \sinh(x - 1) - \cosh(x - 1)$

37. $g(x) = x \operatorname{sech} x$

38. $h(x) = 2 \tanh x - x$

In Exercises 39 and 40, show that the function satisfies the differential equation.

	Function	*Differential Equation*
39.	$y = a \sinh x$	$y''' - y' = 0$
40.	$y = a \cosh x$	$y'' - y = 0$

CAS *Linear and Quadratic Approximations* **In Exercises 41 and 42, use a computer algebra system to find the linear approximation**

$$P_1(x) = f(a) + f'(a)(x - a)$$

and the quadratic approximation

$$P_2(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2} f''(a)(x - a)^2$$

of the function f at $x = a$. Use a graphing utility to graph the function and its linear and quadratic approximations.

41. $f(x) = \tanh x$, $a = 0$

42. $f(x) = \cosh x$, $a = 0$

Catenary **In Exercises 43 and 44, a model for a power cable suspended between two towers is given. (a) Graph the model, (b) find the heights of the cable at the towers and at the midpoint between the towers, and (c) find the slope of the model at the point where the cable meets the right-hand tower.**

43. $y = 10 + 15 \cosh \dfrac{x}{15}$, $-15 \le x \le 15$

44. $y = 18 + 25 \cosh \dfrac{x}{25}$, $-25 \le x \le 25$

In Exercises 45–58, find the integral.

45. $\displaystyle\int \cosh 2x\, dx$

46. $\displaystyle\int \operatorname{sech}^2(-x)\, dx$

47. $\displaystyle\int \sinh(1 - 2x)\, dx$

48. $\displaystyle\int \frac{\cosh \sqrt{x}}{\sqrt{x}}\, dx$

49. $\displaystyle\int \cosh^2(x - 1) \sinh(x - 1)\, dx$

50. $\displaystyle\int \frac{\sinh x}{1 + \sinh^2 x}\, dx$

51. $\displaystyle\int \frac{\cosh x}{\sinh x}\, dx$

52. $\displaystyle\int \operatorname{sech}^2(2x - 1)\, dx$

53. $\displaystyle\int x \operatorname{csch}^2 \frac{x^2}{2}\, dx$

54. $\displaystyle\int \operatorname{sech}^3 x \tanh x\, dx$

55. $\displaystyle\int \frac{\operatorname{csch}(1/x) \coth(1/x)}{x^2}\, dx$

56. $\displaystyle\int \frac{\cosh x}{\sqrt{9 - \sinh^2 x}}\, dx$

57. $\displaystyle\int \frac{x}{x^4 + 1}\, dx$

58. $\displaystyle\int \frac{2}{x\sqrt{1 + 4x^2}}\, dx$

In Exercises 59–64, evaluate the integral.

59. $\displaystyle\int_0^{\ln 2} \tanh x\, dx$

60. $\displaystyle\int_0^1 \cosh^2 x\, dx$

61. $\displaystyle\int_0^4 \frac{1}{25 - x^2}\, dx$

62. $\displaystyle\int_0^4 \frac{1}{\sqrt{25 - x^2}}\, dx$

63. $\displaystyle\int_0^{\sqrt{2}/4} \frac{2}{\sqrt{1 - 4x^2}}\, dx$

64. $\displaystyle\int_0^{\ln 2} 2e^{-x} \cosh x\, dx$

In Exercises 65–74, find the derivative of the function.

65. $y = \cosh^{-1}(3x)$

66. $y = \tanh^{-1}\dfrac{x}{2}$

67. $y = \tanh^{-1}\sqrt{x}$

68. $f(x) = \coth^{-1}(x^2)$

69. $y = \sinh^{-1}(\tan x)$

70. $y = \tanh^{-1}(\sin 2x)$

71. $y = (\operatorname{csch}^{-1} x)^2$

72. $y = \operatorname{sech}^{-1}(\cos 2x), \quad 0 < x < \pi/4$

73. $y = 2x \sinh^{-1}(2x) - \sqrt{1 + 4x^2}$

74. $y = x \tanh^{-1} x + \ln\sqrt{1 - x^2}$

WRITING ABOUT CONCEPTS

75. Discuss several ways in which the hyperbolic functions are similar to the trigonometric functions.

76. Sketch the graph of each hyperbolic function. Then identify the domain and range of each function.

77. Which hyperbolic derivative formulas differ from their trigonometric counterparts by a minus sign?

CAPSTONE

78. Which hyperbolic functions take on only positive values? Which hyperbolic functions are increasing on their domains?

Limits **In Exercises 79–86, find the limit.**

79. $\lim\limits_{x\to\infty} \sinh x$

80. $\lim\limits_{x\to-\infty} \sinh x$

81. $\lim\limits_{x\to\infty} \tanh x$

82. $\lim\limits_{x\to-\infty} \tanh x$

83. $\lim\limits_{x\to\infty} \operatorname{sech} x$

84. $\lim\limits_{x\to-\infty} \operatorname{csch} x$

85. $\lim\limits_{x\to 0} \dfrac{\sinh x}{x}$

86. $\lim\limits_{x\to 0^-} \coth x$

In Exercises 87–96, find the indefinite integral using the formulas from Theorem 5.25.

