Relations

2.1 INTRODUCTION

The reader is familiar with many relations such as "less than," "is parallel to," "is a subset of," and so on. In a certain sense, these relations consider the existence or nonexistence of a certain connection between pairs of objects taken in a definite order. Formally, we define a relation in terms of these "ordered pairs."

An *ordered pair* of elements a and b, where a is designated as the first element and b as the second element, is denoted by (a, b). In particular,

$$(a, b) = (c, d)$$

if and only if $a = c$ and $b = d$. Thus $(a, b) \neq (b, a)$ unless $a = b$. This contrasts with sets where the order of elements is irrelevant; for example, $\{3, 5\} = \{5, 3\}$.

2.2 PRODUCT SETS

Consider two arbitrary sets A and B. The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the *product*, or *Cartesian product*, of A and B. A short designation of this product is $A \times B$, which is read "A cross B." By definition,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

One frequently writes A^2 instead of $A \times A$.

EXAMPLE 2.1 $\mathbf{R}$ denotes the set of real numbers and so $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ is the set of ordered pairs of real numbers. The reader is familiar with the geometrical representation of $\mathbf{R}^2$ as points in the plane as in Fig. 2-1. Here each point P represents an ordered pair (a, b) of real numbers and vice versa; the vertical line through P meets the x-axis at a, and the horizontal line through P meets the y-axis at b. $\mathbf{R}^2$ is frequently called the *Cartesian plane*.

EXAMPLE 2.2 Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a),\ (1, b),\ (1, c),\ (2, a),\ (2, b),\ (2, c)\}$$
$$B \times A = \{(a, 1),\ (b, 1),\ (c, 1),\ (a, 2),\ (b, 2),\ (c, 2)\}$$

Also, $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

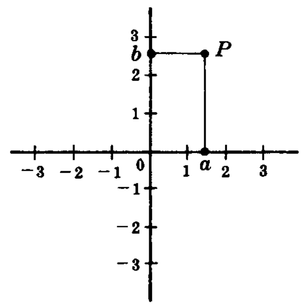


Fig. 2-1

There are two things worth noting in the above examples. First of all $A \times B \neq B \times A$. The Cartesian product deals with ordered pairs, so naturally the order in which the sets are considered is important. Secondly, using $n(S)$ for the number of elements in a set S, we have:

$$n(A \times B) = 6 = 2(3) = n(A)n(B)$$

In fact, $n(A \times B) = n(A)n(B)$ for any finite sets A and B. This follows from the observation that, for an ordered pair (a, b) in $A \times B$, there are $n(A)$ possibilities for a, and for each of these there are $n(B)$ possibilities for b.

Just as we write A^2 instead of $A \times A$, so we write A^n instead of $A \times A \times \cdots \times A$, where there are n factors all equal to A. For example, $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$ denotes the usual three-dimensional space.

2.3 RELATIONS

We begin with a definition.

Definition 2.1: Let A and B be sets. A *binary relation* or, simply, *relation* from A to B is a subset of $A \times B$.

Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B. That is, for each pair $a \in A$ and $b \in B$, exactly one of the following is true:

(i) $(a, b) \in R$; we then say "a is R-related to b", written aRb.

(ii) $(a, b) \notin R$; we then say "a is not R-related to b", written $a \not{R} b$.

If R is a relation from a set A to itself, that is, if R is a subset of $A^2 = A \times A$, then we say that R is a relation *on* A.

The *domain* of a relation R is the set of all first elements of the ordered pairs which belong to R, and the *range* is the set of second elements.

EXAMPLE 2.3

(a) $A = (1, 2, 3)$ and $B = \{x, y, z\}$, and let $R = \{(1, y), (1, z), (3, y)\}$. Then R is a relation from A to B since R is a subset of $A \times B$. With respect to this relation,

$$1Ry,\ 1Rz,\ 3Ry, \quad \text{but} \quad 1\not R x,\ 2\not R x,\ 2\not R y,\ 2\not R z,\ 3\not R x,\ 3\not R z$$

The domain of R is $\{1, 3\}$ and the range is $\{y, z\}$.

