AP Calculus I
Notes 6.3
Separation of Variables

If you have an equation in which the derivative of a function is equal to some other function, we call this a **differential equation**. To solve a differential equation, we must use a variety of techniques to eliminate the differentials. The simplest of these techniques is called **separation of variables**.

Ex. 1: If $x\frac{dy}{dx} = 3$ and $y(1) = 5$, find the equation for y in terms of x. Then, verify this is the solution to the differential equation.

Given $\frac{dy}{dx} \rightarrow y = f(x)$ we need to integrate!

$\int x\frac{dy}{dx} = \int 3$?? need dx out of denominator! So multiply!

$\int x\,dy = \int 3\,dx$?? Can't integrate $x\,dy$, so get x to dx side!

$\int 1\,dy = \int \frac{3}{x}\,dx$ ✓✓ $\rightarrow y + C_1 = 3\ln|x| + C_2$ get y by itself

$C_2 - C_1 = C$

constant − constant is constant? So, only need 1 $+C$!

$y = 3\ln|x| + C \rightarrow y(1) = 5$

$5 = 3\ln 1 + C$, $5 = 0 + C$, $C = 5$

$\boxed{y = 3\ln|x| + 5}$

$x\frac{dy}{dx} = 3$

verify: $dy/dx = 3\frac{1}{x}$, so $x\left(\frac{3}{x}\right) = 3 \ldots\ 3 = 3$ yep!

Ex. 2: If $\frac{dy}{dx} - \frac{3x^2}{y} = 0$ and $y(0) = -2$, find the equation for y in terms of x.

$\frac{dy}{dx} = \frac{3x^2}{y}$

First step is always to multiply by denominator. This will establish where the variables need to go to integrate

$dy = \frac{3x^2}{y}\,dx$

$\int y\,dy = \int 3x^2\,dx$

$\frac{1}{2}y^2 = x^3 + C$

either find C or solve for y first... doesn't matter

use $f(0) = -2$

$\frac{1}{2}(-2)^2 = (0)^3 + C$, $2 = C$

$\frac{1}{2}y^2 = x^3 + 2$

$y^2 = 2x^3 + 4$

$y = \pm\sqrt{2x^3 + 4}$

But.... we want 1 equation!

Since $y(0)$ is -2 (negative!)

$\boxed{y = -\sqrt{2x^3 + 4}}$

Ex. 3: Find the equation of the curve through $(0,2)$ whose slope is always 4 more than its y −coordinate.

$y = f(x)$ $\quad f(0) = 2$ $\quad \frac{dy}{dx} = 4 + y$

$$\frac{dy}{dx} = 4 + y$$

wrong sides! → $dy = (4+y)dx$

$$\int \frac{dy}{4+y} = \int dx$$

$u = 4+y$
$du = dy$

$$\ln|4+y| = x + C$$

$$\ln 6 = 0 + C, \quad C = \ln 6$$

$$e^{\ln|4+y|} = e^{x + \ln 6}$$

$$|4+y| = e^{x+\ln 6}$$

$$4 + y = \pm e^{x+\ln 6}$$

$F(0) > 0$ so positive!

$$4 + y = e^{x+\ln 6}$$

$e^{a+b} = e^a \cdot e^b$

$$\boxed{y = e^{x+\ln 6} - 4 \text{ or } e^x \cdot e^{\ln 6} - 4 = 6e^x - 4}$$

common simplification

Ex. 4: If $y' = \frac{\sqrt{2t+1}}{y^2}$ and given $y(0) = 3$, find an equation for y in terms of t and find $y(4)$.

$\frac{dy}{dt}$

$$dy = \frac{\sqrt{2t+1}}{y^2} dt$$

$$\int y^2 dy = \int (2t+1)^{1/2} dt$$

u-sub!

$u = 2t+1$
$du = 2dt$

$$\frac{1}{3}y^3 = \frac{1}{2} \cdot \frac{2}{3}(2t+1)^{3/2} + C$$

$$\frac{1}{3}y^3 = \frac{1}{3}(2t+1)^{3/2} + C$$

$y(0) = 3$

$$9 = \frac{1}{3} + C$$

$$C = \frac{26}{3}$$

$$\frac{1}{3}y^3 = \frac{1}{3}(2t+1)^{3/2} + \frac{26}{3}$$

$$y^3 = (2t+1)^{3/2} + 26$$

$$\boxed{y = \sqrt[3]{(2t+1)^{3/2} + 26}}$$

$$y(4) = \sqrt[3]{9^{3/2} + 26}$$

$$= \sqrt[3]{27 + 26} = \boxed{\sqrt[3]{53}}$$

Ex. 5: If $\frac{dw}{dt} = \frac{w^2}{4+t^2}$ and $w(2) = -1$, find an equation for w in terms of t.

↳ find $w(t)$

$$dw = \frac{w^2}{4+t^2}\,dt$$

$$\int \frac{dw}{w^2} = \int \frac{1}{4+t^2}\,dt$$

$u^2 = t^2$, $u = t$
$du = dt$
$a^2 = 4$, $a = 2$
arctan!

$\int w^{-2}\,dw$

$$-\frac{1}{w} = \frac{1}{2}\arctan\frac{t}{2} + C$$

$w(2) = -1$

$$1 = \frac{1}{2}\arctan 1 + C$$

$$1 = \frac{1}{2}\cdot\frac{\pi}{4} + C$$

$$C = 1 - \pi/8$$

$$-\frac{1}{w} = \frac{1}{2}\arctan\frac{t}{2} + 1 - \frac{\pi}{8}$$

get out of denominator

$$-1 = w\cdot\left(\frac{1}{2}\arctan\frac{t}{2} + 1 - \frac{\pi}{8}\right)$$

$$\boxed{w = \frac{-1}{\frac{1}{2}\arctan\frac{t}{2} + 1 - \frac{\pi}{8}}}$$

Ex. 6: Which of the following is the solution to the differential equation $\frac{dy}{d\theta} = y\sec^2\theta$ with the initial condition $y\left(\frac{\pi}{4}\right) = -1$.