87. $\displaystyle\int \frac{1}{3 - 9x^2}\,dx$

88. $\displaystyle\int \frac{1}{2x\sqrt{1 - 4x^2}}\,dx$

89. $\displaystyle\int \frac{1}{\sqrt{1 + e^{2x}}}\,dx$

90. $\displaystyle\int \frac{x}{9 - x^4}\,dx$

91. $\displaystyle\int \frac{1}{\sqrt{x}\sqrt{1 + x}}\,dx$

92. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1 + x^3}}\,dx$

93. $\displaystyle\int \frac{-1}{4x - x^2}\,dx$

94. $\displaystyle\int \frac{dx}{(x + 2)\sqrt{x^2 + 4x + 8}}$

95. $\displaystyle\int \frac{1}{1 - 4x - 2x^2}\,dx$

96. $\displaystyle\int \frac{dx}{(x + 1)\sqrt{2x^2 + 4x + 8}}$

In Exercises 97–100, evaluate the integral using the formulas from Theorem 5.25.

97. $\displaystyle\int_3^7 \frac{1}{\sqrt{x^2 - 4}}\,dx$

98. $\displaystyle\int_1^3 \frac{1}{x\sqrt{4 + x^2}}\,dx$

99. $\displaystyle\int_{-1}^1 \frac{1}{16 - 9x^2}\,dx$

100. $\displaystyle\int_0^1 \frac{1}{\sqrt{25x^2 + 1}}\,dx$

In Exercises 101–104, solve the differential equation.

101. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{80 + 8x - 16x^2}}$

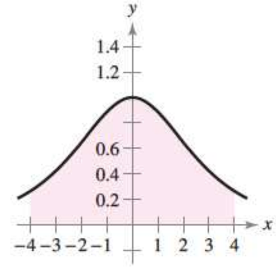
102. $\dfrac{dy}{dx} = \dfrac{1}{(x - 1)\sqrt{-4x^2 + 8x - 1}}$

103. $\dfrac{dy}{dx} = \dfrac{x^3 - 21x}{5 + 4x - x^2}$

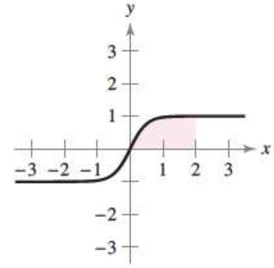
104. $\dfrac{dy}{dx} = \dfrac{1 - 2x}{4x - x^2}$

Area **In Exercises 105–108, find the area of the region.**

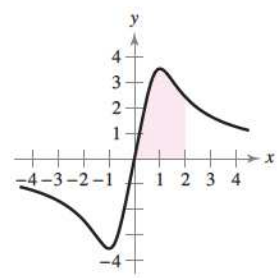
105. $y = \operatorname{sech}\dfrac{x}{2}$



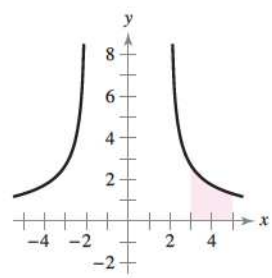
106. $y = \tanh 2x$



107. $y = \dfrac{5x}{\sqrt{x^4 + 1}}$



108. $y = \dfrac{6}{\sqrt{x^2 - 4}}$



In Exercises 109 and 110, evaluate the integral in terms of (a) natural logarithms and (b) inverse hyperbolic functions.

109. $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{x^2 + 1}}$

110. $\displaystyle\int_{-1/2}^{1/2} \frac{dx}{1 - x^2}$

111. ***Chemical Reactions*** Chemicals A and B combine in a 3-to-1 ratio to form a compound. The amount of compound x being produced at any time t is proportional to the unchanged amounts of A and B remaining in the solution. So, if 3 kilograms of A is mixed with 2 kilograms of B, you have

$$\frac{dx}{dt} = k\left(3 - \frac{3x}{4}\right)\left(2 - \frac{x}{4}\right) = \frac{3k}{16}(x^2 - 12x + 32).$$

One kilogram of the compound is formed after 10 minutes. Find the amount formed after 20 minutes by solving the equation

$$\int \frac{3k}{16}\,dt = \int \frac{dx}{x^2 - 12x + 32}.$$

112. *Vertical Motion* An object is dropped from a height of 400 feet.

