Calculus - Chapter 3 Review

Evaluate the following:

1. $\lim_{h\to 0}\dfrac{\sin\left(\dfrac{\pi}{4}+h\right)-\dfrac{1}{\sqrt{2}}}{h}$

2. $\lim_{\Delta x\to 0}\dfrac{3(x+\Delta x)^2+5(x+\Delta x)-3x^2-5x}{\Delta x}$

Use the derivative rules to find the derivative of each of the following functions.

3. $f(x)=7x^8-\dfrac{6}{\sqrt{x}}+\dfrac{5}{8x^4}$

4. $f(x)=3\cot(x)\sec(5x^2)$

5. $g(x)=\dfrac{e^{3x^2+5}}{x^{10}}$

6. $g(x)=\sin^2(\ln x)$

7. An ice cube is melting at a constant rate such that it's sides are changing at a rate of $5\dfrac{mm}{\min}$. Find the rate in which the volume is decreasing at the instant the side length is $s=8mm$.

8. Find all the points on the graph of $y = x^3 + 2x$ where there is a horizontal and a vertical tangent line.

Find $\frac{dy}{dx}$:

9. $4x = \ln\left|4\cos x - x^2 y\right|$

10. $\arctan 2y = e^{2x}$

11. Find $\frac{d^2y}{dx^2}$: $y = \tan(3x)$

12. Find $\frac{d^2y}{dx^2}$: $3x^2 - 4y^2 = 10$

13. Find the equation of the tangent line to $y = \arcsin 2x + 3$ at $x = 0$.

CALCULATOR PAGE - **Use the chart and the functions below to find the derivatives.**

$f(1)=11$	$f'(1)=25$	$g(1)=2$	$g'(1)=4$	$h(1)=-13$	$h'(1)=9$
$f(2)=-3$	$f'(2)=-7$	$g(2)=-4$	$g'(2)=5$	$h(2)=-6$	$h'(2)=8$
$f(3)=1$	$f'(3)=6$	$g(3)=-14$	$g'(3)=\frac{1}{2}$	$h(3)=0$	$h'(3)=-10$

14. $K'(3)$ *if* $K(x)=g(f(x))$

15. $H'(1)$ *if* $H(x)=\ln(h(x))$

Find the intervals of differentiability:

16. $y=\dfrac{x^2-1}{x+4}$

17. $f(x)=\left|x^2-4\right|$

18. $g(t)=\begin{cases} 2\sin t-2t & t<0 \\ \ln\left|t^2+1\right| & t\geq 0 \end{cases}$

19. The temperature T of food put in a freezer is $T=\dfrac{700t}{t^3-4t+10}$ where t is in hours. Find the rate of change of the temperature after $t=3$ hours.

20. If $f(x)=x^3-4$ and $g(x)=f^{-1}(x)$, then what is $g'(4)$?

For questions 21 – 23: A particle is moving along the *x*-axis where, for all values of time *t* for $0 \le t < 2\pi$, the position can be modeled by the function $x(t) = t^2 - 6 + 4\sin t$. Determine the following:

21. Is the speed increasing or decreasing at $t = 2$?

22. What is the <u>average</u> velocity of the particle between the two times in which the acceleration is zero?

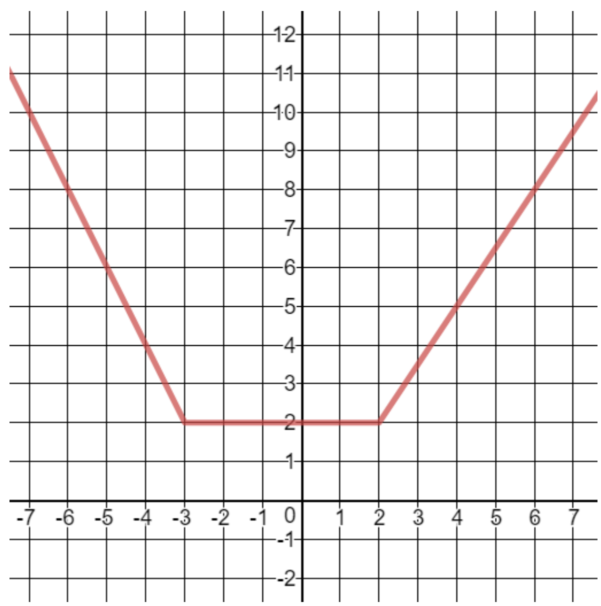
23. What is the position when the velocity is first 3?