(A) $y = -e^{\tan\theta}$

(B) $y = -e^{-1+\tan\theta}$

(C) $y = -e^{(\sec^2\theta - 2\sqrt{2})/3}$

(D) $y = -\sqrt{2\tan\theta - 1}$

$$dy = y\sec^2\theta\,d\theta$$

$$\int\frac{dy}{y} = \int\sec^2\theta\,d\theta$$

$$\ln|y| = \tan\theta + C$$

$$\ln 1 = \tan\pi/4 + C$$

$0 = 1 + C$, $C = -1$

$$\ln|y| = \tan\theta - 1$$

$$|y| = e^{\tan\theta - 1}$$

$$y = \pm e^{\tan\theta - 1}$$

$y(\pi/4) < 0$ so

$$y = -e^{\tan\theta - 1}$$

Ex. 7: Consider the differential equation $\frac{dy}{dx} = y^2(2^x + 2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(1) = -1$.

a) Use the equation of the tangent line to the graph of $f(x)$ at $x = 1$ to approximate $f(2)$.

need point and slope! point: $f(1) = -1$

slope: $\frac{dy}{dx} = y^2(2^x+2)\Big|_{(1,-1)} = 1(2+2) = 4$

line: $\boxed{y+1 = 4(x-1)}$ $\quad f(2) \approx 4(2-1) - 1$

$\boxed{f(2) \approx 3}$

b) Find $\frac{d^2y}{dx^2}$. Use this to determine if $f(x)$ is concave up or down at $x = 1$. Justify your answer.

$\frac{d}{dx}\left[\frac{dy}{dx} = y^2(2^x+2)\right]$ product rule!!

$$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}(2^x+2) + y^2(2^x \ln 2)\Big|_{\substack{x=1\\y=-1}} \qquad \frac{dy}{dx}\Big|_{(1,-1)} = 4$$

implicit!!

$$= 2(-1)(4)(2+2) + (1)(2\ln 2) = -32 + 2\ln 2 < 0$$

$\therefore$ $\boxed{\text{concave down}}$

c) Find $y = f(x)$, the particular solution to the differential equation with $f(1) = -1$. Then, find $f(2)$.

integrate!

$$\int \frac{dy}{dx} = \int y^2(2^x+2) = \int dy = \int y^2(2^x+2)\,dx$$

$$= \int \frac{dy}{y^2} = \int (2^x+2)\,dx \rightarrow -\frac{1}{y} = \frac{2^x}{\ln 2} + 2x + C$$

$(1,-1)$

$$1 = \frac{2}{\ln 2} + 2 + C$$

$$C = -1 - \frac{2}{\ln 2}$$

$$-\frac{1}{y} = \frac{2^x}{\ln 2} + 2x - 1 - \frac{2}{\ln 2}$$

switcheroo!

$$y = \frac{-1}{\frac{2^x}{\ln 2} + 2x - 1 - \frac{2}{\ln 2}}$$

$$f(2) = \frac{-1}{\frac{4}{\ln 2} + 4 - 1 - \frac{2}{\ln 2}}$$

$$f(2) = \frac{-1}{\frac{2}{\ln 2} + 3}$$

AP Calculus I
Notes 6.2
Differential Equations: Growth and Decay

Growth and Decay Models

One of the more common applications to differential equations is modeling growth, especially in the field of science. In many applications, the rate of change of a variable y is proportional to the value of y. If y is a function of time t, the proportionality can be written as:

$$\frac{dy}{dt} = ky$$

If y is a differentiable function of t such that $y > 0$ and $y' = ky$, for some constant k, then $y = Ce^{kt}$. C is the **initial value** of y, and k is the **proportionality constant**.

Proof: $dy = ky\,dt$ * k is a constant

$\int \frac{dy}{y} = \int k\,dt$

$\ln|y| = kt + C$

$|y| = e^{kt+C}$

$|y| = e^{kt} \cdot e^{C}$ * e^{C} is a constant!

$|y| = e^{kt} \cdot C$

$|y| = Ce^{kt}$ *states $y > 0$

$\boxed{y = Ce^{kt}}$ ✓

Ex. 1: The rate of change of y is proportional to y. When $t = 0$, $y = 2$ and when $t = 2$, $y = 4$. What is the value of y when $t = 3$?

phrase means $\frac{dy}{dt} = ky$ "proportional" and therefore solution is $y = Ce^{kt}$

$y = Ce^{kt}$, $(0,2)$ & $(2,4)$

$2 = Ce^{k \cdot 0}$

$2 = Ce^{0}$, $2 = C$

$y = 2e^{kt}$

$4 = 2e^{2k}$, $2 = e^{2k}$

$\ln 2 = 2k$, $k = \frac{1}{2}\ln 2$

$y = 2e^{(\frac{1}{2}\ln 2)t}$

$\boxed{y(3) = 2e^{\frac{3}{2}\ln 2}}$

$dV/dt = k \cdot V^2/\sqrt{t}$

Ex. 2: The rate of change of V is proportional to the quotient of the square of V and the square root of t. If $t = 0, V = 3$ and $t = 1, V = 2$, what is the value of V when $t = 3$?

$$\frac{dV}{dt} = \frac{kV^2}{\sqrt{t}}$$ ← definitely not Ce^{kt}, so must separate variables!

$$\int \frac{dV}{V^2} = \int \frac{k}{\sqrt{t}}\,dt = \int kt^{-1/2}\,dt$$

$$-\frac{1}{V} = 2k\sqrt{t} + C$$

$$-1/3 = 2k\sqrt{0} + C, \quad C = -1/3$$

$$-\frac{1}{V} = 2k\sqrt{t} - 1/3 \quad (1,2)$$

$$-1/2 = 2k - 1/3, \quad 2k = -1/6, \quad k = -1/12$$

$$-\frac{1}{V} = -\frac{1}{6}\sqrt{t} - 1/3$$ switcheroo!

$$V = \frac{-1}{-1/6\sqrt{t} - 1/3} = \frac{6}{\sqrt{t}+2}, \quad \boxed{V(3) = \frac{6}{\sqrt{3}+2}}$$

Ex. 3: Suppose an experimental population of fruit flies increases according to the law of exponential growth. There were 100 flies at the beginning of the experiment and 300 flies after the fourth day. After how many days will there be 1000 of the little buggers present?

means $y = Ce^{kt}$!

$$y = Ce^{kt} \quad (0,100), (4,300), (t,1000)?$$

$$100 = Ce^{k \cdot 0}$$
$$100 = C$$
$$y = 100e^{kt}$$

$$300 = 100e^{4k}$$
$$3 = e^{4k}$$
$$\frac{\ln 3}{4} = k$$

$$y = 100e^{(1/4 \ln 3)t}$$
$$1000 = 100e^{1/4 \ln 3 t}$$
$$10 = e^{1/4 \ln 3 t}$$
$$\ln 10 = \tfrac{1}{4}\ln 3 t$$
$$\boxed{t = \frac{4\ln 10}{\ln 3} = 8.384}$$

Ex. 4:

t	0	2
$f(t)$	4	12

Let $y = f(t)$ be a solution to the differential equation $\frac{dy}{dt} = ky$, where k is a constant. Values of f for selected values of t are given in the table above. Which of the following is an expression for $f(t)$?