(b) Set inclusion $\subseteq$ is a relation on any collection of sets. For, given any pair of set A and B, either $A \subseteq B$ or $A \not\subseteq B$.

(c) A familiar relation on the set $\mathbf{Z}$ of integers is "m divides n." A common notation for this relation is to write $m \mid n$ when m divides n. Thus $6 \mid 30$ but $7 \nmid 25$.

(d) Consider the set L of lines in the plane. Perpendicularity, written "$\perp$," is a relation on L. That is, given any pair of lines a and b, either $a \perp b$ or $a \not\perp b$. Similarly, "is parallel to," written "$\|$," is a relation on L since either $a \parallel b$ or $a \nparallel b$.

(e) Let A be any set. An important relation on A is that of *equality*,

$$\{(a, a) \mid a \in A\}$$

which is usually denoted by "$=$." This relation is also called the *identity* or *diagonal* relation on A and it will also be denoted by Δ_A or simply Δ.

(f) Let A be any set. Then $A \times A$ and $\emptyset$ are subsets of $A \times A$ and hence are relations on A called the *universal relation* and *empty relation*, respectively.

Inverse Relation

Let R be any relation from a set A to a set B. The *inverse* of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs which, when reversed, belong to R; that is,

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}$$

For example, let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$. Then the inverse of

$$R = \{(1, y), (1, z), (3, y)\} \quad \text{is} \quad R^{-1} = \{(y, 1), (z, 1), (y, 3)\}$$

Clearly, if R is any relation, then $(R^{-1})^{-1} = R$. Also, the domain and range of R^{-1} are equal, respectively, to the range and domain of R. Moreover, if R is a relation on A, then R^{-1} is also a relation on A.

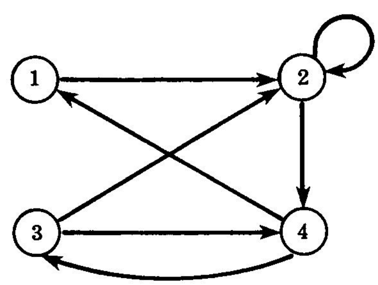
2.4 PICTORIAL REPRESENTATIVES OF RELATIONS

There are various ways of picturing relations.

Directed Graphs of Relations on Sets

There is an important way of picturing a relation R on a finite set. First we write down the elements of the set, and then we draw an arrow from each element x to each element y whenever x is related to y. This diagram is called the *directed graph* of the relation. Figure 2-2(b), for example, shows the directed graph of the following relation R on the set $A = \{1, 2, 3, 4\}$:

$$R = \{(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)\}$$



(b)

Fig. 2-2

Pictures of Relations on Finite Sets

Suppose A and B are finite sets. There are two ways of picturing a relation R from A to B.

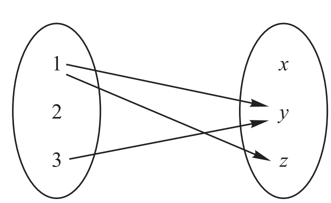
(i) Form a rectangular array (matrix) whose rows are labeled by the elements of A and whose columns are labeled by the elements of B. Put a 1 or 0 in each position of the array according as $a \in A$ is or is not related to $b \in B$. This array is called the *matrix of the relation*.

(ii) Write down the elements of A and the elements of B in two disjoint disks, and then draw an arrow from $a \in A$ to $b \in B$ whenever a is related to b. This picture will be called the *arrow diagram* of the relation.

Figure 2-3 pictures the relation R in Example 2.3(a) by the above two ways.

	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0

(i)



(ii)

$R = \{(1, y),\ (1, z),\ (3, y)\}$

Fig. 2-3

2.5 COMPOSITION OF RELATIONS

Let A, B and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of $A \times B$ and S is a subset of $B \times C$. Then R and S give rise to a relation from A to C denoted by $R \circ S$ and defined by:

$$a(R \circ S)c \text{ if for some } b \in B \text{ we have } aRb \text{ and } bSc.$$

That is ,

$$R \circ S = \{(a, c) \mid \text{there exists } b \in B \text{ for which } (a, b) \in R \text{ and } (b, c) \in S\}$$

The relation $R \circ S$ is called the *composition* of R and S; it is sometimes denoted simply by RS.