24. Sand is being poured onto the ground in the shape of a cone whose height is always twice the radius. When $h = 3$, what is the rate of change of the height if sand is poured at a rate of 16 cubic feet per minute? The volume of a cone is $V = \dfrac{1}{3}\pi r^2 h$.

25. The base of a 50 foot ladder is pulled away from a wall at rate of 2 feet per second. At what rate is the top of the ladder coming down the wall when the base is 14 feet away from the wall?

26. Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, in minutes, is given be a differentiable function C, which is measured in ounces. Using correct units, interpret the meaning of $C'(5) = 2.8$ in terms of the context of the problem.

27. Find the following given the function f to the right:



a) $f'(-4) =$

b) $\dfrac{f(-5) - f(1)}{-5-1} =$

c) $\lim\limits_{h \to 0^+} \dfrac{f(2+h) - f(2)}{h} =$

d) $\lim\limits_{h \to 0^-} \dfrac{f(2+h) - f(2)}{h} =$

e) Find all values of x such that f is continuous but not differentiable. Be sure to explain your reasoning.

Solutions

1. This is the definition of the derivative of $f = \sin x$, evaluated at $x = \frac{\pi}{4}$. So, $f'\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

2. This is the definition of the derivative of $f = 3x^2 + 5x$. So, $f'(x) = 6x + 5$.

3. $f(x) = 7x^8 - 6x^{-\frac{1}{2}} + \frac{5}{8}x^{-4}$ so $f'(x) = 56x^7 + 3x^{-\frac{3}{2}} - \frac{5}{2}x^{-5}$ or $f'(x) = 56x^7 + \frac{3}{\sqrt{x^3}} - \frac{5}{2x^5}$

4. $f(x) = 3\cot(x)\sec(5x^2)$ so $f'(x) = -3\csc^2(x)\sec(5x^2) + 3\cot(x)\sec(5x^2)\tan(5x^2) * 10x$ or

   $f'(x) = 3\sec(5x^2)\left[-\csc^2(x) + 10x\cot(x)\tan(5x^2)\right]$

5. $g(x) = \frac{e^{3x^2+5}}{x^{10}}$ so $g'(x) = \frac{(x^{10})(e^{3x^2+5} * 6x) - (10x^9)(e^{3x^3+5})}{(x^{10})^2}$ or $g'(x) = \frac{6x^{11}e^{3x^2+5} - 10x^9e^{3x^3+5}}{x^{20}} = \frac{2e^{3x^2+5}(3x^2 - 5)}{x^{11}}$

6. $g(x) = \sin^2(\ln x)$ or $g(x) = [\sin(\ln x)]^2$ so $g'(x) = 2[\sin(\ln x)](\cos(\ln x))\left(\frac{1}{x}\right)$ or $g'(x) = \frac{2\sin(\ln x)\cos(\ln x)}{x}$

7. We are given $\frac{ds}{dt} = -5\frac{mm}{\min}$, $s = 8mm$, the volume of a cube is $V = s^3$ and we must find $\frac{dV}{dt}$. So taking the derivative of volume with respect to t, $\frac{dV}{dt} = 3s^2\frac{ds}{dt}$. Plugging everything in gives $\frac{dV}{dt} = -960\frac{mm^3}{\min}$, but since the question is asking at what rate the volume is decreasing, leave your answer as positive.

8. A horizontal tangent line is when $y' = 0$ and a vertical tangent line is when $y' = und.$ So for the horizontal, set $y' = 3x^2 + 2 = 0$, but this ends up with no $x-$values. Also, since $y' = 3x^2 + 2$ is never undefined, there are no vertical tangent lines either. So no points of horizontal or vertical tangent lines.

9. We could try solving for y, but implicit differentiation might be easier. So, differentiate with respect to x.

   $4 = \frac{-4\sin x - \left(2xy + x^2\frac{dy}{dx}\right)}{4\cos x - x^2 y}$ and we must solve for $\frac{dy}{dx}$. So, $4(4\cos x - x^2 y) = -4\sin x - \left(2xy + x^2\frac{dy}{dx}\right)$

   or $16\cos x - 4x^2 y = -4\sin x - 2xy - x^2\frac{dy}{dx}$ or $\frac{16\cos x - 4x^2 y + 4\sin x + 2xy}{-x^2} = \frac{dy}{dx}$.

