Probability

7.1 INTRODUCTION

Probability theory is a mathematical modeling of the phenomenon of chance or randomness. If a coin is tossed in a random manner, it can land heads or tails, but we do not know which of these will occur in a single toss. However, suppose we let s be the number of times heads appears when the coin is tossed n times. As n increases, the ratio $f = s/n$, called the *relative frequency* of the outcome, becomes more stable. If the coin is perfectly balanced, then we expect that the coin will land heads approximately 50% of the time or, in other words, the relative frequency will approach $\frac{1}{2}$. Alternatively, assuming the coin is perfectly balanced, we can arrive at the value $\frac{1}{2}$ deductively. That is, any side of the coin is as likely to occur as the other; hence the chance of getting a head is 1 in 2 which means the probability of getting heads is $\frac{1}{2}$. Although the specific outcome on any one toss is unknown, the behavior over the long run is determined. This stable long-run behavior of random phenomena forms the basis of probability theory.

A probabilistic mathematical model of random phenomena is defined by assigning "probabilities" to all the possible outcomes of an experiment. The reliability of our mathematical model for a given experiment depends upon the closeness of the assigned probabilities to the actual limiting relative frequencies. This then gives rise to problems of testing and reliability, which form the subject matter of statistics and which lie beyond the scope of this text.

7.2 SAMPLE SPACE AND EVENTS

The set S of all possible outcomes of a given experiment is called the *sample space*. A particular outcome, i.e., an element in S, is called a *sample point*. An *event* A is a set of outcomes or, in other words, a subset of the sample space S. In particular, the set $\{a\}$ consisting of a single sample point $a \in S$ is called an *elementary event*. Furthermore, the empty set $\varnothing$ and S itself are subsets of S and so $\varnothing$ and S are also events; $\varnothing$ is sometimes called the *impossible event* or the *null event*.

Since an event is a set, we can combine events to form new events using the various set operations:

(i) $A \cup B$ is the event that occurs iff A occurs *or* B occurs (or both).
(ii) $A \cap B$ is the event that occurs iff A occurs *and* B occurs.
(iii) A^{c}, the complement of A, also written $\overline{A}$, is the event that occurs iff A does *not* occur.

Two events A and B are called *mutually exclusive* if they are disjoint, that is, if $A \cap B = \varnothing$. In other words, A and B are mutually exclusive iff they cannot occur simultaneously. Three or more events are mutually exclusive if every two of them are mutually exclusive.

EXAMPLE 7.1

(a) ***Experiment***: Toss a coin three times and observe the sequence of heads (H) and tails (T) that appears.

The sample space consists of the following eight elements:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Let A be the event that two or more heads appear consecutively, and B that all the tosses are the same:

$$A = \{HHH, HHT, THH\} \quad \text{and} \quad B = \{HHH, TTT\}$$

Then $A \cap B = \{HHH\}$ is the elementary event that only heads appear. The event that five heads appears is the empty set $\varnothing$.

(b) ***Experiment***: Toss a (six-sided) die, pictured in Fig. 7-1(a), and observe the number (of dots) that appear on top.

The sample space S consists of the six possible numbers, that is, $S = \{1, 2, 3, 4, 5, 6\}$. Let A be the event that an even number appears, B that an odd number appears, and C that a prime number appears. That is, let

$$A = \{2, 4, 6\}, \quad B = \{1, 3, 5\}, \quad C = \{2, 3, 5\}$$

Then

$A \cup C = \{2, 3, 4, 5, 6\}$ is the event that an even or a prime number occurs.

$B \cap C = \{3, 5\}$ is the event that an odd prime number occurs.

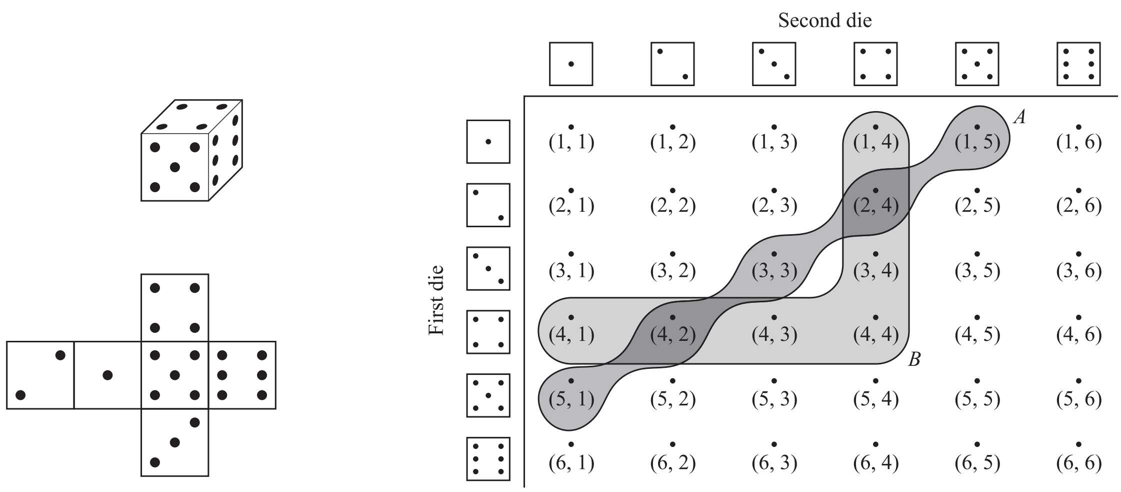
$C^c = \{1, 4, 6\}$ is the event that a prime number does not occur.

Note that A and B are mutually exclusive: $A \cap B = \varnothing$. In other words, an even number and an odd number cannot occur simultaneously.

(c) ***Experiment***: Toss a coin until a head appears, and count the number of times the coin is tossed.

The sample space S of this experiment is $S = \{1, 2, 3, \ldots\}$. Since every positive integer is an element of S, the sample space is infinite.

Remark: The sample space S in Example 7.1(c), as noted, is not finite. The theory concerning such sample spaces lies beyond the scope of this text. Thus, unless otherwise stated, all our sample spaces S shall be finite.



(a) (b)

Fig. 7-1

EXAMPLE 7.2 (Pair of dice) Toss a pair of dice and record the two numbers on the top.

There are six possible numbers, 1, 2, ..., 6, on each die. Thus S consists of the pairs of numbers from 1 to 6, and hence $n(S) = 36$. Figure 7-1(b) shows these 36 pairs of numbers arranged in an array where the rows are labeled by the first die and the columns by the second die.

