Section 3: Solving Equations and Inequalities

3.1 Linear equations and their graphs

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Equations

An **equation** is a statement that two algebraic expressions are equal to each other.

Example:

(i) $2x + 1 = 5x + 3$
(ii) $5y^2 + 4y = 2$

The two sides of the equality sign of an equation are known as the **left side** (l.s) and the **right side** (r.s.) of the equation.

If the two sides of an equation are always equal to each other, irrespective of what values we give to the unknowns, the equation is called an **identity** and the symbol $\equiv$ is used instead of $=$.

Example of an Identity:

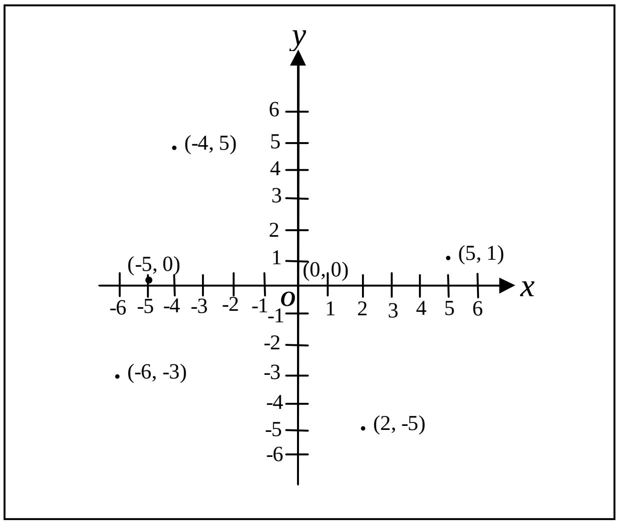
$x^2 + 2x + 1 \equiv (x + 1)^2$

If the two sides of an equation are equal only for a particular value or values of the unknowns, the equation is called an **equation of condition** or simply an **equation.**

Coordinate systems in two dimensions

A **rectangular coordinate system** or **Cartesian coordinate system** consists of two perpendicular straight lines in a plane, which intersect at a point *O*, called the **origin**. The two straight lines are called the **coordinate axes**. The two lines are usually in the horizontal and vertical directions, with the positive direction to the right (of the horizontal line) and the upward direction (of the vertical line) respectively. The horizontal line is usually referred to as the ***x* - axis** and denoted by *x* and the vertical line is usually referred to as the ***y* – axis** and denoted by *y*. The plane in which the two straight lines lie is called the **coordinate plane** or the ***xy* – plane**. The coordinate axes divide the plane into 4 parts called the **1st, 2nd, 3rd and 4th** quadrants and labeled I, II, III and IV respectively. Points on the axes do not belong to any quadrant.

An **ordered pair** of numbers is denoted by (a, b). Each point *P* in an *xy* – plane can be assigned a unique ordered pair (a, b). The number *a* is the ***x* – coordinate** (or **abscissa**) of *P*, and the number *b* is the ***y* - coordinate** (or **ordinate**) of *P*. We say that *P* has coordinates (a, b). Conversely, every ordered pair (a, b) determines a point P in the xy – plane with coordinates a and b. We often refer to the 'point' (a, b), or $P(a, b)$, meaning the point P with x – coordinate a and y – coordinate b. To **plot a point** $P(a, b)$, we locate P in a coordinate plane and represent it by a dot.



Graphs

Suppose W is a set of ordered pairs and (a, b) is an ordered pair belonging to W. Let P be the point on a coordinate plane that has coordinates (a, b). Then the **graph of W** is the set of all such points.

Linear equations

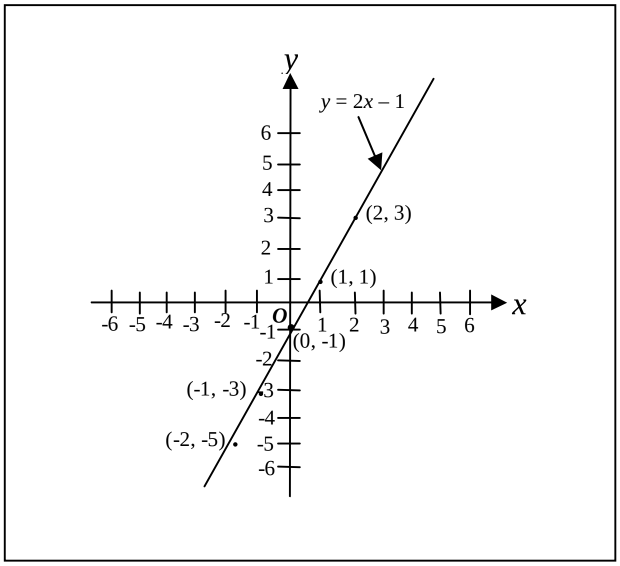
An equation of the form $ax + by + c = 0$ is called a **linear equation** in x and y (Here a, b, c are constants with a and b non-zero). In a linear equation, the power of each of the variables x and y is 1.