10. We could try solving for y, but implicit differentiation might be easier. So, differentiate with respect to x.

    $\frac{1}{1+4y^2} * 2\frac{dy}{dx} = e^{2x} * 2$ or $\frac{2}{1+4y^2}\frac{dy}{dx} = 2e^{2x}$. So multiplying by the reciprocal gives $\frac{dy}{dx} = e^{2x}(1+4y^2)$.

11. This is the second derivative, so $\frac{dy}{dx} = \sec^2(3x) * 3$ or $\frac{dy}{dx} = 3\sec^2(3x)$ or $\frac{dy}{dx} = 3[\sec(3x)]^2$. So the derivative of this is $\frac{d^2y}{dx^2} = 6[\sec(3x)]\sec(3x)\tan(3x) * 3$ or $\frac{d^2y}{dx^2} = 18\sec^2(3x)\tan(3x)$.

12. This is the implicit second derivative, so $6x - 8y\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = \frac{3x}{4y}$. So the derivative of this is

$\frac{d^2y}{dx^2} = \frac{4y*3 - 3x*4\frac{dy}{dx}}{(4y)^2}$. Substituting in $\frac{dy}{dx} = \frac{3x}{4y}$, we get $\frac{d^2y}{dx^2} = \frac{12y - 12x\left(\frac{3x}{4y}\right)}{16y^2}$ or

$\frac{d^2y}{dx^2} = \frac{12y - \left(\frac{36x^2}{4y}\right)}{16y^2}$. Then multiplying all parts by $4y$ leaves $\frac{d^2y}{dx^2} = \frac{48y^2 - 36x^2}{64y^3}$ or

$\frac{d^2y}{dx^2} = \frac{12y^2 - 9x^2}{16y^3}$. (Note: this could be reduced to $\frac{d^2y}{dx^2} = \frac{-30}{16y^3} = \frac{-15}{8y^3}$, but no need to worry about it).

13. We need slope and a point to write the equation of the tangent line. The point is $y(0) = 3$. The slope is the derivative at $x = 0$. So $y' = \frac{2}{\sqrt{1-4x^2}}$ and $y'(0) = 2$. Therefore, $y - 3 = 2(x - 0)$.

14. $K'(x) = g'(f(x)) * f'(x)$, so $K'(3) = g'(f(3)) * f'(3)$ or $K'(3) = g'(1) * f'(3)$ or $K'(3) = 4*6 = 24$.

15. $H'(x) = \frac{1}{h(x)} * h'(x)$, so $H'(1) = \frac{h'(1)}{h(1)}$ or $H'(1) = -\frac{9}{13}$.

16. A function cannot be differentiable if it is not continuous, so $x \neq -4$. Therefore, $(-\infty, -4), (-4, \infty)$.

17. Looking at the graph, there are two sharp turns at $x \neq \pm 2$. Therefore, $(-\infty, -2), (-2, 2), (2, \infty)$.

18. For piecewise functions, we need to check if g is continuous. If it is, then we must check if the slope from the left equals the slope from the right. g is continuous since $\lim_{x \to 0^+} g(t) = \lim_{x \to 0^-} g(t) = g(0) = 0$.

Additionally, $g'(t) = \begin{cases} 2\cos t - 2 & t < 0 \\ \frac{2t}{t^2+1} & t \geq 0 \end{cases}$ and the slopes are 0 from both sides. Therefore, $(-\infty, \infty)$.

19. Rate of change is another phrase meaning derivative. We can let the calculator find that $T'(3) = -49.28$.

20. Since g is the inverse of f, the 4 as the x-value for g is a y-value of f. The derivative of the inverse of f is $g'(4) = (f^{-1})'(x) = \frac{1}{f'(y)}$. So, $4 = x^3 - 4$ or $x = 2$. Since $f' = 3x^2$, $g'(4) = \frac{1}{f'(2)} = \frac{1}{12}$.

21. Speed increasing or decreasing is dependent on the signs of velocity and acceleration. So, $v(t) = x'(t) = 2t + 4\cos t$ and $a(t) = v'(t) = x''(t) = 2 - 4\sin t$. Therefore, $v(2) = 2.335$ and $a(2) = -1.637$ and since they are opposite signs, the speed is decreasing.