Let A be the event that the sum of the two numbers is 6, and let B be the event that the largest of the two numbers is 4. That is, let

$$A = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}, \quad B = \{(1,4), (2,4), (3,4), (4,4), (4,3), (4,2), (4,1)\}$$

Then the event "A and B" consists of those pairs of integers whose sum is 6 and whose largest number is 4 or, in other words, the intersection of A and B. Thus

$$A \cap B = \{(2,4), (4,2)\}$$

Similarly, "A or B," the sum is 6 or the largest is 4, shaded in Fig. 7-1(b), is the union $A \cup B$.

EXAMPLE 7.3 (Deck of cards) A card is drawn from an ordinary deck of 52 cards which is pictured in Fig. 7-2(a).

The sample space S consists of the four *suits*, clubs (C), diamonds (D), hearts (H), and spades (S), where each suit contains 13 cards which are numbered 2 to 10, and jack (J), queen (Q), king (K), and ace (A). The hearts (H) and diamonds (D) are red cards, and the spades (S) and clubs (C) are black cards. Figure 7-2(b) pictures 52 points which represent the deck S of cards in the obvious way. Let E be the event of a *face card*, that is, a Jack (J), Queen (Q), or King (K), and let F be the event of a heart. Then $E \cap F = \{JH, QH, KH\}$, as shaded in Fig. 7-2($b$).

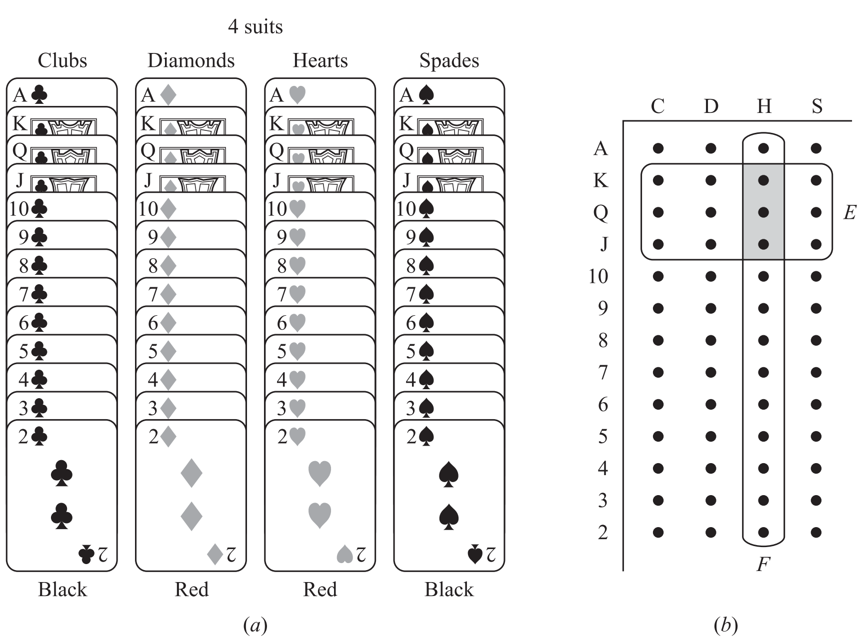


Fig. 7-2

7.3 FINITE PROBABILITY SPACES

The following definition applies.

Definition 7.1: Let S be a finite sample space, say $S = \{a_1, a_2, \ldots, a_n\}$. A *finite probability space*, or *probability model*, is obtained by assigning to each point a_i in S a real number p_i, called the *probability* of a_i satisfying the following properties:

(i) Each p_i is nonnegative, that is, $p_i \geq 0$.

(ii) The sum of the p_i is 1, that is, is $p_1 + p_2 + \cdots + p_n = 1$.

The *probability* of an event A written $P(A)$, is then defined to be the sum of the probabilities of the points in A.

The singleton set $\{a_i\}$ is called an *elementary* event and, for notational convenience, we write $P(a_i)$ for $P(\{a_i\})$.

EXAMPLE 7.4 (Experiment) Suppose three coins are tossed, and the number of heads is recorded. (Compare with the above Example 7.1(a).)

The sample space is $S = \{0, 1, 2, 3\}$. The following assignments on the elements of S define a probability space:

$$P(0) = \tfrac{1}{8}, \quad P(1) = \tfrac{3}{8}, \quad P(2) = \tfrac{3}{8}, \quad P(3) = \tfrac{1}{8}$$

That is, each probability is nonnegative, and the sum of the probabilities is 1. Let A be the event that at least one head appears, and let B be the event that all heads or all tails appear; that is, let $A = \{1, 2, 3\}$ and $B = \{0, 3\}$. Then, by definition,

$$P(A) = P(1) + P(2) + P(3) = \tfrac{3}{8} + \tfrac{3}{8} + \tfrac{1}{8} = \tfrac{7}{8} \quad \text{and} \quad P(B) = P(0) + P(3) = \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{1}{4}$$

Equiprobable Spaces

Frequently the physical characteristics of an experiment suggest that the various outcomes of the sample space be assigned equal probabilities. Such a finite probability space S, where each sample point has the same probability, will be called an *equiprobable space*. In particular, if S contains n points, then the probability of each point is $1/n$. Furthermore, if an event A contains r points, then its probability is $r(1/n) = r/n$. In other words, where $n(A)$ denotes the number of elements in a set A,

$$P(A) = \frac{\text{number of elements in } A}{\text{number of elements in } S} = \frac{n(A)}{n(S)} \quad \text{or} \quad P(A) = \frac{\text{number of outcomes favorable to } A}{\text{total number of possible outcomes}}$$

We emphasize that the above formula for $P(A)$ can only be used with respect to an equiprobable space, and cannot be used in general.

The expression *at random* will be used only with respect to an equiprobable space; the statement "choose a point at random from a set S" shall mean that every sample point in S has the same probability of being chosen.

EXAMPLE 7.5 Let a card be selected from an ordinary deck of 52 playing cards. Let

$$A = \{\text{the card is a spade}\} \quad \text{and} \quad B = \{\text{the card is a face card}\}.$$

We compute $P(A)$, $P(B)$, and $P(A \cap B)$. Since we have an equiprobable space,

$$P(A) = \frac{\text{number of spades}}{\text{number of cards}} = \frac{13}{52} = \frac{1}{4}, \quad P(B) = \frac{\text{number of face cards}}{\text{number of cards}} = \frac{12}{52} = \frac{3}{13}$$

$$P(A \cap B) = \frac{\text{number of spade face cards}}{\text{number of cards}} = \frac{3}{52}$$

Theorems on Finite Probability Spaces

The following theorem follows directly from the fact that the probability of an event is the sum of the probabilities of its points.