When a relation R is composed with itself, then the composition of relation is $R \circ R$.

EXAMPLE 2.4 Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{x, y, z\}$ and let

$$R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\} \quad \text{and} \quad S = \{(b, x), (b, z), (c, y), (d, z)\}$$

Consider the arrow diagrams of R and S as in Fig. 2-4. Observe that there is an arrow from 2 to d which is followed by an arrow from d to z. We can view these two arrows as a "path" which "connects" the element $2 \in A$ to the element $z \in C$. Thus:

$$2(R \circ S)z \quad \text{since } 2Rd \text{ and } dSz$$

Similarly there is a path from 3 to x and a path from 3 to z. Hence

$$3(R \circ S)x \quad \text{and} \quad 3(R \circ S)z$$

No other element of A is connected to an element of C. Accordingly,

$$R \circ S = \{(2, z), (3, x), (3, z)\}$$

Our first theorem tells us that composition of relations is associative.

Theorem 2.1: Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C, and T is a relation from C to D. Then

$$(R \circ S) \circ T = R \circ (S \circ T)$$

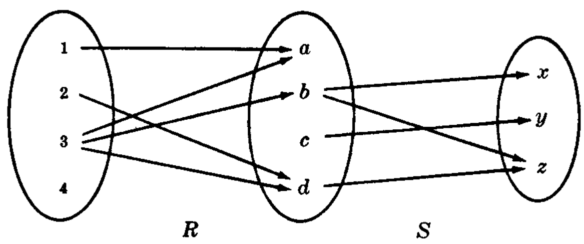


Fig. 2-4

2.6 TYPES OF RELATIONS

This section discusses a number of important types of relations defined on a set A.

Reflexive Relations

A relation R on a set A is *reflexive* if aRa for every $a \in A$, that is, if $(a, a) \in R$ for every $a \in A$. Thus R is not reflexive if there exists $a \in A$ such that $(a, a) \notin R$.

EXAMPLE 2.5 Consider the following five relations on the set $A = \{1, 2, 3, 4\}$:

$$\begin{aligned}
R_1 &= \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\} \\
R_2 &= \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\} \\
R_3 &= \{(1, 3), (2, 1)\} \\
R_4 &= \emptyset, \text{ the empty relation} \\
R_5 &= A \times A, \text{ the universal relation}
\end{aligned}$$

Determine which of the relations are reflexive.

Since A contains the four elements 1, 2, 3, and 4, a relation R on A is reflexive if it contains the four pairs $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$. Thus only R_2 and the universal relation $R_5 = A \times A$ are reflexive. Note that R_1, R_3, and R_4 are not reflexive.

Symmetric and Antisymmetric Relations

A relation R on a set A is *symmetric* if whenever aRb then bRa, that is, if whenever $(a, b) \in R$ then $(b, a) \in R$. Thus R is not symmetric if there exists $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

EXAMPLE 2.7

(a) Determine which of the relations in Example 2.5 are symmetric.

R_1 is not symmetric since $(1, 2) \in R_1$ but $(2, 1) \notin R_1$. R_3 is not symmetric since $(1, 3) \in R_3$ but $(3, 1) \notin R_3$. The other relations are symmetric.

A relation R on a set A is *antisymmetric* if whenever aRb and bRa then $a = b$, that is, if $a \neq b$ and aRb then $b \not R a$. Thus R is not antisymmetric if there exist distinct elements a and b in A such that aRb and bRa.

EXAMPLE 2.8

(a) Determine which of the relations in Example 2.5 are antisymmetric.

R_2 is not antisymmetric since $(1, 2)$ and $(2, 1)$ belong to R_2, but $1 \neq 2$. Similarly, the universal relation R_3 is not antisymmetric. All the other relations are antisymmetric.

