AP Calculus I
Notes 4.1
Extrema on an Interval

Definition of Extrema

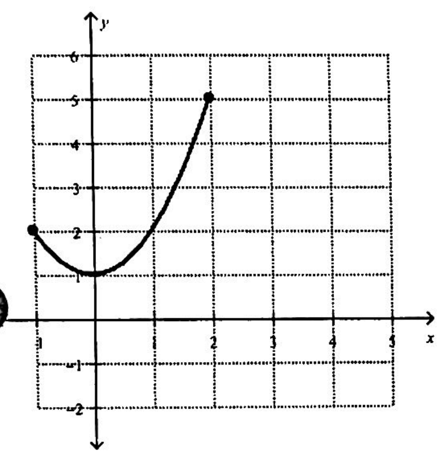
Let f be defined on an interval I containing c.

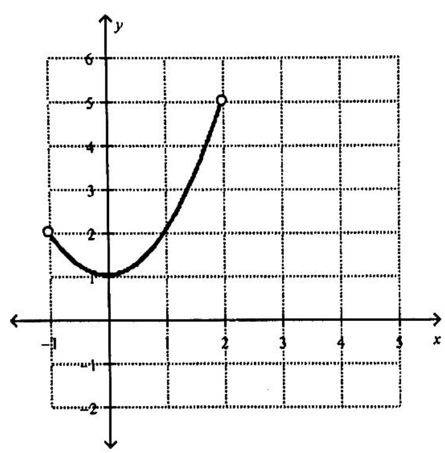
1) $f(c)$ is the **absolute minimum** of f if $f(c) \leq f(x)$ for all x.

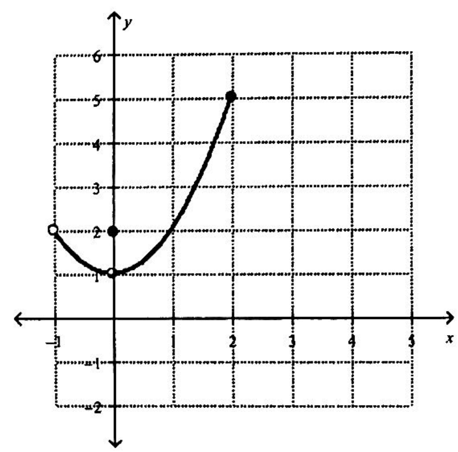
y-vals 2) $f(c)$ is the **absolute maximum** of f if $f(c) \geq f(x)$ for all x.

The minimum and maximum of a function on an interval are the **extreme values**, or **extrema,** of the function on the interval. The overall minimum and maximum of a function are called the **absolute minimum** and **absolute maximum** on the interval. (global)

Ex. 1: Consider the following variations of the function $f(x) = x^2 + 1$:







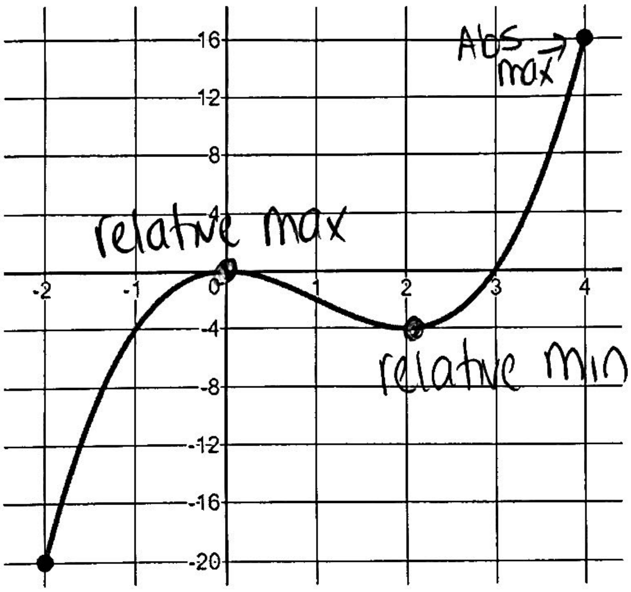
	$f(x) = x^2 + 1$ from $[-1,2]$	$f(x) = x^2 + 1$ from $(-1,2)$	$g(x) = \begin{cases} x^2 + 1, & x \neq 0 \\ 2, & x = 0 \end{cases}$ from $(-1,2]$
Abs Min	1	1	None
Abs Max	5	None	5

The Extreme Value Theorem: Existence Theorem

There is **guaranteed** to be an absolute minimum and an absolute maximum for f on a **closed** interval $[a, b]$ as long as f is continuous on the closed interval $[a, b]$.

Relative Extrema:

Consider the graph below of $f(x) = x^3 - 3x^2$:



Determine if $f(x)$ has absolute extrema from $-2 \leq x \leq 4$.

Since f is continuous on [-2,4] then f is guaranteed to have absolute extrema (max & min)

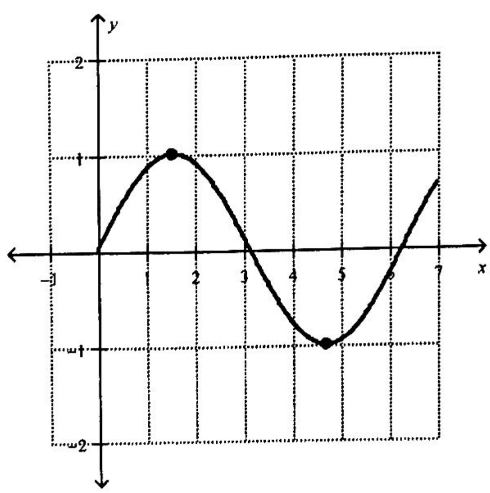
The graph of $f(x) = x^3 - 3x^2$ has a **relative minimum** at the point (2,-4)

The graph of $f(x) = x^3 - 3x^2$ has a **relative maximum** at the point (0,0)

Think of the relative maximum occurring on a hill of the graph and the relative minimum occurring in a valley of the graph.

Ex. 2: Find the value of the derivative, if it exists, at each of the relative extrema below:

a) $f(x) = \sin x$ $\quad f'(x) = \cos x$



rel max $\left(\frac{\pi}{2}, 1\right)$

$f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$

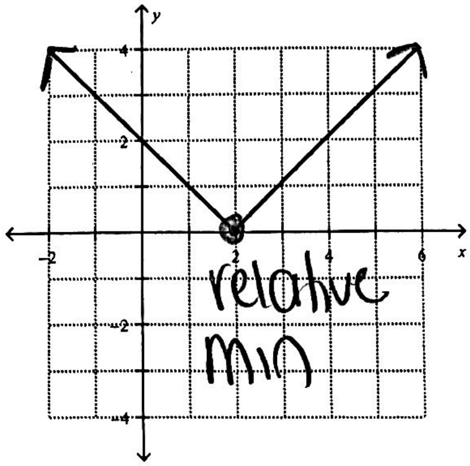
rel min : $\left(\frac{3\pi}{2}, -1\right)$

$f'\left(\frac{3\pi}{2}\right) = \cos\frac{3\pi}{2} = -1$

$f'\left(\frac{3\pi}{2}\right) = 0$

b) $f(x) = |x - 2|$

Relative Max : None



rel min : (2,0)

$$\lim_{x \to 2^-} f' = -1, \quad \lim_{x \to 2^+} f' = 1$$

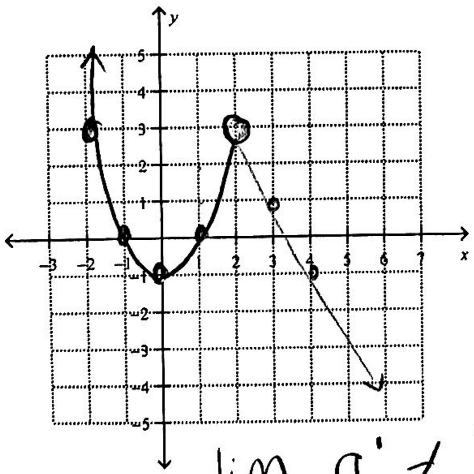
$f'(2)$ Does not exist

(Not Differentiable)

c) $g(x) = \begin{cases} x^2 - 1 & x < 2 \\ 7 - 2x & x \geq 2 \end{cases}$

relative min : $x = 0$

$g'(0) = 2(0) = 0$



Relative max : $x = 2$

$f'(2)$ is undefined

$$\lim_{x \to 2^-} g' \neq \lim_{x \to 2^+} g'$$

$$4 \neq -2$$

Note that in each of these examples of relative extrema, the derivative is either <u>0</u> or <u>undefined</u>

Definition of Critical Number:

Let f be defined at c. If $f'(c) = 0$ or if f' is undefined at c, then c is a **critical number** of f.

Ex. 3: Find the critical values of $f(x) = \sin^2 x + \sin x$ on the interval $[0, 2\pi]$.

Is f' ever undefined? NO

f' is 0 or und.

$$f'(x) = 2\sin x\cos x + \cos x = 0$$

$$\cos x(2\sin x + 1) = 0$$

$$\cos x = 0 \qquad 2\sin x + 1 = 0$$

$$\boxed{x = \pi/2,\ 3\pi/2} \qquad \frac{2\sin x}{2} = \frac{-1}{2}$$

$$\sin x = \frac{-1}{2}$$

S A
T C

$$\boxed{x = \frac{7\pi}{6},\ \frac{11\pi}{6}}$$

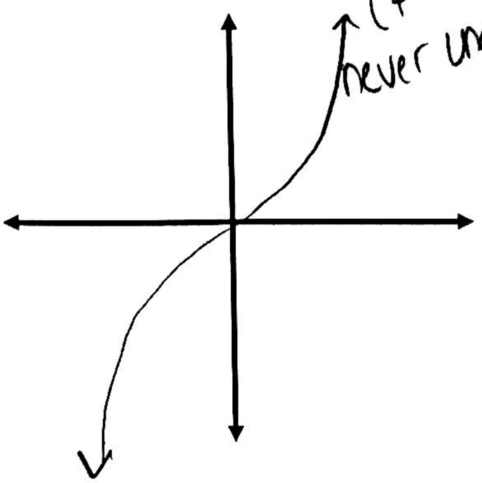
Theorem – Relative Extrema Occur Only at Critical Numbers

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f. However, if $x = c$ is a critical number of f, there is **not necessarily** a relative extrema at $x = c$.