Theorem 7.1: The probability function P defined on the class of all events in a finite probability space has the following properties:

[$\mathbf{P}_1$] For every event A, $0 \leq P(A) \leq 1$.

[$\mathbf{P}_2$] $P(S) = 1$.

[$\mathbf{P}_3$] If events A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$.

The next theorem formalizes our intuition that if p is the probability that an event E occurs, then $1 - p$ is the probability that E does not occur. (That is, if we hit a target $p = 1/3$ of the times, then we miss the target $1 - p = 2/3$ of the times.)

Theorem 7.2: Let A be any event. Then $P(A^c) = 1 - P(A)$.

The following theorem (proved in Problem 7.13) follows directly from Theorem 7.1.

Theorem 7.3: Consider the empty set $\varnothing$ and any events A and B. Then:

(i) $P(\varnothing) = 0$.

(ii) $P(A \backslash B) = P(A) - P(A \cap B)$.

(iii) If $A \subseteq B$, then $P(A) \leq P(B)$.

Observe that Property [$\mathbf{P}_3$] in Theorem 7.1 gives the probability of the union of events in the case that the events are disjoint. The general formula (proved in Problem 7.14) is called the Addition Principle. Specifically:

Theorem 7.4 (Addition Principle): For any events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

EXAMPLE 7.6 Suppose a student is selected at random from 100 students where 30 are taking mathematics, 20 are taking chemistry, and 10 are taking mathematics and chemistry. Find the probability p that the student is taking mathematics or chemistry.

Let $M = \{\text{students taking mathematics}\}$ and $C = \{\text{students taking chemistry}\}$. Since the space is equiprobable,

$$P(M) = \frac{30}{100} = \frac{3}{10}, \quad P(C) = \frac{20}{100} = \frac{1}{5}, \quad P(M \text{ and } C) = P(M \cap C) = \frac{10}{100} = \frac{1}{10}$$

Thus, by the Addition Principle (Theorem 7.4),

$$p = P(M \text{ or } C) = P(M \cup C) = P(M) + P(C) - P(M \cap C) = \frac{3}{10} + \frac{1}{5} - \frac{1}{10} = \frac{2}{5}$$

7.4 CONDITIONAL PROBABILITY

Suppose E is an event in a sample space S with $P(E) > 0$. The probability that an event A occurs once E has occurred or, specifically, the *conditional probability of A given E*. written $P(A|E)$, is defined as follows:

$$P(A|E) = \frac{P(A \cap E)}{P(E)}$$

As pictured in the Venn diagram in Fig. 7-3, $P(A|E)$ measures, in a certain sense, the relative probability of A with respect to the reduced space E.

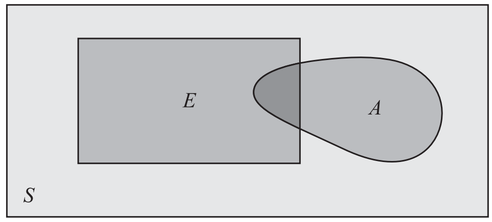


Fig. 7-3

Suppose S is an equiprobable space, and $n(A)$ denotes the number of elements in A. Then:

$$P(A \cap E) = \frac{n(A \cap E)}{n(S)}, \quad P(E) = \frac{n(E)}{n(S)}, \quad \text{and so} \quad P(A|E) = \frac{P(A \cap E)}{P(E)} = \frac{n(A \cap E)}{n(E)}$$

We state this result formally.

Theorem 7.5: Suppose S is an equiprobable space and A and E are events. Then

$$P(A|E) = \frac{\text{number of elements in } A \cap E}{\text{number of elements in } E} = \frac{n(A \cap E)}{n(E)}$$

EXAMPLE 7.7

(a) A pair of fair dice is tossed. The sample space S consists of the 36 ordered pairs (a, b), where a and b can be any of the integers from 1 to 6. (See Example 7.2.) Thus the probability of any point is $\frac{1}{36}$. Find the probability that one of the dice is 2 if the sum is 6. That is, find $P(A|E)$ where:

$$E = \{\text{sum is } 6\} \quad \text{and} \quad A = \{2 \text{ appears on at least one die}\}$$

Now E consists of 5 elements and $A \cap E$ consists of two elements; namely

$$E = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\} \quad \text{and} \quad A \cap E = \{(2, 4), (4, 2)\}$$

By Theorem 7.5, $P(A|E) = 2/5$.

On the other hand A itself consists of 11 elements, that is,

$$A = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (1, 2), (3, 2), (4, 2), (5, 2), (6, 2)\}$$

Since S consists of 36 elements, $P(A) = 11/36$.

(b) A couple has two children; the sample space is $S = \{bb, bg, gb, gg\}$ with probability $\frac{1}{4}$ for each point. Find the probability p that both children are boys if it is known that: (i) at least one of the children is a boy; (ii) the older child is a boy.

(i) Here the reduced space consists of three elements, $\{bb, bg, gb\}$; hence $p = \frac{1}{3}$.

(ii) Here the reduced space consists of only two elements $\{bb, bg\}$; hence $p = \frac{1}{2}$.

Multiplication Theorem for Conditional Probability

Suppose A and B are events in a sample space S with $P(A) > 0$. By definition of conditional probability,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Theorem 7.6 (Multiplication Theorem for Conditional Probability):

$$P(A \cap B) = P(A)P(B|A)$$

The multiplication theorem gives us a formula for the probability that events A and B both occur. It can easily be extended to three or more events $A_1, A_2, \ldots A_m$; that is,

$$P(A_1 \cap A_2 \cap \cdots A_m) = P(A_1) \cdot P(A_2|A_1) \cdots P(A_m|A_1 \cap A_2 \cap \cdots \cap A_{m-1})$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2)$$

EXAMPLE 7.8 A lot contains 12 items of which 4 are defective. Three items are drawn at random from the lot one after the other. Find the probability p that all three are nondefective.

The probability that the first item is nondefective is $\frac{8}{12}$ since 8 of 12 items are nondefective. If the first item is nondefective, then the probability that the next item is nondefective is $\frac{7}{11}$ since only 7 of the remaining 11 items are nondefective. If the first 2 items are nondefective, then the probability that the last item is nondefective is $\frac{6}{10}$ since only 6 of the remaining 10 items are now nondefective. Thus by the multiplication theorem,

$$p = \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} = \frac{14}{55} \approx 0.25$$

7.5 INDEPENDENT EVENTS

Events A and B in a probability space S are said to be *independent* if the occurrence of one of them does not influence the occurrence of the other. More specifically, B is independent of A if $P(B)$ is the same as $P(B|A)$. Now substituting $P(B)$ for $P(B|A)$ in the Multiplication Theorem $P(A \cap B) = P(A)P(B|A)$ yields

$$P(A \cap B) = P(A)P(B).$$

We formally use the above equation as our definition of independence.