Let us now consider the following example of the graph of a linear equation. To draw the graph, we first find the value of y corresponding to several values of x.

Example:

$y = 2x – 1$

x	-2	-1	0	1	2
$2x$	-4	-2	0	2	4
$y = 2x - 1$	-5	-3	-1	1	3



Note:

- The ***x* – intercepts** of a graph are the x coordinates of points at which the graph intersects the x – axis. In the above graph there is only one x – intercept which is $\frac{1}{2}$.
- The ***y* – intercepts** of a graph are the y coordinates of points at which the graph intersects the y – axis. In the above graph, the y – intercept is -1.
- The above is called a **sketch of the graph** (note: it is impossible to plot all the points of the graph)
- Given an equation, $x = a$ and $y = b$ is a **solution** of the equation, if the equation is satisfied when we substitute a for x and b for y. For example, $x = 2$ and $y = 3$ is a solution of the equation $y = 2x - 1$. Solutions of equations can be obtained by considering the points on the graph of the equation.
- For a linear equation $y = mx + c$, the x – intercept can be obtained by substituting $y = 0$, and the y – intercept can be obtained by substituting $x = 0$.

As observed in the above example, linear equations and straight line graphs are related as follows:

The graph of a linear equation is a straight line and, conversely, every straight line is the graph of a linear equation.

Consider the straight line given by the linear equation $y = mx + c$.
Here m is called the **gradient** (slope) of the equation and c is called the **intercept**.

The **slope - intercept form** of an equation: An equation for the line with gradient (slope) m and y – intercept c is: $\boldsymbol{y = mx + c}$.

Suppose the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are on the line $y = mx + c$.

Then

$y_1 = mx_1 + c$ ----------------- (1)

$y_2 = mx_2 + c$ ----------------- (2)

(2) – (1)

$y_2 - y_1 = mx_2 + c - mx_1 - c$

$\therefore\ y_2 - y_1 = m(x_2 - x_1)$

$\therefore \frac{y_2 - y_1}{x_2 - x_1} = m$

> If l is a line that is not parallel to the y – axis, and the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are on the line l, then the gradient m of the line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.
> If l is parallel to the y – axis, then the gradient is not defined.

A **horizontal line** is a line parallel to the x – axis. In this case the gradient of the line is zero.

A **vertical line** is a line parallel to the y – axis. In this case the gradient of the line is not defined.

> (i) The graph of the line $x = a$ is a vertical line with x – intercept a.
> (ii) The graph of the line $y = b$ is a horizontal line with y – intercept b.

Suppose the point $P_1(x_1, y_1)$ lies on the line $y = mx + c$ ----------------- (1)

Then $y_1 = mx_1 + c$ ----------------- (2)

(1) – (2)

$\boldsymbol{y - y_1 = m(x - x_1)}$

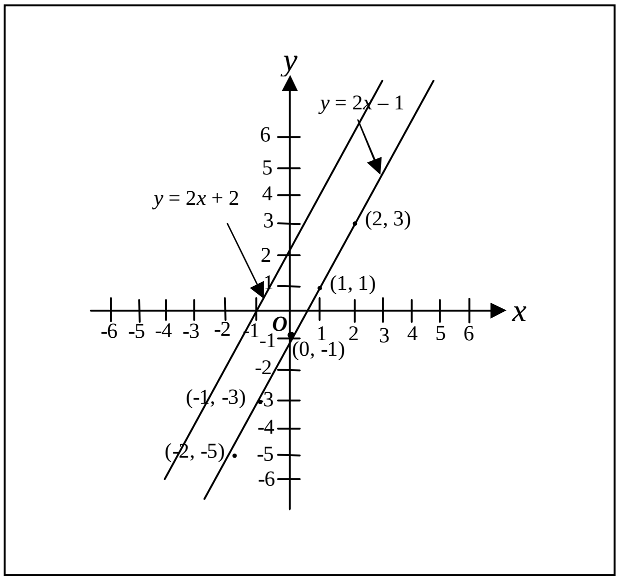
This is called the point-slope form of the equation.

> The **point – slope form** of an equation:
> An equation for the line through the point $P_1(x_1, y_1)$ with gradient m is :
> $\boldsymbol{y - y_1 = m(x - x_1)}$

<u>Result</u>

(i) Two non-vertical lines are parallel if and only if they have the same gradient.

(ii) Two lines with gradients m_1 and m_2 are perpendicular if and only if $m_1m_2 = -1$



3.2 Simple equations

A letter in an equation, the value of which it is required to find, is called an **unknown quantity** of the equation. The process of finding the value of an unknown is called **solving the equation**. A value so found is called a **root** or **solution** of the equation, and we say that the value **satisfies** the equation.

An equation which when reduced to a simple form involves no power of the unknown quantity higher than the first is called a **simple equation**.

Equations can be solved using the following axioms:

1. If $a = b$ and $c = d$, then $a + c = b + d$
2. If $a = b$ and $c = d$, then $a - c = b - d$
3. If $a = b$ and $c = d$, then $a \times c = b \times d$
4. If $a = b$ and $c = d$, then $a \div c = b \div d$

   Note: Here $c, d \neq 0$

Once a solution is obtained for an equation, the accuracy of the solution should be **verified** by substituting the value in for the unknown.

