2.3: Calculating Limits Using the Limit Laws

Learning Objectives

- Recognize the basic limit laws.
- Use the limit laws to evaluate the limit of a function.
- Evaluate the limit of a function by factoring.
- Use the limit laws to evaluate the limit of a polynomial or rational function.
- Evaluate the limit of a function by factoring or by using conjugates.
- Evaluate the limit of a function by using the squeeze theorem.

In the previous section, we evaluated limits by looking at graphs or by constructing a table of values. In this section, we establish laws for calculating limits and learn how to apply these laws. In the Student Project at the end of this section, you have the opportunity to apply these limit laws to derive the formula for the area of a circle by adapting a method devised by the Greek mathematician Archimedes. We begin by restating two useful limit results from the previous section. These two results, together with the limit laws, serve as a foundation for calculating many limits.

Evaluating Limits with the Limit Laws

The first two limit laws were stated previously and we repeat them here. These basic results, together with the other limit laws, allow us to evaluate limits of many algebraic functions.

Basic Limit Results

For any real number a and any constant c,

I. $\lim_{x \to a} x = a$
II. $\lim_{x \to a} c = c$

Example 2.3.1: Evaluating a Basic Limit

Evaluate each of the following limits using "Basic Limit Results."

a. $\lim_{x \to 2} x$
b. $\lim_{x \to 2} 5$

Solution

a. The limit of x as x approaches a is a: $\lim_{x \to 2} x = 2$.
b. The limit of a constant is that constant: $\lim_{x \to 2} 5 = 5$.

We now take a look at the **limit laws**, the individual properties of limits. The proofs that these laws hold are omitted here.

Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Let c be a constant. Then, each of the following statements holds:

- **Sum law for limits**:

$$\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$$

- **Difference law for limits**:

$$\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$

- **Constant multiple law for limits**:

$$\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x) = cL$$

- **Product law for limits**:

$$\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$$

- **Quotient law for limits**:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$$

for $M \neq 0$.

- **Power law for limits**:

$$\lim_{x \to a} \left(f(x)\right)^n = \left(\lim_{x \to a} f(x)\right)^n = L^n$$

for every positive integer n.

- **Root law for limits**:

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$$

We now practice applying these limit laws to evaluate a limit.

Example $2.3.2A$: Evaluating a Limit Using Limit Laws

Use the limit laws to evaluate

$$\lim_{x \to -3} (4x + 2).$$

Solution

Let's apply the limit laws one step at a time to be sure we understand how they work. We need to keep in mind the requirement that, at each application of a limit law, the new limits must exist for the limit law to be applied.

$$\begin{aligned} \lim_{x \to -3} (4x+2) &= \lim_{x \to -3} 4x + \lim_{x \to -3} 2 && \text{Apply the sum law.} \\ &= 4 \cdot \lim_{x \to -3} x + \lim_{x \to -3} 2 && \text{Apply the constant multiple law.} \\ &= 4 \cdot (-3) + 2 = -10. && \text{Apply the basic limit results and simplify.} \end{aligned}$$

Example $2.3.2B$: Using Limit Laws Repeatedly

Use the limit laws to evaluate

$$\lim_{x \to 2} \frac{2x^2 - 3x + 1}{x^3 + 4}.$$

Solution

To find this limit, we need to apply the limit laws several times. Again, we need to keep in mind that as we rewrite the limit in terms of other limits, each new limit must exist for the limit law to be applied.

$$\begin{aligned}
\lim_{x\to 2}\frac{2x^2-3x+1}{x^3+4} &= \frac{\lim\limits_{x\to 2}(2x^2-3x+1)}{\lim\limits_{x\to 2}(x^3+4)} && \text{Apply the quotient law, make sure that } (2)^3+4\neq 0.\\
&= \frac{2\cdot\lim\limits_{x\to 2}x^2-3\cdot\lim\limits_{x\to 2}x+\lim\limits_{x\to 2}1}{\lim\limits_{x\to 2}x^3+\lim\limits_{x\to 2}4} && \text{Apply the sum law and constant multiple law.}\\
&= \frac{2\cdot\left(\lim\limits_{x\to 2}x\right)^2-3\cdot\lim\limits_{x\to 2}x+\lim\limits_{x\to 2}1}{\left(\lim\limits_{x\to 2}x\right)^3+\lim\limits_{x\to 2}4} && \text{Apply the power law.}\\
&= \frac{2(4)-3(2)+1}{(2)^3+4}=\frac{1}{4}. && \text{Apply the basic limit laws and simplify.}
\end{aligned}$$

? Exercise 2.3.2

Use the limit laws to evaluate $\lim\limits_{x\to 6}(2x-1)\sqrt{x+4}$. In each step, indicate the limit law applied.

Hint

Begin by applying the product law.

Answer

$11\sqrt{10}$

Limits of Polynomial and Rational Functions

By now you have probably noticed that, in each of the previous examples, it has been the case that $\lim\limits_{x\to a}f(x)=f(a)$. This is not always true, but it does hold true for all polynomials for any choice of a and for all rational functions at all values of a for which the rational function is defined.