(a) Find the velocity of the object as a function of time (neglect air resistance on the object).

(b) Use the result in part (a) to find the position function.

(c) If the air resistance is proportional to the square of the velocity, then $dv/dt = -32 + kv^2$, where -32 feet per second per second is the acceleration due to gravity and k is a constant. Show that the velocity v as a function of time is $v(t) = -\sqrt{32/k}\tanh\left(\sqrt{32k}\,t\right)$ by performing $\int dv/(32 - kv^2) = -\int dt$ and simplifying the result.

(d) Use the result of part (c) to find $\lim_{t\to\infty} v(t)$ and give its interpretation.

(e) Integrate the velocity function in part (c) and find the position s of the object as a function of t. Use a graphing utility to graph the position function when $k = 0.01$ and the position function in part (b) in the same viewing window. Estimate the additional time required for the object to reach ground level when air resistance is not neglected.

(f) Give a written description of what you believe would happen if k were increased. Then test your assertion with a particular value of k.

Tractrix **In Exercises 113 and 114, use the equation of the tractrix $y = a\,\mathrm{sech}^{-1}(x/a) - \sqrt{a^2 - x^2}$, $a > 0$.**

113. Find dy/dx.

114. Let L be the tangent line to the tractrix at the point P. If L intersects the y-axis at the point Q, show that the distance between P and Q is a.

115. Prove that $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $-1 < x < 1$.

116. Prove that $\sinh^{-1} t = \ln\left(t + \sqrt{t^2 + 1}\right)$.

117. Show that $\arctan(\sinh x) = \arcsin(\tanh x)$.

118. Let $x > 0$ and $b > 0$. Show that $\displaystyle\int_{-b}^{b} e^{xt}\,dt = \frac{2\sinh bx}{x}$.

In Exercises 119–124, verify the differentiation formula.

119. $\dfrac{d}{dx}[\mathrm{sech}^{-1} x] = \dfrac{-1}{x\sqrt{1-x^2}}$

120. $\dfrac{d}{dx}[\cosh^{-1} x] = \dfrac{1}{\sqrt{x^2-1}}$

121. $\dfrac{d}{dx}[\sinh^{-1} x] = \dfrac{1}{\sqrt{x^2+1}}$

122. $\dfrac{d}{dx}[\cosh x] = \sinh x$

123. $\dfrac{d}{dx}[\coth x] = -\mathrm{csch}^2\, x$

124. $\dfrac{d}{dx}[\mathrm{sech}\, x] = -\mathrm{sech}\, x \tanh x$

PUTNAM EXAM CHALLENGE

125. From the vertex $(0, c)$ of the catenary $y = c\cosh(x/c)$, a line L is drawn perpendicular to the tangent to the catenary at a point P. Prove that the length of L intercepted by the axes is equal to the ordinate y of the point P.

126. Prove or disprove that there is at least one straight line normal to the graph of $y = \cosh x$ at a point $(a, \cosh a)$ and also normal to the graph of $y = \sinh x$ at a point $(c, \sinh c)$.

[At a point on a graph, the normal line is the perpendicular to the tangent at that point. Also, $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$.]

SECTION PROJECT

St. Louis Arch

Paul Damien

The Gateway Arch in St. Louis, Missouri was constructed using the hyperbolic cosine function. The equation used for construction was

$y = 693.8597 - 68.7672\cosh 0.0100333x,$

$-299.2239 \le x \le 299.2239$

where x and y are measured in feet. Cross sections of the arch are equilateral triangles, and (x, y) traces the path of the centers of mass of the cross-sectional triangles. For each value of x, the area of the cross-sectional triangle is $A = 125.1406\cosh 0.0100333x$. *(Source: Owner's Manual for the Gateway Arch, Saint Louis, MO, by William Thayer)*

(a) How high above the ground is the center of the highest triangle? (At ground level, $y = 0$.)

(b) What is the height of the arch? (*Hint:* For an equilateral triangle, $A = \sqrt{3}c^2$, where c is one-half the base of the triangle, and the center of mass of the triangle is located at two-thirds the height of the triangle.)

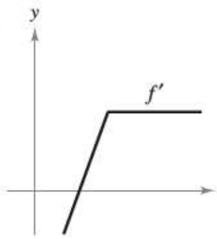
(c) How wide is the arch at ground level?

5 REVIEW EXERCISES

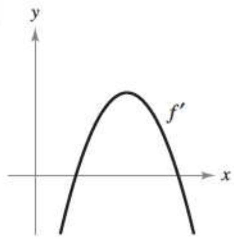
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use the graph of f' to sketch a graph of f. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

1.