Solution is $y = Ce^{kt}$

(A) $4e^{\frac{t}{2}\ln 3}$

(B) $e^{\frac{t}{2}\ln 9} + 3$

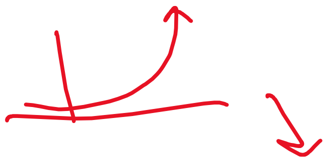
(C) $2t^2 + 4$

(D) $4t + 4$

$$4 = Ce^0, \quad C = 4$$
$$y = 4e^{kt}$$

$$12 = 4e^{2k}$$
$$3 = e^{2k}$$
$$k = \tfrac{1}{2}\ln 3$$

$$y = 4e^{(1/2 \ln 3)t}$$



AP Calculus I
Notes 6.4
Logistic Growth Model

The model $y = Ce^{kt}$ is known as "uninhibited growth". However, in the real world, environmental factors usually inhibit growth. Factors such as water, food, resources, disease and weather cause a population to reach some upper limit L past which growth cannot occur. This upper limit L is known as the **carrying capacity**, which is the idea of how many people/animals/bacteria that can be supported naturally.

The model that is used to describe this type of growth is the **logistic differential equation:**

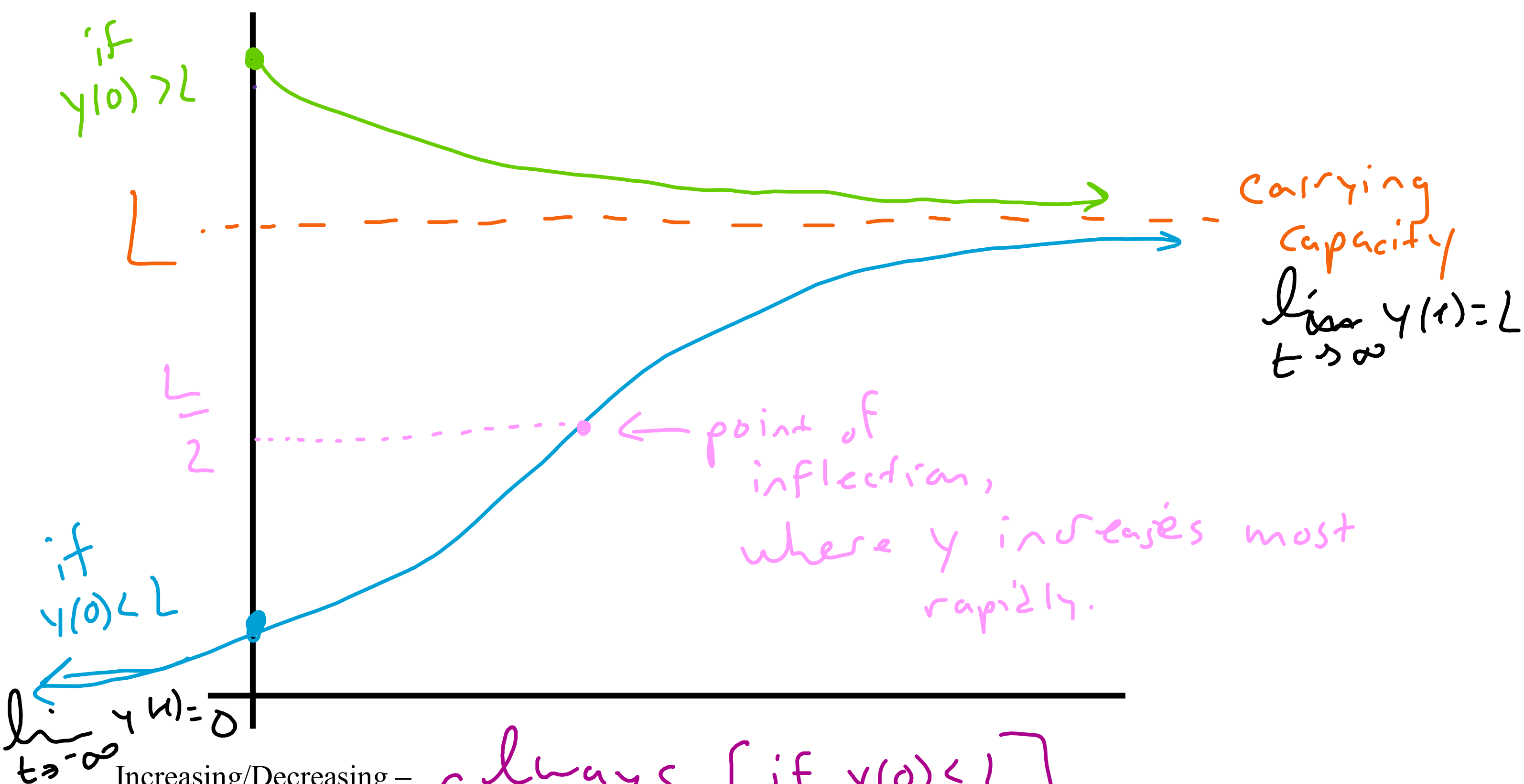
$$\frac{dy}{dt} = k\overset{\text{population}}{y}\underbrace{\left(1 - \frac{y}{L}\right)}_{\text{\% population remaining}}$$

where k and L are both positive constants. The **general solution** of this logistic differential equation is:

we haven't discussed the technique yet to solve...

$$y = \frac{L}{1 + be^{-kt}}$$

The graph of logistic growth models can be found below:



Increasing/Decreasing – always increasing [if $y(0) < L$]

Concave up/Concave down –
$(0, \frac{L}{2})$ "Range" concave up
$(L/2, L)$ "Range" concave down
not t-values. are y-values

Ex. 1: The population of donkeys in Mrsgoughwantedashoutoutinaproblemistan can be modeled by the equation $y = \frac{730}{1+4e^{-1.5t}}$. Find the initial population, the value of k and the carrying capacity. Then, write the equivalent differential equation representing this situation.

Solution: $y = \frac{L}{1+be^{-kt}}$ differential equation: $\frac{dy}{dt} = ky\left(1-\frac{y}{L}\right)$

initial population: $y(0) = \frac{730}{1+4e^{0}} = 146$ donkeys

k: $k = 1.5$ (k represents how "quickly" the S curve happens)

carrying capacity: $L = 730$ donkeys

Differential Equation: $\frac{dy}{dt} = 1.5y\left(1-\frac{y}{730}\right)$

Ex. 2: The population of whales in Mrgoughismehistan can be modeled by $\frac{dy}{dt} = 0.1y\left(1-\frac{y}{200}\right)$. Assuming $y(0) = 40$, find the number of whales at $t = 5$, find $\lim_{t\to-\infty} y(t)$ and find $\lim_{t\to\infty} y(t)$. At what population is the rate of change the greatest? What is that greatest rate of change?