22. Average velocity is the average rate of change of the position function between the two times. We must first find the two times $a(t) = 2 - 4\sin t = 0$ or when $\sin t = \frac{1}{2}$, so $t = \frac{\pi}{6}, \frac{5\pi}{6}$. So the average velocity, or the average rate of change, is $\dfrac{x\left(\frac{5\pi}{6}\right) - x\left(\frac{\pi}{6}\right)}{\frac{5\pi}{6} - \frac{\pi}{6}} = 3.142$

23. Since position is dependent on time, we must find the time $v(t) = 2t + 4\cos t = 3$, which is at $t = 1.64171$. So, $x(1.64171) = 0.685$.

24. We are given $h = 2r, h = 3$ and $\frac{dV}{dt} = 16$ and we are asked to find $\frac{dh}{dt}$. To prevent a product rule due to the h and r, we should substitute in the $h = 2r$. Since we are trying to find $\frac{dh}{dt}$, it would be advantageous to get the volume equation in terms of h, so use $r = \frac{h}{2}$. So, $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$ or $V = \frac{\pi}{12}h^3$. Taking the derivative with respect to t, which is implicit, gives us $\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$. Substituting the necessary values leaves us with $16 = \frac{\pi}{4}(3)^2\frac{dh}{dt}$ or $16 = \frac{9\pi}{4}\frac{dh}{dt}$ or $\frac{dh}{dt} = \frac{64}{9\pi} = 2.264\frac{ft}{\min}$.

25. We are given $\frac{dx}{dt} = 2$ and the hypotenuse is a constant 50 and we are asked to find $\frac{dy}{dt}$ when $x = 14$. The ladder against a wall creates a right triangle, so we can use Pythagorean Theorem to establish the relationship. So $x^2 + y^2 = 50^2$ and taking the derivative with respect to t, which is implicit, gives us $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. Substituting the necessary values leaves us with $2(14)(2) + 2y\frac{dy}{dt} = 0$. Since our objective is to find $\frac{dy}{dt}$, we will need to find y, so we can use the initial equation $x^2 + y^2 = 50^2$ where $x = 14$. Therefore, $14^2 + y^2 = 50^2$, $y^2 = 2500 - 196 = 2304$ or $y = 48$. So, $2(14)(2) + 2(48)\frac{dy}{dt} = 0$ or $56 + 96\frac{dy}{dt} = 0$ or $\frac{dy}{dt} = -\frac{56}{96} = -0.583\frac{ft}{\sec}$.

26. The derivative of a function represents the instantaneous rate of change at a single moment. Therefore, using context, $C'(5) = 2.8$ would say that coffee is dripping into the cup at a rate of 2.8 ounces per minute at $t = 5$ minutes.

27. a) Graphically, $f'(-4)$ represents the slope of the graph of f at $x=-4$. At $x=-4$, f is a line segment, meaning the slope is the derivative. The slope of this line segment is -2, so $f'(-4)=-2$.

b) This expression represents the average rate of change of f from $[-5,1]$ and graphically, it represents the slope of the secant line. So, $\frac{f(-5)-f(1)}{-5-1}=\frac{6-2}{-5-1}=\frac{4}{-6}=-\frac{2}{3}$.

c) This expression represents the definition of the derivative of f at $x=2$ coming in from the right. So, graphically, this is the slope of the line segment to the right of $x=2$. This has a slope of $\frac{3}{2}$.

d) This expression represents the definition of the derivative of f at $x=2$ coming in from the left. So, graphically, this is the slope of the horizontal line segment to the left of $x=2$. This has a slope of 0.

e) This function is continuous everywhere, as there are no holes or vertical asymptotes. Formally, for all values of c, $\lim_{x\to c} f(x)=f(c)$. A function is not differentiable at $x=c$ if the function is not continuous at $x=c$ or if $\lim_{x\to c} f'(x)\neq f'(c)$. Combining the results from c) and d), we can see that $\lim_{x\to 2^+} f'(x)\neq \lim_{x\to 2^-} f'(x)$, meaning that f is not differentiable at $x=2$. We could also identify $x=-2$ as a candidate. $\lim_{x\to -2^-} f'(x)=-2$ is the slope from the left and $\lim_{x\to -2^+} f'(x)=0$ is the slope from the right. So, since $\lim_{x\to -2^+} f'(x)\neq \lim_{x\to -2^-} f'(x)$, meaning that f is not differentiable at $x=-2$. Therefore, f is continuous but not differentiable at $x=-2,2$.