2.



In Exercises 3–12, find the indefinite integral.

3. $\int (4x^2 + x + 3)\,dx$

4. $\int \frac{2}{\sqrt[3]{3x}}\,dx$

5. $\int \frac{x^4 + 8}{x^3}\,dx$

6. $\int \frac{x^4 - 4x^2 + 1}{x^2}\,dx$

7. $\int (2x - 9 \sin x)\,dx$

8. $\int (5 \cos x - 2 \sec^2 x)\,dx$

9. $\int (5 - e^x)\,dx$

10. $\int (t + e^t)\,dt$

11. $\int \frac{5}{x}\,dx$

12. $\int \frac{10}{x}\,dx$

13. Find the particular solution of the differential equation $f'(x) = -6x$ whose graph passes through the point $(1, -2)$.

14. Find the particular solution of the differential equation $f''(x) = 2e^x$ whose graph passes through the point $(0, 1)$ and is tangent to the line $3x - y - 5 = 0$ at that point.

15. ***Velocity and Acceleration*** An airplane taking off from a runway travels 3600 feet before lifting off. The airplane starts from rest, moves with constant acceleration, and makes the run in 30 seconds. With what speed does it lift off?

16. ***Velocity and Acceleration*** The speed of a car traveling in a straight line is reduced from 45 to 30 miles per hour in a distance of 264 feet. Find the distance in which the car can be brought to rest from 30 miles per hour, assuming the same constant deceleration.

17. ***Velocity and Acceleration*** A ball is thrown vertically upward from ground level with an initial velocity of 96 feet per second.

(a) How long will it take the ball to rise to its maximum height?

(b) What is the maximum height?

(c) After how many seconds is the velocity of the ball one-half the initial velocity?

(d) What is the height of the ball when its velocity is one-half the initial velocity?

18. ***Velocity and Acceleration*** Repeat Exercise 17 for an initial velocity of 40 meters per second.

19. Write in sigma notation (a) the sum of the first 10 positive odd integers, (b) the sum of the cubes of the first n positive integers, and (c) $6 + 10 + 14 + 18 + \cdots + 42$.

20. Evaluate each sum for $x_1 = 2, x_2 = -1, x_3 = 5, x_4 = 3$, and $x_5 = 7$.

(a) $\frac{1}{5}\sum_{i=1}^{5} x_i$

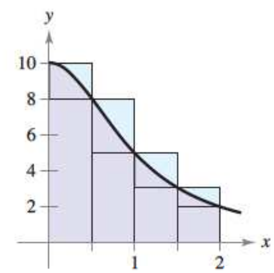
(b) $\sum_{i=1}^{5} \frac{1}{x_i}$

(c) $\sum_{i=1}^{5} (2x_i - x_i^2)$

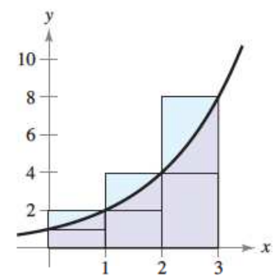
(d) $\sum_{i=2}^{5} (x_i - x_{i-1})$

In Exercises 21 and 22, use upper and lower sums to approximate the area of the region using the given number of subintervals of equal width.

21. $y = \frac{10}{x^2 + 1}$



22. $y = 2^x$



In Exercises 23–26, use the limit process to find the area of the region between the graph of the function and the x-axis over the given interval. Sketch the region.

	Function	*Interval*
23.	$y = 8 - 2x$	$[0, 3]$
24.	$y = x^2 + 3$	$[0, 2]$
25.	$y = 5 - x^2$	$[-2, 1]$
26.	$y = \frac{1}{4}x^3$	$[2, 4]$

27. Use the limit process to find the area of the region bounded by $x = 5y - y^2$, $x = 0$, $y = 2$, and $y = 5$.

28. Consider the region bounded by $y = mx$, $y = 0$, $x = 0$, and $x = b$.

(a) Find the upper and lower sums to approximate the area of the region when $\Delta x = b/4$.

(b) Find the upper and lower sums to approximate the area of the region when $\Delta x = b/n$.

(c) Find the area of the region by letting n approach infinity in both sums in part (b). Show that in each case you obtain the formula for the area of a triangle.

In Exercises 29 and 30, express the limit as a definite integral on the interval $[a, b]$, where c_i is any point in the ith subinterval.

Limit	*Interval*
29. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (2c_i - 3)\,\Delta x_i$	$[4, 6]$
30. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 3c_i(9 - c_i^2)\,\Delta x_i$	$[1, 3]$

In Exercises 31 and 32, sketch the region whose area is given by the definite integral. Then use a geometric formula to evaluate the integral.

