Functions

3.1 INTRODUCTION

One of the most important concepts in mathematics is that of a function. The terms "map," "mapping," "transformation," and many others mean the same thing; the choice of which word to use in a given situation is usually determined by tradition and the mathematical background of the person using the term.

3.2 FUNCTIONS

Suppose that to each element of a set A we assign a unique element of a set B; the collection of such assignments is called *a function* from A into B. The set A is called the *domain* of the function, and the set B is called the *target set* or *codomain*.

Functions are ordinarily denoted by symbols. For example, let f denote a function from A into B. Then we write

$$f: A \to B$$

which is read: "f is a function from A into B," or "f takes (or maps) A into B." If $a \in A$, then $f(a)$ (read: "f of a") denotes the unique element of B which f assigns to a; it is called the *image* of a under f, or the *value* of f at a. The set of all image values is called the *range* or *image* of f. The image of $f: A \to B$ is denoted by $\mathrm{Ran}(f)$, $\mathrm{Im}(f)$ or $f(A)$.

Frequently, a function can be expressed by means of a mathematical formula. For example, consider the function which sends each real number into its square. We may describe this function by writing

$$f(x) = x^2 \quad \text{or} \quad x \mapsto x^2 \quad \text{or} \quad y = x^2$$

In the first notation, x is called a *variable* and the letter f denotes the function. In the second notation, the barred arrow $\mapsto$ is read "goes into." In the last notation, x is called the *independent variable* and y is called the *dependent variable* since the value of y will depend on the value of x.

Remark: Whenever a function is given by a formula in terms of a variable x, we assume, unless it is otherwise stated, that the domain of the function is **R** and the codomain is **R**. (**R** - Real numbers)

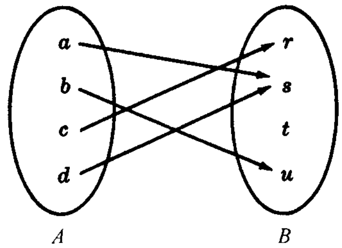

Fig. 3-1

EXAMPLE 3.1

(a) Consider the function $f(x) = x^3$, i.e., f assigns to each real number its cube. Then the image of 2 is 8, and so we may write $f(2) = 8$.

(b) Figure 3-1 defines a function f from $A = \{a, b, c, d\}$ into $B = \{r, s, t, u\}$ in the obvious way. Here

$$f(a) = s, \quad f(b) = u, \quad f(c) = r, \quad f(d) = s$$

The image of f is the set of image values, $\{r, s, u\}$. Note that t does not belong to the image of f because t is not the image of any element under f.

(c) Let A be any set. The function from A into A which assigns to each element in A the element itself is called the *identity function* on A and it is usually denoted by 1_A, or simply 1. In other words, for every $a \in A$,

$$1_A(a) = a.$$

Functions as Relations

There is another point of view from which functions may be considered. First of all, every function $f: A \to B$ gives rise to a relation from A to B called the *graph of f* and defined by

$$\text{Graph of } f = \{(a, b) \mid a \in A, b = f(a)\}$$

Two functions $f: A \to B$ and $g: A \to B$ are defined to be *equal*, written $f = g$, if $f(a) = g(a)$ for every $a \in A$; that is, if they have the same graph. Accordingly, we do not distinguish between a function and its graph. Now, such a graph relation has the property that each a in A belongs to a unique ordered pair (a, b) in the relation. On the other hand, any relation f from A to B that has this property gives rise to a function $f: A \to B$, where $f(a) = b$ for each (a, b) in f. Consequently, one may equivalently define a function as follows:

Definition: A function $f: A \to B$ is a relation from A to B such that each $a \in A$ belongs to a unique ordered pair (a, b) in f.