Counterexample: $f(x) = x^3$ $\quad f'(x) = 3x^2 = 0$

$x = 0$

(f' never und)

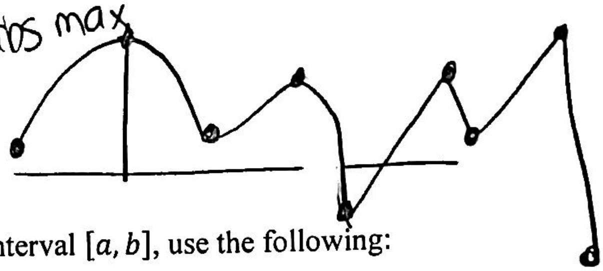


$f(x)$ has a critical value at $\underline{x = 0}$ because

f' = 0 or undef $\qquad f'(0) = 0$

$f(x)$ has a relative extrema at $\underline{\text{none}}$ because

it increases before & after our critical point



Guidelines for Finding Absolute Extrema on a Closed Interval:

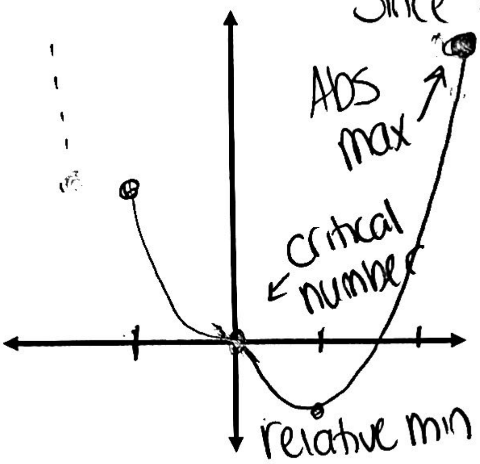
To find the absolute extrema of a continuous function f on a closed interval $[a, b]$, use the following:

1) Find the critical numbers of f in (a, b) and evaluate f at each critical number.
2) Evaluate f at each endpoint of $[a, b]$. The least of these values is the minimum and the greatest is the abs ↑ maximum. abs min

2 candidates for abs extrema → 1) crit. values 2) Endpoint

Ex. 4: Find the absolute extrema of $f(x) = 3x^4 - 4x^3$ on $[-1,2]$. Are the extrema guaranteed?

Since f(x) is continuous, extrema are guaranteed



Abs min

1) Endpts : $x=-1,2$

2) Critical Values : $f'=0$ or und.

$f' = 12x^3 - 12x^2 = 0$

$12x^2(x-1) = 0$

$x=0 \quad x=1$

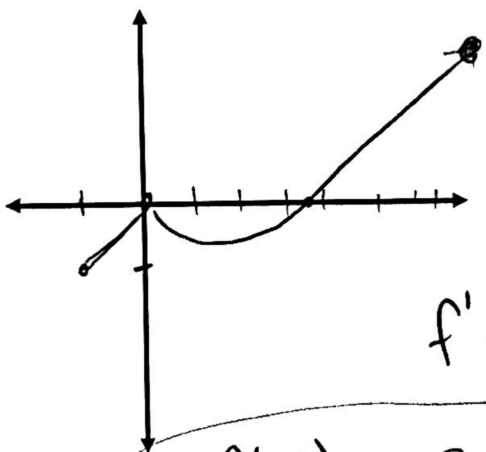
x	f(x)
-1	7
0	0
1	-1
2	16

absolute min

absolute max

Ex. 5: Find the maximum value of $f(x) = 2x - 3x^{\frac{2}{3}}$ on the interval $[-1,8]$, if any.

candidates



1) Endpts $x=-1,8$

2) critical values : $f' = 2 - 2x^{-1/3}$

$= 2 - \frac{2}{\sqrt[3]{x}} = 0$

$2 = \frac{2}{\sqrt[3]{x}}$

$\sqrt[3]{x} = 1$

$x=1$

$f'=0$ @ $x=1$

f' is undef @ $x=0$

$f(-1) = -2-3 = -5$ ← absolute min

$f(0) = 0$

$f(1) = -1$

$f(8) = 16 - 3(4) = 4$ ← absolute max

Summary

a) The Extreme Value Theorem states

if a function is continuous on a closed interval, then an abs max & min are guaranteed

b) A critical number is

where $f' = 0$ or where f' is undefined

c) Absolute extrema can occur at

endpoints or critical values

d) Relative extrema occur at

critical numbers

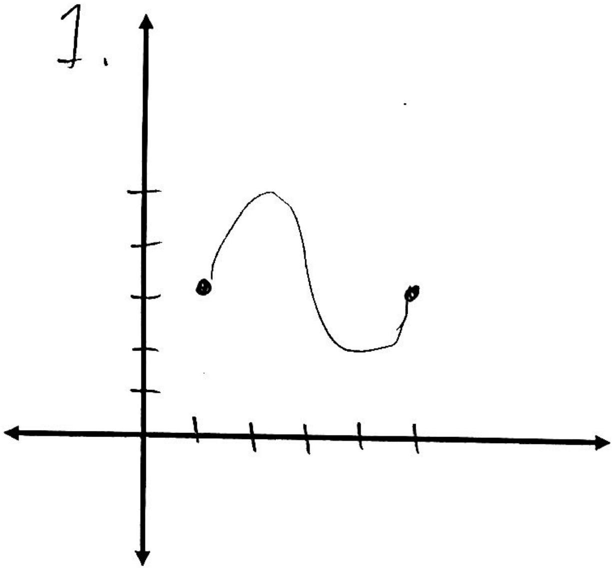
* but critical numbers dont have to be relative extrema)

AP Calculus I
Notes 4.2
Rolle's Theorem and the Mean Value Theorem

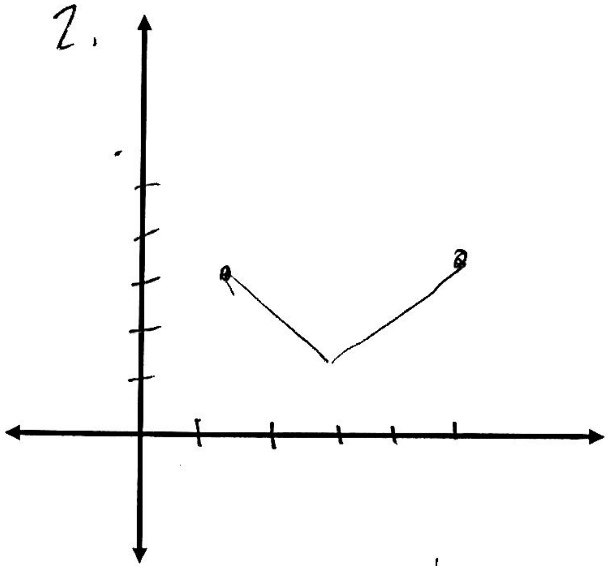
Exploration

Label the points (1,3) and (5,3). Draw any random function that starts at (1,3) and ends at (5,3).

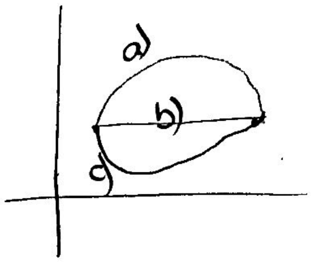
1.



2.



- Is there one point on the graph for which the derivative is zero? Yes, NO
- Could there be more than one? If so, draw an example of this function. ex 1
- Could there be none? If so, draw an example of this function. ex 2



- Describe the continuity and differentiability of the different graphs sketched.

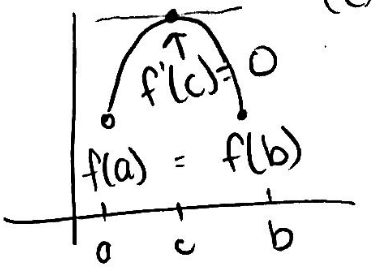
1) cont & differentiable 2) continuous but not differentiable

- Would it be possible to draw a **differentiable** graph so that there *isn't* a point for which the derivative is zero? Explain your reasoning. ✓a) starts off increasing, must decrease to get back to the same value, so slope crosses 0
✓b) slope is always zero
c) same argument as a)

Rolle's Theorem:

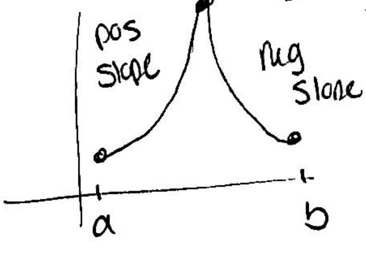
If f is differentiable from (a,b) and f(a) = f(b), conditions that must be met 3 stand

then there must be at least 1 c value such that a<c<b (exist!!) where f'(c)=0. → conclusion

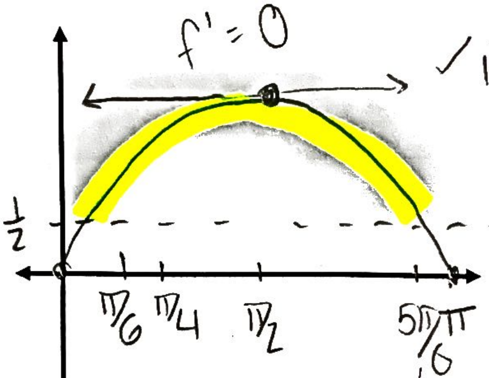


Existence Theorem

not Diff !!



Ex. 1: Determine whether the function $f(x) = \sin x$ over $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ satisfies the conditions of Rolle's Theorem. If f does, then find the guaranteed point(s).



conditions

✓ 1) f is differentiable : $f' = \cos x \rightarrow$ no concerns

2) $f(a) = f(b)$ $\quad f(\pi/6) = f(5\pi/6)$

$\sin \pi/6 = \sin 5\pi/6$

$\frac{1}{2} = \frac{1}{2}$ ✓

3) So Rolles theorem applies, guaranteeing the existence of a point c such that $f'(c) = 0$

4) $f' = \cos x = 0$

$\boxed{x = \pi/2}$

Ex. 2: Find the two zeroes of the function and determine whether Rolle's Theorem can be applied on the interval of the zeros. If Rolle's Theorem can be applied, find the values of c such that $f'(c) = 0$.