Example:

Consider	$5x + 3 = 13$
Subtracting 3 from both sides (axiom 2) we obtain	$5x + 3 - 3 = 13 – 3$
Thus	$5x = 10$
Dividing both sides by 5 (axiom 4) we obtain	$x = 2$

Verification
Substituting $x = 2$ to the left side of the equation we obtain $5(2) + 3 = 13$ which is the right side of the equation. This confirms the accuracy of the solution.

3.3 Equations involving algebraic fractions

Let us now see how simple equations involving algebraic fractions can be solved by considering a few examples.

Example:

(i) $\frac{x}{2} - \frac{x}{6} = 5$

$\frac{3x}{6} - \frac{x}{6} = 5$ (By considering the least common denominator)

$\frac{3x-x}{6} = 5$

$\frac{2x}{6} = 5$

$\frac{x}{3} = 5$

$\therefore\ x = 15$ (By multiplying both sides by 3)

(ii) $\frac{1}{x-2} + \frac{1}{x-5} = 0$

$\frac{(x-5)}{(x-2)(x-5)} + \frac{(x-2)}{(x-5)(x-2)} = 0$ (By considering the least common denominator)

$x – 5 + x – 2 = 0$ (By multiplying both sides by $(x-2)(x-5)$)

$2x = 7$

$\therefore\ x = \frac{7}{2}$

(iii) $\frac{2(x-1)}{3} + \frac{3(x-5)}{4} = -\frac{1}{6}$

$\frac{8(x-1)}{12} + \frac{9(x-5)}{12} = -\frac{2}{12}$

$8x - 8 + 9x - 45 = -2$

$17x = 51$

$\therefore\ x = 3$

3.4 Equations involving absolute values

We recall from the first section that the **absolute value** of the real number x symbolized by $|x|$ is

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

Thus if we consider the number line, the absolute value of a real number is the distance to the number from the origin.

Properties:

(i) For every real number x, $|x| = |-x|$.

(ii) For any two real numbers a, b, $|a.b| = |a|.|b|$ and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ if $b \neq 0$.

(iii) If $|a| = b$, then $a = \pm b$

Solving equations involving absolute values

Example:

(i) $|3t - 4| - 5 = 6$.

$|3t - 4| = 11$

$3t - 4 = 11$ or $3t - 4 = -11$

$3t = 15$ or $3t = -7$

$t = 5$ or $t = -\frac{7}{3}$

Verification:

When $t = 5$, left side = $|3(5) - 4| - 5 = |15 - 4| - 5 = |11| - 5 = 6 =$ right side

When $t = -\frac{7}{3}$,

left side = $\left|3\left(-\frac{7}{3}\right)-4\right| - 5 = |-7-4| - 5 = |-11| - 5 = 11 - 5 = 6 =$ right side

(ii) $\left|\frac{5x-6}{3x}\right| = 5$

$\frac{5x-6}{3x} = 5$ or $\frac{5x-6}{3x} = -5$

$5x - 6 = 15x$ or $5x - 6 = -15x$

$10x = -6$ or $-20x = -6$

$x = -\frac{3}{5}$ or $x = \frac{3}{10}$.

As in the above example, these solutions can be verified by substituting back into the original equation.

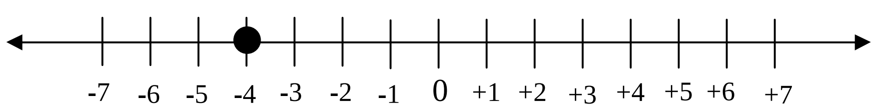
3.5 Inequalities and their solutions

Intervals on the Number Line

Representation of points on the Number Line

When graphing a point on the number line we simply colour in the corresponding point on the number line.

The example below indicates the point $x = -4$



In mathematics we deal with intervals of numbers. An application of intervals is that they are used to describe the solution sets of inequalities.

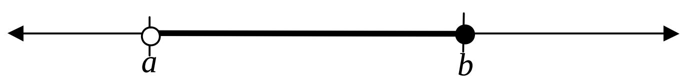
Intervals of finite length

Suppose a, b are real numbers such that $a < b$.

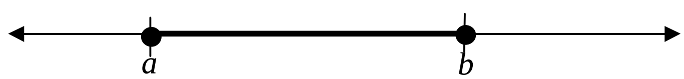
(i) $(a, b) = \{x \mid a < x < b\}$. i.e., the interval (a, b) is the set consisting of all real numbers that lie between a and b. This is called an open interval.

(ii) $[a, b) = \{ x \mid a \leq x < b\}$. i.e., the interval $[a, b)$ is the set consisting of all real numbers that lie between a and b, and the number a. This is called an interval closed on the left and open on the right.