$\frac{dy}{dt} = ky\left(1-\frac{y}{L}\right) \longrightarrow$ so $k = 0.1$, $L = 200$

Solution: $y = \frac{200}{1+be^{-0.1t}}$ $(0, 40)$

$40 = \frac{200}{1+be^{0}}$, $40 = \frac{200}{1+b}$, $1+b = 5$, $b = 4$

$y(t) = \frac{200}{1+4e^{-0.1t}}$ $\boxed{y(5) = 58.375 \text{ whales}}$

population of greatest change is $P = \frac{L}{2} = \boxed{100}$

the greatest change $= \left.\frac{dy}{dt}\right|_{y=100} = 0.1(100)\left(1-\frac{100}{200}\right) = \boxed{5 \text{ whales/time}}$

Ex. 3: Suppose that a population $y(t)$ grows in accordance with the logistic model $\frac{dy}{dt} = 6y - 0.24y^2$.

- What is the carrying capacity of the population?

we are used to form $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$

key parts are "$1 -$" and "$\frac{1y}{L}$"

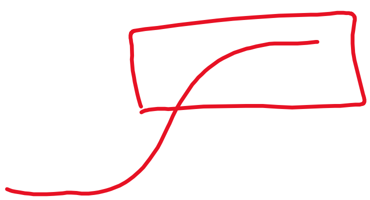
$$\frac{dy}{dt} = 6y(1 - 0.04y) = 6y\left(1 - \frac{4}{100}y\right) = 6y(1 - y/25) \quad \boxed{L = 25}$$

- What is the greatest rate of change of the population?

it occurs when population is $\frac{L}{2} = 25/2$

$$\left.\frac{dy}{dt}\right|_{y=25/2} = 6(25/2)\left(1 - \frac{25/2}{25}\right) = \boxed{37.5}$$

- Over what interval is the graph concave down?



when population is from $(12.5, 25)$

[we don't know t-values though]

Ex. 4: Identify which of the following differential equations are logistic. If so, find the carrying capacity.

a) $\frac{dy}{dx} = 4x - 0.6x^2$ No, needs to be y's

b) $\frac{dy}{dx} = 4y - 0.6y^2$ yes! $\frac{dy}{dx} = 4y(1 - 0.15y) = 4y\left(1 - \frac{15}{100}y\right) = 4y\left(1 - \frac{3}{20}y\right) = 4y\left(1 - \frac{1y}{20/3}\right)$ $\boxed{L = \frac{20}{3}}$

need $1 -$ and only y in numerator

c) $\frac{dC}{dt} = 0.5t\left(3 - \frac{2t}{5}\right)$ No, needs to be in terms of C

d) $\frac{dC}{dt} = 0.5C\left(3 - \frac{2C}{5}\right) = 1.5C\left(1 - \frac{2C}{15}\right) = 1.5C\left(1 - \frac{C}{15/2}\right)$ $\boxed{L = 15/2}$

Ex. 5: AP Problems

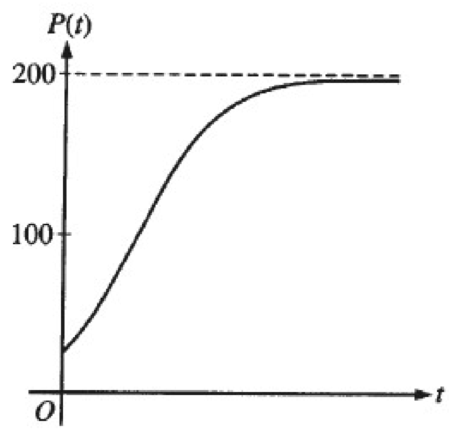
The population $P(t)$ of a species satisfies the logistic differential equation $\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$, where the initial population $P(0) = 3{,}000$ and t is the time in years. What is $\lim_{t\to\infty} P(t)$?

(A) 2,500 (B) 3,000 (C) 4,200 (D) 5,000 (E) 10,000

$\lim_{t\to\infty} P(t)$ is just L! we can identify L

if we get $ky\left(1-\frac{y}{L}\right)$ so $\frac{dP}{dt} = 2P\left(1-\frac{P}{10000}\right)$

$\boxed{L = 10{,}000}$



Which of the following differential equations for a population P could model the logistic growth shown in the figure above?

(A) $\frac{dP}{dt} = 0.2P - 0.001P^2$

(B) $\frac{dP}{dt} = 0.1P - 0.001P^2$

(C) $\frac{dP}{dt} = 0.2P^2 - 0.001P$

(D) $\frac{dP}{dt} = 0.1P^2 - 0.001P$

(E) $\frac{dP}{dt} = 0.1P^2 + 0.001P$

$\frac{2}{10}P - \frac{1}{1000}P^2 = \frac{1}{5}P\left(1 - \frac{1}{200}P\right)$

$\frac{1}{5}P - \frac{1}{1000}P^2$

$L = 200$ ✓✓

wrong form

AP Calculus I
Notes 6.1
Slope Fields and Euler's Method

Remember that a **differential equation** involves one or two variable and a derivative. Solving a differential equation can be very difficult if separation of variables does not work. A way around this is to take a graphical approach to learn about the solution of the differential equation. This approach is to create a **slope field**, which uses the derivative to find the slope at several points to paint a picture of various solutions.

Ex. 1: Consider the differential equation $\frac{dy}{dx} = x(y-1)$.