Definition 7.2: Events A and B are *independent* if $P(A \cap B) = P(A)P(B)$; otherwise they are *dependent*.

We emphasize that independence is a symmetric relation. In particular, the equation

$$P(A \cap B) = P(A)P(B) \quad \text{implies both} \quad P(B|A) = P(B) \quad \text{and} \quad P(A|B) = P(A)$$

EXAMPLE 7.9 A fair coin is tossed three times yielding the equiprobable space

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Consider the events:

$$\begin{aligned} A &= \{\text{first toss is heads}\} = \{HHH, HHT, HTH, HTT\} \\ B &= \{\text{second toss is heads}) = \{HHH, HHT, THH, THT\} \\ C &= \{\text{exactly two heads in a row}\} = \{HHT, THH\} \end{aligned}$$

Clearly A and B are independent events; this fact is verified below. On the other hand, the relationship between A and C and between B and C is not obvious. We claim that A and C are independent, but that B and C are dependent. We have:

$$P(A) = \tfrac{4}{8} = \tfrac{1}{2}, \quad P(B) = \tfrac{4}{8} = \tfrac{1}{2}, \quad P(C) = \tfrac{2}{8} = \tfrac{1}{4}$$

Also,

$$P(A \cap B) = P(\{HHH, HHT\}) = \tfrac{1}{4}, \quad P(A \cap C) = P(\{HHT\}) = \tfrac{1}{8}, \quad P(B \cap C) = P(\{HHT, THH\}) = \tfrac{1}{4}$$

Accordingly,

$$\begin{aligned}
P(A)P(B) &= \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4} = P(A \cap B), && \text{and so } A \text{ and } B \text{ are independent} \\
P(A)P(C) &= \tfrac{1}{2} \cdot \tfrac{1}{4} = \tfrac{1}{8} = P(A \cap C), && \text{and so } A \text{ and } C \text{ are independent} \\
P(B)P(C) &= \tfrac{1}{2} \cdot \tfrac{1}{4} = \tfrac{1}{8} \neq P(B \cap C), && \text{and so } B \text{ and } C \text{ are dependent}
\end{aligned}$$

Frequently, we will postulate that two events are independent, or the experiment itself will imply that two events are independent.

EXAMPLE 7.10 The probability that A hits a target is $\frac{1}{4}$, and the probability that B hits the target is $\frac{2}{5}$. Both shoot at the target. Find the probability that at least one of them hits the target, i.e., that A or B (or both) hit the target.

We are given that $P(A) = \frac{1}{4}$ and $P(B) = \frac{2}{5}$, and we seek $P(A \cup B)$. Furthermore, the probability that A or B hits the target is not influenced by what the other does; that is, the event that A hits the target is independent of the event that B hits the target, that is, $P(A \cap B) = P(A)P(B)$. Thus

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) = \tfrac{1}{4} + \tfrac{2}{5} - \left(\tfrac{1}{4}\right)\left(\tfrac{2}{5}\right) = \tfrac{11}{20}$$

Solved Problems

SAMPLE SPACES AND EVENTS

7.1. Let a coin and a die be tossed; and let the sample space S consists of the 12 elements:

$$S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$$

(a) Express explicitly the following events:

$A = \{\text{heads and an even number}\}$, $B = \{\text{prime number}\}$, $C = \{\text{tails and an odd number}\}$

(b) Express explicitly the events: (i) A or B occurs; (ii) B and C occur: (iii) only B occurs.

(c) Which pair of the events A, B, and C are mutually exclusive?

(a) The elements of A are those elements of S consisting of an H and an even number:

$$A = \{H2, H4, H6\}$$

The elements of B are those points in S whose second component is a prime number (2, 3, or 5):

$$B = \{H2, H3, H5, T2, T3, T5\}$$

The elements of C are those points in S consisting of a T and an odd number; $C = \{T1, T3, T5\}$.

(b) (i) $A \cup B = \{H2, H4, H6, H3, H5, T2, T3, T5\}$
(ii) $B \cap C = \{T3, T5\}$
(iii) $B \cap A^c \cap C^c = \{H3, H5, T2\}$

(c) A and C are mutually exclusive since $A \cap C = \varnothing$.

7.2. A pair of dice is tossed. (See Example 7.2.) Find the number of elements in each event:

(a) $A = \{\text{two numbers are equal}\}$
(b) $B = \{\text{sum is 10 or more}\}$
(c) $C = \{\text{5 appears on the first die}\}$
(d) $D = \{\text{5 appears on at least one die}\}$

Use Fig. 7-1(b) to help count the number of elements in the event.

(a) $A = \{(1, 1), (2, 2), \ldots, (6, 6)\}$, so $n(A) = 6$.

(b) $B = \{(6, 4), (5, 5), (4, 6), (6, 5), (5, 6), (6, 6)\}$, so $n(B) = 6$.

(c) $C = \{(5, 1), (5, 2), \ldots, (5, 6)\}$, so $n(C) = 6$.

(d) There are six pairs with 5 as the first element, and six pairs with 5 as the second element. However, (5, 5) appears in both places. Hence

$$n(D) = 6 + 6 - 1 = 11$$

Alternately, count the pairs in Fig. 7-1(b) which are in D to get $n(D) = 11$.

FINITE EQUIPROBABLE SPACES

7.3 Determine the probability p of each event:

(a) An even number appears in the toss of a fair die;

(b) One or more heads appear in the toss of three fair coins;

(c) A red marble appears in a random drawing of one marble from a box containing four white, three red, and five blue marbles.

Each sample space S is an equiprobable space. Thus, for each event E, use:

$$P(E) = \frac{\text{number of elements in } E}{\text{number of elements in } S} = \frac{n(E)}{n(S)}$$

(a) The event can occur in three ways (2, 4 or 6) out of 6 cases; hence $p = \frac{3}{6} = \frac{1}{2}$.

(b) There are 8 cases:

$$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$$

Only the last case is not favorable; hence $p = 7/8$.

(c) There are $4 + 3 + 5 = 12$ marbles of which three are red; hence $p = \frac{3}{12} = \frac{1}{4}$.