EXAMPLE 3.2

(a) Let $f: A \to B$ be the function defined in Example 3.1 (b). Then the graph of f is as follows:

$$\{(a, s),\ (b, u),\ (c, r),\ (d, s)\}$$

(b) Consider the following three relations on the set $A = \{1, 2, 3\}$:

$$f = \{(1, 3),\ (2, 3),\ (3, 1)\}, \quad g = \{(1, 2),\ (3, 1)\}, \quad h = \{(1, 3),\ (2, 1),\ (1, 2),\ (3, 1)\}$$

f is a function from A into A since each member of A appears as the first coordinate in exactly one ordered pair in f; here $f(1) = 3$, $f(2) = 3$, and $f(3) = 1$. g is not a function from A into A since $2 \in A$ is not the first coordinate of any pair in g and so g does not assign any image to 2. Also h is not a function from A into A since $1 \in A$ appears as the first coordinate of two distinct ordered pairs in h, $(1, 3)$ and $(1, 2)$. If h is to be a function it cannot assign both 3 and 2 to the element $1 \in A$.

Composition Function

Consider functions $f\colon A \to B$ and $g\colon B \to C$; that is, where the codomain of f is the domain of g. Then we may define a new function from A to C, called the *composition* of f and g and written $g \circ f$, as follows:

$$(g \circ f)(a) \equiv g(f(a))$$

That is, we find the image of a under f and then find the image of $f(a)$ under g. This definition is not really new. If we view f and g as relations, then this function is the same as the composition of f and g as relations (see Section 2.6) except that here we use the functional notation $g \circ f$ for the composition of f and g instead of the notation $f \circ g$ which was used for relations.

Consider any function $f\colon A \to B$. Then

$$f \circ 1_A = f \quad \text{and} \quad 1_B \circ f = f$$

where 1_A and 1_B are the identity functions on A and B, respectively.

3.3 ONE-TO-ONE, ONTO, AND INVERTIBLE FUNCTIONS

A function $f: A \to B$ is said to be *one-to-one* (written 1-1) if different elements in the domain A have distinct images. Another way of saying the same thing is that f is *one-to-one* if $f(a) = f(a')$ implies $a = a'$.

A function $f: A \to B$ is said to be an *onto* function if every element of B is the image of some element of A. In other words, $f: A \to B$ is onto if the image of f is the entire codomain, i.e., if $f(A) = B$. In such a case we say that f is a function from A onto B or that f maps A onto B.

A function $f: A \to B$ is *invertible* if its inverse relation f^{-1} is a function from B to A. In general, the inverse relation f^{-1} may not be a function. The following theorem gives simple criteria which tells us when it is.

Theorem 3.1: A function $f: A \to B$ is invertible if and only if f is both one-to-one and onto.

If $f: A \to B$ is one-to-one and onto, then f is called a *one-to-one correspondence* between A and B. This terminology comes from the fact that each element of A will then correspond to a unique element of B and vice versa.

Some texts use the terms *injective* for a one-to-one function, *surjective* for an onto function, and *bijective* for a one-to-one correspondence.

EXAMPLE 3.3 Consider the functions $f_1: A \to B$, $f_2: B \to C$, $f_3: C \to D$ and $f_4: D \to E$ defined by the diagram of Fig. 3-3. Now f_1 is one-to-one since no element of B is the image of more than one element of A. Similarly, f_2 is one-to-one. However, neither f_3 nor f_4 is one-to-one since $f_3(r) = f_3(u)$ and $f_4(v) = f_4(w)$

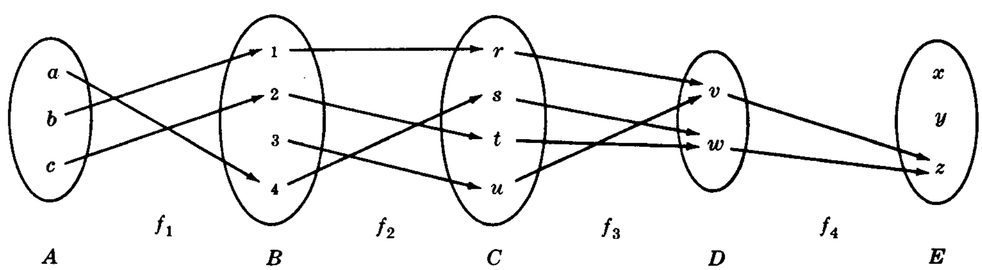

Fig. 3-3

As far as being onto is concerned, f_2 and f_3 are both onto functions since every element of C is the image under f_2 of some element of B and every element of D is the image under f_3 of some element of C, $f_2(B) = C$ and $f_3(C) = D$. On the other hand, f_1 is not onto since $3 \in B$ is not the image under f_1 of any element of A. and f_4 is not onto since $x \in E$ is not the image under f_4 of any element of D.