Remark: The properties of being symmetric and being antisymmetric are not negatives of each other. For example, the relation $R = \{(1, 3), (3, 1), (2, 3)\}$ is neither symmetric nor antisymmetric. On the other hand, the relation $R' = \{(1, 1), (2, 2)\}$ is both symmetric and antisymmetric.

Transitive Relations

A relation R on a set A is *transitive* if whenever aRb and bRc then aRc, that is, if whenever $(a, b), (b, c) \in R$ then $(a, c) \in R$. Thus R is not transitive if there exist $a, b, c \in R$ such that $(a, b), (b, c) \in R$ but $(a, c) \notin R$.

EXAMPLE 2.9

(a) Determine which of the relations in Example 2.5 are transitive.

The relation R_3 is not transitive since $(2, 1), (1, 3) \in R_3$ but $(2, 3) \notin R_3$. All the other relations are transitive.

2.8 EQUIVALENCE RELATIONS

Consider a nonempty set S. A relation R on S is an *equivalence relation* if R is reflexive, symmetric, and transitive. That is, R is an equivalence relation on S if it has the following three properties:

(1) For every $a \in S$, aRa. (2) If aRb, then bRa. (3) If aRb and bRc, then aRc.

The general idea behind an equivalence relation is that it is a classification of objects which are in some way "alike." In fact, the relation "$=$" of equality on any set S is an equivalence relation; that is:

(1) $a = a$ for every $a \in S$. (2) If $a = b$, then $b = a$. (3) If $a = b$, $b = c$, then $a = c$.

EXAMPLE 2.13

(a) Consider the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$ on $S = \{1, 2, 3\}$.

One can show that R is reflexive, symmetric, and transitive, Finally, the relation R is an equivalence relation.

Solved Problems

PRODUCT SETS

2.1. Given: $A = \{1, 2\}$, $B = \{x, y, z\}$, and $C = \{3, 4\}$. Find: $A \times B \times C$.

$A \times B \times C$ consists of all ordered triplets (a, b, c) where $a \in A$, $b \in B$, $c \in C$. These elements of $A \times B \times C$ can be systematically obtained by a so-called tree diagram (Fig. 2-5). The elements of $A \times B \times C$ are precisely the 12 ordered triplets to the right of the tree diagram.

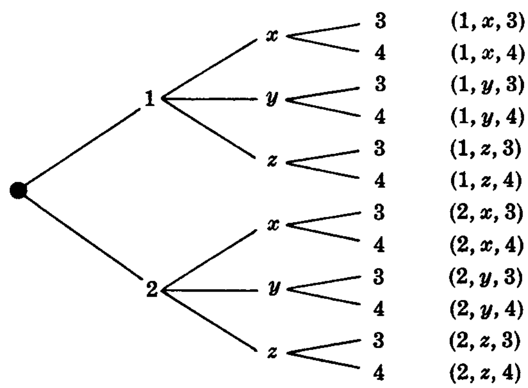


Fig. 2-5

Observe that $n(A) = 2$, $n(B) = 3$, and $n(C) = 2$ and, as expected,

$$n(A \times B \times C) = 12 = n(A) \cdot n(B) \cdot n(C)$$

2.2. Find x and y given $(2x, x + y) = (6, 2)$.

Two ordered pairs are equal if and only if the corresponding components are equal. Hence we obtain the equations

$$2x = 6 \quad \text{and} \quad x + y = 2$$

from which we derive the answers $x = 3$ and $y = -1$.

RELATIONS AND THEIR GRAPHS

2.3. Find the number of relations from $A = \{a, b, c\}$ to $B = \{1, 2\}$.

There are $3(2) = 6$ elements in $A \times B$, and hence there are $m = 2^6 = 64$ subsets of $A \times B$. Thus there are $m = 64$ relations from A to B.

2.4. Given $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$. Let R be the following relation from A to B:

$$R = \{(1, y), (1, z), (3, y), (4, x), (4, z)\}$$

(*a*) Determine the matrix of the relation.