7.4. A single card is drawn from an ordinary deck of 52 cards. (See Fig. 7-2.) Find the probability p that the card is a:

(a) face card (jack, queen or king); (c) face card and a heart;
(b) heart; (d) face card or a heart.

Here $n(S) = 52$.

(a) There are $4(3) = 12$ face cards; hence $p = \dfrac{12}{52} = \dfrac{3}{13}$.

(b) There are 13 hearts; hence $p = \dfrac{13}{52} = \dfrac{1}{4}$.

(c) There are three face cards which are hearts; hence $p = \dfrac{3}{52}$.

(d) Letting $F = \{\text{face cards}\}$ and $H = \{\text{hearts}\}$, we have

$$n(F \cup H) = n(F) + n(H) - n(F \cap H) = 12 + 13 - 3 = 22$$

Hence $p = \frac{22}{52} = \frac{11}{26}$.

7.5. Two cards are drawn at random from an ordinary deck of 52 cards. Find the probability p that: (a) both are spades; (b) one is a spade and one is a heart.

There are $\dbinom{52}{2} = 1326$ ways to draw 2 cards from 52 cards.

(a) There are $\dbinom{13}{2} = 78$ ways to draw 2 spades from 13 spades; hence

$$p = \frac{\text{number of ways 2 spades can be drawn}}{\text{number of ways 2 cards can be drawn}} = \frac{78}{1326} = \frac{3}{51}$$

(b) There are 13 spades and 13 hearts, so there are $13 \cdot 13 = 169$ ways to draw a spade and a heart. Thus $p = \frac{169}{1326} = \frac{13}{102}$.

7.6. Consider the sample space in Problem 7.1. Assume the coin and die are fair; hence S is an equiprobable space. Find:

(a) $P(A)$, $P(B)$, $P(C)$

(b) $P(A \cup B)$, $P(B \cap C)$, $P(B \cap A^{C} \cap C^{C})$

Since S is an equiprobable space, use $P(E) = n(E)/n(S)$. Here $n(S) = 12$. So we need only count the number of elements in the given set.

(a) $P(A) = \frac{3}{12}$, $P(B) = \frac{6}{12}$, $P(C) = \frac{3}{12}$

(b) $P(A \cup B) = \frac{8}{12}$, $P(B \cap C) = \frac{2}{12}$, $P(B \cap A^{C} \cap C^{C}) = \frac{3}{12}$

7.7. A box contains two white socks and two blue socks. Two socks are drawn at random. Find the probability p they are a match (the same color).

There are $\dbinom{4}{2} = 6$ ways to draw two of the socks. Only two pairs will yield a match. Thus $p = \frac{2}{6} = \frac{1}{3}$.

7.8. Five horses are in a race. Audrey picks two of the horses at random, and bets on them. Find the probability p that Audrey picked the winner.

There are $\dbinom{5}{2} = 10$ ways to pick two of the horses. Four of the pairs will contain the winner. Thus $p = \frac{4}{10} = \frac{2}{5}$.

FINITE PROBABILITY SPACES

7.9. A sample space S consists of four elements; that is, $S = \{a_1, a_2, a_3, a_4\}$. Under which of the following functions does S become a probability space?

(*a*) $P(a_1) = \frac{1}{2}$ $P(a_2) = \frac{1}{3}$ $P(a_3) = \frac{1}{4}$ $P(a_4) = \frac{1}{5}$
(*b*) $P(a_1) = \frac{1}{2}$ $P(a_2) = \frac{1}{4}$ $P(a_3) = -\frac{1}{4}$ $P(a_4) = \frac{1}{2}$
(*c*) $P(a_1) = \frac{1}{2}$ $P(a_2) = \frac{1}{4}$ $P(a_3) = \frac{1}{8}$ $P(a_4) = \frac{1}{8}$
(*d*) $P(a_1) = \frac{1}{2}$ $P(a_2) = \frac{1}{4}$ $P(a_3) = \frac{1}{4}$ $P(a_4) = 0$

(*a*) Since the sum of the values on the sample points is greater than one, the function does not define S as a probability space.

(*b*) Since $P(a_3)$ is negative, the function does not define S as a probability space.

(*c*) Since each value is nonnegative and the sum of the values is one, the function does define S as a probability space.

(*d*) The values are nonnegative and add up to one; hence the function does define S as a probability space.

7.10. A coin is weighted so that heads is twice as likely to appear as tails. Find $P(T)$ and $P(H)$.

Let $P(T) = p$; then $P(H) = 2p$. Now set the sum of the probabilities equal to one, that is, set $p + 2p = 1$. Then $p = \frac{1}{3}$. Thus $P(H) = \frac{1}{3}$ and $P(T) = \frac{2}{3}$.

7.11. Suppose A and B are events with $P(A) = 0.6$, $P(B) = 0.3$, and $P(A \cap B) = 0.2$. Find the probability that:

(*a*) A does not occur; (*c*) A or B occurs;
(*b*) B does not occur; (*d*) Neither A nor B occurs.

(*a*) $P(\text{not } A) = P(A^C) = 1 - P(A) = 0.4$.

(*b*) $P(\text{not } B) = P(B^C) = 1 - P(B) = 0.7$.

(*c*) By the Addition Principle,

$$\begin{aligned} P(A \text{ or } B) &= P(A \cup B) = P(A) + P(B) - P(A \cap B) \\ &= 0.6 + 0.3 - 0.2 = 0.7 \end{aligned}$$

(*d*) Recall (DeMorgan's Law) that neither A nor B is the complement of $A \cup B$. Thus:

$$P(\text{neither } A \text{ nor } B) = P((A \cup B)^C) = 1 - P(A \cup B) = 1 - 0.7 = 0.3$$

7.12. Prove Theorem 7.2: $P(A^c) = 1 - P(A)$.

$S = A \cup A^c$ where A and A^c are disjoint. Our result follows from the following:

$$1 = P(S) = P(A \cup A^c) = P(A) + P(A^c)$$

7.13. Prove Theorem 7.3: (i) $P(\varnothing) = 0$; (ii) $P(A \backslash B) = P(A) - P(A \cap B)$; (iii) If $A \subseteq B$, then $P(A) \le P(B)$.

(i) $\varnothing = S^c$ and $P(S) = 1$. Thus $P(\varnothing) = 1 - 1 = 0$.