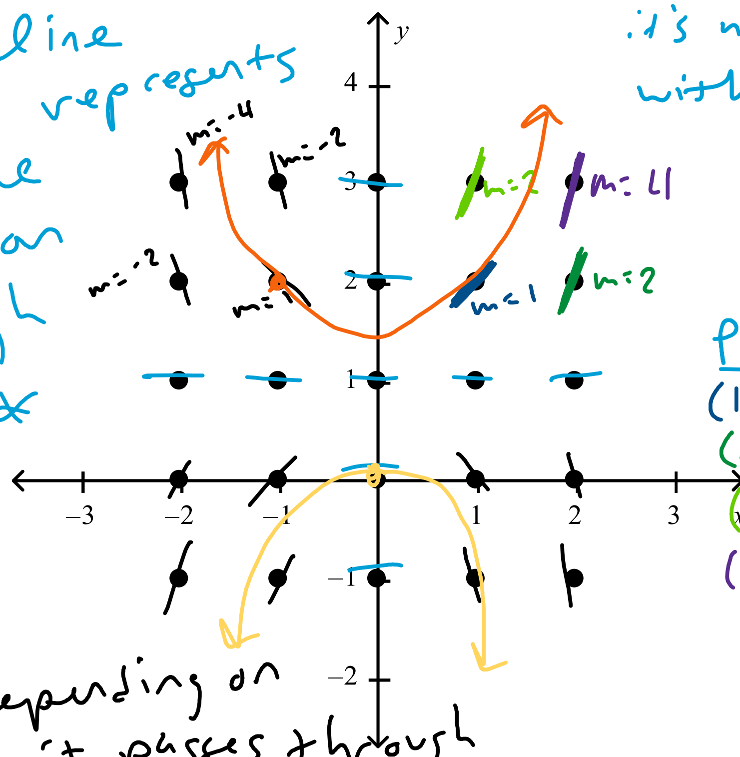
a) Sketch the slope field in the graph below. How many solutions are shown?

* draw little line segment that represents a tangent line if the solution passed through that point *

it's nice to begin with where $\frac{dy}{dx} = 0$

$0 = x(y-1)$

$x = 0$ or $y = 1$



point	dy/dx
(1,2)	$1(2-1) = 1$
(2,2)	$2(2-1) = 2$
(1,3)	$1(3-1) = 2$
(2,3)	$2(3-1) = 4$

↓ rest of points

∞ many solutions... depending on which point it passes through

b) Sketch two different graphs given that the solution passes through $(0,0)$ and $(-1,2)$.

c) Can you find the solution $y = f(x)$ to the differential equation $\frac{dy}{dx}$ that is sketched through $(0,0)$?

yes! separate variables & integrate!

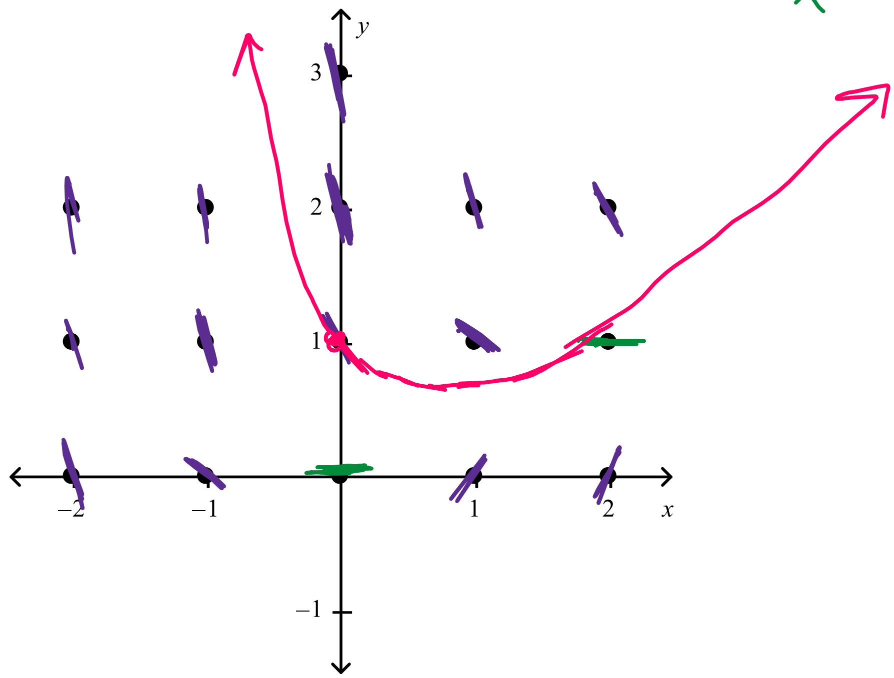
Ex. 2: Consider the differential equation $\frac{dy}{dx} = x - 2y$.

$x - 2y = 0$

$x = 2y$

a) Construct a slope field for the sixteen ordered pairs shown below:

point	slope
$(-2,0)$	$-2-0=-2$
$(-1,1)$	$-1-2=-3$
$(0,2)$	$(0-4)=-4$
$(1,1)$	$1-2=-1$
$(2,0)$	$2-0=2$



b) Sketch the graph of the solution that passes through the point (0,1).

c) Can you find the solution to this differential equation?

$dy = (x-2y)\,dx \ldots$

can't separate!!

Not now (yes in college!)

d) Show that is $y = 3e^{-2x} + \frac{1}{2}x - \frac{1}{4}$ a solution to the differential equation $y' + 2y - x = 0$.

$y' = 3e^{-2x} \cdot -2 + \frac{1}{2} = -6e^{-2x} + \frac{1}{2}$

$\frac{dy}{dx} = x - 2y$

$2y = 6e^{-2x} + x - \frac{1}{2}$

So, $y' + 2y - x = 0$

$\cancel{-6e^{-2x}} + \cancel{\frac{1}{2}} + \cancel{6e^{-2x}} \cancel{+x} - \cancel{\frac{1}{2}} - \cancel{x} = 0$

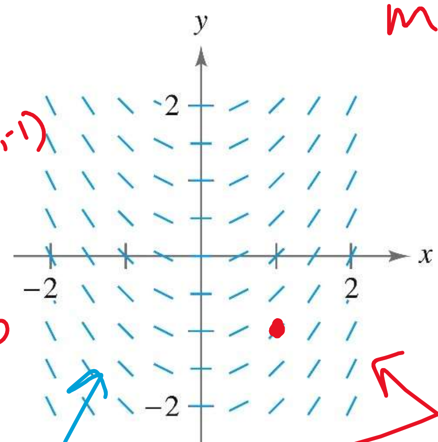
$0 = 0$

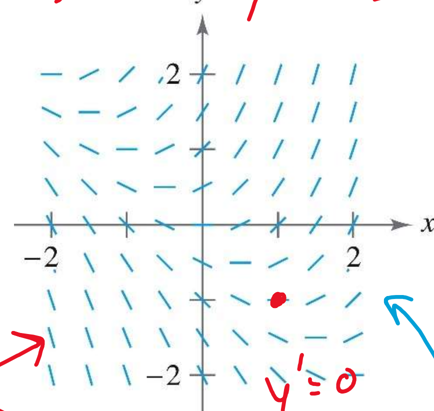
Ex. 3: Match the slope fields to the **differential equations**. Then, match the **solutions** to the slope fields.