Limits of Polynomial and Rational Functions

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$$

$$f(x)=\frac{p(x)}{q(x)}$$

Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then,

$$\lim_{x\to a}p(x)=p(a)$$

$$\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{p(a)}{q(a)}$$

when $q(a)\neq 0$.

To see that this theorem holds, consider the polynomial

$$p(x)=c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0.$$

By applying the sum, constant multiple, and power laws, we end up with

$$\begin{aligned}
\lim_{x\to a}p(x) &= \lim_{x\to a}(c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0)\\
&= c_n\left(\lim_{x\to a}x\right)^n+c_{n-1}\left(\lim_{x\to a}x\right)^{n-1}+\cdots+c_1\left(\lim_{x\to a}x\right)+\lim_{x\to a}c_0\\
&= c_na^n+c_{n-1}a^{n-1}+\cdots+c_1a+c_0\\
&= p(a)
\end{aligned}$$

It now follows from the quotient law that if $p(x)$ and $q(x)$ are polynomials for which $q(a)\neq 0$, then

$$\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{p(a)}{q(a)}.$$

Example 2.3.3: Evaluating a Limit of a Rational Function

Evaluate the $\lim_{x\to 3}\frac{2x^2-3x+1}{5x+4}$.

Solution

Since 3 is in the domain of the rational function $f(x)=\frac{2x^2-3x+1}{5x+4}$, we can calculate the limit by substituting 3 for x into the function. Thus,

$$\lim_{x\to 3}\frac{2x^2-3x+1}{5x+4}=\frac{10}{19}.$$

Exercise 2.3.3

Evaluate $\lim_{x\to -2}(3x^3-2x+7)$.

Hint

Use LIMITS OF POLYNOMIAL AND RATIONAL FUNCTIONS as reference

Answer

−13

Additional Limit Evaluation Techniques

As we have seen, we may evaluate easily the limits of polynomials and limits of some (but not all) rational functions by direct substitution. However, as we saw in the introductory section on limits, it is certainly possible for $\lim_{x\to a}f(x)$ to exist when $f(a)$ is undefined. The following observation allows us to evaluate many limits of this type:

If for all $x\neq a$, $f(x)=g(x)$ over some open interval containing a, then

$$\lim_{x\to a}f(x)=\lim_{x\to a}g(x).$$

To understand this idea better, consider the limit $\lim_{x\to 1}\frac{x^2-1}{x-1}$.

The function

$$f(x)=\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{x-1}$$

and the function $g(x)=x+1$ are identical for all values of $x\neq 1$. The graphs of these two functions are shown in Figure 2.3.1.

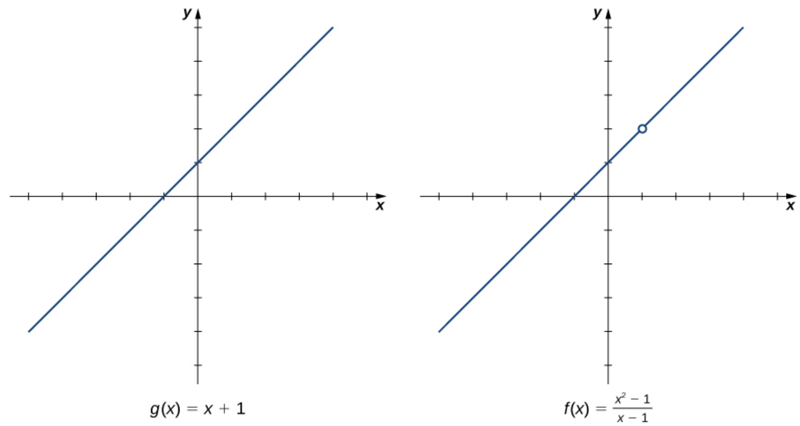

Figure 2.3.1: The graphs of $f(x)$ and $g(x)$ are identical for all $x \neq 1$. Their limits at 1 are equal.

We see that

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x \to 1} (x+1) = 2.$$

The limit has the form $\lim_{x \to a} f(x)/g(x)$, where $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. (In this case, we say that $f(x)/g(x)$ has the indeterminate form $0/0$.) The following Problem-Solving Strategy provides a general outline for evaluating limits of this type.

Problem-Solving Strategy: Calculating a Limit When $f(x)/g(x)$ has the Indeterminate Form $0/0$

1. First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.
2. We then need to find a function that is equal to $h(x) = f(x)/g(x)$ for all $x \neq a$ over some interval containing a. To do this, we may need to try one or more of the following steps:
   a. If $f(x)$ and $g(x)$ are polynomials, we should factor each function and cancel out any common factors.
   b. If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
   c. If $f(x)/g(x)$ is a complex fraction, we begin by simplifying it.
3. Last, we apply the limit laws.

The next examples demonstrate the use of this Problem-Solving Strategy. Example $2.3.4$ illustrates the factor-and-cancel technique; Example $2.3.5$ shows multiplying by a conjugate. In Example $2.3.6$, we look at simplifying a complex fraction.

Example $2.3.4$: Evaluating a Limit by Factoring and Canceling

Evaluate $\lim_{x \to 3} \frac{x^2-3x}{2x^2-5x-3}$.