(*b*) Draw the arrow diagram of R.

(*c*) Find the inverse relation R^{-1} of R.

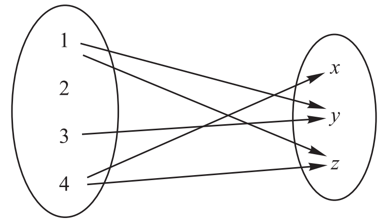
(*d*) Determine the domain and range of R.

(*a*) See Fig. 2-6(*a*) Observe that the rows of the matrix are labeled by the elements of A and the columns by the elements of B. Also observe that the entry in the matrix corresponding to $a \in A$ and $b \in B$ is 1 if a is related to b and 0 otherwise.

(*b*) See Fig. 2.6(*b*) Observe that there is an arrow from $a \in A$ to $b \in B$ iff a is related to b, i.e., iff $(a, b) \in R$.

$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4
\end{array}
\begin{array}{c}
\begin{array}{ccc} x & y & z \end{array} \\
\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}
\end{array}
$$

(a)



(b)

Fig. 2-6

(c) Reverse the ordered pairs of R to obtain R^{-1}:

$$R^{-1} = \{(y, 1), (z, 1), (y, 3), (x, 4), (z, 4)\}$$

Observe that by reversing the arrows in Fig. 2.6(b), we obtain the arrow diagram of R^{-1}.

(d) The domain of R, $\mathrm{Dom}(R)$, consists of the first elements of the ordered pairs of R, and the range of R, $\mathrm{Ran}(R)$, consists of the second elements. Thus,

$$\mathrm{Dom}(R) = \{1, 3, 4\} \quad \text{and} \quad \mathrm{Ran}(R) = \{x, y, z\}$$

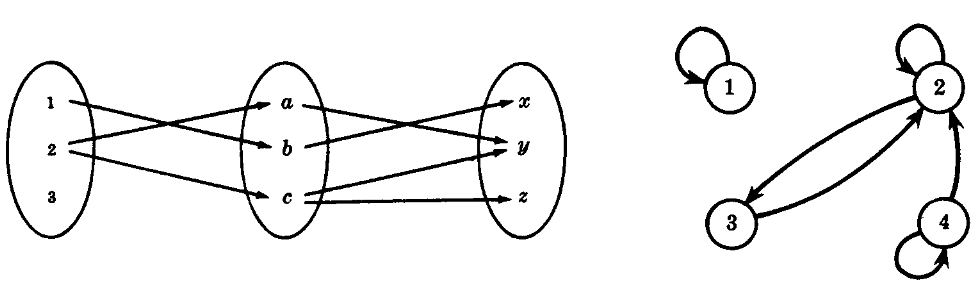
2.5. Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, and $C = \{x, y, z\}$. Consider the following relations R and S from A to B and from B to C, respectively.

$$R = \{(1, b), (2, a), (2, c)\} \quad \text{and} \quad S = \{(a, y), (b, x), (c, y), (c, z)\}$$

(a) Find the composition relation $R \circ S$.

(a) Draw the arrow diagram of the relations R and S as in Fig. 2-7(a). Observe that 1 in A is "connected" to x in C by the path $1 \to b \to x$; hence $(1, x)$ belongs to $R \circ S$. Similarly, $(2, y)$ and $(2, z)$ belong to $R \circ S$. We have

$$R \circ S = \{(1, x), (2, y), (2, z)\}$$



(a) (b)

Fig. 2-7

2.6. Consider the relation $R = \{(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)\}$ on $A = \{1, 2, 3, 4\}$.

(a) Draw its directed graph. (b) Find $R^2 = R \circ R$.

(a) For each $(a, b) \in R$, draw an arrow from a to b as in Fig. 2-7(b).