31. $\int_0^5 (5 - |x - 5|)\,dx$ **32.** $\int_{-6}^{6} \sqrt{36 - x^2}\,dx$

33. Given $\int_4^8 f(x)\,dx = 12$ and $\int_4^8 g(x)\,dx = 5$, evaluate

(a) $\int_4^8 [f(x) + g(x)]\,dx$. (b) $\int_4^8 [f(x) - g(x)]\,dx$.

(c) $\int_4^8 [2f(x) - 3g(x)]\,dx$. (d) $\int_4^8 7f(x)\,dx$.

34. Given $\int_0^3 f(x)\,dx = 4$ and $\int_3^6 f(x)\,dx = -1$, evaluate

(a) $\int_0^6 f(x)\,dx$. (b) $\int_6^3 f(x)\,dx$.

(c) $\int_4^4 f(x)\,dx$. (d) $\int_3^6 -10f(x)\,dx$.

In Exercises 35 and 36, select the correct value of the definite integral.

35. $\int_1^8 \left(\sqrt[3]{x} + 1\right)dx$

(a) $\frac{81}{4}$ (b) $\frac{331}{12}$

(c) $\frac{73}{4}$ (d) $\frac{355}{12}$

36. $\int_1^3 \frac{12}{x^3}\,dx$

(a) $\frac{320}{9}$ (b) $-\frac{16}{3}$

(c) $-\frac{5}{9}$ (d) $\frac{16}{3}$

In Exercises 37–46, use the Fundamental Theorem of Calculus to evaluate the definite integral.

37. $\int_0^8 (3 + x)\,dx$ **38.** $\int_{-3}^{3} (t^2 + 1)\,dt$

39. $\int_{-1}^{1} (4t^3 - 2t)\,dt$ **40.** $\int_{-2}^{-1} (x^4 + 3x^2 - 4)\,dx$

41. $\int_4^9 x\sqrt{x}\,dx$ **42.** $\int_1^2 \left(\frac{1}{x^2} - \frac{1}{x^3}\right)dx$

43. $\int_0^{3\pi/4} \sin\theta\,d\theta$ **44.** $\int_{-\pi/4}^{\pi/4} \sec^2 t\,dt$

45. $\int_0^2 (x + e^x)\,dx$

46. $\int_1^6 \frac{3}{x}\,dx$

In Exercises 47–52, sketch the graph of the region whose area is given by the integral, and find the area.

47. $\int_2^4 (3x - 4)\,dx$ **48.** $\int_0^6 (8 - x)\,dx$

49. $\int_3^4 (x^2 - 9)\,dx$ **50.** $\int_{-2}^{3} (-x^2 + x + 6)\,dx$

51. $\int_0^1 (x - x^3)\,dx$ **52.** $\int_0^1 \sqrt{x}(1 - x)\,dx$

In Exercises 53–56, sketch the region bounded by the graphs of the equations, and determine its area.

53. $y = \frac{4}{\sqrt{x}},\quad y = 0,\quad x = 1,\quad x = 9$

54. $y = \sec^2 x,\quad y = 0,\quad x = 0,\quad x = \frac{\pi}{3}$

55. $y = \frac{2}{x},\quad y = 0,\quad x = 1,\quad x = 3$

56. $y = 1 + e^x,\quad y = 0,\quad x = 0,\quad x = 2$

In Exercises 57 and 58, find the average value of the function over the interval. Find the value(s) of x at which the function assumes its average value, and graph the function.

57. $f(x) = \frac{1}{\sqrt{x}},\quad [4, 9]$ **58.** $f(x) = x^3,\quad [0, 2]$

In Exercises 59–62, use the Second Fundamental Theorem of Calculus to find $F'(x)$.

59. $F(x) = \int_0^x t^2\sqrt{1 + t^3}\,dt$ **60.** $F(x) = \int_1^x \frac{1}{t^2}\,dt$

61. $F(x) = \int_{-3}^{x} (t^2 + 3t + 2)\,dt$ **62.** $F(x) = \int_0^x \csc^2 t\,dt$

In Exercises 63–80, find the indefinite integral.