(ii) As indicated by Fig. 7-6(*a*), $A = (A \backslash B) \cup (A \cap B)$ where $A \backslash B$ and $A \cap B$ are disjoint. Hence

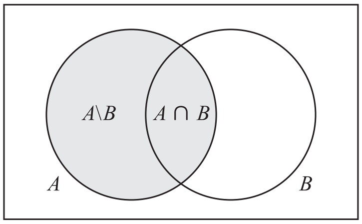
$$P(A) = P(A \backslash B) + P(A \cap B)$$

From which our result follows.

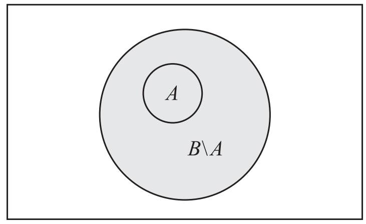
(iii) If $A \subseteq B$, then, as indicated by Fig. 7-6(*b*), $B = A \cup (B \backslash A)$ where A and $B \backslash A$ are disjoint. Hence

$$P(B) = P(A) + P(B \backslash A)$$

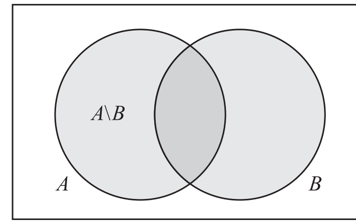
Since $P(B \backslash A) \ge 0$, we have $P(A) \le P(B)$.



(*a*) A is shaded



(*b*) B is shaded



(*c*) $A \cup B$ is shaded

Fig. 7-6

7.14. Prove Theorem 7.4 (Addition Principle): For any events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

As indicated by Fig. 7-6(*c*), $A \cup B = (A \backslash B) \cup B$ where $A \backslash B$ and B are disjoint sets. Thus, using Theorem 7.3(ii),

$$\begin{aligned} P(A \cup B) &= P(A \backslash B) + P(B) = P(A) - P(A \cap B) + P(B) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

CONDITIONAL PROBABILITY

7.15. A pair of fair dice is thrown. (See Fig. 7-1(*b*).) Find the probability that the sum is 10 or greater if: (*a*) 5 appears on the first die; (*b*) 5 appears on at least one die.

(*a*) If a 5 appears on the first die, then the reduced sample space is

$$A = \{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$$

The sum is 10 or greater on two of the six outcomes: (5, 5), (5, 6). Hence $p = \frac{2}{6} = \frac{1}{3}$.

(*b*) If a 5 appears on at least one of the dice, then the reduced sample space has eleven elements.

$$B = \{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (1, 5), (2, 5), (3, 5), (4, 5), (6, 5)\}$$

The sum is 10 or greater on three of the eleven outcomes: (5, 5), (5, 6), (6, 5). Hence $p = \frac{3}{11}$.

7.16. In a certain college town, 25% of the students failed mathematics (M), 15% failed chemistry (C), and 10% failed both mathematics and chemistry. A student is selected at random.

(*a*) If he failed chemistry, find the probability that he also failed mathematics.
(*b*) If he failed mathematics, find the probability that he also failed chemistry.
(*c*) Find the probability that he failed mathematics or chemistry.
(*d*) Find the probability that he failed neither mathematics nor chemistry.

(*a*) The probability that a student failed mathematics, given that he failed chemistry, is

$$P(M|C) = \frac{P(M \cap C)}{P(C)} = \frac{0.10}{0.15} = \frac{2}{3}$$

(*b*) The probability that a student failed chemistry, given that he failed mathematics is

$$P(C|M) = \frac{P(C \cap M)}{P(M)} = \frac{0.10}{0.25} = \frac{2}{5}$$

(*c*) By the Addition Principle (Theorem 7.4),

$$P(M \cup C) = P(M) + P(C) - P(M \cap C) = 0.25 + 0.15 - 0.10 = 0.30$$

(*d*) Students who failed neither mathematics nor chemistry form the complement of the set $M \cup C$, that is, they form the set $(M \cup C)^C$. Hence

$$P((M \cup C)^C) = 1 - P(M \cup C) = 1 - 0.30 = 0.70$$

7.17. A pair of fair dice is thrown. Given that the two numbers appearing are different, find the probability p that: (*a*) the sum is 6; (*b*) an one appears; (*c*) the sum is 4 or less.

There are 36 ways the pair of dice can be thrown, and six of them, (1, 1), (2, 2), ..., (6, 6), have the same numbers. Thus the reduced sample space will consist of $36 - 6 = 30$ elements.

(*a*) The sum 6 can appear in four ways: (1, 5), (2, 4), (4, 2), (5, 1). (We cannot include (3, 3) since the numbers, are the same.) Hence $p = \frac{4}{30} = \frac{2}{15}$.

(*b*) An one can appear in 10 ways: (1, 2), (1, 3), ..., (1, 6) and (2, 1), (3, 1), ..., (6, 1). Therefore $p = \frac{10}{30} = \frac{1}{3}$.

(*c*) The sum of 4 or less can occur in four ways: (3, 1), (1, 3), (2, 1), (1, 2). Thus $p = \frac{4}{30} = \frac{2}{15}$.

7.18. A class has 12 boys and 4 girls. Suppose three students are selected at random from the class. Find the probability p that they are all boys.

The probability that the first student selected is a boy is $12/16$ since there are 12 boys out of 16 students. If the first student is a boy, then the probability that the second is a boy is $11/15$ since there are 11 boys left out of 15 students. Finally, if the first two students selected were boys, then the probability that the third student is a boy is $10/14$ since there are 10 boys left out of 14 students. Thus, by the multiplication theorem, the probability that all three are boys is

$$p = \frac{12}{16} \cdot \frac{11}{15} \cdot \frac{10}{14} = \frac{11}{28}$$

Another Method

There are $C(16, 3) = 560$ ways to select three students out of the 16 students, and $C(12, 3) = 220$ ways to select three boys out of 12 boys; hence

$$p = \frac{220}{560} = \frac{11}{28}$$

Another Method

If the students are selected one after the other, then there are $16 \cdot 15 \cdot 14$ ways to select three students, and $12 \cdot 11 \cdot 10$ ways to select three boys; hence

$$p = \frac{2 \cdot 11 \cdot 10}{16 \cdot 15 \cdot 14} = \frac{11}{28}$$

INDEPENDENCE

7.19. The probability that A hits a target is $\frac{1}{3}$ and the probability that B hits a target is $\frac{1}{5}$. They both fire at the target. Find the probability that:

(*a*) A does not hit the target; (*c*) one of them hits the target;

(*b*) both hit the target; (*d*) neither hits the target.

We are given $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{5}$ (and we assume the events are independent).

(*a*) $P(\text{not } A) = P(A^C) = 1 - P(A) = 1 - \frac{1}{3} = \frac{2}{3}$.