(b) For each pair $(a, b) \in R$, find all $(b, c) \in R$. Then $(a, c) \in R^2$. Thus

$$R^2 = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 2), (4, 3), (4, 4)\}$$

2.7. Let R and S be the following relations on $A = \{1, 2, 3\}$:

$$R = \{(1, 1), (1, 2), (2, 3), (3, 1), (3, 3)\}, \quad S = \{(1, 2), (1, 3), (2, 1), (3, 3)\}$$

Find (a) $R \cup S$, $R \cap S$, R^C; (b) $R \circ S$; (c) $S^2 = S \circ S$.

(a) Treat R and S simply as sets, and take the usual intersection and union. For R^C, use the fact that $A \times A$ is the universal relation on A.

$$\begin{aligned} R \cap S &= \{(1, 2), (3, 3)\} \\ R \cup S &= \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 3)\} \\ R^C &= \{(1, 3), (2, 1), (2, 2), (3, 2)\} \end{aligned}$$

(b) For each pair $(a, b) \in R$, find all pairs $(b, c) \in S$. Then $(a, c) \in R \circ S$. For example, $(1, 1) \in R$ and $(1, 2)$, $(1, 3) \in S$; hence $(1, 2)$ and $(1, 3)$ belong to $R \circ S$. Thus,

$$R \circ S = \{(1, 2), (1, 3), (1, 1), (2, 3), (3, 2), (3, 3)\}$$

(c) Following the algorithm in (b), we get

$$S^2 = S \circ S = \{(1, 1), (1, 3), (2, 2), (2, 3), (3, 3)\}$$

TYPES OF RELATIONS AND CLOSURE PROPERTIES

2.9. Consider the following five relations on the set $A = \{1, 2, 3\}$:

$$R = \{(1, 1), (1, 2), (1, 3), (3, 3)\}, \qquad \emptyset = \text{empty relation}$$
$$S = \{(1, 1)(1, 2), (2, 1)(2, 2), (3, 3)\}, \quad A \times A = \text{universal relation}$$
$$T = \{(1, 1), (1, 2), (2, 2), (2, 3)\}$$

Determine whether or not each of the above relations on A is: (*a*) reflexive; (*b*) symmetric; (*c*) transitive; (*d*) antisymmetric.

(*a*) R is not reflexive since $2 \in A$ but $(2, 2) \notin R$. T is not reflexive since $(3, 3) \notin T$ and, similarly, $\emptyset$ is not reflexive. S and $A \times A$ are reflexive.

(*b*) R is not symmetric since $(1, 2) \in R$ but $(2, 1) \notin R$, and similarly T is not symmetric. S, $\emptyset$, and $A \times A$ are symmetric.

(*c*) T is not transitive since $(1, 2)$ and $(2, 3)$ belong to T, but $(1, 3)$ does not belong to T. The other four relations are transitive.

(*d*) S is not antisymmetric since $1 \neq 2$, and $(1, 2)$ and $(2, 1)$ both belong to S. Similarly, $A \times A$ is not antisymmetric. The other three relations are antisymmetric.

2.10. Give an example of a relation R on $A = \{1, 2, 3\}$ such that:

(*a*) R is both symmetric and antisymmetric.

(*b*) R is neither symmetric nor antisymmetric.

(*c*) R is transitive but $R \cup R^{-1}$ is not transitive.

There are several such examples. One possible set of examples follows:

(*a*) $R = \{(1, 1), (2, 2)\}$; (*b*) $R = \{(1, 2), (2, 3)\}$; (*c*) $R = \{(1, 2)\}$.

2.13. Consider the relation $R = \{(a, a), (a, b), (b, c), (c, c)\}$ on the set $A = \{a, b, c\}$. Find: (*a*) reflexive(R); (*b*) symmetric(R); (*c*) transitive(R).

