Chapter 6 Practice Quiz – Calculator Active

1) Find $y = f(x)$ given $\dfrac{dy}{dx} = y^3(4x+4)$ and $f(-2) = 1$.

2) The rate of change of petroleum entering a barrel is proportional to the product of the square of the amount of petroleum present in the barrel and the amount of empty space in the barrel left to be filled. If p denotes the amount of petroleum present in the barrel and N denotes the total volume of the barrel, which of the following differential equations could be used to model this situation with respect to time t, where k is a positive constant?

a) $\dfrac{dp}{dt} = kp^2$ b) $\dfrac{dp}{dt} = kp^2(N-p)$ c) $\dfrac{dp}{dt} = kp^2(p-N)$ d) $\dfrac{dp}{dt} = kt^2(N-t)$ e) $\dfrac{dp}{dt} = kp^2(N-p^2)$

3) The population of a colony follows on exponential growth pattern. The population started with 1500 citizens and will double every 5 years. How many years, from the start, will it take for the population to reach 10,000?

4) Sea grass grows on a lake. The rate of growth of the grass is $\dfrac{dG}{dt} = kG$, where k is a constant. If the amount of grass is 100 tons initially, and 120 tons after one year, determine how many tons of sea grass will be present after 7 years.

5) Find $f(e)$ given $\dfrac{dy}{dt} = \dfrac{\cos^2 y}{t}$ and $f(1) = 0$.

6) Given the following logistic differential equation $\frac{dP}{dt} = 0.01P(400 - 5P)$, find the $\lim_{t\to\infty} P(t)$, the interval of $P(t)$ where the graph of $P(t)$ is concave down and the $P-$value where $P(t)$ is increasing the fastest.

7) Consider the differential equation $\frac{dy}{dx} = \frac{1}{y(x+1)}$ with $y = f(x)$ and the initial condition $f(0) = 2$.

a) Write the equation of the tangent line to $f(x)$ at $x = 0$ and use this tangent line to estimate $f(1)$.

b) Find the particular solution $y = f(x)$ given the initial condition $f(0) = 2$.

c) Is $f(x)$ increasing or decreasing at $(0, 2)$? Is $f(x)$ concave up or down? Justify.

Answers

1) Separating the variables gives us $\frac{dy}{y^3} = (4x+4)dx$. These both can be integrated, giving us $\frac{-1}{2y^2} = 2x^2 + 4x + C$. Plugging in the initial condition can help us get C, so $\frac{-1}{2} = 0 + C$, $C = \frac{-1}{2}$.

This gives us $\frac{-1}{2y^2} = 2x^2 + 4x - \frac{1}{2}$ and solving for y gives $\frac{-1}{2(2x^2 + 4x - 0.5)} = y^2$ or $y = \sqrt{\frac{-1}{4x^2 + 8x - 1}}$.

2) If p represents the amount present in an N barrel, then the amount left to be filled is $N - p$. So, B.

3) Exponential growth pattern means that this follows the solution $y = Ce^{kt}$ and if the initial population is 1500, then after 5 years, the population is 3000. So, using the first condition, $1500 = Ce^{k(0)}$ or $C = 1500$. Then, to find k, $3000 = 1500e^{k(5)}$, $2 = e^{5k}$, $\frac{\ln 2}{5} = k$. To find how many years it will take to reach 10,000 we will solve for t when $y = 10000$. So, $10000 = 1500e^{0.13863t}$ or $t = 13.685$.

4) This is also exponential growth, so $y = Ce^{kt}$ and if the initial population is 100, then after 1 year, the population is 120. So, using the first condition, $100 = Ce^{k(0)}$ or $C = 100$. Then, to find k, $120 = 100e^{k(1)}$, $1.2 = e^{k}$, $\ln 1.2 = k$. To find how many tons will be present in 7 years, we will solve for y when $t = 7$. So, $y = 100e^{(7)\ln 1.2}$ or 358.318.

5) Separating variables gives us $\frac{dy}{\cos^2 y} = \frac{dt}{t}$. The left integral cannot be done as it is, but with a little rewriting, it can fit one of the rules. So, $\int \sec^2 y\,dy = \int \frac{dt}{t}$, which has the antiderivatives of $\tan y = \ln|t| + C$. Using the initial condition $f(1) = 0$, we get $\tan 0 = \ln 1 + C$, $0 = 0 + C$, $C = 0$. So, $\tan y = \ln|t|$ and solving for y gives us $y = \arctan(\ln|t|)$. So, $f(e) = \arctan(\ln e) = \arctan 1 = \frac{\pi}{4}$.

6) Since this is logistic, the $\lim_{t\to\infty} P(t)$ is the carrying capacity. Therefore, we must rewrite the differential equation to fit the form. Getting "1 – " by factoring out the 400 gives us $\frac{dP}{dt} = 4P\left(1 - \frac{P}{80}\right)$. Therefore, the $\lim_{t\to\infty} P(t) = 80$ and $P(t)$ is growing the fastest at $P = \frac{L}{2}$ or $P = 40$. Knowing the logistic curve, $P(t)$ is concave up and then concave down when $P(t)$ is between $\left(\frac{L}{2}, L\right)$ so $(40, 80)$,

7) a) We need point and slope to write the equation of a tangent line. We are given the point of $f(0) = 2$ and the slope can be found using the given derivative. So, $\left.\frac{dy}{dx}\right|_{(0,2)} = \frac{1}{2}$ and $y - 2 = \frac{1}{2}(x - 0)$. Then, plugging in $x = 1$ gives us $f(1) \approx 2.5$.

b) Separating variables gives us $\int y\,dy = \int \frac{dx}{x+1}$. The right integrand requires a u-sub with $u = x + 1$, $du = 1dx$ or $\int y\,dy = \int \frac{du}{u}$ and $\frac{1}{2}y^2 = \ln|x+1| + C$. Using the condition, $2 = \ln|1| + C$ or $C = 2$. Then, solving for y, $\frac{1}{2}y^2 = \ln|x+1| + 2$, $y^2 = 2\ln|x+1| + 4$ or $y = \sqrt{4 + 2\ln|x+1|}$.

c) Since $\left.\frac{dy}{dx}\right|_{(0,\,2)} = \frac{1}{2} > 0$, then $y = f(x)$ is increasing. To determine concavity, we need the second derivative. So, $\frac{d^2y}{dx^2} = \left(\frac{0 - 1\left(\frac{dy}{dx}(x+1) + y(1) \right)}{\left(y(x+1)\right)^2} \right)$ due to implicit derivatives and a product rule.

Evaluating this at $f(0) = 2$, gives us $\left.\frac{d^2y}{dx^2}\right|_{(0,\,2)} = \frac{0 - \left(\frac{1}{2} + 2 \right)}{(2)^2} < 0$ so, concave down.