(*b*) Since the events are independent,

$$P(A \text{ and } B) = P(A \cap B) = P(A) \cdot P(B) = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$$

(*c*) By the Addition Principle (Theorem 7.4),

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{1}{5} - \frac{1}{15} = \frac{7}{15}$$

(d) We have

$$P(\text{neither } A \text{ nor } B) = P((A \cup B)^{C}) = 1 - P(A \cup B) = 1 - \tfrac{7}{15} = \tfrac{8}{15}$$

7.20. Consider the following events for a family with children:

$$A = (\text{children of both sexes}\}, \quad B = \{\text{at most one boy}\}.$$

(a) Show that A and B are independent events if a family has three children.

(b) Show that A and B are dependent events if a family has only two children.

(a) We have the equiprobable space $S = \{bbb, bbg, bgb, bgg, gbb, gbg, ggb, ggg\}$. Here

$$\begin{array}{lll} A = \{bbg, bgb, bgg, gbb, gbg, ggb\} & \text{and so} & P(A) = \frac{6}{8} = \frac{3}{4} \\ B = \{bgg, gbg, ggb, ggg\} & \text{and so} & P(B) = \frac{4}{8} = \frac{1}{2} \\ A \cap B = \{bgg, gbg, ggb\} & \text{and so} & P(A \cap B) = \frac{3}{8} \end{array}$$

Since $P(A)P(B) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8} = P(A \cap B)$ A and B are independent.

(b) We have the equiprobable space $S = \{bb, bg, gb, gg\}$. Here

$$\begin{array}{lll} A = \{bg, gb\} & \text{and so} & P(A) = \frac{1}{2} \\ B = \{bg, gb, gg\} & \text{and so} & P(B) = \frac{3}{4} \\ A \cap B = \{bg, gb\} & \text{and so} & P(A \cap B) = \frac{1}{2} \end{array}$$

Since $P(A)P(B) \neq P(A \cap B)$, A and B are dependent.

7.21. Box A contains five red marbles and three blue marbles, and box B contains three red and two blue. A marble is drawn at random from each box.

(a) Find the probability p that both marbles are red.

(b) Find the probability p that one is red and one is blue.

(a) The probability of choosing a red marble from A is $\frac{5}{8}$ and from B is $\frac{3}{5}$. Since the events are independent, $P = \frac{5}{8} \cdot \frac{3}{5} = \frac{3}{8}$.

(b) The probability p_1 of choosing a red marble from A and a blue marble from B is $\frac{5}{8} \cdot \frac{2}{5} = \frac{1}{4}$. The probability p_2 of choosing a blue marble from A and a red marble from B is $\frac{3}{8} \cdot \frac{3}{5} = \frac{9}{40}$. Hence $p = p_1 + p_2 = \frac{1}{4} + \frac{9}{40} = \frac{19}{40}$.

7.22. Prove: If A and B are independent events, then A^c and B^c are independent events.

Let $P(A) = x$ and $P(B) = y$. Then $P(A^c) = 1 - x$ and $P(B^c) = 1 - y$. Since A and B are independent. $P(A \cap B) = P(A)P(B) = xy$. Furthermore,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = x + y - xy$$

By DeMorgan's law, $(A \cup B)^c = A^c \cap B^c$; hence

$$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - x - y + xy$$

On the other hand,

$$P(A^c)P(B^c) = (1 - x)(1 - y) = 1 - x - y + xy$$

Thus $P(A^c \cap B^c) = P(A^c)P(B^c)$, and so A^c and B^c are independent.

In similar fashion, we can show that A and B^c, as well as A^c and B, are independent.

Supplementary Problems

SAMPLE SPACES AND EVENTS

7.42. Let A, B, and C be events. Rewrite each of the following events using set notation:

(a) A and B but not C occurs; (c) none of the events occurs;
(b) A or C, but not B occurs; (d) at least two of the events occur.

7.43. A penny, a dime, and a die are tossed.

(a) Describe a suitable sample space S, and find $n(S)$.

(b) Express explicitly the following events:
$A = \{\text{two heads and an even number}\}$
$B = \{2 \text{ appears}\}$
$C = \{\text{exactly one head and an odd number}\}$

(c) Express explicitly the events: (i) A and B; (ii) only B; (iii) B and C.

FINITE EQUIPROBABLE SPACES

7.44. Determine the probability of each event:

(a) An odd number appears in the toss of a fair die.

(b) One or more heads appear in the toss of four fair coins.

(c) One or both numbers exceed 4 in the toss of two fair dice.

7.45. One card is selected at random from 50 cards numbered 1 to 50. Find the probability that the number on the card is;

(a) greater than 10; (c) greater than 10 and divisible by 5;
(b) divisible by 5; (d) greater than 10 or divisible by 5.

7.46. Of 10 girls in a class, three have blue eyes. Two of the girls are chosen at random. Find the probability that:

(a) both have blue eyes; (c) at least one has blue eyes;
(b) neither has blue eyes; (d) exactly one has blue eyes.

7.47. Ten students, A, B, ..., are in a class. A committee of three is chosen at random to represent the class. Find the probability that:

(a) A belongs to the committee; (c) A and B belong to the committee;
(b) B belongs to the committee; (d) A or B belong to the committee.

7.48. Three bolts and three nuts are in a box. Two parts are chosen at random. Find the probability that one is a bolt and one is a nut.

7.49. A box contains two white socks, two blue socks, and two red socks. Two socks are drawn at random. Find the probability they are a match (the same color).

7.50. Of 120 students, 60 are studying French, 50 are studying Spanish, and 20 are studying both French and Spanish. A student is chosen at random. Find the probability that the student is studying: (a) French or Spanish; (b) neither French nor Spanish; (c) only French; (d) exactly one of the two languages.

FINITE PROBABILITY SPACES

7.51. Decide which of the following functions defines a probability space on $S = \{a_1, a_2, a_3\}$:

(a) $P(a_1) = \frac{1}{4}$, $P(a_2) = \frac{1}{3}$, $P(a_3) = \frac{1}{2}$ (c) $P(a_1) = \frac{1}{6}$, $P(a_2) = \frac{1}{3}$, $P(a_3) = \frac{1}{2}$
(b) $P(a_1) = \frac{2}{3}$, $P(a_2) = -\frac{1}{3}$, $P(a_3) = \frac{2}{3}$ (d) $P(a_1) = 0$, $P(a_2) = \frac{1}{3}$, $P(a_3) = \frac{2}{3}$

7.52. A coin is weighted so that heads is three times as likely to appear as tails. Find $P(H)$ and $P(T)$.

7.53. Three students A, B, and C are in a swimming race. A and B have the same probability of winning and each is twice as likely to win as C. Find the probability that: (a) B wins; (b) C wins; (c) B or C wins.