(iii) $(a, b] = \{ x \mid a < x \leq b\}$. i.e., the interval $(a, b]$ is the set consisting of all real numbers that lie between a and b, and the number b. This is called an interval open on the left and closed on the right.

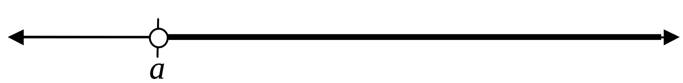


(iv) $[a, b] = \{ x \mid a \leq x \leq b\}$. i.e., the interval $[a, b]$ is the set consisting of all real numbers that lie between a and b, and the numbers a and b. This is called a closed interval.



Intervals of infinite length

(i) $(a, \infty) = \{ x \mid x > a \}$. i.e., the interval (a, ∞) is the set of real numbers greater than a.



(ii) $[a, \infty) = \{ x \mid x \geq a \}$. i.e., the interval $[a, \infty)$ is the set of real numbers greater or equal to a.

(iii) $(-\infty, a) = \{ x \mid x < a \}$. i.e., the interval $(-\infty, a)$ is the set of real numbers less than a.

(iv) $(-\infty, a] = \{ x \mid x \leq a \}$. i.e., the interval $(-\infty, a]$ is the set of real numbers less than or equal to a.

Rules of Inequalities

(i) Trichotomy Property:
For any pair of real numbers a and b, $a < b$ or $a = b$ or $b < a$.

(ii) Transitive Property:
For any real numbers a, b and c; if $a < b$ and $b < c$, then $a < c$.

(iii) Addition Property:
For any real numbers a, b and c; if $a < b$, then $a + c < b + c$.

(iv) Subtraction Property:
For any real numbers a, b and c; if $a < b$, then $a - c < b - c$.

(v) Multiplication Property:
For any real numbers a, b and c;
(a) if $a < b$ and $c > 0$, then $ac < bc$.
(b) if $a < b$ and $c < 0$, then $ac > bc$.

(vi) Division Property:
For any real numbers a, b and c;
(a) if $a < b$ and $c > 0$, then $a \div c < b \div c$.
(b) if $a < b$ and $c < 0$, then $a \div c > b \div c$.

(vii) Power Property:
If $a > b > 0$, then $a^m > b^m$ for $m > 0$.

(viii) Root Property:

If $a > b > 0$, then $\sqrt[m]{a} > \sqrt[m]{b}$ for $m > 0$.

(ix) Reciprocal Property:

If $ab > 0$ and $a < b$, then $\frac{1}{a} > \frac{1}{b}$.

Solutions of inequalities

In this section we will only consider **linear inequalities in one variable**. Linear inequalities in one variable are inequalities involving one variable where the exponent of the variable is 1. The properties of inequalities can be used to solve inequalities. **Solving an inequality** means finding all values of the variable for which the inequality is true. The set of all values which makes the inequality true is the **solution of the inequality**.

Examples of linear inequalities in one variable:

(i) $2x + 1 < -3$
(ii) $2x + 1 > 4x - 3$
(iii) $2x + 1 \geq -4x + 5$

Example:
Solve the following inequalities and illustrate the solution on the number line.

(i) $1 - 2x < 5$
(ii) $4x + 9 \leq 2x - 3$
(iii) $-5 < 3x - 2 \leq 4$

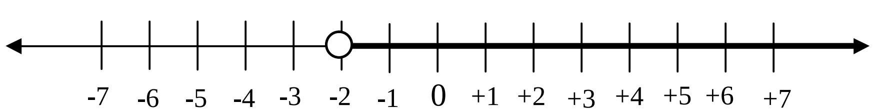
Solution:

(i) $1 - 2x < 5$

$-1 + 1 - 2x < -1 + 5$ (property (iii) of inequalities)

$-2x < 4$

$x > -2$ (property (vi) of inequalities)



(ii) $4x + 9 \leq 2x - 3$

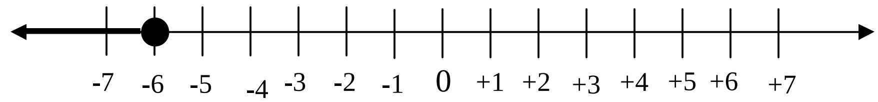
$4x + 9 - 2x \leq 2x - 3 - 2x$ (property (iv))

$2x + 9 \leq -3$

$2x + 9 - 9 \leq -3 - 9$ (property (iv))

$2x \leq -12$

$x \leq -6$ (property (vi))



(iii) $-5 < 3x - 2 \leq 4$

The above inequality can be written as the following two inequalities;

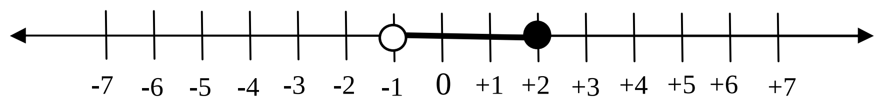
$-5 < 3x - 2$ and $3x - 2 \leq 4$

$-5 < 3x - 2$ and $3x - 2 \leq 4$

$-5 + 2 < 3x$ and $3x \leq 6$

$-3 < 3x$ and $x \leq 2$

$-1 < x$ and $x \leq 2$



Inequalities that involve absolute values

Properties:

(i) For any positive real number a, $|x| < a$ is equivalent to $-a < x < a$.

