Solution of 2-8 sections' last question in Nonlinear Programming 3rd ed, Bazaraa et.

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Introdution

This is the trial solutions to the Book NONLINEAR PROGRAMMING,3rdEd, M.S. Bazaraa et.

Due to the limitation of time, only the first 8 section has been read and hence there would be only offered the 2-8 sections' last question for each section.

2.57

A 3 dimension plane L and its orthogonal complement (a line).

$$L^{\perp} = z[2,1,-1]^T = z' 6^{-0.5}[2,1,-1]^T$$
$$\lambda_2 = [1,2,3]\ 6^{-0.5}[2,1,-1]^T = 6^{-0.5}$$
$$\boldsymbol{x}_2 = 6^{-0.5} * 6^{-0.5}[2,1,-1]^T = \frac{1}{6}[2,1,-1]^T$$
$$\boldsymbol{x}_1 = \boldsymbol{x} - \boldsymbol{x}_2 = [1,2,3]^T - \frac{1}{6}[2,1,-1]^T = \frac{1}{6}[4,11,19]^T$$

3.73

This is about Rvacev Function.

(a)This question is similar to 3.72, there are of course many ways to solve. Without losing generality, set $g_1(x) \geq g_2(x)$.

(a: method I)

$$\begin{aligned}
G_\alpha(\boldsymbol{x}) &= \frac{1}{2}g_1(\boldsymbol{x}) + \frac{1}{2}g_2(\boldsymbol{x}) + \frac{1}{2}\sqrt{g_1^2(\boldsymbol{x}) + g_2^2(\boldsymbol{x}) - 2\alpha g_1(\boldsymbol{x})g_2(\boldsymbol{x})} \\
&= \frac{1}{2}g_1(\boldsymbol{x}) + \frac{1}{2}g_2(\boldsymbol{x}) + \frac{1}{2}\sqrt{(g_1(\boldsymbol{x}) - g_2(\boldsymbol{x}))^2 + 2(1-\alpha)g_1(\boldsymbol{x})g_2(\boldsymbol{x})} \\
&\geq \frac{1}{2}g_1(\boldsymbol{x}) + \frac{1}{2}g_2(\boldsymbol{x}) + \frac{1}{2}(g_1(\boldsymbol{x}) - g_2(\boldsymbol{x})) = g_1(\boldsymbol{x})
\end{aligned}$$

For $g_2(\boldsymbol{x}) \geq 0$, we could get $g_1(\boldsymbol{x}) \geq 0, G_\alpha \geq 0$,

For $g_2(\boldsymbol{x}) < 0$, we could get only when $g_1(\boldsymbol{x}) \geq 0$, $\frac{1}{2}\sqrt{(g_1(\boldsymbol{x}) - g_2(\boldsymbol{x}))^2 + 2(1-\alpha)g_1(\boldsymbol{x})g_2(\boldsymbol{x})} \leq \frac{1}{2}(g_1(\boldsymbol{x}) - g_2(\boldsymbol{x}))$, so that $G \geq 0$. Otherwise,$g_1(\boldsymbol{x}) < 0$, $\frac{1}{2}\sqrt{(g_1(\boldsymbol{x}) - g_2(\boldsymbol{x}))^2 + 2(1-\alpha)g_1(\boldsymbol{x})g_2(\boldsymbol{x})} \geq \frac{1}{2}(g_1(\boldsymbol{x}) - g_2(\boldsymbol{x}))$, so that $G < g_1(\boldsymbol{x}) < 0$.

Therefore,

$$\min G_\alpha(\boldsymbol{x}) = 0 \Leftrightarrow g_1(\boldsymbol{x}) \geq 0$$

(a: method II)

For the Prime Problem:

$$\begin{aligned} &\min G_\alpha(\boldsymbol{x}) \\ &-g_1(\boldsymbol{x}) \leq 0 \\ &\boldsymbol{x} \in \boldsymbol{X} \end{aligned}$$