(*a*) The reflexive closure on R is obtained by adding all diagonal pairs of $A \times A$ to R which are not currently in R. Hence,

$$\text{reflexive}(R) = R \cup \{(b, b)\} = \{(a, a), (a, b), (b, b), (b, c), (c, c)\}$$

(*b*) The symmetric closure on R is obtained by adding all the pairs in R^{-1} to R which are not currently in R. Hence,

$$\text{symmetric}(R) = R \cup \{(b, a), (c, b)\} = \{(a, a), (a, b), (b, a), (b, c), (c, b), (c, c)\}$$

(*c*) The transitive closure on R, since A has three elements, is obtained by taking the union of R with $R^2 = R \circ R$ and $R^3 = R \circ R \circ R$. Note that

$$R^2 = R \circ R = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$$
$$R^3 = R \circ R \circ R = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$$

Hence

$$\text{transitive}(R) = R \cup R^2 \cup R^3 = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$$

2.16. Let R be the following equivalence relation on the set $A = \{1, 2, 3, 4, 5, 6\}$:

$$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$$

Show that R is an equivalence relation.

Supplementary Problems

RELATIONS

2.20. Let $S = \{a, b, c\}$, $T = \{b, c, d\}$, and $W = \{a, d\}$. Find $S \times T \times W$.

2.21. Find x and y where: (a) $(x + 2, 4) = (5, 2x + y)$; (b) $(y - 2, 2x + 1) = (x - 1, y + 2)$.

2.22. Prove: (a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$; (b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

2.23. Consider the relation $R = \{(1, 3), (1, 4), (3, 2), (3, 3), (3, 4)\}$ on $A = \{1, 2, 3, 4\}$.

(a) Find the matrix M_R of R.
(b) Find the domain and range of R.
(c) Find R^{-1}.
(d) Draw the directed graph of R.
(e) Find the composition relation $R \circ R$.
(f) Find $R \circ R^{-1}$ and $R^{-1} \circ R$.

2.24. Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, $C = \{x, y, z\}$. Consider the relations R from A to B and S from B to C as follows:

$$R = \{(1, b), (3, a), (3, b), (4, c)\} \quad \text{and} \quad S = \{(a, y), (c, x), (a, z)\}$$

(a) Draw the diagrams of R and S.
(b) Find the matrix of each relation R, S (composition) $R \circ S$.
(c) Write R^{-1} and the composition $R \circ S$ as sets of ordered pairs.

2.25. Let R and S be the following relations on $B = \{a, b, c, d\}$:

$$R = \{(a, a), (a, c), (c, b), (c, d), (d, b)\} \quad \text{and} \quad S = \{(b, a), (c, c), (c, d), (d, a)\}$$

Find the following composition relations: (a) $R \circ S$; (b) $S \circ R$; (c) $R \circ R$; (d) $S \circ S$.

2.26. Let R be the relation on $\mathbf{N}$ defined by $x + 3y = 12$, i.e. $R = \{(x, y) \mid x + 3y = 12\}$.

(a) Write R as a set of ordered pairs.
(b) Find the domain and range of R.
(c) Find R^{-1}.
(d) Find the composition relation $R \circ R$.

PROPERTIES OF RELATIONS

2.27. Each of the following defines a relation on the positive integers $\mathbf{N}$:

(1) "x is greater than y."
(2) "xy is the square of an integer."
(3) $x + y = 10$
(4) $x + 4y = 10$.

Determine which of the relations are: (a) reflexive; (b) symmetric; (c) antisymmetric; (d) transitive.

2.28. Let R and S be relations on a set A. Assuming A has at least three elements, state whether each of the following statements is true or false. If it is false, give a counterexample on the set $A = \{1, 2, 3\}$:

(a) If R and S are symmetric then $R \cap S$ is symmetric.
(b) If R and S are symmetric then $R \cup S$ is symmetric.
(c) If R and S are reflexive then $R \cap S$ is reflexive.

(d) If R and S are reflexive then $R \cup S$ is reflexive.

(e) If R and S are transitive then $R \cup S$ is transitive.

(f) If R and S are antisymmetric then $R \cup S$ is antisymmetric.

(g) If R is antisymmetric, then R^{-1} is antisymmetric.

(h) If R is reflexive then $R \cap R^{-1}$ is not empty.

(i) If R is symmetric then $R \cap R^{-1}$ is not empty.