Calc

a) $f(x) = (x-3)(x+1)^2$

$0 = (x-3)(x+1)^2$

$x = -1, 3$

Does Rolles theorem apply $[-1, 3]$?

1) f is differentiable (polynomial!)

2) $f(3) = f(-1)$

$0 = 0$

condiditons ✓

$\therefore$ Rolles th applies, guaranteeing a c such that $f'(c) = 0$

$f'(x) = 1(x+1)^2 + 2(x+1)(x-3)$

$0 = (x+1)(x+1+2(x-3))$

$0 = (x+1)(3x-5)$

$x = -1, 5/3$

$x = 5/3$

b) $f(x) = x^{\frac{2}{3}} - 4$

$0 = x^{2/3} - 4$

$(4 = (x^{2/3}))^3$

$64 = x^2$

$x = \pm 8$

conditions

1) $f(-8) = f(8) = 0$

2) Is f differentiable?

$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$

f is not differentiable at $x = 0$ so Rolles theorem doesnt apply

Mean Value Th

Rolles Th

guarantees any slope, not just 0

The Mean Value Theorem:

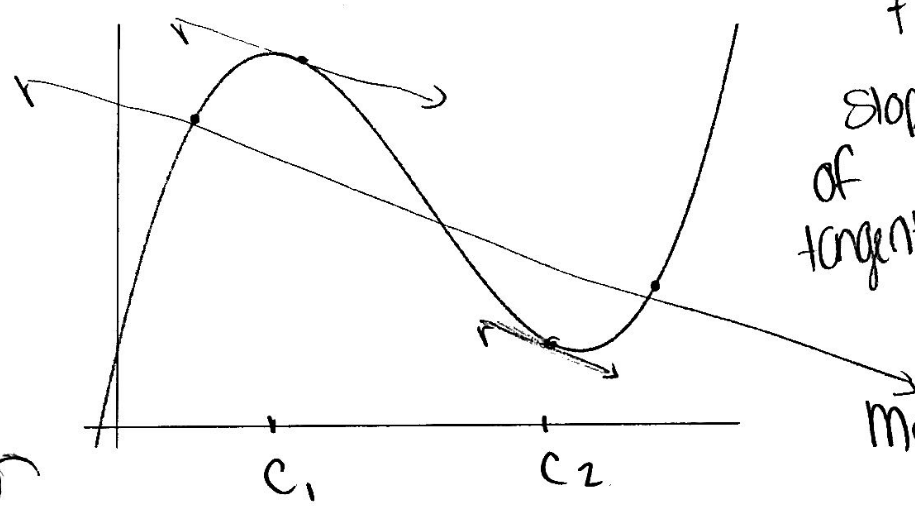
If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number (or multiple) c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

condition

conclusion

In other words, if f is continuous and differentiable, then there is some (or many) number(s) c in (a, b) such that the instantanous rate of change is equal to the average rate of change.

Additionally, Extreme Value, Rolle's and the Mean Value Theorem are <u>**existence**</u> theorems. This means there is **guaranteed** to exist at least one point where the conclusion is met, but you may **not** know where.



$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

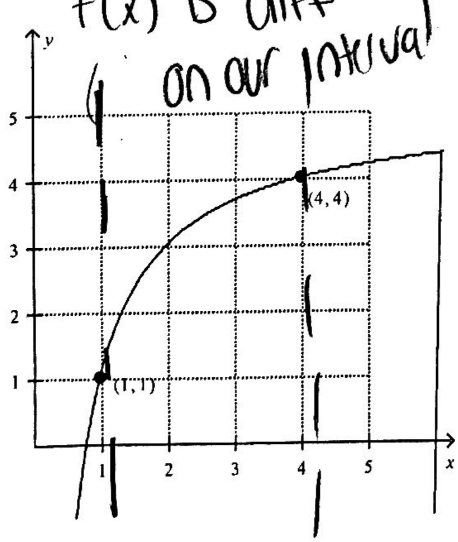
slope of tangent line

slope of secant line

$$m_{sec} = \frac{f(b)-f(a)}{b-a}$$

Ex. 3: Given $f(x) = 5 - \frac{4}{x}$, find all values of c in the open interval $(1,4)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$, if they exist.

$f(x)$ is diff on our interval



Conditions of MVT

1) Differentiable $x \neq 0$ but that is outside our interval

$$f' = \frac{4}{x^2}$$

✓ Mean Value theorem applies

$$f'(c) = \frac{f(4)-f(1)}{4-1}$$

$$f'(c) = \frac{4-1}{3} = 1$$

$$\frac{4}{c^2} = 1 \rightarrow c^2 = 4$$

$c = \cancel{\pm}2$ only pos. 2 on interval

$$\frac{f'(2)-f'(-1)}{2-{-1}}$$

Ex. 4: 2019 New FRQ

Consider the function $f'(x) = x^k - 3x^2 + 1$ where k is a nonzero constant.

a) Let $k = 2$. Explain why there must be a value c, for $-1 < c < 2$, such that $f''(c) = -2$.

conditions

differentiable (polynomial)

$f'(x) = x^2 - 3x^2 + 1$

$= -2x^2 + 1$

$f'(-1) = -1$

$f'(2) = -7$

$\frac{-7-{-1}}{2-{-1}} = f''(c)$

MVT guarantees a c such that $\frac{-6}{3} = -2 = f''(c)$

b) Let $k = 3$. Explain why there must be a value r, for $2 < r < 4$, such that $f'(r) = 0$.

$f'(x) = x^3 - 3x^2 + 1$

$f'(2) = 8 - 12 + 1 = -3$

$f'(4) = 64 - 48 + 1 = 17$

Since the function is continuous with $f'(2) < 0 < f'(4)$ there must exist a point c on $2 < c < 4$ such that $f'(c) = 0$ by the Intermediate Value Th

c) Let $k = 4$. Write an equation for the line $\underbrace{\text{tangent}}_{\text{deriv.}}$ to the $\underbrace{\text{graph of } f'}_{\text{of } f'}$ at the point whose x-coordinate is 2.

$f' = x^4 - 3x^2 + 1$

$f'' = 4x^3 - 6x$

$f''(2) = 4(8) - 12 = 20$

$\rightarrow f'(2) = 16 - 12 + 1$

$f'(2) = 5$

$$\boxed{y - 5 = 20(x - 2)}$$

d) Let $k = 6$. Find the maximum and $\underbrace{\text{minimum value of } f'}_{}$ on the interval [0,2].

use f'' for critical pts

$f'(x) = x^6 - 3x^2 + 1$

$f''(x) = 6x^5 - 6x = 0$

$6x(x^4 - 1) = 0$

$x = 0, -1, 1$

$f'(0) = 1$

$f'(1) = -1$ minimum value

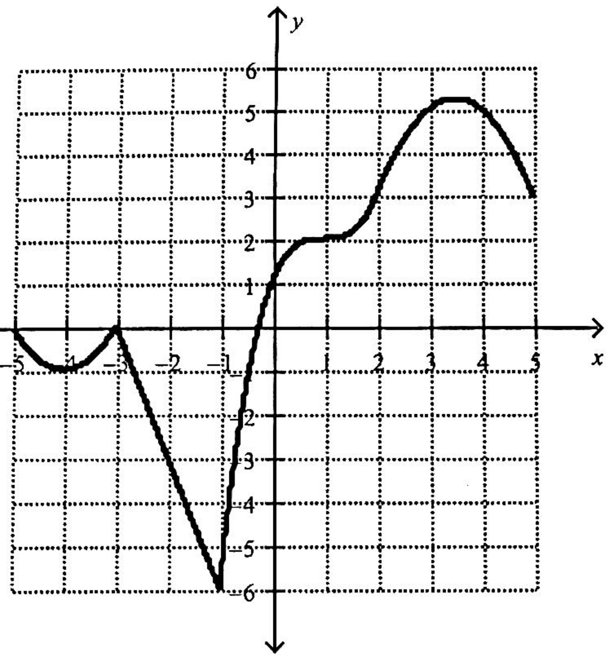
$f'(2) = 53$ maximum value

Extreme value theorem guarantees

AP Calculus I
Notes 4.3
Increasing/ Decreasing Functions and the First Derivative Test

Exploration

Examine the graph of $f(x)$ below:



1. On which intervals is $f(x)$ increasing?

$(-4,-3), (-1,3.5)$

2. On which intervals is $f(x)$ decreasing?

$(-5,-4), (-3,-1), (3.5,5)$

3. What separates these intervals?

critical values
(where $f' = 0$ or undefined)

4. What is true about the slopes of the tangent lines at each value of x in the intervals in which the function is increasing?

$f' > 0$

5. What is true about the slopes of the tangent lines at each value of x in the intervals in which the function is decreasing?

$f' < 0$

Theorem – Test for Increasing and Decreasing Functions

Let f be a function that is continuous on the interval $[a,b]$ and differentiable on the open interval (a,b).

1. If $f'(x) > 0$ for all x in (a,b), then f is increasing on $[a,b]$.
2. If $f'(x) < 0$ for all x in (a,b), then f is decreasing on $[a,b]$.
3. If $f'(x) = 0$ for all x in (a,b), then f is constant on $[a,b]$.

Guidelines for Finding Intervals on Which a Function is Increasing or Decreasing

Let f be continuous on the interval (a, b). To find the intervals on which f is increasing or decreasing:

1) Find the critical values ($f' = 0$ or und)

2) Create a sign chart. Pick test points between critical pts.

3) If $f' > 0$, f is increasing. If $f' < 0$, f is decreasing there

Ex. 1: Find the intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

$f'(x) = 3x^2 - 3x = 0$

$3x(x-1) = 0$ f' never undefined

$x = 0, 1$

plug test points into factored form to get signs

f': + (−1) | 0 | − ($\frac{1}{2}$) | 1 | + (2)

max at 0, min at 1

- f is increasing on $(-\infty, 0), (1, \infty)$ b/c $f' > 0$
- f is decreasing on $(0, 1)$ b/c $f' < 0$

Ex. 2: Find the intervals on which $f(x) = \cos^3 x$ is increasing or decreasing from $[0, 2\pi)$.