Its Lagrangian Dual Problem would be:

$$\begin{aligned} &\max \theta(\mu) \\ &\mu \geq 0 \\ &\theta(\mu) = \inf\{G_\alpha(\boldsymbol{x}) - \mu g_1(\boldsymbol{x}), \boldsymbol{x} \in \boldsymbol{X}\} \end{aligned}$$

$$\theta(\mu) = \inf\{(\frac{1}{2} - \mu_1)g_1(\boldsymbol{x}) + \frac{1}{2}g_2(\boldsymbol{x}) + \frac{1}{2}\sqrt{g_1^2(\boldsymbol{x}) + g_2^2(\boldsymbol{x}) - 2\alpha g_1(\boldsymbol{x})g_2(\boldsymbol{x})}, \boldsymbol{x} \in \boldsymbol{X}\}$$

Then to get the inf, g_1 would be 0, $\theta(\mu) = 0$. As $g_1(\boldsymbol{x}) = 0$, the only restrict of Prime Problem is constraining, so this would no be a saddle point. Therefore:

$$\min G_\alpha(\boldsymbol{x}) = 0 \Leftrightarrow \max \theta = 0 \Leftrightarrow g_1(\boldsymbol{x}) \geq 0$$

(b) For each α,

$$\begin{aligned} G_\alpha(\boldsymbol{x}) &= \frac{1}{2}g_1(\boldsymbol{x}) + \frac{1}{2}g_2(\boldsymbol{x}) + \frac{1}{2}\sqrt{g_1^2(\boldsymbol{x}) + g_2^2(\boldsymbol{x}) - 2\alpha g_1(\boldsymbol{x})g_2(\boldsymbol{x})} \\ &= \frac{1}{2}g_1(\boldsymbol{x}) + \frac{1}{2}g_2(\boldsymbol{x}) + \frac{1}{2}\sqrt{(g_1(\boldsymbol{x}) - \alpha g_2(\boldsymbol{x}))^2 + (1-\alpha^2)g_2(\boldsymbol{x})} \end{aligned}$$

Set $cos\theta = -\alpha, \theta \in [\frac{1}{2}\pi, \pi]$, we could rewrite G as:

$$G_\alpha(\boldsymbol{x}) = \frac{1}{2}g_1(\boldsymbol{x}) + \frac{1}{2}g_2(\boldsymbol{x}) + \frac{1}{2}\sqrt{(g_1(\boldsymbol{x}) - \cos\theta g_2(\boldsymbol{x}))^2 + \sin^2\theta g_2(\boldsymbol{x})}$$

Set $\theta_1 + \theta_2 = \theta$, without losing generality, set $g_1(\boldsymbol{x}) \geq g_2(\boldsymbol{x})$, we get $\theta_1 \in [0, \frac{1}{2}\pi]$, then we could rewrite G as:

$$G_\alpha(\boldsymbol{x}) = \frac{1}{2}g_1(\boldsymbol{x}) + \frac{1}{2}g_2(\boldsymbol{x}) + \frac{1}{2}|g_1(\boldsymbol{x})\cos\theta_1 + g_2(\boldsymbol{x})\cos\theta_2|$$

For $g_1(\boldsymbol{x})\cos\theta_1 + g_2(\boldsymbol{x})\cos\theta_2 \geq 0$ or $g_1(\boldsymbol{x})\cos\theta_1 + g_2(\boldsymbol{x})\cos\theta_2 \leq 0$ $G_\alpha(\boldsymbol{x})$ is differentiable.

$$\begin{aligned} &\alpha = 1 \\ G_\alpha(\boldsymbol{x}) &= \frac{1}{2}g_1(\boldsymbol{x}) + \frac{1}{2}g_2(\boldsymbol{x}) + \frac{1}{2}\sqrt{g_1^2(\boldsymbol{x}) + g_2^2(\boldsymbol{x}) - 2g_1(\boldsymbol{x})g_2(\boldsymbol{x})} \\ &= \frac{1}{2}g_1(\boldsymbol{x}) + \frac{1}{2}g_2(\boldsymbol{x}) + \frac{1}{2}\sqrt{(g_1(\boldsymbol{x}) - g_2(\boldsymbol{x}))^2} \\ &= \max[g_1, g_2] \end{aligned}$$