Solution

Step 1. The function $f(x) = \frac{x^2-3x}{2x^2-5x-3}$ is undefined for $x = 3$. In fact, if we substitute 3 into the function we get $0/0$, which is undefined. Factoring and canceling is a good strategy:

$$\lim_{x \to 3} \frac{x^2-3x}{2x^2-5x-3} = \lim_{x \to 3} \frac{x(x-3)}{(x-3)(2x+1)}$$

Step 2. For all $x \neq 3$, $\frac{x^2-3x}{2x^2-5x-3} = \frac{x}{2x+1}$. Therefore,

$$\lim_{x \to 3} \frac{x(x-3)}{(x-3)(2x+1)} = \lim_{x \to 3} \frac{x}{2x+1}.$$

Step 3. Evaluate using the limit laws:

$$\lim_{x\to 3}\frac{x}{2x+1}=\frac{3}{7}.$$

? Exercise 2.3.4

Evaluate $\lim_{x\to -3}\frac{x^2+4x+3}{x^2-9}$.

Hint

Follow the steps in the Problem-Solving Strategy

Answer

$\frac{1}{3}$

✔ Example 2.3.5: Evaluating a Limit by Multiplying by a Conjugate

Evaluate $\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}$.

Solution

Step 1. $\frac{\sqrt{x+2}-1}{x+1}$ has the form $0/0$ at −1. Let's begin by multiplying by $\sqrt{x+2}+1$, the conjugate of $\sqrt{x+2}-1$, on the numerator and denominator:

$$\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}=\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}\cdot\frac{\sqrt{x+2}+1}{\sqrt{x+2}+1}.$$

Step 2. We then multiply out the numerator. We don't multiply out the denominator because we are hoping that the $(x+1)$ in the denominator cancels out in the end:

$$=\lim_{x\to -1}\frac{x+1}{(x+1)(\sqrt{x+2}+1)}.$$

Step 3. Then we cancel:

$$=\lim_{x\to -1}\frac{1}{\sqrt{x+2}+1}.$$

Step 4. Last, we apply the limit laws:

$$\lim_{x\to -1}\frac{1}{\sqrt{x+2}+1}=\frac{1}{2}.$$

? Exercise 2.3.5

Evaluate $\lim_{x\to 5}\frac{\sqrt{x-1}-2}{x-5}$.

Hint

Follow the steps in the Problem-Solving Strategy

Answer

$\frac{1}{4}$

Example 2.3.6: Evaluating a Limit by Simplifying a Complex Fraction

Evaluate $\lim\limits_{x \to 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1}$.

Solution

Step 1. $\frac{\frac{1}{x+1} - \frac{1}{2}}{x-1}$ has the form $0/0$ at 1. We simplify the algebraic fraction by:

$$\lim_{x \to 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1} = \lim_{x \to 1} \frac{\frac{2-(x+1)}{(x+1)2}}{x-1}$$

Step 2.

$$= \lim_{x \to 1} \frac{2-(x+1)}{2(x+1)(x-1)}$$

Step 3. Then, we simplify the numerator:

$$= \lim_{x \to 1} \frac{-x+1}{2(x+1)(x-1)}$$

Step 4. Now we factor out -1 from the numerator:

$$= \lim_{x \to 1} \frac{-(x-1)}{2(x+1)(x-1)}$$

Step 5. Then, we cancel the common factors of $(x-1)$:

$$= \lim_{x \to 1} \frac{-1}{2(x+1)}.$$

Step 6. Last, we evaluate using the limit laws:

$$\lim_{x \to 1} \frac{-1}{2(x+1)} = -\frac{1}{4}.$$

Exercise 2.3.6

Evaluate $\lim\limits_{x \to -3} \frac{\frac{1}{x+2} + 1}{x+3}$.

Hint

Follow the steps in the Problem-Solving Strategy

Answer

-1

Example $2.3.7$ does not fall neatly into any of the patterns established in the previous examples. However, with a little creativity, we can still use these same techniques.

Example 2.3.7: Evaluating a Limit When the Limit Laws Do Not Apply

Evaluate $\lim_{x \to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)$.

Solution:

Both $1/x$ and $5/x(x-5)$ fail to have a limit at zero. Since neither of the two functions has a limit at zero, we cannot apply the sum law for limits; we must use a different strategy. In this case, we find the limit by performing addition and then applying one of our previous strategies. Observe that

$$\frac{1}{x}+\frac{5}{x(x-5)}=\frac{x-5+5}{x(x-5)}=\frac{x}{x(x-5)}.$$

Thus,

$$\lim_{x \to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)=\lim_{x \to 0}\frac{x}{x(x-5)}=\lim_{x \to 0}\frac{1}{x-5}=-\frac{1}{5}.$$

Exercise 2.3.7

Evaluate $\lim_{x \to 3}\left(\frac{1}{x-3}-\frac{4}{x^2-2x-3}\right)$.

Hint

Use the same technique as Example $2.3.7$. Don’t forget to factor x^2-2x-3 before getting a common denominator.

Answer

$\frac{1}{4}$