$f'(x) = 3\cos^2(x) \cdot -\sin x$

$-3\cos^2 x \sin x = 0$

$x = \pi/2, 3\pi/2$ $x = 0, \pi$

[0 ($\pi/4$) $\pi/2$ ($3\pi/4$) π ($\frac{5\pi}{4}$) $3\pi/2$ $\frac{7\pi}{4}$ 2π)

signs: − − + +

- f is increasing on $(\pi, \frac{3\pi}{2}), (\frac{3\pi}{2}, 2\pi)$ because $f' > 0$
- f is decreasing on $(0, \pi/2), (\pi/2, \pi)$ because $f' < 0$

Ex. 3: The derivative of a function f is given by $f'(x) = e^{\sin x} - \cos x - 1$ for $0 < x < 9$. On what intervals is f decreasing? $\longrightarrow$ f' is negative (f' is below x-axis)

(A) $0 < x < 0.633$ and $4.115 < x < 6.916$

(B) $0 < x < 1.947$ and $5.744 < x < 8.230$

(C) $0.633 < x < 4.115$ and $6.916 < x < 9$

(D) $1.947 < x < 5.744$ and $8.230 < x < 9$

Theorem – The First Derivative Test

If c is a critical number of a function f, then $f(c)$ can be classified as follows:

f changes decreasing to increasing

1. If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a relative min of f.

2. If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a relative maximum of f.

f changes from increasing to decreasing

Ex. 4: Find the relative extrema of $f(x) = xe^{2x}$.

Crit. #s

$f'(x) = 1e^{2x} + 2xe^{2x} = 0$

* f' never undef *

$-\quad +\quad f'$

$-1 \quad -\frac{1}{2} \quad 0$

$e^{2x}(1+2x) = 0$

e never $= 0$

$x = -\frac{1}{2}$

Since f' changes from neg. to positive, f changes from increasing to decreasing meaning we have a local min

Ex. 5: Find the relative extrema of $h(\theta) = (\theta - 2)^3(\theta + 3)^2$.

$h'(\theta) = 3(\theta-2)^2(\theta+3)^2 + 2(\theta+3)'(\theta-2)^3$

$0 = (\theta-2)^2(\theta+3)(3\theta+9+2\theta-4)$

$0 = (\theta-2)^2(\theta+3)(5\theta-5)$

$\theta = 2, -3, 1$

$+ \quad - \quad + \quad +\quad f'$

$-3 \quad 1 \quad 2$

• rel. max at $x=-3$ b/c f' changes from pos to neg

• rel min at $x=1$ b/c f' changes negative to positive

Ex. 6: Let $g(x)$ be a continuous function, where $g < 0$ for all values of x. Find the relative extrema of the function f given $f'(x) = \frac{4g(x)}{\sqrt[3]{x^2-4}}$.

$f' \neq 0$ since $g < 0$

$4g(x) = 0$

$g(x) \neq 0$

f' undef

$\sqrt[3]{x^2-4} = 0$

$x^2 = 4$

$x = \pm 2$

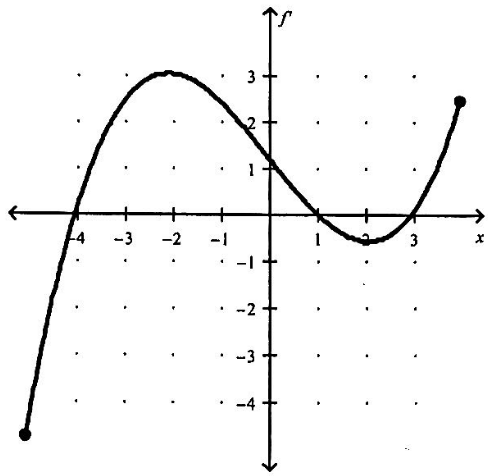
$- \quad + \quad - \quad f'$

$-2 \quad 2$

• relative max at $x=2$ b/c f' changes from positive to negative

• relative max at $x=-2$ b/c f' changes from negative to positive

Ex. 7: Given a graph of f':



a) Is f continuous? Is f differentiable? Justify your answer.

Is f' continuous

f is differentiable b/c f' is continuous, so f is also continuous

b) Over what interval is f increasing/decreasing? Justify your answer.

(-4,1), (3,4) b/c f'>0

(-5,-4), (1,3) because f'<0

c) At what value(s) of x does f have relative mins/maxs? Justify your answer.

$\boxed{x=-4,3}$ f' changes from positive to negative

f' changes from negative to pos. $\boxed{x=1}$

Ex. 8: The following table shows certain values of f', the derivative of the differentiable function f.

x	-2	0	1	3	5	7
f'	4	2	0	-4	3	-5

Which of the following statements are true about the graph of f? Justify your answers.

differentiable means derivative exists

a) There is at least one value of c, such that $-2 < c < 7$, where $f'(c)$ does not exist.

b) f is increasing from $-2 < x < 1$. f' has a positive slope

c) There is a relative maximum of f at $x = 1$. f' changes from pos to neg

d) There is a relative minimum of f at some value $x = c$ such that $3 < c < 5$.

The IVT guarantees a value c such that f'(c)=0 since f(3)<0<f(5) & the deriv changes from - to positive @ that c value, so we know c is a rel. min.

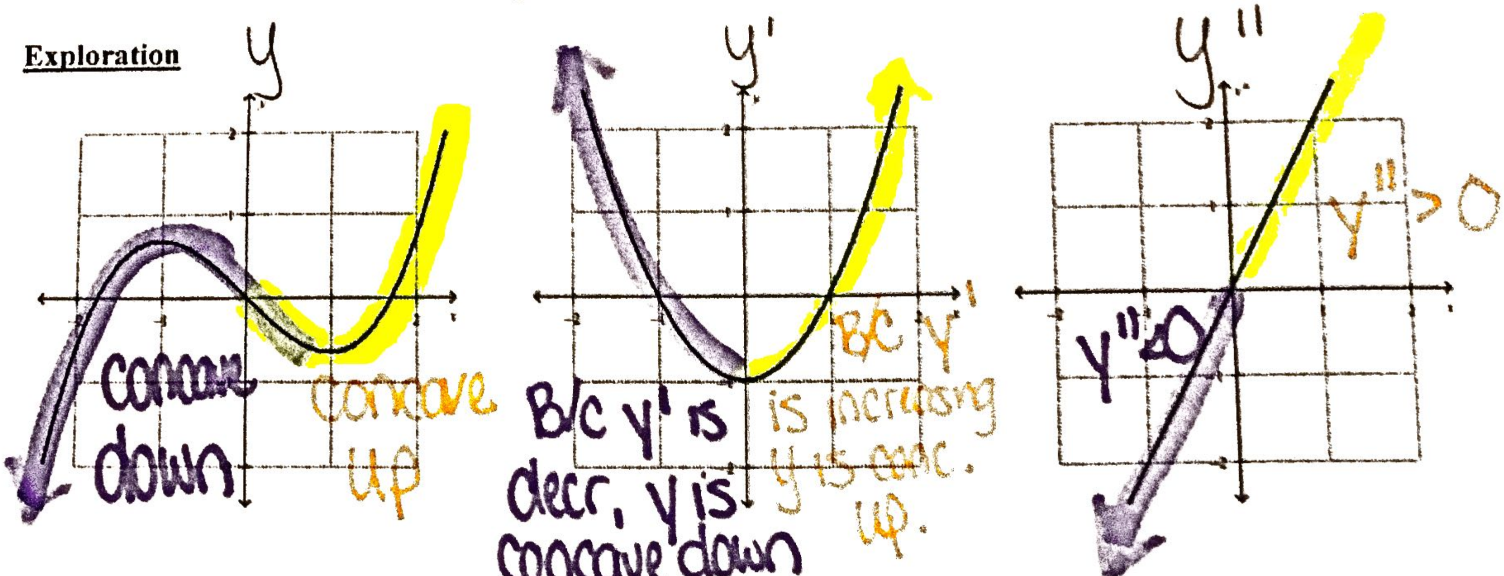
AP Calculus I
Notes 4.4
Concavity and the Second Derivative Test

Definitions of Concavity

Let f be differentiable, the graph of f is

- **Concave upward** if f' is increasing on the interval (the slopes are increasing).
- **Concave downward** if f' is decreasing on the interval (the slopes are decreasing).

Exploration



1. On what interval is the graph of f concave up? $(0, \infty)$

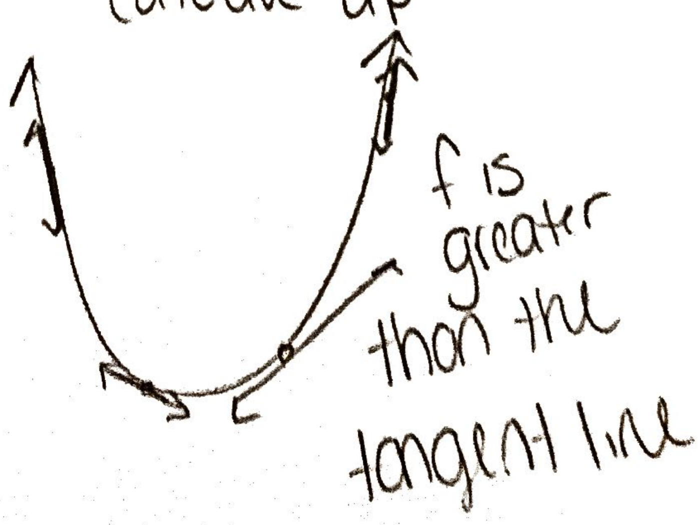
2. On what interval is the graph of f concave down? $(-\infty, 0)$

Think of concave up as a ___cup___ and concave down as a ___frown___.

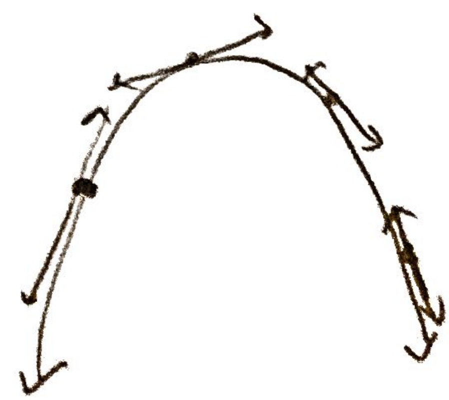
1. Let f be differentiable at c. If the graph of f is **concave upward** at $(c, f(c))$, the graph of f lies *above* the tangent line at $(c, f(c))$ on some open interval containing c.