(c) For $g_1(\boldsymbol{x})\cos\theta_1 + g_2(\boldsymbol{x})\cos\theta_2 \geq 0$,

$$\begin{aligned}
G_\alpha(\boldsymbol{x}) &= \frac{1}{2}g_1(\boldsymbol{x}) + \frac{1}{2}g_2(\boldsymbol{x}) + \frac{1}{2}|g_1(\boldsymbol{x})\cos\theta_1 + g_2(\boldsymbol{x})\cos\theta_2| \\
G_\alpha(\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) &= (\frac{1}{2} + \frac{1}{2}\cos\theta_1)(g_1(\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) + (\frac{1}{2} + \frac{1}{2}\cos\theta_2)g_2(\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2)) \\
&\leq (\frac{1}{2} + \frac{1}{2}\cos\theta_1)(\lambda g_1(\boldsymbol{x}_1) + (1-\lambda)g_1(\boldsymbol{x}_2)) + (\frac{1}{2} + \frac{1}{2}\cos\theta_2)(\lambda g_2(\boldsymbol{x}_1) + (1-\lambda)g_2(\boldsymbol{x}_2)) \\
&\leq \lambda G_\alpha(\boldsymbol{x}_1) + (1-\lambda)G_\alpha(\boldsymbol{x}_2)
\end{aligned}$$

For $g_1(\boldsymbol{x})\cos\theta_1 + g_2(\boldsymbol{x})\cos\theta_2 \leq 0$,

$$\begin{aligned}
G_\alpha(\boldsymbol{x}) &= \frac{1}{2}g_1(\boldsymbol{x}) + \frac{1}{2}g_2(\boldsymbol{x}) + \frac{1}{2}|g_1(\boldsymbol{x})\cos\theta_1 + g_2(\boldsymbol{x})\cos\theta_2| \\
G_\alpha(\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) &= (\frac{1}{2} - \frac{1}{2}\cos\theta_1)(g_1(\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) + (\frac{1}{2} - \frac{1}{2}\cos\theta_2)g_2(\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2)) \\
&\leq (\frac{1}{2} - \frac{1}{2}\cos\theta_1)(\lambda g_1(\boldsymbol{x}_1) + (1-\lambda)g_1(\boldsymbol{x}_2)) + (\frac{1}{2} - \frac{1}{2}\cos\theta_2)(\lambda g_2(\boldsymbol{x}_1) + (1-\lambda)g_2(\boldsymbol{x}_2)) \\
&\leq \lambda G_\alpha(\boldsymbol{x}_1) + (1-\lambda)G_\alpha(\boldsymbol{x}_2)
\end{aligned}$$

Therefore, $G_\alpha(\boldsymbol{x})$ is convex.
For $\alpha \in (-1, 0), cos\theta = -\alpha, \theta \in [0, \frac{1}{2}\pi]$, , $G_\alpha(\boldsymbol{x})$ is still convex.
(d)

$$\begin{aligned}
&\alpha = 1 \\
G_\alpha(\boldsymbol{x}) &= \frac{1}{2}g_1(\boldsymbol{x}) + \frac{1}{2}g_2(\boldsymbol{x}) + \frac{1}{2}\sqrt{g_1^2(\boldsymbol{x}) + g_2^2(\boldsymbol{x}) - 2g_1(\boldsymbol{x})g_2(\boldsymbol{x})} \\
&= \frac{1}{2}g_1(\boldsymbol{x}) + \frac{1}{2}g_2(\boldsymbol{x}) + \frac{1}{2}\sqrt{(g_1(\boldsymbol{x}) - g_2(\boldsymbol{x}))^2} \\
&= \max[g_1, g_2]
\end{aligned}$$

$$\begin{aligned}
G_\alpha(\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) &= \max[g_1(\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2), g_2(\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2)] \\
&\leq \max[\max[g_1(\boldsymbol{x}_1), g_1(\boldsymbol{x}_2)], \max[g_2(\boldsymbol{x}_1), g_2(\boldsymbol{x}_2)]] \\
&= \max[G(\boldsymbol{x}_1), G(\boldsymbol{x}_1)]
\end{aligned}$$

(e) For $g_1(x) = -x^2$, $g_2(x) = -(x-1)^2$, $\max[g_1, g_2] \leq 0$ if set $\max[g_1, g_2] \geq 0$, we set $\max[g_1, g_2] = 0$, meaning $x = 0 or 1$.