63. $\int (3 - x^2)^3\,dx$ **64.** $\int \left(x + \frac{1}{x}\right)^2 dx$

65. $\int \frac{x^2}{\sqrt{x^3 + 3}}\,dx$ **66.** $\int 3x^2\sqrt{2x^3 - 5}\,dx$

67. $\int x(1 - 3x^2)^4\,dx$ **68.** $\int \frac{x + 4}{(x^2 + 8x - 7)^2}\,dx$

69. $\int \sin^3 x \cos x\,dx$ **70.** $\int x \sin 3x^2\,dx$

71. $\int \frac{\cos\theta}{\sqrt{1 - \sin\theta}}\,d\theta$ **72.** $\int \frac{\sin x}{\sqrt{\cos x}}\,dx$

73. $\int \tan^n x \sec^2 x\,dx,\quad n \neq -1$ **74.** $\int \sec 2x \tan 2x\,dx$

75. $\int (1 + \sec \pi x)^2 \sec \pi x \tan \pi x\,dx$ **76.** $\int \cot^4 \alpha \csc^2 \alpha\,d\alpha$

77. $\int xe^{-3x^2}\,dx$ **78.** $\int \frac{e^{1/x}}{x^2}\,dx$

79. $\int (x + 1)5^{(x+1)^2}\,dx$ **80.** $\int \frac{1}{t^2}(2^{-1/t})\,dt$

In Exercises 81–88, evaluate the definite integral. Use a graphing utility to verify your result.

81. $\int_{-2}^{1} x(x^2 - 6)\, dx$

82. $\int_{0}^{1} x^2(x^3 - 2)^3\, dx$

83. $\int_{0}^{3} \frac{1}{\sqrt{1+x}}\, dx$

84. $\int_{3}^{6} \frac{x}{3\sqrt{x^2 - 8}}\, dx$

85. $2\pi \int_{0}^{1} (y + 1)\sqrt{1 - y}\, dy$

86. $2\pi \int_{-1}^{0} x^2\sqrt{x + 1}\, dx$

87. $\int_{0}^{\pi} \cos \frac{x}{2}\, dx$

88. $\int_{-\pi/4}^{\pi/4} \sin 2x\, dx$

89. ***Precipitation*** The normal monthly precipitation in Portland, Oregon can be approximated by the model

$R = 2.880 + 2.125 \sin(0.578t + 0.745)$

where R is measured in inches and t is the time in months, with $t = 0$ corresponding to January 1. *(Source: U.S. National Oceanic and Atmospheric Administration)*

(a) Write and evaluate an integral to approximate the normal annual precipitation.

(b) Approximate the average monthly precipitation during the months of September and October.

90. ***Respiratory Cycle*** After exercising for a few minutes, a person has a respiratory cycle for which the rate of air intake v, in liters per second, is $v = 1.75 \sin(\pi t/2)$, where t is time in seconds. Find the volume, in liters, of air inhaled during one cycle by integrating the function over the interval $[0, 2]$.

In Exercises 91–95, use the Trapezoidal Rule and Simpson's Rule with $n = 4$, and use the integration capabilities of a graphing utility, to approximate the definite integral. Compare the results.

91. $\int_{2}^{3} \frac{2}{1 + x^2}\, dx$

92. $\int_{0}^{1} \frac{x^{3/2}}{3 - x^2}\, dx$

93. $\int_{0}^{\pi/2} \sqrt{x} \cos x\, dx$

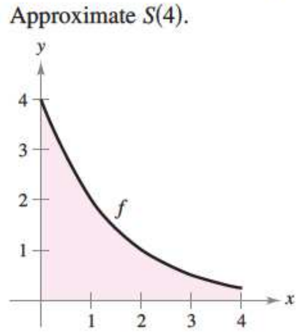
94. $\int_{0}^{\pi} \sqrt{1 + \sin^2 x}\, dx$

95. $\int_{-1}^{1} e^{-x^2}\, dx$

96. Let $I = \int_0^4 f(x)\, dx$, where f is shown in the figure. Let $L(n)$ and $R(n)$ represent the Riemann sums using the left-hand endpoints and right-hand endpoints of n subintervals of equal width. (Assume n is even.) Let $T(n)$ and $S(n)$ be the corresponding values of the Trapezoidal Rule and Simpson's Rule.

(a) For any n, list $L(n)$, $R(n)$, $T(n)$, and I in increasing order.

(b) Approximate $S(4)$.



In Exercises 97–106, find or evaluate the integral.

97. $\int \frac{1}{7x - 2}\, dx$

98. $\int \frac{x}{x^2 - 1}\, dx$

99. $\int \frac{\sin x}{1 + \cos x}\, dx$

100. $\int \frac{\ln \sqrt{x}}{x}\, dx$

101. $\int_{1}^{4} \frac{2x + 1}{2x}\, dx$

102. $\int_{1}^{e} \frac{\ln x}{x}\, dx$

103. $\int_{0}^{\pi/3} \sec \theta\, d\theta$

104. $\int_{0}^{\pi/4} \tan\left(\frac{\pi}{4} - x\right) dx$

105. $\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\, dx$

106. $\int \frac{e^{2x}}{e^{2x} + 1}\, dx$

In Exercises 107–114, find the indefinite integral.