2.29. Suppose R and S are relations on a set A, and R is antisymmetric. Prove that $R \cap S$ is antisymmetric.

EQUIVALENCE RELATIONS

2.30. Prove that if R is an equivalence relation on a set A, than R^{-1} is also an equivalence relation on A.

2.31. Let $S = \{1, 2, 3, \ldots, 18, 19\}$. Let R be the relation on S defined by "xy is a square," (a) Prove R is an equivalence relation.

Answers to Supplementary Problems

2.20. $\{(a, b, a), (a, b, d), (a, c, a), (a, c, d), (a, d, a), (a, d, d), (b, b, a), (b, b, d), (b, c, a), (b, c, d), (b, d, a), (b, d, d), (c, b, a), (c, b, d), (c, c, a), (c, c, d), (c, d, a), (c, d, d)\}$

2.21. (a) $x = 3$, $y = -2$; (b) $x = 2$, $y = 3$.

2.23. (a) $M_R = [0, 0, 1, 1;\ 0, 0, 0, 0;\ 0, 1, 1, 1; 0, 0, 0, 0]$;

(b) Domain $= \{1, 3\}$, range $= \{2, 3, 4\}$;

(c) $R^{-1} = \{(3, 1), (4, 1), (2, 3), (3, 3), (4, 3)\}$;

(d) See Fig. 2-8(a);

(e) $R\,o\,R = \{(1, 2), (1, 3), (1, 4), (3, 2), (3, 3), (3, 4)\}$.

2.24. (a) See Fig. 2-8(b);
(b) $R = [0, 1, 0;\ 0, 0, 0;\ 1, 1, 0;\ 0, 0, 1]$,
$S = [0, 1, 1;\ 0, 0, 0;\ 1, 0, 0]$,
$R \circ S = [0, 0, 0;\ 0, 0, 0;\ 0, 1, 1;\ 1, 0, 0]$;
(c) $\{(b, 1), (a, 3), (b, 3), (c, 4)\}$, $\{(3, y), (3, z), (4, x)\}$.

2.25. (a) $R \circ S = \{(a, c), (a, d), (c, a), (d, a)\}$
(b) $S \circ R = \{(b, a), (b, c), (c, b), (c, d), (d, a), (d, c)\}$
(c) $R \circ R = \{(a, a), (a, b), (a, c), (a, d), (c, b)\}$
(d) $S \circ S = \{(c, c), (c, a), (c, d)\}$

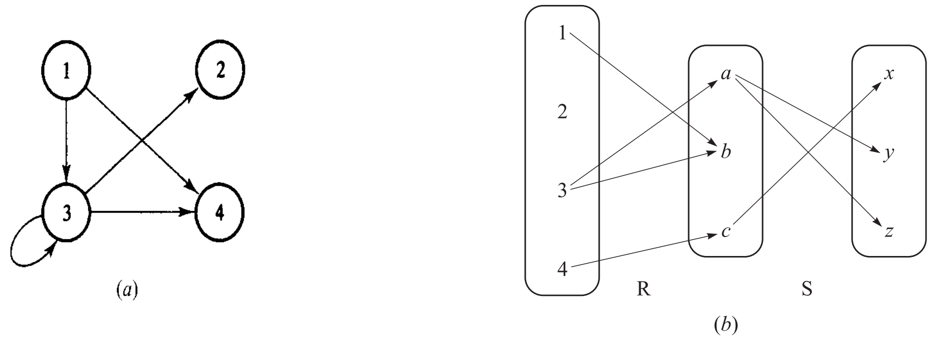


Fig. 2-8

2.26. (a) $\{(9, 1), (6, 2), (3, 3)\}$; (b) (i) $\{9, 6, 3\}$, (ii) $\{1, 2, 3\}$, (iii) $\{(1, 9), (2, 6), (3, 3)\}$; (c) $\{(3, 3)\}$.

2.27. (a) None; (b) (2) and (3); (c) (1) and (4); (d) all except (3).

2.28. All are true except: (e) $R = \{(1, 2)\}$, $S = \{(2, 3)\}$; (f) $R = \{(1, 2)\}$, $S = \{(2, 1)\}$.