2. Let f be differentiable at c. If the graph of f is **concave downward** at $(c, f(c))$, the graph of f lies *below* the tangent line at $(c, f(c))$ on some open interval containing c.

concave up



concave down



Theorem – Test for Concavity

1. Since concave up means f' is increasing and increasing means positive slope / derivative

the deriv of f' is positive $(f')' > 0 \rightarrow f'' > 0$ or f'' above x-axis

2. Since concave down means f' is decreasing and decreasing means negative slope / derivative

$(f')' < 0$, $f'' < 0$, f'' is below x axis

Three definitions of Concave up (and down):

1) f looks like a cup 2) f' is increasing 3) $f'' > 0$

Ex. 1: Determine the intervals on which the graph of $f(x) = e^{-x^2/2}$ is concave upward or downward.

a number line / sign chart will help us tell the sign

$f' = -xe^{-\frac{1}{2}x^2}$

$f''(x) = -1e^{-\frac{1}{2}x^2} + -xe^{-\frac{1}{2}x^2} \cdot -x$

$f''(x) = -e^{-\frac{1}{2}x^2} + x^2 e^{-\frac{1}{2}x^2} = 0$

never $= 0$ ← $e^{-\frac{1}{2}x^2}(-1 + x^2) = 0$ → $x^2 = 1$, $x = \pm 1$

$f'' > 0$ + $f'' < 0$ −   +

-2   -1   0   1

- f is concave up $(-\infty, -1), (1, \infty)$ b/c $f'' > 0$
- f is concave down on $(-1, 1)$ b/c $f'' < 0$

Ex. 2: Determine the open intervals in which the graph of $f(x) = 2\ln|x^2 + 1| + 3\arctan x$ is concave upward or concave downward.

$f'(x) = 2\left(\frac{2x}{x^2+1}\right) + \frac{3}{1+x^2}$

$= \frac{4x}{x^2+1} + \frac{3}{x^2+1}$

$f'(x) = \frac{4x+3}{x^2+1}$

$f''(x) = \frac{(x^2+1)(4) - (4x+3)(2x)}{(x^2+1)^2}$

$f''(x) = \frac{4x^2 + 4 - 8x^2 - 6x}{(x^2+1)^2} = 0$

$(-4x^2 - 6x - 4) = 0$

$-2(2x^2 + 3x - 2) = 0$

$-2(2x-1)(x+2) = 0$

$x = \frac{1}{2}, -2$

−   +   −

-3   -2   0   ½

conc. up $(-2, \frac{1}{2})$ b/c $f'' > 0$

conc. down $(-\infty, -2), (\frac{1}{2}, \infty)$ b/c $f'' < 0$

$f'' \to$ points of inflection

Theorem – Points of Inflection $f' \to$ relative extrema

A point of inflection occurs when the concavity changes from upward or downward or vice versa. So, the point of inflection occurs when f'' changes from positive to negative or vice versa. Also, the point of inflection occurs when f' changes from increasing to decreasing or vice versa. If $(c, f(c))$ is a point of inflection of a graph of f, then either $f''(c) = 0$ or $f''(c)$ is undefined.

Ex. 3: Determine the points of inflection and discuss the concavity of the graph of $f(x) = x^5 - 5x^4 + x$.

f'' changes sign

f'' never undef

$f'(x) = 5x^4 - 20x^3 + 1$

$f''(x) = 20x^3 - 60x^2 = 0$

$20x^2(x-3) = 0$

$x = 0, 3$

f'': $-$ $-$ $+$

(-2) 0 (1) 3 (5)

• point of inflection at $x=3$ b/c f'' changes $-$ to $+$

• concave up $(3, \infty)$ b/c $f'' > 0$

• concave down $(-\infty, 0), (0, 3)$ b/c $f'' < 0$

Ex. 4: Find the points of inflection of $f(x)$, where $x > -1$, given $f'(x) = x\sqrt{x+1}$.

$f'(x) = 1\sqrt{x+1} + \frac{1}{2}(x+1)^{-1/2} \cdot x$

$f'(x) = \sqrt{x+1} + \dfrac{x}{2\sqrt{x+1}}$

$f' = \dfrac{2(x+1)}{2\sqrt{x+1}} + \dfrac{x}{2\sqrt{x+1}}$

$f' = \dfrac{3x+1}{2\sqrt{x+1}} = 0 \to 3x = -1$

$x = -1/3$

$\to$ only undefined @ $x = -1$ which is excl.

$-$ $+$

(-1) $-1/3$ (0)

f has a point of inflection at $x = -1/3$ b/c f'' changes from negative to positive.

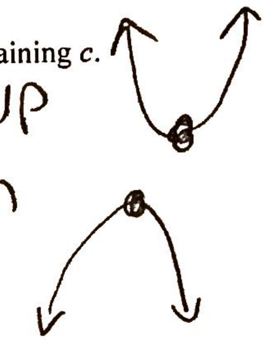
First Deriv Test : f' $+ \wedge - \vee +$ at c_1, c_2

Concavity also allows us to determine relative extrema of a function.

Theorem – The Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an interval containing c.

1. If $f''(c) > 0$, then $f(c)$ is a relative min of f. f is concave up
2. If $f''(c) < 0$, then $f(c)$ is a relative max of f. f is concave down
3. If $f''(c) = 0$, the test fails. In such cases, use the First Derivative Test.



Ex. 5: Find the relative extrema of the function $B(t) = -3t^4 + 6t^2$ using the second derivative test.

2nd derivative test

← never und.

1) Find critical values : $B' = -12t^3 + 12t = 0$

$-12t(t^2-1) = 0$

2) $B'' = -36t^2 + 12$

← $t = 0, \pm 1$

$B''(0) = 12 > 0$ conc. up → rel. min.

$B''(-1) = -24 < 0$ conc down → rel max

$B''(1) = -24 < 0$ conc down

1st deriv test

$+ \quad - \quad +$ at -1, 0

$x=0$ rel min

$x=\pm 1$ rel max ✓

Ex. 6: For all x in the closed interval [2,5], the function f has a positive first derivative and a negative second derivative. Which of the following could be the table of values for f?

x	$f(x)$
2	7
3	9
4	12
5	16

+2, +3, +4

$f' > 0$

$f'' > 0$

f' is positive and increasing

x	$f(x)$
2	7
3	9
4	11
5	13

$f' > 0$

$f'' = 0$

x	$f(x)$
2	7
3	11
4	14
5	16

+4, +3, +2

$f' > 0$

$f'' < 0$

f' is decreasing

x	$f(x)$
2	16
3	14
4	11
5	7

neg 1st

x	$f(x)$
2	16
3	13
4	10
5	7

neg 1st

AP Calculus I
Graph of f'

Quick Definition Review

A function $f(x)$ is increasing when $f'(x)$ positive.

- If the graph shown is $f(x)$, graph is going up.
- If the graph shown is $f'(x)$, graph is above x-axis.

A function $f(x)$ is decreasing when $f'(x)$ negative.

- If the graph shown is $f(x)$, graph is going down.
- If the graph shown is $f'(x)$, graph is below x-axis.

A function $f(x)$ has a local maximum when $f'(x)$ changes from positive to neg

- If the graph shown is $f(x)$, a hill.
- If the graph shown is $f'(x)$, crosses the x axis from above → below

A function $f(x)$ has a local minimum when $f'(x)$ changes from negative to pos

- If the graph shown is $f(x)$, a valley.
- If the graph shown is $f'(x)$, crosses the x-axis from below to above

A function $f(x)$ has a point of inflection when $f''(x)$ changes sign.

- If the graph shown is $f(x)$, changes conc. up to conc down or vice versa
- If the graph shown is $f'(x)$, changes from increasing to decreasing or vice versa
- If the graph shown is $f''(x)$, crosses the x-axis.

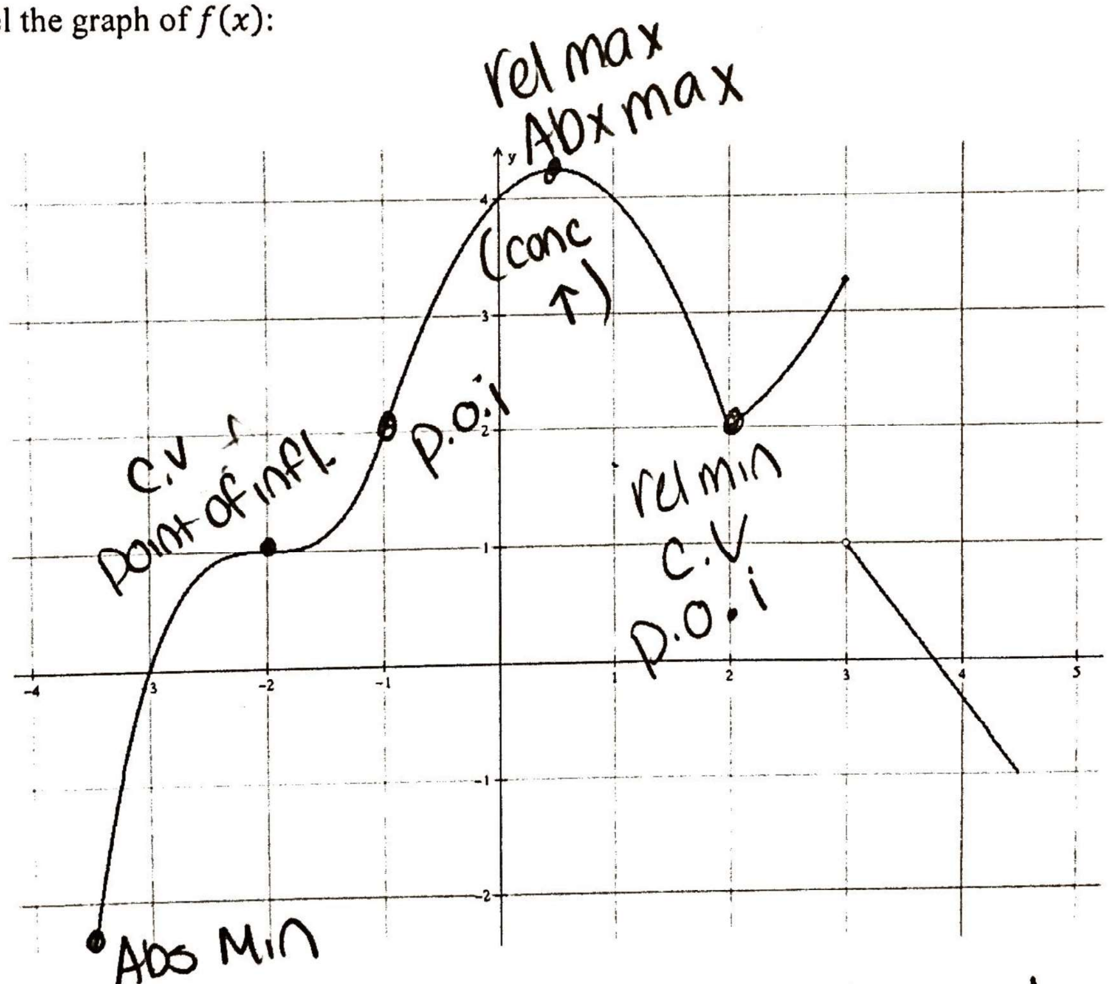
A function $f(x)$ is concave up when $f''(x)$ is positive.