(ii) For real numbers a and b with $b > 0$, $|x - a| < b$ is equivalent to $-b < x - a < b$

(iii) For any positive real number a, $|x| > a$ is equivalent to $x > a$ or $x < -a$.

(iv) For real numbers a and b with $b > 0$, $|x - a| > b$ is equivalent to $x - a > b$ or $x - a < -b$.

Example:

Solve the following inequalities

(i) $|3x + 5| < 2$

(ii) $|5 - 7x| \geq 9$

Solution:

(i) $|3x + 5| < 2$

$-2 < 3x + 5 < 2$ (property (ii))

$-2 < 3x + 5$ and $3x + 5 < 2$

$-7 < 3x$ and $3x < -3$

$-\frac{7}{3} < x$ and $x < -1$

$-\frac{7}{3} < x < -1$.

Thus the solution set is $\{x \mid -\frac{7}{3} < x < -1\}$ or $(-\frac{7}{3}, -1)$

(ii) $|5-7x| \geq 9$

$5 - 7x \geq 9$ or $5 - 7x \leq -9$ (property (iv))

$-7x \geq 4$ or $-7x \leq -14$

$x \leq -\frac{4}{7}$ or $x \geq 2$

Thus the solution is $\{x \mid x \leq -\frac{4}{7}$ or $x \geq 2\}$

3.6 Quadratic equations

An equation which contains the square of the unknown quantity but no higher power is called a **quadratic equation** or an equation of the second degree.

An equation of the form $ax^2 + bx + c = 0$ where a, b and c are real numbers and $a \neq 0$ is called a **general quadratic equation** in the variable x.

A quadratic equation written in the form $ax^2 + bx + c = 0$ (where the right side is equal to 0) is said to be in **standard form**.

Solving Quadratic Equations by Factoring

The solution of a quadratic equation by factoring depends on the following result:

If the product of two numbers is zero, then one (or both) of the numbers is zero;
i.e., if $a \times b = 0$ then $a = 0$ or $b = 0$ (or both $a = 0$ and $b = 0$)

Example:
If $(x - 3)(x + 2) = 0$ then $x - 3 = 0$ or $x + 2 = 0$; i.e., $x = 3$ or $x = -2$

To solve quadratic equations by factoring

1. Write the quadratic equation in the standard form $ax^2 + bx + c = 0$
2. Factor $ax^2 + bx + c$
3. Set each factor equal to 0 and solve the resulting linear equations
4. Check the answers by substituting back into the original quadratic equation

Example:

(i) Solve $x^2 - 3x - 10 = 0$

(ii) Solve $9x^2 + 12x = -4$

(iii) Find the lengths of the sides of a right triangle of which the hypotenuse is respectively 1cm and 8cm longer than the other two sides.

Solution:

(i) $x^2 - 3x - 10 = 0$

$(x - 5)(x + 2) = 0$ by factoring

$x - 5 = 0$ or $x + 2 = 0$ by setting each factor equal to zero

$x = 5$ or $x = -2$ by solving for x

Thus the solutions of the quadratic equation $x^2 - 3x - 10 = 0$ are $x = 5$ and $x = -2$.
To verify these solutions we substitute $x = 5$ and $x = -2$ back into the original equation.

$x = 5$; Left side $= (5)^2 - 3(5) - 10 = 25 - 15 - 10 = 0 =$ right side

$x = -2$; Left side $= (-2)^2 - 3(-2) - 10 = 4 + 6 - 10 = 0 =$ right side

(ii) $9x^2 + 12x = -4$

$9x^2 + 12x + 4 = 0$ by adding 4 on both sides to obtain the general form

$(3x + 2)(3x + 2) = 0$ by factoring

$3x + 2 = 0$ or $3x + 2 = 0$ by setting each factor equal to zero

$x = -\frac{2}{3}$ or $x = -\frac{2}{3}$ by solving for x

Here the solution to the equation is a unique value $x = -\frac{2}{3}$. This is always the case when the factors are identical and we say that the root $-\frac{2}{3}$ is of multiplicity two.

As in question (i) we can verify the solution $x = -\frac{2}{3}$ by substituting it back into the original equation.

(iii) Let the length of the hypotenuse be denoted by x cm. Then the other two sides are of length $(x - 1)$ cm and $(x - 8)$ cm respectively. Therefore, by Pythagoras' theorem,

$(x - 1)^2 + (x - 8)^2 = x^2$

Expanding the equation we obtain

$x^2 - 2x + 1 + x^2 - 16x + 64 = x^2$

Therefore,
$x^2 - 18x + 65 = 0$
$(x - 13)(x - 5) = 0$
$x - 13 = 0$ or $x - 5 = 0$
$x = 13$ or $x = 5$
If $x = 5$ then $x - 8 = -3$. Since the length of a side cannot be a negative value, $x = 5$ cannot be an answer.
Therefore $x = 13$, $x - 1 = 12$ and $x - 8 = 5$; i.e., the lengths of the three sides are respectively 13cm, 12cm and 5cm.