Thus f_1 is one-to-one but not onto, f_3 is onto but not one-to-one and f_4 is neither one-to-one nor onto. However, f_2 is both one-to-one and onto, i.e., is a one-to-one correspondence between A and B. Hence f_2 is invertible and f_2^{-1} is a function from C to B.

Solved Problems

FUNCTIONS

3.1. Let $X = \{1, 2, 3, 4\}$. Determine whether each relation on X is a function from X into X.

(a) $f = \{(2, 3), (1, 4), (2, 1), (3, 2), (4, 4)\}$

(b) $g = \{(3, 1), (4, 2), (1, 1)\}$

(c) $h = \{(2, 1), (3, 4), (1, 4), (2, 1), (4, 4)\}$

Recall that a subset f of $X \times X$ is a function $f: X \to X$ if and only if each $a \in X$ appears as the first coordinate in exactly one ordered pair in f.

(a) No. Two different ordered pairs (2, 3) and (2, 1) in f have the same number 2 as their first coordinate.

(b) No. The element $2 \in X$ does not appear as the first coordinate in any ordered pair in g.

(c) Yes. Although $2 \in X$ appears as the first coordinate in two ordered pairs in h, these two ordered pairs are equal.

3.3. Let $A = \{a, b, c\}$, $B = \{x, y, z\}$, $C = \{r, s, t\}$. Let $f: A \to B$ and $g: B \to C$ be defined by:

$$f = \{(a, y)(b, x), (c, y)\} \quad \text{and} \quad g = \{(x, s), (y, t), (z, r)\}.$$

Find: (a) composition function $g \circ f: A \to C$; (b) $\text{Im}(f)$, $\text{Im}(g)$, $\text{Im}(g \circ f)$.

(a) Use the definition of the composition function to compute:

$$(g \circ f)(a) = g(f(a)) = g(y) = t$$
$$(g \circ f)(b) = g(f(b)) = g(x) = s$$
$$(g \circ f)(c) = g(f(c)) = g(y) = t$$

That is $g \circ f = \{(a, t), (b, s), (c, t)\}$.

(b) Find the image points (or second coordinates):

$$\text{Im}(f) = \{x, y\}, \quad \text{Im}(g) = \{r, s, t\}, \quad \text{Im}(g \circ f) = \{s, t\}$$

3.4. Let $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$. Find the formula for the composition function $g \circ f$.

Compute $g \circ f$ as follows: $(g \circ f)(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - 2 = 4x^2 + 4x - 1$.

Observe that the same answer can be found by writing

$$y = f(x) = 2x + 1 \quad \text{and} \quad z = g(y) = y^2 - 2$$

and then eliminating y from both equations:

$$z = y^2 - 2 = (2x + 1)^2 - 2 = 4x^2 + 4x - 1$$

ONE-TO-ONE, ONTO, AND INVERTIBLE FUNCTIONS

3.5. Let the functions $f: A \to B$, $g: B \to C$, $h: C \to D$ be defined by Fig. 3-9. Determine if each function is: (a) onto, (b) one-to-one, (c) invertible.

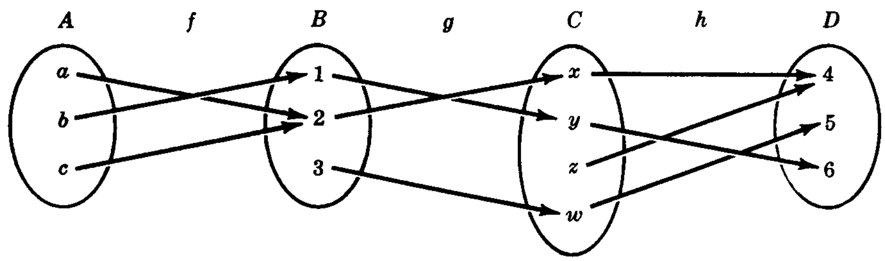

Fig. 3-9

(a) The function $f: A \to B$ is not onto since $3 \in B$ is not the image of any element in A.