- If the graph shown is $f(x)$, looks like a cup / curve above tangent line
- If the graph shown is $f'(x)$, graph is increasing.
- If the graph shown is $f''(x)$, above the x-axis.

A function $f(x)$ is concave down when $f''(x)$ is negative.

- If the graph shown is $f(x)$, looks like a frown / curve is below tengent line
- If the graph shown is $f'(x)$, graph is decreasing.
- If the graph shown is $f''(x)$, below the x-axis.

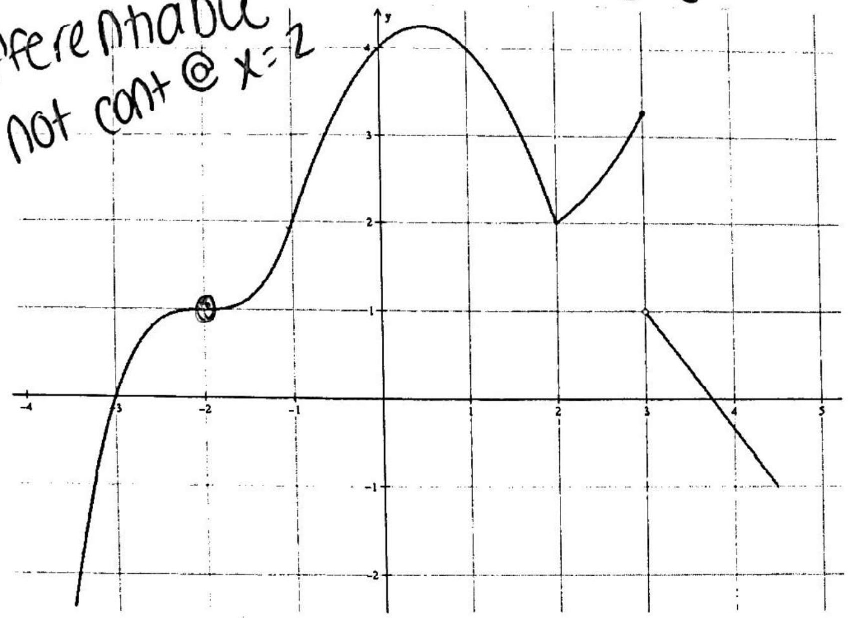
Ex. 1: Label the graph of $f(x)$:



concavity: down | up + down —— up | zero |

Ex. 2: Now, let's make the same graph a graph of $f'(x)$. every y-value on the graph is a slope

* f' isnt differentiable @ x=2, so f'' not cont @ x=2



a) Find the value(s) of x for which $f(x)$ is not differentiable. Is $f(x)$ continuous there?

f'(x) not continuous

Maybe!

X=3 → Abrupt change in sl

b) Find $f(-2)$, $f'(-2)$, and $f''(-2)$, if possible.

Dont Knot

f'(-2)=1 y value of f' @ x=-2

f''(-2)=0 Slope of f' @ x=-2

c) Find the interval(s) in which $f(x)$ is increasing/decreasing. Justify your answer.

f incr: f'>0 (f' is above x-axis)

(-3, 3.75)

f'<0 or f' is below x-axis

(-3.5, -3), (3.75, 4.5)

d) Find the interval(s) in which $f(x)$ is concave up/concave down. Justify your answer.

f''>0 → f' increasing

(-3.5, .5), (2, 3)

f''<0 f' is decreasing

(.5, 2), (3, 4.5)

e) Find all relative extrema and points of inflection of $f(x)$. Justify your answer.

rel min: X=-3 b/c f' changes from neg to pos

relative max: X=3.75 b/c f' changes from pos to neg

POI: X=.5, 2, 3 b/c the slope of f' changes sign

$f'>0$ $f'<0$ $f''>0$

Ex. 3: Given the function $f(x) = \frac{1}{3}x^3 - 4x$, find the intervals of increasing, decreasing, concave up, $f''<0$ concave down, relative maxima, relative minima, and points of inflection. Then, use the graph to verify all your answers.

$f'(x) = x^2 - 4 = 0$

$x^2 = 4$

$x = \pm 2$

f': $+$ $\wedge$ $-$ $\vee$ $+$ at (-3) -2 (0) 2 (3)

$f''(x) = 2x = 0$

$x = 0$ f'': $-$ $+$ at 0

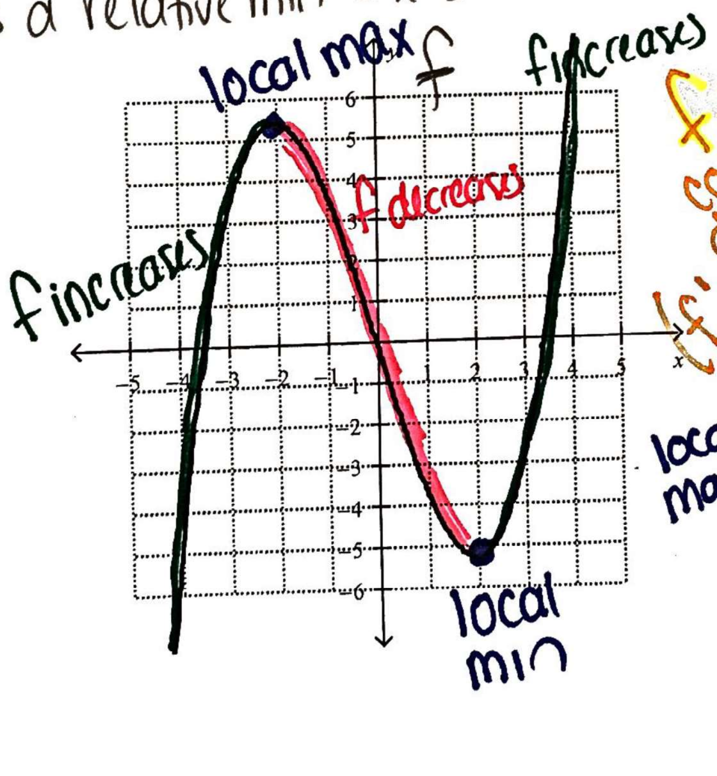
- f has a point of inflection at $x=0$ because f'' changes sign
- f is concave up on $(0,\infty)$ b/c $f''>0$
- f is concave down on $(-\infty,0)$ b/c $f''<0$

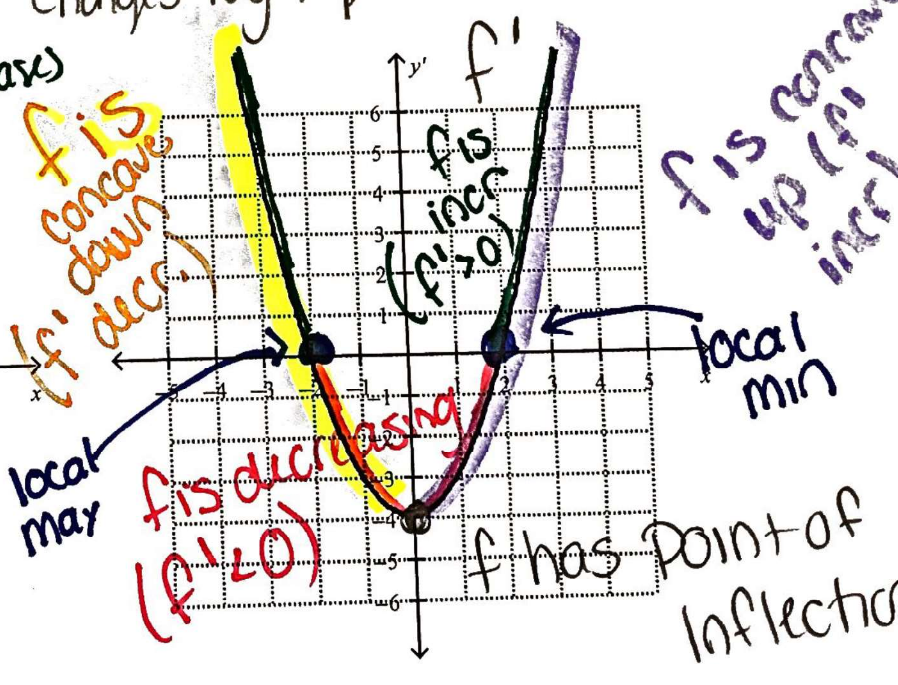
f is increasing $(-\infty,-2)\cup(2,\infty)$ b/c $f'>0$

f is decreasing $(-2,2)$ b/c $f'<0$

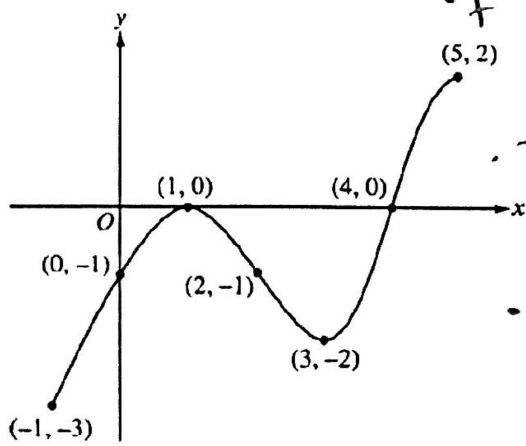
f has a relative max @ $x=-2$ b/c f' changes pos. to neg.

f has a relative min @ $x=2$ b/c f' changes neg to pos.