107. $\int \frac{1}{e^{2x} + e^{-2x}}\, dx$

108. $\int \frac{1}{3 + 25x^2}\, dx$

109. $\int \frac{x}{\sqrt{1 - x^4}}\, dx$

110. $\int \frac{1}{16 + x^2}\, dx$

111. $\int \frac{x}{16 + x^2}\, dx$

112. $\int \frac{4 - x}{\sqrt{4 - x^2}}\, dx$

113. $\int \frac{\arctan(x/2)}{4 + x^2}\, dx$

114. $\int \frac{\arcsin 2x}{\sqrt{1 - 4x^2}}\, dx$

115. ***Harmonic Motion*** A weight of mass m is attached to a spring and oscillates with simple harmonic motion. By Hooke's Law, you can determine that

$$\int \frac{dy}{\sqrt{A^2 - y^2}} = \int \sqrt{\frac{k}{m}}\, dt$$

where A is the maximum displacement, t is the time, and k is a constant. Find y, the vertical distance, as a function of t, given that $y = 0$ when $t = 0$.

116. ***Think About It*** Sketch the region whose area is given by $\int_0^1 \arcsin x\, dx$. Then find the area of the region. Explain how you arrived at your answer.

In Exercises 117 and 118, find the derivative of the function.

117. $y = 2x - \cosh \sqrt{x}$

118. $y = x \tanh^{-1} 2x$

In Exercises 119 and 120, find the indefinite integral.

119. $\int \frac{x}{\sqrt{x^4 - 1}}\, dx$

120. $\int x^2 \operatorname{sech}^2 x^3\, dx$

P.S. PROBLEM SOLVING

1. Let $L(x) = \int_1^x \frac{1}{t}\,dt,\ x > 0.$

(a) Find $L(1)$.

(b) Find $L'(x)$ and $L'(1)$.

(c) Use a graphing utility to approximate the value of x (to three decimal places) for which $L(x) = 1$.

(d) Prove that $L(x_1x_2) = L(x_1) + L(x_2)$ for all positive values of x_1 and x_2.

2. Let $F(x) = \int_2^x \sin t^2\,dt.$

(a) Use a graphing utility to complete the table.

x	0	1.0	1.5	1.9	2.0
$F(x)$					

x	2.1	2.5	3.0	4.0	5.0
$F(x)$					

(b) Let $G(x) = \frac{1}{x-2}F(x) = \frac{1}{x-2}\int_2^x \sin t^2\,dt$. Use a graphing utility to complete the table and estimate $\lim_{x\to 2} G(x)$.

x	1.9	1.95	1.99	2.01	2.1
$G(x)$					

(c) Use the definition of the derivative to find the exact value of $\lim_{x\to 2} G(x)$.

3. The **Fresnel function** S is defined by the integral

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt.$$

(a) Graph the function $y = \sin\left(\frac{\pi x^2}{2}\right)$ on the interval $[0, 3]$.

(b) Use the graph in part (a) to sketch the graph of S on the interval $[0, 3]$.

(c) Locate all relative extrema of S on the interval $(0, 3)$.

(d) Locate all points of inflection of S on the interval $(0, 3)$.

4. Galileo Galilei (1564–1642) stated the following proposition concerning falling objects:

The time in which any space is traversed by a uniformly accelerating body is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest speed of the accelerating body and the speed just before acceleration began.

Use the techniques of this chapter to verify this proposition.

5. The graph of the function f consists of the three line segments joining the points $(0, 0)$, $(2, -2)$, $(6, 2)$, and $(8, 3)$. The function F is defined by the integral

$$F(x) = \int_0^x f(t)\,dt.$$

(a) Sketch the graph of f.

(b) Complete the table.

x	0	1	2	3	4	5	6	7	8
$F(x)$									

(c) Find the extrema of F on the interval $[0, 8]$.

(d) Determine all points of inflection of F on the interval $(0, 8)$.

6. A car travels in a straight line for 1 hour. Its velocity v in miles per hour at six-minute intervals is shown in the table.

t (hours)	0	0.1	0.2	0.3	0.4	0.5
v (mi/h)	0	10	20	40	60	50

t (hours)	0.6	0.7	0.8	0.9	1.0
v (mi/h)	40	35	40	50	65

(a) Produce a reasonable graph of the velocity function v by graphing these points and connecting them with a smooth curve.

(b) Find the open intervals over which the acceleration a is positive.

(c) Find the average acceleration of the car (in miles per hour squared) over the interval $[0, 0.4]$.

(d) What does the integral $\int_0^1 v(t)\,dt$ signify? Approximate this integral using the Trapezoidal Rule with five subintervals.