Solving quadratic equations by the method of completing the square

The equation $x^2 = 16$ is an instance of the simplest form of quadratic equations.
Taking the square root on both sides we obtain $\pm x = \pm 4$.
From this we obtain $x = 4$ or $x = -4$.
The solution to the equation $(x - 3)^2 = 16$ can be found in a similar manner by taking the square root on both sides.
$x - 3 = \pm 4$
$x - 3 = 4$ or $x - 3 = -4$
$x = 7$ or $x = -1$.

Quadratic equations that are not in the above form can be put into this form by a technique called **completing the square**; i.e., we make the left side of the equation a perfect square.

Example:

$x^2 + 14x + 24 = 0$
$x^2 + 14x + 24 - 24 = 0 - 24$
$x^2 + 14x = -24$
$x^2 + 14x + \left(\frac{14}{2}\right)^2 = -24 + \left(\frac{14}{2}\right)^2$
$x^2 + 14x + 49 = -24 + 49$
$(x + 7)^2 = 25$
$x + 7 = \pm 5$
$x + 7 = 5$ or $x + 7 = -5$
$x + 7 - 7 = 5 - 7$ or $x + 7 - 7 = -5 - 7$
$x = -2$ or $x = -12$

The solutions can be verified by substituting $x = -2$, $x = -12$ back into the original equation.

We will illustrate the steps involved in solving quadratic equations by completing the square with another example.

Example:

$$3x^2 - 12x - 36 = 0$$

Step 1:
Divide both sides of the equation by the coefficient of x^2 (if it is not already equal to 1)

$$\frac{3x^2}{3} - \frac{12x}{3} - \frac{36}{3} = 0$$

$$x^2 - 4x - 12 = 0$$

Step 2:
Move the constant term to the right side of the equal sign and all terms containing x's to the left side.

$$x^2 - 4x = 12$$

Step 3:
Complete the square on the left side by adding the square of one half the coefficient of the linear term to both sides.

$$x^2 - 4x + \left(\frac{-4}{2}\right)^2 = 12 + \left(\frac{-4}{2}\right)^2$$

$$x^2 - 4x + 4 = 12 + 4$$
$$x^2 - 4x + 4 = 16$$
$$(x - 2)^2 = 16$$

Step 4:
Take the square root of both sides and solve for x

$$x - 2 = \pm 4$$
$$x - 2 = 4 \text{ or } x - 2 = -4$$
$$x = 6 \text{ or } x = -2$$

Therefore, the roots of the equation $3x^2 - 12x - 36 = 0$ are $x = 6$ and $x = -2$

The Quadratic Formula

We will construct a formula called the **quadratic formula** which can be used to obtain the roots of any quadratic equation, by solving the general quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$, using the method of completing the square.

$$ax^2+bx+c=0$$

$$x^2+\frac{b}{a}x+\frac{c}{a}=\frac{0}{a}$$ Step 1, divide each term by the coefficient *a* of x^2

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$ Step 2, subtract the constant term from both sides

$$x^2+\frac{b}{a}x+\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2=-\frac{c}{a}+\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2$$ Step 3, complete the square

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{-4ac+b^2}{4a^2}$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$ Step 4, take the square root of both sides and solve for *x*

$$x=-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a} \text{ or } x=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ or } x=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Therefore, the solutions of $ax^2+bx+c=0$ are

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } x=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

The quadratic formula for solving $ax^2+bx+c=0$ where $a \neq 0$, is

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Example:

(i) Solve $7x^2 - 6x - 1 = 0$

(ii) If the distance *s* (in metres) travelled by a cyclist in *t* seconds is given by $s = t^2 - 5t$, how long will he take to travel a distance of 36 m?

Solution:

(i) For the equation $7x^2 - 6x - 1 = 0$, $a = 7$, $b = -6$ and $c = -1$.
Substituting these values into the quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$ we obtain

$$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(7)(-1)}}{2(7)}$$

$$x = \frac{6 \pm \sqrt{36+28}}{14}$$

$$x = \frac{6 \pm \sqrt{64}}{14} = \frac{6 \pm 8}{14}$$

$x = \frac{14}{14}$ or $x = \frac{-2}{14} = -\frac{1}{7}$

Therefore, $x = 1$ and $x = -\frac{1}{7}$ are the solutions of the quadratic equation $7x^2 - 6x - 1 = 0$.

(ii) Since $s = 36$ m, the quadratic equation to be solved is $36 = t^2 - 5t$
The general form of this equation is $t^2 - 5t - 36 = 0$
Here $a = 1$, $b = -5$ and $c = -36$

Substituting these values into the quadratic formula $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ we obtain

$$t = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-36)}}{2(1)} = \frac{5 \pm \sqrt{25+144}}{2(1)} = \frac{5 \pm \sqrt{169}}{2}$$

Thus $t = \frac{5+13}{2} = 9$ or $t = \frac{5-13}{2} = -4$.
Since t can take only positive values, the time the cyclist takes to travel a distance of 36 m is 9 s.