Ex. 4:



Graph of f'

- f decreases from $x=-1$ to $x=4$ b/c $f'<0$.
- Then f increases $(4,5)$ b/c $f'>0$
- So → Absolute min at $x=4$

The figure above shows the graph of f', the derivative of the function f, on the closed interval $-1 \leq x \leq 5$. The graph of f' has horizontal tangent lines at $x = 1$ and $x = 3$. The function f is twice differentiable with $\boxed{f(2) = 6}$.

a) Find the x –coordinate of each of the points of inflection of the graph of $f(x)$. Give a reason for your answer.

$x = 1, 3$ Where the slope of f' changes sign

b) Is $f(4) < 6$? Justify your response.

$f(2) = 6$ ↳ $f(x)$ decreases on $(2,4)$ b/c $f' < 0$, so yes $f(4) < 6$

c) At what value of x does f attain its minimum value on the closed interval $-1 \leq x \leq 5$? Show the analysis that leads to your answer.

Candidates: critical pts (where $f' = 0$ or undef)

$x = 1, 4$

endpoints $x = -1, 5$

d) Let g be the function defined by $g(x) = xf(-2x) + \ln |f'(-3x)|$. Find an equation for the line tangent to the graph of g at $x = -1$.

$$g'(x) = 1\, f(-2x) + f'(-2x) \cdot -2 \cdot x + \frac{1}{f'(-3x)} \cdot f''(-3x) \cdot -3$$

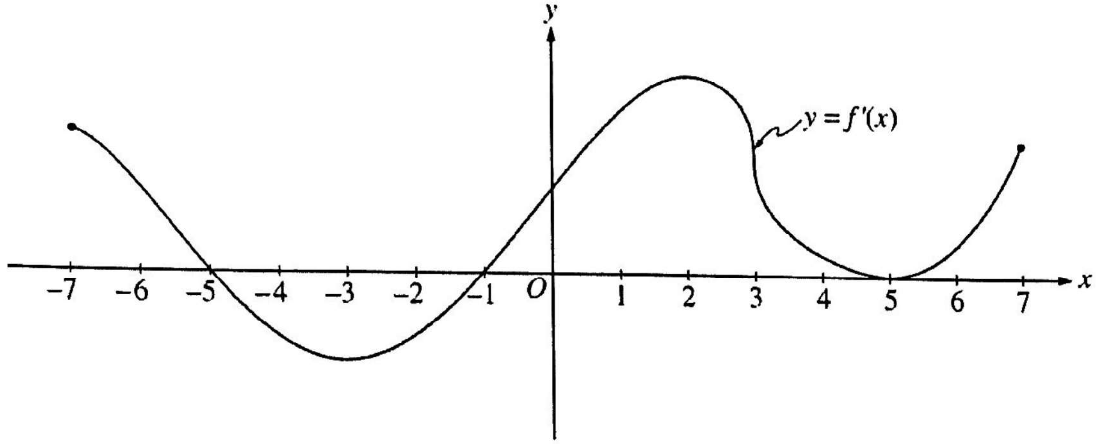
$$g(-1) = 1\, f(2) + \ln(f'(3)) = 6 + \ln(2)$$

$$g'(-1) = f(2) + 2 \cdot f'(2) + {}^{-}3 \cdot \frac{f''(3)}{f'(3)}$$

$$g'(-1) = 6 + 2(-1) + \cancel{\frac{-3 \cdot 0}{}} = 4$$

$$y - (6 + \ln 2) = 4(x + 1)$$

Ex. 5:



The figure above shows the graph of f', the derivative of the function f, for $-7 \leq x \leq 7$. The graph of f' has horizontal tangent lines at $x = -3$, $x = 2$ and $x = 5$, and a vertical tangent line at $x = 3$.

a) Find all values of x, for $-7 < x < 7$, at which f attains a relative min. Justify your answer.

rel min → f' changes from negative (below x) to positive (above x) @ $\boxed{x=-1}$

b) Find all values of x, for $-7 < x < 7$, at which f attains a relative max. Justify your answer.

rel max → f' changes from positive to negative @ $x=-5$

c) Find all values of x, for $-7 < x < 7$, at which $f''(x) < 0$.

$f''<0 \rightarrow f'(x)$ is decreasing $(-7,-3),(2,3),(3,5)$

d) At which value of x, for $-7 \leq x \leq 7$, does f attains its absolute maximum. Justify your answer.

candidates: crit values & endpts: $x=-7,-5,-1,7$ → plug into f (but no

f increases (briefly) from $(-7,-5)$ b/c $f'>0$. the, f decreas

from $(-5,-1)$ b/c $f'<0$. finally f increases from

$(-1,5),(5,7)$ b/c $f'>0$. Since f increases more

than it decreases, the absolute max is at the end:

$\boxed{x=7}$

Ex. 6: The function f is continuous on the closed interval $[-3,3]$ such that $f(-3) = 4$ and $f(3) = 1$. The functions f' and f'' have the properties given in the table below.

x	$-3 < x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x < 3$
$f'(x)$	Positive	Fails to Exist	Negative	0	Negative
$f''(x)$	Positive	Fails to Exist	Positive	0	Negative

a) At what value of x does f attain its relative minimum and maximum value, if any?

rel min → f' changes from negative to positive → none

rel max → f' changes from positive to negative → x = -1

VT

VT

olles

IVT

b) Is f guaranteed a value of c, where $-3 \le c \le 3$, such that $f(c) \ge f(x)$? If so, find the value of $x = c$, if possible. Show the analysis that leads to your answer. abs max

yes, the extreme value theorem guarantees the existence, since f(x) is continuous on [-3,3]

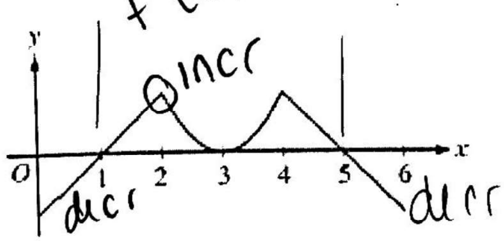
Value of c : f(x) increases (-3,-1) b/c f'>0. After it either decreases or has a slope of 0. This makes the absolute max x=-

c) Find the x –coordinate of each point of inflection of the graph of f. Justify your answer.

at x = 1 f(x) has a p.o.i b/c f'' = 0 and changes sign

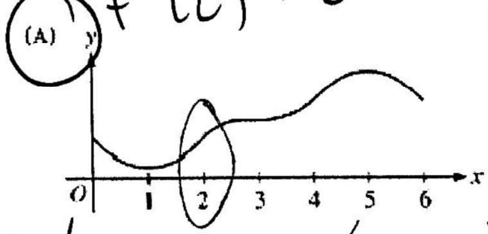
Ex. 7: The graph of f' is shown below. Which of the following could be the graph of f?

f'(2)=2

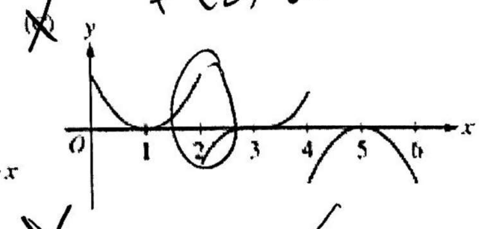


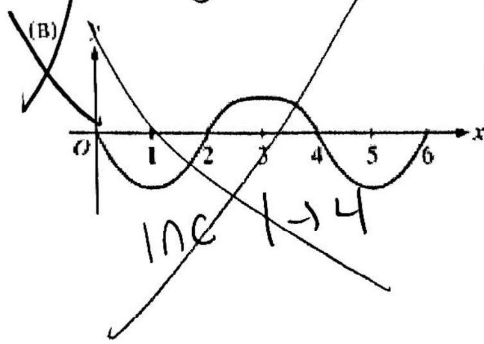
Graph of f'

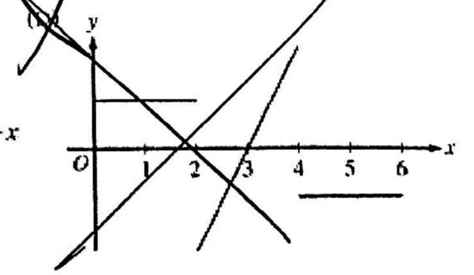
f'(2) > 0



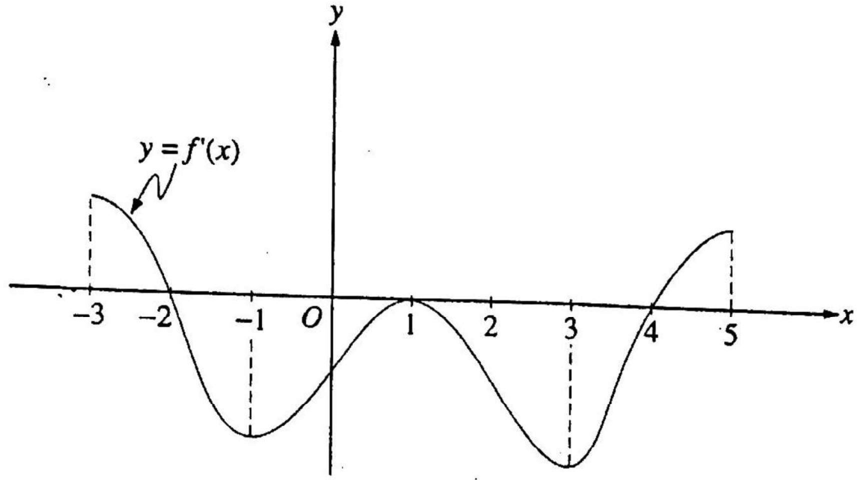
f'(2) undefined







Ex. 8:



The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $-3 < x < 5$.

a) For what values of x does f have a critical value on $-3 < x < 5$? Determine whether each is a relative maximum, relative minimum or neither. Justify your answer.

- Critical values @ $x = -2, 1, 4$ b/c $f' = 0$
- $x = -2$ is a local max b/c f' changes from positive to negative
- $x = 4$ is a local max b/c f' changes from negative to positive

b) On what intervals is the graph of f concave upward? Justify your answer.

f is concave up $(-1, 1), (3, 5)$ b/c f' is increasing

c) Given $f(4) = \pi$ and $g(x) = -2e^{f(x)}$, write the equation of the tangent line to g at $x = 4$.