(e) Approximate the acceleration at $t = 0.8$.

7. The **Two-Point Gaussian Quadrature Approximation** for f is

$$\int_{-1}^1 f(x)\,dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

(a) Use this formula to approximate

$$\int_{-1}^1 \cos x\,dx.$$

Find the error of the approximation.

(b) Use this formula to approximate

$$\int_{-1}^1 \frac{1}{1+x^2}\,dx.$$

(c) Prove that the Two-Point Gaussian Quadrature Approximation is exact for all polynomials of degree 3 or less.

8. Prove that $\int_0^x f(t)(x-t)\,dt = \int_0^x \left(\int_0^t f(v)\,dv \right) dt.$

9. Prove that $\int_a^b f(x)f'(x)\,dx = \frac{1}{2}([f(b)]^2 - [f(a)]^2).$

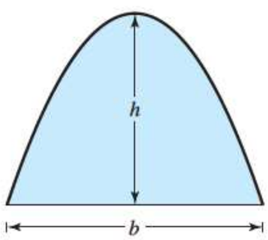
10. Use an appropriate Riemann sum to evaluate the limit

$$\lim_{n\to\infty} \frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}}{n^{3/2}}.$$

11. Use an appropriate Riemann sum to evaluate the limit

$$\lim_{n\to\infty} \frac{1^5 + 2^5 + 3^5 + \cdots + n^5}{n^6}.$$

12. Archimedes showed that the area of a parabolic arch is equal to $\frac{2}{3}$ the product of the base and the height (see figure).



(a) Graph the parabolic arch bounded by $y = 9 - x^2$ and the x-axis. Use an appropriate integral to find the area A.

(b) Find the base and height of the arch and verify Archimedes' formula.

(c) Prove Archimedes' formula for a general parabola.

13. Suppose that f is integrable on $[a, b]$ and $0 < m \le f(x) \le M$ for all x in the interval $[a, b]$. Prove that

$$m(a-b) \le \int_a^b f(x)\,dx \le M(b-a).$$

Use this result to estimate $\int_0^1 \sqrt{1+x^4}\,dx$.

14. Verify that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

by showing the following.

(a) $(1+i)^3 - i^3 = 3i^2 + 3i + 1$

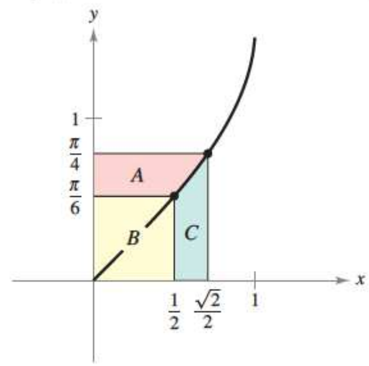
(b) $(n+1)^3 = \sum_{i=1}^{n}(3i^2 + 3i + 1) + 1$

(c) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

15. Prove that if f is a continuous function on a closed interval $[a, b]$, then

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx.$$

16. Consider the three regions A, B, and C determined by the graph of $f(x) = \arcsin x$, as shown in the figure.



(a) Calculate the areas of regions A and B.

(b) Use your answer in part (a) to evaluate the integral

$$\int_{1/2}^{\sqrt{2}/2} \arcsin x\,dx.$$

(c) Use your answer in part (a) to evaluate the integral

$$\int_1^3 \ln x\,dx.$$

(d) Use your answer in part (a) to evaluate the integral

$$\int_1^{\sqrt{3}} \arctan x\,dx.$$

17. Use integration by substitution to find the area under the curve $y = \dfrac{1}{\sqrt{x} + x}$ between $x = 1$ and $x = 4$.

18. Use integration by substitution to find the area under the curve $y = \dfrac{1}{\sin^2 x + 4\cos^2 x}$ between $x = 0$ and $x = \pi/4$.

19. (a) Use a graphing utility to compare the graph of the function $y = e^x$ with the graph of each given function.

(i) $y_1 = 1 + \frac{x}{1!}$

(ii) $y_2 = 1 + \frac{x}{1!} + \frac{x^2}{2!}$

(iii) $y_3 = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$

(b) Identify the pattern of successive polynomials in part (a) and extend the pattern one more term. Compare the graph of the resulting polynomial function with the graph of $y = e^x$.

(c) What do you think this pattern implies?

20. Let f be continuous on the interval $[0, b]$ such that $f(x) + f(b-x) \neq 0$. Show that

$$\int_0^b \frac{f(x)}{f(x) + f(b-x)}\,dx = \frac{b}{2}.$$

Use this result to evaluate $\int_0^1 \dfrac{\sin x}{\sin(1-x) + \sin x}\,dx.$