3.7 Simultaneous equations

When two or more equations are satisfied by the same values of the unknown quantities they are called **simultaneous equations**.

Example:

Consider the equation

$5x - 4y = 1$ -------------------- (i)

that contains two unknown quantities which are denoted by x and y respectively.
Adding $4y$ to both sides we obtain

$5x - 4y + 4y = 1 + 4y$

i.e., $5x = 1 + 4y$

Therefore by dividing both sides by 5 we obtain

$x = \frac{1+4y}{5}$ ------------------ (ii)

We see that by substituting different values for y we obtain different values for x such that the pair of values satisfies the given equation.

For example, when $y = 1$, we obtain $x = \frac{1+4(1)}{5} = 1$ from (ii).

Substituting $y = 1$, $x = 1$ in the original equation (i) we obtain
left side $= 5(1) - 4(1) = 5 - 4 = 1 =$ right side
Thus $x = 1$, $y = 1$ is a solution of the equation.

When $y = 6$, $x = \frac{1+4(6)}{5} = 5$ from (ii).
Substituting $y = 6$, $x = 5$ in the original equation we obtain
left side $= 5(5) - 4(6) = 25 - 24 = 1 =$ right side
Therefore $x = 5$, $y = 6$ is also a solution of the equation.

We can obtain several pairs of values for x and y that satisfy the given equation by continuing in this manner.

Now suppose we have a second equation of the same kind such as
$x + 2y = 3$ -------------------------- (iii)
From this we obtain $x = 3 - 2y$ --------------------(iv)

If we wish to find the solution to both equations
$5x - 4y = 1$ -------------------- (i)
and $x + 2y = 3$ -------------------- (iii)
then the values of x we obtain from $x = \frac{1+4y}{5}$ and $x = 3 - 2y$ should be identical.

Thus $\frac{1+4y}{5} = 3 - 2y$.
Multiplying both sides by 5 we obtain $1 + 4y = 15 - 10y$.
Thus $4y + 10y = 15 - 1$
Therefore $14y = 14$ and hence $y = 1$.
Substituting this value into (iv) we obtain $x = 3 - 2(1) = 1$.

Earlier we verified that $x = 1$, $y = 1$ is a solution of the equation $5x - 4y = 1$ ------------ (i)
Substituting these values into the equation $x + 2y = 3$ --------------------(iii)
we obtain
left side $= 1 + 2(1) = 3 =$ right side.

Thus the solution to the pair of simultaneous equations
$5x - 4y = 1$ ------------ (i)
$x + 2y = 3$ ------------- (iii)
is $x = 1$, $y = 1$.

We note here that since the two equations are simultaneously true, any equation formed by combining them using the axioms for solving equations will be satisfied by the values of x and y which satisfy the original equation.

In this section we will be considering how to solve pairs of linear simultaneous equations in 2 unknowns using the following methods

(i) Elimination Method
(ii) Substitution Method
(iii) Graphical Method

Elimination Method

In the elimination method we eliminate one of the unknown quantities by multiplying one or both the equations by appropriate constants and adding them together or subtracting one from the other.

Example:

(i) $2x - 5y = 3$ ------------------- (a)
$4x – 2y = -2$ ------------------- (b)

(ii) $2x - 5y = 3$ ------------------- (a)
$3x – 2y = -1$ ------------------- (b)

(iv) Half the sum of two numbers equals 20 and three times their difference equals 18. Find the two numbers.

Solution:

(i) $2x - 5y = 3$ ------------------- (a)
$4x – 2y = -2$ ------------------- (b)

Multiplying equation (a) by -2 we obtain
$-4x + 10y = -6$ ------------------- (c)

Adding the left sides of equations (b) and (c) together and the right sides together (axiom 1) we obtain
$8y = - 8$
Therefore $y = -1$
Substituting $y = -1$ into equation (a) and solving for x we obtain
$2x – 5(-1) = 3$
$2x + 5 = 3$
$2x = -2$
$x = -1$

Note: We would obtain the same value $x = -1$, if we substitute $y = -1$ into equation (b) instead of into equation (a).

We verify the solution $x = -1$, $y = -1$ by substituting these values back into the equations (a) and (b).