makes line segments "picture painted"

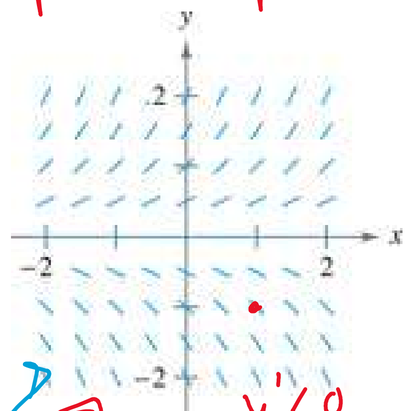
pt: (1,-1)

$y' > 0$





$y' = 0$



$y' < 0$

a) $y' = x + y$ → 0   b) $y' = 2x$ → 2   c) $y' = y$ → -1

d) $y = x^2$ e) $y = e^x$ f) $y = e^x - x - 1$

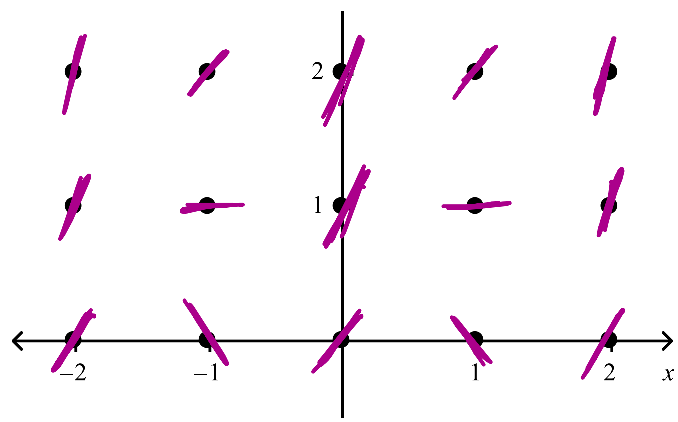
parabola

line and exponential?

Ex. 4: Given the differential equation $\dfrac{dy}{dx} = y + \cos \pi x$,

a) Construct a slope field for the fifteen ordered pairs shown:

plug in points!



b) Is the graph of y increasing/decreasing and concave up/down at $(-1,2)$? Justify your answer.

$$\left.\frac{dy}{dx}\right|_{(-1,2)} = 2 + \cos(-\pi) = 1 > 0 \quad \therefore \text{increasing}$$

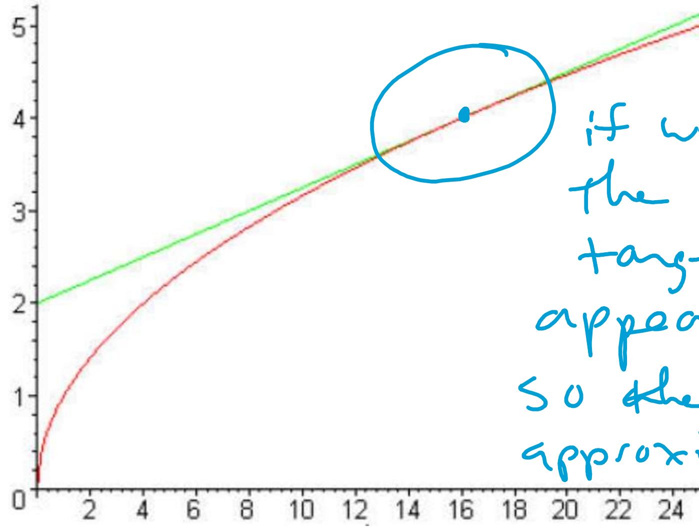
$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + -\sin \pi x \cdot \pi = y + \cos \pi x - \pi \sin \pi x$$

$$\left.\frac{d^2y}{dx^2}\right|_{(-1,2)} = 2 + \cos(-\pi) - \pi \sin(-\pi) = 1 > 0$$

$\therefore$ concave up

Tangent Line Approximations and Euler's Method

We will now look at two different ways to approximate a solution given a difficult function or differential equation. The first is using a **tangent line approximation**, also known as a **local linear approximation**.



if we zoom in, the curve and tangent line appear similar... so the line can approximate the curve

Ex. 5: Approximate the value of $g(0.1)$ using the tangent line to $g(x) = 3x + \sqrt{4 + 5\cos x}$ at $x = 0$.

point: $g(0) = 0 + \sqrt{4+5} = 3$

slope: $g'(x) = 3 + \frac{1}{2}(4+5\cos x)^{-1/2} \cdot -5\sin x = 3 - \frac{5\sin x}{2\sqrt{4+5\cos x}}$

$g(0.1) = 3.29583$

$g'(0) = 3 - 0 = 3$

Line: $y - 3 = 3(x-0)$, $y = 3x + 3 \approx g(x)$

$\boxed{g(0.1) \approx 3.3}$

Ex. 6: Approximate the value of $f(1.02)$ using the tangent line to $f(x) = 5 - 3xe^{2x-2}$ at $x = 1$. Is this approximation an over or underestimate of $f(1.02)$? Explain your reasoning.

$f(1.02)$: 1.815

$f(1) = 5 - 3e^0 = 2$ $\quad f'(x) = -3e^{2x-2} - 3xe^{2x-2} \cdot 2$

$f'(1) = -3 - 6 = -9$

$y - 2 = -9(x-1)$

so $f(x) \approx y = -9(x-1) + 2$, $f(1.02) \approx -9(1.02-1) + 2$

$\approx \boxed{1.82}$

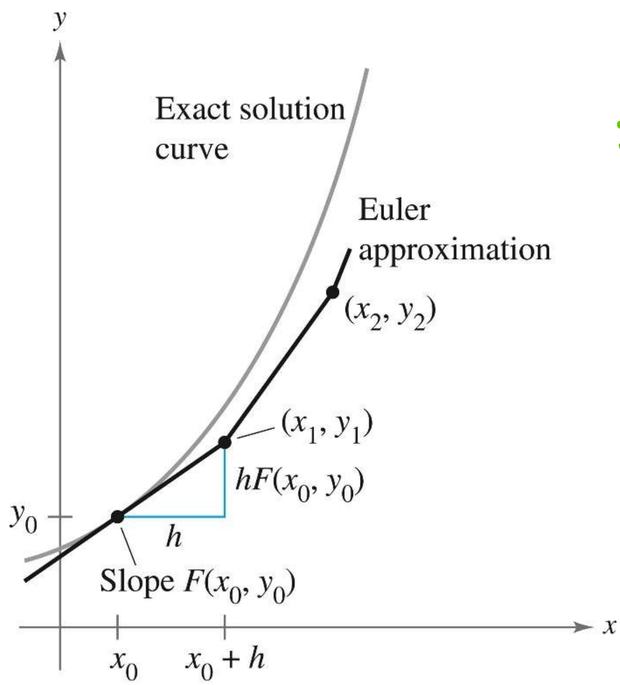
Tangent lines being over/under depends on concavity!