7.54. Consider the following probability distribution:

Outcome x	1	2	3	4	5
Probability $P(x)$	0.2	0.4	0.1	0.1	0.2

Consider the events $A = \{\text{even number}\}$, $B = \{2, 3, 4, 5\}$, $C = \{1, 2\}$. Find:
(a) $P(A)$, $P(B)$, $P(C)$; (b) $P(A \cap B)$, $P(A \cap C)$, $P(B \cap C)$.

7.55. Suppose A and B are events with $P(A) = 0.7$, $P(B) = 0.5$, and $P(A \cap B) = 0.4$. Find the probability that:
(a) A does not occur; (c) A but not B occurs;
(b) A or B occurs; (d) neither A nor B occurs.

CONDITIONAL PROBABILITY, INDEPENDENCE

7.56. A fair die is tossed. Consider events $A = \{2, 4, 6\}$, $B = \{1, 2\}$, $C = \{1, 2, 3, 4\}$. Find:
(a) $P(A \text{ and } B)$ and $P(A \text{ or } C)$, (c) $P(A|C)$ and $P(C|A)$
(b) $P(A|B)$ and $P(B|A)$ (d) $P(B|C)$ and $P(C|B)$
Decide whether the following are independent: (i) A and B; (ii) A and C; (iii) B and C.

7.57. A pair of fair dice is tossed. If the numbers appearing are different, find the probability that: (a) the sum is even; (b) the sum exceeds nine.

7.58. Let A and B be events with $P(A) = 0.6$, $P(B) = 0.3$, and $P(A \cap B) = 0.2$. Find:
(a) $P(A \cup B)$; (b) $P(A|B)$; (c) $P(B|A)$.

7.59. Let A and B be events with $P(A) = 1/3$, $P(B) = \frac{1}{4}$, and $P(A \cup B) = \frac{1}{2}$.
(a) Find $P(A|B)$ and $P(B|A)$. (b) Are A and B independent?

7.60. Let A and B be events with $P(A) = 0.3$, $P(A \cup B) = 0.5$, and $P(B) = p$. Find p if:
(a) A and B are mutually disjoint; (b) A and B are independent; (c) A is a subset of B.

7.61. Let A and B be independent events with $P(A) = 0.3$ and $P(B) = 0.4$. Find:
(a) $P(A \cap B)$ and $P(A \cup B)$; (b)$P(A|B)$ and $P(B|A)$.

7.62. In a country club, 60% of the women play tennis, 40% play golf, and 20% play both tennis and golf. A woman is chosen at random.

(a) Find the probability that she plays neither tennis nor golf.

(b) If she plays tennis, find the probability that she plays golf.

(c) If she plays golf, find the probability that she plays tennis.

7.63. Box A contains six red marbles and two blue marbles, and box B contains two red and four blue. A marble is drawn at random from each box.

(a) Find the probability p that both marbles are red.

(b) Find the probability p that one is red and one is blue.

7.64. The probability that A hits a target is $\frac{1}{4}$ and the probability that B hits a target is $\frac{1}{3}$.

(a) If each fires twice, what is the probability that the target will be hit at least once?

(b) If each fires once and the target is hit only once, what is the probability that A hits the target?

7.65. Three fair coins are tossed. Consider the events:
$A = \{\text{all heads or all tails}\}$, $B = \{\text{at least two heads}\}$, $C = \{\text{at most two heads}\}$.
Of the pairs (A, B), (A, C), and (B, C), which are independent? Which are dependent?

7.66. Find $P(B|A)$ if: (a) A is a subset of B; (b) A and B are mutually exclusive. (Assume $P(A) > 0$.)

Answers to Supplementary Problems

The notation $[x_1, \ldots, x_n;\ f(x_1), \ldots, f(x_n)]$ will be used for the distribution $f = \{(x_i, f(X_i))\}$.

7.42. (a) $A \cap B \cap C^C$; (c) $(A \cup B \cup B)^C = A^C \cap B^C C^C$;
(b) $(A \cup C) \cap B^C$; (d) $(A \cap B) \cup (A \cap C) \cup (B \cap C)$.

7.43. (a) $n(S) = 24$; $S = \{H, T\} \times \{H, T\} \times \{1, 2, \ldots, 6\}$
(b) $A = \{HH2, HH4, HH6\}$; $B = \{HH2, HT2, TH2, TT2\}$; $C = \{HT1, HT3, HT5, TH1, TH3, TH5\}$
(c) (i) $HH2$; (ii) $HT2$, $TH2$, $TT2$; (iii) $\varnothing$.

7.44. (a) $3/6$; (b) $15/16$; (c) $20/36$.

7.45. (a) $40/50$; (b) $10/50$; (c) $8/50$; (d) $42/50$.

7.46. (a) $1/15$; (b) $7/15$; (c) $8/15$; (d) $7/15$.

7.47. (a) $3/10$; (b) $3/10$; (c) $1/15$; (d) $8/15$.

7.48. $3/5$.

7.49. $1/5$.

7.50. (a) 3/4; (b) 1/4; (c) 1/3; (d) 7/12.

7.51. (c) and (d).

7.52. $P(H) = 3/4$; $P(T) = 1/4$.

7.53. (a) $2/5$; (b) $1/5$; (c) $3/5$.

7.54. (a) 0.6, 0.8, 0.5; (b) 0.5, 0.7, 0.4.

7.55. (a) 0.3; (b) 0.8; (c) 0.3; (d) 0.2.

7.56. (a) 1/6, 5/6; (b) 1/2, 1/3; (c) 1/2, 2/3; (d) 1/2, (i) Yes; (ii) yes (iii) no.

7.57. (a) $12/30$; (b) $4/30$.

7.58. (a) 0.7; (b) $2/3$; (c) $1/3$.

7.59. (a) $1/3$, 1/4; (b) yes.

7.60. (a) 0.2; (b) $2/7$; (c) 0.5.

7.61. (a) 0.12, 0.58; (b) $3/10$, $4/10$.

7.62. (a) 20%; (b) $1/3$; (c) 1/2.

7.63. (a) 1/4; (b) $7/12$.

7.64. (a) 3/4; (b) $1/3$.

7.65. Only (A, B) are independent.

7.66. (a) l, (b) 0.