The function $g: B \to C$ is not onto since $z \in C$ is not the image of any element in B.

The function $h: C \to D$ is onto since each element in D is the image of some element of C.

(b) The function $f: A \to B$ is not one-to-one since a and c have the same image 2.

The function $g: B \to C$ is one-to-one since 1, 2 and 3 have distinct images.

The function $h: C \to D$ is not one-to-one since x and z have the same image 4.

(c) No function is one-to-one and onto; hence no function is invertible.

3.8. Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x - 3$. Now f is one-to-one and onto; hence f has an inverse function f^{-1}. Find a formula for f^{-1}.

Let y be the image of x under the function f:

$$y = f(x) = 2x - 3$$

Consequently, x will be the image of y under the inverse function f^{-1}. Solve for x in terms of y in the above equation:

$$x = (y + 3)/2$$

Then $f^{-1}(y) = (y + 3)/2$. Replace y by x to obtain

$$f^{-1}(x) = \frac{x + 3}{2}$$

which is the formula for f^{-1} using the usual independent variable x.

Supplementary Problems

FUNCTIONS

3.27. Let $W = \{a, b, c, d\}$. Decide whether each set of ordered pairs is a function from W into W.

(a) $\{(b, a), \quad (c, d), \quad (d, a), \quad (c, d) \quad (a, d)\}$
(b) $\{(d, d), \quad (c, a), \quad (a, b), \quad (d, b)\}$
(c) $\{(a, b), \quad (b, b), \quad (c, d), \quad (d, b)\}$
(d) $\{(a, a), \quad (b, a), \quad (a, b), \quad (c, d)\}$

3.28. Let $V = \{1, 2, 3, 4\}$. For the following functions $f\colon V \to V$ and $g\colon V \to V$, find:
(a) $f \circ g$; (b), $g \circ f$; (c) $f \circ f$:

$$f = \{(1, 3),\ (2, 1),\ (3, 4),\ (4, 3)\} \quad \text{and} \quad g = \{(1, 2),\ (2, 3),\ (3, 1),\ (4, 1)\}$$

3.29. Find the composition function $h \circ g \circ f$ for the functions in Fig. 3-9.

ONE-TO-ONE, ONTO, AND INVERTIBLE FUNCTIONS

3.30. Determine if each function is one-to-one.

(a) To each person on the earth assign the number which corresponds to his age.
(b) To each country in the world assign the latitude and longitude of its capital.
(c) To each book written by only one author assign the author.
(d) To each country in the world which has a prime minister assign its prime minister.

3.31. Let functions f, g, h from $V = \{1, 2, 3, 4\}$ into V be defined by: $f(n) = 6 - n$, $g(n) = 3$, $h = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$. Decide which functions are:

(a) one-to-one; (b) onto; (c) both; (d) neither.

3.34. Let $f\colon \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 3x - 7$. Find a formula for the inverse function $f^{-1}\colon \mathbf{R} \to \mathbf{R}$.

Answers to Supplementary Problems

3.27. (a) Yes; (b) No; (c) Yes; (d) No.

3.28. (a) $\{(1, 1), (2, 4), (3, 3), (4, 3)\}$;
(b) $\{(1, 1), (2, 2), (3, 1), (4, 1)\}$;
(c) $\{(1, 4), (2, 3), (3, 3), (4, 4)\}$.

3.29. $\{(a, 4), (b, 6), (c, 4)\}$

3.30. (a) No, (b) yes, (c) no, (d) yes.

3.31. (a) f, h; (b) f, h; (c) f, h; (d) g.

3.34. $f^{-1}(x) = (x + 7)/3$