$f'' = -3e^{2x-2} \cdot 2 - 6e^{2x-2} - 6xe^{2x-2} \cdot 2$

$f'' = e^{2x-2}(-6-6-12x) = e^{2x-2}(-12-12x)$

from $1 < x < 1.02$, $f'' < 0$, so f is concave down, so this is an $\boxed{\text{over}}$estimate

Euler's Method is a numerical approach to approximating the particular solution to a differential equation that may be too difficult to separate given a "starting point" to the original function.



small intervals of tangent lines that you re-evaluate after some Δx interval

The starting point in this situation is (x_0, y_0) and from this starting point, you can proceed in the direction of the curve using the slope. Using a small step h or dx, move along the tangent line until you arrive at the point (x_1, y_1). If you think of (x_1, y_1) as the new starting point, you can repeat this process to obtain a second point (x_2, y_2) and so on…

$dx = \frac{0.2-0}{2} = 0.1$

Ex. 7: Approximate the value of $y(0.2)$ given $y(0) = 2$ and $\frac{dy}{dx} = y + x$ using 2 steps. Can you find the actual value of $y(0.2)$? Is this an over or under approximation for $y(0.2)$?

end of 1 is start of next

x	y	dx	$dy = (y+x)dx$	$(x+dx, y+dy)$
0	2	0.1	$(2+0)(0.1) = 0.2$	$(0+0.1, 2+0.2) = (0.1, 2.2)$
0.1	2.2	0.1	$(2.2+0.1)(0.1) = 0.23$	$(0.2, 2.43)$

$\boxed{y(0.2) \approx 2.43}$

To find actual, we need to separate & integrate… can't do here

over/under is concavity! $\frac{d^2y}{dx^2} = \frac{dy}{dx} + 1 = y + x + 1$

from $0 < x < 0.2$, $\frac{d^2y}{dx^2} > 0$ so

f is concave up $\therefore$ $\boxed{\text{underestimate}}$

$f'(t) = dy/dt$ $\Delta t = \frac{1-0}{2} = 0.5$ step size

Ex. 8: Use Euler's method for $y = f(t)$, starting at $f(0) = 9$ with 2 steps of equal size, to approximate $f(1)$ given $f'(t) = \frac{y}{3}(6 - y)$. Then, solve the differential equation to find $f(1)$.

t	y	dt	$dy = \frac{y}{3}(6-y)dt$	$(t+dt, y+dy)$
0	9	0.5	$\frac{9}{3}(6-9)(0.5) = -4.5$	$(0.5, 4.5)$
0.5	4.5	0.5	$\frac{4.5}{3}(6-4.5)(0.5) = 1.125$	$(1, 5.625)$

approximate $\boxed{f(1) \approx 5.625}$

Doesn't this look familiar??? .... LOGISTIC!

$$\frac{dy}{dt} = \frac{6y}{3}\left(1 - \frac{y}{6}\right) = 2y\left(1 - \frac{y}{6}\right) \xrightarrow[L=6]{k=2} y = \frac{6}{1+be^{-2t}}$$

$$9 = \frac{6}{1+be^{0}}, \quad 9 = \frac{6}{1+b} \qquad y = \frac{6}{1 - \frac{1}{3}e^{-2t}} \qquad \boxed{y(1) = 6.283}$$

$$1+b = 6/9, \quad b = -1/3$$

exact!

Ex. 9: Approximate the value of $f(1.7)$ given $f(2) = 5$ and $\frac{dy}{dx} = 2y + x$ using 3 steps of equal size.

x	y	dx	$dy = (2y+x)dx$	$(x+dx, y+dy)$
2	5	-0.1	$(10+2)(-0.1) = -1.2$	$(1.9, 3.8)$
1.9	3.8	-0.1	$(7.6+1.9)(-0.1) = -0.95$	$(1.8, 2.85)$
1.8	2.85	-0.1	$(5.7+1.8)(-0.1) = -0.75$	$(1.7, 2.1)$

$\Delta x = \frac{1.7-2}{3} = -0.1$

$\boxed{f(1.7) \approx 2.1}$

t	0	1	2	4	6	8	10	13
$f(t)$	7.5	21	27.5	32	45	57	70	100

BC2: The functions f [redacted] are continuous for all $t \geq 0$. Selected values for the function f and [redacted] are shown above. For $t \geq 0$ the function f is increasing.

Part III: The weight of a human male $W(t)$, in pounds, that is t years old satisfies the logistic differential equation $\frac{dW}{dt} = \frac{W}{5}\left(1 - \frac{W}{200}\right)$ where $W(t) = f(t)$.

(**a**) Approximate $W'(9)$. Using correct units, interpret the meaning of $W'(9)$ in context of the problem.

(**b**) Find the time t, in years, when the human male growing the fastest. What is rate that he is growing, in pounds per year, at the time when he is growing the fastest?

(**c**) Find $\frac{d^2W}{dt^2}$ in terms of W. Evaluate $\frac{d^2W}{dt^2}$ when the male weighs 50 pounds.

(**d**) Use Euler's method, with two steps of equal size starting at $t = 13$, to approximate $W(17)$.

(**e**) The weight of the male, in pounds, can also be modeled by the function M, given by $M(x) = \frac{1000}{75 - x}$, where x is the height of the male, in inches, and $x < 70$. When the male weighs 100 pounds, his height is increasing at a rate of 0.85 inches per year. According to this model, what is the rate of change of the weight of the male, in pounds per year, at the time when he weighs 100 pounds?

a) $W'(9) \approx \frac{W(10)-W(8)}{10-8} = \boxed{\frac{70-57}{10-8}}$ The approximate change in weight of a human male that is 9 yrs old is $\frac{13}{2}$ pounds/yr

b) Since logistic, growing fastest at $L/2 = \frac{200}{2} = 100$

$f(t) = 100$ at $\boxed{t = 13}$

$\left.\frac{dW}{dt}\right|_{(13,100)} = \frac{100}{5}\left(1 - \frac{100}{200}\right) = \boxed{10}$

c) $\frac{d^2W}{dt^2} = \frac{1}{5}\frac{dW}{dt}\left(1 - \frac{W}{200}\right) + \frac{W}{5}\left(-\frac{1}{200}\frac{dW}{dt}\right)\Big|_{W=50} \rightarrow \left.\frac{dW}{dt}\right|_{W=50} = (10)\left(\frac{3}{4}\right)$

$= \boxed{\frac{1}{5}\left(\frac{15}{2}\right)\left(1 - \frac{50}{200}\right) + 10\left(\frac{-1}{200}\right)\left(\frac{15}{2}\right)}$

d)

t	W	dt	$dW = \frac{W}{5}(1 - W/200)\,dt$	$(t+dt, W+dW)$
13	100	2	$20(1/2)(2) = 20$	(15, 120)
15	120	2	$24(3/5)(2) = 19.2$	(17, 139.2)

$\boxed{W(17) \approx 139.2}$

e) $\frac{dM}{dt} = ?$ when $M = 100$?