$g(4) = -2e^{f(4)}$

$= -2e^{\pi}$

$\boxed{y = 2e^{\pi}}$

$g'(x) = -2e^{f(x)} \cdot f'(x)$

$g'(4) = -2e^{f(4)} \cdot f'(4)$

$g'(4) = 2e^{\pi} \cdot 0 = 0$

e) Given g as defined above, for what value(s) of x does g have a relative maximum? Justify your answer.

g has a relative max when g' changes from positive to negative at $x = 4$

$+ \quad - $

4

$g'(x) = -2e^{f(x)} \cdot f'(x)$

neg pos ?

Optimization

Monday, November 02, 2020 8:37 AM

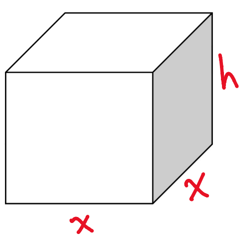
AP Calculus I
Notes 4.7
Optimization

One of the most common applications of calculus involves determining maximum and minimum values.

Problem-Solving Strategy for Applied Minimum and Maximum Problems

1) Assign symbols to all *given* quantities and quantities *to be determined*. When feasible, make a sketch.
2) Write a **primary equation** for the quantity that is to be maximized (or minimized).
3) Reduce the primary equation to one having a *single independent variable*. This may involve the use of **secondary equations** relating the independent variables of the primary equation.
4) Differentiate the equation and then solve for the *critical values*
5) Determine the desired maximum or minimum value depending on the situation.

Ex. 1: A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?



$SA = x^2 + 4xh$

① $SA = 108\text{ in}^2 = x^2 + 4xh$ ← secondary

② Maximize: Volume $= x^2 h$ ← primary

③ Solve secondary for h and sub in to primary to rewrite primary in a single variable

$108 = x^2 + 4xh$

$4xh = 108 - x^2$

$h = \dfrac{108 - x^2}{4x}, \; x \neq 0$

$V = x^2 h = x^2\left(\dfrac{108 - x^2}{4x}\right) = \dfrac{108x^2 - x^4}{4x} = \dfrac{108x^2}{4x} - \dfrac{x^4}{4x}$

$V = 27x - \frac{1}{4}x^3$

④ $\dfrac{dV}{dx} = 27 - \frac{3}{4}x^2$

$0 = 27 - \frac{3}{4}x^2$

$\frac{4}{3} \cdot \frac{3}{4}x^2 = 27 \cdot \frac{4}{3}$

$x^2 = 36$

$x = \pm 6$

⑤ V': + | − on (1) 6 (10)

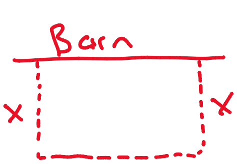
So x = 6 will give the absolute maximum for the volume.

$h = \dfrac{108 - x^2}{4x}$

$\dfrac{108 - (6)^2}{4(6)} = \dfrac{72}{24} = 3$

The box of maximum volume will have length and width of 6 in. and height of 3 in.

Ex. 2: A rectangular field, bounded on one side by a building, is to be fenced in on the other three sides. If 3000 feet of fence is to be used, find the dimensions of the largest field that can be fenced in.



① $3000 = x + y + x \rightarrow 3000 = 2x + y$ ← secondary equation

$y = 3000 - 2x$

② $A = xy$ ← maximize

$\to y = 3000 - 2x$

② $A = xy \leftarrow$ maximize

③ $A = x(3000-2x) = 3000x - 2x^2$

④ $\frac{dA}{dx} = 3000 - 4x$

$0 = 3000 - 4x$

$4x = 3000$

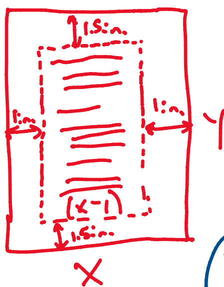
$x = 750$ ft.

$x = 750$ ft.

$y = 3000 - 2(750) = 1500$ ft.

A': $+$ | $-$; (1) 750 (1000)

Ex. 3: A rectangular page is to contain 54 square inches of print. The margins at the top and bottom of the page are to be 1.5 inches, and the margins on the left and right are to be 1 inch. What should the dimensions of the page be so that the least amount of paper is used?



① $54 = (x-2)(y-3) \leftarrow$ secondary $\to \frac{54}{y-3} = x-2 \to x = \frac{54}{y-3} + 2$

② $A = xy \leftarrow$ primary

③ $A = \left(\frac{54}{y-3} + 2\right)y = \frac{54y}{y-3} + 2y$

④ $\frac{dA}{dy} = \frac{(y-3)(54) - (54y)(1)}{(y-3)^2} + 2$ crit. number at $y=3$ (makes A' undefined)

$0 = \frac{\cancel{54y} - 162 - \cancel{54y}}{(y-3)^2} + 2$

$-2 = \frac{-162}{(y-3)^2}$

$-2(y-3)^2 = -162$

$(y-3)^2 = 81$

$\to y - 3 = \pm 9$

$y-3 = 9$, $y = 12$ $y-3 = -9$, $\cancel{y = -6}$

⑤ A': $-$ | $-$ | $+$; (1) 3 (10) 12 (100)

$y = 12$ in.

$x = \frac{54}{12-3} + 2 = 6 + 2 = 8$ in.

Ex. 4: The product of two positive numbers is 200. Minimize the sum of the first and twice the second.

x - first
y - second

(1) $xy = 200 \xrightarrow{\text{secondary}} x = \frac{200}{y}$

(2) $S = x + 2y$

(3) $S = \frac{200}{y} + 2y$

$S = 200y^{-1} + 2y$

~~crit. num = 0~~ (4) $S' = -200y^{-2} + 2$

$0 = -\frac{200}{y^2} + 2$

$-2 = -\frac{200}{y^2}$

$-2y^2 = -200$

$y^2 = 100$

$y = \pm 10$

(5) S': $-$ | $+$; (1) 10 (100)

$x = \frac{200}{10} = 20$

Ex. 5: What is the radius of a cylindrical soda can with volume of 512 cubic inches that will use the minimum material? (Formulas for a cylinder: $V = \pi r^2 h$ and $SA = 2\pi r^2 + 2\pi r h$)

$512 = \pi r^2 h \longrightarrow h = \frac{512}{\pi r^2}$

$SA = 2\pi r^2 + 2\pi r h$

$= 2\pi r^2 + 2\cancel{\pi}\cancel{r}\left(\frac{512}{\cancel{\pi} r^{\cancel{2}}}\right)$

$SA = 2\pi r^2 + 1024r^{-1}$

crit# $r = 0$

$SA' = 4\pi r - 1024r^{-2}$

$0 = 4\pi r - \frac{1024}{r^2}$

$\frac{1024}{r^2} = 4\pi r$

$1024 = 4\pi r^3$

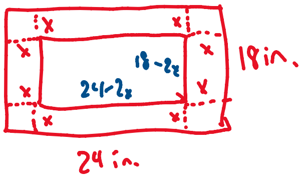
$\sqrt[3]{\frac{256}{\pi}} = \sqrt[3]{r^3}$

$r = \sqrt[3]{\frac{256}{\pi}} = 4\sqrt[3]{\frac{4}{\pi}}$

$-$ | $+$; 0 (1) $\sqrt[3]{\frac{256}{\pi}} \approx 4.335$ (10)

$r = \sqrt[3]{\frac{256}{\pi}} \approx 4.335\text{ in.}$

Ex. 6: Max wants to make a box with no lid from a rectangular sheet of cardboard that is 18 inches by 24 inches. The box is to be made by cutting a square of side x from each corner of the sheet and folding up the sides. Find the value of x that maximizes the volume of the box.



$h = x$

$l = 24 - 2x$

$w = 18 - 2x$

$V = (24-2x)(18-2x)\cdot x$

$V = (432 - 36x - 48x + 4x^2)x$

$V = 4x^3 - 84x^2 + 432x$

$V' = 12x^2 - 168x + 432$

$0 = 12(x^2 - 14x + 36)$

$x = \dfrac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(36)}}{2(1)}$

$= \dfrac{14 \pm \sqrt{52}}{2} = \dfrac{14 \pm 2\sqrt{13}}{2} = 7 \pm \sqrt{13}$

36
3 12
-3 -12
-6 -6
-4, -9

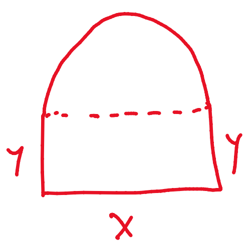
$+ \quad - \quad +$

(1) $7-\sqrt{13}$ (10) $\cancel{7+\sqrt{13}}$ (100)

≈ 3.394 $\quad \approx \cancel{10.606}$

$x = 7 - \sqrt{13}$ in.

Ex. 7: A Norman window has the shape of a rectangle with a semicircle on top. If the perimeter of the window is 30 feet, find the dimensions of the window that allows the greatest amount of light in.



$30 = 2y + x + \frac{1}{2}(\pi x)$

$2y = 30 - x - \frac{\pi}{2}x$

$y = 15 - \frac{1}{2}x - \frac{\pi}{4}x$

$\text{Area} = xy + \frac{1}{2}\left(\pi\left(\frac{1}{2}x\right)^2\right)$

$= xy + \frac{\pi}{8}x^2$

$= x\left(15 - \frac{1}{2}x - \frac{\pi}{4}x\right) + \frac{\pi}{8}x^2$

$= 15x - \frac{1}{2}x^2 - \frac{2\pi}{8}x^2 + \frac{\pi}{8}x^2$

$= 15x - \frac{1}{2}x^2 - \frac{\pi}{8}x^2$

$A' = 15 - x - \frac{\pi}{4}x$

$0 = 15 - x - \frac{\pi}{4}x$

$x + \frac{\pi}{4}x = 15$

$x(1 + \pi/4) = 15$

$x = \dfrac{15}{1+\pi/4}$

A' $\quad + \quad -$

(1) $\quad \dfrac{15}{1+\frac{\pi}{4}} \quad$ (10)

≈ 8.401

$x = \dfrac{15}{1+\pi/4}$

$y = 15 - \frac{1}{2}\left(\dfrac{15}{1+\pi/4}\right) - \frac{\pi}{4}\left(\dfrac{15}{1+\pi/4}\right)$