4.50

(a) $\boldsymbol{\eta}(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{x}_2 - \boldsymbol{x}_1$
(b)

$$\begin{aligned}
&\min f(\boldsymbol{x}) \\
&\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0} \\
&\boldsymbol{x} \in \boldsymbol{X}
\end{aligned}$$

KKT condition:

$$\begin{aligned}\nabla f(\bar{\boldsymbol{x}}) + \boldsymbol{\mu}^T \nabla \boldsymbol{g}(\bar{\boldsymbol{x}}) &= \mathbf{0} \\ \boldsymbol{\mu}^T \boldsymbol{g}(\bar{\boldsymbol{x}}) &= \mathbf{0} \\ \boldsymbol{\mu} &\geq \mathbf{0}\end{aligned}$$

For $i \in I = \{i : g_i(\boldsymbol{x}) = 0\}$, $\mu_i > 0$. For $j \in J = \{j : g_j(\boldsymbol{x}) > 0\}$, $\mu_j = 0$.
For feasible solutions:

$$\begin{aligned}g_i(\bar{\boldsymbol{x}}) &= 0 \\ \Rightarrow g_i(\bar{\boldsymbol{x}}) &\geq g_i(\boldsymbol{x}) \\ \Rightarrow \nabla g_i(\bar{\boldsymbol{x}})^T \boldsymbol{\eta}_i(\boldsymbol{x}, \bar{\boldsymbol{x}}) &\leq 0 \\ \Rightarrow \nabla \boldsymbol{g}(\bar{\boldsymbol{x}})^T \boldsymbol{\eta}(\boldsymbol{x}, \bar{\boldsymbol{x}}) &\leq 0 \\ \Rightarrow \nabla f(\bar{\boldsymbol{x}})^T \boldsymbol{\eta}_f(\boldsymbol{x}, \bar{\boldsymbol{x}}) &\geq 0 \ (\because \nabla f(\boldsymbol{x}) + \mu_i \nabla \boldsymbol{g}_i(\boldsymbol{x}) + \cdots = \mathbf{0}) \\ \Rightarrow f(\bar{\boldsymbol{x}}) &\leq f(\boldsymbol{x})\end{aligned}$$

(c) same as (b).

5.22

(a)

$$\begin{aligned}T &= cl\{a(\boldsymbol{x} - \bar{\boldsymbol{x}}) : \boldsymbol{x} \in \{\boldsymbol{x} : \boldsymbol{g}(\boldsymbol{x}) \leq \mathbf{0}\}\} \\ T_* &= \{\boldsymbol{y} : \boldsymbol{y}(\boldsymbol{x} - \bar{\boldsymbol{x}}) \geq 0 : \boldsymbol{x} \in \{\boldsymbol{x} : \boldsymbol{g}(\boldsymbol{x}) \leq \mathbf{0}\}\} \\ \Xi &= \{\boldsymbol{y} : \boldsymbol{y} = \nabla f(\bar{\boldsymbol{x}}), \nabla f(\bar{\boldsymbol{x}})(\boldsymbol{x} - \bar{\boldsymbol{x}}) \geq 0 : \boldsymbol{x} \in \{\boldsymbol{x} : \boldsymbol{g}(\boldsymbol{x}) \leq \mathbf{0}\}\} \\ \therefore T_* &= \Xi\end{aligned}$$

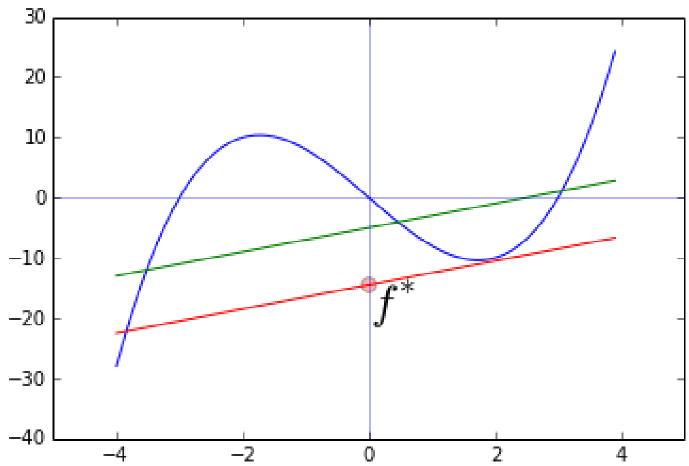
(b) KKT condition:

$$\begin{aligned}\nabla f(\bar{\boldsymbol{x}}) + \boldsymbol{\mu}^T \nabla \boldsymbol{g}(\bar{\boldsymbol{x}}) &= \mathbf{0} \\ \boldsymbol{\mu}^T \boldsymbol{g}(\bar{\boldsymbol{x}}) &= \mathbf{0} \\ \boldsymbol{\mu} &\geq \mathbf{0}\end{aligned}$$

$$\begin{aligned}\text{if } D\Xi = G'_*, \\ \nabla f(\bar{\boldsymbol{x}})^T (\boldsymbol{x} - \bar{\boldsymbol{x}}) \geq 0 \text{ for all } \nabla g_i(\bar{\boldsymbol{x}})^T (\boldsymbol{x} - \bar{\boldsymbol{x}}) \leq 0 \\ \Rightarrow \nabla f(\bar{\boldsymbol{x}}) + \mu_i^T \nabla \boldsymbol{g}_i(\bar{\boldsymbol{x}}) + \cdots = \mathbf{0} \text{ for } g_i(\bar{\boldsymbol{x}}) = 0,\ \mu_i \geq 0 \\ \text{set } \boldsymbol{\mu}^T \boldsymbol{g}(\bar{\boldsymbol{x}}) = \mathbf{0} \\ \mu_i = \mathbf{0} \text{ for } g_i(\bar{\boldsymbol{x}}) < 0 \\ \therefore \begin{cases}\nabla f(\bar{\boldsymbol{x}}) + \boldsymbol{\mu}^T \nabla \boldsymbol{g}(\bar{\boldsymbol{x}}) = \mathbf{0} \\ \boldsymbol{\mu}^T \boldsymbol{g}(\bar{\boldsymbol{x}}) = \mathbf{0} \\ \boldsymbol{\mu} \geq \mathbf{0}\end{cases}\end{aligned}$$

6.46

(a) For a given μ as the slide of the lines, f^* is the minimum of the interception.



(b)

$$
\begin{aligned}
& f^*(\lambda\boldsymbol{\mu}_1 + (1-\lambda)\boldsymbol{\mu}_2) \\
&= \inf\{f(\boldsymbol{x}) - \lambda\boldsymbol{\mu}_1^T\boldsymbol{x} + (1-\lambda)\boldsymbol{\mu}_2^T\boldsymbol{x}\} \\
&= \inf\{f(\boldsymbol{x}) - \lambda\boldsymbol{\mu}_1^T\boldsymbol{x} + (1-\lambda)\boldsymbol{\mu}_2^T\boldsymbol{x}\} \\
&= \inf\{\lambda(f(\boldsymbol{x}) - \boldsymbol{\mu}_1^T\boldsymbol{x}) + (1-\lambda)(f(\boldsymbol{x}) - \boldsymbol{\mu}_2^T\boldsymbol{x})\} \\
&\geq \inf\{\lambda\inf(f(\boldsymbol{x}) - \boldsymbol{\mu}_1^T\boldsymbol{x}) + (1-\lambda)\inf(f(\boldsymbol{x}) - \boldsymbol{\mu}_2^T\boldsymbol{x})\} \\
&= \inf\{\lambda f^*(\boldsymbol{\mu}_1) + (1-\lambda)f^*(\boldsymbol{\mu}_2)\} \\
&\therefore f^*(\lambda\boldsymbol{\mu}_1 + (1-\lambda)\boldsymbol{\mu}_2) > \inf\{\lambda f^*(\boldsymbol{\mu}_1) + (1-\lambda)f^*(\boldsymbol{\mu}_2)\} \\
&\therefore f^* \text{ is concave.}
\end{aligned}
$$

same method for $g_*(\mu)$

(c)