$dx/dt = 0.85$ $\quad 100 = \frac{1000}{75 - x}$, $x = 65$

$\frac{dM}{dt} = \frac{-1000(-1)\frac{dx}{dt}}{(75-x)^2} = \boxed{\frac{1000(0.85)}{(75-65)^2}}$

No Calculator!

At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

(b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

a) point: $W(0) = 1400$

Slope: $\frac{dW}{dt} = \frac{1}{25}(1400-300)$

Line: $W - 1400 = \frac{1}{25}(1400-300)(t-0)$

$$\boxed{W(1/4) \approx \frac{1}{25}(1400-300)(1/4 - 0) + 1400 \approx 1411}$$

b) $\frac{d^2W}{dt^2} = \frac{1}{25}\frac{dW}{dt} = \boxed{\frac{1}{25}\cdot\frac{1}{25}(W-300)}$

Since W is always above 1400, $\frac{d^2W}{dt^2}$ is always positive, so W is concave up, so this is an $\boxed{\text{underestimate}}$

c) $dW = \frac{1}{25}(W-300)\,dt$

$$\int \frac{dW}{W-300} = \int \frac{1}{25}\,dt$$

$u = W - 300$

$du = 1\,dW$

$\ln|W-300| = \frac{1}{25}t + C$

$\ln 1100 = 0 + C$

$C = \ln 1100$

$\ln|W-300| = \frac{1}{25}t + \ln 1100$

$|W-300| = e^{\frac{1}{25}t + \ln 1100}$

* $W(0) = 1400 > 0$, so + *

$W - 300 = e^{\frac{1}{25}t + \ln 1100}$

$$\boxed{W = 300 + e^{\frac{1}{25}t + \ln 1100} = 300 + 1100e^{\frac{1}{25}t}}$$

Non-Calculator At time $t = 0$, a marshmallow is taken out of a campfire in pursuit to create the perfect s'more. The internal temperature of the golden-roasted marshmallow is $68°C$ at time $t = 0$, and the internal temperature of the marshmallow is greater than $20°C$ at all times $t > 0$. The internal temperature of the marshmallow at time t minutes can be modeled by the function M that satisfies the differential equation $\frac{dM}{dt} = -\frac{1}{3}(M - 20)$, where $M(t)$ is measured in degrees Celsius $(°C)$ and $M(0) = 68$.

a) Write an equation for the line tangent to the graph of M at $t = 0$. Use this equation to approximate the internal temperature of the marshmallow at time $t = 2$.

point: $M(0)=68$

slope: $\frac{dM}{dt} = -\frac{1}{3}(68-20) = -16$

$\boxed{M - 68 = -16(t-0)}$

$\boxed{M(2) \approx -16(2-0)+68 \quad M(2) \approx 36°C}$

b) Use $\frac{d^2M}{dt^2}$ to determine whether your answer in part a) is an underestimate or an overestimate of the internal temperature of the marshmallow at time $t = 2$. Explain your reasoning.

$\frac{d^2M}{dt^2} = -\frac{1}{3}\frac{dM}{dt} = \frac{1}{9}(M-20)$. since $M>20$, then $\frac{d^2M}{dt^2}$ is always positive.

So M is always concave up, so $M(2)\approx 36$ is an $\boxed{\text{Underestimate}}$

c) Find an expression for $M(t)$. Based on this model, what is the internal temperature of the marshmallow at time $t = 2$?

$dM = -\frac{1}{3}(M-20)dt$

$\int \frac{dM}{M-20} = \int -\frac{1}{3}dt$

$u = M-20$ ✓
$du = dM$

$\ln|M-20| = -\frac{1}{3}t + C$

use $M(0)=68$

$\ln 48 = C$

$\ln|M-20| = -\frac{1}{3}t + \ln 48$

$|M-20| = e^{-1/3t + \ln 48}$

$M(t) > 20$ so positive

$M - 20 = 48e^{-1/3t}$

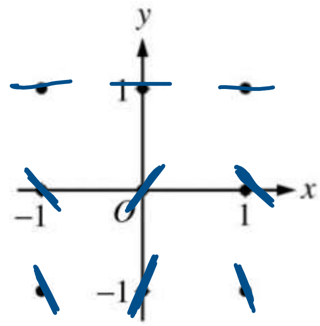
$\boxed{M = 48e^{-1/3t}+20}$

$\boxed{M(2) = 44.644°C}$

Consider the differential equation $\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.

(Note: Use the axes provided in the exam booklet.)



(b) There is a horizontal line with equation $y = c$ that satisfies this differential equation. Find the value of c.

(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 0$.

b) Horizontal line means $\frac{dy}{dx} = 0$, this will happen when $(y-1)^2 = 0$, so $\boxed{y = 1 = c}$

c) $dy = (y-1)^2 \cos(\pi x)\, dx$

$$\int \frac{dy}{(y-1)^2} = \frac{1}{\pi}\int \cos \pi x \, \pi \, dx$$

$u = y-1$
$du = 1dy$ ✓

$u = \pi x$
$du = \pi\, dx$

$$\frac{-1}{y-1} = \frac{1}{\pi}\sin(\pi x) + C$$

$f(1) = 0$

$$1 = \cancel{\frac{1}{\pi}\sin \pi} + C$$

$c = 1$

$$\frac{-1}{y-1} = \frac{1}{\pi}\sin(\pi x) + 1$$

Switcheroo!

$$y - 1 = \frac{-1}{\frac{1}{\pi}\sin(\pi x) + 1}$$

$$\boxed{y = 1 + \frac{-1}{\frac{1}{\pi}\sin(\pi x) + 1} \cdot \frac{\pi}{\pi} \quad \text{or} \quad y = 1 + \frac{-\pi}{\sin(\pi x) + \pi}}$$