Equation (a): left side = 2(-1) – 5(-1) = -2 + 5 = 3 = right side
Equation (b): left side = 4(-1) – 2(-1) = -4 + 2 = -2 = right side

(ii) $2x - 5y = 3$ ------------------- (a)
$3x - 2y = -1$ ------------------- (b)

Multiplying (a) by 3 and (b) by 2 we obtain
$6x - 15y = 9$ ------------------- (c)
$6x - 4y = -2$ ------------------- (d)

Subtracting (c) from (d) we obtain
$-4y - (-15y) = -2 - 9$
Therefore
$11y = -11$ and hence $y = -1$
Substituting this value into equation (a) we obtain
$2x - 5(-1) = 3$
$2x + 5 = 3$
$2x + 5 - 5 = 3 - 5$
$2x = -2$
$x = -1$

We now verify the solution $x = -1$, $y = -1$ by substituting them back into the equations (a) and (b).
Equation (a): left side = 2(-1) – 5(-1) = -2 + 5 = 3 = right side
Equation (b): left side = 3(-1) – 2(-1) = -3 + 2 = -1 = right side

(iii) Let the two numbers be x and y respectively. Then the two simultaneous equations are
$\frac{1}{2}(x+y) = 20$ ----------------- (a)
$3(x - y) = 18$ ------------------ (b)
Multiplying equation (a) by 2 and equation (b) by $\frac{1}{3}$ we obtain
$x + y = 40$ ---------------- (c)
$x - y = 6$ ------------------ (d)
Adding equations (c) and (d) we obtain
$2x = 46$
Thus $x = 23$
Substituting $x = 23$ into equation (d) we obtain
$23 - y = 6$
Therefore $y = 17$.
The accuracy of the solution $x = 23$, $y = 17$ can be verified as before by substituting these values back into the equations (a) and (b).

Substitution Method

In this method we use one equation to write one of the unknowns in terms of the other and then substitute this into the other equation, thus eliminating one unknown.

Example:

(i) Three pencils and four pens cost Rs. 72. A pencil and three pens cost Rs. 44. Determine the cost of a pencil and a pen.

(ii) The sum of two numbers is 34 while their difference is 10. What are the two numbers?

Solution:

(i) Suppose the cost of a pencil is Rs. x and the cost of a pen is Rs. y. Then we get the following simultaneous equations

$3x + 4y = 72$ ----------------- (a)

$x + 3y = 44$ ----------------- (b)

From (b) we obtain $x = 44 – 3y$.

Substituting this into (a) we obtain

$3(44 – 3y) + 4y = 72$

This is an equation in only the variable y.

$132 – 9y + 4y = 72$

$132 – 5y = 72$

Thus $5y = 132 – 72 = 60$

Hence $y = 12$.

Substituting this into equation (a) we obtain

$3x + 4(12) = 72$

$3x + 48 = 72$

$3x = 24$

$x = 8$

Therefore the cost of a pencil is Rs. 8 and the cost of a pen is Rs. 12.

Note: The accuracy of the solution $x = 8$, $y = 12$ can be verified by substituting these values back into the equations (a) and (b),

(ii) Let the two numbers be x and y respectively. Then we obtain the simultaneous equations

$x + y = 34$ ----------------- (a)

$x – y = 10$ ----------------- (b)

From equation (b) we obtain $x = y + 10$.

Substituting this value for x in equation (a) we obtain

$(y + 10) + y = 34$

$2y + 10 = 34$

$2y = 24$

$y = 12$

Substituting this in equation (b) we obtain
$x - 12 = 10$
Thus $x = 22$
The accuracy of the solution $x = 22$, $y = 12$ can be easily verified by substituting the values back into the equations (a) and (b).

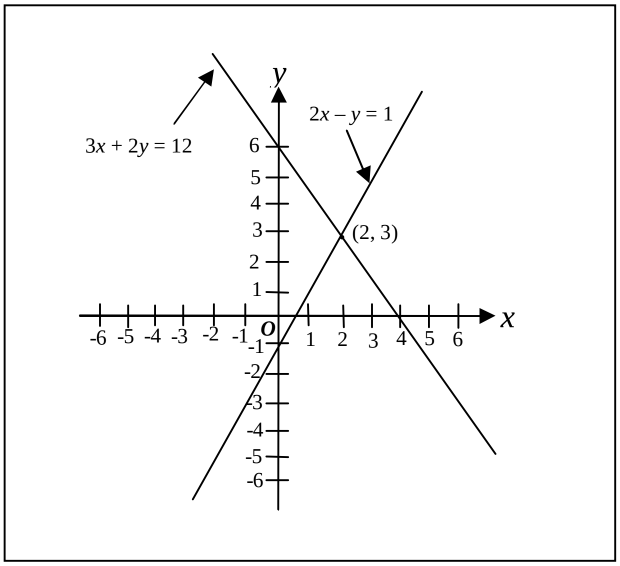
Note: Problem (ii) could have been solved more easily by the method of elimination by adding equations (a) and (b), thus eliminating the unknown y. Therefore, given a situation, the student should discern which method would be more appropriate to use.

Graphical Method

Pairs of linear simultaneous equations in two unknowns can also be solved by considering the intersection point of the two graphs.

Example:

$2x - y = 1$
$3x + 2y = 12$



It is clear from the above figure that the coordinates of the intersection point of the two straight lines is (2, 3). This point lies on both the lines, and hence satisfy both linear equations $2x - y = 1$ and $3x + 2y = 12$. Hence the solution of the pair of simultaneous equations is $x = 2$ and $y = 3$.