$$
\begin{aligned}
& \inf\{f(\boldsymbol{x}) - g(\boldsymbol{x}) : \boldsymbol{x}\} \\
&= \inf\{f(\boldsymbol{x}) - \boldsymbol{\mu}^T\boldsymbol{x} - (g(\boldsymbol{x}) - \boldsymbol{\mu}^T\boldsymbol{x}) : \boldsymbol{x}\} \\
&\geq \inf\{f(\boldsymbol{x}) - \boldsymbol{\mu}^T\boldsymbol{x} : \boldsymbol{x}\} + \inf\{-(g(\boldsymbol{x}) + \boldsymbol{\mu}^T\boldsymbol{x}) : \boldsymbol{x}\} \text{ stands for all } \boldsymbol{\mu} \\
&\therefore \inf\{f(\boldsymbol{x}) - g(\boldsymbol{x}) : \boldsymbol{x}\} \geq \sup\{\inf\{f(\boldsymbol{x}) - \boldsymbol{\mu}^T\boldsymbol{x} : \boldsymbol{x}\} - \sup\{\boldsymbol{\mu}^T\boldsymbol{x}) - (g(\boldsymbol{x}) : \boldsymbol{x}\}\} \\
& \inf\{f(\boldsymbol{x}) - g(\boldsymbol{x}) : \boldsymbol{x}\} \geq \sup\{f^*(\boldsymbol{\mu}) - g^*(\boldsymbol{\mu})\}
\end{aligned}
$$

(d,e)

$$
\begin{aligned}
\min_x f(\boldsymbol{x}) - g(\boldsymbol{x}) \\
-f(\boldsymbol{x}) + \boldsymbol{\mu}^T\boldsymbol{x} + f^*(\boldsymbol{\mu}) \leq 0 \\
g(\boldsymbol{x}) - \boldsymbol{\mu}^T\boldsymbol{x} - g^*(\boldsymbol{\mu}) \leq 0
\end{aligned}
$$

Dual:

$$\begin{aligned}
&\max \theta(\boldsymbol{v}) \\
&\boldsymbol{v} \geq 0 \\
&\theta(\boldsymbol{v}) = \inf\{f(\boldsymbol{x}) - g(\boldsymbol{x}) + \boldsymbol{v}_1(-f(\boldsymbol{x}) + \boldsymbol{\mu}^T\boldsymbol{x} + f^*(\boldsymbol{\mu})) + \boldsymbol{v}_2(g(\boldsymbol{x}) - \boldsymbol{\mu}^T\boldsymbol{x} - g^*(\boldsymbol{\mu})) : \boldsymbol{x}\} \\
&\theta(\boldsymbol{v}) = v_1 f^*(\boldsymbol{\mu}) - v_2 g^*(\boldsymbol{\mu}) + \inf\{(v_1 - 1)(-f(\boldsymbol{x}) + \boldsymbol{\mu}^T\boldsymbol{x}) + (v_2 - 1)(g(\boldsymbol{x}) - \boldsymbol{\mu}^T\boldsymbol{x})) : \boldsymbol{x}\} \\
&\max \theta(\boldsymbol{v}) = f^*(\boldsymbol{\mu}) - g^*(\boldsymbol{\mu}), \\
&\inf(f(\boldsymbol{x}) - g(\boldsymbol{x})) = \sup(f^*(\boldsymbol{\mu}) - g^*(\boldsymbol{\mu}))(\because f - g, f^* - g^* \text{ are convex, Strong Duality Theorem})
\end{aligned}$$

7.24

(a)

$$\begin{aligned}
\min\ & -3x_1 - 2x_2 \\
s.t. & -x_1^2 + x_2 + 1 \leq 0 \\
& 2x_1 + 3x_2 \leq 6 \\
& x_1, x_2 \geq 0
\end{aligned}$$

$$\begin{aligned}
\min\ & -3x_1 - 2x_2 \\
s.t. & -x_1^2 + x_2 + 1 \leq 0 \\
& \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix} \boldsymbol{x} \leq \begin{bmatrix} 6 \\ 0 \end{bmatrix}
\end{aligned}$$

```
Initial:
Z_1=Polyhedron({0,2},({0,-2},{1.5,1})
Step 1:
linear programming result: x_1={1.5,1}
g(x_1)>0
Step 2:
barx_1={1.430,1.046}
Z_2=Polyhedron({0,2},({0,-2},{1.2139,0.4279},{1.430,1.046})
Step 1:
linear programming result: x_2={1.430,1.046}
g(x_2)<=0
Result:{1.430,1.046}
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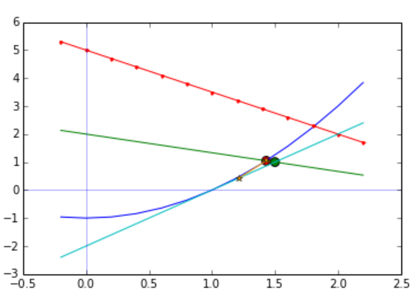

(b)B should be pseudo-convex to facilitate slacking into linear programming problem.
(c)

1. $\{\boldsymbol{x}\}$ $\{\bar{\boldsymbol{x}}\}$ are in the R^q, R^r respectively.

2. For all Z, if $\boldsymbol{x} \in \boldsymbol{f}(Z)$, $\boldsymbol{x} \in Z$

3. G is a closed map

4. Given $\boldsymbol{x} \notin \{g(x) \leq 0\} and Z, where \boldsymbol{x} \in \boldsymbol{f}(Z), \bar{\boldsymbol{x}} \in g(x) \leq 0$, implying that $x \notin \nabla g(\bar{\boldsymbol{x}})(\boldsymbol{x} - \bar{\boldsymbol{x}}) \geq 0$ and $Z \cap \{\boldsymbol{x} : \nabla g(\bar{\boldsymbol{x}})(\boldsymbol{x} - \bar{\boldsymbol{x}}) \geq 0\} \neq \emptyset$

For any k, we have $\nabla g(\bar{\boldsymbol{x}}_{\boldsymbol{k}})(\boldsymbol{x}_l - \bar{\boldsymbol{x}}_{\boldsymbol{k}}) \geq 0$, $l \geq k+1$ Before we end the program, we would have $\nabla g(\bar{\boldsymbol{x}})(\boldsymbol{x} - \bar{\boldsymbol{x}}) \geq 0$.
$\therefore \boldsymbol{x} \in \{g(x) \leq 0\}$, it is the optimal.

8.60

$\bar{\boldsymbol{x}} = (-2, 3, 1, 2)^T$, $X = \{\boldsymbol{x} : \boldsymbol{x}^T \mathbf{1} = 1, \mathbf{0} \leq \boldsymbol{x} \leq \mathbf{1}\}$.
Initialization: set $(\bar{\boldsymbol{x}}^{\mathbf{0}}, \boldsymbol{I}^0, \boldsymbol{l}^0, \boldsymbol{\mu}^0, \beta^0) = ((-2,3,1,2)^T, \{1,2,3,4\}, \mathbf{0}, \mathbf{1}, 1)$
Step 1:

$$\hat{x}_i^0 = \bar{x}_i^0 + \frac{1-4}{4} * 1$$
$$\hat{\boldsymbol{x}}^0 = \bar{\boldsymbol{x}}^0 - 3/4\mathbf{1} = (-2.75, 2.25, 0.25, 1.25)$$

Step 2:

$$\gamma = 1 + 2.75 + 1.5 = 5.25 > \beta$$
$$J_3 = \{1\}, J_4 = \emptyset$$
$$x_1^* = 0$$
$$I^2 = 2, 3, 4$$
$$(\bar{\boldsymbol{x}}^{\mathbf{1}}, \boldsymbol{I}^1, \boldsymbol{l}^1, \boldsymbol{\mu}^1, \beta^1) = ((3,1,2)^T, \{2,3,4\}, \mathbf{0}, (1, 0.25, 1), 1)$$

Step 1:

$$\hat{\boldsymbol{x}}^1 = (3,1,2)^T + \frac{1-6}{3} * 1 = (\frac{4}{3}, -\frac{2}{3}, \frac{1}{3})$$

Step 2:

$$\gamma_1 = 1 + \frac{2}{3} + \frac{1}{3} = 2 > \beta$$
$$J_3 = \{3\}, J_4 = \emptyset$$
$$x_3^* = 0$$
$$I^1 = 2, 4$$
$$(\bar{\boldsymbol{x}}^{\mathbf{2}}, \boldsymbol{I}^2, \boldsymbol{l}^2, \boldsymbol{\mu}^2, \beta^2) = ((\frac{4}{3}, \frac{1}{3})^T, \{2,4\}, \mathbf{0}, (1, \frac{1}{3}), 1)$$

Step 1:

$$\hat{\boldsymbol{x}}^2 = (\frac{4}{3}, \frac{1}{3})^T - \frac{1}{3}\mathbf{1} = (1, 0)$$

Result: $x^* = (0, 1, 0, 0)$